Analysis II

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1 Uniform Convergence

For a subset $E \subseteq \mathbb{R}$, have a sequence $f_n : E \to \mathbb{R}$. What does it mean for the sequence (f_n) to converge? The most basic notion for any $x \in E$ require that the sequence of real numbers $f_n(x)$ to converge in $\mathbb{R}$. If this holds we can defined a new function $f : E \to \mathbb{R}$ by setting each value to the limit of the function.

Definition. (Pointwise limit) We say that (f_n) converges *pointwise* if for all x in its domain we have that

$$f(x) = \lim_{n\to\infty} f_n(x)$$

converges. We write that $f_n \to f$ pointwise.

Are properties such as continuity, differentiability integrability, preserved in the limit? We'll use an example to show that continuity is not preserved.

We can see this by taking a sequence of functions which converge to a step function by taking tighter and tighter curvers which get steeper and steeper. For example take,

$$f_n : [-1,1] \to \mathbb{R}, \quad f_n(x) = x^{\frac{1}{2n+1}}.$$

So in the limit we get that

$$f_n(x) \to f(x) = \begin{cases} 1 & 0 < x \leq 1 \\ 0 & x = 0 \\ -1 & -1 \leq x < 0 \end{cases}$$

which is not continious.

For an example where integability is not preserved, let $q_1, q_2, q_3, \ldots$ be an enumeration of $\mathbb{Q} \cap [0,1]$ and define

$$f_n(x) = \begin{cases} 1 & x \in \{q_1, \ldots, q_n\} \\ 0 & \text{otherwise} \end{cases}$$

so we get $f_n(x)$ continious everywhere on $[0,1]$ apart from a finite number of points, then f_n is integrable on $[0,1]$ (IA Analysis I). But,

$$\lim_{n\to\infty} f_n(x) = \mathbf{1}_{\mathbb{Q}}(x)$$

which we know is not integrable.

If $f_n \to f$ pointwise, f_n integrable, f integrable, does it follow that $\int f_n \to \int f$? (Spoiler: No) For example take f_n to be a 'spike' with height n and width $\frac{2}{n}$, concretely,

$$f_n(x) = \begin{cases} n^2 x & 0 \leq x \leq \frac{1}{n} \\ n^2(\frac{2}{n} - x) & \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \text{otherwise} \end{cases}$$

So the integral of f_n over $[0,1]$ is 1, but we can see that f_n converges pointwise to zero. So $\int_0^1 f_n \to 1$ but $\int_0^1 f \to 0$.

So we need a better (stronger) notion for the convergence of a sequence of functions. We can't use something too strong, such as $f_n \to f$ if f_n is eventually f for large enough n. We've got to find something inbetween. This is uniform convergence.

Definition. (Uniform convergence) Let $f_n, f : E \to \mathbb{R}$, for $n \in \mathbb{N}$. We say that (f_n) converges *uniformly* on E if the following holds. For all $\varepsilon > 0$, $\exists N = N(\varepsilon)$ such that for every $n \geq N$ and for every $x \in E$ we have that $|f_n(x) - f(x)| < \varepsilon$.

Remark. This statement is equivalent to the following,

$$\forall \varepsilon > 0, \exists N = N(\varepsilon), \text{ s.t. } \forall n \geq N, \sup_{x \in E} |f_n(x) - f(x)| < \varepsilon.$$

Comparing this to pointwise convergence, $\forall x \in E$ and $\forall \varepsilon > 0$, $\exists N = N(\varepsilon, x)$ such that $n \geq N \implies |f_n(x) - f(x)| < \varepsilon$. So we can change our N value for each individual x. However we can't in uniform convergence, which makes this is stronger statement.

Hence we see Uniform convergence $\implies$ Pointwise convergence. This gives a nice way to compute uniform limits. If a function doesn't converge pointwise then we know it doesn't converge uniformly. If we know a sequence of functions converges pointwise to some limit function, then this function must be the limit of the uniform limit, if it exists.

Definition. (Uniformly Cauchy) Let $f_n : E \to \mathbb{R}$ be a sequence of functions. We say that (f_n) is *uniformly Cauchy* on E if

$$\forall \varepsilon > 0, \exists N = N(\varepsilon) \text{ s.t. } n, m \geq N \implies \sup_{x \in E} |f_n(x) - f_m(x)| < \varepsilon.$$

Theorem. (Cauchy criterion for uniform convergence) Let (f_n) be a sequence of functions with $f_n : E \to \mathbb{R}$. The (f_n) converges uniformly on E if and only if (f_n) is uniformly Cauchy on E.

