Methods

Notes made by Finley Cooper

28th November 2025

Contents

1 Fourier Series

1.1 Motivation

In 1807 J. Fourier was studying head conduction along a metal rod. This lead him to study 2π-periodic functions i.e. functions $f : \mathbb{R} \to \mathbb{R}$ was such that $f(\theta + 2\pi) = f(\theta)$ for all $\theta \in \mathbb{R}$ then he found that if

$$f(\theta) = \sum_{n\in\mathbb{Z}} \hat{f}_n e^{in\theta}$$

$$\begin{aligned}\frac{1}{2\pi}\int_{-n}^{n} e^{i\lambda x}\hat{f}(\lambda)\,\mathrm{d}\lambda - f(x) &= \int_{-\infty}^{\infty} \sin(ny)\frac{f(x+y)-f(x)}{\pi y}\,\mathrm{d}y \\ &= \int_{-\infty}^{\infty} \sin(ny)F(y;x)\,\mathrm{d}y \\ &= \frac{1}{n}\int_{-\infty}^{\infty} \cos(ny)F_y(y;x)\,\mathrm{d}y \to 0.\end{aligned}$$

□

$$V = \{f : \mathbb{R} \to \mathbb{C} : \text{ with } f \text{ a "nice" function, } f(\theta + L) = f(\theta), \forall\theta \in \mathbb{R}\}.$$

Note for $f \in V$ need only to consider values of f taken in an interval of length L, i.e. $[0, L)$ or $(-\frac{L}{2}, \frac{L}{2}]$ since periodicity covers elsewhere.

We can introduce an inner product on V with

$$\langle f, g\rangle = \int_0^1 f(\theta)\overline{g(\theta)}\mathrm{d}\theta.$$

This gives the associated norm,

$$||f|| = \sqrt{\langle f, f\rangle}.$$

For $n \in \mathbb{Z}$ consider $e_n \in V$ defined by $e_n(\theta) = e^{2\pi in\theta/L}$.

$$\langle e_n, e_m\rangle = \int_0^L e^{2\pi i(n-m)\theta/L}\,\mathrm{d}\theta = L\,\delta_{nm}.$$

So $\{e_n\}$ are orthogonal and $||e_n||^2 = L$ for each $n \in \mathbb{Z}$. This looks like IA Vectors and Matrices.

Recall that if v_N is N-dim vector space equipped with usual inner product and $\{e_n\}_{n=1}^N$ are orthogonal with $|e_n| = L$, then for each $x \in V$ we can write $x = \sum_{n=1}^N \hat{x}_n e_n$ for some $\{\hat{x}_n\}$. To find $\{\hat{x}_n\}$ take the inner product of both sides with e_m. So

$$(x, e_m) = \sum_{n=1}^N \hat{x}_n(e_n \cdot e_m) = L\hat{x}_m$$

i.e

$$\hat{x}_n = \frac{1}{L}(x \cdot e_n).$$

Now could this work on V? V is not finite dimensional so it's not obvious. Every subset of $\{e_n\}$ is linearly indepedent. Ignoring this for now we assume that for all $f \in V$ we can write f in our basis $\{e_n\}$. Then

$$f(\theta) = \sum_n \hat{f}_n e_n(\theta),$$

So taking the inner product as before

$$\langle f, e_m \rangle = \sum_n \hat{f}_n \langle e_n, e_m \rangle$$

so using the delta as before

$$= L\hat{f}_m$$

i.e.

$$\hat{f}_n = \frac{1}{L}\langle f, e_n \rangle = \frac{1}{L}\int_0^1 f(\theta) e^{-2\pi i n\theta/L} \mathrm{d}\theta$$

Definition. (Complex Fourier series) For an L-periodic $f : \mathbb{R} \to \mathbb{C}$ define its *complex Fourier series* by

$$\sum_n \hat{f}_n e^{2\pi i n\theta/L}$$

where

$$\hat{f}_n = \frac{1}{L}\int_0^1 f(\theta) e^{-2\pi i n\theta/L} \mathrm{d}\theta$$

are called the complex Fourier coefficients. We will write for $f \in V$

$$f(\theta) \sim \sum_n \hat{f}_n e^{2\pi i n\theta/L}$$

to mean the series on the right corresponds to complex Fourier series for the function on the left.

We'd like to replace the $\sim$ symbol with equality, but we require a bit more than that.

If we split the complex Fourier series into the parts $\{n = 0\} \cup \{n > 0\} \cup \{n < 0\}$ we get

$$\sum_n \hat{f}_n e^{2\pi i n\theta/L} = \hat{f}_0 + \sum_{n=1}^{\infty} \hat{f}_n \left[\cos\left(\frac{2\pi n\theta}{L}\right) + i\sin\left(\frac{2\pi n\theta}{L}\right)\right] + \sum_{n=1}^{\infty} \hat{f}_{-n} \left[\cos\left(\frac{2\pi n\theta}{L}\right) - i\sin\left(\frac{2\pi n\theta}{L}\right)\right].$$

Definition. (Fourier series) For $f : \mathbb{R} \to \mathbb{C}$ an L-periodic function define its *Fourier series* by

$$\frac{1}{L}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n\theta}{L}\right) + b_n \sin\left(\frac{2\pi n\theta}{L}\right)\right]$$

where

$$a_n = \frac{2}{L}\int_0^L f(\theta)\cos\left(\frac{2\pi n\theta}{L}\right) \mathrm{d}\theta$$

and

$$b_n = \frac{2}{L}\int_0^L f(\theta)\sin\left(\frac{2\pi n\theta}{L}\right) \mathrm{d}\theta$$

are called the Fourier cofficients for f.

If we set

$$c_n(\theta) = \cos\left(\frac{2\pi n\theta}{L}\right),$$
$$s_n(\theta) = \sin\left(\frac{2\pi n\theta}{L}\right),$$

then we can show, for $m, n \geq 1$ that $\langle c_n, c_m\rangle = \langle s_n, s_m\rangle = \frac{L}{2}\delta_{mn}$ and

$$\langle c_n, 1\rangle = \langle s_m, 1\rangle = \langle c_n, s_m\rangle = 0.$$

So we have that $\{1, c_n, c_n\}$ is orthogonal set in V.

For an example take $f : \mathbb{R} \to \mathbb{R}$, 1-periodic, such that $f(\theta) = \theta(1-\theta)$ on $[0, 1)$. For $n \neq 0$ we have

$$\hat{f}_n = \int_0^1 \theta(1-\theta)e^{-2\pi in\theta}\,\mathrm{d}\theta.$$

Integrating by parts (or using a standard Fourier integral computation) yields

$$\hat{f}_n = -\frac{1}{2(\pi n)^2}, \qquad n \neq 0,$$

and

$$\hat{f}_0 = \int_0^1 (\theta - \theta^2)\,\mathrm{d}\theta = \frac{1}{6}.$$

Hence

$$f(\theta) \sim \frac{1}{6} - \sum_{n\neq 0} \frac{e^{2\pi in\theta}}{2(\pi n)^2}.$$

so the sine terms cancel in the sum giving just cosine terms as we expect since our f function is even.

1.2 Convergence of Fourier series

This subject is extremely subtle.

Definition. For $f : \mathbb{R} \to \mathbb{C}$ an L-periodic function we defined the *partial Fourier series* as

$$\begin{aligned}(S_N f)(\theta) &= \sum_{|n|<N} \hat{f}_n e^{2\pi in\theta/L} \\ &= \frac{1}{2}a_0 + \sum_{n=1}^{N}\left[a_n \cos\left(\frac{2\pi n\theta}{L}\right) + b_n \sin\left(\frac{2\pi n\theta}{L}\right)\right]\end{aligned}$$

Natural to ask if $(S_N f) \to f$. For this we need to specify what type of functional convergence we're looking at. Pointwise? Uniform? Maybe they converge in the idea of our new norm?

$$||S_N f - f|| = \sqrt{\int_0^L |(S_N f)(\theta) - f(\theta)|^2 \mathrm{d}\theta} \to 0$$

. For simplicity, we will only consider pointwise convergence.

Proposition. Let $f : \mathbb{R} \to \mathbb{C}$ be an L-periodic function for which on $[0, L)$ we have the following,

(i) f has finitely many discontinuities.

(ii) f has finitely many local maxima and minima.

Then for each $\theta \in [0, 1)$ we have

$$\begin{aligned}\frac{\theta_+ + \theta_-}{2} &= \lim_{n\to\infty} (S_N f)(\theta)\\ &= \sum_n \hat{f}_n e^{2\pi i n\theta/L}\end{aligned}$$

where $f(\theta_\pm) = \lim_{\varepsilon\to 0^+} f(\theta \pm \varepsilon)$. So at the points of continuity the Fourier series gives back the original function, and at points of discontunity the Fourier series gives back the average of the function at the disconunity neighbourhood.

