Methods

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1 Fourier Series

1.1 Motivation

In 1807 J. Fourier was studying head conduction along a metal rod. This lead him to study 2π-periodic functions i.e. functions $f:\mathbb{R}\to\mathbb{R}$ was such that $f(\theta+2\pi)=f(\theta)$ for all $\theta\in\mathbb{R}$ then he found that if

$$f(\theta)=\sum_{n\in\mathbb{Z}}\hat{f}_n e^{in\theta}$$

then you can write down the coefficients $\{\hat{f}_n\}$ via the formula

$$\hat{f}_n=\frac{1}{2\pi}\int_0^{2\pi}f(\theta)e^{-in\theta}\mathrm{d}\theta,\quad n\in\mathbb{Z}.$$

And Fourier believed that this worked for any 2π-periodic function f. So computing each $\{\hat{f}_n\}$ and construcuted the sum as above, then it would return the original function. He was wrong.

1.2 Modern Treatment

Introduce a vector space V of L-periodic functions. Hence

$$V=\{f:\mathbb{R}\to\mathbb{C}:\text{ with } f \text{ a "nice" function, } f(\theta+L)=f(\theta),\forall\theta\in\mathbb{R}\}.$$

Note for $f\in V$ need only to consider values of f taken in an interval of length L, i.e. $[0,L)$ or $(-\frac{L}{2},\frac{L}{2}]$ since periodicity covers elsewhere.

We can introduce an inner product on V with

$$\langle f,g\rangle=\int_0^1 f(\theta)\overline{g(\theta)}\mathrm{d}\theta.$$

This gives the associated norm,

$$||f||=\sqrt{\langle f,f\rangle}.$$

For $n\in\mathbb{Z}$ consider $e_n\in V$ defined by $e_n(\theta)=e^{2\pi in\theta/L}$.

$$\langle e_n,e_m\rangle=\int_0^L e^{2\pi i(n-m)\theta/L}\,\mathrm{d}\theta=L\,\delta_{nm}.$$

So $\{e_n\}$ are orthogonal and $||e_n||^2=L$ for each $n\in\mathbb{Z}$. This looks like IA Vectors and Matrices.

Recall that if v_N is N-dim vector space equipped with usual inner product and $\{e_n\}_{n=1}^N$ are orthogonal with $|e_n|=L$, then for each $x\in V$ we can write $x=\sum_{n=1}^N\hat{x}_ne_n$ for some $\{\hat{x}_n\}$. To find $\{\hat{x}_n\}$ take the inner product of both sides with e_m. So

$$(x,e_m)=\sum_{n=1}^N\hat{x}_n(e_n\cdot e_m)=L\hat{x}_m$$

i.e

$$\hat{x}_n=\frac{1}{L}(x\cdot e_n).$$

Now could this work on V? V is not finite dimensional so it's not obvious. Every subset of $\{e_n\}$ is linearly indepedent. Ignoring this for now we assume that for all $f \in V$ we can write f in our basis $\{e_n\}$. Then

$$f(\theta) = \sum_n \hat{f}_n e_n(\theta),$$

So taking the inner product as before

$$\langle f, e_m \rangle = \sum_n \hat{f}_n \langle e_n, e_m \rangle$$

so using the delta as before

$$= L\hat{f}_m$$

i.e.

$$\hat{f}_n = \frac{1}{L}\langle f, e_n \rangle = \frac{1}{L}\int_0^1 f(\theta)e^{-2\pi i n\theta/L}\mathrm{d}\theta$$

Definition. (Complex Fourier series) For an L-periodic $f : \mathbb{R} \to \mathbb{C}$ define its *complex Fourier series* by

$$\sum_n \hat{f}_n e^{2\pi i n\theta/L}$$

where

$$\hat{f}_n = \frac{1}{L}\int_0^1 f(\theta)e^{-2\pi i n\theta/L}\mathrm{d}\theta$$

are called the complex Fourier coefficients. We will write for $f \in V$

$$f(\theta) \sim \sum_n \hat{f}_n e^{2\pi i n\theta/L}$$

to mean the series on the right corresponds to complex Fourier series for the function on the left.

We'd like to replace the $\sim$ symbol with equality, but we require a bit more than that.

If we split the complex Fourier series into the parts $\{n = 0\} \cup \{n > 0\} \cup \{n < 0\}$ we get

$$\sum_n \hat{f}_n e^{2\pi i n\theta/L} = \hat{f}_0 + \sum_{n=1}^{\infty} \hat{f}_n \left[\cos\left(\frac{2\pi n\theta}{L}\right) + i\sin\left(\frac{2\pi n\theta}{L}\right)\right] + \sum_{n=1}^{\infty} \hat{f}_{-n}\left[\cos\left(\frac{2\pi n\theta}{L}\right) - i\sin\left(\frac{2\pi n\theta}{L}\right)\right].$$

Definition. (Fourier series) For $f : \mathbb{R} \to \mathbb{C}$ an L-periodic function define its *Fourier series* by

$$\frac{1}{L}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{2\pi n\theta}{L}\right) + b_n\sin\left(\frac{2\pi n\theta}{L}\right)\right]$$

where

$$a_n = \frac{2}{L}\int_0^L f(\theta)\cos\left(\frac{2\pi n\theta}{L}\right)\mathrm{d}\theta$$

and

$$b_n = \frac{2}{L}\int_0^L f(\theta)\sin\left(\frac{2\pi n\theta}{L}\right)\mathrm{d}\theta$$

are called the Fourier cofficients for f.

