# 2 Partial Automorphisms and the work of Kocay

## 2.1 Partial permutations

We extend the traditional definition of a function to that of "partial" functions, eventually focusing on "partial" bijections from a set to itself. In the process, we develop the cycle structure of a partial permutation $\pi$ and show that a partial permutation on a set $X$ partitions the set into five classes, $U, D, R, T, P$. The class $U$ is the set of elements on which neither $\pi$ nor $\pi^{-1}$ is defined. $D$ is the set (partial "domain") on which $\pi$ is defined but $\pi^{-1}$ is not; $R$ is the set (partial "range") on which $\pi^{-1}$ is defined but $\pi$ is not; $P$ is the set on which $\pi$ acts as a true permutation. We prove that if $X$ is finite, a partial permutation gives a bijection from $D$ to $R$ passing, possibly through an intermediate set of elements, $T$.

We are especially interested in applying partial functions to graph reconstruction.

We begin with the standard definition of a function.

**Definition.** A **function** $f : A \to B$ (from $A$ into $B$) is a subset of $A \times B$ which satisfies the following two properties:

1. if $a \in A$ then there exists $b \in B$ such that $(a, b) \in f$,
2. if $((a, b_1) \in f) \wedge ((a, b_2) \in f)$ then $b_1 = b_2$.

By the second property, above, if $(a, b) \in f$ then we may write $b = f(a)$. The first property tells us that for every $a \in A$, $f(a)$ exists.

**Definition.** A **partial function** $f$ from set $A$ into $B$ is a subset of $A \times B$ which is required to satisfy only the second condition above: if $((a, b_1) \in f) \wedge ((a, b_2) \in f)$ then $b_1 = b_2$.

If $f : A \to B$ and $g : B \to C$ are partial functions then we may create the partial function $g \circ f : A \to C$ by defining

$$(a, c) \in g \circ f \iff \exists b \in B \text{ such that } ((a, b) \in f, (b, c) \in C).$$

The function $g \circ f$ is the **composition** of $f$ with $g$.

A partial function $f$ allows the possibility that there are elements $a \in A$ such that $f(a)$ does not exist. The concept of "partial" function generalizes the idea of a function; it merely allows "input" elements from $A$ to have no defined output. The standard definitions of domain, range, injection, surjection and permutation (normally described in a first introduction to functions) generalize naturally to partial functions.

**Definition.** The **domain** of a partial function is the set $\mathrm{Dom}(f) \subseteq A$ on which $f$ is defined, that is,

$$\mathrm{Dom}(f) := \{a \in A : \exists b \in B, (a, b) \in f\}.$$

A partial function $f : A \to B$ is a function if and only $\mathrm{Dom}(f) = A$.

**Definitions.** The **range** of a partial function is the set $\mathrm{Rng}(f) \subseteq B$ of images of $A$ under $f$:

$$\mathrm{Rng}(f) := \{b \in A : \exists a \in B, (a, b) \in f\}.$$

A partial function $f$ is a **partial injection** if

$$((a_1, b) \in f) \wedge ((a_2, b) \in f) \implies a_1 = a_2.$$

If $f : A \to B$ is a partial injection then $f^{-1} := \{(b, a) : (a, b) \in f\}$ is a partial function from $B$ into $A$.

A partial injection $f : A \to B$ is **partial permutation** on $A$ if $B = A$. A partial function $f : A \to B$ is a **partial surjection** if $\mathrm{Rng}(f) = B$.

If a partial function $f : A \to B$ is in fact a (genuine) function, then we will drop the word "partial", so that single terms like "injection" and "permutation" and "surjection" have their standard meanings for the class of functions.

If $g : X \to X$ and $f : X \to X$ are partial functions then $\mathrm{Dom}(g \circ f) \subseteq \mathrm{Dom}(f) \subseteq X$. We write $f^2$ for $(f \circ f)$ and recursively define, for natural number $n$, $f^{n+1} := f^n \circ f$.

**Lemma 1.**

Suppose $f : X \to X$ is a partial function and suppose there exists an integer $n$ such that $\mathrm{Dom}(f^{n+1}) = \mathrm{Dom}(f^n)$. Write $P := \mathrm{Dom}(f^{n+1})$. Then $f|_P$ is a surjection from $P$ onto $P$. If $P$ is finite then $f|_P$ is a permutation.

The set $P$ is the maximal set on which $f$ acts like a (genuine) permutation. Note that if $X$ is finite then there will *always* be some integer $n$ such that $\mathrm{Dom}(f^{n+1}) = \mathrm{Dom}(f^n)$.

Let $f$ be a partial permutation $f : X \to X$ and suppose that there exists a natural number $n$ such that

$$P := \mathrm{Dom}(f^{n+1}) = \mathrm{Dom}(f^n).$$

Write

$$D := \mathrm{Dom}(f) - \mathrm{Rng}(f),$$

$$R := \mathrm{Dom}(f) - \mathrm{Dom}(f).$$

(Thus $D$ is the set of elements of $X$ for which $f$ is defined but for which $f^{-1}$ is not. Similarly $R$ is the set of elements of $X$ for which $f$ is not defined but $f^{-1}$ is.) Set

$$T := (\mathrm{Dom}(f) \cap \mathrm{Rng}(f)) - P.$$

The elements of $T$ are those which "temporarily" involve $f$; if $t \in T$ then $f(t)$ exists, as does $f^{-1}(t)$ but there is some integer $n$ for which neither $f^n(t)$ nor $(f^{-1})^n(t)$ exist.

Let $U$ represent the elements $u$ of $X$ for which both $f(u)$ and $f^{-1}(u)$ are undefined. In this way, we partition the set X into mutually disjoint subsets:

$$X = U \sqcup D \sqcup R \sqcup T \sqcup P. \tag{1}$$

We could avoid the set $U$ by presuming that $f$ acts as the identity on any element not listed in the domain or range. For this reason, it will be convenient to assume $U = \emptyset$. We will also assume, from here on, that $X$ is finite.

