Topological Spaces

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1 Topologies

1.1 Definitions

We denote $\mathcal{P}(X)$ as the power set of X.

Definition. (Topology) Let X be a set. A *topology* on X is a collection of sets $T \subseteq \mathcal{P}(X)$ such that
  (i) $\emptyset, X \in T$,
  (ii) T is closed under (possibly uncountable) unions.
  (iii) T is closed under finite intersections.

A set X with a topology T is called a *topological space* of X. An element of X is called a *point* and elements of T are called *open sets*. If $x \in U \in T$ we say U is an open neighbourhood of x. Strictly we should always denote (X, T) for a topological space, but when T is clear, we just write X for the topological space.

Definition. (Continuity) If (X, T_X) and (Y, T_Y) are topological spaces then a function $f : X \to Y$ is called *continuous* if for $U \in T_Y$, $f^{-1}(U) \in T_X$.

Definition. (Homeomorphism) A function $f : (X, T_X) \to (Y, T_Y)$ is a *homeomorphism* if it is continuous and has a continuous inverse.

Definition. If $T \subseteq T'$ are topologies on X then we say that T is *coarser* and T' is *finer*. The identity function $d : (X, T) \to (X, T')$ is continuous.

1.2 Topologies from metrics

If (X, d) is a metric space, recall that a subset $U \subseteq X$ is called *open* if for every point $x \in U$ there exists a $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$.

Proposition. If T_d is the subset of X which are open under the metric d, then (X, T_d) is a topological space. We will call this the topology on X induced by the metric d.

Proof. Tautologically we have that $\emptyset \in T_d$. Clearly we have that $X \in T_d$ too. Let $\{U_\alpha\}_{\alpha \in I}$ be a collection of open sets in T_d with a (possibly uncountable) index set I. Let

$$x \in \bigcup_{\alpha \in I} U_\alpha.$$

Then $x \in U_\beta$ for some $\beta \in I$, so U_β is open hence there exists a $\varepsilon > 0$ such that $B_\varepsilon \subseteq U_\beta \subseteq \bigcup_{\alpha \in I} U_\alpha$, hence $\bigcup_{\alpha \in I} U_\alpha$ is open.

Now suppoes that I is finite, and $x \in \bigcap_{\alpha \in I} U_\alpha$. For each α there exists a $\varepsilon_\alpha > 0$ such that $B_{\varepsilon_\alpha}(x) \subseteq U_\alpha$. Take $\varepsilon = \inf_{\alpha \in I} \varepsilon_\alpha$, so $B_\varepsilon(x) \subseteq B_{\varepsilon_\alpha}(x) \subseteq U_\alpha$ for all α, hence we have that $B_\varepsilon(x) \subseteq \bigcap_{\alpha \in I} U_\alpha$ so it's open. Hence T is a topology. $\square$

Now we have lots of examples we can use for topological spaces. For example we have that topology induced by the Euclidean metric on $\mathbb{R}^d$ which we will call the Euclidean topology. For any $X \subseteq \mathbb{R}^d$ we can have a topology induced by the Euclidean metric too, like $\mathbb{Q}$, $[0,1], (0,1)$.

Proposition. If we have two metric spaces $(X, d_X), (Y, d_Y)$ and we have $f : X \to Y$, the f is continuous in the metric space sense if and only if it is continuous in the topological space sense (with the topologies induced by the metric d_X and d_Y respectively).

Proof. Let $f : X \to Y$ be continuous in the metric space sense. Let U be an open set in T_{d_Y} so we need to show that $f^{-1}(U)$ is open. Let $x \in f^{-1}(U)$, so $f(x) \in U$. Hence there exists an $\varepsilon > 0$ such that $B_\varepsilon(f(x)) \subseteq U$. So since f is continuous there exists a $\delta > 0$ such that if $d_X(x, x') < \delta$, then $d_Y(f(x), f(x')) < \varepsilon$. Hence $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$. So $B_\delta(x) \in f^{-1}(U)$, hence $f^{-1}(U)$ is open.

Now let's do the converse and suppose that $f : X \to Y$ is continuous in the topological sense. Fix some $x \in X$ and $\varepsilon > 0$. Consider $B_\varepsilon(f(x))$ which is open in Y. Then $f^{-1}(B_\varepsilon(f(x)))$ is in T_{d_X}. It contains x so there exists a $\delta > 0$ such that $x \in B_\delta(x)$, so

$$d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \varepsilon.$$

So f is continuous in the metric sense. $\square$

Definition. Let (X, T) be a topological space and $x_1, x_2, \dots \in X$ say. We say that x_n convergences to x if for every open neighbourhood U of x there exists a N such that $x_n \in U$ for all $n \geq N$.

Proposition. If (X, d) is a metric space with topology T_d then a sequence (x_n) converges in the metric sense if and only if it converges in the topological sense.

Proof. Suppose it convergens in the metric sense to x. Then for all $\varepsilon > 0$ there exists a N such that for all $n \geq N$ we have that $x_n \in B_\varepsilon(x)$. If U is a neighourhood of x then there there is some ε such that the ball of radius ε centred at x is contained in U. Conversely if (x_n) converges in the topological sense to x, let $\varepsilon > 0$ and consider the open ball centred at x with radius ε. Now $B_\varepsilon(x)$ is an open neighbourhood of x so there exists an integer N such that $x_n \in B_\varepsilon(x)$ for all $n > N$. Hence (x_n) converges to x in the metric sense. $\square$

Consider $\mathbb{R}$ and $(0,1)$ with the Euclidean metric and topology. Then the two spaces are related, by the function $(0,1) \to \mathbb{R}$ by $\tan^{-1} x$ which is invertible. Hence we say the two spaces are homeomorphic, and $\mathbb{R} \cong (0,1)$. However the two spaces are not isometric since $\mathbb{R}$ is not complete under the Euclidean metric and $(0,1)$ is not. Hence the property of completeness is not a topological property: it is a property induced by the metric.

Definition. (Discrete topology) Let X be a set. The *discrete* topology is the topology $T_{\text{discrete}} = \mathcal{P}(X)$ (so every set is open).

Remark. Any function from (X, T_{discrete}) to any space si continuous. This toplogy can be induced by the discrete metric, where $d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$. So $B_{\frac{1}{2}}(x) = \{x\}$ so $\{x\}$ is open, hence all the sets are open.

Definition. (Indiscrete topology) Let X be a set. The *indiscrete* topology $T_{\text{indiscrete}} = \{\emptyset, X\}$ (as little as possible sets are open).