Proof. Suppose that (f_n) is a sequence converging uniformly in E to some function f. Given some $\varepsilon > 0$, there is a N such that $\sup_{x \in E} |f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$. By the triangle inequality $\forall x \in E$, picking $n, m \geq N$,

$$\begin{aligned}
|f_n(x) - f_m(x)| &\leq |f_n(x) - f(x)| + |f_m(x) - f(x)| \\
&\leq \sup_E |f_n - f| + \sup_E |f_m - f| \\
&< \varepsilon + \varepsilon \\
&< 2\varepsilon
\end{aligned}$$

hence (f_n) is uniformly Cauchy.
For the converse, suppose that (f_n) is a sequence uniformly Cauchy in E. Then the sequence of real numbers $(f_n(x))$ is Cauchy so by IA Analysis I, this sequence has a limit, call it $f(x)$. So (f_n) converges pointwise to f. Now we check that $f_n \to f$ uniformly on E. Pick any $\varepsilon > 0$ and note that by the hypothesis that (f_n) is uniformly Cauchy, there exists a number N such that for all $n, m \geq N$ we have $|f_n(x) - f_m(x)| < \varepsilon$. Fix $n \geq N$ and let $m \to \infty$ in this. So since $f_m(x)$ converges to $f(x)$ pointwise, we get that

$$|f_n(x) - f(x)| \leq \varepsilon$$

hence (f_n) converges uniformly in E. □

For an example consider $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = \frac{x}{n}$. So $f_n \to 0$ pointwise on $\mathbb{R}$. But $|f_n - 0|$ is unbounded so the suprenum doesn't exist so f_n does not converge uniformly on $\mathbb{R}$. However if we restrict the domain of f_n to $[-a, a]$ then we get uniform convergence.

Theorem. (Continuity is preserved under uniform limits) Let $f_n, f : [a, b] \to \mathbb{R}$. Suppose that (f_n) converges to f uniformly on $[a, b]$. If $x \in [a, b]$ is such that f_n is continuous at x for all $n \in \mathbb{N}$, then f is continuous at x.

Proof. Let $\varepsilon > 0$ by uniform convergence of $f_n \to f$ we have some $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\sup_{y \in [a,b]} |f_n(y) - f(y)| < \varepsilon$$

. By continuity of f_N at x we have $\delta = \delta(N, x, \varepsilon) > 0$ s.t. $y \in [a, b], |x - y| < \delta \implies |f_N(y) - f_N(x)| < \varepsilon$.
Then $y \in [a, b], |x - y| < \delta$ we] have

$$\begin{aligned} |f(y) - f(x)| &\leq |f(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)| \\ &< \varepsilon + \varepsilon + \varepsilon \\ &< 3\varepsilon \end{aligned}$$

Hence f is continuous at x. $\square$

It is instructive to see where this proof goes wrong if we only assume that (f_n) converges to f pointwise.

Corollary. (Uniform limits of continuous functions are continuous) If $f_n, f : [a, b] \to \mathbb{R}$, and $f_n \to f$ uniformly on $[a, b]$ and if f_n is continuous on $[a, b]$ for every n then f is continuous on $[a, b]$.

Proof. Immediate from the previous theorem. $\square$

From now on we will denote $C([a, b]) = \{f : [a, b] \to \mathbb{R} : f \text{ is continuous on } [a, b]\}$.

Theorem. Let (f_n) be a uniformly Cauchy sequence of functions in $C([a, b])$ the it converges to a function in $C([a, b])$.

Proof. Trivial from our theorems earlier proved. $\square$

Theorem. (Uniform convergence implies convergence of integrals) For $f_n, f : [a, b] \to \mathbb{R}$ be such that f_n, f are bounded and integrable on $[a, b]$. If $f_n \to f$ uniformly on $[a, b]$ then

$$\int_a^b f_n(x)\mathrm{d}x \to \int_a^b f(x)\mathrm{d}x$$

Remark. The assumption that f is integrable is redundant. We will see later that integrability of f_n implies that f is integrable if $f_n \to f$ uniformly

Proof.

$$\begin{aligned}\left|\int_a^b f_n(x)\mathrm{d}x - \int_a^b f(x)\mathrm{d}x\right| &= \left|\int_a^b f_n(x) - f(x)\mathrm{d}x\right| \\ &\le \int_a^b |f_n(x) - f(x)|\mathrm{d}x \\ &\le \sup_{x\in[a,b]} |f_n(x) - f(x)|(b-a) \to 0\end{aligned}$$

by assumption.

