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Partial Automorphisms and the Reconstruction of Bi-degreed Graphs

W. L. Kocay

ABSTRACT We examine the reconstruction of graphs with only two degrees, k and k-1. We use a method of partial automorphisms [2] to show that in a graph G, if k=3 and there are only two vertices of degree two, then G is in most cases reconstructible. In the cases for which we cannot prove the reconstructibility of G, we determine much of the structure of G.

Although this is a very special case, the method is sufficiently general that it can likely be applied to many other situations.

1. Introduction.

The reconstruction of graphs with only two degrees, k and k-1, is a long-outstanding problem in reconstruction theory [1], and it is often thought that if the reconstruction conjecture is false, then this is where counter-examples are likely to be found.

In this paper, we use a technique of Godsil and Kocay [2] to show that in a graph G, if k=3 and there are only two vertices of degree two, then G is in most cases reconstructible. In the case for which we cannot prove the reconstructibility of G, we determine much of the structure of G.

Let G be a simple graph and let $u,v \in V(G)$. We denote an edge joining u and v by uv. Thus u and v are adjacent in G if $uv \in E(G)$, and non-adjacent if $uv \notin E(G)$. If $uv \notin E(G)$, we write $G+uv$ for the graph obtained from G by adding the edge uv. Similarly if $uv \in E(G)$, we can remove it to get $G-uv$.

We may also delete one or more vertices of G to get vertex-deleted subgraphs of G. In this case we write $G-u$ when one vertex is deleted, and $G-\{u,v\}$ when several are deleted.

Let G and H be graphs with all vertices of degree three but two, which have degree two, and suppose that G and H are reconstructions of each other, i.e., there exists a <u>hypomorphism</u> $\phi: V(G) \to V(H)$, a bijection such that $G-u \cong H-\phi(u)$, for every $u \in V(G)$.

Let $u,v \in V(G)$ be vertices of degree two. Then $G+uv$ is a 3-regular graph. We show that H is isomorphic to a subgraph of $G+uv$, and that $G+uv$ has a non-trivial automorphism p. In many cases $p^{\ell}(G) \cong H$, for some ℓ, so that $G \cong H$, proving that G is in fact reconstructible.

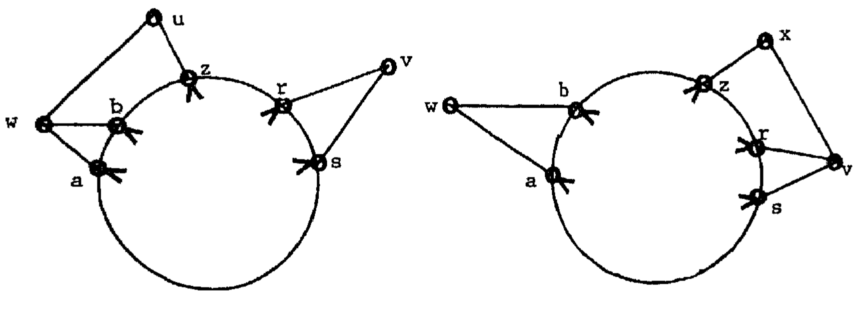
This is an illustration of what seems to be a general technique: given graphs G and H which are reconstructions of each other, embed them both into a larger graph which has a non-trivial automorphism group. Use the symmetries of the larger graph to show that in fact $G \cong H$, thereby proving that G is reconstructible.

2. The Isomorphism p.

Now let G and H be as above and suppose that $u \in V(G)$ and $x \in V(H)$ are vertices of degree two such that $G-u \cong H-x$. Without loss of generality, we can use the isomorphism $G-u \cong H-x$ to give a common labelling to $G-u$ and $H-x$, and take $G-u = H-x$, so that $V(G-u) = V(H-x) = V$.

Let $v \in V$ be such that $\deg_G v = 2$. If $uv \in E(G)$, then G is easily obtained from $G-u$ by rejoining u to the unique vertices of degree one and two in $G-u$, since $\deg_G u = 2$. Thus we can assume that $uv \notin E(G)$.

Let $uw, uz \in E(G)$, where $v \neq w \neq z \neq v$. Since there are only two vertices of degree two in G, it must be that H is formed from $H-x$ $(=G-u)$ by joining x to v and z, say. Then $\deg_H w = \deg_G v = 2$, and $G-v \cong H-w$, since G and H are reconstructions of each other. Thus H is obtained from G by altering just one edge. This is depicted in the following illustration.



G H

Figure 2.1

Obviously $G+uv \cong H+xw$. We will show that in fact $G+uv$ has a non-trivial automorphism that will often guarantee that $G \cong H$.

We have labelled the vertices adjacent to u, w, x, and v by the letters a, b, r, s, and z. Although we must have $a \neq b$ and $r \neq s$, we do not discount the possibility that $a=z$, $r=z$, $b=r$, etc. However we can always assume that $a \neq r$ and $b \neq s$.

It will be convenient to work with the graph $\Gamma = G \cup H$, illustrated below. Without loss of generality, we can assume that Γ is connected.