Remark. A function from any space to $(X, T_{\text{indiscrete}})$ is continuous. This topology does not come from a metric unless X is a singleton set. This is because if $x \neq y$ then $d(x, y) = \varepsilon > 0$, so $y \notin B_\varepsilon(x)$ and since y is arbitrary, then $B_\varepsilon(x) = \{x\} = X$.

Let $X = \{o, c\}$. Then let $T = \{\emptyset, \{o, c\}, \{o\}\}$ be a topology of X. This is called the Sierpinski space. It as the property that every sequence converges to c. A continuous function $f : T \to (X, T_{\text{Sierpinski}})$ is exactly an open subset of Y.

Let $X = \mathbb{R}$ we'll define the right order topology on X as

$$T_{\text{ord}} = \{(a, \infty) \mid -\infty \leq a \leq \infty\}.$$

Let $\{(a, \infty)\}_{a \in I}$ be a collect of elements of T_{ord}. Then

$$\bigcup_{a \in I} (a, \infty) = (\inf_{a \in I} a, \infty) \in T_{\text{ord}}.$$

Similarly for finite I,

$$\bigcap_{a \in I} (a, \infty) = (\max_{a \in I} a, \infty) \in T_{\text{ord}}$$

1.3 Bases and subbases

Definition. (Basis) Let T be a topology of X. A *basis*, $B \subseteq T$ for T is a subcollection such that every element of T is a union of elements in B.

Definition. (Subbasis) Let T be a topology of X. A *subbasis*, $S \subseteq T$ for T is a subcollection such that every element of T is a union of sets which are finite intersections of elements of S.

Lemma. Let $f : (X, T_X) \to (Y, T_Y)$ and $S \subseteq T_Y$ is a subbasis. If $f^{-1}(U)$ is open for all $U \in S$ then f is continuous.

Proof. If $V \subseteq T_Y$, then $V = \bigcup_{a \in I} V_a$ where $V_a \in \bigcap_{b \in J_a} U_{a,b}$ with $U_{a,b} \in S$ and J_a finite. Then

$$f^{-1}(V) = \bigcup_{a \in I} V_a = \bigcup_{a \in I} \left(\bigcap_{b \in J_a} f^{-1}(U_{a,b}) \right) \in T_X,$$

by the axioms of the topology. $\square$

Consider the Euclidean topology on $\mathbb{R}^n$. The collection $B = \{B_r(x) \mid x \in \mathbb{R}^n, r > 0\}$ is a basis. Likewise the collection of n-cubes everywhere are also a basis. Interestingly the set $QB \subseteq B$ with balls at rational points with rational radii is also a basis. This is interesting since QB is countable while B is uncountable and $\mathcal{P}(\mathbb{R}^n)$ is $\aleph_2$.

Definition. (Closed set) Let (X, T) be a topological space. A subset $C \subseteq X$ is *closed* if $X \setminus C \in T$.

Proposition. Let (X, T) be a topological space and $\mathcal{F} = \{C \subseteq X \mid C \text{ closed}\}$. Then
(i) $\emptyset, X \in \mathcal{F}$;
(ii) $\mathcal{F}$ is closed under (possibly uncountable) intersections;
(iii) $\mathcal{F}$ is closed under finite unions.

Proposition. A function $f : X \to Y$ between topological spaces is continuous if and only if the preimage of every closed set is closed.

Definition. Let (X, T) be a topological space. Let $A \subseteq X$ be a subset of X. Then
(i) The closure $\bar{A}$ is the smallest (by inclusion) closed set containing A so

$$\bar{A} = \bigcap_{S \text{ closed}, A \subseteq S} S.$$

(ii) We say that A is dense in X if $A = \bar{A}$.
(iii) The interior $\mathring{A}$ is the largest open set contained in A so

$$\mathring{A} = \bigcup_{S \text{ open}, S \subseteq A} S.$$

Definition. (Limit point) Let X be a topological space and $A \subseteq X$. A *limit point* of A is a point in X which is a limit of a sequence in A.

Proposition. If C is a closed subset of (X, T), then the limit points of C lie in C.

Proof. Let $\{x_n\}$ be a sequence in C with limit x_∞. If $x_\infty \notin C$, then $x_\infty \in X \setminus C$ which is open. Then if $x_n \to x_\infty$ then we should have that $x_n \in X \setminus C$ for $n \geq N$ but $x_n \in C$ so $x_n \notin X \setminus C$ which is a contradiction. $\square$

Corollary. A limit point of a A lies in $\bar{A}$.

For an example $\overline{\mathbb{Q}} = \mathbb{R}$ since any real number is a limit of a sequence of rational numbers. We have that $\overline{(0,1)} = [0,1]$ too. The cocountable topology on $\mathbb{R}$ is the topology $T_{\text{countable}} = \{\emptyset\} \cup \{\mathbb{R} \setminus C \mid C \text{ countable}\}$ Let $\{x_n\}$ be a sequence in $\mathbb{R}$, for $x \in \mathbb{R}$ consider $\{x\} \cup \{\mathbb{R} - \{x_n\}\}$ is open and contains x. If $x_n \to x$, then x_n must be in a U for all $n \geq \mathbb{N}$ so $x_n = x$ for all $n \geq N$. Hence the convergent sequences are exactly the eventually constant sequences with the limits being the value they are eventually constant to. So the limit points of a set A are A under this topology. However almost all A is not closed. For example $(0, 1)$ is not closed since $\mathbb{R} \setminus (0, 1)$ is not countable. But the closure of $(0, 1)$ must be closed, so it must be $\mathbb{R}$ hence the sense of limit points and closure are actually two very different properties in topology instead of metric spaces.

1.4 Hausdorff spaces

Definition. (Hausdorff) A space (X, T) is *Hausdorff* if for $x \neq y \in X$ there are open neighbourhoods $x \in U$, $y \in V$ with $U \cap V = \emptyset$.

Remark. This is the notion that points are seperated by open sets.

Lemma. If the topology T is induced by a metric then it is Hausdorff.

Proof. If $x \neq y$ then $d(x, y) = s > 0$. So consider $U = B_{s/2}(x)$ and $V = B_{s/2}(y)$. The triangle inequality shows that $U \cap V = \emptyset$ and we know all balls are open. $\square$

Proposition. If a space is Hausdorff then a sequence in X has at most 1 limit.

Proof. Let (x_n) be a sequence in X. Suppose it has limits $y \neq z \in X$. Let U and V be disjoint local neighbourhoods sfor y and z respectively. Then $x_n \in U$ for all $n \geq N_1$ and $x_n \in V$ for all $n \geq N_2$. So if we take that $N = \max\{N_1, N_2\}$ then for all $n \geq N$, we have that $x_n \in U \cap V$ which is empty, hence we have a contradiction. $\square$

Proposition. If (X, T) is Hausorff then points are closed.

