Fluid Dynamics

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1 Kinematics

1.1 Streamlines and pathlines

There are two natural ways to think of flow.

(i) A stationary observer watching flow go past. This is the Eulerian perspective. This is the approach used through this course. We define a velocity field (continuum field) $\mathbf{u}(\mathbf{x}, t)$.

(ii) A moving observing, travelling along with the flow. This is the Lagrangian perspective.

Definition. (Streamlines) These are curves that are everywhere parallel to the flow at a given instant.

Remark. The streamline that goes through $\mathbf{x}_0$ at time t_0 is given parametrically as $\mathbf{x} = \mathbf{x}(s, \mathbf{x}_0, t_0)$ and

$$\frac{d\mathbf{x}}{ds} = \mathbf{u}(\mathbf{x}, t_0)$$

(with $\mathbf{x} = \mathbf{x}_0$ at $s = 0$).

The set of streamlines shows the direction of flow a *given* instant a time (all fluid particle at one given time). Take the example $\mathbf{u} = (1, t)$. So at $t = 0$ we have $\mathbf{u} = (1,)$ so the streamlines are horizontal lines. At $t = 1$ we have $\mathbf{u} = (1, 1)$, so the streamlines are diagonal.

Definition. (Pathlines) A *pathline* is the trajectory of a fluid particle (a very small bit of fluid). The pathline $\mathbf{x} = \mathbf{x}(t, \mathbf{x}_0)$ of a fluid which is at $\mathbf{x}_0$ at $t = 0$ is such that

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t)$$

with $\mathbf{x}(0, \mathbf{x}_0) = \mathbf{x}_0$.

Again if we take $\mathbf{u} = (1, t)$ we get

$$\begin{cases} \frac{dx}{dt} = 1 \\ \frac{dy}{dt} = t \end{cases} \rightarrow \begin{cases} x = x_0 + t \\ y = y_0 + \frac{t^2}{2} \end{cases}$$

which describes the path $y - y_0 = \frac{1}{2}(x - x_0)^2$.

Remark. Pathlines are often called "Lagrangian trajectories". The applications are very useful to characterise transport (infecious diseases and pollution simulations).

If the flow is *steady* (so $\mathbf{u}$ does not depend on time). Then pathlines and streamlines are the same.

1.2 The material derivative

We will characterise the rate of change of "stuff" moving with a fluid. Consider a quantity $F(\mathbf{x}, t)$ in a fluid flow (intution is F is temperature). We want to measure how the temperature chnages as we move through the field F along the flow. Let compute the rate of change of (in time) seen by a moving observer. We will call this $\frac{DF}{Dt}$. Take a small time interval δt. Then

$$\begin{aligned}\delta F &= F(\mathbf{x} + \delta\mathbf{x}, t + \delta t) - F(\mathbf{x}, t) \\ &= \delta t \frac{\partial F}{\partial t} + (\delta\mathbf{x} \cdot \nabla)F + (\text{higher order terms}).\end{aligned}$$

We have that $\delta\mathbf{x} = \mathbf{u}\delta t$, so

$$\frac{\delta F}{\delta t} = \frac{DF}{Dt} = \frac{\partial F}{\partial t} + (\mathbf{u} \cdot \nabla)F.$$

We have the derivative and the convected derivative. This should be thought of as moving along gradients of a field.

1.3 Conservation of mass

Consider the flow through a straight rigid pipe with constant cross section. Suppose we have a $\mathbf{u}_{\text{in}}$ and a $\mathbf{u}_{\text{out}}$. Can we have $\mathbf{u}_{\text{in}} \neq \mathbf{u}_{\text{out}}$? For a gas, yes we can since they can be compressed. For a fluid, we cannot, since they are incompressible.

Define $\rho(\mathbf{x}, t)$ as the mass density with $[\rho] = \frac{\mathrm{M}}{\mathrm{L}^3}$. We want a relation between ρ and $\mathbf{u}$. Consider a fixed volume V and compute the rate of change of its mass, M.

$$M = \int_V \rho \mathrm{d}V$$

Assume that mass can only change due to the flow of mass across the boundary surface ∂V. Take a small surface element δA with normal $\mathbf{n}$. The volume out of V during δt is $(\mathbf{u} \cdot \mathbf{n})\delta A \delta t$. Hence the mass out is $\rho(\mathbf{u} \cdot \mathbf{n})\delta A \delta t$, so we get that

$$\frac{dM}{dt} = -\int_{\partial V} \rho(\mathbf{u} \cdot \mathbf{n})\mathrm{d}A.$$

The divergence theorem will allow us to rewrite this as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{u}) = 0.$$

We know from IA Vector Calculus that $\nabla \cdot (\rho\mathbf{u}) = \rho\nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla\rho$, so we can write that

$$\frac{D\rho}{Dt} = -\rho\nabla \cdot \mathbf{u}.$$

Definition. (Incompressible) A fluid flow is *incompressible* if $\frac{D\rho}{Dt} = 0$.

