

Topological Spaces

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1 Topologies

1.1 Definitions

We denote $\mathcal{P}(X)$ as the power set of X .

Definition. (Topology) Let X be a set. A *topology* on X is a collection of sets $T \subseteq \mathcal{P}(X)$ such that

- (i) $\emptyset, X \in T$,
- (ii) T is closed under (possibly uncountable) unions.
- (iii) T is closed under finite intersections.

A set X with a topology T is called a *topological space* of X . An element of X is called a *point* and elements of T are called *open sets*. If $x \in U \in T$ we say U is an open neighbourhood of x . Strictly we should always denote (X, T) for a topological space, but when T is clear, we just write X for the topological space.

Definition. (Continuity) If (X, T_X) and (Y, T_Y) are topological spaces then a function $f : X \rightarrow Y$ is called *continuous* if for $U \in T_Y$, $f^{-1}(U) \in T_X$.

Definition. (Homeomorphism) A function $f : (X, T_X) \rightarrow (Y, T_Y)$ is a *homeomorphism* if it is continuous and has a continuous inverse.

Definition. If $T \subseteq T'$ are topologies on X then we say that T is *coarser* and T' is *finer*. The identity function $d : (X, T) \rightarrow (X, T')$ is continuous.

1.2 Topologies from metrics

If (X, d) is a metric space, recall that a subset $U \subseteq X$ is called *open* if for every point $x \in U$ there exists a $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$.

Proposition. If T_d is the subset of X which are open under the metric d , then (X, T_d) is a topological space. We will call this the topology on X induced by the metric d .

Proof. Tautologically we have that $\emptyset \in T_d$. Clearly we have that $X \in T_d$ too. Let $\{U_\alpha\}_{\alpha \in I}$ be a collection of open sets in T_d with a (possibly uncountable) index set I . Let

$$x \in \bigcup_{\alpha \in I} U_\alpha.$$

Then $x \in U_\beta$ for some $\beta \in I$, so U_β is open hence there exists a $\varepsilon > 0$ such that $B_\varepsilon \subseteq U_\beta \subseteq \bigcup_{\alpha \in I} U_\alpha$, hence $\bigcup_{\alpha \in I} U_\alpha$ is open.

Now suppose that I is finite, and $x \in \bigcap_{\alpha \in I} U_\alpha$. For each α there exists a $\varepsilon_\alpha > 0$ such that $B_{\varepsilon_\alpha}(x) \subseteq U_\alpha$. Take $\varepsilon = \inf_{\alpha \in I} \varepsilon_\alpha$, so $B_\varepsilon(x) \subseteq B_{\varepsilon_\alpha}(x) \subseteq U_\alpha$ for all α , hence we have that $B_\varepsilon(x) \subseteq \bigcap_{\alpha \in I} U_\alpha$ so it's open. Hence T is a topology. \square

Now we have lots of examples we can use for topological spaces. For example we have that topology induced by the Euclidean metric on \mathbb{R}^d which we will call the Euclidean topology. For any $X \subseteq \mathbb{R}^d$ we can have a topology induced by the Euclidean metric too, like \mathbb{Q} , $[0, 1]$, $(0, 1)$.

Proposition. If we have two metric spaces (X, d_X) , (Y, d_Y) and we have $f : X \rightarrow Y$, the f is continuous in the metric space sense if and only if it is continuous in the topological space sense (with the topologies induced by the metric d_X and d_Y respectively).

Proof. Let $f : X \rightarrow Y$ be continuous in the metric space sense. Let U be an open set in T_{d_Y} so we need to show that $f^{-1}(U)$ is open. Let $x \in f^{-1}(U)$, so $f(x) \in U$. Hence there exists an $\varepsilon > 0$ such that $B_\varepsilon(f(x)) \subseteq U$. So since f is continuous there exists a $\delta > 0$ such that if $d_X(x, x') < \delta$, then $d_Y(f(x), f(x')) < \varepsilon$. Hence $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$. So $B_\delta(x) \in f^{-1}(U)$, hence $f^{-1}(U)$ is open.

Now let's do the converse and suppose that $f : X \rightarrow Y$ is continuous in the topological sense. Fix some $x \in X$ and $\varepsilon > 0$. Consider $B_\varepsilon(f(x))$ which is open in Y . Then $f^{-1}(B_\varepsilon(f(x)))$ is in T_{d_X} . It contains x so there exists a $\delta > 0$ such that $x \in B_\delta(x)$, so

$$d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \varepsilon.$$

So f is continuous in the metric sense. □

Definition. Let (X, T) be a topological space and $x_1, x_2, \dots \in X$ say. We say that x_n converges to x if for every open neighbourhood U of x there exists a N such that $x_n \in U$ for all $n \geq N$.

Proposition. If (X, d) is a metric space with topology T_d then a sequence (x_n) converges in the metric sense if and only if it converges in the topological sense.

