Topological Spaces

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Contents

1 Topologies

1.1 Definitions

We denote $\mathcal{P}(X)$ as the power set of X.

Definition. (Topology) Let X be a set. A *topology* on X is a collection of sets $T \subseteq \mathcal{P}(X)$ such that
(i) $\emptyset, X \in T$,
(ii) T is closed under (possibly uncountable) unions.
(iii) T is closed under finite intersections.

A set X with a topology T is called a *topological space* of X. An element of X is called a *point* and elements of T are called *open sets*. If $x \in U \in T$ we say U is an open neighbourhood of x. Strictly we should always denote (X, T) for a topological space, but when T is clear, we just write X for the topological space.

Definition. (Continuity) If (X, T_X) and (Y, T_Y) are topological spaces then a function $f : X \to Y$ is called *continuous* if for $U \in T_Y$, $f^{-1}(U) \in T_X$.

Definition. (Homeomorphism) A function $f : (X, T_X) \to (Y, T_Y)$ is a *homeomorphism* if it is continuous and has a continuous inverse.

Definition. If $T \subseteq T'$ are topologies on X then we say that T is *coarser* and T' is *finer*. The identity function $d : (X, T) \to (X, T')$ is continuous.

1.2 Topologies from metrics

If (X, d) is a metric space, recall that a subset $U \subseteq X$ is called *open* if for every point $x \in U$ there exists a $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$.

Proposition. If T_d is the subset of X which are open under the metric d, then (X, T_d) is a topological space. We will call this the topology on X induced by the metric d.

Proof. Tautologically we have that $\emptyset \in T_d$. Clearly we have that $X \in T_d$ too. Let $\{U_\alpha\}_{\alpha \in I}$ be a collection of open sets in T_d with a (possibly uncountable) index set I. Let

$$x \in \bigcup_{\alpha \in I} U_\alpha.$$

Then $x \in U_\beta$ for some $\beta \in I$, so U_β is open hence there exists a $\varepsilon > 0$ such that $B_\varepsilon \subseteq U_\beta \subseteq \bigcup_{\alpha \in I} U_\alpha$, hence $\bigcup_{\alpha \in I} U_\alpha$ is open.

Now suppoes that I is finite, and $x \in \bigcap_{\alpha \in I} U_\alpha$. For each α there exists a $\varepsilon_\alpha > 0$ such that $B_{\varepsilon_\alpha}(x) \subseteq U_\alpha$. Take $\varepsilon = \inf_{\alpha \in I} \varepsilon_\alpha$, so $B_\varepsilon(x) \subseteq B_{\varepsilon_\alpha}(x) \subseteq U_\alpha$ for all α, hence we have that $B_\varepsilon(x) \subseteq \bigcap_{\alpha \in I} U_\alpha$ so it's open. Hence T is a topology. $\square$

Now we have lots of examples we can use for topological spaces. For example we have that topology induced by the Euclidean metric on $\mathbb{R}^d$ which we will call the Euclidean topology. For any $X \subseteq \mathbb{R}^d$ we can have a topology induced by the Euclidean metric too, like $\mathbb{Q}$, $[0, 1], (0, 1)$.

Proposition. If we have two metric spaces $(X, d_X), (Y, d_Y)$ and we have $f : X \to Y$, the f is continuous in the metric space sense if and only if it is continuous in the topological space sense (with the topologies induced by the metric d_X and d_Y respectively).

Proof. Let $f : X \to Y$ be continuous in the metric space sense. Let U be an open set in T_{d_Y} so we need to show that $f^{-1}(U)$ is open. Let $x \in f^{-1}(U)$, so $f(x) \in U$. Hence there exists an $\varepsilon > 0$ such that $B_\varepsilon(f(x)) \subseteq U$. So since f is continuous there exists a $\delta > 0$ such that if $d_X(x, x') < \delta$, then $d_Y(f(x), f(x')) < \varepsilon$. Hence $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$. So $B_\delta(x) \in f^{-1}(U)$, hence $f^{-1}(U)$ is open.

Now let's do the converse and suppose that $f : X \to Y$ is continuous in the topological sense. Fix some $x \in X$ and $\varepsilon > 0$. Consider $B_\varepsilon(f(x))$ which is open in Y. Then $f^{-1}(B_\varepsilon(f(x)))$ is in T_{d_X}. It contains x so there exists a $\delta > 0$ such that $x \in B_\delta(x)$, so

$$d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \varepsilon.$$

So f is continuous in the metric sense. $\square$

Definition. Let (X, T) be a topological space and $x_1, x_2, \dots \in X$ say. We say that x_n convergences to x if for every open neighbourhood U of x there exists a N such that $x_n \in U$ for all $n \geq N$.

Proposition. If (X, d) is a metric space with topology T_d then a sequence (x_n) converges in the metric sense if and only if it converges in the topological sense.

Proof. Suppose it convergens in the metric sense to x. Then for all $\varepsilon > 0$ there exists a N such that for all $n \geq N$ we have that $x_n \in B_\varepsilon(x)$. If U is a neighourhood of x then there there is some ε such that the ball of radius ε centred at x is contained in U. Conversely if (x_n) converges in the topological sense to x, let $\varepsilon > 0$ and consider the open ball centred at x with radius ε. Now $B_\varepsilon(x)$ is an open neighbourhood of x so there exists an integer N such that $x_n \in B_\varepsilon(x)$ for all $n > N$. Hence (x_n) converges to x in the metric sense. $\square$

Consider $\mathbb{R}$ and $(0, 1)$ with the Euclidean metric and topology. Then the two spaces are related, by the function $(0, 1) \to \mathbb{R}$ by $\tan^{-1} x$ which is invertible. Hence we say the two spaces are homeomorphic, and $\mathbb{R} \cong (0, 1)$. However the two spaces are not isometric since $\mathbb{R}$ is not complete under the Euclidean metric and $(0, 1)$ is not. Hence the property of completeness is not a topological property: it is a property induced by the metric.