It is well known that if $f$ is a permutation acting on a finite set $P$ then there is an essentially unique way to write $f$ as a product of cycles. This **cycle decomposition** of a permutation can be extended to a similar cycle decomposition of a partial permutation acting on a finite set $X$.

Given an element $d \in D$, we consider the sequence $d, f(d), f^2(d), ...$ Since $X$ is finite, this sequence must eventually end at a member of $R$. So there exists an integer $k$ and element $r \in R$ such that the sequence has length $k + 1$: $d, f(d), f^2(d), ..., f^{k-1}(d), r = f^k(d)$. (We write $[d, ..., r]$ for this sequence.) The endpoints of the sequence $[d, ..., r]$ are in $D$ and $R$; the internal elements $f^j(d)$, $1 \le j \le k-1$ lie in $T$. Indeed, this sequence provides a one-to-one correspondence between $D$ and $R$.

Each element of $D$ (and each corresponding element of $R$) provides a sequence

$$d, f(d), f^2(d), ..., f^{k-1}(d), r = f^k(d).$$

(The length $k$ will vary with $d$ (and $r$).) Each element of $T$ is in exactly one such sequence. Given $d$ and $r = f^k(d)$ we call the sequence $f(d), f^2(d), ..., f^{k-1}(d)$ the **thread** through $T$ from $d$ to $r$.

We summarize our progress to this point.

**Lemma 2.**
Let $\pi$ be a partial permutation acting on a finite set $X$. Then $\pi$ partitions $X$ into the five sets, $P, D, T, R, U$ described above equation 1. Furthermore $|D| = |R|$ and powers of $\pi$ provide a bijection from $D$ onto $R$ If we order $D = \{d_1, d_2 ..., d_m\}$ then the elements of $T$ may be uniquely labeled as $t_{i,j} := f^j(d_i), 1 \leq i \leq m$.

Suppose the cardinality of $D$ (and the cardinality of $R$) is an integer $m$. Just as we may partition $P$ into the orbits of $f$, we may partition $D \sqcup T \sqcup R$ into $m$ sequences beginning at a member of $D$, passing through $T$ and ending at a member of $R$. Let us order $D = \{d_1, d_2 ..., d_m\}$ and, given this ordering, similarly order $R = \{r_1, r_2, ..., r_m\}$ with the understanding that a sequence starting at $d_i$ ends at $r_i$ We may then extend $f$ to a permutation by $f(r_i) = d_i$. In this way, we extend $f$ to a permutation of $X$ and the sequence

$$[d_i, f(d_i), f^2(d_i), ..., f^{k-1}(d_i), r_i = f^k(d_i)]$$

extends to the $k + 1-$cycle

$$(d_i, f(d_i), f^2(d_i), ..., f^{k-1}(d_i), r_i).$$

**Corollary 3.**
Let $\pi$ be a partial permutation acting on a finite set $X$ and partition $X$ as in equation 1. We may uniquely extend $\pi$ to a permutation $\hat{\pi}$ on $X$ by replacing sequences $[d, ..., r]$ with permutations $(d, ..., r)$ (so that $\hat{\pi}(r) = d$) and by requiring that $\hat{\pi}$ act as the identity map on $U$.

**Some examples.**

1. Suppose
$$f = \{(1, 2), (2, 3), (3, 1), (4, 5), (5, 4), (6, 7), (7, 8), (8, 9), (10, 11)\}$$
is a partial permutation of $X := \{1, 2, 3, ..., 11\}$.

   In this case the elements 9 and 11 are not in the domain and 6 and 10 are not in the range. So $D = \{6, 10\}$ and $R = \{9, 11\}$. If one begins at the element 6, the function $f$ traverses through $T = \{f(6) = 7, f(7) = 8\}$, before ending at 9. On the remaining five elements, $P = \{1, 2, 3, 4, 5\}$, the function $f$ acts as a permutation The cycle/sequence decomposition for $f$ is
$$(1, 2, 3), (4, 5), [6, 7, 8, 9], [10, 11].$$

2. Suppose $f$ acting on $X = [1..12]$ is given by the table below.
$$\pi = \begin{bmatrix} 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \hline 6 & 6 & 10 & 7 & 1 & 9 & 11 & 12 & 8 & 2 \end{bmatrix}.$$
   Note that $1, 2 \in R$ and since $3, 4$ do not show up in the second column, they must be in $D$. The cycle/sequence decomposition for $f$ is
$$(8, 9, 11), [3, 6, 7, 1], [4, 5, 10, 12, 2].$$
   and so $P = \{8, 9, 11\}, T = \{6, 7, 5, 10, 12\}, D = \{3, 4\}, R = \{1, 2\}$.

3. The previous problem would not significantly change if, instead of the previous table, we claimed that $f$ acts on $X = [1..15]$. In this case, the numbers 13, 14, 15 are neither in the domain nor the range of $f$ and so we could assign them to $U$ and set $U = \{13, 14, 15\}$.

**Exercises on Partial Permutations**

1. Suppose $f = \{(1,2),(2,3),(3,1),(4,7),(5,6),(7,8),(8,9),(9,11)\}$ is a partial permutation of $X := \{1,2,3,...,11\}$.

   (a) Write out the $f$ in our cycle notation.

   (b) Identify the sets $P, T, D, R$ and $U$.

2. Redo problem 1 with $f = \{(1,2),(2,3),(3,1),(5,6),(7,8),(8,9),(9,11)\}$

3. Redo problem 1 with the new assumption that $X := \{1,2,3,...,15\}$.

4. Construct a partial permutation $f$ with $P = \{1,3,5,6,8\}, D = \{2,4,7\}, R = \{11,12,13\}$ and $T = \{9,10\}$. Assume $U$ is empty.

5. How many *different* solutions are available in problem 4??

## 2.2 Graphs that share two cards

We now apply the concept of partial permutation to some problems in graph theory, examining partial automorphisms of graphs.

We suppose $\Gamma(V, E)$ is a graph with adjacency relation $\sim$.

