Markov Chains

Notes made by Finley Cooper

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Contents

1 Markov Chains

1.1 The Markov property

Throughout all our random variables and random processes will be assumed to be defined on an appropiate underlying probablity space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition. (Markov chain) A discrete-time Markov chain is a sequence $\underline{\overline{X}} = (X_n)_{n\geq 0}$ of random variables taking values in the same discrete countable state space I, such that:

$$\mathbb{P}\left(X_{n+1} = x_{n+1} | X_0 = x_0, \ldots, X_n = x_n\right) = \mathbb{P}\left(X_{n+1} = x_{n+1} | X_n = x_n\right) \quad \forall n \geq 0.$$

If $\mathbb{P}\left(X_{n+1} = y | X_n = x\right)$ is indepedent of n for all x, y then we call $\underline{\overline{X}}$ a time-homogeneous Markov chain. For this course all Markov chains are time-homogeneous with a countable state space.

Definition. (Transition matrix) We define the transition matrix P as the matrix

$$P(x, y) = P_{xy} = \mathbb{P}\left(X_{n+1} = y | X_n = x\right).$$

Note that P is a stochastic matrix i.e. $P_{xy} \geq 0$ for all x, y and the sum of each row is 1. For example take the simple Markov chain with $I = \{0, 1\}$ moving from 0 to 1 w.p. α and moving from 1 to 0 w.p. β, so

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

We say that $\underline{\overline{X}} = (X_n)$ is a Markov chain with transition matrix P with initial distribution λ if $\lambda = (\lambda_n)$ is a distribution and I is such that $\mathbb{P}\left(X_0 = x\right) = \lambda_{something}$, for all $x \in I$, P is the transition matrix of $\underline{\overline{X}}$ i.e.

$$\mathbb{P}\left(X_{n+1} = y | X_n = x, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\right) = P_{xy}$$

for all $i_0, \ldots, i_{n-1} \in I$. Then $\underline{\overline{X}} \sim \text{Markov}(\lambda, P)$

Theorem. $\underline{\overline{X}} = (X_n)$ is Markov(λ, P) on I iff

$$\mathbb{P}\big(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) = \lambda_{i_0} p_{i_0, i_1}, \ldots p_{i_{n-1}, i_n}\big)$$

Proof. Exercise.