Definition. (Discrete topology) Let X be a set. The *discrete* topology is the topology $T_{\text{discrete}} = \mathcal{P}(X)$ (so every set is open).

Remark. Any function from (X, T_{discrete}) to any space si continuous. This toplogy can be induced by the discrete metric, where $d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$. So $B_{\frac{1}{2}}(x) = \{x\}$ so $\{x\}$ is open, hence all the sets are open.

Definition. (Indiscrete topology) Let X be a set. The *indiscrete* topology $T_{\text{indiscrete}} = \{\emptyset, X\}$ (as little as possible sets are open).

Remark. A function from any space to $(X, T_{\text{indiscrete}})$ is continuous. This topology does not come from a metric unless X is a singleton set. This is because if $x \neq y$ then $d(x, y) = \varepsilon > 0$, so $y \notin B_\varepsilon(x)$ and since y is arbitrary, then $B_\varepsilon(x) = \{x\} = X$.

Let $X = \{o, c\}$. Then let $T = \{\emptyset, \{o, c\}, \{o\}\}$ be a topology of X. This is called the Sierpinski space. It as the property that every sequence converges to c. A continuous function $f : T \to (X, T_{\text{Sierpinski}})$ is exactly an open subset of Y.

Let $X = \mathbb{R}$ we'll define the right order topology on X as

$$T_{\text{ord}} = \{(a, \infty) \mid -\infty \leq a \leq \infty\}.$$

Let $\{(a, \infty)\}_{a \in I}$ be a collect of elements of T_{ord}. Then

$$\bigcup_{a \in I} (a, \infty) = (\inf_{a \in I} a, \infty) \in T_{\text{ord}}.$$

Similarly for finite I,

$$\bigcap_{a \in I} (a, \infty) = (\max_{a \in I} a, \infty) \in T_{\text{ord}}$$

1.3 Bases and subbases

Definition. (Basis) Let T be a topology of X. A *basis*, $B \subseteq T$ for T is a subcollection such that every element of T is a union of elements in B.

Definition. (Subbasis) Let T be a topology of X. A *subbasis*, $S \subseteq T$ for T is a subcollection such that every element of T is a union of sets which are finite intersections of elements of S.

Lemma. Let $f : (X, T_X) \to (Y, T_Y)$ and $S \subseteq T_Y$ is a subbasis. If $f^{-1}(U)$ is open for all $U \in S$ then f is continuous.

Proof. If $V \subseteq T_Y$, then $V = \bigcup_{a \in I} V_a$ where $V_a \in \bigcap_{b \in J_a} U_{a,b}$ with $U_{a,b} \in S$ and J_a finite. Then

$$f^{-1}(V) = \bigcup_{a \in I} V_a = \bigcup_{a \in I} \left(\bigcap_{b \in J_a} f^{-1}(U_{a,b}) \right) \in T_X,$$

by the axioms of the topology. $\square$

Consider the Euclidean topology on $\mathbb{R}^n$. The collection $B = \{B_r(x) \mid x \in \mathbb{R}^n, r > 0\}$ is a basis. Likewise the collection of n-cubes everywhere are also a basis. Interestingly the set $QB \subseteq B$ with balls at rational points with rational radii is also a basis. This is interesting since QB is countable while B is uncountable and $\mathcal{P}(\mathbb{R}^n)$ is $\aleph_2$.

Definition. (Closed set) Let (X, T) be a topological space. A subset $C \subseteq X$ is *closed* if $X \setminus C \in T$.

Proposition. Let (X, T) be a topological space and $\mathcal{F} = \{C \subseteq X \mid C \text{ closed}\}$. Then
(i) $\emptyset, X \in \mathcal{F}$;
(ii) $\mathcal{F}$ is closed under (possibly uncountable) intersections;
(iii) $\mathcal{F}$ is closed under finite unions.

Proposition. A function $f : X \to Y$ between topological spaces is continuous if and only if the preimage of every closed set is closed.

Definition. Let (X, T) be a topological space. Let $A \subseteq X$ be a subset of X. Then
(i) The closure $\bar{A}$ is the smallest (by inclusion) closed set containing A so

$$\bar{A} = \bigcap_{S \text{ closed}, A \subseteq S} S.$$

(ii) We say that A is dense in X if $A = \bar{A}$.
(iii) The interior $\mathring{A}$ is the largest open set contained in A so

$$\mathring{A} = \bigcup_{S \text{ open}, S \subseteq A} S.$$

Definition. (Limit point) Let X be a topological space and $A \subseteq X$. A *limit point* of A is a point in X which is a limit of a sequence in A.

Proposition. If C is a closed subset of (X, T), then the limit points of C lie in C.

Proof. Let $\{x_n\}$ be a sequence in C with limit x_∞. If $x_\infty \notin C$, then $x_\infty \in X \setminus C$ which is open. Then if $x_n \to x_\infty$ then we should have that $x_n \in X \setminus C$ for $n \geq N$ but $x_n \in C$ so $x_n \notin X \setminus C$ which is a contradiction. $\square$

Corollary. A limit point of a A lies in $\bar{A}$.

For an example $\overline{\mathbb{Q}} = \mathbb{R}$ since any real number is a limit of a sequence of rational numbers. We have that $\overline{(0, 1)} = [0, 1]$ too. The cocountable topology on $\mathbb{R}$ is the topology $T_{\text{countable}} = \{\emptyset\} \cup \{\mathbb{R} \setminus C \mid C \text{ countable}\}$ Let $\{x_n\}$ be a sequence in $\mathbb{R}$, for $x \in \mathbb{R}$ consider $\{x\} \cup \{\mathbb{R} - \{x_n\}\}$ is open and contains x. If $x_n \to x$, then x_n must be in a U for all $n \geq \mathbb{N}$ so $x_n = x$ for all $n \geq N$. Hence the convergent sequences are exactly the eventually constant sequences with the limits being the value they are eventually constant to. So the limit points of a set A are A under this topology. However almost all A is not closed. For example $(0, 1)$ is not closed since $\mathbb{R} \setminus (0, 1)$ is not countable. But the closure of $(0, 1)$ must be closed, so it must be $\mathbb{R}$ hence the sense of limit points and closure are actually two very different properties in topology instead of metric spaces.

