Methods

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1 Fourier Series

1.1 Motivation

In 1807 J. Fourier was studying head conduction along a metal rod. This lead him to study 2π-periodic functions i.e. functions $f : \mathbb{R} \to \mathbb{R}$ was such that $f(\theta + 2\pi) = f(\theta)$ for all $\theta \in \mathbb{R}$ then he found that if

$$f(\theta) = \sum_{n \in \mathbb{Z}} \hat{f}_n e^{in\theta}$$

then you can write down the coefficients $\{\hat{f}_n\}$ via the formula

$$\hat{f}_n = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-in\theta} \mathrm{d}\theta, \quad n \in \mathbb{Z}.$$

And Fourier believed that this worked for any 2π-periodic function f. So computing each $\{\hat{f}_n\}$ and construcuted the sum as above, then it would return the original function. He was wrong.

1.2 Modern Treatment

Introduce a vector space V of L-periodic functions. Hence

$$V = \{f : \mathbb{R} \to \mathbb{C} : \text{ with } f \text{ a "nice" function, } f(\theta + L) = f(\theta), \forall \theta \in \mathbb{R}\}.$$

Note for $f \in V$ need only to consider values of f taken in an interval of length L, i.e. $[0, L)$ or $(-\frac{L}{2}, \frac{L}{2}]$ since periodicity covers elsewhere.

We can introduce an inner product on V with

$$\langle f, g \rangle = \int_0^1 f(\theta) \overline{g(\theta)} \mathrm{d}\theta.$$

This gives the associated norm,

$$||f|| = \sqrt{\langle f, f \rangle}.$$

For $n \in \mathbb{Z}$ consider $e_n \in V$ defined by $e_n(\theta) = e^{2\pi in\theta/L}$.

$$\langle e_n, e_m \rangle = \int_0^L e^{2\pi i(n-m)\theta/L} \, \mathrm{d}\theta = L \, \delta_{nm}.$$

So $\{e_n\}$ are orthogonal and $||e_n||^2 = L$ for each $n \in \mathbb{Z}$. This looks like IA Vectors and Matrices.

Recall that if v_N is N-dim vector space equipped with usual inner product and $\{e_n\}_{n=1}^N$ are orthogonal with $|e_n| = L$, then for each $x \in V$ we can write $x = \sum_{n=1}^N \hat{x}_n e_n$ for some $\{\hat{x}_n\}$. To find $\{\hat{x}_n\}$ take the inner product of both sides with e_m. So

$$(x, e_m) = \sum_{n=1}^N \hat{x}_n (e_n \cdot e_m) = L\hat{x}_m$$

i.e

$$\hat{x}_n = \frac{1}{L}(x \cdot e_n).$$

Now could this work on V? V is not finite dimensional so it's not obvious. Every subset of $\{e_n\}$ is linearly indepedent. Ignoring this for now we assume that for all $f \in V$ we can write f in our basis $\{e_n\}$. Then

$$f(\theta) = \sum_n \hat{f}_n e_n(\theta),$$

So taking the inner product as before

$$\langle f, e_m \rangle = \sum_n \hat{f}_n \langle e_n, e_m \rangle$$

so using the delta as before

$$= L\hat{f}_m$$

i.e.

$$\hat{f}_n = \frac{1}{L}\langle f, e_n \rangle = \frac{1}{L}\int_0^1 f(\theta) e^{-2\pi i n\theta/L} \mathrm{d}\theta$$

Definition. (Complex Fourier series) For an L-periodic $f : \mathbb{R} \to \mathbb{C}$ define its *complex Fourier series* by

$$\sum_n \hat{f}_n e^{2\pi i n\theta/L}$$

where

$$\hat{f}_n = \frac{1}{L}\int_0^1 f(\theta) e^{-2\pi i n\theta/L} \mathrm{d}\theta$$

are called the complex Fourier coefficients. We will write for $f \in V$

$$f(\theta) \sim \sum_n \hat{f}_n e^{2\pi i n\theta/L}$$

to mean the series on the right corresponds to complex Fourier series for the function on the left.

We'd like to replace the $\sim$ symbol with equality, but we require a bit more than that.

If we split the complex Fourier series into the parts $\{n = 0\} \cup \{n > 0\} \cup \{n < 0\}$ we get

$$\sum_n \hat{f}_n e^{2\pi i n\theta/L} = \hat{f}_0 + \sum_{n=1}^{\infty} \hat{f}_n \left[\cos\left(\frac{2\pi n\theta}{L}\right) + i\sin\left(\frac{2\pi n\theta}{L}\right)\right] + \sum_{n=1}^{\infty} \hat{f}_{-n} \left[\cos\left(\frac{2\pi n\theta}{L}\right) - i\,\mathrm{si}\right.$$

Definition. (Fourier series) For $f : \mathbb{R} \to \mathbb{C}$ an L-periodic function define its *Fourier series* by

$$\frac{1}{L}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n\theta}{L}\right) + b_n \sin\left(\frac{2\pi n\theta}{L}\right)\right]$$

where

$$a_n = \frac{2}{L}\int_0^L f(\theta)\cos\left(\frac{2\pi n\theta}{L}\right)\mathrm{d}\theta$$

and

$$b_n = \frac{2}{L} \int_0^L f(\theta) \sin\left(\frac{2\pi n\theta}{L}\right) \mathrm{d}\theta$$

are called the Fourier cofficients for f.

If we set

$$c_n(\theta) = \cos\left(\frac{2\pi n\theta}{L}\right),$$
$$s_n(\theta) = \sin\left(\frac{2\pi n\theta}{L}\right),$$

then we can show, for $m, n \geq 1$ that $\langle c_n, c_m \rangle = \langle s_n, s_m \rangle = \frac{L}{2}\delta_{mn}$ and

$$\langle c_n, 1 \rangle = \langle s_m, 1 \rangle = \langle c_n, s_m \rangle = 0.$$

So we have that $\{1, c_n, c_n\}$ is orthogonal set in V.

For an example take $f : \mathbb{R} \to \mathbb{R}$, 1-periodic, such that $f(\theta) = \theta(1-\theta)$ on $[0, 1)$. For $n \neq 0$ we have

$$\hat{f}_n = \int_0^1 \theta(1-\theta)e^{-2\pi i n\theta} \, \mathrm{d}\theta.$$

Integrating by parts (or using a standard Fourier integral computation) yields

$$\hat{f}_n = -\frac{1}{2(\pi n)^2}, \qquad n \neq 0,$$

and

$$\hat{f}_0 = \int_0^1 (\theta - \theta^2) \, \mathrm{d}\theta = \frac{1}{6}.$$

Hence

$$f(\theta) \sim \frac{1}{6} - \sum_{n \neq 0} \frac{e^{2\pi i n\theta}}{2(\pi n)^2}.$$

so the sine terms cancel in the sum giving just cosine terms as we expect since our f function is even.