Analysis II

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1 Uniform Convergence

For a subset $E \subseteq \mathbb{R}$, have a sequence $f_n : E \to \mathbb{R}$. What does it mean for the sequence (f_n) to converge? The most basic notion for any $x \in E$ require that the sequence of real numbers $f_n(x)$ to converge in $\mathbb{R}$. If this holds we can defined a new function $f : E \to \mathbb{R}$ by setting each value to the limit of the function.

Definition. (Pointwise limit) We say that (f_n) converges *pointwise* if for all x in its domain we have that

$$f(x) = \lim_{n\to\infty} f_n(x)$$

converges. We write that $f_n \to f$ pointwise.

Are properties such as continuity, differentiability integrability, preserved in the limit? We'll use an example to show that continuity is not preserved.

We can see this by taking a sequence of functions which converge to a step function by taking tighter and tighter curvers which get steeper and steeper. For example take,

$$f_n : [-1, 1] \to \mathbb{R}, \quad f_n(x) = x^{\frac{1}{2n+1}}.$$

So in the limit we get that

$$f_n(x) \to f(x) = \begin{cases} 1 & 0 < x \leq 1 \\ 0 & x = 0 \\ -1 & -1 \leq x < 0 \end{cases}$$

which is not continious.

For an example where integability is not preserved, let $q_1, q_2, q_3, \ldots$ be an enumeration of $\mathbb{Q} \cap [0, 1]$ and define

$$f_n(x) = \begin{cases} 1 & x \in \{q_1, \ldots, q_n\} \\ 0 & \text{otherwise} \end{cases}$$

so we get $f_n(x)$ continious everywhere on $[0, 1]$ apart from a finite number of points, then f_n is integrable on $[0, 1]$ (IA Analysis I). But,

$$\lim_{n\to\infty} f_n(x) = \mathbf{1}_{\mathbb{Q}}(x)$$

which we know is not integrable.

If $f_n \to f$ pointwise, f_n integrable, f integrable, does it follow that $\int f_n \to \int f$? (Spoiler: No) For example take f_n to be a 'spike' with height n and width $\frac{2}{n}$, concretely,

$$f_n(x) = \begin{cases} n^2 x & 0 \leq x \leq \frac{1}{n} \\ n^2(\frac{2}{n} - x) & \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \text{otherwise} \end{cases}$$

So the integral of f_n over $[0, 1]$ is 1, but we can see that f_n converges pointwise to zero. So $\int_0^1 f_n \to 1$ but $\int_0^1 f \to 0$.

So we need a better (stronger) notion for the convergence of a sequence of functions. We can't use something too strong, such as $f_n \to f$ if f_n is eventually f for large enough n. We've got to find something inbetween. This is uniform convergence.

Definition. (Uniform convergence) Let $f_n, f : E \to \mathbb{R}$, for $n \in \mathbb{N}$. We say that (f_n) converges *uniformly* on E if the following holds. For all $\varepsilon > 0$, $\exists N = N(\varepsilon)$ such that for every $n \geq N$ and for every $x \in E$ we have that $|f_n(x) - f(x)| < \varepsilon$.

Remark. This statement is equivalent to the following,

$$\forall \varepsilon > 0, \exists N = N(\varepsilon), \text{ s.t. } \forall n \geq N, \sup_{x \in E} |f_n(x) - f(x)| < \varepsilon.$$

Comparing this to pointwise convergence, $\forall x \in E$ and $\forall \varepsilon > 0$, $\exists N = N(\varepsilon, x)$ such that $n \geq N \implies |f_n(x) - f(x)| < \varepsilon$. So we can change our N value for each individual x. However we can't in uniform convergence, which makes this is stronger statement.

Hence we see Uniform convergence $\implies$ Pointwise convergence. This gives a nice way to compute uniform limits. If a function doesn't converge pointwise then we know it doesn't converge uniformly. If we know a sequence of functions converges pointwise to some limit function, then this function must be the limit of the uniform limit, if it exists.