**Definitions.**

An **automorphism** of $\Gamma$ is a permutation $f$ of $V$ with the property that $a$ is adjacent to $b$ (written $a \sim b$) if and only if $f(a)$ is adjacent to $f(b)$.

A **partial automorphism** of $\Gamma$ is a partial permutation with the property that *if* $a$ and $b$ are in the domain of $f$ then $a \sim b \iff f(a) \sim f(b)$. (This does not quite fit the definition of Kocay, who assumes that $f$ can be extended to an automorphism of a larger graph.)

A **card** of a graph is a subgraph induced by deleting a vertex.

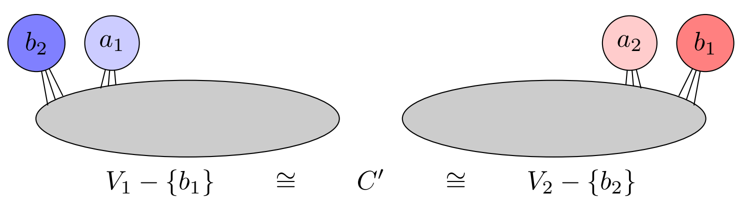
Suppose $\Gamma_1(V_1, E_1)$ and $\Gamma_2(V_2, E_2)$ are two graphs, each of order $n$ with a common card $C$. This means that there are vertices $a_1 \in V_1$ and $a_2 \in V_2$ and graph isomorphisms $\alpha_1$ and $\alpha_2$,

$$\alpha_1 : (V_1 - \{a_1\}) \to C \text{ and } \alpha_2 : (V_2 - \{a_2\}) \to C. \tag{2}$$

So $\alpha_2^{-1}\alpha_1$ is an automorphism from $V_1 - \{a_1\}$ to $V_2 - \{a_2\}$.

Suppose also that $\Gamma_1$ and $\Gamma_2$ share a second common card $C'$. This means that there are vertices $b_1 \in V_1$ and $b_2 \in V_2$ and graph isomorphisms $\beta_1$ and $\beta_2$,

$$\beta_1 : (V_1 - \{b_1\}) \to C' \text{ and } \beta_2 : (V_2 - \{b_2\}) \to C'. \tag{3}$$




Now $\pi := \beta_2^{-1}\beta_1$ is a graph automorphism from $V_1 - \{b_1\}$ on the left in the drawing above to $V_2 - \{b_2\}$ on the right. Thus $\pi$ is a partial automorphism with $D = \{a_1, b_2\}$ and $R = \{a_2, b_1\}$.

The partial automorphism $\pi$, with its two threads, have been created by our assumption that $\Gamma_1$ and $\Gamma_2$ share two vertex deleted subgraphs. $C$ and $C'$. We may clean up our notation a bit by giving preference to our first card $C$ and assuming that $C$ provides us with a labeling of the vertices of the graph $\Gamma_1 \cup \Gamma_2$. By this, I mean that we identify $V_1 - \{a_1\}$ and $V_2 - \{a_2\}$ with $C$ and so the functions $\alpha_1, \alpha_2$ are the identity maps on $C$. We will described $C$ as the **base card** of what follows.

Given base card $C$, we can redo all of the above, defining

$$\pi := \beta_2^{-1}\beta_1 \tag{4}$$

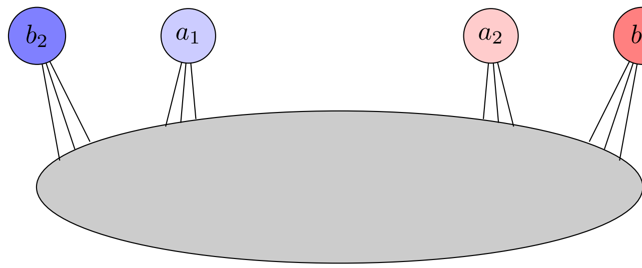
We explore the threads the partial permutation $\pi$ creates as it traces $\{b_2, a_1\}$ through to $\{b_1, a_2\}$. $\pi$ is a partial automorphism as it acts on everything but $b_1, a_2$ and $\pi^{-1}$ is also a partial automorphism as it acts on everything but $b_2, a_1$.

I will write this

$$\pi : \{a_1, b_2\} \rightsquigarrow \{a_2, b_1\}. \tag{5}$$

We will call the pair of cards $C$ and $C'$ "crossing" if $a_1 \rightsquigarrow b_1$ and $b_2 \rightsquigarrow a_2$. The card pair $C$ and $C'$ are "parallel" if $a_1 \rightsquigarrow a_2$ and $b_2 \rightsquigarrow b_1$.

For the partial automorphism $\pi$, $D = \{a_1, b_2\}, R = \{a_2, b_1\}$. The threads $T$ and the set $P$, on which $\pi$ is a permutation, are unknown. We attempt to identify these in several specific cases.

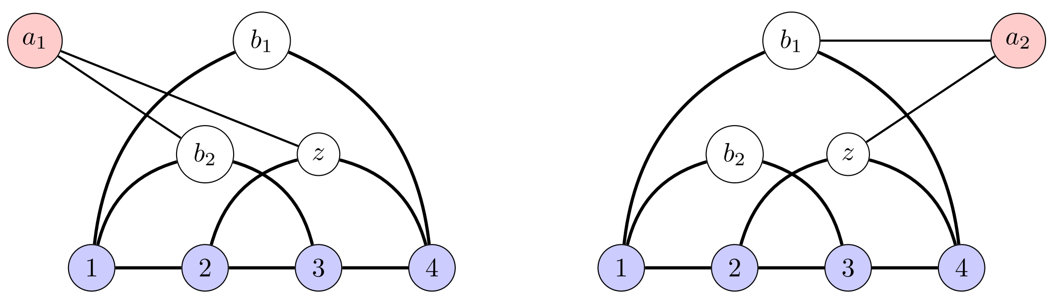



$\Gamma = \Gamma_1 \cup \Gamma_2$

In this picture, the original vertex set $V_1$ consists of everything but $a_2$; the original vertex set $V_2$ consists of everything but $a_1$. The card $C$ is everything but the two vertices $a_1$ and $a_2$ and the card $C'$ is isomorphic to two graphs, the one obtained by deleting vertices $b_1$ and $a_2$ and the one obtained by deleting $a_1$ and $b_2$.