Proof. Let $x \in X$. We want to show that $\{x\} = \overline{\{x\}}$. Let $y \neq x$. Let U, V be disjoint neighbourhoods of x and y respectively. We know that $x \in X \setminus V$ which is closed. Hence $\overline{\{x\}} \subseteq X \setminus V$. But $y \notin V$, so y is not in the closure of $\{x\}$ hence the closure of $\{x\}$ is just $\{x\}$, so $\{x\}$ is closed. $\square$

Let's see an example. Let X be an infinite set and consider the cofinite topology on X. Take two non-empty open sets, so

$$(X \setminus F) \cap (X \setminus F') = X \setminus (F \cup F')$$

which is non-empty since $F \cup F'$ is finite and X is infinite so the set on the RHS is non-empty hence the space is not Hausdorff.

1.5 Defining new topologies on existing ones

We have three main ways to define new topologies when given a topology already.

1.5.1 The subspace topology

Definition. (Subset topology) Let (X, T_X) be a topological space. Let $Y \subseteq X$ a subset. The *subset topology* on Y is

$$T\mid_Y = \{Y \cap U \mid U \in T\}.$$

Definition. (Subspace) A subspace of (X, T) is a subset equipped with the subspace topology.

Proposition. The subset topology is a topology.

Proof. Simple exercise of the axioms. □

Proposition. The inclusion map $\iota : (Y, T\mid_Y) \to (X, T)$ is continuous. In fact $T\mid_Y$ is the constant topology on Y such that the inclusion map is continuous.

Proof. Let $U \in T$ then $\iota^{-1}(U) = U \cap Y \in T\mid_Y$ by definition. So it is continuous. Suppose $\iota : (Y, T') \to (X, T)$ is continuous. For $U \in T$, $\iota^{-1}(U) \in T'$ so $T\mid_Y \subseteq T'$. □

A further point of view, a function $f : (z, T_z) \to (Y, T\mid_Y)$ is continuous if and only if $\iota \circ f$ is continuous.

Lemma. (Gluing Lemma) Let $f : X \to Y$ be a function between topological spaces.

(i) If $\{U_\alpha\}_{\alpha \in I}$ are open subsets which cover X and each $f\mid_{U_\alpha} : U_\alpha \to Y$ are continuous (where U_α is given the subspace topology) then f is continuous.

(ii) If $\{C_\alpha\}_{\alpha \in I}$ is a finite collection of closed sets containing X and $f\mid_{C_\alpha} : C_\alpha \to Y$ is continuous for each $a \in I$ then f is continuous.

Proof. Let $V \subseteq Y$ be open. We want to show that $f^{-1}(V)$ is open. We know that

$$\begin{aligned} f^{-1}(V) = (^{-1}V) \cap X &= f^{-1}(V) \cap \left(\bigcup_{\alpha \in I} U_\alpha \right) \\ &= \bigcup_{\alpha \in I} f^{-1}(V) \cap U_\alpha \end{aligned}$$

Since $f\mid_{U_\alpha}$ are continuous, we have that $f^{-1}\mid_{U_\alpha}$ is open in U_α in the subspace topology. So there exists a W open in X such that $f^{-1}\mid_{U_\alpha}(V) = U_\alpha \cap W$ hence this is the intersection on open subsets of X so is open in X, hence since the union of open subsets is open $f^{-1}(V)$ is open, so f continuous.

The second part can be proved the same using the closed set definition of continuity. □

If (X, d) is a metric space with topology T_d and $Y \subseteq X$ then $T_d\mid_Y$ is the topology induced by $d\mid_Y$.

1.5.2 The quotient topology

Definition. (Quotient topology) Let (X, T_X) be a topological space, $\sim$ an equivalence relation on X and $X/\sim$ is the set of equivalence classes, and $\pi : X \to X/\sim$ the equivalence map. The *quotient toplogy* on $X/\sim$ is

$$T_{X/\sim} = \{U \subset X/\sim \mid \pi^{-1}(U) \in T_X\}.$$

Proposition. $T_{X/\sim}$ is indeed a topology.

Proof. $\emptyset = \pi^{-1}(\emptyset) \in T_X$ so $\emptyset \in T_{X/\sim}$. $X = \pi^{-1}(X/\sim) \in T_X$ so $X/\sim \in T_{X/\sim}$. Let $\{U_\alpha\}$ be a collection of sets of $T_{X/\sim}$, then

$$\pi^{-1}\left(\bigcup_{\alpha\in I} U_\alpha\right) = \bigcup_{\alpha\in I} \pi^{-1}(U_\alpha),$$

and $\pi^{-1}(U_\alpha) \in T_X$, so the union is too. Hence $\bigcup_{\alpha\in I} U_\alpha \in T_{X/\sim}$. We have a similar proof for finite intersections. □

Proposition. The quotient map $\pi : (X, T_X) \to (X/\sim, T_{X/\sim})$ is continuous and $T_{X/\sim}$ is the finest topology for which this is true.

Proof. This is a tautology. □

An alternative characterisation of the quotient topology is that $f : X/\sim \to Y$ is continuous if and only if $f \circ \pi : X \to Y$ is continuous.

Definition. For a continuous function $g : (X, T_X) \to (Y, T_Y)$ is a *quotient map* if it surjective and $U \in T_Y \iff g^{-1}(Y) \in T_X$.
Given, this construct $\sim$ on X by $x \sim x' \iff g(x) = g(x')$. There is an induced function $G : X/\sim \to Y$ sending $G([x]) = g(x)$.

Remark. This function G is a bijection and continuous with a continuous inverse. This means that G is a homeomorphism, so $X/\sim \cong Y$.

Let's see an example on $\mathbb{R}$. Consider $x \sim y \iff x - y \in \mathbb{Z}$. What is $\mathbb{R}/\sim$? Consider $f : \mathbb{R} \to \mathbb{R}^2$ defined by $x \to (\sin(2\pi x), \cos(2\pi x))$. This is a continuous map so $f : \mathbb{R} \to S^1 \subseteq \mathbb{R}^2$ is also continuous and surjective. By periodicity $x \sim y \iff f(x) = f(y)$, so we get $F : \mathbb{R}/\sim \to S^1$ which we can check is a homeomorphism.