We call functions which properties (i) and (ii) Dirichlet functions. For now on assume all functions are Dirichlet functions so that $\sim$ means that the series on the RHS coincides with the function on the LHS at points of continuity and to the average at points of discontinuity.

Proof. We'll prove the proposition only for functions in $C^\infty(\mathbb{R})$ (actually $C^1(\mathbb{R})$ will do. Assume *wlog* that $L = 2\pi$. Examine $\lim S_N f(\theta_0)$ for some $\theta_0 \in [0, 2\pi)$. By replacing $f(\theta)$ with $f(\theta + \theta_0)$ can assume that $\theta_0 = 0$ *wlog*.

$$\begin{aligned}(S_N f)(\theta) &= \sum_{|n|\le N} \hat{f}_n e^{in\cdot\theta}\\ &= \sum_{|n|\le N} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\mathrm{d}\theta\right)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\left[\sum_{|n|\le N} e^{-in\theta}\right]\mathrm{d}\theta\end{aligned}$$

We can sum the series as a geometric series, so

$$e^{-iN\theta}\sum_{n=0}^{2N} e^{-in\theta} = \frac{\sin[(N+\frac{1}{2})\theta]}{\sin\left(\frac{\theta}{2}\right)}$$

when $\theta \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ and the sum is $2N + 1$ when $\theta \in 2\pi\mathbb{Z}$.

Define the *Dirichlet Kernal* as

$$D_N(\theta) = \begin{cases}\frac{\sin[(N+\frac{1}{2})\theta]}{\sin\left(\frac{\theta}{2}\right)} & \theta \in \mathbb{R} \setminus 2\pi\mathbb{Z}\\ 2N+1 & \text{otherwise}\end{cases}$$

For each $N \ge 0$,

(i) D_N is continiuous, even 2π perioidic

(ii) $\int_{-\pi}^{\pi} D_N(\theta)\mathrm{d}\theta = 2\pi$

Property (ii) follows by intergrating $\sum$ termwise, only 1 is non-zero. This means that

$$f(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(\theta)f(\theta)\mathrm{d}\theta$$

So

$$S_N(f)(0) = f(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(\theta)[f(\theta) - f(0)]\mathrm{d}\theta$$

now set $F(\theta) = \frac{\theta}{\sin\left(\frac{\theta}{2}\right)}\left[\frac{f(\theta)-f(0)}{\theta}\right]$ so we get

$$(S_N f)(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin[(N + \frac{1}{2})\theta]F(\theta)\mathrm{d}\theta$$

Note that $\theta \to F(\theta)$ is smooth since

$$\frac{f(\theta) - f(0)}{\theta} = \frac{1}{\theta}\int_0^{\theta} f'(t)\mathrm{d}t = \frac{1}{\theta}\int_0^1 f'(\tau\theta)\theta\mathrm{d}\tau$$

Hence integrating by parts gives that

$$\begin{aligned}(S_N f)(0) - f(0) &= \frac{1}{N+\frac{1}{2}}\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos[(N + \frac{1}{2})\theta]F'(\theta)\mathrm{d}\theta \\ &\to 0 \text{ as } N \to \infty\end{aligned}$$

For an example consider the function

$$f(\theta) = \begin{cases} +1 & 0 \le \theta < \pi \\ -1 & -\pi \le \theta < 0 \end{cases}$$

Since f is odd, $a_n = 0$ for each n and

$$\begin{aligned} b_n &= \frac{2}{2\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\mathrm{d}\theta \\ &= \frac{2}{\pi}\int_0^{\pi}\sin(n\theta)\mathrm{d}\theta \\ &= \frac{2}{n\pi}[1 - (-1)^n] \end{aligned}$$

Thus

$$f(\theta) \sim \frac{4}{\pi}\sum_{n \text{ odd}} \frac{\sin(n\theta)}{n}$$

1.3 Peridoic extensions: Cosine and sine series

Given a function $f : [0, L) \to \mathbb{C}$ we can define $2L$-periodic even/odd extensions called f_{even}, f_{odd}. Define,

$$f_{even}(\theta) = \begin{cases} f(\theta) & \theta \in [0, L) \\ f(-\theta) & \theta \in [-L, 0) \end{cases}$$

and

$$f_{odd}(\theta) = \begin{cases} f(\theta) & \theta \in [0, L) \\ -f(-\theta) & \theta \in [-L, 0) \end{cases}$$

. Note that $f(\theta) = f_{even}(\theta) = f_{odd}(\theta)$ if $\theta \in [0, L)$.

$$
\begin{aligned}
f_{even}(\theta) &\sim \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2\pi n\theta}{2L}\right) \\
A_n &= \frac{2}{2L}\int_{-L}^{L} f_{even}(\theta) cos\left(\frac{2\pi n\theta}{2L}\right) \mathrm{d}\theta \\
&= \frac{2}{L}\int_{0}^{L} f(\theta) cos\left(\frac{2\pi\theta}{L}\right) \mathrm{d}\theta
\end{aligned}
$$

simiarly we have that

$$
\begin{aligned}
f_{odd}(\theta) &\sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{2\pi n\theta}{2L}\right) \\
B_n &= \frac{2}{L}\int_{0}^{L} f(\theta)\sin\left(\frac{n\pi\theta}{L}\right) \mathrm{d}\theta
\end{aligned}
$$

Definition. For $f : [0, L) \to \mathbb{C}$ define its *cosine* and *sine* series by

$$
\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi\theta}{L}\right), \quad \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi\theta}{L}\right)
$$

where A_n and B_n defined as before.

For an example consider $f(\theta) = 1$ on $[0, \pi)$. For the sine series,

$$
B_n = \frac{2}{\pi}\int_{0}^{\pi} \sin(n\theta)\mathrm{d}\theta = \frac{2}{n\pi}(1 - (-1)^n)
$$

On the interval $(0, \pi)$ we get that $1 = f(\theta) = f_{odd}(\theta) = 4\sum_{n\in\mathbb{N}} \frac{\sin(n\theta)}{n\pi}$. Whereas for the cosine series we get that

$$
A_0 = 2, \quad A_n = 0 \quad n \geq 1
$$

So for $\theta \in [0, \pi)$ we get that $f_{even} = \frac{1}{2} \cdot 2 = 1 = f(\theta)$.

1.4 Regularity and decay of Fourier coefficients

A true but non-examinable fact is that if $g : [a, b] \to \mathbb{C}$ is integrable on $[a, b]$ and $\lambda \in \mathbb{R}$ then

$$
\int_{a}^{b} e^{-i\lambda\theta} g(\theta)\mathrm{d}\theta \to 0 \text{ as } |\lambda| \to \infty.
$$

IF $f : \mathbb{R} \to \mathbb{C}$ is a L-periodic function and integrable on $[0, L)$ then

$$
\hat{f} = \frac{1}{L}\int_{0}^{L} e^{-2\pi i n\theta/L} f(\theta)\mathrm{d}\theta
$$

so taking $\lambda = \frac{2\pi n}{L}$ gives that $\hat{f}_n \to 0$ as $n \to \infty$ by the Riemann-Lebesgue lemma. Also

$$
a_n = \hat{f}_n + \hat{f}_{-n} \quad b_n = i(\hat{f}_n - \hat{f}_{-n}),
$$

both go to zero as $n \to \infty$.

Suppose that f is L-periodic and $f \in C^k(\mathbb{R})$.

$$\begin{aligned}
\hat{f}_n &= \frac{1}{L}\int_0^L e^{-2\pi i n\theta/L} f(\theta)\mathrm{d}\theta \\
&= -\frac{1}{L}\left(\frac{L}{2\pi i n}\right) f(\theta) e^{-2\pi i n\theta/L}\Big|_{\theta=0}^{L} + \left(\frac{L}{2\pi i n}\right)\frac{1}{L}\int_0^L e^{-2\pi i n\theta/L} f'(\theta)\mathrm{d}\theta \\
&= -\frac{L}{2\pi i n}\left[\frac{f(L^-) - f(0^+)}{L}\right] + \frac{L}{2\pi i n}\frac{1}{L}\int_0^L e^{-2\pi i n\theta} f'(\theta)\mathrm{d}\theta
\end{aligned}$$

Since f is periodic and continuously differentiable we have that

$$f(0^+) = f(L^+) = f(L^-)$$

hence the boundary term cancels so repeating we get that

$$\hat{f}_n = \left(\frac{L}{2\pi i n}\right)^k \frac{1}{L}\int_0^L e^{-2\pi i n\theta/L} f^{(k)}(\theta)\mathrm{d}\theta$$

and the integral is $o(1)$ by the Rieman-Lebesgue lemma.

So we get that if f is $C^k(\mathbb{R})$ then $\hat{f}_n = o\left(\frac{1}{n^k}\right)$ as $|n| \to \infty$.