If the partial automorphism $\pi$ can be extended to an automorphism of $\Gamma$ by mapping $a_1$ to $a_2$ then the graph $\Gamma_1$ is reconstructible.

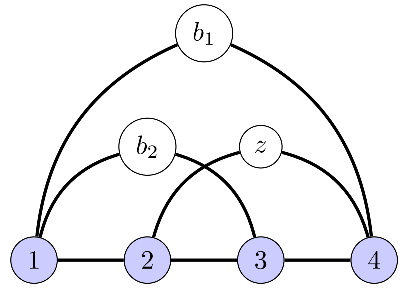
Here is a specific example in which the two graphs $\Gamma_1$ and $\Gamma_2$ are not isomorphic.



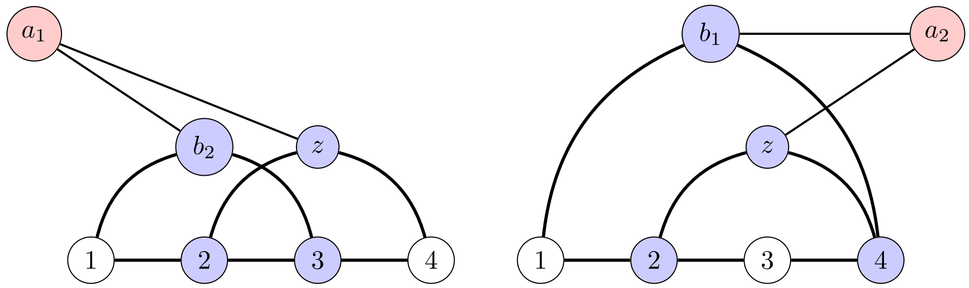

$\Gamma_1$ $\Gamma_2$

The graph, $\Gamma_2$, on the right, has a 4-cycle $b_2 - 1 - 2 - 3 - b_2$ containing a bump ($b_2$) while the all 4-cycles in the graph on the left are made up of cubic vertices.

The card $C$ is




The card $C'$ is both




An isomorphism $\pi$, mapping the graph on the left to the graph on the right, without mapping $a_1$ to $a_2$ is

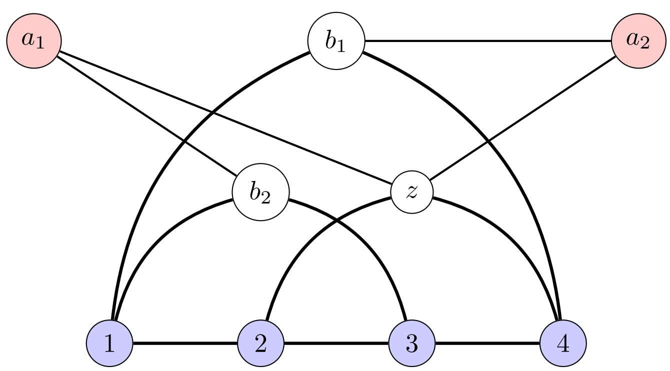
$$\pi = \begin{bmatrix} a_1 & b_2 & 3 & 2 & z & 1 & 4 \\ \hline 1 & b_1 & 4 & z & 2 & a_2 & 3 \end{bmatrix}.$$

In this case, the graphs are small enough that we can list all isomorphisms. There are two. Here is the other.

$$\pi_2 = \begin{bmatrix} a_1 & b_2 & 3 & 2 & z & 1 & 4 \\ \hline 1 & 2 & z & 4 & b_1 & 3 & a_2 \end{bmatrix}.$$

We may view either of these isomorphisms as a partial permutation on the larger graph on nine vertices.

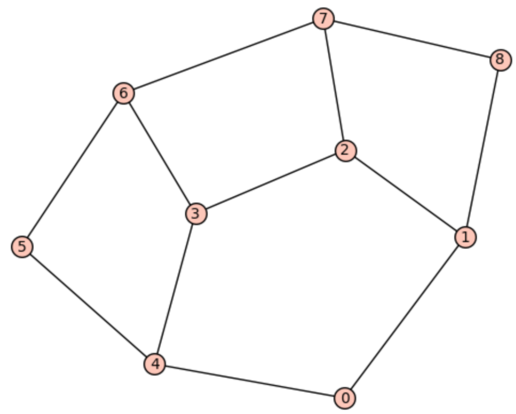
We may construct $\Gamma_1 \cup \Gamma_2$ as follows. The partial automorphisms $\pi_1$ and $\pi_2$ act on this graph.




The partial automorphism $\pi_1$ has cycle structure $(2, z)(3, 4)[a_1, 1, a_2][b_2, b_1]$. It is a "parallel" partial automorphism, sending $a_1$ to $a_2$. Later we will denote this partial automorphism as type $P(2, 1)$. On the other hand, the partial automorphism $\pi_2$ has cycle structure $[a_1, 1, 3, z, b_1][b_2, 2, 4, a_2]$. It is a "crossing" partial automorphism, sending $a_1$ to $b_1$. We will later say this partial automorphism has type $C(4, 3)$.

**Exercises on Partial Automorphism of Kocay Graphs**

1. The graph below is the card of two different graphs, $\Gamma_1$ and $\Gamma_2$.




   Reconstruct the two graphs $\Gamma_1$ and $\Gamma_2$ and show that they are not isomorphic.

2. Show that the card above appears *twice* in the decks of $\Gamma_1$ and $\Gamma_2$.

3. Using the two cards from problem 2, find all partial automorphisms of $\Gamma_1 \cup \Gamma_2$.

# 3 Bidegree graphs

We consider an example described by William Kocay in several papers in the early 1980s.