Γ:

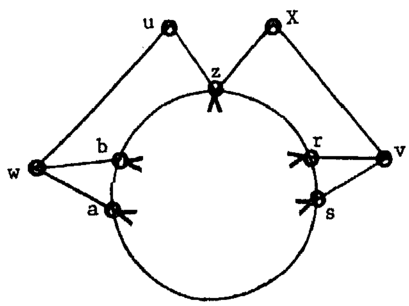


Figure 2.2

Let p: $G\text{-}v \rightarrow H\text{-}w$ be an isomorphism. Notice that p maps $V(\Gamma)-\{x,v\}$ to $V(\Gamma)-\{u,w\}$. Since p is one-to-one and onto, this immediately gives the following.

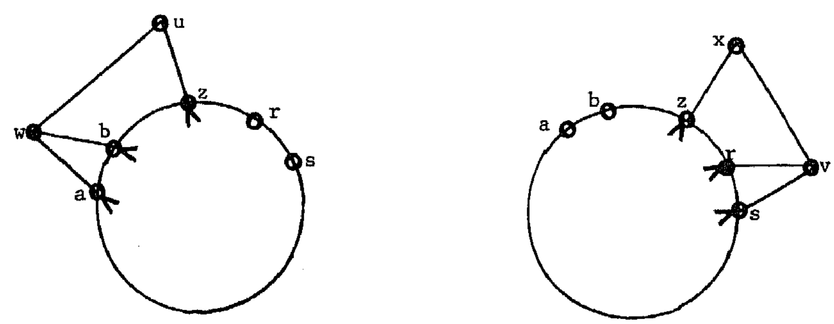
<u>2.1. Lemma</u>. There exist positive integers k_1 and k_2 such that either

i) $p^{k_1}(u) = v$ and $p^{k_2}(w) = x$; or

ii) $p^{k_1}(u) = x$ and $p^{k_2}(w) = v$.

<u>Proof</u>. Consider $p(u)$. Either $p(u) \in \{x,v\}$, or we can find $p^2(u)$. We continue like this, iterating p, until we find $p^{k_1}(u) \in \{x,v\}$ for some positive integer k_1. Similarly $p^{k_2}(w) \in \{x,v\}$. □

G–v and H–w are illustrated below.



G–v H–w

<u>Figure 2.3</u>

Now the only vertices of degree two in $G-v$ are u,r, and s, and those in $H-w$ are a,b, and x. This gives the following.

2.2. Lemma $\quad p\{u,r,s\} = \{x,a,b\}$.

We consider the two cases of Lemma 2.1 separately.

3. $p^{k_1}(u)=v$ and $p^{k_2}(w)=x$

We assume throughout this section that $p^{k_1}(u)=v$ and $p^{k_2}(w)=x$. We prove that in this case we must always have $G \cong H$, so that G is reconstructible. The proof proceeds by a sequence of lemmas.

3.1. Lemma $\quad k_1 \le k_2 + 1$.

Proof. Suppose that $k_1 \ge k_2 + 2$. First note that $k_1 \ne 1$, so that $p(u) \ne v$. By Lemma 2.2 we can take $p(u) = a \ne v$. We know that $uw \in E(G-v)$, so that $p^{k_2}(uw) \in E(\Gamma)$, by the definition of p. But $p^{k_2}(uw) = p^{k_2}(u)x$. Now $p^{k_2}(u) \ne p^{k_1}(u)=v$. It follows that $p^{k_2}(u)=z$, since x is adjacent only to v and z. However we also know that $wa \in E(G-v)$ so that $p^{k_2}(wa) \in E(\Gamma)$. But $p^{k_2}(wa) = xp^{k_2+1}(u)$. Thus x is adjacent to three distinct vertices: $z=p^{k_2}(u)$, $p^{k_2+1}(u)$, and $v=p^{k_1}(u)$. This is a contradiction, since $\deg_\Gamma x = 2$. Thus $k_1 \le k_2 + 1$. $\square$

It will be convenient to denote the sets $\{u,p(u),p^2(u),\ldots,v=p^{k_1}(u)\}$ and $\{w,p(w),p^2(w),\ldots,x=p^{k_2}(w)\}$ by $[u,v]$ and $[w,x]$, respectively. These sets have a natural order specified by p, and we shall use this order to take smaller intervals, such as $[u,p^{k_1-1}(u)]$, $[p(u),x]$, etc.

3.2. Lemma $\quad$ If $k_1 = k_2+1$, then $V(\Gamma)=[u,v]\cup[w,x]$.

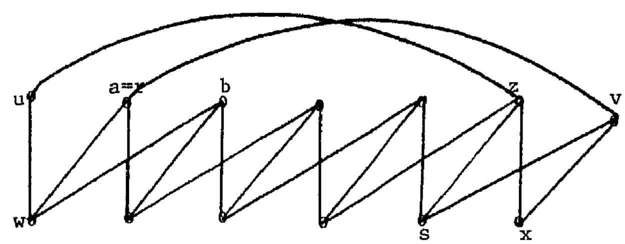
Proof. By the proof of Lemma 3.1, we see that $z=p^{k_2}(u)$, so that $p(z)=v$. Note that the following sequence of edges are all edges of Γ: uw, $p(uw)$, $p^2(uw),\ldots$, $p^{k_2}(uw)=zx$. These edges match the vertices of

$[w,x]$ to those of $[u,z]$. Similarly the following are all in $E(\Gamma)$: wa, $p(wa)$, $p^2(wa)$, $\ldots$, $p^{k_2}(wa) = xv$. They match the vertices of $[w,x]$ to $[a,v]$. This accounts for two edges incident with each vertex of $[a,z]$ and $[w,x]$, and one edge with u and v.