If we set

$$c_n(\theta) = \cos\left(\frac{2\pi n\theta}{L}\right),$$
$$s_n(\theta) = \sin\left(\frac{2\pi n\theta}{L}\right),$$

then we can show, for $m, n \geq 1$ that $\langle c_n, c_m\rangle = \langle s_n, s_m\rangle = \frac{L}{2}\delta_{mn}$ and

$$\langle c_n, 1\rangle = \langle s_m, 1\rangle = \langle c_n, s_m\rangle = 0.$$

So we have that $\{1, c_n, c_n\}$ is orthogonal set in V.

For an example take $f : \mathbb{R} \to \mathbb{R}$, 1-periodic, such that $f(\theta) = \theta(1-\theta)$ on $[0, 1)$. For $n \neq 0$ we have

$$\hat{f}_n = \int_0^1 \theta(1-\theta)e^{-2\pi i n\theta}\,\mathrm{d}\theta.$$

Integrating by parts (or using a standard Fourier integral computation) yields

$$\hat{f}_n = -\frac{1}{2(\pi n)^2}, \qquad n \neq 0,$$

and

$$\hat{f}_0 = \int_0^1 (\theta - \theta^2)\,\mathrm{d}\theta = \frac{1}{6}.$$

Hence

$$f(\theta) \sim \frac{1}{6} - \sum_{n\neq 0}\frac{e^{2\pi i n\theta}}{2(\pi n)^2}.$$

so the sine terms cancel in the sum giving just cosine terms as we expect since our f function is even.

1.3 Convergence of Fourier series

This subject is extremely subtle.

Definition. For $f : \mathbb{R} \to \mathbb{C}$ an L-periodic function we defined the *partial Fourier series* as

$$\begin{aligned}(S_N f)(\theta) &= \sum_{|n|<N}\hat{f}_n e^{2\pi i n\theta/L}\\ &= \frac{1}{2}a_0 + \sum_{n=1}^{N}\left[a_n\cos\left(\frac{2\pi n\theta}{L}\right) + b_n\sin\left(\frac{2\pi n\theta}{L}\right)\right]\end{aligned}$$

Natural to ask if $(S_N f) \to f$. For this we need to specify what type of functional convergence we're looking at. Pointwise? Uniform? Maybe they converge in the idea of our new norm?

$$||S_N f - f|| = \sqrt{\int_0^L |(S_N f)(\theta) - f(\theta)|^2 \mathrm{d}\theta} \to 0$$

. For simplicity, we will only consider pointwise convergence.

Proposition. Let $f : \mathbb{R} \to \mathbb{C}$ be an L-periodic function for which on $[0, L)$ we have the following,

(i) f has finitely many discontinuities.

(ii) f has finitely many local maxima and minima.

Then for each $\theta \in [0, 1)$ we have

$$\begin{aligned} \frac{\theta_+ + \theta_-}{2} &= \lim_{n\to\infty} (S_N f)(\theta) \\ &= \sum_n \hat{f}_n e^{2\pi i n\theta/L} \end{aligned}$$

where $f(\theta_\pm) = \lim_{\varepsilon\to 0^+} f(\theta \pm \varepsilon)$. So at the points of continuity the Fourier series gives back the original function, and at points of discontunity the Fourier series gives back the average of the function at the disconunity neighbourhood.

We call functions which properties (i) and (ii) Dirichlet functions. For now on assume all functions are Dirichlet functions so that $\sim$ means that the series on the RHS coincides with the function on the LHS at points of continuity and to the average at points of discontinuity.

Proof. We'll prove the proposition only for functions in $C^\infty(\mathbb{R})$ (actually $C^1(\mathbb{R})$ will do. Assume *wlog* that $L = 2\pi$. Examine $\lim S_N f(\theta_0)$ for some $\theta_0 \in [0, 2\pi)$. By replacing $f(\theta)$ with $f(\theta + \theta_0)$ can assume that $\theta_0 = 0$ *wlog.*

$$\begin{aligned} (S_N f)(\theta) &= \sum_{|n|\le N} \hat{f}_n e^{in\cdot\theta} \\ &= \sum_{|n|\le N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} \mathrm{d}\theta \right) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \left[\sum_{|n|\le N} e^{-in\theta} \right] \mathrm{d}\theta \end{aligned}$$

We can sum the series as a geometric series, so

$$e^{-iN\theta} \sum_{n=0}^{2N} e^{-in\theta} = \frac{\sin[(N + \frac{1}{2})\theta]}{\sin\left(\frac{\theta}{2}\right)}$$

when $\theta \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ and the sum is $2N + 1$ when $\theta \in 2\pi\mathbb{Z}$.