Proof. Suppose it converges in the metric sense to x . Then for all $\varepsilon > 0$ there exists a N such that for all $n \geq N$ we have that $x_n \in B_\varepsilon(x)$. If U is a neighbourhood of x then there is some ε such that the ball of radius ε centred at x is contained in U . Conversely if (x_n) converges in the topological sense to x , let $\varepsilon > 0$ and consider the open ball centred at x with radius ε . Now $B_\varepsilon(x)$ is an open neighbourhood of x so there exists an integer N such that $x_n \in B_\varepsilon(x)$ for all $n > N$. Hence (x_n) converges to x in the metric sense. □

Consider \mathbb{R} and $(0, 1)$ with the Euclidean metric and topology. Then the two spaces are related, by the function $(0, 1) \rightarrow \mathbb{R}$ by $\tan^{-1} x$ which is invertible. Hence we say the two spaces are homeomorphic, and $\mathbb{R} \cong (0, 1)$. However the two spaces are not isometric since \mathbb{R} is not complete under the Euclidean metric and $(0, 1)$ is not. Hence the property of completeness is not a topological property: it is a property induced by the metric.

Definition. (Discrete topology) Let X be a set. The *discrete* topology is the topology $T_{\text{discrete}} = \mathcal{P}(X)$ (so every set is open).

Remark. Any function from (X, T_{discrete}) to any space is continuous. This topology can be induced by the discrete metric, where $d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$. So $B_{\frac{1}{2}}(x) = \{x\}$ so $\{x\}$ is open, hence all

the sets are open.

Definition. (Indiscrete topology) Let X be a set. The *indiscrete* topology $T_{\text{indiscrete}} = \{\emptyset, X\}$ (as little as possible sets are open).

Remark. A function from any space to $(X, T_{\text{indiscrete}})$ is continuous. This topology does not come from a metric unless X is a singleton set. This is because if $x \neq y$ then $d(x, y) = \varepsilon > 0$, so $y \notin B_\varepsilon(x)$ and since y is arbitrary, then $B_\varepsilon(x) = \{x\} = X$.

Let $X = \{o, c\}$. Then let $T = \{\emptyset, \{o, c\}, \{o\}\}$ be a topology of X . This is called the Sierpinski space. It has the property that every sequence converges to c . A continuous function $f : T \rightarrow (X, T_{\text{Sierpinski}})$ is exactly an open subset of Y .

Let $X = \mathbb{R}$ we'll define the right order topology on X as

$$T_{\text{ord}} = \{(a, \infty) \mid -\infty \leq a \leq \infty\}.$$

Let $\{(a, \infty)\}_{a \in I}$ be a collection of elements of T_{ord} . Then

$$\bigcup_{a \in I} (a, \infty) = (\inf_{a \in I} a, \infty) \in T_{\text{ord}}.$$

Similarly for finite I ,

$$\bigcap_{a \in I} (a, \infty) = (\max_{a \in I} a, \infty) \in T_{\text{ord}}$$

1.3 Bases and subbases

Definition. (Basis) Let T be a topology of X . A *basis*, $B \subseteq T$ for T is a subcollection such that every element of T is a union of elements in B .

Definition. (Subbasis) Let T be a topology of X . A *subbasis*, $S \subseteq T$ for T is a subcollection such that every element of T is a union of sets which are finite intersections of elements of S .

Lemma. Let $f : (X, T_X) \rightarrow (Y, T_Y)$ and $S \subseteq T_Y$ is a subbasis. If $f^{-1}(U)$ is open for all $U \in S$ then f is continuous.

Proof. If $V \subseteq T_Y$, then $V = \bigcup_{a \in I} V_a$ where $V_a \in \bigcap_{b \in J_a} U_{a,b}$ with $U_{a,b} \in S$ and J_a finite. Then

$$f^{-1}(V) = \bigcup_{a \in I} V_a = \bigcup_{a \in I} \left(\bigcap_{b \in J_a} f^{-1}(U_{a,b}) \right) \in T_X,$$

by the axioms of the topology. □

Consider the Euclidean topology on \mathbb{R}^n . The collection $B = \{B_r(x) \mid x \in \mathbb{R}^n, r > 0\}$ is a basis. Likewise the collection of n -cubes everywhere are also a basis. Interestingly the set $QB \subseteq B$ with balls at rational points with rational radii is also a basis. This is interesting since QB is countable while B is uncountable and $\mathcal{P}(\mathbb{R}^n)$ is \aleph_2 .

Definition. (Closed set) Let (X, T) be a topological space. A subset $C \subseteq X$ is *closed* if $X \setminus C \in T$.

Proposition. Let (X, T) be a topological space and $\mathcal{F} = \{C \subseteq X \mid C \text{ closed}\}$. Then

- (i) $\emptyset, X \in \mathcal{F}$;
- (ii) \mathcal{F} is closed under (possibly uncountable) intersections;
- (iii) \mathcal{F} is closed under finite unions.

Proposition. A function $f : X \rightarrow Y$ between topological spaces is continuous if and only if the preimage of every closed set is closed.

Definition. Let (X, T) be a topological space. Let $A \subseteq X$ be a subset of X . Then

- (i) The closure \bar{A} is the smallest (by inclusion) closed set containing A so

$$\bar{A} = \bigcap_{S \text{ closed}, A \subseteq S} S.$$

- (ii) We say that A is dense in X if $A = \bar{A}$.
- (iii) The interior \mathring{A} is the largest open set contained in A so

$$\mathring{A} = \bigcup_{S \text{ open}, S \subseteq A} S.$$

Definition. (Limit point) Let X be a topological space and $A \subseteq X$. A *limit point* of A is a point in X which is a limit of a sequence in A .

Proposition. If C is a closed subset of (X, T) , then the limit points of C lie in C .

