Electromagnetism

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1 Introduction

1.1 Charges and currents

Electric charge is a physical property of elementary particles. It is:

(i) A signed quantity, it can either be positive, negative, or zero.

(ii) It is quantised to integer multiplies of the elementary charge.

(iii) It is a conserved quantity even if particles are created or destroyed.

By convention the electron has charge $-e$, the proton has charge $+e$ and the neutron has no charge. On macroscopic scales, the number of particles is so large that charge can be considered to have a continuous electric charge density $\rho(\mathbf{x}, t)$. The total charge in a volume V is then

$$Q = \int_V \rho \mathrm{d}V.$$

The *electric current density* $\mathbf{J}(\mathbf{x}, t)$ is the flux of electric charge per unit area. The current folowing through a surface S is

$$I = \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}.$$

Consider a time-independent volume V with boundary S. Since charge is conserved, we have that

$$\begin{aligned}
\frac{dQ}{dt} &= -I \\
\frac{d}{dt}\int_V \rho \mathrm{d}V + \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} &= 0 \\
\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} \right) \mathrm{d}V &= 0
\end{aligned}$$

Since this is true for any V, we have that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0.$$

This *equation of charge conservation* has the typical form of a conservation law.

The discrete charge distribution of a single particle of charge q_i; and position vector $\mathbf{x}_i(t)$, is

$$\begin{aligned}
\rho &= q_i \delta(\mathbf{x} - \mathbf{x}_i(t)), \\
\mathbf{J} &= q_i \dot{\mathbf{x}}_i \delta(\mathbf{x} - \mathbf{x}_i(t)).
\end{aligned}$$

For N particles, it is

$$\begin{aligned}
\rho &= \sum_{i=1}^{N} q_i \delta(\mathbf{x} - \mathbf{x}_i(t)) \\
\mathbf{J} &= \sum_{i=1}^{N} q_i \dot{\mathbf{x}}_i \delta(\mathbf{x} - \mathbf{x}_i(t)).
\end{aligned}$$

As an exercise we can see that these satisfy the equation of charge conservation.

1.2 Fields and forces

Electromagnetism is a *field theory*.

Charged particles don't interact directly, but rather by generating fields around them, which are then experienced by other charged particles. In general we have two time-dependent vector fields, the electric field $\mathbf{E}(\mathbf{x}, t)$, and the magnetic field $\mathbf{B}(\mathbf{x}, t)$.

The *Lorentz force* on a particle of charge q and velocity v is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

1.3 Maxwell's equations

In this course we will explore some consequences of Maxwell's equations.

Definition. (Maxwell's equations)

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right).
\end{aligned}$$

Remark. We have some properties about these equations.

- Coupled linear PDEs in space and time,
- Involve two positive constants:
  (i) ε_0 (vacuum permittivity)
  (ii) μ_0 (vacuum permeability)
- Charges (ρ) and currents ($\mathbf{J}$) are the sources of electromagnetic fields.
- Each equation is an equivalent integral form (see later) related via the divergece or Stokes' theorem.
- These are the *vacuum* equations that apply on microscopic scales or in a vacuum. A related macroscopic version applies in media (Part II Electrodynamics).
- The equations of consistent with each other and with charge conservation. We will show this now.
  (i) Taking the divergence of the third equation, this agrees with the time derivative of the second equation.
  (ii) For charge conversation, we have that

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} &= \frac{\partial}{\partial t} (\varepsilon_0 \nabla \cdot \mathbf{E}) + \nabla \cdot \left(-\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\mu_0} \nabla \times \mathbf{B} \right) \\
&= 0.
\end{aligned}$$

1.4 Units

The SI unit of electric charge is the coulomb (C). The elementary charge is exactly

$$e = 1.602\ 176\ 634 \times 10^{-19}\ \text{C}.$$

The SI unit of electric current is the ampere or amp (A) which is equal to 1 C s^{-1}.

The SI base units needed in electromagnetism and then the second, metre, kilogram, and ampere. From the Lorentz force law we see that the units of **E** and **B** must be

$$\text{kg m s}^{-3}\text{A}^{-1} \quad \text{and} \quad \text{kg s}^{-2}\text{A}^{-1}.$$

We sometimes refer to the units of **B** as the *Telsa* (T).