Define the *Dirichlet Kernal* as

$$D_N(\theta) = \begin{cases} \frac{\sin[(N+\frac{1}{2})\theta]}{\sin\left(\frac{\theta}{2}\right)} & \theta \in \mathbb{R} \setminus 2\pi\mathbb{Z} \\ 2N + 1 & \text{otherwise} \end{cases}$$

For each $N \geq 0$,

(i) D_N is continiuous, even 2π perioidic

(ii) $\int_{-\pi}^{\pi} D_N(\theta)\mathrm{d}\theta = 2\pi$

Property (ii) follows by intergrating $\sum$ termwise, only 1 is non-zero. This means that

$$f(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(\theta)f(\theta)\mathrm{d}\theta$$

So

$$S_N(f)(0) = f(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(\theta)[f(\theta) - f(0)]\mathrm{d}\theta$$

now set $F(\theta) = \frac{\theta}{\sin\left(\frac{\theta}{2}\right)}\left[\frac{f(\theta)-f(0)}{\theta}\right]$ so we get

$$(S_N f)(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin[(N+\frac{1}{2})\theta]F(\theta)\mathrm{d}\theta$$

Note that $\theta \to F(\theta)$ is smooth since

$$\frac{f(\theta) - f(0)}{\theta} = \frac{1}{\theta}\int_0^{\theta} f'(t)\mathrm{d}t = \frac{1}{\theta}\int_0^1 f'(\tau\theta)\theta\mathrm{d}\tau$$

Hence integrating by parts gives that

$$\begin{aligned}(S_N f)(0) - f(0) &= \frac{1}{N+\frac{1}{2}}\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos[(N+\frac{1}{2})\theta]F'(\theta)\mathrm{d}\theta \\ &\to 0 \text{ as } N \to \infty\end{aligned}$$

For an example consider the function

$$f(\theta) = \begin{cases} +1 & 0 \leq \theta < \pi \\ -1 & -\pi \leq \theta < 0 \end{cases}$$

Since f is odd, $a_n = 0$ for each n and

$$\begin{aligned}b_n &= \frac{2}{2\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\mathrm{d}\theta \\ &= \frac{2}{\pi}\int_0^{\pi}\sin(n\theta)\mathrm{d}\theta \\ &= \frac{2}{n\pi}[1-(-1)^n]\end{aligned}$$

Thus

$$f(\theta) \sim \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin(n\theta)}{n}$$

1.4 Peridoic extensions: Cosine and sine series

Given a function $f : [0, L) \to \mathbb{C}$ we can define $2L$-periodic even/odd extensions called f_{even}, f_{odd}. Define,

$$f_{even}(\theta) = \begin{cases} f(\theta) & \theta \in [0, L) \\ f(-\theta) & \theta \in [-L, 0) \end{cases}$$

and

$$f_{odd}(\theta) = \begin{cases} f(\theta) & \theta \in [0, L) \\ -f(-\theta) & \theta \in [-L, 0) \end{cases}$$

. Note that $f(\theta) = f_{even}(\theta) = f_{odd}(\theta)$ if $\theta \in [0, L)$.

$$\begin{aligned} f_{even}(\theta) &\sim \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2\pi n\theta}{2L}\right) \\ A_n &= \frac{2}{2L}\int_{-L}^{L} f_{even}(\theta) cos\left(\frac{2\pi n\theta}{2L}\right) \mathrm{d}\theta \\ &= \frac{2}{L}\int_{0}^{L} f(\theta) cos\left(\frac{2\pi\theta}{L}\right) \mathrm{d}\theta \end{aligned}$$

simiarly we have that

$$\begin{aligned} f_{odd}(\theta) &\sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{2\pi n\theta}{2L}\right) \\ B_n &= \frac{2}{L}\int_{0}^{L} f(\theta) \sin\left(\frac{n\pi\theta}{L}\right) \mathrm{d}\theta \end{aligned}$$

Definition. For $f : [0, L) \to \mathbb{C}$ define its *cosine* and *sine* series by

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi\theta}{L}\right), \quad \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi\theta}{L}\right)$$

where A_n and B_n defined as before.

For an example consider $f(\theta) = 1$ on $[0, \pi)$. For the sine series,

$$B_n = \frac{2}{\pi}\int_0^{\pi} \sin(n\theta)\mathrm{d}\theta = \frac{2}{n\pi}(1 - (-1)^n)$$

On the interval $(0, \pi)$ we get that $1 = f(\theta) = f_{odd}(\theta) = 4\sum_{n\in\mathbb{N}} \frac{\sin(n\theta)}{n\pi}$. Whereas for the cosine series we get that

$$A_0 = 2, \quad A_n = 0 \quad n \geq 1$$

So for $\theta \in [0, \pi)$ we get that $f_{even} = \frac{1}{2} \cdot 2 = 1 = f(\theta)$.

1.5 Regularity and decay of Fourier coefficients

A true but non-examinable fact is that if $g : [a, b] \to \mathbb{C}$ is integrable on $[a, b]$ and $\lambda \in \mathbb{R}$ then

$$\int_a^b e^{-i\lambda\theta} g(\theta)\mathrm{d}\theta \to 0 \text{ as } |\lambda| \to \infty.$$

IF $f : \mathbb{R} \to \mathbb{C}$ is a L-periodic function and integrable on $[0, L)$ then

$$\hat{f} = \frac{1}{L} \int_0^L e^{-2\pi i n\theta/L} f(\theta)\mathrm{d}\theta$$

so taking $\lambda = \frac{2\pi n}{L}$ gives that $\hat{f}_n \to 0$ as $n \to \infty$ by the Riemann-Lebesgue lemma. Also

$$a_n = \hat{f}_n + \hat{f}_{-n} \quad b_n = i(\hat{f}_n - \hat{f}_{-n}),$$

both go to zero as $n \to \infty$.

