

# Quantum Mechanics

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# 1 Historical Introduction

## 1.1 Classical mechanics

Classical mechanics is based on two distinct concepts.

- (i) Particles - Point-like objects which have positions and velocities fully determined by Newton's second law as a function of time.  $m\ddot{x}(t) = F(x(t), \dot{x}(t))$ . Once  $x(t_0), \dot{x}(t_0)$  is known, the solution of the equation gives the position and velocity at all times. Particles can collide, scatter, but never interfere.
- (ii) Waves - Spread out objects which are functions of time and position. Periodic in  $t/x$ . Propagation is determined by the wave equation which is

$$\frac{\partial^2 f(x, t)}{\partial t^2} - c^2 \frac{\partial^2 f(x, t)}{\partial x^2} = \nu$$

We have solution  $f_{\pm}(x, t) = A_{\pm} \exp[i(kx - \omega t)]$ . And the solutions obey the dispersion relation  $\omega = ck$ . We have that  $\omega$  is the angular frequency related to  $\lambda$  (wavelength) by  $\nu = \frac{\omega}{2\pi}$  and  $\lambda = \frac{2\pi c}{\omega} = \frac{c}{\nu}$ . Waves interfere, causing constructive interference when in-phase and destructive interference when out of phase.

However we have some programs with classical mechanics.

- (i) Light behaving like particles
  - (a) Black body radiation (NE)
  - (b) Photoelectric effect
  - (c) Compton scattering (NE)
- (ii) Stability of atom
- (iii) Particles behaving like waves
  - (a) De Broglie Principle
  - (b) Electron diffraction

## 1.2 Photoelectric effect

This is the experiment when light hits a metal surface and causes electrons to be emitted. They used monochromatic radiation with fixed wavelength and changed the intensity of light and the wavelength. The classical expectation were:

- (i) Incident light carries  $E \propto I = |A|^2$ . They proposed that as the intensity increases there is enough energy to break the bond of the electron with the metal atoms, causing it to be released.
- (ii) The emission rate of the electrons should be constant over intensity past a certain point.

However what they actually observed was surprising.

- (i) Below given  $\omega$  there was no emission of electrons.
- (ii) The velocity (i.e. KE) depended on  $\omega$  not on  $I$ .
- (iii) The emission rate increased with intensity.

In 1905 Einstein used these observations to propose the following

- (i) Light comes in small quanta (now called photons).
- (ii) Each photon carries a small packet of energy,  $E$ .

$$E = \hbar\omega, \quad \mathbf{p} = \hbar\mathbf{k}$$

where

$$\hbar = \frac{h}{2\pi}$$

where  $h$  is the Planck constant.

- (iii) The interaction seen in the photoelectric effect was caused by each photon interacting with each photon in a *one-to-one* interaction. So we have that

$$\text{Kinetic energy of } e^- = \text{Kinetic energy of } \gamma - \text{binding energy of the metal}$$

we write this equation as

$$E_{min} = 0 = \hbar\omega_{min} - \phi.$$

So the kinetic energy of the electrons is directly proportion to  $\omega$ , not  $I$  which we expected in classical mechanics. Instead increasing the intensity increases the emission rate as we have more photons interacting with each electron.

### 1.3 Atomic spectra

In 1897 Thomson plum-pudding model purposed a uniform distribution of positive charge, with negatively charged electrons inside.

Later in 1908, Rutherford performed his gold foil experiment, showing Rutherford scattering (large angle scattering), hence proving that the atom was mostly empty space, and the positively charged part was concentrated at the centre of the atom. This couldn't work since the electrons would radiate energy in an orbit and the electrons would collapse on the nucleus. Also it didn't explain the spectra of light emitted by atoms in a set of discrete values of

$$w_{mn} = 2\pi c R_0 \left( \frac{1}{n^2} - \frac{1}{m^2} \right) \quad n, m \in \mathbb{N}.$$

But in 1913 Bohr showed that the electron orbits are quantised so that the  $L$  (orbital angular model) takes only these values

$$L_n = n\hbar$$

So it was proposed that  $L$  was quantised hence so is  $r, v$ , and  $E$ .