1.4 Hausdorff spaces

Definition. (Hausdorff) A space (X, T) is *Hausdorff* if for $x \neq y \in X$ there are open neighbourhoods $x \in U$, $y \in V$ with $U \cap V = \emptyset$.

Remark. This is the notion that points are seperated by open sets.

Lemma. If the topology T is induced by a metric then it is Hausdorff.

Proof. If $x \neq y$ then $d(x, y) = s > 0$. So consider $U = B_{s/2}(x)$ and $V = B_{s/2}(y)$. The triangle inequality shows that $U \cap V = \emptyset$ and we know all balls are open. □

Proposition. If a space is Hausdorff then a sequence in X has at most 1 limit.

Proof. Let (x_n) be a sequence in X. Suppose it has limits $y \neq z \in X$. Let U and V be disjoint local neighbourhoods sfor y and z respectively. Then $x_n \in U$ for all $n \geq N_1$ and $x_n \in V$ for all $n \geq N_2$. So if we take that $N = \max\{N_1, N_2\}$ then for all $n \geq N$, we have that $x_n \in U \cap V$ which is empty, hence we have a contradiction. □

Proposition. If (X, T) is Hausorff then points are closed.

Proof. Let $x \in X$. We want to show that $\{x\} = \overline{\{x\}}$. Let $y \neq x$. Let U, V be disjoint neighbourhoods of x and y respectively. We know that $x \in X \setminus V$ which is closed. Hence $\overline{\{x\}} \subseteq X \setminus V$. But $y \notin V$, so y is not in the closure of $\{x\}$ hence the closure of $\{x\}$ is just $\{x\}$, so $\{x\}$ is closed. □

Let's see an example. Let X be an infinite set and consider the cofinite topology on X. Take two non-empty open sets, so

$$(X \setminus F) \cap (X \setminus F') = X \setminus (F \cup F')$$

which is non-empty since $F \cup F'$ is finite and X is infinite so the set on the RHS is non-empty hence the space is not Hausdorff.

1.5 Defining new topologies on existing ones

We have three main ways to define new topologies when given a topology already.

1.5.1 The subspace topology

Definition. (Subset topology) Let (X, T_X) be a topological space. Let $Y \subseteq X$ a subset. The *subset topology* on Y is

$$T\,|_Y = \{Y \cap U \mid U \in T\}.$$

Definition. (Subspace) A subspace of (X, T) is a subset equipped with the subspace topology.

Proposition. The subset topology is a topology.

Proof. Simple exercise of the axioms. $\square$

Proposition. The inclusion map $\iota : (Y, T\,|_Y) \to (X, T)$ is continuous. In fact $T\,|_Y$ is the constant topology on Y such that the inclusion map is continuous.

Proof. Let $U \in T$ then $\iota^{-1}(U) = U \cap Y \in T\,|_Y$ by definition. So it is continuous. Suppose $\iota : (Y, T') \to (X, T)$ is continuous. For $U \in T$, $\iota^{-1}(U) \in T'$ so $T\,|_Y \subseteq T'$. $\square$

A further point of view, a function $f : (z, T_z) \to (Y, T\,|_Y)$ is continuous if and only if $\iota \circ f$ is continuous.

Lemma. (Gluing Lemma) Let $f : X \to Y$ be a function between topological spaces.

(i) If $\{U_\alpha\}_{\alpha \in I}$ are open subsets which cover X and each $f\,|_{U_\alpha} : U_\alpha \to Y$ are continuous (where U_α is given the subspace topology) then f is continuous.

(ii) If $\{C_\alpha\}_{\alpha \in I}$ is a finite collection of closed sets containing X and $f\,|_{C_\alpha} : C_\alpha \to Y$ is continuous for each $a \in I$ then f is continuous.

Proof. Let $V \subseteq Y$ be open. We want to show that $f^{-1}(V)$ is open. We know that

$$\begin{aligned} f^{-1}(V) = ({}^{-1}V) \cap X = f^{-1}(V) \cap \left(\bigcup_{\alpha \in I} U_\alpha \right) \\ = \bigcup_{\alpha \in I} f^{-1}(V) \cap U_\alpha \end{aligned}$$

Since $f\,|_{U_\alpha}$ are continuous, we have that $f^{-1}\,|_{U_\alpha}$ is open in U_α in the subspace topology. So there exists a W open in X such that $f^{-1}\,|_{U_\alpha}(V) = U_\alpha \cap W$ hence this is the intersection on open subsets of X so is open in X, hence since the union of open subsets is open $f^{-1}(V)$ is open, so f continuous.

The second part can be proved the same using the closed set definition of continuity. $\square$

If (X, d) is a metric space with topology T_d and $Y \subseteq X$ then $T_d\,|_Y$ is the topology induced by $d\,|_Y$.

1.5.2 The quotient topology

Definition. (Quotient topology) Let (X, T_X) be a topological space, $\sim$ an equivalence relation on X and $X/\sim$ is the set of equivalence classes, and $\pi : X \to X/\sim$ the equivalence map. The *quotient toplogy* on $X/\sim$ is

$$T_{X/\sim} = \{U \subset X/\sim \mid \pi^{-1}(U) \in T_X\}.$$

Proposition. $T_{X/\sim}$ is indeed a topology.