1.5 Termwise differentiation

Suppose f is L-periodic continuously differentiable on $[0, L)$ with $f' = g$ thne g is continuous on $[0, L)$ so

$$\begin{aligned}
\hat{g}_n &= \frac{1}{L}\int_0^L e^{-2\pi i n\theta/L} f'(\theta)\mathrm{d}\theta \\
&= \frac{f(L^-) - f(0^+)}{L} + \left(\frac{2\pi i n}{L}\right)\frac{1}{L}\int_0^L e^{-2\pi i n\theta/L} f(\theta)\mathrm{d}\theta
\end{aligned}$$

If f is continuous on $\mathbb{R}$ then by periodicity we have that

$$f(0^+) = f(L^+) = f(L^-)$$

so that

$$\hat{g}_n = \left(\frac{2\pi i n}{L}\right)\hat{f}_n$$

i.e.

$$f'(\theta) = g(\theta) \sim \sum_n \left(\frac{2\pi i n}{L}\right)\hat{f}_n e^{2\pi i n\theta/L}$$

1.6 Parseval's theorem

If we have that

$$f(\theta) \sim \sum_n \hat{f}_n e_n(\theta)$$

and

$$g(\theta) \sim \sum_n \hat{g}_n e_n(\theta)$$

then taking the inner product of both function we get that

$$\begin{aligned}\langle f, g\rangle &= \sum_{n,m} \hat{f}_n \overline{\hat{g}_n} \langle e_n, e_m\rangle \\ &= L \sum_n \hat{f}_n \overline{\hat{g}_n}\end{aligned}$$

finally that

$$\frac{1}{L}\int_0^L f(\theta)\overline{g(\theta)}\mathrm{d}\theta = \sum_n \hat{f}_n \overline{\hat{g}_n}$$

and when f and g are the same we get that

$$\frac{1}{L}\int_0^L |f(\theta)|^2 \mathrm{d}\theta = \sum_n |\hat{f}_n|^2$$

2 Sturm-Liouville Theory

2.1 Abstract eigenvalues problem

Recall from IA Vectors and Matrices that a linear map $A : V_N \to V_n$ was called *Hermitian* if $A^\dagger = A$ or equivalently we have that

$$\mathbf{x} \cdot (A\mathbf{y}) = (A\mathbf{x}) \cdot \mathbf{x}$$

for all $\mathbf{x}, \mathbf{y} \in V_N$.

They had properties where all eigenvalues are real, eigenvectors with distinct eigenvalues were orthogonal, and that we could pick an porthogonal set of eigenvectors $\{\mathbf{v}_i\}_{i=1}^N$ such that for each $\mathbf{x} \in V_n$ we have that

$$\mathbf{x} = \sum_{i=1}^N \hat{\mathbf{x}}_i \mathbf{v}_i$$

where

$$\hat{\mathbf{x}}_i = \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{v}_i|^2}.$$

But now we're in $N = \infty$, we can't assume everything we've learnt so far.

Use a vector space of nice functions, $f : [a, b] \to \mathbb{C}$ with an inner product

$$\langle f, g\rangle_w = \int_a^b f(x)\overline{g(x)}w(x)\mathrm{d}x.$$

where w is real valued and $w > 0$ on (a, b). We call w the *weight function* associated with the inner product. This gives an associated norm

$$||f|| = \sqrt{\langle f, f\rangle_w}$$

when $w(x) = 1$ we just write $\langle \cdot, \cdot \rangle$.

Definition. (Self-adjoint) A linear differential operator, L, is said to be *self-adjoint* on $(V, \langle \cdot, \cdot \rangle_w)$ if

$$\langle Ly_1, y_2 \rangle_w = \langle y_1, Ly_2 \rangle_w \quad \forall y_1, y_2 \in V.$$

Definition. (Eigenfunction/value) For $(y, \lambda) \in (V \backslash \{0\} \times \mathbb{C}$ is an *eigenfunction, eigenvalue* pair for L if $Ly = \lambda y$.

Proposition. If L is self-adjoint on $(V, \langle \cdot, \cdot, \rangle_w)$ then:

(i) Eigenvalues are real,

(ii) eigenfunctions with distinct eigenvalues are orthogonal,

(iii) there exists a complete orthogonal set of eigenfunctions $\{y_n\}_{n=1}^{\infty}$ i.e. for each $f \in V$ we can write,

$$f = \sum_{n=1}^{\infty} \hat{f}_n y_n$$

where

$$\hat{f}_n = \frac{\langle f, y_n \rangle_w}{||y_n||_w^2}$$

Proof. (For (i)) If $Ly = \lambda y$ with $y \neq 0$ then

$$\begin{aligned}(\lambda - \overline{\lambda})||y||_w^2 &= \langle \lambda y, y \rangle_w - \langle y, \lambda y \rangle_w \\ &= \langle Ly, y \rangle_w - \langle y, Ly \rangle_w \\ &= 0 \implies \lambda = \overline{\lambda}\end{aligned}$$

(For (ii)) If $Ly_1 = \lambda_1 y_1, Ly_2 = \lambda_2 y_2$ with $\lambda_1 \neq \lambda_2$,

$$\begin{aligned}(\lambda_1 - \lambda_2)\langle y_1, y_2 \rangle_w &= \langle \lambda_1 y_1, y_2 \rangle_w - \langle y_1, \lambda_2 y_2 \rangle_w \\ &= \langle Ly_1, y_2 \rangle - \langle y_1, Ly_2 \rangle_w \\ &= 0 \implies \langle y_1, y_2 \rangle_w = 0\end{aligned}$$

The third statement is too hard to prove for this course. ☐

We will study problems of the form

$$\begin{cases} Ly = \lambda y a < x < b \\ y \text{ satisfies some boundary conditions at } x = a, b \end{cases} \tag{2.1}$$

Definition. (Sturm-Liouville operator) We say that L is a *Sturm-Liouville operator* on (a, b) if it has the form

$$\begin{aligned}L &= \frac{1}{w}\left[-\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}\cdot}{\mathrm{d}x}\right) + q\cdot\right] \\ &= \frac{1}{w}\left[-p\frac{\mathrm{d}^2\cdot}{\mathrm{d}x^2} - p^2\frac{\mathrm{d}\cdot}{\mathrm{d}x} + q\cdot\right]\end{aligned}$$

where p, q, w are real valued and $p, w > 0$ on (a, b). We call w the *weight function.*

See that $Ly = \lambda y$ is equivialent to

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}y}{\mathrm{d}x}\right) + qy = \lambda wy \quad a < x < b.$$

We will enforce boundary conditions by stipulating that y belongs to a suitable vector space of functions that appropriate behaviour at the boundaries.

Definition. (Singular) For a Sturm-Liiouville operator on (a, b) say an endpoint $c \in \{a, b\}$ is *singular* if $p(c) = 0$ and *non-singular* otherwise.

We will impose real homogeneous boundary conditions of the form

$$c \in \{a, b\} \quad \alpha_c y(c) + \beta_c y'(c) = 0$$

at each non-singular endpoint, :w for $\alpha_c, \beta_c \in \mathbb{R}$ and $\alpha_c^2 + \beta_c^2 \neq 0$.

We will work on generic vector spaces of the form

$V = \Big\{y \in C^2[a, b] : y$ satisfies real homogeneous boundary conditions at each non-singular endpoint $\Big\}$

Let's look at the example

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[(1 - x^2)\frac{\mathrm{d}y}{\mathrm{d}x}\right] = \lambda y \quad -1 < x < 1.$$

So we have that $p = (1 - x^2), q = 0, w = 1$. Then $x = \pm 1$ both singular. Take $V = \{y \in C^2[a, b]\}$ then

$$\langle f, g\rangle_w = \int_{-1}^{1} f(x)\overline{g(x)}\mathrm{d}x.$$

Proposition. If L is a Sturm-Lionville operator on (a, b) with weight function w then if $y_1, y_0 \in C^2[a, b]$ we have that

$$\langle Ly_1, y_2\rangle_w - \langle y_1, Ly_2\rangle_w = p(x)W(y_1, \overline{y_2})(x)\Big|_a^b$$

where W is the Wronskian.

So if $y_1, y_2 \in V$ then L is self-adjoint on $(V, \langle\cdot,\cdot\rangle_w)$.