1.1 Differentation and uniform convergence

This is more subtle if $f_n \to f$ uniformly on some interval and if f_n are differentiable it does not follow that

(i) That f is differentiable.

(ii) Even if f is differentiable that $f_n'(x) \to f(x)$.

We can view this in the example of $f_n : [-1,1] \to \mathbb{R}$ with $f_n(x) = |x|^{1+\frac{1}{n}}$. Hence we have that

$$\lim_{x\to 0} \frac{f_n(x) - f_n(0)}{x} = \lim_{x\to 0} \operatorname{sgn}(x^{\frac{1}{n}}) = 0$$

So f_n is differentialbe at 0 with $f_n(0) = 0$ and clearly f_n is differentiable everywhere where $x = 0$ too. We can check that $f_n \to |x|$ uniformly. But $|x|$ is not differentiable at $x = 0$.

Now consider the example $f_n : \mathbb{R} \to \mathbb{R}$ with

$$f_n(x) = \frac{\sin(nx)}{\sqrt{n}}.$$

So $f_n \to 0$ uniformly on $\mathbb{R}$. So we have a differentiable limit but $f_n'(x) = \sqrt{n}\cos(nx)$ which is not convergent as $n \to \infty$. So we don't have $f_n'(x) \to f'(x)$ pointwise on $\mathbb{R}$.

Theorem. Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of differentiable functions (at the end points this means that the one-sided derivative exists). Suppose that:

(i) $f_n' \to g$ uniformly for some function $g : [a,b] \to \mathbb{R}$.

(ii) For some $c \in [a,b]$ the sequence $(f_n(c))$ converges.

Then (f_n) converges uniformly to some function $f : [a,b] \to \mathbb{R}$ where f is differentiable everywhere on $[a,b]$ and $f'(x) = g(x)$ for all $x \in [a,b]$.

This proves that

$$\left(\lim_{n\to\infty} f_n\right)' = \lim_{n\to\infty} f_n'$$

i.e. we can exchange the derivative and limit in this case.

Remark. If we assume that f_n' are continuous, then the proof is more straightforward and can be based on the fundamental theorem of calculus.

Proof. By the mean value theorem applied to the difference $(f_n - f_m)$ we have that for any $x \in [a,b]$

$$\begin{aligned} f_n(x) - f_m(x) &= f_n(c) - f_m(c) + (x-c)(f_n - f_m)'(x_{n,m}) \\ \implies |f_n(x) - f_m(x)| &\le |f_n(c) - f_m(c)| + (b-a)|f_n'(x_{n,m}) - f_m'(x_{n,m})| \\ \implies \sup |f_n - f_m| &< |f_n(c) - f_m(c)| + (b-a)\sup |f_n' - f_m'| \to 0 \end{aligned}$$

as $n \to \infty$. So (f_n) is uniformly Cauchy and hence there is an $f : [a,b] \to \mathbb{R}$ s.t. $f_n \to f$ uniformly.

For the next part fix some $y \in [a,b]$. Define

$$h(x) = \begin{cases} \frac{f(x)-f(y)}{x-y} & x \neq y \\ g(y) & x = y \end{cases}$$

Now we only have to estabilish that h is continuous at y to show that f is differentiable at y with $f'(y) = g(y)$. Let

$$h_n(x) = \begin{cases} \frac{f_n(x)-f_n(y)}{x-y} & x \neq y \\ f_n'(y) & x = y \end{cases}$$

then since f_n is differentiable at y we see that h_n is continuous on $[a,b]$. The pointwise limit of (h_n) is h almost by definition since $f_n' \to g$ at $x = y$. Since the uniform limit of sequence of continuous functions is continuous, we just need to show that (h_n) is uniformly Cauchy on $[a,b]$ since the limit must be h since it converges pointwise to h.

$$h_n(x) - h_m(x) = \begin{cases} \frac{(f_n-f_m)(x)-(f_n-f_m)(y)}{x-y} & x \neq y \\ (f_n' - f_m')(y) & x = y \end{cases}.$$

By the mean value theorem,

$$h_n(x) - h_m(x) = \begin{cases} (f_n - f_m)'(x_{n,m}) \text{ for some } x_{n,m} \text{ between } x \text{ and } y & x \neq y \\ (f_n - f_m)'(y) & x = y \end{cases}$$

$$\sup_{[a,b]} |h_n - h_m| \le \sup_{[a,b]} |f_n' - f_m'| \to 0$$

as $n, m \to \infty$. So (h_n) is uniformly Cauchy so we're done. $\square$

Remark. f_n' need not be continuous consider

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

the f is differentiable on $[-1,1]$ with f'(x) not continuous at $x = 0$ and we can take $f_n(x) = f(x)$ for all n (or $f_n(x) = f(x) + \frac{x}{n}$.