Now take the example $X = \mathbb{R} \times \{0, 1\} \subseteq \mathbb{R}^2$ with the standard subspace topology. Let $(x, i) \sim (y, j) \iff (x, i) = (y, j)$ or $x = y \neq 0$. We can then think of $X/\sim$ is a line with two origins. We cannot draw $X/\sim$ since it is not Hausdorff. Any neighbourhood of $[(0, 0)]_\sim$ intersects any neighbourhood of $[(1, 0)]_\sim$ so not Hausdorff. Hence it is not subspace of any Euclidean space.

1.5.3 The product topology

For sets X, Y the projections functions are

$$\begin{aligned} \pi_X : & X \times Y \to X \\ & (x, y) \to x \end{aligned}$$

and

$$\begin{aligned} \pi_Y : & X \times Y \to Y \\ & (x, y) \to y \end{aligned}$$

Definition. (Product topology) Let (X, T_X) and (Y, T_Y) be topological spaces. Then *product topology* on $X \times Y$ consists of open sets $U \subseteq X \times Y$ such that for $(x, y) \in U$ there is a $V \in T_X$ and $W \in T_Y$ such that $(x,y) \in V \times W \in U$.

Proposition. This indeed is a topology and the sets $V \times W$ are a basis for $T_{X\times Y}$.

Proof. Tautologically, we have that $\emptyset \in T_{X\times Y}$. Taking $V = X, W = Y$ we have that $X \times Y \in T_{X\times Y}$. For a collection $\{U_\alpha\}_{\alpha\in I}$ of elements of $T_{X\times Y}$, let $(x,y) \in \bigcup_{\alpha\in I} U_\alpha$. Then $(x,y) \in U_\beta$ for $\beta \in I$ so there exists neighbourhoods of x, y with their product a subset of $U_\beta \subseteq \bigcup_{\alpha\in I} U_\alpha \in T_{X\times Y}$. If I is finite and $(x,y) \in \bigcap_{\alpha\in I} U_\alpha$. Then $(x,y) \in V_\alpha \times W_\alpha \subseteq U_\alpha$ for each $\alpha \in I$. So $(x,y) \in (\bigcap_\alpha V_\alpha) \times (\bigcap_\alpha W_\alpha) \in \bigcap_\alpha U_\alpha$ and since these intersections are finite, these intersections are open. □

Proposition. The projection maps

$$\pi_X : (X \times Y, T_{X\times Y}) \to (X, T_X) \qquad \pi_Y : (X \times Y, T_{X\times Y}) \to (Y, T_Y)$$

are continuous and $T_{X\times Y}$ is the coarsest topology for which this is true.

Proof. Let $V \in T_X$. Then $\pi_X^{-1}(V) = V \times Y$, so this is open. Hence π_X, π_Y are continuous.

Suppose that T' is a topology on $X \times Y$ such that π_X and π_Y are continuous, then $\pi_X^{-1}(V) = V \times Y$ is open and $\pi_Y^{-1}(W) = X \times W$ is open. So $V \times W$ is open in T', so $T_{X\times Y} \subseteq T'$. □

The universal property of the product topology is that the function

$$f : (Z, T_Z) \to (X \times Y, T_{X\times Y})$$

is continuous if and only if $\pi_X \circ f : (Z, T_Z) \to (X, T_X)$ and $\pi_Y \circ f : (Z, T_Z) \to (Y, T_Y)$ are continuous. Equivalently f is componentwise continuous if and only if it is componentwise continuous.

2 Connectivity

2.1 Connected and disconnected

We know from IA Analysis I, if $f : [0,1] \to \mathbb{R}$ is continuous, $f(0) < 0 < f(1)$ then $f(t) = 0$ for some $t \in [0,1]$. This is a statement about continuous functions, but also about the interval $[0,1]$. For example if we change the interval to $[0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ then this does not satisfy the intermediate value theorem. The property of the interval we're using is connectedness.

Definition. (Disconnected) A topological space X is *disconnected* if $X = U \cup V$ for U, V disjoint nonempty open sets.

Definition. (Connected) A topological space is *connected* if it is not disconnected.

If $X = U \cup V$ is disconnected, then U and V are both open and also both closed.

Any set with the coarse topology is connected, due to the lack of non-trivial open sets. A set with the discrete topology is disconnected, if it has more than 1 point, since every set is open, so the result is trivial.

The set $X = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1] \subseteq \mathbb{R}$ is disconnected since $[0, \frac{1}{2})$ is open in X and $(\frac{1}{2}, 1]$ is open in X too. They are disjoint, hence X is disconnected.

Proposition. A space X is disconnected if and only if, there is a continuous surjection $f : X \to \{0, 1\}$ where $\{0, 1\}$ is equipped with the discrete topology.

Proof. Suppose that X is disconnected. So $X = U \cup V$ disjoint. The define f such that

$$f(x) = \begin{cases} 0 & x \in U \\ 1 & x \in V \end{cases}.$$

This is well-defined since U and V are disjoint. Since U and V are non-empty, the function is surjective. The preimage of $\{0\}$ and $\{1\}$ are U and V respectively which we know is open. And the preimage of $\{0, 1\}$ and $\emptyset$ are clearly open, so f is continuous.

Conversely suppose that f is continuous. Then define $U = f^{-1}(\{0\})$ and $V = \int f(\{1\})$. So since f is continuous, U and V are open. Clearly U and V is disjoint and non-empty since f is surjective. We have that $X = U \cup V$ since $X = f^{-1}(\{0, 1\}) = f^{-1}(0) \cup f^{-1}(1) = U \cup V$. $\square$

Theorem. The spaces $[0, 1]$, $[0, 1)$, $(0, 1)$ are all connected.

Proof. Let's just consider $[0, 1]$, the rest of the proves are similar. If it was disconnected, then there is a continuous surjection

$$f : [0, 1] \to \{0, 1\} \subseteq \mathbb{R}.$$

Then

$$f(\cdot) - \frac{1}{2} : [0, 1] \to \mathbb{R}$$

is continuous and takes the values $\pm\frac{1}{2}$ only. By the intermediate value theorem, we should have that f takes the value 0 which is a contradiction hence $[0, 1]$ is connected. $\square$

Theorem. (Generalised intermediate value theorem) Let X be a connected topological space and $f : X \to \mathbb{R}$ continuous. If there exists $x_0, x_1 \in X$ such that $f(x_0) < 0 < f(x_1)$ then there exists a $x_2 \in X$ such that $f(x_2) = 0$.