Suppose that f is L-periodic and $f \in C^k(\mathbb{R})$.

$$\begin{aligned}
\hat{f}_n &= \frac{1}{L} \int_0^L e^{-2\pi i n\theta/L} f(\theta)\mathrm{d}\theta \\
&= -\frac{1}{L} \left(\frac{L}{2\pi i n}\right) f(\theta) e^{-2\pi i n\theta/L} \Big|_{\theta=0}^{L} + \left(\frac{L}{2\pi i n}\right) \frac{1}{L} \int_0^L e^{-2\pi i n\theta/L} f'(\theta)\mathrm{d}\theta \\
&= -\frac{L}{2\pi i n} \left[\frac{f(L^-) - f(0^+)}{L}\right] + \frac{L}{2\pi i n} \frac{1}{L} \int_0^L e^{-2\pi i n\theta} f'(\theta)\mathrm{d}\theta
\end{aligned}$$

Since f is periodic and continuously differentiable we have that

$$f(0^+) = f(L^+) = f(L^-)$$

hence the boundary term cancels so repeating we get that

$$\hat{f}_n = \left(\frac{L}{2\pi i n}\right)^k \frac{1}{L} \int_0^L e^{-2\pi i n\theta/L} f^{(k)}(\theta)\mathrm{d}\theta$$

and the integral is $o(1)$ by the Rieman-Lebesgue lemma.

So we get that if f is $C^k(\mathbb{R})$ then $\hat{f}_n = o\left(\frac{1}{n^k}\right)$ as $|n| \to \infty$.

1.6 Termwise differentiation

Suppose f is L-periodic continuously differentiable on $[0, L)$ with $f' = g$ thne g is continuous on $[0, L)$ so

$$\begin{aligned}
\hat{g}_n &= \frac{1}{L} \int_0^L e^{-2\pi i n\theta/L} f'(\theta)\mathrm{d}\theta \\
&= \frac{f(L^-) - f(0^+)}{L} + \left(\frac{2\pi i n}{L}\right) \frac{1}{L} \int_0^L e^{-2\pi i n\theta/L} f(\theta)\mathrm{d}\theta
\end{aligned}$$

If f is continuous on $\mathbb{R}$ then by periodicity we have that

$$f(0^+) = f(L^+) = f(L^-)$$

so that

$$\hat{g}_n = \left(\frac{2\pi i n}{L}\right) \hat{f}_n$$

i.e.

$$f'(\theta) = g(\theta) \sim \sum_n \left(\frac{2\pi i n}{L}\right) \hat{f}_n e^{2\pi i n\theta/L}$$

1.7 Parseval's theorem

If we have that

$$f(\theta) \sim \sum_n \hat{f}_n e_n(\theta)$$

and

$$g(\theta) \sim \sum_n \hat{g}_n e_n(\theta)$$

then taking the inner product of both function we get that

$$\begin{aligned}\langle f, g\rangle &= \sum_{n,m} \hat{f}_n \overline{\hat{g}_n} \langle e_n, e_m\rangle \\ &= L \sum_n \hat{f}_n \overline{\hat{g}_n}\end{aligned}$$

finally that

$$\frac{1}{L}\int_0^L f(\theta)\overline{g(\theta)}\mathrm{d}\theta = \sum_n \hat{f}_n \overline{\hat{g}_n}$$

and when f and g are the same we get that

$$\frac{1}{L}\int_0^L |f(\theta)|^2 \mathrm{d}\theta = \sum_n |\hat{f}_n|^2$$

2 Sturm-Liouville Theory

2.1 Abstract eigenvalues problem

Recall from IA Vectors and Matrices that a linear map $A : V_N \to V_n$ was called *Hermitian* if $A^\dagger = A$ or equivalently we have that

$$\mathbf{x} \cdot (A\mathbf{y}) = (A\mathbf{x}) \cdot \mathbf{x}$$

for all $\mathbf{x}, \mathbf{y} \in V_N$.

They had properties where all eigenvalues are real, eigenvectors with distinct eigenvalues were orthogonal, and that we could pick an porthogonal set of eigenvectors $\{\mathbf{v}_i\}_{i=1}^N$ such that for each $\mathbf{x} \in V_n$ we have that

$$\mathbf{x} = \sum_{i=1}^N \hat{\mathbf{x}}_i \mathbf{v}_i$$

where

$$\hat{\mathbf{x}}_i = \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{v}_i|^2}.$$

But now we're in $N = \infty$, we can't assume everything we've learnt so far.