*Proof.*  $\mathbf{L} = m_e \mathbf{v} \times \mathbf{r}$  so we have that

$$|\mathbf{L}| = L = M_e v r \implies v = \frac{L}{m_e r} \implies v_n = n \frac{\hbar}{m_e r}$$

$$\mathbf{F}^{\text{Coulomb}} = -\frac{e^2}{4\pi\epsilon_0} \frac{1}{r^2} \mathbf{e}_r$$

and we know that

$$|\mathbf{F}^{\text{Coulomb}}| = m_e a_r$$

so

$$\frac{e^2}{4\pi\epsilon_0} \frac{1}{r^2} = m_e \frac{v^2}{r} \implies r_h = \frac{4\pi\epsilon_0 \hbar^2}{m_e e^2} n^2$$

so finally

$$r_0 = \frac{4\pi\epsilon_0}{m_e e^2} \hbar^2$$

similarly the energy is quantised since

$$E = \frac{1}{2} m_e v^2 - \frac{e^2}{4\pi\epsilon_0} \frac{1}{r} \implies E_n = -\frac{e^2}{8\pi\epsilon_0 r_0} \frac{1}{n^2}.$$

The lowest energy state is called the Ground level,  $E_1$  and the excited levels  $E_2, E_3, \dots$  get closer to each other as energy increases.

We have that

$$\omega_{mn} = \frac{\Delta E_{mn}}{\hbar} = 2\pi c \left( \frac{e^2}{4\pi\epsilon_0 \hbar c} \right)^2 \left( \frac{1}{n^2} - \frac{1}{m^2} \right)$$

where  $\omega_{mn}$  is the angular frequency of light associated with an electron moving from energy state  $m$  to level  $n$ . This is the Bohr prediction from Rydberg constant  $R_0$ .

## 1.4 The wave-like behaviour of particles

In 1923 De Broglie hypothesised that each particle of any mass is associated with a wave having angular frequency,  $\omega$  given by

$$\omega = \frac{E}{\hbar}$$

. In 1927 this was observed by showing electron scattering off crystals which observed an interference pattern consistent with De Broglie's hypothesis.

## 2 Foundation of Quantum Mechanics

In quantum mechanics instead of a vector, we have a state represented by the letter  $\psi$ . The basis is a continuous basis  $\{\mathbf{x}\} \rightarrow \psi(\mathbf{x}, t)$ . Instead of a vector space  $\mathbb{C}^n$  we have a space of square-integrable functions,  $L^2(\mathbb{R}^3)$ . We define an inner product of two states,  $\psi, \phi \in L^2(\mathbb{R}^3)$  as

$$(\psi, \phi) = \int_{\mathbb{R}^3} \phi(\mathbf{x}, t) \psi(\mathbf{x}, t) d^3t.$$

An operator  $\hat{O}$  sends a state  $\psi$  to a state  $\phi$  as shown,

$$\phi = \hat{O}\psi.$$

### 2.1 Wave function and probabilistic interpretation

In classical mechanics  $\mathbf{x}$  and  $\dot{\mathbf{x}}$  determine the dynamics of a particle in a deterministic way.

However in quantum mechanics the state  $\psi(\mathbf{x}, t)$  determines the dynamics of particles in a probabilistic way.

**Definition.** (State of a particle) We say that  $\psi$  is the *state* of a particle, where  $\psi(\mathbf{x}, t)$  is the complex coefficient of  $\psi$  in the continuous basis of  $\mathbf{x}$  at a given time  $t$ . i.e.  $\psi(\mathbf{x}, t)$  is  $\psi$  in the  $\mathbf{x}$  representation and is called a wave function.

$$\psi(\mathbf{x}, t) : \mathbb{R}^3 \rightarrow \mathbb{C}$$

that satisfies mathematical properties dictated by its physical interpretation.