Proof. Consider the open sets $U = f^{-1}((-\infty, 0)), V = f^{-1}((0, \infty))$. f is continuous, so U, V are open. We know that x_0, x_1 exist hence U, V are non-empty. If $f(x)$ is never zero, then $X = U \cup V$ disjoint and open so X is disconnected. But X is connected hence $f^{-1}(0)$ is non-empty, so pick $x_2 \in f^{-1}(0)$, so $f(x_2) = 0$. $\square$

Proposition. Let $f : X \to Y$ be a continuous surjection. Then X connected implies that Y is connected.

Proof. Let's show the contrapositive. Suppose that Y is disconnected. Then we have some $h : Y \to \{0, 1\}$ continuous and surjective. So

$$h \circ f : X \to \{0, 1\}$$

is also continuous and surjective, hence X is disconnceted. $\square$

Corollary. If X is connected and $f : X \to Y$ is continuous then $\mathrm{im}(f)$ is connected.

Proof. Apply the proposition to $f : X \to \operatorname{im} f$.

For example if X is a connected space and $\sim$ is an equivalence relation then $\pi : X \to X/\sim$ is a continuous surjection so $X/\sim$ is connected.

Lemma. If $f : X \to Y$ is a homeomorphism and $Z \subseteq X$, then $f\mid_Z : Z \to \mathrm{im}(f\mid_Z)$ is a homeomorphism.

Proof. Obvious. $\square$

Let's use this to show that $[0, 1]$ is not homeomorphic to $(0, 1)$. Suppose they are. So we have a homeomorphism $f : [0, 1] \to (0, 1)$. Let's now restrict f to $(0, 1]$. Then by the lemma we know that $f\mid_{(0,1]}$ is a homeomorphism with

$$f\mid_{(0,1]} : (0, 1] \to (0, 1) \setminus \{f(0)\}$$

for some $0 < f(0) < 1$. But $(0, 1]$ is connected and $(0, 1) \setminus \{f(0)\} = (0, f(0)) \cup (f(0), 1)$ so $(0, 1) \setminus \{f(0)\}$ is disconnected which is a contradiction.

We can do a similar process to show that S^1 is not homeomorphic to $\mathbb{R}$. We know that S^1 is connected since it is a quotient space of $\mathbb{R}$ and $\mathbb{R}$ is connected since $\mathbb{R} \cong (0, 1)$. Suppose that S^1 is homeomorphic to $\mathbb{R}$. Then remove the point $(1, 0) \in S^1$ and consider the restricted homeomorphism between the new spaces. $\mathbb{R}$ is no longer connected since $\mathbb{R} \setminus \{f(1, 0)\} = (-\infty, f(1, 0)) \cup (f(1, 0), \infty)$, but $S^1 \setminus \{f(1, 0)\}$ is connected since it's homeomorphic to $(0, 1)$.

Proposition. Let $\{X_\alpha\}_{\alpha\in I}$ be a collection of subspaces of X. Suppose that each X_α is connected and $\bigcap_{\alpha\in I} X_\alpha \neq \emptyset$. Then $\bigcup_{\alpha\in I} X_\alpha$ is connected.

Proof. Let X be the union of all the sets. If it were disconnected then $X = U \cup V$ with U, V open, so $U \cap X_\alpha, V \cap X_\alpha$ are disjoint open subsets covering X_α. Since X_α is connected one of them must be zero so $X_\alpha \subseteq U$ or $X_\alpha \subseteq V$. This holds for each X_α but as $\bigcap_{\alpha\in I} X_\alpha \neq \emptyset$ they are all in U or all in V hence X is U or X is V so one of them is empty. $\square$

Corollary. If X and Y are connected then so is $X \times Y$.

Proof. Suppose X, Y both non-empty (since the empty set is connected). Choose $x \in X$. Consider

$$C_y = \{x\} \times Y \cup X \times \{y\}.$$

The sets $\{x\} \times Y$ and $X \times \{y\}$ intersect in (x, y) and pieces are connected by assumption so C_y is connected. Now observe that

$$X \times Y = \bigcup_{y\in Y} C_y$$

the intersection of all of C_y is $\{x\} \times Y$ which is non-empty hence the proposition applies and $X \times Y$ is connected.

2.2 Path-connectedness

Definition. (Path) If X is a topological space and $x_0, x_1 \in X$ a *path* between them is a continuous $\gamma : [0,1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

Definition. (Path connected) A topological space is *path connected* if for any two points in the space, there is a path between them.

The spaces $(0,1), [0,1], (0,1]$ are all path-connected just by taking the line $\gamma(t) = (1-t)x_0 + tx_1$. For the same reason $\mathbb{R}^n$ is path-connected and any *convex* subset $X \subseteq \mathbb{R}^n$ is too.

Now let's let $X = \mathbb{R}^2 \setminus \{0\}$ be the punctured plane. We do the same again if our linear path doesn't go through the (missing) origin. If it does, just take a circular path instead.

Proposition. A path-connected space is connected.

We'll prove the contrapositive. If X is not connected, we have a continuous surjective function $f : X \to \{0,1\}$ Let x_0, x_1 be such that $f(x_0) = 0, f(x_1) = 1$ and suppose that X is path connected so there is a path connecting x_0, x_1. So $f \circ \gamma$ is a surjective and continuous map from $[0,1] \to \{0,1\}$ hence $[0,1]$ is disconnected which is a contradiciton. $\square$

We can now show that $\mathbb{R}^n \ncong \mathbb{R}$ for $n > 1$. If it were then $\mathbb{R}^n \setminus \{0\} \cong \mathbb{R} \setminus \{0\}$. But the RHS is disconnected and the LHS is path-connected, contradiction.

Proposition. If X and Y are path-connected then $X \times Y$ is path-connected.

Proof. Omitted.

2.2.1 Path components

Let X be a space. Define an equivalence relation on X, $\sim$, defined by

$$x \sim y \iff \text{there exists a path from } x \text{ to } y.$$

Lemma. $\sim$ is indeed an equivalence relation.

Proof.

(i) $x \sim x$ since we can take the path $\gamma(t) = x$.

(ii) If $x \sim y$ then we have a path $\gamma(t)$ connecting x and y. Then we can take the path $\gamma'(t) = \gamma(1-t)$ which goes from y to x. Hence $y \sim x$.

(iii) If $x \sim y$ and $y \sim z$, there there is a path γ' connecting x to y and γ' connecting y to z. Define

$$(\gamma' \cdot \gamma)(t) = \begin{cases} \gamma(2t) & 0 \le t \le \frac{1}{2} \\ \gamma'(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}.$$

This is a well defined function since $\gamma(1) = \gamma'(0)$. It is continuous by the gluing lemma applied to $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ so it is a path between x and z, so $x \sim z$.