Use a vector space of nice functions, $f : [a, b] \to \mathbb{C}$ with an inner product

$$\langle f, g\rangle_w = \int_a^b f(x)\overline{g(x)}w(x)\mathrm{d}x.$$

where w is real valued and $w > 0$ on (a, b). We call w the *weight function* associated with the inner product. This gives an associated norm

$$||f|| = \sqrt{\langle f, f \rangle_w}$$

when $w(x) = 1$ we just write $\langle \cdot, \cdot \rangle$.

Definition. (Self-adjoint) A linear differential operator, L, is said to be *self-adjoint* on $(V, \langle \cdot, \cdot \rangle_w)$ if

$$\langle Ly_1, y_2 \rangle_w = \langle y_1, Ly_2 \rangle_w \quad \forall y_1, y_2 \in V.$$

Definition. (Eigenfunction/value) For $(y, \lambda) \in (V \setminus \{0\} \times \mathbb{C}$ is an *eigenfunction, eigenvalue* pair for L if $Ly = \lambda y$.

Proposition. If L is self-adjoint on $(V, \langle \cdot, \cdot, \rangle_w)$ then:

(i) Eigenvalues are real,

(ii) eigenfunctions with distinct eigenvalues are orthogonal,

(iii) there exists a complete orthogonal set of eigenfunctions $\{y_n\}_{n=1}^{\infty}$ i.e. for each $f \in V$ we can write,

$$f = \sum_{n=1}^{\infty} \hat{f}_n y_n$$

where

$$\hat{f}_n = \frac{\langle f, y_n \rangle_w}{||y_n||_w^2}$$

Proof. (For (i)) If $Ly = \lambda y$ with $y \neq 0$ then

$$\begin{aligned}
(\lambda - \overline{\lambda})||y||_w^2 &= \langle \lambda y, y \rangle_w - \langle y, \lambda y \rangle_w \\
&= \langle Ly, y \rangle_w - \langle y, Ly \rangle_w \\
&= 0 \implies \lambda = \overline{\lambda}
\end{aligned}$$

(For (ii)) If $Ly_1 = \lambda_1 y_1, Ly_2 = \lambda_2 y_2$ with $\lambda_1 \neq \lambda_2$,

$$\begin{aligned}
(\lambda_1 - \lambda_2)\langle y_1, y_2 \rangle_w &= \langle \lambda_1 y_1, y_2 \rangle_w - \langle y_1, \lambda_2 y_2 \rangle_w \\
&= \langle Ly_1, y_2 \rangle - \langle y_1, Ly_2 \rangle_w \\
&= 0 \implies \langle y_1, y_2 \rangle_w = 0
\end{aligned}$$

The third statement is too hard to prove for this course. □

We will study problems of the form

$$\begin{cases} Ly = \lambda y a < x < b \\ y \text{ satisfies some boundary conditions at } x = a, b \end{cases} \tag{2.1}$$

Definition. (Sturm-Liouville operator) We say that L is a *Sturm-Liouville operator* on (a, b) if it has the form

$$\begin{aligned} L &= \frac{1}{w}\left[-\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}\bullet}{\mathrm{d}x}\right) + q\bullet\right] \\ &= \frac{1}{w}\left[-p\frac{\mathrm{d}^2\bullet}{\mathrm{d}x^2} - p^2\frac{\mathrm{d}\bullet}{\mathrm{d}x} + q\bullet\right] \end{aligned}$$

where p, q, w are real valued and $p, w > 0$ on (a, b). We call w the *weight function.*

See that $Ly = \lambda y$ is equvialent to

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}y}{\mathrm{d}x}\right) + qy = \lambda wy \quad a < x < b.$$

We will enforce boundary conditions by stipulating that y belongs to a suitable vector space of functions that appropriate behaviour at the boundaries.

Definition. (Singular) For a Sturm-Liiouville operator on (a, b) say an endpoint $c \in \{a, b\}$ is *singular* if $p(c) = 0$ and *non-singular* otherwise.

We will impose real homogeneous boundary conditions of the form

$$c \in \{a, b\} \quad \alpha_c y(c) + \beta_c y'(c) = 0$$

at each non-singular endpoint, :w for $\alpha_c, \beta_c \in \mathbb{R}$ and $\alpha_c^2 + \beta_c^2 \neq 0$.

We will work on generic vector spaces of the form

$$V = \Big\{ y \in C^2[a, b] : y \text{ satisfies real homogeneous boundary conditions at each non-singular endpoint } \Big\}$$

Let's look at the example

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[(1 - x^2)\frac{\mathrm{d}y}{\mathrm{d}x}\right] = \lambda y \quad -1 < x < 1.$$

So we have that $p = (1 - x^2), q = 0, w = 1$. Then $x = \pm 1$ both singular. Take $V = \{y \in C^2[a, b]\}$ then

$$\langle f, g\rangle_w = \int_{-1}^{1} f(x)\overline{g(x)}\mathrm{d}x.$$

Proposition. If L is a Sturm-Lionville operator on (a, b) with weight function w then if $y_1, y_0 \in C^2[a, b]$ we have that

$$\langle Ly_1, y_2\rangle_w - \langle y_1, Ly_2\rangle_w = p(x)W(y_1, \overline{y_2})(x)\Big|_a^b$$

where W is the Wronskian.

So if $y_1, y_2 \in V$ then L is self-adjoint on $(V, \langle\cdot,\cdot\rangle_w)$.