Proof.

$$\begin{aligned}
&\int_a^b \frac{1}{w}\left[-(py')' + qy_1\right]\bar{y_2}w\mathrm{d}x - \int_a^b y_1\frac{1}{w}\left[-(p\bar{y_2})' + q\bar{y_2}\right]w\mathrm{d}x \\
&= \int_a^b \left[y_1(p\bar{y_2})' - \bar{y_2}(py_1')'\right]\mathrm{d}x \\
&= \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)W(y_1, \bar{y_2})(x)\right]\mathrm{d}x \\
&= p(x)W(y_1, \bar{y_2})(x)\Big|_a^b.
\end{aligned}$$

Now assume that $y_1, y_2 \in V$. If $x = c \in \{a, b\}$ is singular then $p(c) = 0$ hence $p(c)W(y_1, \bar{y_2})(c) = 0$. If $c \in \{a, b\}$ non-singular then y_1, y_2 satisfy boundary conditions of the form

$$\alpha_c y(c) + \beta_c y'(c) = 0, \quad \alpha_c, \beta_c \in \mathbb{R}, \alpha_c^2 + \beta_c^2 \neq 0.$$

Since $\alpha_c, \beta_c \in \mathbb{R}$ we know that $\bar{y}$ also satisfies the same boundary conditions hence

$$\begin{pmatrix} y_1(c) & y_1'(c) \\ \bar{y_2}(c) & \bar{y_2}'(c) \end{pmatrix} \begin{pmatrix} \alpha_c \\ \beta_c \end{pmatrix} = 0$$

So the determinate of the matrix on the left is zero because α_c and β_c don't both equal zero hence $W(y_1, \bar{y}_2)(c) = \det(\cdots) = 0$ Hence we have that

$$\langle Ly_1, y_2 \rangle_w - \langle y_1, Ly_2 \rangle_w = 0$$

for all $y_1, y_2 \in V$. $\square$

2.2 Sturm-Lioville Eigenvalue problems

We'll be studying problems of the form

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[p\frac{\mathrm{d}y}{\mathrm{d}x}\right] + qy = \lambda y \quad y \in V$$

where

$$V = \left\{y \in C^2[a, b] : y \text{ satisfies real homogeneous BCs at each non-ingular end point}\right\}$$

Equip V with an inner product with a weight function as before. Assume elements of V are real-valued *wlog* since if $y = u + iv$ and $Ly = \lambda y$ then we can split up into

$$Lu = \lambda u, \quad Lv = \lambda v$$

since $p, q, w, \lambda \in \mathbb{R}$. So

$$\langle y_1, y_2 \rangle_w = \int_a^b y_1(x) y_2(x) \mathrm{d}x.$$

Since L is self-adjoint, we know there exists $(y_n, \lambda_n) \in (V \setminus \{0\}) \times \mathbb{R}$ such that $Ly_n = \lambda_n y_n$ with $\langle y_n, y_m \rangle_w = 0$ if $\lambda_n \neq \lambda_m$ and for $f \in V$ we have

$$f(x) = \sum_{n=1}^{\infty} \hat{f}_n y_n(x)$$
$$\hat{f}_n = \frac{\langle f, y_n \rangle_w}{||y_n||_w^2}$$

are the generalised Fourier coefficients of f. It will also be the cases that $\lambda_1 < \lambda_2 < \cdots$ with $\lambda_n \to \infty$ as $n \to \infty$. Let's look at an example. Take

$$\begin{cases} -y'' = \lambda y & 0 < x < L \\ y(0) = y(L) = 0 \end{cases}$$

so $p = w = 1$ and $q = 0$ and $V = \{y \in C^2[0, L] : y(0) = y(L) = 0\}$.

Solving $y''+\lambda y=0$ then $y=A\sin\left(\sqrt{\lambda}x\right)+B\cos\left(\sqrt{\lambda}x\right)$. If $\lambda\leq 0$ we only get the trivial solution, so we must have that $\lambda>0$. If we use $y(0)=0 \implies B=0$ and $y(L)=0 \implies A\sin\left(\sqrt{\lambda}L\right)=0$ so other than the trivial solution, we have that

$$\sqrt{\lambda}=\frac{n\pi}{L}, \quad n=1,2,\dots$$

So

$$\lambda=\left(\frac{n\pi}{L}\right)^2, \quad y_n(x)=\sin\left(\frac{n\pi x}{L}\right).$$

We can see that $\lambda_n\to\infty$ and $\lambda_1<\lambda_2<\cdots$ and

$$\begin{aligned}\langle y_n,y_m\rangle &= \int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\mathrm{d}x\\ &= \frac{L}{2}\delta_{nm}\end{aligned}$$

For $f\in V$,

$$\begin{aligned}f(x) &= \sum_{n=1}^{\infty}\hat{f}_n\sin\left(\frac{n\pi x}{L}\right)\\ \hat{f}_n &= \frac{\langle f,y_n\rangle}{||y_n||^2}\\ &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x\end{aligned}$$

We have re-derived the Fourier sine series from the previous section.

2.3 Reduction to Sturn-Lionville form

Consider a general eigenvalue problem of the form

$$\alpha(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\beta(x)\frac{\mathrm{d}y}{\mathrm{d}x}+\gamma(x)y+\lambda y=0$$

with $\alpha(x)>0$. Divide the equation by $\alpha(x)$ and multiply by

$$I(x)=\exp\left[\int^x\frac{\beta(t)}{\alpha(t)}\mathrm{d}t\right]$$

Proposition. The equation

$$\alpha(x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\beta(x)\frac{\mathrm{d}y}{\mathrm{d}x}+\gamma(x)y+\lambda y=0$$

is equivalent to

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[p\frac{\mathrm{d}y}{\mathrm{d}x}\right]+qy=\lambda wy$$

where

(i) $p(x)=I(x)$,

(ii) $q(x) = -\frac{I(x)\gamma(x)}{\alpha(x)}$,
(iii) $w(x) = \frac{I(x)}{\alpha(x)}$.

Proof.

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[p\frac{\mathrm{d}y}{\mathrm{d}x}\right] + qy - \lambda wy = I\left[-\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\beta(x)}{\alpha(x)}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{\gamma(x)}{\alpha(x)}y - \frac{\lambda y}{\alpha(x)}\right]$$

since $I > 0$ we get that the equation is zero if and only if $LHS = 0$. □

For an example consider

$$\begin{cases} y'' = 2y' + \lambda y = 0 & 0 < x < 1 \\ y(0) = y'(1) = 0 \end{cases}$$

So we have that

$$I(x) = \exp\left[\int^x -\frac{2}{1}\right] = e^{-2x}$$

So the ODE becomes

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{-2x}\frac{\mathrm{d}y}{\mathrm{d}x}\right] = \lambda e^{-2x}y.$$

So we get e^{-2x} as our weight function.

To solve put $y \propto e^{-\alpha x} \implies \alpha = 1 \pm \sqrt{1-\lambda}$ So if $\lambda \neq 1$ we'll get solutions of the form

$$y = e^x\left[Ae^{x\sqrt{1-\lambda}} + Be^{-x\sqrt{1-\lambda}}\right]$$

We need $1 - \lambda < 0$ for non-trivial solutions.

We can see $y(0 = 0 \implies B = 0$ and $y'(1) = 0 \implies Ae\left[\sin\mu + \mu\cos\mu\right] = 0$ where $\mu^2 = \lambda - 1$ and $\mu > 0$ *wlog*. So $\tan\mu = -\mu$. By plotting the graph we can see we have infinitely many solutions for the equation. Call $\mu_1, \mu_2, \ldots$ so we have $\lambda_n = 1 + \mu_n^2$. From the graph we have that $\mu_n \to \infty$ hence $\lambda_n \to \infty$. The corresponding eigenfunctions are

$$y_n(x) = e^x\sin(\mu_n x), \quad n = 1, 2, \ldots$$

Check that $\langle y_n, y_m\rangle \propto \delta_{nm}$. For $n \neq m$

$$\begin{aligned}
\langle y_n, y_m\rangle_w &= \int_0^1 e^x\sin(\mu_n x)e^x\sin(\mu_m x)e^{-2x}\mathrm{d}x \\
&= \int_0^1 \sin(\mu_n x)\sin(\mu_m x)\mathrm{d}x \\
&= \frac{1}{2}\int_0^1 \left[\cos((\mu_n - \mu_m)x) - cos((\mu_n + \mu_m)x)\right]\mathrm{d}x \\
&\vdots \\
&= 0
\end{aligned}$$

(ommitting a large amount of the algebra.)