Hence $\sim$ is an equivalence relation. $\square$

Definition. (Path components) The *path components* of X are the equivalence classes of $\sim$.

Claim. Each component is path-connected.

Proof. Suppose that x, y are in the same path component, so there exists a path γ connecting x and y. Since $\gamma\mid_{[0,t]}\colon [0,t] \to X$ is a path from $\gamma(0)$ to $\gamma(t$ every point on the path is in the path component, hence the path lies in the equivalence class. $\square$

2.2.2 Connected components

For a space X, define $\approx$ by $x \approx y$ if and only if there exists a subset $C \subseteq X$ connected with $x, y \in C$.

Lemma. $\approx$ is an equivalence relation.

Proof.

(i) $\{x\}$ is connected, so $x \approx x$.

(ii) Definition is symmetric so $x \approx y \iff y \approx x$.

(iii) If $x \approx y$ and $y \approx z$ then there are connected subsets C_1 containing x, y and C_2 containing y, z. Then let $C = C_1 \cup C_2$ which is connected since the intersection contains y so non-empty and contains x, z so $x \approx z$.

Hence $\approx$ is an equivalence relation. $\square$

Definition. (Connected components) Let the *connected components* of X are the equivalence classes of $\approx$.

Proposition. The connected components are connected.

Proof. Let $C \subseteq X$ be a connected component. Suppose that $f : C \to \{0, 1\}$ is a surjective continuous function. Let $f(x_0) = 0$ and $f(x_1) = 1$. As $x_0 \approx x_1$, there exists a connected space $D \subseteq X$ with $x_0, x_1 \in D$. If $d \in D$, then $d \approx x_0$, hence $D \subseteq C$. But then $f\mid_D\colon D \to \{0, 1\}$ is a continuous surjective function so D is disconnected, contradiction, hence C is connected. $\square$

For example $X = (-\infty, 0) \cup (-, \infty)$ has connected components $(-\infty, 0), (0, \infty)$ which are also the path-components.

For $\mathbb{Q} \subset \mathbb{R}$ the connected compoments are the singletons, so are the path-components.

Let's look at the *Topologist's sine curve*, which is defined as

$$S = \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \in \mathbb{R} \mid 0 < x \leq 1 \right\}.$$

This is not closed and it's closure is

$$\overline{S} = S \cup (\{0\} \times [-1, 1]).$$

We want to show that $\overline{S}$ is connected but *not* path connected. Since S is the image of $(0, 1]$ under $x \to \left(x, \sin\left(\frac{1}{x}\right)\right)$ hence S is connected.

Lemma. The closure of a connected subspace is connected.

Proof. Let $C \subset X$ be a connected subspace and suppose that $\overline{C} = U \cup V$ is disconnected, so U, V are non-empty, disjoint open sets of $\overline{C}$. Thus U, V are also closed in $\overline{C}$. Hence they are closed in X too. Consider $C \cap U$ and $C \cap V$. This is a cover of C by disjoint open subsets. So one of them is empty, so $C \subset U$ without loss of generality. But since U is closed in X, $\overline{C} \subset U$ too, so $\overline{C} \cap V = \emptyset$. This contradicts the fact that $U \cup V$ was a disconnection. □

Hence we have that $\overline{S}$ is connected. Note that $\{0\} \times [-1, 1]$ and S are both path connected, so to show that $\overline{S}$ is not path connected we need to show that there is no $\gamma : [0, 1] \to \overline{S}$ with $\gamma(0) \in \{0\} \times [-1, 1]$ and $\gamma(1) \in S$. Suppose that γ is such a path. As $\{0\} \times [-1, 1]$ is closed, $\gamma^{-1}(\{0\} \times [-1, 1])$ is closed, so it contains its supremum t. Then $\gamma \mid_{[t,1]}$ is a new path with $\gamma(t) \in \{0\} \times [-1, 1]$ and $\gamma(s < t) \in S$. SO we replace γ by $\gamma \mid_{[t,1]}$. So we can write $\gamma(s) = (x(s), y(s))$ for $x, y : [0, 1] \to \mathbb{R}$ continuous. For $s > 0$ we have that $y(s) = \sin\left(\frac{1}{x(s)}\right)$. Our goal is to find a sequence $s_n \to 0$ in $[0, 1]$ such that $y(s_n) = (-1)^n$. Then this is a contradiction since it should converge to $y(0)$ since y is continuous. For each n choose a $0 = x(0) < a < x\left(\frac{1}{n}\right)$ such that $\sin\left(\frac{1}{u}\right) = (-1)^n$. By the intermediate value theorem we have that $u = x(s_n)$ for some $0 < s_n < \frac{1}{n}$. So $y(s_n) = (-1)^n$ as required. □

2.3 Compactness

Definition. (Cover) A collection $\mathcal{X} \subset P(X)$ is a *cover of* X if for each $x \in X$ there is a $S \in \mathcal{X}$ with $x \in S$.

Definition. (Open cover) An *open cover* of X is a cover consisting of open sets.

Definition. (Subcover) A *subcover* of $\mathcal{X}$ is a $\mathcal{X}' \subseteq \mathcal{X}$ which is also a cover.

Let's give a definition now which generalised the idea of being closed and bounded without need of a metric.

Definition. (Compact) A topological space X is *compact* if every open cover has a finite subcover.

$\mathbb{R}$ is not compact by considering the open sets $\{(n-1, n+1) \subset \mathbb{R} \mid n \in \mathbb{Z}\}$ which clearly is an open cover but has no finite subcover since there isn't even a subcover, since each $n \in \mathbb{Z}$ is only contained in exactly one open interval in the cover.

If X is a topological space with finitely many points then it is compact.

If $X = \{0\} \cup \{\frac{1}{n}\}$ for $n = 1, 2, \ldots$. Then X is compact. Let $\mathcal{U}$ be an open cover of X. The point $0 \in X$ lies in some $U_0 \in \mathcal{U}$. As this is open, there exists an $\varepsilon > 0$ such that $0 \in B_\varepsilon(0) = (-\varepsilon, \varepsilon) \subset U_0$. Thus $\frac{1}{n} \in U_0$ for $0 < \frac{1}{n} < \varepsilon$ for $n > \frac{1}{\varepsilon}$. The finitely many points $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{m}$ that do not satisfy this, lie in open sets $1 \in U_1, \frac{1}{2} \in U_2, \ldots, \frac{1}{m} \in U_m$. Hence $\{U_i\}_{i=0}^m$ is a finite subcover.