This is then equivalent to $\nabla \cdot \mathbf{u} = 0$ which is the equivalent condition we'll use for the course.

For this course we will assume that ρ is constant. This means as a consequence that $\nabla \cdot \mathbf{u} = 0$.

1.4 Kinematic boundary condition

Consider the material boundary, with unit norm $\mathbf{n}$, of a body of fluid has a given velocity $\mathbf{U}(\mathbf{x}, t)$. At a point $\mathbf{x}$ on the boundary, the fluid velocity relative to the surface is $\mathbf{u} - \mathbf{U}$. Applying mass conversation on the interface over a small surface element δA in time δt. So

$$\rho(\mathbf{u} - \mathbf{U}) \cdot \mathbf{n}\delta A \delta t = 0.$$

Hence we require $\mathbf{u} \cdot n = \mathbf{U} \cdot \mathbf{n}$ at the interface. This is the kinematic boundary condition.

Remark. $\mathbf{n}$ occurs on both sides, hence we don't need $\mathbf{n}$ to be a unit vector.

We have some consequences of this condition.

(i) If the boundary is fixed, $\mathbf{U} = 0$ implies that $\mathbf{u} \cdot \mathbf{n} = 0$. This is called the no penetration condition.

(ii) Consider an air/water interface (free surface). Suppose the surface is defined by $z = \xi(x, y, t)$. Then can think of the free space as $F(x, y, z, t) = 0$ where $F(x, y, z, t) = z - (x, y, t)$. So $\mathbf{n}$ is perp to $\nabla F = (-\xi_x, -\xi_y, 1)$. Then if $\mathbf{u} = (u, v, w)$ so $\mathbf{U} = (0, 0, \xi_t)$. Then the kinematic boundary condition becomes $-u\xi_x - c\xi_y + w = \xi_t$, so $w = \xi_t + u\xi_x + v\xi_y = \frac{D\xi}{Dt}$. This is equivalent to $\frac{DF}{Dt} = 0$.

1.5 Streamfunction for 2D incompressible flow

We know that $\nabla \cdot \mathbf{u} = 0$ which is equivalent to there existing a vector potential $\mathbf{A}$ such that $\mathbf{u} = \nabla \times \mathbf{A}$. In 2D if $\mathbf{u} = (u, v, 0)$ then $\mathbf{A} = (0, 0, \psi(x, y))$. So

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}.$$

We call ψ a *streamfunction*. Looking at dimensions we have that $[\psi] = \mathrm{L}^2\ \mathrm{T}^{-1}$. Now we'll see an example.

Let $\mathbf{u} = (y, x)$ (which we can see is incompressible) so

$$\frac{\partial \psi}{\partial y} = u = y$$

, hence $\psi = \frac{1}{2}y^2 + f(x)$. We also have that $-\frac{\partial \psi}{\partial x} = -f'(x) = x$, so $\psi = \frac{1}{2}(y^2 - x^2) + C$.

We have some properties about the streamfunction,

(i) Streamlines are given by $\psi = \text{constant}$.

(ii) $|\mathbf{u}| = |\nabla \psi|$, so the flow is faster if the streamlines are closer together.

(iii) If we take two points $\mathbf{x}_0, \mathbf{x}_1$, then then the volume flux crossing the line between $\mathbf{x}_0$ and $\mathbf{x}_1$ is

$$\int_{\mathbf{x}_0}^{\mathbf{x}_1} \mathbf{u} \cdot \mathbf{n} \mathrm{d}\ell = \psi(\mathbf{x}_1) - \psi(\mathbf{x}_0).$$

(iv) ψ is constant at rigid boundaries.

We can do the same in polar coordinates. So $\mathbf{u} = (u_r(r, \theta), u_\theta(r, \theta), 0)$. We have that

$$\mathbf{u} = \nabla \times \mathbf{A} = \left(\frac{1}{r}\frac{\partial \psi}{\partial \theta}, -\frac{\partial \psi}{\partial r}, 0\right),$$

so we can check that $\nabla \cdot \mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial}{\partial \theta} = 0$

2 Dynamics of inviscid flow

2.1 Surface and volume forces

There a two types of forces exerted on a fluid.

(i) Forces proportional to the volume (gravity);

(ii) Forces proportional to the surface area (pressure, viscous stresses).

We'll first look at the first type, called volume forces. We'll denote $F(\mathbf{x}, t)\delta V$ as the force acting on a small volume element ∂V. Let's take gravity as an example, so $\mathbf{F} = \rho\mathbf{g}$. Often we have that $\mathbf{F}$ is conservative, so $\mathbf{F} = -\nabla\chi$ for some function χ (we know gravity is $\chi = \rho g z$).