## 3.1 A regular graph less one edge

A regular graph of degree $k$ is reconstructible. But what if we remove an edge? Is the new graph reconstructible? This simple changes seems to give a very hard problem.

We suppose a graph $\Gamma_1(V_1, E_1)$ which has $n$ vertices of degree $k$ and has exactly two vertices of degree $k-1$, call them $a_1$ and $b_1$. We assume that $a_1$ and $b_1$ are not adjacent (else the graph would be easy to reconstruct.)

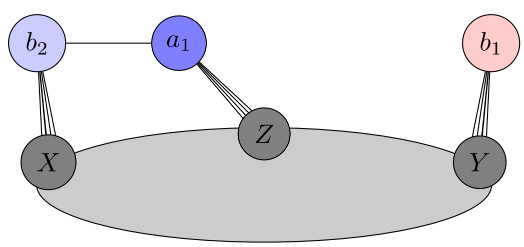
Suppose that this graph is a counterexample to the graph reconstruction conjecture. Then there exists a graph $\Gamma_2(V_2, E_2)$ which has $n$ vertices of degree $k$ and has exactly two vertices of degree $k-1$, $a_2$ and $b_2$ such that the card $C$ obtained by deleting vertex $a_2$ from $\Gamma_2$ is the same as the card obtained by deleting $a_1$ from $\Gamma_1$. Let us identify the card $C$ in both graphs, so the $V_1 - \{a_1\} = V_2 - \{a_2\}$.

Furthermore, if this is a counterexample to the graph reconstruction conjecture then the card obtained by deleting vertex $b_1$ from $\Gamma_1$, call it $C'$, is isomorphic the card obtained by deleting $b_2$ from $\Gamma_2$.

The card $C$ obtained by deleting $a_1$ has $k$ vertices of degree $k-1$. These are the "short" vertices, vertices with too few edges. All but one of these "short" vertices is adjacent to $a_1$ in $\Gamma_1$. And all but one of these "short" vertices is adjacent to $a_2$ in $\Gamma_2$.

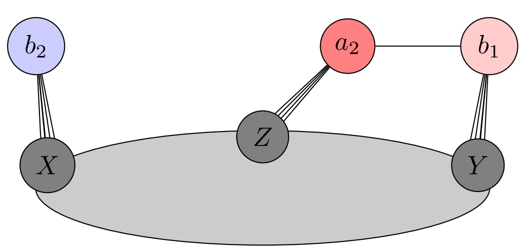
If $\Gamma_1$ and $\Gamma_2$ are nonisomorphic then $a_1$ is adjacent to $k-2$ vertices $z_1, z_2, ..., z_{k-2}$ that are also adjacent to $a_2$. The short vertex in the card $C$ which is not adjacent to $a_1$ must be the "true" bump $b_1$.

Here is a representation of the graph $\Gamma_1$ with short vertices $a_1$ and $b_1$. If we remove the vertex $a_1$ we have a graph with $k$ short vertices, represented here by $b_1$, $b_2$ and the set $Z$. The set $Z$ has size $k-2$ and represents somewhat anonymous vertices adjacent to $a_1$. The set $X = \{x_0, x_1, ..., x_{k-2}\}$ represents the $k-1$ vertices (other than $a_1$) that are adjacent to $b_2$. Similarly, the set $Y = \{x_0, x_1, ..., x_{k-2}\}$ represents the $k-1$ vertices adjacent to $b_1$. It is possible that $X, Y$ and $Z$ have some vertices in common.




The graph $\Gamma_1$.

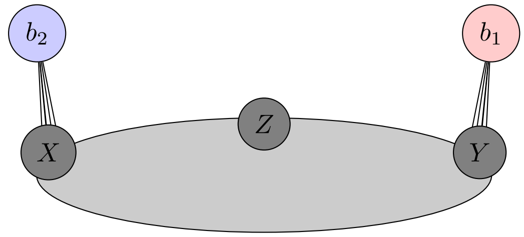
The graph $\Gamma_2$, is drawn below.



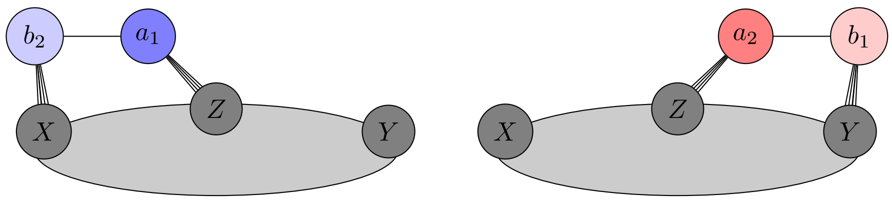

The graph $\Gamma_2$

Note that if $\Gamma_1$ is not isomorphic to $\Gamma_2$ then the simple change of the single edge $a_1 \sim b_2$ to $a_2 \sim b_1$ has created a nonisomorphic graph.

The two graphs $\Gamma_1$ and $\Gamma_2$ have the same card $C$.




If the graphs $\Gamma_1$ and $\Gamma_2$ have a second card $C' \cong \Gamma_1 - \{b_1\} \cong \Gamma_2 - \{b_2\}$ then the two graphs below are isomorphic.




The two versions of $C'$

The automorphism $\pi$ maps the graph on the left onto the graph on the right.

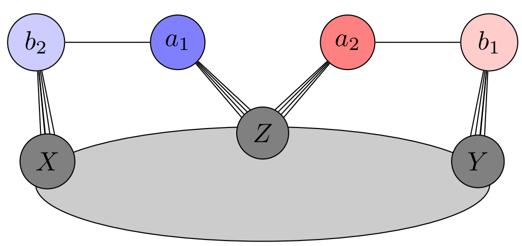
The short vertices (vertices of degree $k-1$) in the graph on the left are $a_1$ and the members of $Y$. The short vertices in the graph on the right are $a_2$ and the members of $X$.