Born's rule or the probabilistic interpretation for a particle described by a state  $\psi$  to sit at  $\mathbf{x}$  at given time  $t$  is

$$\rho(\mathbf{x}, t) \propto |\psi(\mathbf{x}, t)|^2 = \psi^*(\mathbf{x}, t)\psi(\mathbf{x}, t)$$

where

$$\rho(\mathbf{x}, t)dV = \text{probability that the particle sits in the some small volume } dV \text{ centred at } \mathbf{x}.$$

From this we get properties on  $\psi$  as follows:

- (i)  $\int_{\mathbb{R}^3} |\psi(\mathbf{x}, t)|^2 d^3x = \nu \leq \infty$  with  $\nu \in \mathbb{R}$  and  $\nu \neq 0$ .
- (ii) Because the total probability has to be equal to 1 the wavefunction must be normalised to 1. So

$$\bar{\psi}(\mathbf{x}, t) = \frac{1}{\sqrt{\nu}} \psi(\mathbf{x}, t)$$

which integrates to 1 over  $\mathbb{R}^3$ , giving that  $\psi(\mathbf{x}, t) = |\bar{\psi}(\mathbf{x}, t)|^2$ .

*Remark.* Often we drop the  $\bar{\psi}$  notation and just write  $\psi$ , and normalise at the end.

If  $\tilde{\psi}(\mathbf{x}, t) = e^{i\alpha} \psi(\mathbf{x}, t)$  with  $\alpha \in \mathbb{R}$  then we have that

$$|\tilde{\psi}(\mathbf{x}, t)|^2 = |\psi(\mathbf{x}, t)|^2$$

so  $\psi$  and  $\tilde{\psi}$  are equivalent states.

## 2.2 The Hilbert space

The set of all square-integrable functions in  $\mathbb{R}^3$  is called the *Hilbert space* denoted by  $\mathcal{H}$  or  $L^2(\mathbb{R}^3)$ .

**Theorem.** If  $\psi_1, \psi_2 \in \mathcal{H}$  then for  $a_1, a_2 \in \mathbb{C}$  we have that

$$a_1\psi_1 + a_2\psi_2 \in \mathcal{H}$$

*Proof.*

So by the triangle inequality for complex numbers, if  $z_1, z_2 \in \mathbb{C}$  we have that  $|z_1 + z_2| \leq |z_1| + |z_2|$ . So setting

$$z_1 = a_1\psi_1(\mathbf{x}, t)$$

$$z_2 = a_2\psi_2(\mathbf{x}, t)$$

we get that

$$\begin{aligned}
\int_{\mathbb{R}^3} |\psi(\mathbf{x}, t)|^2 d^3x &= \int_{\mathbb{R}^3} |a_1\psi_1(\mathbf{x}, t) + a_2\psi_2(\mathbf{x}, t)|^2 d^3x \\
&\leq \int_{\mathbb{R}^3} (|a_1\psi_1(\mathbf{x}, t)| + |a_2\psi_2(\mathbf{x}, t)|)^2 d^3x \\
&= \int_{\mathbb{R}^3} (|a_1\psi_1(\mathbf{x}, t)|^2 + |a_2\psi_2(\mathbf{x}, t)|^2 + 2|a_1\psi_1||a_2\psi_2|) d^3x \\
&\leq \int_{\mathbb{R}^3} (2|a_1\psi_1(\mathbf{x}, t)|^2 + 2|a_2\psi_2(\mathbf{x}, t)|^2) d^3x \\
&= 2|a_1|^2\nu_1 + 2|a_2|^2\nu_2 \leq \infty \quad \square
\end{aligned}$$

### 2.3 Inner product in $\mathcal{H}$

**Definition.** Inner product in  $\mathcal{H}$  is defined as

$$(\psi, \phi) = \int_{\mathbb{R}^3} \psi(\mathbf{x}, t)\phi(\mathbf{x}, t)d^3x.$$

**Theorem.** If  $\psi, \phi \in \mathcal{H}$  then their inner product exists.

*Proof.*

$$\begin{aligned}
\int_{\mathbb{R}^3} |\psi(\mathbf{x}, t)|^2 d^3x &= \nu_1 \leq \infty \\
\int_{\mathbb{R}^3} |\phi(\mathbf{x}, t)|^2 d^3x &= \nu_2 \leq \infty
\end{aligned}$$

So we have that

$$|(\psi, \phi)| = \left| \int_{\mathbb{R}^3} \psi^*(\mathbf{x}, t)\phi(\mathbf{x}, t)d^3x \right|$$