Proof. Let $\{x_n\}$ be a sequence in C with limit x_∞ . If $x_\infty \notin C$, then $x_\infty \in X \setminus C$ which is open. Then if $x_n \rightarrow x_\infty$ then we should have that $x_n \in X \setminus C$ for $n \geq N$ but $x_n \in C$ so $x_n \notin X \setminus C$ which is a contradiction. \square

Corollary. A limit point of a A lies in \bar{A} .

For an example $\overline{\mathbb{Q}} = \mathbb{R}$ since any real number is a limit of a sequence of rational numbers. We have that $\overline{(0, 1)} = [0, 1]$ too. The cocountable topology on \mathbb{R} is the topology $T_{\text{cocountable}} = \{\emptyset\} \cup \{\mathbb{R} \setminus C \mid C \text{ countable}\}$. Let $\{x_n\}$ be a sequence in \mathbb{R} , for $x \in \mathbb{R}$ consider $\{x\} \cup \{\mathbb{R} - \{x_n\}\}$ is open and contains x . If $x_n \rightarrow x$, then x_n must be in a U for all $n \geq N$ so $x_n = x$ for all $n \geq N$. Hence the convergent sequences are exactly the eventually constant sequences with the limits being the value they are eventually constant to. So the limit points of a set A are A under this topology. However almost all A is not closed. For example $(0, 1)$ is not closed since $\mathbb{R} \setminus (0, 1)$ is not countable. But the closure of $(0, 1)$ must be closed, so it must be \mathbb{R} hence the sense of limit points and closure are actually two very different properties in topology instead of metric spaces.

1.4 Hausdorff spaces

Definition. (Hausdorff) A space (X, T) is *Hausdorff* if for $x \neq y \in X$ there are open neighbourhoods $x \in U, y \in V$ with $U \cap V = \emptyset$.

Remark. This is the notion that points are separated by open sets.

Lemma. If the topology T is induced by a metric then it is Hausdorff.

Proof. If $x \neq y$ then $d(x, y) = s > 0$. So consider $U = B_{s/2}(x)$ and $V = B_{s/2}(y)$. The triangle inequality shows that $U \cap V = \emptyset$ and we know all balls are open. \square

Proposition. If a space is Hausdorff then a sequence in X has at most 1 limit.

Proof. Let (x_n) be a sequence in X . Suppose it has limits $y \neq z \in X$. Let U and V be disjoint local neighbourhoods of y and z respectively. Then $x_n \in U$ for all $n \geq N_1$ and $x_n \in V$ for all $n \geq N_2$. So if we take that $N = \max\{N_1, N_2\}$ then for all $n \geq N$, we have that $x_n \in U \cap V$ which is empty, hence we have a contradiction. \square

Proposition. If (X, T) is Hausdorff then points are closed.

Proof. Let $x \in X$. We want to show that $\{x\} = \overline{\{x\}}$. Let $y \neq x$. Let U, V be disjoint neighbourhoods of x and y respectively. We know that $x \in X \setminus V$ which is closed. Hence $\overline{\{x\}} \subseteq X \setminus V$. But $y \notin V$, so y is not in the closure of $\{x\}$ hence the closure of $\{x\}$ is just $\{x\}$, so $\{x\}$ is closed. \square

Let's see an example. Let X be an infinite set and consider the cofinite topology on X . Take two non-empty open sets, so

$$(X \setminus F) \cap (X \setminus F') = X \setminus (F \cup F')$$

which is non-empty since $F \cup F'$ is finite and X is infinite so the set on the RHS is non-empty hence the space is not Hausdorff.

1.5 Defining new topologies on existing ones

We have three main ways to define new topologies when given a topology already.

1.5.1 The subspace topology

Definition. (Subset topology) Let (X, T_X) be a topological space. Let $Y \subseteq X$ a subset. The *subset topology* on Y is

$$T|_Y = \{Y \cap U \mid U \in T\}.$$

Definition. (Subspace) A subspace of (X, T) is a subset equipped with the subspace topology.

Proposition. The subset topology is a topology.

Proof. Simple exercise of the axioms. \square

Proposition. The inclusion map $\iota : (Y, T|_Y) \rightarrow (X, T)$ is continuous. In fact $T|_Y$ is the constant topology on Y such that the inclusion map is continuous.

Proof. Let $U \in T$ then $\iota^{-1}(U) = U \cap Y \in T|_Y$ by definition. So it is continuous. Suppose $\iota : (Y, T') \rightarrow (X, T)$ is continuous. For $U \in T$, $\iota^{-1}(U) \in T'$ so $T|_Y \subseteq T'$. \square

A further point of view, a function $f : (z, T_z) \rightarrow (Y, T|_Y)$ is continuous if and only if $\iota \circ f$ is continuous.

Lemma. (Gluing Lemma) Let $f : X \rightarrow Y$ be a function between topological spaces.

- (i) If $\{U_\alpha\}_{\alpha \in I}$ are open subsets which cover X and each $f|_{U_\alpha} : U_\alpha \rightarrow Y$ are continuous (where U_α is given the subspace topology) then f is continuous.
- (ii) If $\{C_\alpha\}_{\alpha \in I}$ is a finite collection of closed sets containing X and $f|_{C_\alpha} : C_\alpha \rightarrow Y$ is continuous for each $\alpha \in I$ then f is continuous.