If the two graphs $\Gamma_1$ and $\Gamma_2$ are not isomorphic then $\pi$ cannot map $a_1$ to $a_2$. It must then map $a_1$ to some member of $X$ only $a_2$ and the members of $X$ are the short vertices in the graph on the right. Let's assume, without loss of generality, that $x_0 := \pi(a_1)$.

Similarly, $\pi$ must map some member of $Y$ to $a_2$, say $\pi(y_0) = a_2$. Label the remaining $k-2$ members of $X$ and $Y$ so that $\pi(y_i) = x_i$.

$$\pi = \left[\begin{array}{ccc} a_1 & y_i & y_0 \\ \hline x_0 & x_i & a_2 \end{array}\right].$$

Kocay will draw all of this together in one large graph and view $\pi$ as a partial autmorphism.




$\Gamma_1 \cup \Gamma_2$

We wish to trace the threads created by following $a_1$ and $b_2$. Since $a_1$ and $b_2$ are adjacent, then their images under $\pi$ are also adjacent. Since $D = \{a_1, b_2\}$ and $R = \{a_2, b_1\}$, eventually the threads starting at $a_1$ and $b_2$ must end up at $a_2$ and $b_1$.

## 3.2 Bumps and cubics

We explore this by first setting $k = 3$ and following the 1982b paper, "Partial automorphisms and the reconstruction of bidegreed graphs", *Congressus Numeratium.*[1]

Consider the graph $\Gamma_1$ which has degrees two and three and has exactly two vertices of degree 2, $a_1$ and $b_1$. As before, we suppose that $a_1$ and $b_1$ are not adjacent.

We focus on the cards created by short vertices. We remove the vertex $a_1$ and attempt to reconstruct the graph. We do this by introducing a new vertex $a_2$ of degree two and build the graph $\Gamma = \Gamma_1 \cup \{a_2\}$. If after joining $a_2$ to two vertices in $\Gamma_1 - \{a_1\}$, $\Gamma_2$ is isomorphic to $\Gamma_1$ then we have reconstructed $\Gamma_1$.

The vertex $a_1$ has two neighbors in $\Gamma_1$, call them $b_2$ and $z$, and so the graph $\Gamma_1 - \{a_1\}$ has exactly three vertices of degree two, the vertex $b_1$ which was originally of degree 2, and the two vertices, $b_2$ and $z$ whose degree dropped to 2 when $a_1$ was deleted. Now if $\Gamma_2$ is not isomorphic to $\Gamma_1$ then $a_2$ is not adjacent to both $b_2$ and $z$. Let us assume then that $a_2$ is adjacent to $z$ and $b_1$.

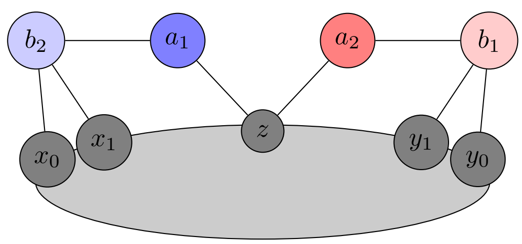
Let $x_0, x_1$ be the neighbors of $b_2$. Let $y_0, y_1$ be the neighbors of $b_1$.[2] In $\Gamma - \{b_1\}$, $y_0$ and $y_1$ have degree 2 and in $\Gamma - \{b_2\}$, $x_0$ and $x_1$ have degree 2. The partial automorphism $\pi : \Gamma_1 - \{b_1\} \to \Gamma_2 - \{b_2\}$ must map the set $\{a_1, y_0, y_1\}$ of vertices of degree two in $\Gamma_1 - \{b_1\}$ to $\{a_2, x_0, x_1\}$, the set of vertices of degree two in $\Gamma_2 - \{b_2\}$. If $\Gamma_1, \Gamma_2$ is a counterexample to the GRC then $\pi$ will not map $a_1$ to $a_2$. We may assume then, without loss of generality, that

$$\pi(a_1) = x_0, \pi(y_1) = x_1, \pi(y_0) = a_2,$$

that is, in two-row function notation

$$\pi = \begin{bmatrix} a_1 & y_1 & y_0 \\ \hline x_0 & x_1 & a_2 \end{bmatrix}.$$

Note that $b_2$ is adjacent to both $a_1$ and $x_0 = \pi(a_1)$. Similarly $b_1$ is adjacent to both $y_0$ and $a_2 = \pi(y_0)$.




$\Gamma_1 \cup \Gamma_2$

A strategy here will be to consider the two threads created by $a_1$ and $b_2$. Let $P$ be the the vertices that are *not* in these two threads. (Kocay calls this set $V'$. It is the set on which $\pi$ acts as a true automorphism.) I think that the larger $P$, the less control we have on our reconstruction.

Consider the two threads

$$[a_1, x_0 = \pi(a_1), \pi^2(a_1), \pi^3(a_1), \ldots, \pi^{k_1}(a_1)]$$

$$[b_2, \pi(b_2), \pi^2(b_2), \pi^3(b_2), \ldots, \pi^{k_2-1}(b_2) = y_0, \pi^{k_2}(b_2)]$$

Since $a_1$ is adjacent to $b_2$ then for $j \le \min\{k_1, k_2\}$, $\pi^j(a_1)$ is adjacent to $\pi^j(b_2)$. Similarly, since $b_2$ is adjacent to $a = \pi(a_1)$ then for $j \le \min\{k_1 - 1, k_2\}$, $\pi^j(b_2)$ is adjacent to $\pi^{j+1}(a_1)$. Thus we have a path starting at $a_1$, snaking its way back and forth $(a_1, b_2, x_0 = \pi(a_1), \pi(b_2)...)$ across the two threads.

The partial permutation $\pi^{-1}$ also creates a path through the threads, beginning with $a_2, b_1, y_0$ and going in the opposite direction of the first path.

[1]Kocay uses letters $v, u, r, s, w, x, a, b, z$ whereas I use $b_1, a_1, y_1, y_0, b_2, a_2, x_0, x_1, z$, respectively. Kocay's partial automorphism $p$ is the automorphism $\pi$ here.