2.4 Legendre's Equation

Consider an eigenvalue problem defined as

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x}\right] = \lambda y \quad -1 < x < 1$$

So $p = 1 - x^2$, $q = 0$, $w = 1$. Since both endpoints are singular, work on $V = C^2[-1,1]$. Since $x = 0$ is a regular point we can look for solutions in the form

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

By subbing in, we get that

$$a_{n+2} = \left[\frac{n(n+1)-\lambda}{(n+1)(n+2)}\right] a_n$$

Which gives two linearly independent solutions.

$$y_0 = a_0\left[1 + \frac{(-\lambda)x^2}{2!} + \frac{(-\lambda)(6-\lambda)x^3}{4!} + \cdots\right]$$
$$y_1 = a_1\left[x + \frac{(2-\lambda)x^3}{3!} + \frac{(2-\lambda)(12-\lambda)x^5}{5!} + \cdots\right].$$

Note that y_0 collapses if $\lambda = 0, 6$. In general if $\lambda = k(k+1)$ for $k = 0, 1, 2, \ldots$ either y_0 or y_1 gives a polynomial. What if $\lambda \neq k(k+1)$? Since the ratio $\left|\frac{a_{n+2}}{a_n}\right| \to 1$ we know that both series will converge on $|x| < 1$. This doesn't tell us about $y(\pm 1)$. Let's look at y_0 only, y_1 is treated similiar. Let $A_n = a_{2n}$, so

$$\frac{A_n}{A_{n+1}} = \frac{(2n+1)(2n+2)}{2n(2n+1)-\lambda} = 1 + \frac{1}{n} + \varepsilon_n$$

where $|\varepsilon_n| \leq M/n^2$, $M = M(\lambda) > 0$. In particular the RHS true for n sufficiently large, say $n \geq N$. So $\{A_n\}$ have the same sign for $n \geq N$. Using $e^x > 1 + x$ for all $x \in \mathbb{R}$ we get that

$$\frac{|A_n|}{|A_{n+1}|} \leq e^{1/n} + |\varepsilon_n|$$
$$\implies |A_{n+1}| \geq \frac{e^{-1/n}|A_n|}{1 + e^{-1/n}|\varepsilon_n|}$$
$$\geq \frac{e^{-1/n}|A_n|}{1 + |\varepsilon_n|} \geq e^{-1/n}|A_n|e^{-|2n}$$

So for $n \geq N$ we can repeat to get that

$$|A_{n+1}| \geq |A_n| \exp\left[-\left(\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{N}\right) - (|\varepsilon_n| + \cdots + |\varepsilon_N|)\right]$$

hence we have that

$$|A_{n+1}| \geq |A_N| e^{-H_n + H_{N-1}} e^{-M\pi^2/6}.$$

Since $H_n \leq \log n + 2\gamma$

$$\begin{aligned} |A_{n+1}| &\geq |A_N| e^{H_{N-1} - M\pi^2/6} e^{-\log n - 2\gamma} \\ &> \frac{c}{n+1} \end{aligned}$$

Hence we have that

$$y_0 = \sum_{n \leq N} A_n x^{2n} + \sum_{n > N} A_n x^{2n}$$

and since $\{A_n\}$ have the same sign, assume all positive *wlog.* Note that

$$\begin{aligned} \sum_{n>N} A_n x^{2n} > c \sum_{n=1}^{\infty} \frac{x^{2n}}{n} - \sum_{n \leq N} \frac{x^{2n}}{n}. \\ = C \left[\log \left(\frac{1}{1-x^2} \right) - \text{ (some polynomial in x)} \right] \to \infty \quad \text{as } x \to \pm 1. \end{aligned}$$

So $y_0 \notin V$ so we must have that $\lambda_k = k(k+1)$. This gives an even polynomial of degree k from y_0. Make normalisation so $y(1) = 1$ choosing a_0 and a_1 accordingly then the solutions are called Legendre polynomials.

2.5 Bessel's Equation

Fix an integer $n \geq 0$. Consider the eigenvalue problem

$$-\frac{\mathrm{d}}{\mathrm{d}r} \left[r \frac{\mathrm{d}y}{\mathrm{d}x} \right] + \frac{m^2}{r} y = \lambda r y$$

with $0 < r < 1$ and $y(1) = 0$. We have $p = r, q = \frac{m^2}{r}, w = r$. Expanding out derivatives gives that

$$r^2 y'' + r y' + (\lambda r^2 - m^2) y = 0.$$

Set $z = \sqrt{\lambda} r$ (we can show that $\lambda > 0$). Set $R(z) = y(r)$. This gives that

$$z^2 R'' + z R' + (z^2 - m^2) R = 0 \quad 0 < z < \sqrt{\lambda}, R(\sqrt{\lambda}) = 0$$

. This is *Bessel's equation of order m*. Since $x = 0$ is a regular singular point, get can get solutions in the form

$$z \to z^{\sigma} \sum_{n=0}^{\infty} a_n z^n$$

by Fuch's theorem. We get two linearly independent solutions only one of which is non-singular as $z \to 0$. Label the cooresponding solution $R = J_m(z)$. We can show that

$$J_m(z) = \left(\frac{z}{2} \right)^m \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+m)!} \left(\frac{z}{2} \right)^{2n}.$$

These are *Bessel functions of the first kind* of order m. We can show that $J_n(z)$ has infinitely many zeros on the z axis, we label them j_{mk}. Since we require that $J_m(\sqrt{\lambda}) = 0$, solutions to the equation are

$$y_k(r) = J_m(j_{mk} r), \quad \lambda_k = j_{mk}^2$$

for $k = 1, 2, 3, \ldots$

2.6 Inhomogeneous Problems

Let L be a Sturm-Liouville operator. Consider problems of the form

$$\text{find } y \in V : Ly = f \in V.$$

wlog, $w = 1$. Let $\{y_k\}$ be normalised eigenfunctions of L. Be completeness we can write that

$$\begin{aligned} y = \sum A_k y_K, \quad f &= \sum B_k y_k \\ \implies \sum_{k=1}^{\infty} (\lambda_k A_k - B_k) y_k &= 0 \\ \implies \lambda_k A_k &= B_k \quad k = 0, 1, 2, \ldots \end{aligned}$$

So if $\lambda_k \neq 0$, $A_k = B_k/\lambda_k$, we get have

$$y(x) = \sum_{k=1}^{\infty} \frac{B_k}{\lambda_k} y_k(x), \quad B_k = \int_a^b f(\xi) y_k(\xi) \mathrm{d}\xi$$

Putting the B_k into y and changing sums and integrals we get that

$$y(x) = \int_a^b G(x; \xi) f(\xi) \mathrm{d}\xi$$

where

$$G(x; \xi) = \sum_{k=1}^{\infty} \frac{y_k(\xi) y_k(x)}{\lambda_k}$$

is called the Green's function.

3 Linear PDEs and Seperation of Variables

3.1 Superposition

We will be interested in solving boundary value problems (BVP) and initial boundary value problems (IBVP)

$$(\dagger) \begin{cases} P\psi(\mathbf{x}) = 0 & \mathbf{x} \in \Omega \\ \text{some B.Cs} & \mathbf{x} \in \partial\Omega \end{cases}$$

$$(\dagger\dagger) \begin{cases} Q\phi(\mathbf{x}, t) = 0 & (\mathbf{x}, t) \in \Omega \times (0, T) \\ \text{some I.Cs} & (\mathbf{x}, t) \in \Omega \times \{t = 0\} \\ \text{some B.Cs} & (\mathbf{x}, t) \in \partial\Omega \times (0, T) \end{cases}$$

where P, Q are *linear* partial differentiable operators and Ω will be bounded on an open subset of $\mathbb{R}^n$ for $n = 1, 2, 3$.

Remark. We can split $(\dagger\dagger)$ into

$$\begin{cases} Q\phi_1(\mathbf{x}, t) = 0 \quad Q\phi_2(\mathbf{x}, t) = 0 & (\mathbf{x}, t) \in \cdots \\ \text{I.Cs} = 0 \quad \text{some I.Cs} & \vdots \\ \text{Some B.Cs} \quad \text{B.Cs} = & (\mathbf{x}, t) \in \cdots \end{cases}$$

In this course, Ω will always be (possibly after a change of variables) a line/rectangle/cuboid. So *wlog* we can always deal with B.Cs that are zero everywhere apart from on one endpoint/edge/-face. For example

$$(\dagger\dagger\dagger) = \begin{cases} P\psi(\mathbf{x}) = 0 & \mathbf{x} \in (0,1)\times(0,1) \\ \psi = f_1 & \text{ on side } i \text{ for } i = 1,2,3,4 \end{cases}$$

We could look at 4 problems for $\{\psi_i\}_{i=1}^4$

$$\begin{cases} P\psi_i(\mathbf{x}) = 0 & \mathbf{x} \in (0,1)\times(0,1) \\ \psi_i = 0 & \text{on side } \neq i \\ \psi_i = f_i & \text{on side } = i \end{cases}$$

Then $\psi = \psi_1 + \cdots + \psi_4$ solves $(\dagger\dagger\dagger)$.

3.2 Laplace's Equation

Recall for $\varphi : \mathbb{R}^n \to \mathbb{R}$, Laplace's equation is

$$\Delta\varphi = 0$$

where $\Delta = \nabla\cdot\nabla = \nabla^2$. So on the Cartesian coordinates,

$$\Delta = \frac{\partial^2}{\partial x^2} + \cdots + \frac{\partial^2}{\partial z^2}.$$

We say that φ is *harmonic* if $\Delta\varphi = 0$. Harmonic functions are always infinitely differentiable. Let's look at an example.