From Maxwell's equations we can work out the units of ε_0 and μ_0. The values of these constants can be calculated via experimentation as

$$\begin{aligned} \varepsilon_0 &= 8.854 \cdots \times 10^{-12}\ \text{kg}^{-1}\text{m}^{-3}\text{s}^4\ \text{A}^2 \\ \mu_0 &= 1.256 \cdots \times 10^{-6}\ \text{kg m s}^{-2}\text{A}^{-2} \end{aligned}$$

The speed of light is exactly

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 299\ 792\ 458\ \text{m s}^{-1}.$$

2 Electrostatics

In a time-independent situation, Maxwell's equations reduce to

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= 0 \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} \end{aligned}$$

Now **E** and **B** are decoupled so we can study them seperately. Electrostatics is the study of the electric field generated by a stationary charge distribution. We'll be looking at

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \qquad \nabla \times \mathbf{E} = 0.$$

2.1 Gauss' Law

Consider a closed surface S enclosing a volume V. Integrate over V and use the divergence theorem to obtain Gauss' law which is

$$\int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q}{\varepsilon_0},$$

Where

$$Q = \int_V \rho \mathrm{d}V$$

is the total charge in V. Gauss' law is the integral version of the first of Maxwell's equations and is valid generally. We get that electric flux is proportional to the total charge enclosed.

In special situations we use Gauss' law together with symmetry to deduce from ρ, by choosing the *Gaussian surface* S appropriately.

2.1.1 Spherical symmetry

Consider a spherically symmetric charge distribution, $\rho(r)$ in spherical polar coordinates with total charge Q contained within an outer radius R. To have spherical symmetry, the electric field should have the form

$$\mathbf{E} = E(r)\mathbf{e}_r.$$

This will satisfy $\nabla \times \mathbf{E} = 0$ as required.

To find $E(r)$ apply Gauss' law to a sphere of radius r. If $r > R$ then we get that

$$\begin{aligned}\int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} &= E(r) \int_S \mathbf{e}_r \cdot \mathrm{d}\mathbf{S} \\ &= E(r) \int_S \mathrm{d}S \\ &= E(r) 4\pi r^2 = \frac{Q}{\varepsilon_0}.\end{aligned}$$

Thus

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\mathbf{e}_r.$$

So the external electric field of a spherically symmetric body depends only on the total charge, and is equivalent to a point charge at the origin with all of the charge. The Lorentz force on a particle of charge q in $r > R$ is

$$\mathbf{F} = q\mathbf{E} = \frac{Qq}{4\pi\varepsilon_0 r^2}\mathbf{e}_r.$$

This is the *Coulomb force* between charge particles. The force is repulsive if the charges have the same sign and attractive if the charges have different sign.

In the limit as $R \to 0$ we obtain the electric field at a *point charge* Q, corresponding to

$$\rho = Q\delta(\mathbf{x}).$$

There is a close analogy between the Coulomb force and the gravitational force between massive particles, recall from IA Dynamics and Relativity that

$$\mathbf{F} = -\frac{GMm}{r^2}\mathbf{e}_r.$$

Both involve an inverse-square law and the product of the charges, however there are some differences.

(i) While gravity is always attractive, electric forces can be repulsive or attractive;

(ii) Gravity is very much weaker, due to the much smaller constant of proportionality.

For example if we consider two protons, the ratio of the electric to gravitational force is 10^{36}. On the atom scale, gravity is irrelevant. But the $+$ and $-$ charges balance so accurately, that they cancle on the planetary scale, and gravity is much more dominant.

2.1.2 Cylindrical symmetry

Consider a cylindrically symmetric charge distribution, with $\rho(r)$ in cylindrical polar coordinates with totaly charge λ per unit length contained within an outer radius R. To have cylindrical symmetry again we have that

$$\mathbf{E} = E(r)\mathbf{e}_r.$$

Again this will satisfy $\nabla \times \mathbf{E} = 0$. To find $E(r)$, apply Gauss' law to a cylinder of radius r arbitrary length L.

If $r > R$ then

$$\begin{aligned}\int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} &= E(r) \int_S \mathbf{e}_r \cdot \mathrm{d}\mathbf{S} \\ &= E(r) \int_S \mathrm{d}S \\ &= E(r) 2\pi r L = \frac{\lambda L}{\varepsilon_0}.\end{aligned}$$

Thus we have that

$$\mathbf{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\mathbf{e}_r.$$

In the limit as $R \to 0$ we obtain the electric field of a line charge λ per unit length, corresponding to $\rho = \lambda\delta(x)\delta(y)$.

2.1.3 Planar symmetry

For a planar charge distribution, we have a charge density of $\rho(z)$ in Cartesian coordinates with total charge σ per unit area contained within a region $-d < z < d$ of thinkness $2d$.

We will assume reflective symmetry, so $\rho(z)$ is even.

To have planar symmetry, we have $\mathbf{E} = E(z)\mathbf{e}_z$. Again we have that $\nabla \times \mathbf{E} = 0$. The reflectional symmetry implies that $E(-z) = -E(z)$.