Definition. (Uniformly Cauchy) Let $f_n : E \to \mathbb{R}$ be a sequence of functions. We say that (f_n) is *uniformly Cauchy* on E if

$$\forall \varepsilon > 0, \exists N = N(\varepsilon) \text{ s.t. } n, m \geq N \implies \sup_{x \in E} |f_n(x) - f_m(x)| < \varepsilon.$$

Theorem. (Cauchy criterion for uniform convergence) Let (f_n) be a sequence of functions with $f_n : E \to \mathbb{R}$. The (f_n) converges uniformly on E if and only if (f_n) is uniformly Cauchy on E.

Proof. Suppose that (f_n) is a sequence converging uniformly in E to some function f. Given some $\varepsilon > 0$, there is a N such that $\sup_{x \in E} |f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$. By the triangle inequality $\forall x \in E$, picking $n, m \geq N$,

$$\begin{aligned}
|f_n(x) - f_m(x)| &\leq |f_n(x) - f(x)| + |f_m(x) - f(x)| \\
&\leq \sup_E |f_n - f| + \sup_E |f_m - f| \\
&< \varepsilon + \varepsilon \\
&< 2\varepsilon
\end{aligned}$$

hence (f_n) is uniformly Cauchy.
For the converse, suppose that (f_n) is a sequence uniformly Cauchy in E. Then the sequence of real numbers $(f_n(x))$ is Cauchy so by IA Analysis I, this sequence has a limit, call it $f(x)$. So (f_n) converges pointwise to f. Now we check that $f_n \to f$ uniformly on E. Pick any $\varepsilon > 0$ and note that by the hypothesis that (f_n) is uniformly Cauchy, there exists a number N such that for all $n, m \geq N$ we have $|f_n(x) - f_m(x)| < \varepsilon$. Fix $n \geq N$ and let $m \to \infty$ in this. So since $f_m(x)$ converges to $f(x)$ pointwise, we get that

$$|f_n(x) - f(x)| \leq \varepsilon$$

hence (f_n) converges uniformly in E. □

For an example consider $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = \frac{x}{n}$. So $f_n \to 0$ pointwise on $\mathbb{R}$. But $|f_n - 0|$ is unbounded so the suprenum doesn't exist so f_n does not converge uniformly on $\mathbb{R}$. However if we restrict the domain of f_n to $[-a, a]$ then we get uniform convergence.

Theorem. (Continuity is preserved under uniform limits) Let $f_n, f : [a, b] \to \mathbb{R}$. Suppose that (f_n) converges to f uniformly on $[a, b]$. If $x \in [a, b]$ is such that f_n is continuous at x for all $n \in \mathbb{N}$, then f is continuous at x.

Proof. Let $\varepsilon > 0$ by uniform convergence of $f_n \to f$ we have some $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\sup_{y \in [a,b]} |f_n(y) - f(y)| < \varepsilon$$

. By continuity of f_N at x we have $\delta = \delta(N, x, \varepsilon) > 0$ s.t. $y \in [a, b], |x - y| < \delta \implies |f_N(y) - f_N(x)| < \varepsilon$.
Then $y \in [a, b], |x - y| < \delta$ we] have

$$\begin{aligned} |f(y) - f(x)| &\leq |f(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)| \\ &< \varepsilon + \varepsilon + \varepsilon \\ &< 3\varepsilon \end{aligned}$$

Hence f is continuous at x. $\square$

It is instructive to see where this proof goes wrong if we only assume that (f_n) converges to f pointwise.

Corollary. (Uniform limits of continuous functions are continuous) If $f_n, f : [a, b] \to \mathbb{R}$, and $f_n \to f$ uniformly on $[a, b]$ and if f_n is continuous on $[a, b]$ for every n then f is continuous on $[a, b]$.

Proof. Immediate from the previous theorem. $\square$

From now on we will denote $C([a, b]) = \{f : [a, b] \to \mathbb{R} : f \text{ is continuous on } [a, b]\}$.

Theorem. Let (f_n) be a uniformly Cauchy sequence of functions in $C([a, b])$ the it converges to a function in $C([a, b])$.

Proof. Trivial from our theorems earlier proved. $\square$