Proof. $\emptyset = \pi^{-1}(\emptyset) \in T_X$ so $\emptyset \in T_{X/\sim}$. $X = \pi^{-1}(X/\sim) \in T_X$ so $X/\sim \in T_{X/\sim}$. Let $\{U_\alpha\}$ be a collection of sets of $T_{X/\sim}$, then

$$\pi^{-1}\left(\bigcup_{\alpha \in I} U_\alpha\right) = \bigcup_{\alpha \in I} \pi^{-1}(U_\alpha),$$

and $\pi^{-1}(U_\alpha) \in T_X$, so the union is too. Hence $\bigcup_{\alpha \in I} U_\alpha \in T_{X/\sim}$. We have a similar proof for finite intersections. □

Proposition. The quotient map $\pi : (X, T_X) \to (X/\sim, T_{X/\sim})$ is continuous and $T_{X/\sim}$ is the finest topology for which this is true.

Proof. This is a tautology. □

An alternative characterisation of the quotient topology is that $f : X/\sim \to Y$ is continuous if and only if $f \circ \pi : X \to Y$ is continuous.

Definition. For a continuous function $g : (X, T_X) \to (Y, T_Y)$ is a *quotient map* if it surjective and $U \in T_Y \iff g^{-1}(Y) \in T_X$.
Given, this construct $\sim$ on X by $x \sim x' \iff g(x) = g(x')$. There is an induced function $G : X/\sim \to Y$ sending $G([x]) = g(x)$.

Remark. This function G is a bijection and continuous with a continuous inverse. This means that G is a homeomorphism, so $X/\sim \cong Y$.

Let's see an example on $\mathbb{R}$. Consider $x \sim y \iff x - y \in \mathbb{Z}$. What is $\mathbb{R}/\sim$? Consider $f : \mathbb{R} \to \mathbb{R}^2$ defined by $x \to (\sin(2\pi x), \cos(2\pi x))$. This is a continuous map so $f : \mathbb{R} \to S^1 \subseteq \mathbb{R}^2$ is also continuous and surjective. By periodicity $x \sim y \iff f(x) = f(y)$, so we get $F : \mathbb{R}/\sim \to S^1$ which we can check is a homeomorphism.

Now take the example $X = \mathbb{R} \times \{0, 1\} \subseteq \mathbb{R}^2$ with the standard subspace topology. Let $(x, i) \sim (y, j) \iff (x, i) = (y, j)$ or $x = y \neq 0$. We can then think of $X/\sim$ is a line with two origins. We cannot draw $X/\sim$ since it is not Hausdorff. Any neighbourhood of $[(0, 0)]_\sim$ intersects any neighbourhood of $[(1, 0)]_\sim$ so not Hausdorff. Hence it is not subspace of any Euclidean space.

1.5.3 The product topology

For sets X, Y the projections functions are

$$\begin{aligned} \pi_X : & X \times Y \to X \\ & (x, y) \to x \end{aligned}$$

and

$$\begin{aligned} \pi_Y : & X \times Y \to Y \\ & (x, y) \to y \end{aligned}$$

Definition. (Product topology) Let (X, T_X) and (Y, T_Y) be topological spaces. Then *product topology* on $X \times Y$ consists of open sets $U \subseteq X \times Y$ such that for $(x, y) \in U$ there is a $V \in T_X$ and $W \in T_Y$ such that $(x, y) \in V \times W \in U$.

Proposition. This indeed is a topology and the sets $V \times W$ are a basis for $T_{X\times Y}$.

Proof. Tautologically, we have that $\emptyset \in T_{X\times Y}$. Taking $V = X, W = Y$ we have that $X \times Y \in T_{X\times Y}$. For a collection $\{U_\alpha\}_{\alpha\in I}$ of elements of $T_{X\times Y}$, let $(x, y) \in \bigcup_{\alpha\in I} U_\alpha$. Then $(x, y) \in U_\beta$ for $\beta \in I$ so there exists neighbourhoods of x, y with their product a subset of $U_\beta \subseteq \bigcup_{\alpha\in I} U_\alpha \in T_{X\times Y}$. If I is finite and $(x, y) \in \bigcap_{\alpha\in I} U_\alpha$. Then $(x, y) \in V_\alpha \times W_\alpha \subseteq U_\alpha$ for each $\alpha \in I$. So $(x, y) \in (\bigcap_\alpha V_\alpha) \times (\bigcap_\alpha W_\alpha) \in \bigcap_\alpha U_\alpha$ and since these intersections are finite, these intersections are open. $\square$

Proposition. The projection maps

$$\pi_X : (X \times Y, T_{X\times Y}) \to (X, T_X) \qquad \pi_Y : (X \times Y, T_{X\times Y}) \to (Y, T_Y)$$

are continuous and $T_{X\times Y}$ is the coarsest topology for which this is true.

Proof. Let $V \in T_X$. Then $\pi_X^{-1}(V) = V \times Y$, so this is open. Hence π_X, π_Y are continuous.

Suppose that T' is a topology on $X\times Y$ such that π_X and π_Y are continuous, then $\pi_X^{-1}(V) = V\times Y$ is open and $\pi_Y^{-1}(W) = X \times W$ is open. So $V \times W$ is open in T', so $T_{X\times Y} \subseteq T'$. $\square$

The universal property of the product topology is that the function

$$f : (Z, T_Z) \to (X \times Y, T_{X\times Y})$$

is continuous if and only if $\pi_X \circ f : (Z, T_Z) \to (X, T_X)$ and $\pi_Y \circ f : (Z, T_Z) \to (Y, T_Y)$ are continuous. Equivalently f is componentwise continuous if and only if it is componentwise continuous.

2 Connectivity

2.1 Connected and disconnected

We know from IA Analysis I, if $f : [0, 1] \to \mathbb{R}$ is continuous, $f(0) < 0 < f(1)$ then $f(t) = 0$ for some $t \in [0, 1]$. This is a statement about continuous functions, but also about the interval $[0, 1]$. For example if we change the interval to $[0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ then this does not satisfy the intermediate value theorem. The property of the interval we're using is connectedness.