By Lemma 2.2 we can take $p(s)=x$. (this includes the possibility that $w=s$). Now $vs \in E(\Gamma)$, so that the following sequence of edges all belong to $E(\Gamma)$: vs, $p^{-1}(vs)$, $p^{-2}(vs), \ldots$, $p^{-k_2+1}(vs) = p^2(u)w$. This matches the vertices $[w,s]$ to $[p^2(u),v]$, so that every vertex now has degree three except for a and v, which have degree two, and u, which has degree one. But $uz \in E(\Gamma)$, so that $p(uz)=av \in E(\Gamma)$. This gives a and v degree three, u degree two, and z degree four, which accounts for all the edges of Γ. Therefore $V(\Gamma)=[u,v]\cup[w,x]$. □

Γ is drawn below in the case $k_1=6$ and $k_2=5$. p is represented by a shift to the right. Notice that when $k_1=k_2+1$, we must have $a=r$, since $wp(a)=wp^2(u) \in E(\Gamma)$. Therefore $p(a)=b$, so that $a=r$, by Lemma 2.2.

Γ:



$k_1=6$, $k_2=5$

Figure 3.1

We now define θ: $G+uv \to G+uv$, an extension of p, as follows: If $y \in V(G)\setminus\{v,s\}$, then $\theta(y)=p(y)$. We also set $\theta(v)=w$ and $\theta(s)=u$.

3.3. Lemma θ is an automorphism of $G+uv$.

Proof. Since p is an isomorphism, we need only check the vertices v and s. In $G+uv$, v is adjacent to u,s, and $r(=a)$. Thus we must have $\theta(v)=w$ adjacent to $\theta(u)=p(u)=a$, $\theta(s)=u$, and $\theta(r)=p(r)=b$. This is indeed the case.

Similarly s is adjacent to $p^{-1}(z),z$, and v in $G+uv$. We must have $\theta(s)=u$ adjacent to $\theta(p^{-1}(z))=z$, $\theta(z)=p(z)=v$, and $\theta(v)=w$, which shows that θ is an automorphism as required. □

3.4. Theorem If $k_1 = k_2 + 1$ is odd, then $G+uv$ is a prism. If k_1 is even, then $G+uv$ is a Möbius ladder.

Proof. Let k_1 be odd. Consider the path given by the following sequence of adjacent vertices w, $p^2(u)$, $p^2(w)$, $p^4(u)$, ... Since k_1 is odd it must end in $p^{k_1-1}(u)=z$. We complete it to a circuit as follows: z, u, w. The remaining vertices also form a circuit: $p(w)$, $p^3(u)$, $p^3(w)$, $p^5(u)$, ..., $p^{k_1}(u)=v$, $r=p(u)$, $p(w)$. The circuits have the same length $(k_1-1)/2$ and corresponding vertices are matched by the edges $wp(u)$, $p^2(w)p^3(u)$,... to form a prism.

If k_1 is even, then the sequence $w,p^2(u)$, $p^2(w)$, $p^4(u)$,... ends in $p^{k_1}(u)=v$. We continue it as follows: v, $r=p(u)$, $p(w)$, $p^3(u)$, $p^3(w)$,...,z,u,w. Instead of two circuits with corresponding vertices matched, we have altered two edges in the circuits to make one long circuit, i.e., we now have a Möbius ladder. □

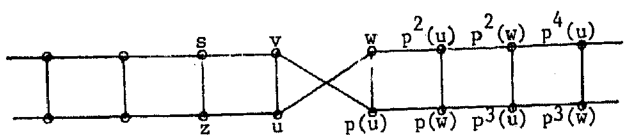
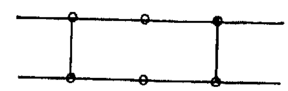


Figure 3.2.

It is clear from Figure 3.2 that $G+uv \cong H+xw$, and that the isomorphism takes u to x. Thus $G = (G+uv)-uv$ and $H \cong (G+uv)-uw$. However, by Theorem 3.4 we see that the edges uv and uw lie in different orbits of the group generated by θ: uv is one of the "matching" edges and uw is one of the "circuit" edges. Thus $G \not\cong H$ But then it is easy to see that G and H are not in fact reconstructions of each other.

<u>3.5. Theorem</u> If $k_1 = k_2+1$, then G is reconstructible.

<u>Proof.</u> Since G is got by removing a "matching" edge of $G+uv$, most vertex deleted subgraphs of G will have an induced subgraph isomorphic to a 6-cycle.



<u>Figure 3.3</u>

No vertex-deleted subgraph of H will have this induced subgraph. But G and H were supposed to be reconstructions of each other. $\square$

<u>3.6. Lemma</u> $k_1 \geq k_2-1$.