[2]Kocay instead labels $x_0, x_1, y_0, y_1$ as $a, b, s, r$.

**Case 1.** (The crossing case.)[3] In this case the partial automorphism $\pi$ acts as follows, $(a_1, b_2) \rightsquigarrow (b_1, a_2)$, that is iteration of $\pi$ maps $a_1$ to $b_1$ and $b_2$ to $a_2$.

We write $[a_1, ..., b_1]$ for the set of vertices

$$\{a_1, \pi(a_1), \pi^2(a_1), ..., \pi^{k_1}(a_1) = b_1\}.$$

Similarly we write $[b_2, ..., a_2]$ for the set of vertices (thread)

$$\{b_2, \pi(b_2), \pi^2(b_2), ..., \pi^{k_2}(b_2) = a_2\}.$$

Now $b_2$ is adjacent to both $a_1$ and $x_0 = \pi(a_1)$. The partial automorphism $\pi$ preserves these adjacencies.

The vertex $a_1$ is adjacent to $b_2$ and $z$.

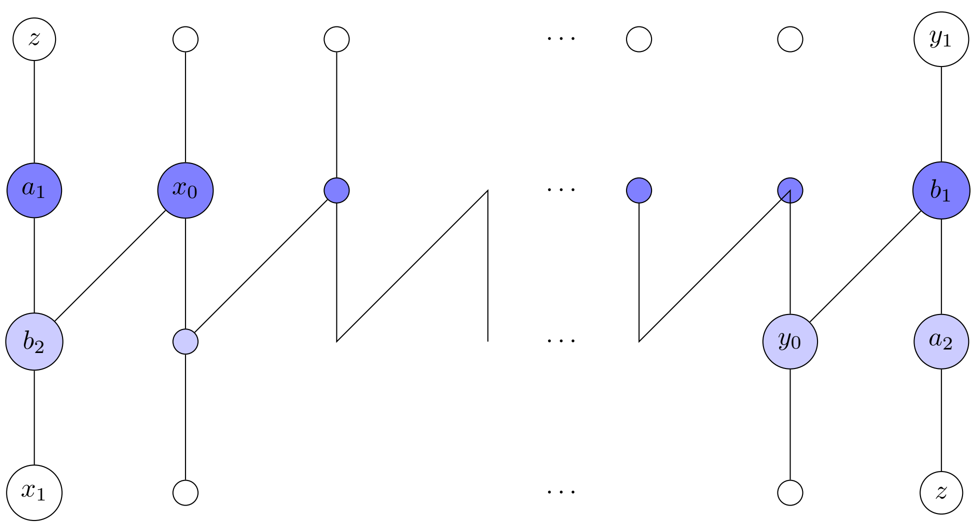
The action of the partial automorphism $\pi$ on the vertices of $\Gamma_1 \cup \Gamma_2$ is essentially something like this. In dark blue is the thread

$$[a_1, \pi(a_1) = x_0, ..., \pi^{k_1}(a_1) = b_1].$$

In light blue is the thread

$$[b_2, \pi(b_1), ..., \pi^{k_2-1}(b_2) = y_0, \pi^{k_2}(b_2) = a_2].$$

Since $\pi$ and $\pi^{-1}$ are partial automorphisms, they preserve the edges in the thread and so we have a zigzag path winding through the two threads.



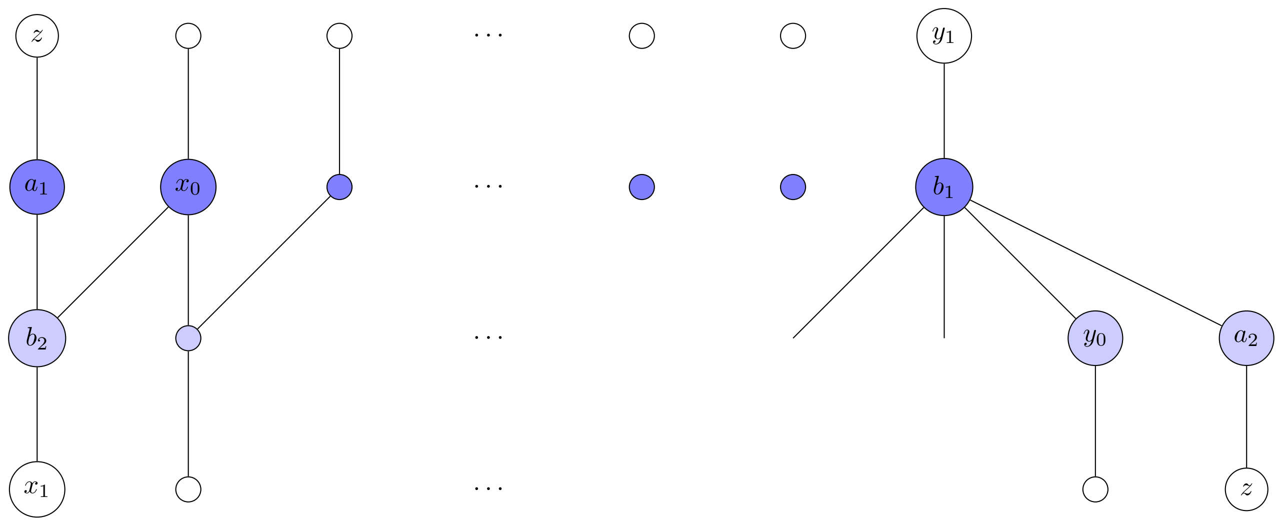

This picture assumes that $z$ and $y_1$ are not in the two threads but as we will see, we may indeed need to have these two vertices down among the two threads.

This picture has been drawn as if the two threads have the same length. In this case, the vertex $z$ (appearing twice in this picture) is in a large cycle of length $2k + 1$.