Hence by the integral Schwartz inequality we have that

$$\leq \sqrt{\int_{\mathbb{R}^3} |\psi(\mathbf{x}, t)|^2 d^3x \cdot \int_{\mathbb{R}^3} |\phi(\mathbf{x}, t)|^2 d^3x}$$

which is

$$\leq \sqrt{\nu_1\nu_2} \leq \infty$$

We have some properties of the inner product

- (i)  $(\psi, \phi) = (\phi, \psi)^*$
- (ii) Antilinear in the first entry and linear in the second entry so we have for  $a_1, a_2 \in \mathbb{C}$  that

$$\begin{aligned}
(a_1\psi_1 + a_2\psi_2, \phi) &= a_1^*(\psi_1, \phi) + a_2^*(\psi_2, \phi) \\
(\psi, a_1\phi_1 + a_2\phi_2) &= a_1(\psi, \phi_1) + a_2(\psi, \phi_2)
\end{aligned}$$

(iii) Inner product of  $\psi \in \mathcal{H}$  is positive i.e. that

$$(\psi, \psi) = \int_{\mathbb{R}^3} |\psi(\mathbf{x}, t)|^2 d^3x > 0$$

**Definition.** (Norm) The *Norm* of a wavefunction  $\psi$  is the real function

$$||\psi|| = \sqrt{(\psi, \psi)}.$$

**Definition.** (Normalised) For a wavefunction  $\psi$  we say that  $\psi$  is normalised if

$$||\psi|| = 1$$

**Definition.** (Orthonormal) A set of wavefunctions  $\{\psi_n\}$  is called *orthonormal* if

$$(\psi_m, \psi_n) = \delta_{mn} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

**Definition.** (Complete) A set of wavefunctions  $\{\psi_n\}$  is *complete* if  $\forall \phi \in \mathcal{H}$  we can write  $\phi$  as a linear combination of  $\{\psi_n\}$ .

**Lemma.** If  $\{\psi_n\}$  forms a complete and orthonormal basis of  $\mathcal{H}$  then

$$c_n = (\psi_n, \phi)$$

*Proof.*

$$\begin{aligned} (\psi_n, \phi) &= \left( \psi_n, \sum_{m=0}^{\infty} c_m \psi_m \right) \\ &= \sum_{m=0}^{\infty} c_m (\psi_n, \psi_m) \\ &= \sum_{m=0}^{\infty} c_m \delta_{nm} \\ &= c_n \end{aligned}$$

## 2.4 Time-dependent Schrodinger equation

We recap that the first postulate of quantum mechanics is that

$$\rho(\mathbf{x}, t) d^3x = |\psi(\mathbf{x}, t)|^2 d^3x$$

The second postulate of quantum mechanics is Schrodinger's equation

$$i\hbar \frac{\partial \psi(\mathbf{x}, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(\mathbf{x}, t) + V(\mathbf{x}) \psi(\mathbf{x}, t)$$



where  $\nabla^2$  is the Laplacian, and  $\nu(\mathbf{x}) \in \mathbb{R}$  is called the potential.

We can see that once  $\psi(\mathbf{x}, t_0)$  is known we have that  $\psi(\mathbf{x}, t)$  is known.

We also have an asymmetry in space and time so the equation is non-relativistic so only holds for speeds not near the speed of light.

The equation is complicated, so let's try and see a heuristic interpretation of the equation. The solution of the wave equation is something like

$$\exp[i(kx - \omega t)]$$

and for an electromagnetic wave we have that  $\omega = ck$ . If we plug the solution to the wave equation into the Schrodinger equation with  $\nu = 0$  we find that  $\omega \propto k^2$ . This is consistent with De Broglie's formula for the wavelength of a particle

$$k = \frac{p}{\hbar}, \quad \omega = \frac{E}{\hbar}$$

For a free particle with  $\nu = 0$  we have that  $E = \frac{p^2}{2m}$ , so using De Broglie's formula,

$$\omega = \frac{p^2}{2m\hbar} = \frac{\hbar}{2m} k^2$$

hence  $\omega \propto k^2$ . So the dispersion relation for a wavefunction is different from the dispersion relation of an electromagnetic wave for with  $\omega \propto k$ . So the free wave solution to TDSE is

$$\sim \exp\left[i\left(kx - \frac{\hbar k^2}{2m}t\right)\right].$$

We want to verify two properties of the TDSE.

- (i) If  $\psi$  obeys the TDSE then normalisation is constant.