<u>Proof.</u> Suppose that $k_1 \leq k_2-2$. First note that $k_2 \geq 3$, so that $p(w) \neq x$. By Lemma 2.2 we can take $p(s)=x$, where $p^{k_2-1}(w) = s \neq w$. We know that vs, $vx \in E(\Gamma)$, so that $p^{-k_1}(vs)$, $p^{-k_1}(vx) \in E(\Gamma)$, i.e., $u\,p^{k_2-k_1-1}(w)$, $u\,p^{k_2-k_1}(w) \in E(\Gamma)$. But $uw \in E(\Gamma)$, too, so that $\deg_\Gamma u \geq 3$, a contradiction. Therefore $k_1 \geq k_2-1$. $\square$

<u>3.7. Theorem</u> If $k_1 = k_2-1$, then G is reconstructible.

Proof. This is really the same case as Lemma 3.2. and Theorem 3.4. Simply substitute u for x, $p(u)$ for $p^{-1}(x)$, $p^2(u)$ for $p^{-2}(x)$, etc. Substitute p^{-1} for p, and interchange k_1 and k_2 to get the same situation. By Theorem 3.5., G is reconstructible. □

We have still to consider the case $k_1 = k_2$. We define $V' = V(\Gamma)\backslash\{[u,v]\cup[w,x]\}$. By Lemma 2.2 we can assume that $p(u) = a$, and $p(s) = x$, where we include the possiblity that $a=v$ or $w=s$.

3.8. Lemma If $k_1 = k_2$, then $z \in V'$.

Proof. Suppose that $z \in [u,v]$. Write $z = p^{\ell}(u)$ where $1 \le \ell < k_1$. Now $xz \in E(\Gamma)$ so that $p^{-\ell}(xz)=p^{-\ell}(x)u \in E(\Gamma)$. Now u is adjacent only to w and z, and $p^{-\ell}(x) \neq z$. Therefore $p^{-\ell}(x)=w$, or $\ell=k_1$, a contradiction. Therefore $z \notin [u,v]$. Similarly $z \notin [w,x]$. It follows that $z \in V'$. □

3.9. Lemma If $k_1 = k_2$ then $b,r \in V'$, $z = p^{k_1}(b)$, and $r = p^{k_1}(z)$.

Proof. First note that p maps V' onto itself. Hence p is an automorphism of the induced subgraph $\Gamma[V']$ of Γ. We know that $z \in V'$ and that $uz \in E(\Gamma)$. Hence $p^{\ell}(uz)\in E(\Gamma)$ for every ℓ such that $1 \le \ell \le k_1$. In particular $p^{k_1}(uz) = vp^{k_1}(z) \in E(\Gamma)$. It follows that $p^{k_1}(z)\in V'$ and that $p^{k_1}(z) = r$, since v is adjacent to x, r, and s, and $s, x \in [w,x]$.

Similarly $xz \in E(\Gamma)$ so that $p^{-\ell}(xz) \in E(\Gamma)$ for $1 \le \ell \le k_1$. But then $p^{-k_1}(xz) = w\, p^{-k_1}(z) \in E(\Gamma)$, where $p^{-k_1}(z) \in V'$. It follows that $b = p^{-k_1}(z)$, or $z = p^{k_1}(b)$, since w is adjacent to a, b, and u, and $a,u \in [u,v]$. □

The structure of Γ is now largely determined. We have the vertices $[w,x]$ matched to the vertices $[u,v]$ by the edges uw, $p(uw)$, $p^2(uw)$,... $p^{k_1}(uw)$. Also the vertices $[w,s]$ are matched to the vertices $[a,v]$ by the edges wa, $p(wa)$,... $p^{k_1-1}(wa)$. The vertices $[w,x]$ are matched to vertices in V' by the edges uz, $p(uz)$, $p^2(uz)$,...,

$p^{k_1}(uz) = vr$, by Lemma 3.9. This accounts for all edges incident with $[u,v] \cup [w,x]$. This situation is illustrated below in the case $k_1 = k_2 = 5$. p is represented by a shift to the right.

G:

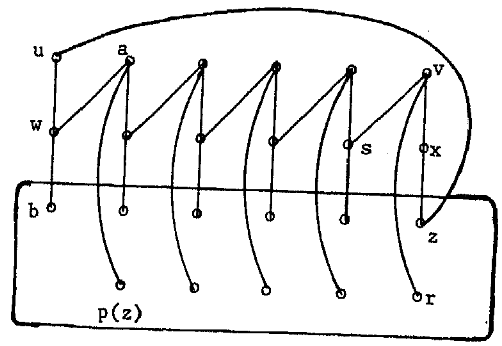


Figure 3.4

We now define a map $\theta : G + uv \to G + uv$ as before. If $y \in V(G) \setminus \{v,s\}$, then $\theta(y) = p(y)$. Also $\theta(v) = w$ and $\theta(s) = u$.

3.10 Lemma. θ is an automorphism of $G + uv$.

Proof. We need only check the vertices v and s. The vertices adjacent to v are u, r, and s. Thus, we must have $\theta(v) = w$ adjacent to $\theta(u) = p(u) = 1$, $\theta(r) = p(r) = b$, and $\theta(s) = u$. Similarly s is adjacent to $p^{-1}(v)$, v, and $p^{-1}(a)$. Thus $\theta(s) = u$ must be adjacent to $\theta(p^{-1}(v)) = v$, $\theta(v) = w$, and $\theta(p^{-1}(z)) = z$, which is the case. It follows that θ is an automorphism. □

The action of θ on $G + uv$ is illustrated in the diagram following. θ is represented by a clockwise rotation. The case $k_1 = k_2 = 4$ is shown.