Definition. (Disconnected) A topological space X is *disconnected* if $X = U \cup V$ for U, V disjoint nonempty open sets.

Definition. (Connected) A topological space is *connected* if it is not disconnected.

If $X = U \cup V$ is disconnected, then U and V are both open and also both closed.

Any set with the coarse topology is connected, due to the lack of non-trivial open sets. A set with the discrete topology is disconnected, if it has more than 1 point, since every set is open, so the result is trivial.

The set $X = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1] \subseteq \mathbb{R}$ is disconnected since $[0, \frac{1}{2})$ is open in X and $(\frac{1}{2}, 1]$ is open in X too. They are disjoint, hence X is disconnected.

Proposition. A space X is disconnected if and only if, there is a continuous surjection $f : X \to \{0, 1\}$ where $\{0, 1\}$ is equipped with the discrete topology.

Proof. Suppose that X is disconnected. So $X = U \cup V$ disjoint. The define f such that

$$f(x) = \begin{cases} 0 & x \in U \\ 1 & x \in V \end{cases}.$$

This is well-defined since U and V are disjoint. Since U and V are non-empty, the function is surjective. The preimage of $\{0\}$ and $\{1\}$ are U and V respectively which we know is open. And the preimage of $\{0, 1\}$ and $\emptyset$ are clearly open, so f is continuous.

Conversely suppose that f is continuous. Then define $U = f^{-1}(\{0\})$ and $V = \int f(\{1\})$. So since f is continuous, U and V are open. Clearly U and V is disjoint and non-empty since f is surjective. We have that $X = U \cup V$ since $X = f^{-1}(\{0, 1\}) = f^{-1}(0) \cup f^{-1}(1) = U \cup V$. $\square$

Theorem. The spaces $[0, 1]$, $[0, 1)$, $(0, 1)$ are all connected.

Proof. Let's just consider $[0, 1]$, the rest of the proves are similar. If it was disconnected, then there is a continuous surjection

$$f : [0, 1] \to \{0, 1\} \subseteq \mathbb{R}.$$

Then

$$f(\cdot) - \frac{1}{2} : [0, 1] \to \mathbb{R}$$

is continuous and takes the values $\pm\frac{1}{2}$ only. By the intermediate value theorem, we should have that f takes the value 0 which is a contradiction hence $[0, 1]$ is connected. $\square$

Theorem. (Generalised intermediate value theorem) Let X be a connected topological space and $f : X \to \mathbb{R}$ continuous. If there exists $x_0, x_1 \in X$ such that $f(x_0) < 0 < f(x_1)$ then there exists a $x_2 \in X$ such that $f(x_2) = 0$.

Proof. Consider the open sets $U = f^{-1}((-\infty, 0)), V = f^{-1}((0, \infty))$. f is continuous, so U, V are open. We know that x_0, x_1 exist hence U, V are non-empty. If $f(x)$ is never zero, then $X = U \cup V$ disjoint and open so X is disconnected. But X is connected hence $f^{-1}(0)$ is non-empty, so pick $x_2 \in f^{-1}(0)$, so $f(x_2) = 0$. $\square$

Proposition. Let $f : X \to Y$ be a continuous surjection. Then X connected implies that Y is connected.

Proof. Let's show the contrapositive. Suppose that Y is disconnected. Then we have some $h : Y \to \{0,1\}$ continuous and surjective. So

$$h \circ f : X \to \{0,1\}$$

is also continuous and surjective, hence X is disconnceted. $\square$

Corollary. If X is connected and $f : X \to Y$ is continuous then $\mathrm{im}(f)$ is connected.

Proof. Apply the proposition to $f : X \to \mathrm{im}\, f$.

For example if X is a connected space and $\sim$ is an equivalence relation then $\pi : X \to X/\sim$ is a continuous surjection so $X/\sim$ is connected.

Lemma. If $f : X \to Y$ is a homeomorphism and $Z \subseteq X$, then $f\mid_Z : Z \to \mathrm{im}(f\mid_Z)$ is a homeomorphism.

Proof. Obvious. $\square$

Let's use this to show that $[0,1]$ is not homeomorphic to $(0,1)$. Suppose they are. So we have a homeomorphism $f : [0,1] \to (0,1)$. Let's now restrict f to $(0,1]$. Then by the lemma we know that $f\mid_{(0,1]}$ is a homeomorphism with

$$f\mid_{(0,1]} : (0,1] \to (0,1) \setminus \{f(0)\}$$

for some $0 < f(0) < 1$. But $(0,1]$ is connected and $(0,1) \setminus \{f(0)\} = (0, f(0)) \cup (f(0), 1)$ so $(0,1) \setminus \{f(0)\}$ is disconnected which is a contradiction.

We can do a similar process to show that S^1 is not homeomorphic to $\mathbb{R}$. We know that S^1 is connected since it is a quotient space of $\mathbb{R}$ and $\mathbb{R}$ is connected since $\mathbb{R} \cong (0,1)$. Suppose that S^1 is homeomorphic to $\mathbb{R}$. Then remove the point $(1,0) \in S^1$ and consider the restricted homeomorphism between the new spaces. $\mathbb{R}$ is no longer connected since $\mathbb{R} \setminus \{f(1,0)\} = (-\infty, f(1,0)) \cup (f(1,0), \infty)$, but $S^1 \setminus \{f(1,0)\}$ is connected since it's homeomorphic to $(0,1)$.

Proposition. Let $\{X_\alpha\}_{\alpha \in I}$ be a collection of subspaces of X. Suppose that each X_α is connected and $\bigcap_{\alpha \in I} X_\alpha \neq \emptyset$. Then $\bigcup_{\alpha \in I} X_\alpha$ is connected.