To find $E(Z)$ for $z > 0$ apply Gauss' law to a "Gaussian pillbox" of height $2z$ and arbitrary area A. If $z > d$ then

$$\begin{aligned}\int_S \mathbf{E} \cdot S &= E(z)A - E(-z)A \\ &= 2E(z)A \\ &= \frac{\sigma A}{\varepsilon_0}\end{aligned}$$

Thus we have that

$$\mathbf{E} = \begin{cases} \frac{\sigma}{2\varepsilon_0}\mathbf{e}_z & z < d \\ -\frac{\sigma}{2\varepsilon_0}\mathbf{e}_z & z < -d \end{cases}.$$

In the limit as $d \to 0$ we obtain the electric field of a *surface charge* σ per unit area, corresponding to $\rho = \sigma\delta(z)$.

2.1.4 Surface charge and discontinuity

Let $\mathbf{n}$ be a unit vector normal to the charged surface, pointing from region 1 to region 2. In our example we have that $\mathbf{n} = \mathbf{e}_z$. This discontinuity in $\mathbf{E}$ is given by

$$[\mathbf{n} \cdot \mathbf{E}] = \frac{\sigma}{\varepsilon_0}$$

where σ is the surface charge density and

$$[X] = X_2 - X_1$$

denotes a discontinuity between regions 1 and 2.

The tangential components are continuous:

$$[\mathbf{n} \times \mathbf{E}] = 0.$$

And these two equations apply to any surface surface even if it's curved and non-uniform.

2.2 The electrostatic potential

For a general $\rho(\mathbf{x})$ we cannot determine $\mathbf{E}(\mathbf{x})$ using Gauss' law alone. We'll need to use the Maxwell equation $\nabla \times \mathbf{E} = 0$. This implies that E is irrotational so it has an *electrostatic* potential $\Phi(\mathbf{x})$, such that

$$\mathbf{E} = -\nabla\Phi.$$

Definition. (Potential difference) The *potential difference* or voltage between two points $\mathbf{x}_1$ and $\mathbf{x}_2$ is

$$\Phi(\mathbf{x}_2) - \Phi(\mathbf{x}_1) = \int \mathrm{d}\Phi = -\int_{\mathbf{x}_1}^{\mathbf{x}_2} \mathbf{E} \cdot \mathrm{d}\mathbf{x}$$

and is path independent since $\nabla \times \mathbf{E} = 0$ is zero and the region is simply connected, so the field is conservative .

Definition. (Electric force) The *electric force* on a particle of charge q is

$$\mathbf{F} = q\mathbf{E} = -q\nabla\Phi.$$

Remark. This is a conservative force associated with the potential energy

$$U(\mathbf{x}) = q\Phi(\mathbf{x}).$$

Recall that the first Maxwell equation implies that Φ satisfies Poisson's equation, so

$$-\nabla^2\Phi = \frac{\rho}{\varepsilon_0}.$$

So we have the solution (from IB Methods) as (over all space with boundary conditions that $\Phi \to 0$ as $|\mathbf{x}| \to \infty$).

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon} \int \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \mathrm{d}^3\mathbf{x}'.$$

This is the convolution of $\rho(\mathbf{x})$ with the potential of a unit point charge (which relates to our Green's function from IB Methods) $\frac{1}{4\pi\varepsilon|\mathbf{x}|}$. Namely it is the solution to

$$-\nabla^2\Phi = \frac{\delta(\mathbf{x})}{\varepsilon_0}$$

satisfying $\Phi \to 0$ as $|\mathbf{x}| \to \infty$. Note that is unaffected if we add an arbitrary constant to Φ (this makes sense since Φ measures a potential difference between two points so increasing the charge uniformly doesn't change). We usually choose this such that $\Phi \to 0$ as $|\mathbf{x}| \to \infty$. If $\rho(\mathbf{x})$ does not decay sufficiently rapidly this may not be possible. For example if we have a line charge $E_r \propto \frac{1}{r}$, so we have that $\Phi \propto \log r$ which doesn't go to zero as $r \to \infty$.