G + uv :

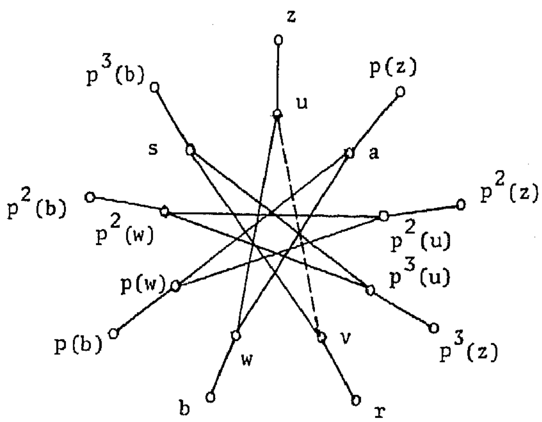


Figure 3.5

3.11 Theorem. If $k_1 = k_2$ then $G \cong H$.

We know that $G = (G+uv) - uv$ and that $H \cong (G+uv) - uw$, by Figure 2.2. Notice that $\theta^{k_1+1}(uv) = \theta^{k_1+1}(u)\theta^{k_1+1}(v) = \theta(v)\theta^{k_1}(w) = w\theta(s) = wu$, so that uv and uw lie in the same orbit of the group generated by θ. Therefore $(G+uv) - uv \cong (G+uv) - uw$, or $G \cong H$, as required. □

The diagram makes it graphically clear that $G \cong H$. This works because $k_1 + k_2 + 1$ is an odd number. In Section 5, we shall see that sometimes $k_1 + k_2 + 1$ can be even, in which case θ is not sufficient to guarantee that $G \cong H$. We call the isomorphism $p: G - v \to H - w$ a partial automorphism of G+uv since it can be extended to an automorphism of G+uv.

Notice that the proof that G is reconstructible has relied on only two things: the degree sequence of G, and the two vertex-deleted subgraphs corresponding to the vertices of degree two. This was sufficient to force G to be reconstructible. Considering a third vertex-deleted subgraph would introduce another partial automorphism t. We expect that the combined symmetries of p and t would force $G \cong H$ in the remaining case, but this seems to be a difficult thing to do.

4. $p^{k_1}(u) = x$ and $p^{k_2}(w) = v$.

We assume throughout this section that $p^{k_1}(u) = x$ and $p^{k_2}(w) = v$. By Lemma 2.1 this is the remaining case to consider. We prove that except for a small number of special exceptions, we must have $k_1 = k_2 + 1$.

4.1 Lemma. $k_1 \leq k_2 + 2$. If $k_1 = k_2 + 2$, then $k_1 = 3$ and $k_2 = 1$.

Proof. Suppose that $k_1 \geq k_2 + 3$. Then $k_1 \geq 4$, and by Lemma 2.2, we can take $p(u) = a$, and $p(s) = x$. Now wu and $wa \in E(\Gamma)$ so that $p^{k_2}(wu) = vp^{k_2}(u) \in E(\Gamma)$ and $p^{k_2}(wa) = vp^{k_2+1}(u) \in E(\Gamma)$. But then v is joined to $p^{k_2}(u)$, $p^{k_2+1}(u)$, $s = p^{k_1-1}(u)$, and $x = p^{k_1}(u)$. If $k_1 \geq k_2 + 3$, this is impossible, since v is adjacent only to v, s, and x. Therefore $k_1 \leq k_2 + 2$. If $k_1 = k_2 + 2$, this is possible only if $s = p^{k_2+1}(u)$ and $r = p^{k_2}(u)$. By Lemma 2.2 this requires that $b = s = p^{k_2+1}(u)$. But since $vx \in E(\Gamma)$, we have $p^{-k_2}(vx) = wp^2(u) \in E(\Gamma)$. Thus w is adjacent to u, $a = p(u)$, and $p^2(u)$. It follows that $p^2(u) = b = s = p^{k_2+1}(u)$, or $k_2 = 1$ and $k_1 = 3$, as required. □

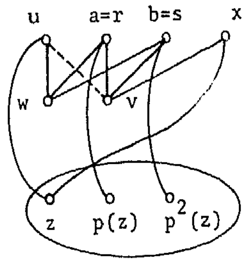
As in Section 3, we set $V' = V(\Gamma)\setminus\{[u,x]\cup[w,v]\}$.

4.2 Theorem. If $k_1 = 3$ and $k_2 = 1$, then $G \cong H$.

Proof. By the proof of Lemma 4.1, we have $p(u) = a = r$ and $p^2(u) = s = b$. We know that w is adjacent to u, a, and b, and v is adjacent to $a = r$, $b = s$, and x. Thus all edges adjacent to w and v are accounted for.