Proof. Let X be the union of all the sets. If it were disconnected then $X = U \cup V$ with U, V open, so $U \cap X_\alpha, V \cap X_\alpha$ are disjoint open subsets covering X_α. Since X_α is connected one of them must be zero so $X_\alpha \subseteq U$ or $X_\alpha \subseteq V$. This holds for each X_α but as $\bigcap_{\alpha \in I} X_\alpha \neq \emptyset$ they are all in U or all in V hence X is U or X is V so one of them is empty. $\square$

Corollary. If X and Y are connected then so is $X \times Y$.

Proof. Suppose X, Y both non-empty (since the empty set is connected). Choose $x \in X$. Consider

$$C_y = \{x\} \times Y \cup X \times \{y\}.$$

The sets $\{x\} \times Y$ and $X \times \{y\}$ intersect in (x, y) and pieces are connected by assumption so C_y is connected. Now observe that

$$X \times Y = \bigcup_{y \in Y} C_y$$

the intersection of all of C_y is $\{x\} \times Y$ which is non-empty hence the proposition applies and $X \times Y$ is connected.

2.2 Path-connectedness

Definition. (Path) If X is a topological space and $x_0, x_1 \in X$ a *path* between them is a continuous $\gamma : [0,1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

Definition. (Path connected) A topological space is *path connected* if for any two points in the space, there is a path between them.

The spaces $(0,1), [0,1], (0,1]$ are all path-connected just by taking the line $\gamma(t) = (1-t)x_0 + tx_1$. For the same reason $\mathbb{R}^n$ is path-connected and any *convex* subset $X \subseteq \mathbb{R}^n$ is too.

Now let's let $X = \mathbb{R}^2 \setminus \{0\}$ be the punctured plane. We do the same again if our linear path doesn't go through the (missing) origin. If it does, just take a circular path instead.

Proposition. A path-connected space is connected.

We'll prove the contrapositive. If X is not connected, we have a continuous surjective function $f : X \to \{0,1\}$ Let x_0, x_1 be such that $f(x_0) = 0, f(x_1) = 1$ and suppose that X is path connected so there is a path connecting x_0, x_1. So $f \circ \gamma$ is a surjective and continuous map from $[0,1] \to \{0,1\}$ hence $[0,1]$ is disconnected which is a contradiciton. □

We can now show that $\mathbb{R}^n \ncong \mathbb{R}$ for $n > 1$. If it were then $\mathbb{R}^n \setminus \{0\} \cong \mathbb{R} \setminus \{0\}$. But the RHS is disconnected and the LHS is path-connected, contradiction.

Proposition. If X and Y are path-connected then $X \times Y$ is path-connected.

Proof. Omitted.

2.2.1 Path components

Let X be a space. Define an equivalence relation on X, $\sim$, defined by

$$x \sim y \iff \text{there exists a path from } x \text{ to } y.$$

Lemma. $\sim$ is indeed an equivalence relation.

Proof.

(i) $x \sim x$ since we can take the path $\gamma(t) = x$.

(ii) If $x \sim y$ then we have a path $\gamma(t)$ connecting x and y. Then we can take the path $\gamma'(t) = \gamma(1-t)$ which goes from y to x. Hence $y \sim x$.

(iii) If $x \sim y$ and $y \sim z$, there there is a path γ' connecting x to y and γ' connecting y to z. Define

$$(\gamma' \cdot \gamma)(t) = \begin{cases} \gamma(2t) & 0 \le t \le \frac{1}{2} \\ \gamma'(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}.$$

This is a well defined function since $\gamma(1) = \gamma'(0)$. It is continuous by the gluing lemma applied to $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ so it is a path between x and z, so $x \sim z$.

Hence $\sim$ is an equivalence relation. $\square$

Definition. (Path components) The *path components* of X are the equivalence classes of $\sim$.

Claim. Each component is path-connected.

Proof. Suppose that x, y are in the same path component, so there exists a path γ connecting x and y. Since $\gamma\mid_{[0,t]}: [0, t] \to X$ is a path from $\gamma(0)$ to $\gamma(t$ every point on the path is in the path component, hence the path lies in the equivalence class. $\square$

2.2.2 Connected components

For a space X, define $\approx$ by $x \approx y$ if and only if there exists a subset $C \subseteq X$ connected with $x, y \in C$.

Lemma. $\approx$ is an equivalence relation.

Proof.

(i) $\{x\}$ is connected, so $x \approx x$.

(ii) Definition is symmetric so $x \approx y \iff y \approx x$.

(iii) If $x \approx y$ and $y \approx z$ then there are connected subsets C_1 containing x, y and C_2 containing y, z. Then let $C = C_1 \cup C_2$ which is connected since the intersection contains y so non-empty and contains x, z so $x \approx z$.

Hence $\approx$ is an equivalence relation. $\square$

Definition. (Connected components) Let the *connected components* of X are the equivalence classes of $\approx$.

Proposition. The connected components are connected.

Proof. Let $C \subseteq X$ be a connected component. Suppose that $f : C \to \{0, 1\}$ is a surjective continuous function. Let $f(x_0) = 0$ and $f(x_1) = 1$. As $x_0 \approx x_1$, there exists a connected space $D \subseteq X$ with $x_0, x_1 \in D$. If $d \in D$, then $d \approx x_0$, hence $D \subseteq C$. But then $f\mid_D: D \to \{0, 1\}$ is a continuous surjective function so D is disconnected, contradiction, hence C is connected. $\square$

For example $X = (-\infty, 0) \cup (-, \infty)$ has connected components $(-\infty, 0), (0, \infty)$ which are also the path-components.

For $\mathbb{Q} \subset \mathbb{R}$ the connected compoments are the singletons, so are the path-components.

Let's look at the *Topologist's sine curve*, which is defined as

$$S = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \in \mathbb{R} \mid 0 < x \leq 1 \right\}.$$

This is not closed and it's closure is

$$\overline{S} = S \cup (\{0\} \times [-1, 1]).$$

We want to show that $\overline{S}$ is connected but *not* path connected. Since S is the image of $(0, 1]$ under $x \to \left(x, \sin\left(\frac{1}{x}\right)\right)$ hence S is connected.