The space $Y = \{\frac{1}{n} \mid n = 1, 2, \ldots\} \subseteq \mathbb{R}$ is *not* compact. This is because it is an infinite set and the topology is discrete since $\left\{\frac{1}{n}\right\} = Y \cap \left(\frac{1}{n+1}, \frac{1}{n+1}\right)$. Hence the cover by each of the points has no subcover at all.

Theorem. $[0,1]$ is compact.

Proof. Let $\mathcal{U}$ be an open cover of $[0,1]$. Consider

$$A = \{a \in [0,1] \mid \text{there is a finite } \mathcal{U}' \subset \mathcal{U} \text{ whose union contains } [0,a]\}.$$

If $0 \leq a \leq b$ then $[0,a] \subseteq [0,b]$. So if $b \in A$ then $a \in A$. A is not non-empty since $0 \in A$ since $[0,0] = \{0\}$ has some $U \in \mathcal{U}$ containing 0. Hence by completeness A has a supremum, so set $\alpha = \sup A \in [0,1]$. We want to show that $\alpha = 1$. Let $\alpha \in U_\alpha \in \mathcal{U}$. If $\alpha < 1$ then there is an $\varepsilon > 0$ such that $(a-, a+\varepsilon) \subset [0,1] \subset U_\alpha$. So as $\alpha - \varepsilon < \alpha$, there must be a $\alpha - \varepsilon \leq a < \varepsilon$ with $a \in A$. Hence $\alpha - \varepsilon \in A$ too. But then $[0, \alpha - \varepsilon]$ has a finite cover by elements of $\mathcal{U}$. Adding U_α we have a finite subcover of $[0, \alpha + \varepsilon/2]$ hence this contradicts the fact that α is a supremum, so $\alpha = 1$ and $[0,1]$ is compact. $\square$

Let's look at some consequences of compactness.

Proposition. If X is compact and $C \subseteq X$ is closed, then C is compact.

Proof. Open sets in C are of the form $C \cap U_\alpha$ for U_α open in X. Suppose we have $\{C \cap T_\alpha \mid \alpha \in I\}$ which is an open cover of C. Then $\{X \setminus C\} \cup \{U_\alpha \mid \alpha \in I\}$ is an open cover X. So as X is compact, this has a finite subcover. It can be taken to have the form $\{X \setminus C\} \cap \{U_\alpha \mid \alpha \in I'\}$ for $I' \subseteq I$ finite. As these cover X, their intersections with C cover C, so $\{C \cap U_\alpha \mid \alpha \in I'\}$ is a finite subcover of C, hence C is compact. $\square$

Proposition. If X is Hausdorff and $C \subseteq X$ is compact, then C is closed.

Proof. We will show that $U = X \setminus C$ is open. So for $x \in U$, we have to find an open $x \in U_\alpha \subseteq U$. For each $y \in C$ use Hausdorffness to find disjoint open sets $y \in V_y$ and $x \in W_y$. Then $\{C \cap V_y \mid y \in C\}$ is an open cover of C. As C is copmact, we can find $y_1, \ldots, y_n \in C$ such that $\{C \cap V_y \mid i = 1, \ldots, n\}$ cover C. Let

$$U_x = \bigcap_{i=1}^{n} W_{y_j}.$$

This contains x and is a finite intersection of open sets, so it is open. As W_{y_i} is disjoint from V_{j_i}, $\bigcap_{i=1}^n W_{y_i}$ is disjoint from $\bigcap_{i=1}^n V_{y_i} \supseteq C$, so U_α is also disjoint fro C. So $\alpha \in U_\alpha \subseteq U = X \setminus C$. Doing this for each $x \in U$ shows that U is open. $\square$

Proposition. If X is compact and $f : X \to Y$ is continuous, then $f(X)$ is a compact subspace of Y.

Proof. We can suppose that f is surjective by replacing Y with $f(X)$. If $\mathcal{U}$ is an open cover of Y consider $f^{-1}(\mathcal{U}) = \{f^{-1}(U) \mid U \in \mathcal{U}\}$. This is an open cover of X, so by compactness it has a finite subcover $f^{-1}(U_1), \ldots, f^{-1}(U_n)$. We claim that $U_1, \ldots, U_n$ covers Y. This is because we can write

$$X = \bigcup_{i=1}^{n} f^{-1}(U_i)$$

so

$$f(X) \subseteq \bigcup_{i=1}^{n} U_i \subseteq Y,$$

hence we have

$$\bigcup_{i=1}^{n} f(U_i) = Y$$

so Y is compact. □

Corollary. If $f : X \to Y$ is a continuous bijection from a compact space X to a Hausdorff space Y, then it is a homeomorphism.

Proof. We only need to show that $f^{-1} : Y \to X$ is continuous. We will use the closed set characterisation of continuity. Let $C \subseteq X$ be closed. We want $(f^{-1})^{-1}(C) = f(C)$ closed. We know that C is closed and X is compact hence C is compact. Hence $f(C)$ is compact and by Hausdorffness, $f(C)$ is closed, so f is a homeomorphism. □

Definition. (Sequentially compact) A metric space (X, d) is *sequientially compact* if every sequence in X has a convergent subsequence.

The metric d gives a topology T_d on X.

Lemma. (Lebesgue's number lemma) Let (X, d) be sequentially compact and $\mathcal{U} \subset T_d$ be an open cover. Then there is a $\delta > 0$ such that each $B_\delta(X)$ lies inside some element of $\mathcal{U}$.

Proof. Suppose not. Then for each $n = 1, 2, \ldots,$ there exists a point $xx_n \in X$ such that $B_{\frac{1}{n}}(x_n)$ is *not* contained in any element of $\mathcal{U}$. By sequentially compactness there is a convergent subsequence $x_{n_i} \to x_\infty$. Let $x_\infty \in U \in \mathcal{U}$ and $\varepsilon > 0$ be such that $B_\varepsilon(x_\infty) \subseteq U$. For all $i \gg 0$ we have that $x_{n_i} \in B_{\varepsilon/2}(x_\infty)$ and $\frac{1}{n_i} < \frac{\varepsilon}{2}$. The triangle inequality gives a contradiction from here. So there is such a δ called the Lebesgue number for the cover $\mathcal{U}$. □

Theorem. The metric space (X, d) is sequentially comapct if and only if (X, T_d) is compact.