We have a shorter proof of the above theorem, assuming that (f_n') are continuous in addition to the hypothesis. For any $x \in [a,b]$ we can write

$$f_n(x) = f_n(c) + \int_c^x f_n'(t)\mathrm{d}t$$

by FTC. Then

$$\begin{aligned}|f_n(x) - f_m(x)| &= \left| f_n(c) - f_m(c) + \int_c^x (f_n'(c) - f_m'(c))\mathrm{d}t \right| \\ &\le |f_n(c) - f_m(c)| + \sup_{t\in[a,b]} |f_n'(t) - f_m'(t)|(b-a) \to 0\end{aligned}$$

as $n, m \to \infty$. So (f_n) is uniformly Cauchy, hence converges uniformly.

Note that

$$\int_c^x f_n'(t)\mathrm{d}t \to \int_c^x g(t)\mathrm{d}t$$

by uniform convergence of $f_n' \to g$ which implies g is continuous and hence also integrable. We can let $n \to \infty$ the first equation for $f_n(x)$ which gives that

$$f(x) = f(c) + \int_c^x g(x)\mathrm{d}t$$

So we can take the derivative of both sides giving that $f'(x) = g(x) = \lim f_n'(x)$. □

Proposition. If $f_n, g_n : E \to \mathbb{R}$ with $f_n \to f$ uniformly on E and $g_n \to g$ uniformly on E then $f_n + g_n$ converges uniformly to $f + g$ on E, and if $h : E \to \mathbb{R}$ is a bounded function then $hf_n \to hf$ uniformly on E also.

Proof. On the example sheet.

2 Series of functions

Definition. (Convergence of a series of functions) Let $g_n : E \to \mathbb{R}$ for $n \in \mathbb{N}$ then write

$$f_n = \sum_{j=1}^{n} g_j$$

defined pointwise. Then we say that that,

(i) The series of functions $\sum_{n=1}^{\infty} g_n$ is convergent at a point $x \in E$ if the sequence of partial sums $(f_n(x))$ converges.
(ii) The series of functions $\sum_{n=1}^{\infty} g_n$ uniformly on E if the sequence (f_n) converges uniformly on E.
(iii) $\sum_{n=1}^{\infty} g_n$ converges absolutely at $x \in E$ if the series $\sum_{n=1}^{\infty} |g_n(x)|$ converges.
(iv) $\sum_{n=1}^{\infty} g_n$ converges absolutely uniformly on E if $\sum_{n=1}^{\infty} |g_n|$ converges uniformly on E.

We know from IA Analysis I that absolutely convergence $\implies$ convergence for a sequences in $\mathbb{R}$. From this we have that if $\sum_{n=1}^{\infty} g_n$ converges absolutely at a point $x \in E$ then $\sum_{n=1}^{\infty} g_n$ converges at x. Similiar to this we have the following proposition relating absolute uniform convergence and uniform convergence.

Proposition. (Absolute uniform convergence implies uniform convergence) If $g_n : E \to \mathbb{R}$ and if $\sum_{n=1}^{\infty} g_n$ converges absolutely uniformly on E then $\sum_{n=1}^{\infty} g_n$ converges uniformly on E.

Proof. Let $f_n = \sum_{i=1}^{n} g_i$ Then

$$\begin{aligned} |f_n(x) - f_m(x)| &= \left| \sum_{i=m+1}^{n} g(i) \right| \\ &= \sum_{i=m+1}^{n} |g_i(x)| = h_n(x) - h_m(x), \quad \text{where } h_n(x) = \sum_{i=1}^{n} |g_i(x)| \\ \sup_{x\in E} |f_n(x) - f_m(x)| \leq \sup_{x\in E} |h_n(x) - h_m(x)| \to 0 \end{aligned}$$

as $n, m \to \infty$ so (f_n) converges uniformly on E. □

Remark. Uniform convergence and absolute pointwise convergence aren't enough to conclude that the series convergence absolutely uniformly.