If $\mathbf{u} : \mathbb{R}^3 \to \mathbb{R}^3$ is the velocity of an incompressible fluid (so $\nabla\cdot\mathbf{u} = 0$) that is irrotational, i.e. $\nabla\times\mathbf{u} = 0$ then we can solve Laplace's equation. Since $\mathbf{u}$ is irrotational on the whole of $\mathbb{R}^3$ there exists a scalar potential such that $\mathbf{u} = \nabla\varphi$. Then $\Delta\varphi = \nabla\cdot(\nabla\varphi) = \nabla\cdot\mathbf{u} = 0$.

We will consider BVPs of the following

$$\begin{cases} \Delta\varphi(\mathbf{x}) = 0 & \mathbf{x} \in \Omega \\ B\varphi = f(\mathbf{x}) & \mathbf{x} \in \partial\Omega \end{cases}$$

where $B\varphi \equiv \varphi$ (Dirichlet) or $B\varphi = \frac{\partial\varphi}{\partial\mathbf{n}}$ (Neumann), or even $B\varphi = \varphi + \frac{\partial\varphi}{\partial\mathbf{n}}$ (Robin).

3.2.1 Serpation of variables on the square

Consider

$$\begin{cases} \varphi_{xx} + \varphi_{yy} = 0 & (x,y) \in (0,1)\times(0,1) \\ \varphi(x,y) = 0 & \text{on } x = 0, x = 1, y = 0 \\ \varphi(x,1) = f(x) & \text{otherwise} \end{cases}$$

Try a separable solution of the form $\varphi = X(x)Y(y)$. Then we get that

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

Dividing through by XY gives that

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0,$$

hence there exists a $\lambda \in \mathbb{R}$ such that

$$\frac{X''}{X} = -\lambda, \quad \frac{Y''}{Y} = \lambda.$$

Since $\varphi = 0$ at $x = 0, 1$ looking just at the X-equation we get that

$$\begin{cases} X'' + \lambda X = 0 & 0 < x < 1 \\ X(0) = 0 \\ X(1) = 0 \end{cases}.$$

This is a Sturm-Liouville problem. So there exists $(X_n, \lambda_n)_{n=1}^{\infty}$ solutions with $\lambda_1 < \lambda_2 < \cdots$ and $\langle X_n, X_m \rangle = \int_0^1 X_n(x) X_m(x) \mathrm{d}x \propto \delta_{nm}$.

We check that we require $\lambda > 0$ for non-trivial solutions. General solutions are

$$X(x) = A \sin\left(\sqrt{\lambda} x\right) + B \cos\left(\sqrt{\lambda} x\right)$$

with $X(0) = 0 \implies B = 0$ and $X(1) = 0 \implies A \sin\left(\sqrt{\lambda}\right) = 0 \implies \lambda = \lambda_n = (n\pi)^2$. So we get that $X_n(x) = \sin(n\pi x)$ with eigenvalues $\lambda_n = (n\pi)^2$ and that $\langle X_n, X_m \rangle = \frac{1}{2}\delta_{nm}$. The Y problem becomes $Y'' - (n\pi)^2 Y = 0$ with $Y(0) = 0$ which gives that

$$Y = A \sinh(n\pi y) + B \cosh(n\pi y).$$

Now $Y(0) = 0 \implies B = 0$, so $Y_n(y) = A_n \sinh(n\pi y)$. So we have functions $\{\varphi_n\}_{n=1}^{\infty}$ with

$$\varphi_n(x, y) = A_n \sin(n\pi x) \sinh(n\pi y)$$

and each satisfies $\Delta \varphi_n = 0$ in $(0,1) \times (0,1)$ and $\varphi_n = 0$ on $x = 0, x = 1, y = 0$. So same is true for their sum

$$\varphi(x, y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \sinh(n\pi y).$$

We still want that $\varphi(x, 1) = f(x)$ so we set

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \sinh(n\pi)$$

we need to find the A_ns to use get equality.

By orthogonality

$$\begin{aligned} \langle f, X_n \rangle &= \sum_{n=0}^{\infty} A_n \langle X_n, X_m \rangle \sinh(m\pi) \\ &= \frac{A_n}{2} \sinh(n\pi) \end{aligned}$$

So our final solution is

$$\varphi(x, y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) \sinh(n\pi y)$$

where

$$\begin{aligned} A_n &= \frac{2}{\sinh(n\pi)} \langle f, X_n \rangle \\ &= \frac{2}{\sinh(n\pi)} \int_0^1 f(x) \sin(n\pi x) \mathrm{d}x \end{aligned}$$

3.2.2 Seperation of variables in a disc/annulus

We want to solve Laplace's equation in the region $r_1 < |\mathbf{x}| < r_2$ in the (x, y) plane. Use plane polar coordinates (r, θ) so that Laplace's equation becomes

$$\Delta\varphi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} = 0.$$

Look for seperable solutions in the form $\varphi = R(r)\Theta(\theta)$. So

$$\frac{r(rR')'}{R} + \frac{\Theta''}{\Theta} = 0$$

so there exists a $\lambda \in \mathbb{R}$ such that

$$\begin{aligned} r(rR')' &= \lambda R \\ \Theta'' + \lambda\Theta &= 0. \end{aligned}$$

The solution to the Θ-equation is going to be

$$\Theta(\theta) = \begin{cases} A\cos\left(\sqrt{\lambda}\theta\right) + B\sin\left(\sqrt{\lambda}\theta\right) & \lambda > 0 \\ A\theta + B & \lambda = 0 \\ A\cosh\left(\sqrt{\lambda}\theta\right) + B\sinh\left(\sqrt{\lambda}\theta\right) & \lambda < 0 \end{cases}.$$

We need $\Theta(\theta + 2\pi) = \Theta(\theta)$ for all θ. This forces Θ to be a constant or $\lambda > 0$ with $\lambda = n^2$ for $n \in \mathbb{N} \cup \{0\}$. So

$$\begin{aligned} \Theta_0(\theta) &= A \\ \Theta_n(\theta) &= C_n\cos(n\theta) + D_n\sin(n\theta) \end{aligned}$$

Plug $\lambda = n^2$ into the R-equation. So

$$r(rR')' = n^2 R$$

For $n = 0$ we have that rR' is constant, integrating gives that

$$R_0(r) = A + B\log r.$$

And for $n > 0$ we try a solution in the form $R(r) = r^\alpha$. Plugging this gives that $\alpha^2 = n^2$ so $\alpha = \pm n$,

$$R_n(r) = A_n r^n + B_n r^{-n}$$

so we have the general solution

$$\varphi(r, \theta) = A + B\log r + \sum_{r=1}^{\infty}\left[A_n r^n + B_n r^{-n}\right]\left[C_n\cos(n\theta) + D_n\sin(n\theta)\right].$$

If the point $r = 0$ belongs to our domain we have to throw out the r^{-n} and $\log r$ terms since they're not defined. So we must take $B = B_n = 0$ for each n.

Let's see an example.
Solve

$$\Delta\varphi = 0 \quad r_1 < r < r_2$$

with $\varphi = 0$ on $r = r_1$ and $\varphi = f(\theta)$ on $r = r_2$. We can repeat analysis but require $R_n(r) = 0$ where $r = r_1$ for $n = 0, 1, 2, \ldots$. Then $R_0(r) = \log\left(\frac{r}{r\,1}\right)$ and $R(r) = \left(\frac{r}{r_1}\right)^n - \left(\frac{r_1}{r}\right)^n$ to get solutions of the form

$$\varphi(r,\theta) = C_0 \log\left(\frac{r}{r_1}\right) + \sum_{n=1}^{\infty}\left[\left(\frac{r}{r_1}\right)^n - \left(\frac{r_1}{r}\right)\right][A_n\cos(n\theta) + B_n\sin(n\theta)].$$

The boundary condition $r = r_2$ gives that

$$f(\theta) = C_0 \log\left(\frac{r_2}{r_1}\right) + \sum_{n=1}^{\infty}\left[\left(\frac{r_2}{r_1}\right)^n - \left(\frac{r_1}{r_2}\right)\right][A_n\cos(n\theta) + B_n\sin(n\theta)].$$

which can be written as

$$f(\theta) = \frac{1}{n}a_0 + \sum_{n=0}^{\infty}[a_n\cos(n\theta) + b_n\sin(n\theta)]$$

i.e. the right hand side should be a Fourier series for f. So a_n and b_n are given by

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\mathrm{d}\theta \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\mathrm{d}\theta. \end{aligned}$$

For a more general problem, where φ takes the values of a function $g(\theta)$ on the interior $r = r_1$, we write that $\varphi = \varphi_1 + \varphi_2$ where $\varphi_i = 0$ on r_i and use superposition to split our task up for two Fourier series computations.