Proof. Let $V \subseteq Y$ be open. We want to show that $f^{-1}(V)$ is open. We know that

$$\begin{aligned} f^{-1}(V) &= (\iota^{-1}V) \cap X = f^{-1}(V) \cap \left(\bigcup_{\alpha \in I} U_\alpha \right) \\ &= \bigcup_{\alpha \in I} f^{-1}(V) \cap U_\alpha \end{aligned}$$

Since $f|_{U_\alpha}$ are continuous, we have that $f^{-1}|_{U_\alpha}$ is open in U_α in the subspace topology. So there exists a W open in X such that $f^{-1}|_{U_\alpha}(V) = U_\alpha \cap W$ hence this is the intersection on open subsets of X so is open in X , hence since the union of open subsets is open $f^{-1}(V)$ is open, so f continuous.

The second part can be proved the same using the closed set definition of continuity. \square

If (X, d) is a metric space with topology T_d and $Y \subseteq X$ then $T_d|_Y$ is the topology induced by $d|_Y$.

1.5.2 The quotient topology

Definition. (Quotient topology) Let (X, T_X) be a topological space, \sim an equivalence relation on X and X/\sim is the set of equivalence classes, and $\pi : X \rightarrow X/\sim$ the equivalence map. The *quotient topology* on X/\sim is

$$T_{X/\sim} = \{U \subset X/\sim \mid \pi^{-1}(U) \in T_X\}.$$

Proposition. $T_{X/\sim}$ is indeed a topology.

Proof. $\emptyset = \pi^{-1}(\emptyset) \in T_X$ so $\emptyset \in T_{X/\sim}$. $X = \pi^{-1}(X/\sim) \in T_X$ so $X/\sim \in T_{X/\sim}$. Let $\{U_\alpha\}$ be a collection of sets of $T_{X/\sim}$, then

$$\pi^{-1}\left(\bigcup_{\alpha \in I} U_\alpha\right) = \bigcup_{\alpha \in I} \pi^{-1}(U_\alpha),$$

and $\pi^{-1}(U_\alpha) \in T_X$, so the union is too. Hence $\bigcup_{\alpha \in I} U_\alpha \in T_{X/\sim}$. We have a similar proof for finite intersections. \square

Proposition. The quotient map $\pi : (X, T_X) \rightarrow (X/\sim, T_{X/\sim})$ is continuous and $T_{X/\sim}$ is the finest topology for which this is true.

Proof. This is a tautology. \square

An alternative characterisation of the quotient topology is that $f : X/\sim \rightarrow Y$ is continuous if and only if $f \circ \pi : X \rightarrow Y$ is continuous.

Definition. For a continuous function $g : (X, T_X) \rightarrow (Y, T_Y)$ is a *quotient map* if it surjective and $U \in T_Y \iff g^{-1}(U) \in T_X$. Given, this construct \sim on X by $x \sim x' \iff g(x) = g(x')$. There is an induced function $G : X/\sim \rightarrow Y$ sending $G([x]) = g(x)$.

Remark. This function G is a bijection and continuous with a continuous inverse. This means that G is a homeomorphism, so $X/\sim \cong Y$.

Let's see an example on \mathbb{R} . Consider $x \sim y \iff x - y \in \mathbb{Z}$. What is \mathbb{R}/\sim ? Consider $f : \mathbb{R} \rightarrow \mathbb{R}^2$ defined by $x \mapsto (\sin(2\pi x), \cos(2\pi x))$. This is a continuous map so $f : \mathbb{R} \rightarrow S^1 \subseteq \mathbb{R}^2$ is also continuous and surjective. By periodicity $x \sim y \iff f(x) = f(y)$, so we get $F : \mathbb{R}/\sim \rightarrow S^1$ which we can check is a homeomorphism.

Now take the example $X = \mathbb{R} \times \{0, 1\} \subseteq \mathbb{R}^2$ with the standard subspace topology. Let $(x, i) \sim (y, j) \iff (x, i) = (y, j)$ or $x = y \neq 0$. We can then think of X/\sim is a line with two origins. We cannot draw X/\sim since it is not Hausdorff. Any neighbourhood of $[(0, 0)]_\sim$ intersects any neighbourhood of $[(1, 0)]_\sim$ so not Hausdorff. Hence it is not subspace of any Euclidean space.

1.5.3 The product topology

For sets X, Y the projections functions are

$$\begin{aligned} \pi_X : X \times Y &\rightarrow X \\ (x, y) &\mapsto x \end{aligned}$$

and

$$\begin{aligned} \pi_Y : X \times Y &\rightarrow Y \\ (x, y) &\mapsto y \end{aligned}$$

Definition. (Product topology) Let (X, T_X) and (Y, T_Y) be topological spaces. Then *product topology* on $X \times Y$ consists of open sets $U \subseteq X \times Y$ such that for $(x, y) \in U$ there

is a $V \in T_X$ and $W \in T_Y$ such that $(x, y) \in V \times W \in U$.

Proposition. This indeed is a topology and the sets $V \times W$ are a basis for $T_{X \times Y}$.