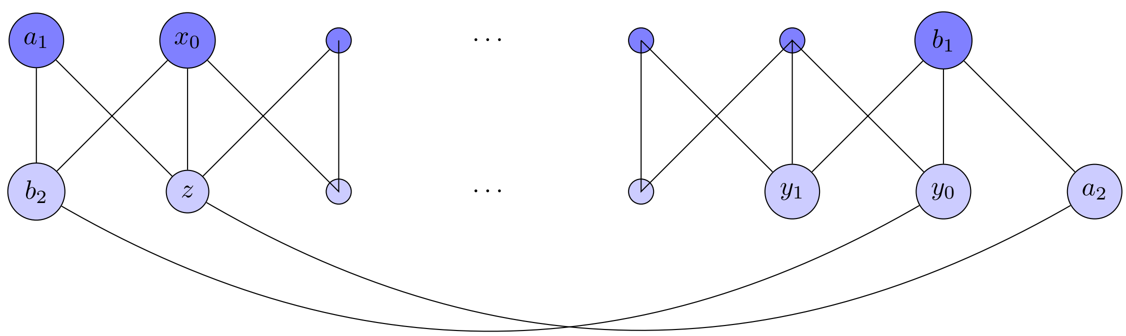
Need the lengths, $k_1$ and $k_2$ of the two threads be equal? Let us suppose that $k_2 > k_1$. (If $k_1 > k_2$, we may replace $\pi$ and $\pi^{-1}$ and consider the threads running from right to left, mapping $b_1$ eventually to $a_1$ and $a_2$ to $b_2$.)

Suppose $k_2 \geq k_1+2$. Then both the partial automorphisms $\pi$ and $\pi^{-1}$ preserve edges Then our threads have this form and $b_1$ is adjacent to too many vertices. Since $a_1$ and $b_2$ are adjacent, $b_1 = \pi^{k_2}(a_1)$ it is adjacent to $\pi^{k_2}(b_2)$. Since $x_0 = \pi(a_1$ is adjacent to $b_2$ then $b_1 = \pi^{k_2-1}(x_0)$ it is adjacent to $\pi^{k_2-1}(b_2)$. But $b_1$ is already adjacent to $y_0 = \pi k_2 - 1(b_2)$ and also adjacent to $a_2$ and so $b_1$, a vertex of degree 3 in $\Gamma_1 \cup \Gamma_2$, is adjacent to at least four vertices.

[3]This case is section 3 of Kocay's 1982b paper.

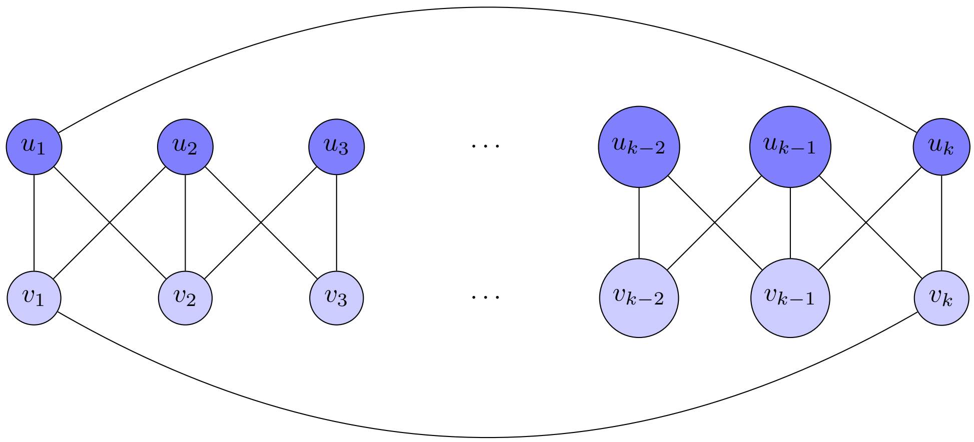



What if $k_2 = k_1 + 1$? In this case, by a similar argument, $b_1$ is adjacent to $y_0, a_2$ and $\pi^{-1}(y_0)$.

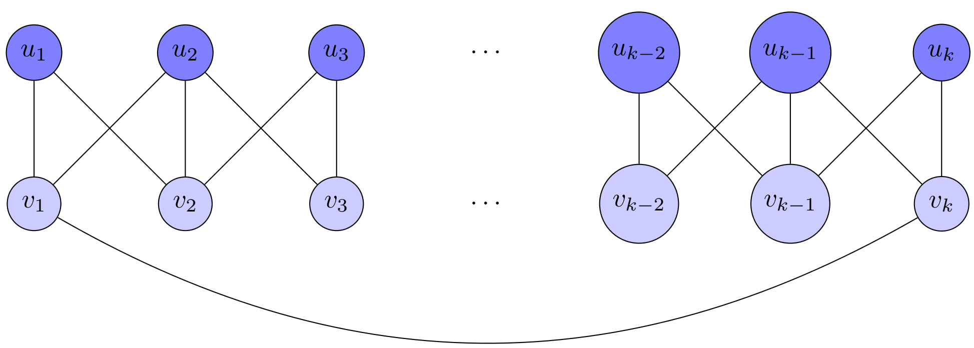



Now the two threads induce vertices of degree three and since $b_1 \sim a_2$ then $a_1 \sim \pi^{-k_2}(a_2) = \pi(b_2)$. Thus $z = \pi(b_2)$ and the vertex $y_0$ is adjacent to $b_2$. The graph $\Gamma_1 \cup \Gamma_2$ *is* exactly the union of two threads.

We can identify this graph. Create a cubic graph as follows. The vertices of the cubic graph are $u_1, u_2, ...u_k$ and $v_1, v_2, ..., v_k$. For all $i$, join $u_i$ to $v_i$. For $i$ in the interval $(1, k)$, join $u_i$ to $v_{i-1}, v_{i+1}$ and join $v_i$ to $u_{i-1}, u_{i+1}$. Join $u_1$ to $u_k$ and $v_1$ to $v_k$.




The graph $\Gamma_1$ is the graph created by removing the edge from $u_1$ to $u_k$.




**Exercise.** Show that this graph is reconstructible.