If $z = a$, then we have $a(=r)$ adjacent to u, x, w, and v. Furthermore $p(uz) = ab \in E(\Gamma)$. Thus $\deg_\Gamma z = 5$, a contradiction. Similarly $z \neq b$. It follows that $z \in V'$. Since $uz \in E(\Gamma)$, we have $ap(z)$, $bp^2(z) \in E(\Gamma)$, where $p(z)$, $p^2(z) \in V'$. But then $p^3(uz) =$

$xp^3(z) \in E(\Gamma)$, so that $p^3(z) = z$. This is illustrated below.



$k_1 = 3$ and $k_2 = 1$.

Figure 4.1

We define $\theta : G + uv \to G + uv$ as follows. If $y \in V(G)\backslash\{s,v\}$, then $\theta(y) = p(y)$. Also $\theta(s) = u$ and $\theta(v) = w$. It is easy to see that θ is an automorphism of $G + uv$. But $\theta^3(uv) = uw$. It follows that $G = (G + uv) - uv \cong (G + uv) - uw \cong H$, as required. □

By Lemma 4.1 and Theorem 4.2, it follows that we can take $k_1 \le k_2 + 1$. In Lemma 4.3 and Theorem 4.4, we show that we can also take $k_1 \ge k_2$.

4.3 Lemma. $k_1 \ge k_2 - 1$. If $k_1 = k_2 - 1$, then $k_1 = 1$ and $k_2 = 2$.

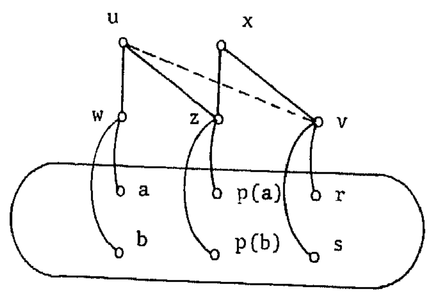
Proof. Suppose that $k_1 \le k_2 - 2$. Then $k_2 \ge 3$. Since $uw \in E(\Gamma)$, it follows that $p^{k_1}(uw) = xp^{k_1}(w) \in E(\Gamma)$. Therefore $p^{k_1}(w) = z$. If $k_1 \ge 2$, then we can take $p(u) = a$. Since $wa \in E(\Gamma)$, we have $p^{k_1-1}(wa) = p^{k_1-1}(w)x \in E(\Gamma)$. This is impossible, since x is adjacent only to v and $z = p^{k_1}(w)$. Hence, $k_1 = 1$, and $p(w) = z$.

Now $uz \in E(\Gamma)$ so that $p(uz) = xp^2(w) \in E(\Gamma)$. If $k_2 \ge 3$, then x is joined to $z = p(w)$, $p^2(w)$, and v, a contradiction. Therefore $k_2=2$, and the lemma follows. □

4.4 Theorem. If $k_1 = 1$ and $k_2 = 2$, then $G \cong H$.

Proof. By Lemma 4.3 we have $p(w) = z$ and $p(z) = v$. It is easy to see that $a, b, r, s \in V'$. We know that w is joined to a and b. Hence z is joined to $p(a), p(b) \in V'$ and v is joined to $p^2(a), p^2(b) \in V'$. Therefore $\{r,s\} = \{p^2(a), p^2(b)\}$, and $p\{r,s\} = \{a,b\}$.

If we define $\theta : G+uv \to G+uv$ by $\theta(u) = u$, $\theta(v) = w$, and $\theta(y) = p(y)$ for $y \in V(G)\backslash\{u,v\}$, then it is easy to see that θ is an automorphism of $G+uv$ for which $\theta(uv) = uw$, from which it follows that $G \cong H$. $\square$



$k_1 = 1$ and $k_2 = 2$.

Figure 4.2

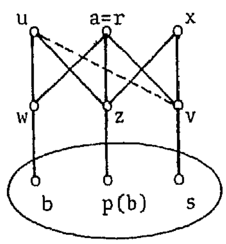
4.5 Lemma. If $k_1 = k_2$, the $k_1 \leq 2$.

Proof. If $k_1 = 2$, we can take $p(u) = a$ and $p^{-1}(x) = r$, by Lemma 2.2. Since $wa, vr \in E(\Gamma)$, we have $p^{k_1-1}(wa) = p^{-1}(v)x \in E(\Gamma)$ and $p^{-k_1+1}(vr) = p(w)\ u \in E(\Gamma)$. This requires that $p^{-1}(v) = z = p(w)$, which implies that $k_1 = k_2 = 2$. The lemma follows. $\square$

4.6 Theorem. If $k_1 = k_2$, then $G \cong H$.

Proof. By Lemma 4.5 there are two cases to consider. We take $k_1 = k_2 = 2$ first.

If $k_1 = k_2 = 2$, then by the proof of Lemma 4.5, we have $p(w)=z$, $p(z)=v$, and $p(u)=a=r$ and $p(r)=x$. This is illustrated below.



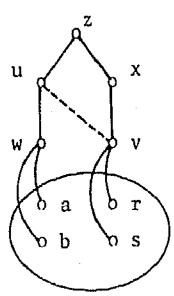
$k_1 = k_2 = 2$

Figure 4.3

We must have $b \in V'$. Since $wb \in E(\Gamma)$, we also have edges $zp(b)$ and $vp^2(b)$, where $p^2(b) = s$, and $p(s) = b$, since p maps V' to itself.