Proof. Tautologically, we have that $\emptyset \in T_{X \times Y}$. Taking $V = X, W = Y$ we have that $X \times Y \in T_{X \times Y}$. For a collection $\{U_\alpha\}_{\alpha \in I}$ of elements of $T_{X \times Y}$, let $(x, y) \in \bigcup_{\alpha \in I} U_\alpha$. Then $(x, y) \in U_\beta$ for $\beta \in I$ so there exists neighbourhoods of x, y with their product a subset of $U_\beta \subseteq \bigcup_{\alpha \in I} U_\alpha \in T_{X \times Y}$. If I is finite and $(x, y) \in \bigcap_{\alpha \in I} U_\alpha$. Then $(x, y) \in V_\alpha \times W_\alpha \subseteq U_\alpha$ for each $\alpha \in I$. So $(x, y) \in (\bigcap_\alpha V_\alpha) \times (\bigcap_\alpha W_\alpha) \in \bigcap_\alpha U_\alpha$ and since these intersections are finite, these intersections are open. \square

Proposition. The projection maps

$$\pi_X : (X \times Y, T_{X \times Y}) \rightarrow (X, T_X) \quad \pi_Y : (X \times Y, T_{X \times Y}) \rightarrow (Y, T_Y)$$

are continuous and $T_{X \times Y}$ is the coarsest topology for which this is true.

Proof. Let $V \in T_X$. Then $\pi_X^{-1}(V) = V \times Y$, so this is open. Hence π_X, π_Y are continuous.

Suppose that T' is a topology on $X \times Y$ such that π_X and π_Y are continuous, then $\pi_X^{-1}(V) = V \times Y$ is open and $\pi_Y^{-1}(W) = X \times W$ is open. So $V \times W$ is open in T' , so $T_{X \times Y} \subseteq T'$. \square

The universal property of the product topology is that the function

$$f : (Z, T_Z) \rightarrow (X \times Y, T_{X \times Y})$$

is continuous if and only if $\pi_X \circ f : (Z, T_Z) \rightarrow (X, T_X)$ and $\pi_Y \circ f : (Z, T_Z) \rightarrow (Y, T_Y)$ are continuous. Equivalently f is componentwise continuous if and only if it is componentwise continuous.

2 Connectivity

2.1 Connected and disconnected

We know from IA Analysis I, if $f : [0, 1] \rightarrow \mathbb{R}$ is continuous, $f(0) < 0 < f(1)$ then $f(t) = 0$ for some $t \in [0, 1]$. This is a statement about continuous functions, but also about the interval $[0, 1]$. For example if we change the interval to $[0, \frac{1}{2}] \cup (\frac{1}{2}, 1]$ then this does not satisfy the intermediate value theorem. The property of the interval we're using is connectedness.

Definition. (Disconnected) A topological space X is *disconnected* if $X = U \cup V$ for U, V disjoint nonempty open sets.

Definition. (Connected) A topological space is *connected* if it is not disconnected.

If $X = U \cup V$ is disconnected, then U and V are both open and also both closed.

Any set with the coarse topology is connected, due to the lack of non-trivial open sets. A set with the discrete topology is disconnected, if it has more than 1 point, since every set is open, so the result is trivial.

The set $X = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1] \subseteq \mathbb{R}$ is disconnected since $[0, \frac{1}{2})$ is open in X and $(\frac{1}{2}, 1]$ is open in X too. They are disjoint, hence X is disconnected.

Proposition. A space X is disconnected if and only if, there is a continuous surjection $f : X \rightarrow \{0, 1\}$ where $\{0, 1\}$ is equipped with the discrete topology.

Proof. Suppose that X is disconnected. So $X = U \cup V$ disjoint. Then define f such that

$$f(x) = \begin{cases} 0 & x \in U \\ 1 & x \in V \end{cases}.$$

This is well-defined since U and V are disjoint. Since U and V are non-empty, the function is surjective. The preimage of $\{0\}$ and $\{1\}$ are U and V respectively which we know is open. And the preimage of $\{0, 1\}$ and \emptyset are clearly open, so f is continuous.

Conversely suppose that f is continuous. Then define $U = f^{-1}(\{0\})$ and $V = f^{-1}(\{1\})$. So since f is continuous, U and V are open. Clearly U and V is disjoint and non-empty since f is surjective. We have that $X = U \cup V$ since $X = f^{-1}(\{0, 1\}) = f^{-1}(0) \cup f^{-1}(1) = U \cup V$. \square

Theorem. The spaces $[0, 1]$, $[0, 1)$, $(0, 1)$ are all connected.

Proof. Let's just consider $[0, 1]$, the rest of the proves are similar. If it was disconnected, then there is a continuous surjection

$$f : [0, 1] \rightarrow \{0, 1\} \subseteq \mathbb{R}.$$

Then

$$f(\cdot) - \frac{1}{2} : [0, 1] \rightarrow \mathbb{R}$$

is continuous and takes the values $\pm \frac{1}{2}$ only. By the intermediate value theorem, we should have that f takes the value 0 which is a contradiction hence $[0, 1]$ is connected. \square

Theorem. (Generalised intermediate value theorem) Let X be a connected topological space and $f : X \rightarrow \mathbb{R}$ continuous. If there exists $x_0, x_1 \in X$ such that $f(x_0) < 0 < f(x_1)$ then there exists a $x_2 \in X$ such that $f(x_2) = 0$.