Lemma. The closure of a connected subspace is connected.

Proof. Let $C \subset X$ be a connected subspace and suppose that $\overline{C} = U \cup V$ is disconnected, so U, V are non-empty, disjoint open sets of $\overline{C}$. Thus U, V are also closed in $\overline{C}$. Hence they are closed in X too. Consider $C \cap U$ and $C \cap V$. This is a cover of C by disjoint open subsets. So one of them is empty, so $C \subset U$ without loss of generality. But since U is closed in X, $\overline{C} \subset U$ too, so $\overline{C} \cap V = \emptyset$. This contradicts the fact that $U \cup V$ was a disconnection. $\square$

Hence we have that $\overline{S}$ is connected. Note that $\{0\} \times [-1, 1]$ and S are both path connected, so to show that $\overline{S}$ is not path connected we need to show that there is no $\gamma : [0, 1] \to \overline{S}$ with $\gamma(0) \in \{0\} \times [-1, 1]$ and $\gamma(1) \in S$. Suppose that γ is such a path. As $\{0\} \times [-1, 1]$ is closed, $\gamma^{-1}(\{0\} \times [-1, 1])$ is closed, so it contains its supremum t. Then $\gamma \mid_{[t,1]}$ is a new path with $\gamma(t) \in \{0\} \times [-1, 1]$ and $\gamma(s < t) \in S$. SO we replace γ by $\gamma \mid_{[t,1]}$. So we can write $\gamma(s) = (x(s), y(s))$ for $x, y : [0, 1] \to \mathbb{R}$ continuous. For $s > 0$ we have that $y(s) = \sin\left(\frac{1}{x(s)}\right)$. Our goal is to find a sequence $s_n \to 0$ in $[0, 1]$ such that $y(s_n) = (-1)^n$. Then this is a contradiction since it should converge to $y(0)$ since y is continuous. For each n choose a $0 = x(0) < a < x\left(\frac{1}{n}\right)$ such that $\sin\left(\frac{1}{u}\right) = (-1)^n$. By the intermediate value theorem we have that $u = x(s_n)$ for some $0 < s_n < \frac{1}{n}$. So $y(s_n) = (-1)^n$ as required. $\square$

2.3 Compactness

Definition. (Cover) A collection $\mathcal{X} \subset P(X)$ is a *cover of* X if for each $x \in X$ there is a $S \in \mathcal{X}$ with $x \in S$.

Definition. (Open cover) An *open cover* of X is a cover consisting of open sets.

Definition. (Subcover) A *subcover* of $\mathcal{X}$ is a $\mathcal{X}' \subseteq \mathcal{X}$ which is also a cover.

Let's give a definition now which generalised the idea of being closed and bounded without need of a metric.

Definition. (Compact) A topological space X is *compact* if every open cover has a finite subcover.

$\mathbb{R}$ is not compact by considering the open sets $\{(n-1, n+1) \subset \mathbb{R} \mid n \in \mathbb{Z}\}$ which clearly is an open cover but has no finite subcover since there isn't even a subcover, since each $n \in \mathbb{Z}$ is only contained in exactly one open interval in the cover.

If X is a topological space with finitely many points then it is compact.

If $X = \{0\} \cup \{\frac{1}{n}\}$ for $n = 1, 2, \ldots$. Then X is compact. Let $\mathcal{U}$ be an open cover of X. The point $0 \in X$ lies in some $U_0 \in \mathcal{U}$. As this is open, there exists an $\varepsilon > 0$ such that $0 \in B_\varepsilon(0) = (-\varepsilon, \varepsilon) \subset U_0$. Thus $\frac{1}{n} \in U_0$ for $0 < \frac{1}{n} < \varepsilon$ for $n > \frac{1}{\varepsilon}$. The finitely many points $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{m}$ that do not satisfy this, lie in open sets $1 \in U_1, \frac{1}{2} \in U_2, \ldots, \frac{1}{m} \in U_m$. Hence $\{U_i\}_{i=0}^m$ is a finite subcover.

The space $Y = \{\frac{1}{n} \mid n = 1, 2, \ldots\} \subseteq \mathbb{R}$ is *not* compact. This is because it is an infinite set and the topology is discrete since $\left\{\frac{1}{n}\right\} = Y \cap \left(\frac{1}{n+1}, \frac{1}{n+1}\right)$. Hence the cover by each of the points has no subcover at all.

Theorem. $[0,1]$ is compact.

Proof. Let $\mathcal{U}$ be an open cover of $[0,1]$. Consider

$$A = \{a \in [0,1] \mid \text{there is a finite } \mathcal{U}' \subset \mathcal{U} \text{ whose union contains } [0,a]\}.$$

If $0 \le a \le b$ then $[0,a] \subseteq [0,b]$. So if $b \in A$ then $a \in A$. A is not non-empty since $0 \in A$ since $[0,0] = \{0\}$ has some $U \in \mathcal{U}$ containing 0. Hence by completeness A has a supremum, so set $\alpha = \sup A \in [0,1]$. We want to show that $\alpha = 1$. Let $\alpha \in U_\alpha \in \mathcal{U}$. If $\alpha < 1$ then there is an $\varepsilon > 0$ such that $(a-, a+\varepsilon) \subset [0,1] \subset U_\alpha$. So as $\alpha - \varepsilon < \alpha$, there must be a $\alpha - \varepsilon \le a < \varepsilon$ with $a \in A$. Hence $\alpha - \varepsilon \in A$ too. But then $[0, \alpha - \varepsilon]$ has a finite cover by elements of $\mathcal{U}$. Adding U_α we have a finite subcover of $[0, \alpha + \varepsilon/2]$ hence this contradicts the fact that α is a supremum, so $\alpha = 1$ and $[0,1]$ is compact. $\square$

Let's look at some consequences of compactness.

Proposition. If X is compact and $C \subseteq X$ is closed, then C is compact.