3.2.3 Seperation of variables on a ball/shell (anti-symmetric case)

We want to solve $\Delta\varphi = 0$ in the region $a < |\mathbf{x}| < b$ in $\mathbb{R}^3$ but under the restriction that the problem is symmetric about the z-axis. We'll work in spherical polar coordinates, so

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial\varphi}{\partial r}\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\varphi}{\partial\theta}\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2} = 0.$$

Here the last term vanishes using symmetry about the z-axis. Look for a solution in the form $\varphi = R(r)\Theta(\theta)$ so

$$\frac{[r^2R']'}{R} + \frac{1}{\Theta\sin\theta}[\sin\theta\cdot\theta']' = 0$$

so there exists a $\lambda \in \mathbb{R}$ such that

$$[r^2R']' = \lambda R - [\sin\theta\cdot\Theta']' = \lambda\sin\theta\cdot\Theta$$

which then can be solved using the techniques discussed earlier.

3.3 Wave Equation

Consider a taut string, under constant tension τ clamped at ends $x = 0$ and $x = L$. Let $y = y(x,t)$ denote vertical displacement of the string. Assume that oscillations are transverse and that the slope $|y_x| \ll 1$. Note that

$$\begin{aligned} |y(x,t)| &= \left| \int_0^x y_x(s,t)\mathrm{d}s \right| \\ &\leq \int_0^L |y_x|\mathrm{d}s \ll 1. \end{aligned}$$

Take a diagram of a string with tension τ arcing upwards. Let there be two points x_A and x_B with midpoint x. Let θ_A and θ_B be the subtended angles from the string to the horizontal respectively. Then resolving forces horizontally we get that (given no transverse motion) that

$$\tau \cos\theta_B - \tau\cos\theta_A = 0.$$

This is consistant since that $\tan\theta_A = \left(\frac{\partial y}{\partial x}\right)_A$ so we get that $\cos\theta_A = \frac{1}{\sqrt{1+(\frac{\partial y}{\partial x})^2_A}} \approx 1$. Also the mass of the string is proportional to its length so

$$\int_{x_A}^{x_B} \mathrm{d}s = \int_{x_A}^{x_B} \sqrt{1+y_x^2}\mathrm{d}x \approx \delta x.$$

So reasonable to assume that mass is $\mu\delta x$ where $\mu > 0$ is density. Resolving vertically and using Newton's second law gives that

$$\frac{\mu\delta x}{\tau}\frac{\partial^2 y}{\partial x^2} = \frac{\tau\sin\theta_B}{\tau\cos\theta_B} - \frac{\tau\sin\theta_A}{\tau\cos\theta_A} = \left(\frac{\partial y}{\partial x}\right)_B - \left(\frac{\partial y}{\partial x}\right)_A.$$

Divide by τ to ge that $\tau = \tau\cos\theta_A = \tau\cos\theta_B$ to leading order. Hence by MVT we get that

$$\frac{\mu}{\tau}\delta x\frac{\partial^2 y}{\partial^2 t} = \frac{\partial^2 y}{\partial x^2}\delta x.$$

where the first partial is evaluated at x and the second is evaluated at some $\xi \in (x_A, x_B)$. Divide by δx and take $\delta x \to 0$ and $\xi \to x$ giving that

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}, \quad c^2 = \frac{\tau}{\mu}.$$

call c the wave speed. We have boundary conditions given by $y(0,t) = y(L,t) = 0$ and Newton's second law gives initial conditions that $y(x,0) = f(x)$ and $y_t(x,0) = g(x)$.

3.3.1 Waves on a string

Solve IBVP

$$\begin{cases} y_tt - c^2 y_x x = 0 & (x,t) = (0,L)\times(0,\infty) \\ y(0,t) = 0 & t \in (0,\infty) \\ y(L,t) = 0 & t \in (0,\infty) \\ y(x,0) = f(x) & x \in (0,L) \\ y_t(x,0) = g(x) & x \in (0,L) \end{cases}$$

Try a solution in the form $y = X(x)T(t)$ with $X(0) = 0$ and $X(L) = 0$. We get that

$$\frac{\ddot{T}}{c^2 T} = \frac{X''}{X}$$

so there must exists some $\lambda \in \mathbb{R}$ such that we have that

$$\begin{aligned} X'' + \lambda X &= 0 \\ X(0) = X(L) &= 0 \end{aligned}$$

and

$$\ddot{T} + \lambda c^2 T = 0.$$

The solutions to the X equation is a S-L problem, which has known solutions with

$$X_n(x) = \sin\Big(\frac{n\pi x}{L}\Big), \quad \lambda_n = \Big(\frac{n\pi}{L}\Big)^2, \quad n = 1, 2, \dots$$

The T-equation becomes

$$\ddot{T} + \Big(\frac{n\pi c}{L}\Big)^2 T = 0$$

which we solve as

$$T_n(t) = A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)$$

so by superposition,

$$y(x,t) = \sum_{n=1}^{\infty} \sin\Big(\frac{n\pi x}{L}\Big) \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right]$$

solves the wave equation with $y(0,t) = y(L,t) = 0$. The initial conditions gives that

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\Big(\frac{n\pi x}{L}\Big), \quad g(x) = \sum_{n=1}^{\infty} \Big(\frac{n\pi c}{L}\Big) B_n \sin\Big(\frac{n\pi x}{L}\Big).$$

which look like the sine series. By orthogonality $\langle X_n, X_m \rangle = \frac{L}{2}\delta_{nm}$, so

$$\begin{aligned} A_n &= \frac{2}{L}\int_0^L f(x) \sin\Big(\frac{n\pi x}{L}\Big) \mathrm{d}x \\ B_n &= \left(\frac{L}{n\pi c}\right) \frac{2}{L}\int_0^L g(x) \sin\Big(\frac{n\pi x}{L}\Big) \mathrm{d}x \end{aligned}$$

3.3.2 Waves on a drum

The higher dimensional analogue of the wave equation is

$$\frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = \Delta\varphi$$

Solving the IBVP on a drum,

$$\Omega = \{(r,\theta) : 0 \le r < 1, 0 \le \theta < 2\pi\}$$

$$\begin{cases} \varphi_{tt} - c^2\Delta\varphi = 0 & \Omega \times (0,\infty) \\ \varphi = 0 & \partial\Omega \times (0,\infty) \\ \varphi = f & \Omega \times t = 0 \\ \varphi_t = g & \Omega \times t = 0 \end{cases}$$

For simplicity assume that $f = f(r)$, $g = g(r)$. So that $\varphi = \varphi(r,t)$. Wave equation becomes

$$\frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}.$$

Where the last term is zero since the solution is independent of θ. So we can try a solution in the form $\varphi = R(r)T(t)$. Hence

$$\frac{\ddot{T}}{c^2 T} = \frac{[rR']'}{rR}.$$

So there exists a $\lambda \in \mathbb{R}$ such that

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}r}\left[r\frac{\mathrm{R}}{\mathrm{r}}\right] &= \lambda r R \qquad 0 < r < 1, R(1) = 0 \\ \ddot{T} + c^2\lambda T &= 0. \end{aligned}$$

So the R-equation is the Bessel problem with $m = 0$. This gives solutions

$$R_k(r) = J_0(j_{0k}r), \quad \lambda_k = j_{0k}^2.$$

The T-equation becomes

$$\ddot{T} + (cj_{0k})^2 T = 0$$

so we get a solution in the form

$$\varphi(r,t) = \sum_{k=1}^{\infty} J_0(j_{0k}r)\left[A_k\cos(j_{0k}ct) + B_k\sin(j_{0k}ct)\right]$$

which solves the PDE and boundary conditions. The initial conditions are

$$\begin{aligned} f(r) &= \sum_{k=1}^{\infty} A_k J_0(j_{0k}r) \\ g(r) &= \sum_{k=1}^{\infty} B_k j_{0k} c J_0(j_{0k}r) \end{aligned}$$

Recall that $\langle R_k, R_\ell\rangle_w = \int_0^1 J_0(j_{0k}r)J_0(j_{0\ell r})r\mathrm{d}r = \frac{1}{2}J_0'(j_{0k})^2\delta_{k\ell}$, so by orthogonality we get that

$$\begin{aligned} A_k &= \frac{2}{J_0'(j_{0k})}^2 \int_0^1 f(r)J(j_{0k}r)r\mathrm{d}r \\ B_k &= \frac{2}{J_0'(j_{0k})^2}\frac{1}{cj_{0k}}\int_0^1 g(r)J_0(j_{0k}r)r\mathrm{d}r \end{aligned}$$

Missed Lecture - 03.11.25

3.4 The Heat Equation

The temperature of a conductive material, $\varphi(\mathbf{x},t)$ satisfies the *heat equation* which is

$$\frac{\partial \varphi}{\partial t} = \kappa\Delta\varphi$$

where $\kappa > 0$ is a constant. We are interested in the IBVP

$$(\dagger\dagger) \begin{cases} \varphi_t - \kappa\Delta\varphi = 0 & \Omega \times (0,\infty) \\ \varphi = 0 & \partial\Omega \times (0,\infty) \\ \varphi = f & \Omega \times \{t = 0\} \end{cases}.$$

We'll try a solution in the form $\varphi = T(t)\psi(\mathbf{x})$. Plugging in, for some $\mu \in \mathbb{R}$ we get that

$$\dot{T} + \kappa\mu = 0$$

and

$$-\Delta\psi = \mu\psi$$

. We get that $T(t) = e^{-\kappa\mu t}$. We can see that T does not vanish on the boundary (unless T is trivial), hence we can impose the vanishing boundary condition onto ψ. So we're now solving

$$\begin{cases} -\Delta\psi = \mu\psi & \mathbf{x} \in \Omega \\ \psi = 0 & \mathbf{x} \in \partial\Omega \end{cases}.$$

Solutions to this depend on the geometry of Ω.