2.2.1 Point charge

The potential due to a point charge q at the origin is

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0|\mathbf{x}|} = \frac{q}{4\pi\varepsilon_0 r}.$$

2.2.2 Electric dipole

Two equal and opposite charges at different positions. Without loss of generality consider charges $-q$ t $\mathbf{x} = 0$ and $+q$ at $\mathbf{x} = \mathbf{d}$. The potential due to the dipole is

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0}\left(-\frac{1}{|\mathbf{x}|} + \frac{1}{|\mathbf{x}-\mathbf{d}|}\right)$$

Apply Taylor's theorem for a scalar field,

$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + (\mathbf{h}\cdot\nabla)f(\mathbf{x}) + \frac{1}{2}(\mathbf{h}\cdot\nabla)^2 f(\mathbf{x}) + O(||\mathbf{h}||^2).$$

So we get that

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\varepsilon_0}\left(-\frac{1}{r} + \frac{1}{r} - (\mathbf{d}\cdot\nabla)\frac{1}{r} + O(|\mathbf{d}|^2)\right) = \frac{q\mathbf{d}\cdot\mathbf{x}}{4\pi\varepsilon_0|\mathbf{x}|^3} + O(|\mathbf{d}|^2).$$

In the limit as $|\mathbf{d}| \to 0$ with $q\mathbf{d}$ finite, we obtain a *point dipole* with *electric dipole moment*

$$\mathbf{p} = q\mathbf{d}.$$

which has potential

$$\Phi(\mathbf{x}) = \frac{\cdot\mathbf{x}}{4\pi\varepsilon_0|\mathbf{x}|^3}$$

and electric field

$$\mathbf{E} = -\nabla\Phi = \frac{3(\mathbf{p}\cdot\mathbf{x})\mathbf{x} - |\mathbf{x}|^2\mathbf{p}}{4\pi\varepsilon_0|\mathbf{x}|^5}.$$

In spherical polar coordinates aligned with $\mathbf{p} = p\mathbf{e}_z$. So

$$\Phi = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}.$$

Then we get that
$$E_r = -\frac{\partial \Phi}{\partial r} = \frac{2p\cos(\theta)}{4\pi\varepsilon_0 r^3}$$
and
$$E_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}.$$
From our alignment we have that $E_\phi = 0$.

Remark. Note that

(i) Φ and $\mathbf{E}$ are not spherically symmetric.

(ii) They decrease more rapidly with r than a point charge since the dipole are nearly cancelling eachother out.

A point dipole $\mathbf{p}$ at the origin corresponds to
$$\rho(\mathbf{x}) = -\mathbf{p} \cdot \nabla\delta(\mathbf{x}),$$
So we can find the associated potential Φ as
$$\Phi(\mathbf{x}) = \mathbf{p} \cdot \nabla \left(\frac{1}{4\pi\varepsilon_0|\mathbf{x}|}\right).$$

2.2.3 Field lines and equipotentials

Electric field lines are the integral curves of $\mathbf{E}$ being tangent to $\mathbf{E}$ everywhere. Since we have that $\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$, field lines begin on positive charges and end on negative charges. In electrostatics, $\mathbf{E} = -\nabla\Phi$, so field lines are perpendicular to the equipotential surfaces of which Φ are constant.

2.2.4 Dipole in an external field

Consider a dipole $\mathbf{p}$ in an external field $\mathbf{E}_{\text{external}} = -\nabla\Phi$ generated by distinct charges. With $-q$ at $\mathbf{x}$ and $+q$ and $\mathbf{x}+\mathbf{d}$, the potential energy at the dipole due to the external field is
$$\begin{aligned} U &= -q\Phi(\mathbf{x}) + Q\Phi(\mathbf{x}+\mathbf{d}) \\ &= q(\mathbf{d}\cdot\nabla)\Phi(\mathbf{x}) + O(|\mathbf{d}|^2) \end{aligned}$$

In the limit at the point dipole,
$$U = \mathbf{p}\cdot\nabla\Phi = -\mathbf{p}\cdot\mathbf{E}_{\text{external}}$$
and is minimised when $\mathbf{p}$ is aligned with $\mathbf{E}_{\text{external}}$.

2.2.5 Multipole expansion

For a general charge distribution $\rho(\mathbf{x})$ confined to a ball $\{V : |\mathbf{x}| < R\}$,
$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}^3\mathbf{x}'.$$

We'll look at the external potential at $\mathbf{x}$ with $\mathbf{x} \notin V$. Expand

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{r} - (\mathbf{x}' \cdot \nabla)\frac{1}{r} + \frac{1}{2}(\mathbf{x}' \cdot \nabla)^2 \frac{1}{r} + O(|\mathbf{x}'|^3).$$

Which is

$$= \frac{1}{r}\left[1 + \frac{\mathbf{x}' \cdot \mathbf{x}}{r^2} + \frac{3(\mathbf{x}' \cdot \mathbf{x})^2 - |\mathbf{x}'|^2|\mathbf{x}|^2}{2r^4} + O\left(\frac{R^3}{r^3}\right)\right]$$

This leads to the *multipole expansion* of the potential,

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{\mathbf{p} \cdot \mathbf{x}}{r^2} + \frac{1}{2}\frac{Q_{ij}x_i x_j}{r^5} + \cdots\right).$$