Proof. Consider the open sets $U = f^{-1}((-\infty, 0))$, $V = f^{-1}((0, \infty))$. f is continuous, so U, V are open. We know that x_0, x_1 exist hence U, V are non-empty. If $f(x)$ is never zero, then $X = U \cup V$ disjoint and open so X is disconnected. But X is connected hence $f^{-1}(0)$ is non-empty, so pick $x_2 \in f^{-1}(0)$, so $f(x_2) = 0$. \square

Proposition. Let $f : X \rightarrow Y$ be a continuous surjection. Then X connected implies that Y is connected.

Proof. Let's show the contrapositive. Suppose that Y is disconnected. Then we have some $h : Y \rightarrow \{0, 1\}$ continuous and surjective. So

$$h \circ f : X \rightarrow \{0, 1\}$$

is also continuous and surjective, hence X is disconnected. \square

Corollary. If X is connected and $f : X \rightarrow Y$ is continuous then $\text{im}(f)$ is connected.

Proof. Apply the proposition to $f : X \rightarrow \text{im } f$.

For example if X is a connected space and \sim is an equivalence relation then $\pi : X \rightarrow X/\sim$ is a continuous surjection so X/\sim is connected.

Lemma. If $f : X \rightarrow Y$ is a homeomorphism and $Z \subseteq X$, then $f|_Z : Z \rightarrow \text{im}(f|_Z)$ is a homeomorphism.

Proof. Obvious. \square

Let's use this to show that $[0, 1]$ is not homeomorphic to $(0, 1)$. Suppose they are. So we have a homeomorphism $f : [0, 1] \rightarrow (0, 1)$. Let's now restrict f to $(0, 1]$. Then by the lemma we know that $f|_{(0, 1]}$ is a homeomorphism with

$$f|_{(0, 1]} : (0, 1] \rightarrow (0, 1) \setminus \{f(0)\}$$

for some $0 < f(0) < 1$. But $(0, 1]$ is connected and $(0, 1) \setminus \{f(0)\} = (0, f(0)) \cup (f(0), 1)$ so $(0, 1) \setminus \{f(0)\}$ is disconnected which is a contradiction.

We can do a similar process to show that S^1 is not homeomorphic to \mathbb{R} . We know that S^1 is connected since it is a quotient space of \mathbb{R} and \mathbb{R} is connected since $\mathbb{R} \cong (0, 1)$. Suppose that S^1 is homeomorphic to \mathbb{R} . Then remove the point $(1, 0) \in S^1$ and consider the restricted homeomorphism between the new spaces. \mathbb{R} is no longer connected since $\mathbb{R} \setminus \{f(1, 0)\} = (-\infty, f(1, 0)) \cup (f(1, 0), \infty)$, but $S^1 \setminus \{f(1, 0)\}$ is connected since it's homeomorphic to $(0, 1)$.

Proposition. Let $\{X_\alpha\}_{\alpha \in I}$ be a collection of subspaces of X . Suppose that each X_α is connected and $\bigcap_{\alpha \in I} X_\alpha \neq \emptyset$. Then $\bigcup_{\alpha \in I} X_\alpha$ is connected.

Proof. Let X be the union of all the sets. If it were disconnected then $X = U \cup V$ with U, V open, so $U \cap X_\alpha, V \cap X_\alpha$ are disjoint open subsets covering X_α . Since X_α is connected one of them must be zero so $X_\alpha \subseteq U$ or $X_\alpha \subseteq V$. This holds for each X_α but as $\bigcap_{\alpha \in I} X_\alpha \neq \emptyset$ they are all in U or all in V hence X is U or X is V so one of them is empty. \square

Corollary. If X and Y are connected then so is $X \times Y$.

Proof. Suppose X, Y both non-empty (since the empty set is connected). Choose $x \in X$. Consider

$$C_y = \{x\} \times Y \cup X \times \{y\}.$$

The sets $\{x\} \times Y$ and $X \times \{y\}$ intersect in (x, y) and pieces are connected by assumption so C_y is connected. Now observe that

$$X \times Y = \bigcup_{y \in Y} C_y$$

the intersection of all of C_y is $\{x\} \times Y$ which is non-empty hence the proposition applies and $X \times Y$ is connected.

2.2 Path-connectedness

Definition. (Path) If X is a topological space and $x_0, x_1 \in X$ a *path* between them is a continuous $\gamma : [0, 1] \rightarrow X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

Definition. (Path connected) A topological space is *path connected* if for any two points in the space, there is a path between them.

The spaces $(0, 1)$, $[0, 1]$, $(0, 1]$ are all path-connected just by taking the line $\gamma(t) = (1-t)x_0 + tx_1$. For the same reason \mathbb{R}^n is path-connected and any *convex* subset $X \subseteq \mathbb{R}^n$ is too.

Now let's let $X = \mathbb{R}^2 \setminus \{0\}$ be the punctured plane. We do the same again if our linear path doesn't go through the (missing) origin. If it does, just take a circular path instead.

Proposition. A path-connected space is connected.

We'll prove the contrapositive. If X is not connected, we have a continuous surjective function $f : X \rightarrow \{0, 1\}$. Let x_0, x_1 be such that $f(x_0) = 0, f(x_1) = 1$ and suppose that X is path connected so there is a path connecting x_0, x_1 . So $f \circ \gamma$ is a surjective and continuous map from $[0, 1] \rightarrow \{0, 1\}$ hence $[0, 1]$ is disconnected which is a contradiction. \square

We can now show that $\mathbb{R}^n \not\cong \mathbb{R}$ for $n > 1$. If it were then $\mathbb{R}^n \setminus \{0\} \cong \mathbb{R} \setminus \{0\}$. But the RHS is disconnected and the LHS is path-connected, contradiction.

