Electromagnetism

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Contents

1 Introduction

1.1 Charges and currents

Electric charge is a physical property of elementary particles. It is:

(i) A signed quantity, it can either be positive, negative, or zero.

(ii) It is quantised to integer multiplies of the elementary charge.

(iii) It is a conserved quantity even if particles are created or destroyed.

By convention the electron has charge $-e$, the proton has charge $+e$ and the neutron has no charge. On macroscopic scales, the number of particles is so large that charge can be considered to have a continuous electric charge density $\rho(\mathbf{x}, t)$. The total charge in a volume V is then

$$Q = \int_V \rho \mathrm{d}V.$$

The *electric current density* $\mathbf{J}(\mathbf{x}, t)$ is the flux of electric charge per unit area. The current folowing through a surface S is

$$I = \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}.$$

Consider a time-independent volume V with boundary S. Since charge is conserved, we have that

$$\begin{aligned} \frac{dQ}{dt} &= -I \\ \frac{d}{dt}\int_V \rho \mathrm{d}V + \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} &= 0 \\ \int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} \right) \mathrm{d}V &= 0 \end{aligned}$$

Since this is true for any V, we have that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$

This *equation of charge conservation* has the typical form of a conservation law.

The discrete charge distribution of a single particle of charge q_i; and position vector $\mathbf{x}_i(t)$, is

$$\begin{aligned} \rho &= q_i \delta(\mathbf{x} - \mathbf{x}_i(t)), \\ \mathbf{J} &= q_i \dot{\mathbf{x}}_i \delta(\mathbf{x} - \mathbf{x}_i(t)). \end{aligned}$$

For N particles, it is

$$\begin{aligned} \rho &= \sum_{i=1}^{N} q_i \delta(\mathbf{x} - \mathbf{x}_i(t)) \\ \mathbf{J} &= \sum_{i=1}^{N} q_i \dot{\mathbf{x}}_i \delta(\mathbf{x} - \mathbf{x}_i(t)). \end{aligned}$$

As an exercise we can see that these satisfy the equation of charge conservation.

1.2 Fields and forces

Electromagnetism is a *field theory.*

Charged particles don't interact directly, but rather by generating fields around them, which are then experienced by other charged particles. In general we have two time-dependent vector fields, the electric field $\mathbf{E}(\mathbf{x}, t)$, and the magnetic field $\mathbf{B}(\mathbf{x}, t)$.

The *Lorentz force* on a particle of charge q and velocity v is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

1.3 Maxwell's equations

In this course we will explore some consequences of Maxwell's equations.

Definition. (Maxwell's equations)

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right).
\end{aligned}$$

Remark. We have some properties about these equations.

- Coupled linear PDEs in space and time,
- Involve two positive constants:
  (i) ε_0 (vacuum permittivity)
  (ii) μ_0 (vacuum permeability)
- Charges (ρ) and currents ($\mathbf{J}$) are the sources of electromagnetic fields.
- Each equation is an equivalent integral form (see later) related via the divergece or Stokes' theorem.
- These are the *vacuum* equations that apply on microscopic scales or in a vacuum. A related macroscopic version applies in media (Part II Electrodynamics).
- The equations of consistent with each other and with charge conservation. We will show this now.
  (i) Taking the divergence of the third equation, this agrees with the time derivative of the second equation.
  (ii) For charge conversation, we have that

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} &= \frac{\partial}{\partial t} \left(\varepsilon_0 \nabla \cdot \mathbf{E} \right) + \nabla \cdot \left(-\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\mu_0} \nabla \times \mathbf{B} \right) \\
&= 0.
\end{aligned}$$

1.4 Units

The SI unit of electric charge is the coulomb (C). The elementary charge is exactly

$$e = 1.602\ 176\ 634 \times 10^{-19}\ \mathrm{C}.$$

The SI unit of electric current is the ampere or amp (A) which is equal to 1 C s^{-1}.

The SI base units needed in electromagnetism and then the second, metre, kilogram, and ampere. From the Lorentz force law we see that the units of $\mathbf{E}$ and $\mathbf{B}$ must be

$$\mathrm{kg\ m\ s^{-3}A^{-1}} \quad \text{and} \quad \mathrm{kg\ s^{-2}A^{-1}}.$$

We sometimes refer to the units of $\mathbf{B}$ as the *Telsa* (T).

From Maxwell's equations we can work out the units of ε_0 and μ_0. The values of these constants can be calculated via experimentation as

$$\begin{aligned} \varepsilon_0 &= 8.854\cdots \times 10^{-12}\ \mathrm{kg^{-1}m^{-3}s^4\ A^2} \\ \mu_0 &= 1.256\cdots \times 10^{-6}\ \mathrm{kg\ m\ s^{-2}A^{-2}} \end{aligned}$$

The speed of light the exactly

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 299\ 792\ 458\ \mathrm{m\ s^{-1}}.$$