Proof. Suppose that (X, d) is not sequentially compact. Then there is a sequence t_n in X with no convergent subsequence. i.e. for each $x \in X$ there is an open $U_x \ni x$ which contains only finitely many (t_i). Then $\mathcal{U} = \{U_x \mid x \in X\}$ is an open cover of (X, T_d). If $\mathcal{U}$ had a finite subcover, then as each U_x contains only finitely many values that the sequence (t_i) takes. But this is a contradiction since (t_i) does not have a convergent subsequence. Hence $\mathcal{U}$ doesn't have a finite subcover, so (X, T_d) is not compact.

Suppose now that (X, d) is sequentially comapct and $\mathcal{U}$ is an open cover of (X, T_d). Let δ be a Lebesgue number for this cover by the lemma. We want to show that there is a finite set of points $A \subseteq X$ such that

$$X = \bigcup_{a \in A} B_\delta(a).$$

Suppose not. Then for every finite set A, there is a point $x \in X$ which lies at least δ away for all $a \in A$. So we can find a sequence $x_1, x_2, \ldots$ such that $d(x_i, x_j) \geq \delta$ for all $i \neq j$. This property means it has no convergent subsequence which is a contradiction since (X, d) is sequentially compact. Hence A does exist. Now each $B_\delta(a)$ lies in some $U_a \in \mathcal{U}$ since δ is a Lebesgue number, hence $\{U_a \mid a \in A\}$ is a finite subcover of $\mathcal{U}$. □

Corollary. (Heine-Borel Theorem) A subspcae $X \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof. From IB Analysis II a subspace of $\mathbb{R}^n$ is sequentially compact if and only if it is closed and bounded. Hence the result follows. □

Corollary. (Extreme value theorem) If X is a compact topological space and $f : X \to \mathbb{R}$ is continuous then there are $a, b \in x$ such that

$$f(a) \leq f(x) \leq f(b) \qquad \forall x \in X.$$

Proof. the set $f(X0 \subseteq \mathbb{R}$ is compact so by Heine-Borel it is closed and bounded. So it contains its supremum, $A = f(a)$ and it's infimum $B = f(b)$. So $f(X) \subseteq [A, B]$ so we're done. □

Theorem. If X and Y are compact, then the product space $X \times Y$ is compact.

Proof. We first prove the following lemma which only uses the fact that Y is compact.

Lemma. (Tube lemma) If Y is compact, $x_0 \in X$ and $\{x_0\} \times Y \subseteq W$ is an open neighbourhood then there is an open neighbourhood $U_{x_0} \ni x_0$ such that $U_{x_0} \times Y \subseteq W$.

Proof. We can cover $\{x_0\} \times Y$ by its intersection with open sets of the form $U \times V$ with $U \times V \subseteq W$. We can do this as W is open and the sets of the form $U \times B$ are a basis for the product topology. As this slice $\{x_0\} \times Y$ is compact, this cover has a finite subcover, so there are open sets

$$U_1 \times V_1, \ldots, U_n \times V_n \subseteq W$$

whose intersection with $\{x_0\} \times Y$ cover it. Let

$$U_{x_0} = \bigcap_{i=1}^{n} U_i$$

which is an open neighbourhood of x_0. To see that $U_{x_0} \times Y \subseteq W$, consider a point $(x, y) \in U_{x_0} \times Y$. Note (x_0, y) lies on the slice, so lies in some $U_i \times V_i$. So $y \in V_i$. But $x \in U_{x_0} \subseteq U_i$ so $(x, y) \in U_i \times V_i \subseteq W$. ☐

Now back to the theorem. To show $X \times Y$ is compact let W be an open cover of $X \times Y$. For each $x_0 \in X$, $\{x_0\} \times Y$ can be covered by (its intersection with) finitely many $W_1, \dots, W_n \in W$. So $\{x_0\} \times Y \subseteq \bigcup_{i=1}^n W_i$. By the tube lemma there is a neighbourhood $U_{x_0} \ni x_0$ such that $U_{x_0} \times Y \subseteq \bigcup_{i=1}^n W_i$, so $U_{x_0} \times Y$ can be covered by finitely many elements of W. Now $\{U_{x_0} \mid x_0 \in X\}$ is an open cover of X, so has a finite subcover since X is comapct. Say $U_{x_1}, \dots, U_{x_n}$ is a finite subcover. So $X \times Y$ is covered by $U_{x_1} \times Y, \dots, U_{x_n} \times Y$. As each of these can be covered by finitely-many elements of W, so can the whole of $X \times Y$. ☐

Corollary. $[0, 1]^n$ is compact.

Proof. $[0, 1]$ is compact, so the result follows by induction, using the theorem. ☐

We can also prove Heine-Borel without using any analysis: it is a purely topological consequence.

Corollary. (Heine-Borel) A subspace $X \subseteq R^n$ is compact if and only if it is closed and bounded.

First let X be closed and bounded. Then $X \subseteq [-N, N]^n$ for some $N > 0$. By the corollary $[-N, N]^n$ is compact and X is a closed subspace, hence X is compact.

Conversely suppose that X is a compact then using Hausdorffness we have that it is closed. To see its bounded, consider the open cover $\{X \cap (-N, N)^N \mid N \in \mathbb{N}\}$. So by compactness this must have a finite subcover, and in fact looking at the cover it can be covered by a singular set, say $X \subseteq (-N, N)^n$ for some $N \in \mathbb{N}$. Hence X is bounded. So it's closed and bounded. ☐

2.3.1 The finite intersection property

We say a collection $\{S_\alpha \mid \alpha \in I\}$ of subsets of a fixed set S has the *finite intersection property* if every finite intersection of S_α is non-empty.

Theorem. A space X is compact if every collection of closed subsets $\{C_\alpha \mid \alpha \in I\}$ having the finite intersection property has

$$\bigcap_{\alpha \in I} C_\alpha \neq \emptyset.$$

Proof. We translate the definition of compactness to closed sets from open sets using the complement. This takes unions to intersections and covers to empty intersections. And if we then take the contrapositive i.e.

$$\bigcap_{\alpha \in I} C_\alpha = \emptyset \implies \text{some finite intersection is empty.}$$

This can be expanded on for a full proof, but it's just linguistical. ☐

2.3.2 The Cantor space

Define a subspace $C \subset [0,1]$ inductively. First set $C_0 = [0,1]$. We define $C_1 = [0,1] - \left(\frac{1}{3}, \frac{2}{3}\right)$, removing the open middle third. We contine this inductively, removing the middle third from each closed interval, so

$$C_n = C_{n-1} \setminus \bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right).$$

We then define the *Cantor space* as

$$C = \bigcap_{k=0}^{\infty} C_n \subset [0,1].$$

We know that C_n is closed for all $n \in \mathbb{N}$, hence C is closed. Clearly bounded hence C is compact.