Linear Algebra

Notes by Finley Cooper

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1 Vector Spaces

For this lecture course, $\mathbb{F}$ will always be field.

Definition. (Vector Space) A $\mathbb{F}$-*vector space* (or a vector space over $\mathbb{F}$) is an abelian group $(V, +, \mathbf{0})$ equipped with a function

$$\begin{aligned}\mathbb{F} \times V &\to V \\ (\lambda, v) &\to v\end{aligned}$$

which we call scalar multiplication such that $\forall v, w \in V, \forall \lambda, \mu \in \mathbb{F}$

(i) $(\lambda + \mu)v = \lambda v + \mu v$
(ii) $\lambda(v + w) = \lambda v + \lambda w$
(iii) $\lambda(\mu v) = (\lambda \mu)v$
(iv) $1 \cdot v = v \cdot 1 = v$

Remember that $\mathbf{0}$ and 0 are not the same thing. 0 is an element in the field $\mathbb{F}$ and $\mathbf{0}$ is the additive identity in V.

For an example consider $\mathbb{F}^n$ n-dimensional column vectors with entries in $\mathbb{F}$. We also have the example of a vector space $\mathbb{C}^n$ which is a complex vector space, but also a real vector space (taking either $\mathbb{C}$ or $\mathbb{R}$ as the underlying scalar field).

We also can see that $M_{m \times n}(\mathbb{F})$ form a vector space with m rows and n columns.
For any non-empty set X, we denote $\mathbb{F}^X$ as the space of functions from X to $\mathbb{F}$ equipped with operations such that:

$$\begin{aligned}f + g \text{ is given by } (f + g)(x) &= f(x) + g(x) \\ \lambda f \text{ is given by } (\lambda f)(x) &= \lambda f(x)\end{aligned}$$

Proposition. For all $v \in V$ we have that $0 \cdot v = \mathbf{0}$ and $(-1) \cdot v = -v$ where $-v$ denotes the additive inverse of v.

Proof. Trivial.

Definition. (Subspace) A *subspace* of a $\mathbb{F}$-vector space V is a subset $U \subseteq V$ which is a $\mathbb{F}$-vector space itself under the same operations as V. Equivalently, $(U, +)$ is a subgroup of $(V, +)$ and $\forall \lambda \in \mathbb{F}$, $\forall u \in U$ we have that $\lambda u \in U$.

Remark. Axioms (i)-(iv) are always automatically inherited into all subspaces.

Proposition. (Subspace test) Let V be a $\mathbb{F}$-vector space and $U \subseteq V$ then U is a subspace of V if and only if,

(i) U is nonempty.
(ii) $\forall \lambda \in \mathbb{F}$ and $\forall u, w \in U$ we have that $u + \lambda w \in U$.

Proof. If U is a subspace then U satisfies (i) and (ii) since it contains 0 and is closed. Conversely suppose that $U \subseteq V$ satisfies (i) and (ii). Taking $\lambda = -1$ so $\forall u, w \in V$, $u - w \in U$ hence $(U, +)$ is a subgroup of $(V,+)$ by the subgroup test. Finally taking $u = \mathbf{0}$ so we have that $\forall w \in U, \forall \lambda \in \mathbb{F}$ we have that $\lambda w \in U$. So U is a subspace of V. $\square$

We notate U by $U \leq V$.

For some examples

(i)
$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x + y + z = t \right\} \subseteq \mathbb{R}^3,$$
for fixed $t \in \mathbb{R}$ is a subspace of $\mathbb{R}^3$ iff $t = 0$.

(ii) Take $\mathbb{R}^{\mathbb{R}}$ as all the functions from $\mathbb{R}$ to $\mathbb{R}$ then the set of continuous functions is a subspace.

(iii) Also we have that $C^\infty(\mathbb{R})$, the set of infintely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$ and the subspace of continuous functions.

(iv) A further subspace of all of those subspaces is the set of polynomial functions.

Lemma. For $U, W \leq V$ we have that $U \cap W \leq V$.

Proof. We'll use the subspace test. Both U, W are subspaces so they contain $\mathbf{0}$ hence $\mathbf{0} \in U \cap W$ so $U \cap W$ is nonempty. Secondly take $x, y \in U \cap W$ with $\lambda \in \mathbb{F}$. Then $U \leq V$ and $x, y \in U$ so $x + \lambda y \in U$. Similarly with W so $x + \lambda y \in W$ hence we have that $x + \lambda y \in U \cap W$ hence $U \cap W \leq V$ $\square$

Remark. This does not apply for subspaces, in fact from IA Groups, we know it doesn't even hold for the underlying abelian group.

Definition. (Subspace sum) For $U, W \leq V$, the *subspace sum* of U, W is
$$U + W = \{u + w : u \in U, w \in W\}.$$

Lemma. If $U, W \leq V$ then $U + W \leq V$.

Proof. Simple application of the subspace test.

Remark. $U + W$ is the smallest subgroup of U, W in terms of inclusion, i.e. if K is such that $U \subseteq K$ and $W \subseteq K$ then $U + W \subseteq K$.

Definition. (Linear map) For V, W $\mathbb{F}$-vector spaces. A *linear map* from V to W is a group homomorphism, φ, from $(V,+)$ to $(W,+)$ such that $\forall v \in V$
$$\varphi(\lambda v) = \lambda \varphi(v)$$

Equivalently to show any function $\alpha : V \to W$ is a linear map we just need to show that $\forall u, w \in V$, $\forall \lambda \in \mathbb{F}$ we have
$$\alpha(u + \lambda w) = \alpha(u) + \lambda \alpha(w).$$

For some examples of linear maps

(i) $V = \mathbb{F}^n, W = \mathbb{F}^m$ $A \in M_{m\times n}(\mathbb{F})$. Then let $\alpha : V \to W$ be given by $\alpha(v) = Av$. Then α is linear.

(ii) $\alpha : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ defined by taking the derivative.

(iii) $\alpha : C(\mathbb{R}) \to \mathbb{R}$ defined by taking the integral from 0 to 1.

(iv) X any nonempty set, $x_0 \in X$,

$$\begin{aligned} \alpha : \mathbb{F}^X &\to \mathbb{F} \\ f &\to f(x_0) \end{aligned}$$

(v) For any V, W the identity mapping from V to V is linear and so is the zero map from V to W.

(vi) The composition of two linear maps is linear.

(vii) For a non-example squaring in $\mathbb{R}$ is not linear. Similiarly adding constants is not linear, since linear maps preserve the zero vector.