Proof.

$$\begin{aligned}
&\int_a^b \frac{1}{w}\left[-(py')' + qy_1\right]\bar{y_2}w\mathrm{d}x - \int_a^b y_1\frac{1}{w}\left[-(p\bar{y_2})' + q\bar{y_2}\right]w\mathrm{d}x \\
&= \int_a^b \left[y_1(p\bar{y_2})' - \bar{y_2}(py_1')'\right]\mathrm{d}x \\
&= \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)W(y_1,\bar{y_2})(x)\right]\mathrm{d}x \\
&= p(x)W(y_1,\bar{y_2})(x)\Big|_a^b.
\end{aligned}$$

Now assume that $y_1, y_2 \in V$. If $x = c \in \{a, b\}$ is singular then $p(c) = 0$ hence $p(c)W(y_1, \bar{y_2})(c) = 0$. If $c \in \{a, b\}$ non-singular then y_1, y_2 satisfy boundary conditions of the form

$$\alpha_c y(c) + \beta_c y'(c) = 0, \quad \alpha_c, \beta_c \in \mathbb{R}, \alpha_c^2 + \beta_c^2 \neq 0.$$

Since $\alpha_c, \beta_c \in \mathbb{R}$ we know that $\bar{y}$ also satisfies the same boundary conditions hence

$$\begin{pmatrix} y_1(c) & y_1'(c) \\ \bar{y_2}(c) & \bar{y_2}'(c) \end{pmatrix}\begin{pmatrix} \alpha_c \\ \beta_c \end{pmatrix} = 0$$

So the determinate of the matrix on the left is zero because α_c and β_c don't both equal zero hence $W(y_1, \bar{y}_2)(c) = \det(\cdots) = 0$ Hence we have that

$$\langle Ly_1, y_2\rangle_w - \langle y_1, Ly_2\rangle_w = 0$$

for all $y_1, y_2 \in V$. □

2.2 Sturm-Lioville Eigenvalue problems

We'll be studying problems of the form

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[p\frac{\mathrm{d}y}{\mathrm{d}x}\right] + qy = \lambda y \quad y \in V$$

where

$$V = \left\{y \in C^2[a,b] : y \text{ satisfies real homogeneous BCs at each non-ingular end point}\right\}$$

Equip V with an inner product with a weight function as before. Assume elements of V are real-valued *wlog* since if $y = u + iv$ and $Ly = \lambda y$ then we can split up into

$$Lu = \lambda u, \quad Lv = \lambda v$$

since $p, q, w, \lambda \in \mathbb{R}$. So

$$\langle y_1, y_2\rangle_w = \int_a^b y_1(x)y_2(x)\mathrm{d}x.$$

Since L is self-adjoint, we know there exists $(y_n, \lambda_n) \in (V \setminus \{0\}) \times \mathbb{R}$ such that $Ly_n = \lambda_n y_n$ with $\langle y_n, y_m\rangle_w = 0$ if $\lambda_n \neq \lambda_m$ and for $f \in V$ we have

$$f(x) = \sum_{n=1}^{\infty} \hat{f}_n y_n(x)$$

$$\hat{f}_n = \frac{\langle f, y_n\rangle_w}{||y_n||_w^2}$$

are the generalised Fourier coefficients of f. It will also be the cases that $\lambda_1 < \lambda_2 < \cdots$ with $\lambda_n \to \infty$ as $n \to \infty$. Let's look at an example. Take

$$\begin{cases} -y'' = \lambda y & 0 < x < L \\ y(0) = y(L) = 0 \end{cases}$$

so $p = w = 1$ and $q = 0$ and $V = \{y \in C^2[0, L] : y(0) = y(L) = 0\}$.

Solving $y'' + \lambda y = 0$ then $y = A\sin\left(\sqrt{\lambda}x\right) + B\cos\left(\sqrt{\lambda}x\right)$. If $\lambda \leq 0$ we only get the trivial solution, so we must have that $\lambda > 0$. If we use $y(0) = 0 \implies B = 0$ and $y(L) = 0 \implies A\sin\left(\sqrt{\lambda}L\right) = 0$ so other than the trivial solution, we have that

$$\sqrt{\lambda} = \frac{n\pi}{L}, \quad n = 1, 2, \ldots$$

So

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \quad y_n(x) = \sin\left(\frac{n\pi x}{L}\right).$$

We can see that $\lambda_n \to \infty$ and $\lambda_1 < \lambda_2 < \cdots$ and

$$\begin{aligned} \langle y_n, y_m \rangle &= \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) \mathrm{d}x \\ &= \frac{L}{2}\delta_{nm} \end{aligned}$$

For $f \in V$,

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \hat{f}_n \sin\left(\frac{n\pi x}{L}\right) \\ \hat{f}_n &= \frac{\langle f, y_n \rangle}{||y_n||^2} \\ &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x \end{aligned}$$

We have re-derived the Fourier sine series from the previous section.