It is easily verified that $\theta: G+uv \to G+uv$, define by $\theta(r)=u$, $\theta(v)=w$, and $\theta(y)=p(y)$ for $y \in V(G)\setminus\{r,v\}$, is an automorphism of $G+uv$. Since $\theta^4(uv)=uw$, it follows as before that $G \cong H$.

The case $k_1 = k_2 = 1$ is illustrated below.



$k_1 = k_2 = 1$

Figure 4.4

We must have $z, a, b, r, s \in V'$. Since $uz \in E(\Gamma)$, we have $p(uz) = xp(z) \in E(\Gamma)$, or z is fixed by p. We also have $p\{a,b\} = \{r,s\}$ and $p\{r,s\} = \{a,b\}$. If we define $\theta: G+uv \to G+uv$ by $\theta(u)=u$, $\theta(v)=w$, $\theta(y)=y$ for $y \in V(G)\setminus\{u,v\}$, then θ is an automorphism of $G+uv$ such that $\theta(uv)=uw$. As before, we have $G \cong H$. □

5. $k_1 = k_2+1$

We have shown that if $k_1 \neq k_2+1$, then $G \cong H$, i.e., G is reconstructible. If $k_1 = k_2+1$, then we are not, in general, able to prove that $G \cong H$, using only the degree sequence and the two vertex-deleted subgraphs corresponding to the vertices of degree two. We can, however, determine quite a lot of the structure of G and H.

We assume throughout this section that $p^{k_1}(u) = x$ and $p^{k_2}(w) = v$, where $k_1 = k_2+1$.

We begin with some lemmas.

5.1 Lemma If $z \in [u,x]$, then k_1 is even and $z = p^{k_1/2}(u)$.

Proof. Write $z = p^{\ell}(u)$, where $1 \le \ell < k_1$. Since $uz, xz \in E(\Gamma)$, it follows that $p^{k_1-\ell}(uz) = p^{k_1-\ell}(u)x \in E(\Gamma)$. Therefore $p^{k_1-\ell}(u)=z=p^{\ell}(u)$, so that $k_1 = 2\ell$, as required. □

By Lemma 2.2 we can take $p(u)=a$ and $p(r)=x$. This includes the possibility that $a=r$. We can now take $p(s)=b$. Thus $s \in [u,x]$ if and only if b is.

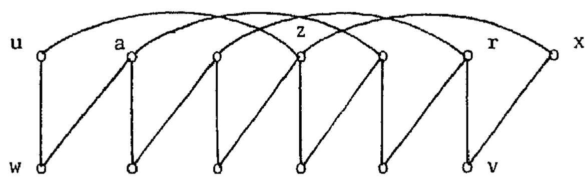
5.2 Lemma If $s,b \in [w,v]$ then k_1 is even and $s = p^{k_1/2-1}(w)$ and $b = p^{k_1/2}(w)$.

Proof. Write $b = p^{\ell}(w)$, where $1 \le \ell \le k_2$. Now $wb \in E(\Gamma)$ so that $p^{k_2-\ell}(wb) = p^{k_2-\ell}(w)v \in E(\Gamma)$. Therefore $s = p^{k_2-\ell}(w)$. But $s = p^{\ell-1}(w)$ so that $k_2-\ell = \ell-1$, or $k_1 = 2\ell$, as required. □

5.3 Lemma $z \in [w,v]$ if and only if $s,\ b \in [u,x]$.

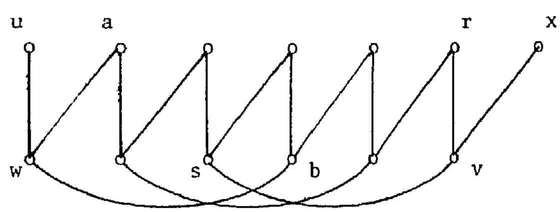
Proof. Write $z = p^{\ell}(w)$, where $1 \le w < k_2$. Now $uz \in E(\Gamma)$, so that $p^{k_2-\ell}(uz) = p^{k_2-\ell}(u)\ v \in E(\Gamma)$. Since $k_2 = k_1 - 1$, we know that $p^{k_2-\ell}(u) \neq r = p^{-1}(x)$, so that $s = p^{k_2-\ell}(u)$. Therefore $b = p(s) = p^{k_1-\ell}(u)$, i.e., we have shown that $z \in [w,v]$ implies that $s,b \in [u,x]$. The converse is similarly proved. □

The three situations considered by Lemmas 5.1, 5.2, and 5.3 are illustrated below.



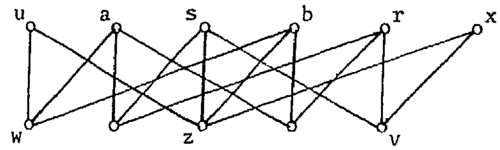
$z \in [u,x]$

Figure 5.1



$s,\ b \in [w,v]$

Figure 5.2



$z \in [w,v]$ and $s,\ b \in [u,x]$

Figure 5.3

As before we define $\theta: G+uv \to G+uv$ as follows. If $y \in V(G)\setminus\{r,v\}$, then $\theta(y) = p(y)$. Also $\theta(r)=u$ and $\theta(v)=w$.