Theorem. (Weierstrass M-test) Let $g_n : E \to \mathbb{R}$ be a sequence of functions and suppose that $\exists M_n$ such that

$$\sup_{x\in E} |g_n(x)| \leq M_n$$

and that

$$\sum_{n=1}^{\infty} M_n$$

converges. Then

$$\sum_{n=1}^{\infty} g_n$$

converges absolutely uniformly on E.

Proof. Let

$$h_n(x) = \sum_{j=1}^{n} |g_n(x)|$$

for $n > m$,

$$\begin{aligned} h_n(x) - h_m(x) = \sum_{j=m+1}^{n} |g_j(x)| \leq \sum_{j=k+1}^{n} M_j &= \sum_{j=1}^{n} M_j - \sum_{j=1}^{m} M_j \\ \implies \sup_{x\in E} |h_n(x) - h_m(x)| \leq \left| \sum_{j=1}^{n} M_j - \sum_{j=1}^{m} M_j \right| &\quad \forall n, m \end{aligned}$$

by assumption the right hand side $\to 0$ since $\sum_{j=1}^{\infty} M_j$ is convergent, hence (h_n) is uniformly Cauchy hence converges uniformly.

2.1 Power series

We'll now specialise to the case where $g_n(x) = c_n(x-a)^n$ for $a, c_n \in \mathbb{R}$. This gives a real power series.

Theorem. (Radius of convergence) Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a real power series then there exists a $R \in [0, \infty]$ called the *radius of convergence* of the power series such that

(i) If $|x-a| < R$ then the power series converges absolutely.
(ii) If $|x-a| > R$ then the power series diverges.
(iii) R is given by
$$R = \frac{1}{\limsup_{n\to\infty} |c_n|^{\frac{1}{n}}}$$
where if the limit is zero, then $R = \infty$.
(iv) For any $r \in (0, R)$ we have the power series converges uniformly on $[a-r, a+r]$, in particular the function that the power series converges to is continuous on $(a - R, a+R)$.

Proof. The proof for (i), (ii), and (iii) are in IA Analysis I. We'll just prove (iv). Note first that the power series converges absolutely at $x = a + r$ i.e. we have that

$$\sum_{n=0}^{\infty} |c_n| r^n$$

is convergent. Since $|c_n(x-a)^n| \le |c_n| r^n$ for any $x \in [a-r, a+r]$ we can apply the Weierstrass M-test with $M_n = |c_n| r^n$ to conclude that the series

$$\sum_{n=0}^{\infty} c_n(x-a)^n \to f$$

converges absolutely uniformly on $[a-r, a+r]$. It follows that f is continuous. at any point in $(a-R, a+R)$ by picking r small enough.

Remark. (Boundary behaviour. Let

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

with power series boundary R with $0 < R < \infty$. If the power series converges at one of the boundary points of the interval of convergence, say at $x = a + R$ i.e. $\sum_{n=0}^{\infty} c_n R^n$ is convergent then
$$\lim_{x\to a+R} f(x) = \sum_{n=0}^{\infty} c_n R^n$$
so f extends to $(a-R, a+R]$ as a continuous function.

Moreover, under the same conditions that $\sum_{n=0}^{\infty} c_n R^n$ converges we have that the series converges uniformly on $[a-r, a+r]$ for any $r \in (0, R)$. Same discussion applies at the endpoint $a - R$.

Theorem. (Differentation of power series) Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a power series with radius of convergent $R > 0$. Let

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

defined on $(a-R, a+R)$. We have the following

(i) The derived series

$$\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

has radius of convergent R.

(ii) f is differentiable on $(a-R, a+R)$ with

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} \quad \forall x \in (a-R, a+R)$$

Proof.

$$\limsup_{n\to\infty}(n|c_n|)^{\frac{1}{n}} = \limsup_{n\to\infty}|c_n|^{\frac{1}{n}}$$

since we have that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$. So we have (i).

Define $f_n(x) = \sum_{j=0}^{j} c_j(x-a)^j$ is clearly differentiable on $\mathbb{R}$ with $f_n'(x) = \sum_{j=1}^{n} jc_j(x-a)^{j-1}$. By (i) we have that $f_n'(x)$ converges uniformly on $[a-r, a+r]$ for all $r < R$ and $f_n(a) = c_0 \forall n$ so $(f_n(a))$ converges. So the limit is differentiable in $[a-r, a+r]$, with

$$f'(x) = \lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \sum_{j=1}^{n} jc_j(x-a)^{j-1}$$

. □