*Proof.* In lecture notes, very tedious algebra.

- (ii) Probability is conserved. *Proof.* Follows from the same proof as above.

## 2.5 Measurements in Quantum Mechanics

How do we extract information from a state  $\psi$ ?

**Definition.** (Observable) A property is *observable* if it is described by a particle's state  $\psi$  and can be measured.

So a measurement is the expectation of an operator acting on  $\psi$ . Let's give a heuristic derivation of  $\hat{x}$  and  $\hat{p}$  in one dimension.

From the probabilistic interpretation if we want to measure the position of the particle

$$\begin{aligned} \langle x \rangle &= \int_{-\infty}^{\infty} x |\psi(x, t)|^2 dx \\ &= \int_{-\infty}^{\infty} \psi^*(x, t) x \psi(x, t) dx \end{aligned}$$

The expectation value of an observable property is the mean of an infinite series of measurements performed on particles on the same state  $\psi$ . So if we want to measure  $p$  instead,

$$\begin{aligned}
\langle p \rangle &= m \frac{d\langle x \rangle}{dt} = m \frac{d}{dt} \int_{-\infty}^{\infty} \psi^*(x, t) x \psi(x, t) dx \\
&= m \int_{-\infty}^{\infty} x \frac{\partial}{\partial t} (\psi^* \psi) dx \\
&= \frac{i\hbar m}{2m} \int_{-\infty}^{\infty} x \frac{\partial}{\partial x} \left( \psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x} \right) dx \\
&= -\frac{i\hbar}{2} \int_{-\infty}^{\infty} \left( \psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x} \right) dx \\
&= -i\hbar \int_{-\infty}^{\infty} \psi^* \frac{\partial \psi}{\partial x} dx \\
&= \int_{-\infty}^{\infty} \psi^* \left( -i\hbar \frac{\partial}{\partial x} \right) \psi dx.
\end{aligned}$$

Heuristically, position is given by the operator  $x$  and momentum is given by the operator  $-i\hbar \frac{\partial}{\partial x}$ .

## 2.6 Hermitian operators

An example of an operator is the finite differential operators in the form

$$\sum_{n=0}^N q_n(x) \frac{\partial^n}{\partial x^n}$$

with  $q_n(x)$  a polynomial. Our position and momentum operators both form in the this class.

We also have translation operators sending  $\psi(x) \rightarrow \psi(x - a)$  and parity operations  $\psi(x) \rightarrow \psi(-x)$ .

**Definition.** (Hermitian conjugate) For an operator  $\hat{O}$ , its *Hermitian conjugate*  $\hat{O}^\dagger$  is the operator such that

$$(\hat{O}^\dagger \psi_1, \psi_2) = (\psi_1, \hat{O} \psi_2)$$

for all  $\psi_1, \psi_2 \in \cdot$ .

*Remark.* We can see clearly that

- (i)  $(a_1 \hat{A}_1 + a_2 \hat{A}_2)^\dagger = a_1^* \hat{A}_1^\dagger + a_2^* \hat{A}_2^\dagger$  for all  $a_1, a_2 \in \mathbb{C}$ .
- (ii)  $(\hat{A} \hat{B})^\dagger = \hat{B}^\dagger \hat{A}^\dagger$ .

**Definition.** (Hermitian) An operator  $\hat{O}$  is *Hermitian* if

$$\hat{O} = \hat{O}^\dagger.$$

All physical quantities in quantum mechanics are represented by Hermitian operators. It's not too hard to check all the operators we've covered so far are Hermitian, including, position, momentum, kinetic energy, potential energy, and total energy.

**Theorem.** The eigenvalues of Hermitian operators are real.

*Proof.* Suppose  $\hat{A}$  is Hermitian with an eigenvalue  $a \in \mathbb{C}$  and  $\psi$  with  $\|\psi\| = 1$ . Then

$$(\psi, \hat{A}\psi) = (\psi, a\psi) = a(\psi, \psi) = a\|\psi\|^2 = a.$$

But  $\hat{A}$  is Hermitian so

$$(\psi, \hat{A}\psi) = (\hat{A}\psi, \psi) = (a\psi, \psi) = a^*(\psi, \psi) = a^* = a$$

so  $a \in \mathbb{R}$ . □