2.3 Reduction to Sturn-Lionville form

Consider a general eigenvalue problem of the form

$$\alpha(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \beta(x)\frac{\mathrm{d}y}{\mathrm{d}x} + \gamma(x)y + \lambda y = 0$$

with $\alpha(x) > 0$. Divide the equation by $\alpha(x)$ and multiply by

$$I(x) = \exp\left[\int^x \frac{\beta(t)}{\alpha(t)}\mathrm{d}t\right]$$

Proposition. The equation

$$\alpha(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \beta(x)\frac{\mathrm{d}y}{\mathrm{d}x} + \gamma(x)y + \lambda y = 0$$

is equivalent to

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[p\frac{\mathrm{d}y}{\mathrm{d}x}\right] + qy = \lambda w y$$

where
(i) $p(x) = I(x)$,
(ii) $q(x) = -\frac{I(x)\gamma(x)}{\alpha(x)}$,
(iii) $w(x) = \frac{I(x)}{\alpha(x)}$.

Proof.

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[p\frac{\mathrm{d}y}{\mathrm{d}x}\right] + qy - \lambda w y = I\left[-\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\beta(x)}{\alpha(x)}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{\gamma(x)}{\alpha(x)}y - \frac{\lambda y}{\alpha(x)}\right]$$

since $I > 0$ we get that the equation is zero if and only if $LHS = 0$. □

For an example consider

$$\begin{cases} y'' = 2y' + \lambda y = 0 \quad 0 < x < 1 \\ y(0) = y'(1) = 0 \end{cases}$$

So we have that

$$I(x) = \exp\left[\int^x -\frac{2}{1}\right] = e^{-2x}$$

So the ODE becomes

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{-2x}\frac{\mathrm{d}y}{\mathrm{d}x}\right] = \lambda e^{-2x}y.$$

So we get e^{-2x} as our weight function.

To solve put $y \propto e^{-\alpha x} \implies \alpha = 1 \pm \sqrt{1-\lambda}$ So if $\lambda \neq 1$ we'll get solutions of the form

$$y = e^x\left[Ae^{x\sqrt{1-\lambda}} + Be^{-x\sqrt{1-\lambda}}\right]$$

We need $1 - \lambda < 0$ for non-trivial solutions.

We can see $y(0 = 0 \implies B = 0$ and $y'(1) = 0 \implies Ae\left[\sin\mu + \mu\cos\mu\right] = 0$ where $\mu^2 = \lambda - 1$ and $\mu > 0$ *wlog*. So $\tan\mu = -\mu$. By plotting the graph we can see we have infinitely many solutions for the equation. Call $\mu_1, \mu_2, \ldots$ so we have $\lambda_n = 1 + \mu_n^2$. From the graph we have that $\mu_n \to \infty$ hence $\lambda_n \to \infty$. The corresponding eigenfunctions are

$$y_n(x) = e^x \sin(\mu_n x), \quad n = 1, 2, \ldots$$

Check that $\langle y_n, y_m \rangle \propto \delta_{nm}$. For $n \neq m$

$$\begin{aligned}
\langle y_n, y_m \rangle_w &= \int_0^1 e^x \sin(\mu_n x) e^x \sin(\mu_m x) e^{-2x} \mathrm{d}x \\
&= \int_0^1 \sin(\mu_n x) \sin(\mu_m x) \mathrm{d}x \\
&= \frac{1}{2} \int_0^1 \left[\cos((\mu_n - \mu_m)x) - cos((\mu_n + \mu_m)x)\right] \mathrm{d}x \\
&\ \vdots \\
&= 0
\end{aligned}$$

(ommitting a large amount of the algebra.)

2.4 Legendre's Equation

Consider an eigenvalue problem defined as

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x}\right] = \lambda y \quad -1 < x < 1$$

So $p = 1 - x^2$, $q = 0$, $w = 1$. Since both endpoints are singular, work on $V = C^2[-1, 1]$. Since $x = 0$ is a regular point we can look for solutions in the form

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

By subbing in, we get that

$$a_{n+2} = \left[\frac{n(n+1) - \lambda}{(n+1)(n+2)}\right] a_n$$

Which gives two linearly independent solutions.

$$\begin{aligned}
y_0 &= a_0 \left[1 + \frac{(-\lambda)x^2}{2!} + \frac{(-\lambda)(6-\lambda)x^3}{4!} + \cdots\right] \\
y_1 &= a_1 \left[x + \frac{(2-\lambda)x^3}{3!} + \frac{(2-\lambda)(12-\lambda)x^5}{5!} + \cdots\right].
\end{aligned}$$

Note that y_0 collapses if $\lambda = 0, 6$. In general if $\lambda = k(k+1)$ for $k = 0, 1, 2, \ldots$ either y_0 or y_1 gives a polynomial. What if $\lambda \neq k(k+1)$? Since the ratio $\left|\frac{a_{n+2}}{a_n}\right| \to 1$ we know that both series will converge on $|x| < 1$. This doesn't tell us about $y(\pm 1)$. Let's look at y_0 only, y_1 is treated similiar. Let $A_n = a_{2n}$, so

$$\frac{A_n}{A_{n+1}} = \frac{(2n+1)(2n+2)}{2n(2n+1) - \lambda} = 1 + \frac{1}{n} + \varepsilon_n$$

where $|\varepsilon_n| \leq M/n^2$, $M = M(\lambda) > 0$. In particular the RHS true for n sufficiently large, say $n \geq N$. So $\{A_n\}$ have the same sign for $n \geq N$. Using $e^x > 1 + x$ for all $x \in \mathbb{R}$ we get that

$$\begin{aligned}
\frac{|A_n|}{|A_{n+1}|} &\leq e^{1/n} + |\varepsilon_n| \\
\implies |A_{n+1}| &\geq \frac{e^{-1/n}|A_n|}{1 + e^{-1/n}|\varepsilon_n|} \\
&\geq \frac{e^{-1/n}|A_n|}{1 + |\varepsilon_n|} \geq e^{-1/n}|A_n|e^{-|2n}
\end{aligned}$$