Proposition. If X and Y are path-connected then $X \times Y$ is path-connected.

Proof. Omitted.

2.2.1 Path components

Let X be a space. Define an equivalence relation on X , \sim , defined by

$$x \sim y \iff \text{there exists a path from } x \text{ to } y.$$

Lemma. \sim is indeed an equivalence relation.

Proof.

- (i) $x \sim x$ since we can take the path $\gamma(t) = x$.
- (ii) If $x \sim y$ then we have a path $\gamma(t)$ connecting x and y . Then we can take the path $\gamma'(t) = \gamma(1-t)$ which goes from y to x . Hence $y \sim x$.
- (iii) If $x \sim y$ and $y \sim z$, then there is a path γ' connecting x to y and γ' connecting y to z . Define

$$(\gamma' \cdot \gamma)(t) = \begin{cases} \gamma(2t) & 0 \leq t \leq \frac{1}{2} \\ \gamma'(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

This is a well defined function since $\gamma(1) = \gamma'(0)$. It is continuous by the gluing lemma applied to $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ so it is a path between x and z , so $x \sim z$.

Hence \sim is an equivalence relation. \square

Definition. (Path components) The *path components* of X are the equivalence classes of \sim .

Claim. Each component is path-connected.

Proof. Suppose that x, y are in the same path component, so there exists a path γ connecting x and y . Since $\gamma|_{[0,t]}: [0, t] \rightarrow X$ is a path from $\gamma(0)$ to $\gamma(t)$ every point on the path is in the path component, hence the path lies in the equivalence class. \square

2.2.2 Connected components

For a space X , define \approx by $x \approx y$ if and only if there exists a subset $C \subseteq X$ connected with $x, y \in C$.

Lemma. \approx is an equivalence relation.

Proof.

- (i) $\{x\}$ is connected, so $x \approx x$.
- (ii) Definition is symmetric so $x \approx y \iff y \approx x$.
- (iii) If $x \approx y$ and $y \approx z$ then there are connected subsets C_1 containing x, y and C_2 containing y, z . Then let $C = C_1 \cup C_2$ which is connected since the intersection contains y so non-empty and contains x, z so $x \approx z$.

Hence \approx is an equivalence relation. \square

Definition. (Connected components) Let the *connected components* of X are the equivalence classes of \approx .

Proposition. The connected components are connected.

Proof. Let $C \subseteq X$ be a connected component. Suppose that $f: C \rightarrow \{0, 1\}$ is a surjective continuous function. Let $f(x_0) = 0$ and $f(x_1) = 1$. As $x_0 \approx x_1$, there exists a connected space $D \subseteq X$ with $x_0, x_1 \in D$. If $d \in D$, then $d \approx x_0$, hence $D \subseteq C$. But then $f|_D: D \rightarrow \{0, 1\}$ is a continuous surjective function so D is disconnected, contradiction, hence C is connected. \square

For example $X = (-\infty, 0) \cup (0, \infty)$ has connected components $(-\infty, 0), (0, \infty)$ which are also the path-components.

For $\mathbb{Q} \subset \mathbb{R}$ the connected components are the singletons, so are the path-components.

Let's look at the *Topologist's sine curve*, which is defined as

$$S = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \in \mathbb{R} \mid 0 < x \leq 1 \right\}.$$

This is not closed and it's closure is

$$\overline{S} = S \cup (\{0\} \times [-1, 1]).$$

We want to show that \overline{S} is connected but *not* path connected. Since S is the image of $(0, 1]$ under $x \rightarrow (x, \sin(\frac{1}{x}))$ hence S is connected.

Lemma. The closure of a connected subspace is connected.

Proof. Let $C \subset X$ be a connected subspace and suppose that $\overline{C} = U \cup V$ is disconnected, so U, V are non-empty, disjoint open sets of \overline{C} . Thus U, V are also closed in \overline{C} . Hence they are closed in X too. Consider $C \cap U$ and $C \cap V$. This is a cover of C by disjoint open subsets. So one of them is empty, so $C \subset U$ without loss of generality. But since U is closed in X , $\overline{C} \subset U$ too, so $\overline{C} \cap V = \emptyset$. This contradicts the fact that $U \cup V$ was a disconnection. \square

Hence we have that \overline{S} is connected. Note that $\{0\} \times [-1, 1]$ and S are both path connected, so to show that \overline{S} is not path connected we need to show that there is no $\gamma : [0, 1] \rightarrow \overline{S}$ with $\gamma(0) \in \{0\} \times [-1, 1]$ and $\gamma(1) \in S$. Suppose that γ is such a path. As $\{0\} \times [-1, 1]$ is closed, $\gamma^{-1}(\{0\} \times [-1, 1])$ is closed, so it contains its supremum t . Then $\gamma|_{[t,1]}$ is a new path with $\gamma(t) \in \{0\} \times [-1, 1]$ and $\gamma(s < t) \in S$. SO we replace γ by $\gamma|_{[t,1]}$. So we can write $\gamma(s) = (x(s), y(s))$ for $x, y : [0, 1] \rightarrow \mathbb{R}$ continuous. For $s > 0$ we have that $y(s) = \sin\left(\frac{1}{x(s)}\right)$. Our goal is to find a sequence $s_n \rightarrow 0$ in $[0, 1]$ such that $y(s_n) = (-1)^n$. Then this is a contradiction since it should converge to $y(0)$ since y is continuous. For each n choose a $0 = x(0) < a < x\left(\frac{1}{n}\right)$ such that $\sin\left(\frac{1}{a}\right) = (-1)^n$. By the intermediate value theorem we have that $u = x(s_n)$ for some $0 < s_n < \frac{1}{n}$. So $y(s_n) = (-1)^n$ as required. \square