Proof. Open sets in C are of the form $C \cap U_\alpha$ for U_α open in X. Suppose we have $\{C \cap T_\alpha \mid \alpha \in I\}$ which is an open cover of C. Then $\{X \setminus C\} \cup \{U_\alpha \mid \alpha \in I\}$ is an open cover X. So as X is compact, this has a finite subcover. It can be taken to have the form $\{X \setminus C\} \cap \{U_\alpha \mid \alpha \in I'\}$ for $I' \subseteq I$ finite. As these cover X, their intersections with C cover C, so $\{C \cap U_\alpha \mid \alpha \in I'\}$ is a finite subcover of C, hence C is compact. $\square$

Proposition. If X is Hausdorff and $C \subseteq X$ is compact, then C is closed.

Proof. We will show that $U = X \setminus C$ is open. So for $x \in U$, we have to find an open $x \in U_\alpha \subseteq U$. For each $y \in C$ use Hausdorffness to find disjoint open sets $y \in V_y$ and $x \in W_y$. Then $\{C \cap V_y \mid y \in C\}$ is an open cover of C. As C is copmact, we can find $y_1, \ldots, y_n \in C$ such that $\{C \cap V_y \mid i = 1, \ldots, n\}$ cover C. Let

$$U_x = \bigcap_{i=1}^{n} W_{y_j}.$$

This contains x and is a finite intersection of open sets, so it is open. As W_{y_i} is disjoint from V_{j_i}, $\bigcap_{i=1}^n W_{y_i}$ is disjoint from $\bigcap_{i=1}^n V_{y_i} \supseteq C$, so U_α is also disjoint fro C. So $\alpha \in U_\alpha \subseteq U = X \setminus C$. Doing this for each $x \in U$ shows that U is open. $\square$

Proposition. If X is compact and $f : X \to Y$ is continuous, then $f(X)$ is a compact subspace of Y.

Proof. We can suppose that f is surjective by replacing Y with $f(X)$. If $\mathcal{U}$ is an open cover of Y consider $f^{-1}(\mathcal{U}) = \{f^{-1}(U) \mid U \in \mathcal{U}\}$. This is an open cover of X, so by compactness it has a finite subcover $f^{-1}(U_1), \dots, f^{-1}(U_n)$. We claim that $U_1, \dots, U_n$ covers Y. This is because we can write

$$X = \bigcup_{i=1}^{n} f^{-1}(U_i)$$

so

$$f(X) \subseteq \bigcup_{i=1}^{n} U_i \subseteq Y,$$

hence we have

$$\bigcup_{i=1}^{n} f(U_i) = Y$$

so Y is compact. □

Corollary. If $f : X \to Y$ is a continuous bijection from a compact space X to a Hausdorff space Y, then it is a homeomorphism.

Proof. We only need to show that $f^{-1} : Y \to X$ is continuous. We will use the closed set characterisation of continuity. Let $C \subseteq X$ be closed. We want $(f^{-1})^{-1}(C) = f(C)$ closed. We know that C is closed and X is compact hence C is compact. Hence $f(C)$ is compact and by Hausdorffness, $f(C)$ is closed, so f is a homeomorphism. □

Definition. (Sequentially compact) A metric space (X, d) is *sequientially compact* if every sequence in X has a convergent subsequence.

The metric d gives a topology T_d on X.

Lemma. (Lebesgue's number lemma) Let (X, d) be sequentially compact and $\mathcal{U} \subset T_d$ be an open cover. Then there is a $\delta > 0$ such that each $B_\delta(X)$ lies inside some element of $\mathcal{U}$.

Proof. Suppose not. Then for each $n = 1, 2, \dots,$ there exists a point $xx_n \in X$ such that $B_{\frac{1}{n}}(x_n)$ is *not* contained in any element of $\mathcal{U}$. By sequentially compactness there is a convergent subsequence $x_{n_i} \to x_\infty$. Let $x_\infty \in U \in \mathcal{U}$ and $\varepsilon > 0$ be such that $B_\varepsilon(x_\infty) \subseteq U$. For all $i \gg 0$ we have that $x_{n_i} \in B_{\varepsilon/2}(x_\infty)$ and $\frac{1}{n_i} < \frac{\varepsilon}{2}$. The triangle inequality gives a contradiction from here. So there is such a δ called the Lebesgue number for the cover $\mathcal{U}$. □

Theorem. The metric space (X, d) is sequentially comapct if and only if (X, T_d) is compact.

Proof. Suppose that (X, d) is not sequentially compact. Then there is a sequence t_n in X with no convergent subsequence. i.e. for each $x \in X$ there is an open $U_x \ni x$ which contains only finitely many (t_i). Then $\mathcal{U} = \{U_x \mid x \in X\}$ is an open cover of (X, T_d). If $\mathcal{U}$ had a finite subcover, then as each U_x contains only finitely many values that the sequence (t_i) takes. But this is a contradiction since (t_i) does not have a convergent subsequence. Hence $\mathcal{U}$ doesn't have a finite subcover, so (X, T_d) is not compact.

Suppose now that (X, d) is sequentially comapct and $\mathcal{U}$ is an open cover of (X, T_d). Let δ be a Lebesgue number for this cover by the lemma. We want to show that there is a finite set of points $A \subseteq X$ such that

$$X = \bigcup_{a \in A} B_\delta(a).$$

Suppose not. Then for every finite set A, there is a point $x \in X$ which lies at least δ away for all $a \in A$. So we can find a sequence $x_1, x_2, \ldots$ such that $d(x_i, x_j) \geq \delta$ for all $i \neq j$. This property means it has no convergent subsequence which is a contradiction since (X, d) is sequentially compact. Hence A does exist. Now each $B_\delta(a)$ lies in some $U_a \in \mathcal{U}$ since δ is a Lebesgue number, hence $\{U_a \mid a \in A\}$ is a finite subcover of $\mathcal{U}$. $\square$