So for $n \geq N$ we can repeat to get that

$$|A_{n+1}| \geq |A_n| \exp\left[-\left(\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{N}\right) - (|\varepsilon_n| + \cdots + |\varepsilon_N|)\right]$$

hence we have that

$$|A_{n+1}| \geq |A_N|e^{-H_n + H_{N-1}}e^{-M\pi^2/6}.$$

Since $H_n \leq \log n + 2\gamma$

$$\begin{aligned}
|A_{n+1}| &\geq |A_N|e^{H_{N-1} - M\pi^2/6}e^{-\log n - 2\gamma} \\
&> \frac{c}{n+1}
\end{aligned}$$

Hence we have that

$$y_0 = \sum_{n \leq N} A_n x^{2n} + \sum_{n > N} A_n x^{2n}$$

and since $\{A_n\}$ have the same sign, assume all positive *wlog*. Note that

$$\begin{aligned}
\sum_{n>N} A_n x^{2n} &> c\sum_{n=1}^{\infty} \frac{x^{2n}}{n} - \sum_{n \leq N} \frac{x^{2n}}{n}. \\
&= C\left[\log\left(\frac{1}{1-x^2}\right) - \text{ (some polynomial in x)}\right] \to \infty \quad \text{as } x \to \pm 1.
\end{aligned}$$

So $y_0 \notin V$ so we must have that $\lambda_k = k(k+1)$. This gives an even polynomial of degree k from y_0. Make normalisation so $y(1) = 1$ choosing a_0 and a_1 accordingly then the solutions are called Legendre polynomials.

2.5 Bessel's Equation

Fix an integer $n \geq 0$. Consider the eigenvalue problem

$$-\frac{\mathrm{d}}{\mathrm{d}r}\left[r\frac{\mathrm{d}y}{\mathrm{d}x}\right] + \frac{m^2}{r}y = \lambda r y$$

with $0 < r < 1$ and $y(1) = 0$. We have $p = r, q = \frac{m^2}{r}, w = r$. Expanding out derivatives gives that

$$r^2 y'' + r y' + (\lambda r^2 - m^2)y = 0.$$

Set $z = \sqrt{\lambda}r$ (we can show that $\lambda > 0$). Set $R(z) = y(r)$. This gives that

$$z^2 R'' + zR' + (z^2 - m^2)R = 0 \quad 0 < z < \sqrt{\lambda}, R(\sqrt{\lambda}) = 0$$

. This is *Bessel's equation of order m*. Since $x = 0$ is a regular singular point, get can get solutions in the form

$$z \to z^\sigma \sum_{n=0}^{\infty} a_n z^n$$

by Fuch's theorem. We get two linearly independent solutions only one of which is non-singular as $z \to 0$. Label the cooresponding solution $R = J_m(z)$. We can show that

$$J_m(z) = \left(\frac{z}{2}\right)^m \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+m)!} \left(\frac{z}{2}\right)^{2n}.$$

These are *Bessel functions of the first kind* of order m. We can show that $J_n(z)$ has infinitely many zeros on the z axis, we label them j_{mk}. Since we require that $J_m(\sqrt{\lambda}) = 0$, solutions to the equation are

$$y_k(r) = J_m(j_{mk}r), \quad \lambda_k = j_{mk}^2$$

for $k = 1, 2, 3, \ldots$

2.6 Inhomogeneous Problems

Let L be a Sturm-Liouville operator. Consider problems of the form

$$\text{find } y \in V : Ly = f \in V.$$

wlog, $w = 1$. Let $\{y_k\}$ be normalised eigenfunctions of L. Be completeness we can write that

$$\begin{aligned} y = \sum A_k y_K, \quad f = \sum B_k y_k \\ \implies \sum_{k=1}^{\infty} (\lambda_k A_k - B_k) y_k = 0 \\ \implies \lambda_k A_k = B_k \quad k = 0, 1, 2, \ldots \end{aligned}$$

So if $\lambda_k \neq 0$, $A_k = B_k/\lambda_k$, we get have

$$y(x) = \sum_{k=1}^{\infty} \frac{B_k}{\lambda_k} y_k(x), \quad B_k = \int_a^b f(\xi) y_k(\xi) \mathrm{d}\xi$$

Putting the B_k into y and changing sums and integrals we get that

$$y(x) = \int_a^b G(x;\xi) f(\xi) \mathrm{d}\xi$$

where

$$G(x;\xi) = \sum_{k=1}^{\infty} \frac{y_k(\xi) y_k(x)}{\lambda_k}$$

is called the Greens function.