2.3 Compactness

Definition. (Cover) A collection $\mathcal{X} \subset P(X)$ is a *cover* of X if for each $x \in X$ there is a $S \in \mathcal{X}$ with $x \in S$.

Definition. (Open cover) An *open cover* of X is a cover consisting of open sets.

Definition. (Subcover) A *subcover* of \mathcal{X} is a $\mathcal{X}' \subseteq \mathcal{X}$ which is also a cover.

Let's give a definition now which generalised the idea of being closed and bounded without need of a metric.

Definition. (Compact) A topological space X is *compact* if every open cover has a finite subcover.

\mathbb{R} is not compact by considering the open sets $\{(n-1, n+1) \subset \mathbb{R} \mid n \in \mathbb{Z}\}$ which clearly is an open cover but has no finite subcover since there isn't even a subcover, since each $n \in \mathbb{Z}$ is only contained in exactly one open interval in the cover.

If X is a topological space with finitely many points then it is compact.

If $X = \{0\} \cup \{\frac{1}{n}\}$ for $n = 1, 2, \dots$. Then X is compact. Let \mathcal{U} be an open cover of X . The point $0 \in X$ lies in some $U_0 \in \mathcal{U}$. As this is open, there exists an $\varepsilon > 0$ such that $0 \in B_\varepsilon(0) = (-\varepsilon, \varepsilon) \subset U_0$. Thus $\frac{1}{n} \in U_0$ for $0 < \frac{1}{n} < \varepsilon$ for $n > \frac{1}{\varepsilon}$. The finitely many points $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{m}$ that do not satisfy this, lie in open sets $1 \in U_1, \frac{1}{2} \in U_2, \dots, \frac{1}{m} \in U_m$. Hence $\{U_i\}_{i=0}^m$ is a finite subcover.

The space $Y = \{\frac{1}{n} \mid n = 1, 2, \dots\} \subseteq \mathbb{R}$ is *not* compact. This is because it is an infinite set and the topology is discrete since $\{\frac{1}{n}\} = Y \cap (\frac{1}{n+1}, \frac{1}{n+1})$. Hence the cover by each of the points has no subcover at all.

Theorem. $[0, 1]$ is compact.

Proof. Let \mathcal{U} be an open cover of $[0, 1]$. Consider

$$A = \{a \in [0, 1] \mid \text{there is a finite } \mathcal{U}' \subset \mathcal{U} \text{ whose union contains } [0, a]\}.$$

If $0 \leq a \leq b$ then $[0, a] \subseteq [0, b]$. So if $b \in A$ then $a \in A$. A is not non-empty since $0 \in A$ since $[0, 0] = \{0\}$ has some $U \in \mathcal{U}$ containing 0. Hence by completeness A has a supremum, so set $\alpha = \sup A \in [0, 1]$. We want to show that $\alpha = 1$. Let $\alpha \in U_\alpha \in \mathcal{U}$. If $\alpha < 1$ then there is an $\varepsilon > 0$ such that $(\alpha - \varepsilon, \alpha + \varepsilon) \subset [0, 1] \subset U_\alpha$. So as $\alpha - \varepsilon < \alpha$, there must be a $\alpha - \varepsilon \leq a < \varepsilon$ with $a \in A$. Hence $\alpha - \varepsilon \in A$ too. But then $[0, \alpha - \varepsilon]$ has a finite cover by elements of \mathcal{U} . Adding U_α we have a finite subcover of $[0, \alpha + \varepsilon/2]$ hence this contradicts the fact that α is a supremum, so $\alpha = 1$ and $[0, 1]$ is compact. \square

Let's look at some consequences of compactness.

Proposition. If X is compact and $C \subseteq X$ is closed, then C is compact.

Proof. Open sets in C are of the form $C \cap U_\alpha$ for U_α open in X . Suppose we have $\{C \cap U_\alpha \mid \alpha \in I\}$ which is an open cover of C . Then $\{X \setminus C\} \cup \{U_\alpha \mid \alpha \in I\}$ is an open cover of X . So as X is compact, this has a finite subcover. It can be taken to have the form $\{X \setminus C\} \cap \{U_\alpha \mid \alpha \in I'\}$ for $I' \subseteq I$ finite. As these cover X , their intersections with C cover C , so $\{C \cap U_\alpha \mid \alpha \in I'\}$ is a finite subcover of C , hence C is compact. \square

Proposition. If X is Hausdorff and $C \subseteq X$ is compact, then C is closed.

Proof. We will show that $U = X \setminus C$ is open. So for $x \in U$, we have to find an open $x \in U_\alpha \subseteq U$.