5.4 Lemma θ is an automorphism of $G+uv$.

Proof. We need only check the vertices r and v. The vertices adjacent to r are always v, $p^{-1}(v)$, and $p^{-1}(z)$, since $p(r)=x$. Thus $\theta(r)=u$ must be adjacent to $\theta(v)=w$, $\theta(p^{-1}(v)) = v$, and $\theta(p^{-1}(z)) = z$. This is the case.

The vertices adjacent to v are always u, r, and s. Thus $\theta(v)=w$ must be adjacent to $\theta(u)=a$, $\theta(r)=u$, and $\theta(s)=b$. This is also the case, so that θ is an automorphism. □

The action of θ is most easily seen in the following diagram. We illustrate the case $z, s, b \in V'$, $k_1 = 6$ and $k_2 = 5$. θ is represented by a double clockwise rotation.

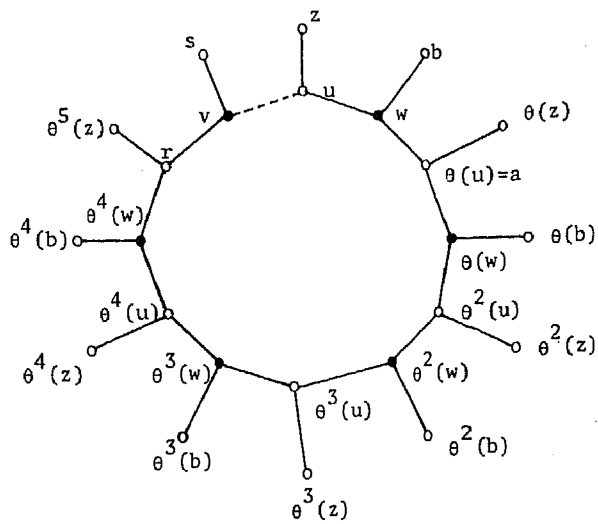


Figure 5.4

The graph $G+uv$ always contains a cycle of even length $2k_1$. The orbits of u and w under the group generated by θ form a bipartition of the cycle. But uv and uw fall into different orbits of edges. Unless the graph admits other symmetries, too, we cannot deduce that $G \cong H$.

Notice that the graph G satisfies $G-wa \cong G-r\theta^{-1}(v)$, and $G-\theta(w)\theta^2(u) \cong G-\theta^{-1}(r)\theta^{-2}(v)$, etc., i.e., it has pairs of edges which are likely pseudo-similar (see [2] and [3]). Similarly H satisfies $H-a\theta(w) \cong H-rv$, etc. Moreover the vertex-deleted subgraph $G'=G-u(=H-x)$ satisfies $G'-a \cong G'-r$, and $G'-\theta(a) \cong G'-\theta^{-1}(r)$, etc., i.e., it has pairs of vertices which are likely pseudo-similar ([2], [3]). Similarly, $H' = H-w$ ($\cong G-v$) satisfies $H'-v \cong H'-\theta(w)$, etc.

It is interesting to note that if G is a non-reconstructible graph, then both G and its vertex-deleted subgraphs are likely to have many pseudo-similar edges and/or vertices.

5.5 Theorem. If $z \in [u,x]$ and $s, b \in [w,v]$, then $G \cong H$.

Proof. First notice that $G+uv$ is a cycle of length $2k_1$, with all the main diagonals present, since $z = p^{k_1/2}(u)$ and $b = p^{k_1/2}(w)$, by Lemmas 5.1 and 5.2. It is then easy to see that uv and uw are similar edges in $G+uv$, since they are adjacent edges along the cycle. Thus $G = (G+uv)-uv \cong (G+uv)-uw \cong H$. □

In the case $z \in [w,v]$ and $s, b \in [u,x]$, we have $V(\Gamma) = [u,x] \cup [w,v]$. $G+uv$ is a bipartite graph consisting of a cycle of length $2k_1$, with some chords, which are not necessarily main diagonals, present. If we write $z = p^{\ell}(w)$, where $1 \le w < k_2$, then depending on the values of ℓ and k_1, we will sometimes have $G \cong H$, and sometimes $G \not\cong H$. The author does not immediately see how to prove G reconstructible in this case. Perhaps some such similar G is non-reconstructible.

In the preceding constructions, we have used the structure of the graph local to the vertices u, v, w, and x in order to prove reconstructibility. We have $G-u = H-x$ and $G-v \cong H-w$. We must also have the vertex-deleted subgraphs $\{G-w, G-z, G-r, G-s\}$ isomorphic to $\{H-v, H-z, H-a, H-b\}$ in some order. This would introduce another partial isomorphism t. Most probably the combined symmetries of p and t would enable us to prove that the graphs of this section are reconstructible. However this would require a more detailed structure of the graph local to a, b, r, and s, and this would result in a great multiplication of cases to be considered.

It seems to the author that the method of partial automorphisms is a powerful one for obtaining structural information about a graph which is assumed to be non-reconstructible.

Lastly, we mention that similar calculations have been done for the case in which G has two vertices of degree k-1 and the rest of degree k. These are too long to include here, and can be found in [4].

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