3.4.1 Heat conduction on a square sheet

Take $\Omega = \{(x,y) \in (0,L) \times (0,L)\}$. Try a solution seperable in x and y, $\psi = X(x)Y(y)$. So we get that

$$\frac{X''}{X} + \frac{Y''}{Y} + \mu = 0.$$

So there exists a $\lambda \in \mathbb{R}$ such that $X'' + \lambda X = 0$ and $Y'' + (\mu - \lambda)Y = 0$. We also have the requirement that $X(0) = X(L) = Y(0) = Y(L) = 0$. The X-equation gives that

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

and the Y-equation gives that

$$Y = A\sin\left(y\sqrt{\mu - \lambda_n}\right) + B\cos\left(y\sqrt{\mu - \lambda_n}\right)$$

So $Y(0) = 0 \implies B = 0$ and $Y(L) = 0$ gives that

$$\mu_{mn} - \lambda_n = \left(\frac{m\pi}{L}\right)^2$$

so

$$Y_n(y) = \sin\left(\frac{m\pi y}{L}\right), \quad \mu_{mn} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{L}\right)^2$$

so by superposition we get that

$$\varphi(x,y,t) = \sum_{m,n=1}^{\infty} A_{mn} e^{-\kappa\mu_{mn}t} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{L}\right)$$

satisfies the heat equation and the boundary condition $\varphi = 0$ on $\partial\Omega$. Now to impose the initial condition,

$$f(x,y) = \sum_{n,m=1}^{\infty} A_{mn} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{L}\right).$$

So using $\langle X_n, X_k\rangle = \frac{1}{2}\delta_{nk}$ we get that

$$A_{mn} = \frac{4}{L^2}\int_0^L \mathrm{d}x \int_0^L \mathrm{d}y f(x,y)\sin\Big(\frac{n\pi x}{L}\Big)\sin\Big(\frac{m\pi y}{L}\Big).$$

3.4.2 Heat flow down a pipe

In cylindrical polars,

$$\Omega = \{(\rho,\phi,z) : 0 \leq \rho < 1, 0 \leq \phi < 2\pi, 0 < z < L\}$$

Want to solve

$$(\dagger\dagger)\begin{cases} \varphi_t - \kappa\Delta\varphi = 0 & \Omega\times(0,\infty) \\ \varphi = 0 & \partial\Omega\times(0,\infty \\ \varphi = f & \Omega\times\{t=0\} \end{cases}.$$

For simplicity we'll assume that $f = f(\rho, z)$. We'll look for solutions $\varphi = T(t)\psi(\rho,z)$. So

$$\begin{aligned} &-\Delta\psi = \mu\varphi \\ \implies &\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{\partial^2\psi}{\partial z^2} + \mu\psi = 0 \end{aligned}$$

We'll try $\psi(\rho,z) = P(\rho)Z(z)$. So there exists a $\lambda\in\mathbb{R}$ such that

$$-\frac{1}{\rho P}[\rho P']' = \lambda, \quad Z'' + (\mu-\lambda)Z = 0.$$

i.e.

$$-\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\rho\frac{\mathrm{d}P}{\mathrm{d}\rho}\right] = \lambda\rho P \quad 0 < p < 1, \quad P(1) = 0$$

This is Bessel's problem with $m = 0$.

$$\begin{aligned} P_k(\rho) = J_0(j_{0k}\rho), \quad k = 1,2,\ldots \\ \lambda_k = j_{0k}^2. \end{aligned}$$

This Z-equation becomes

$$Z'' + (\mu-\lambda_k)Z = 0, \quad Z(0) = Z(L) = 0$$

Hence we get that

$$Z_n(t) = \sin\Big(\frac{n\pi z}{L}\Big)$$

with $\mu_{kn} - \lambda_k = \left(\frac{n\pi}{L}\right)^2$ for $n = 1,2,\ldots$. By superposition we get that

$$\varphi(\rho,z,t) = \sum_{n,k=1}^{\infty} B_{nk}e^{-\kappa\mu_{kn}t}J_0(j_{0k}\rho)\sin\Big(\frac{n\pi z}{L}\Big)$$

is a solution to the heat equation, and the boundary condition $\varphi = 0$ on $\partial\Omega$. For the initial conditions we need,

$$f(\rho,t) = \sum_{n,k=1}^{\infty} B_{nk}J_0(j_{0k}\rho)\sin\Big(\frac{n\pi z}{L}\Big).$$

Recall that

$$\int_0^1 J_0(j_{0k}\rho)J_0(j_{0\ell}\rho)\rho\mathrm{d}\rho = \frac{1}{2}J_0'(j_{0k})^2\delta_{k\ell}$$

We get that

$$B_{nk} = \frac{4}{LJ_0'(j_{0k})^2}\int_0^L \mathrm{d}\rho\int_0^L \mathrm{d}z\rho f(\rho z)J_0(j_{0k}\rho)\sin\Big(\frac{n\pi z}{L}\Big)$$

3.4.3 Heat loss and uniqueness

Recall again our problem

$$(\dagger\dagger)\begin{cases}\varphi_t - \kappa\Delta\varphi = 0 & \Omega\times(0,\infty)\\ \varphi = 0 & \partial\Omega\times(0,\infty\\ \varphi = f & \Omega\times\{t=0\}\end{cases}.$$

is the solution to ($\dagger\dagger$) unique? Define the energy Q of the system as

$$Q(t) = \frac{1}{2}\int_\Omega \varphi(\mathbf{x},t)^2\mathrm{d}V.$$

Then

$$\begin{aligned}Q'(t) &= \int_\Omega \varphi_t\varphi\mathrm{d}V\\ &= \int_\Omega \kappa\varphi\Delta\varphi\mathrm{d}V\\ &= \kappa\int_\Omega\left[\nabla\cdot(\varphi\nabla\varphi) - |\nabla\varphi|^2\right]\mathrm{d}V\\ &= \kappa\int_{\partial\Omega}\varphi\frac{\partial\varphi}{\partial\mathbf{n}}\mathrm{d}S - \kappa\int_\Omega|\nabla\varphi|^2\mathrm{d}V\end{aligned}$$

The first integral is zero by the boundary condition, and the second integral is clearly non-negative, so $Q'(t)\le 0$. So $Q(t)\le Q(0) = \frac{1}{2}\int_\Omega f(\mathbf{x})^2\mathrm{d}V$.

Proposition. The solution to the problem in ($\dagger\dagger$) is unique.

Proof. Suppose φ_1,φ_2 satisfy ($\dagger\dagger$) and set $\varphi=\varphi_1-\varphi_2$. Then φ satisfies ($\dagger\dagger$) with $f=0$. So

$$\int_\Omega \varphi(\mathbf{x},t)^2\mathrm{d}V \le Q(0) = 0$$

hence $\varphi(\mathbf{x},t)=0$ on $\Omega\times(0,\infty)$ hence $\varphi_1=\varphi_2$. □

4 Inhomogeneous Problems and Green's Functions

4.1 The Dirac Delta Function

We say this mysterious function $\delta(x)$ in IA Differential Equations. It has the properties that

$$\begin{aligned}\delta(x) &= 0 \quad \text{for } x\neq 0\\ \forall\varepsilon>0 &\quad \int_{-\varepsilon}^{\varepsilon}\delta(x)\mathrm{d}x = 1\end{aligned}$$

We can make this (slightly) rigorous by definiting a sequence of functions

$$\delta_n(x) = \begin{cases} \frac{n}{\alpha} \exp\left[-\frac{1}{1-n^2x^2}\right] & |x| < \frac{1}{n} \\ 0 & |x| \geq \frac{1}{n} \end{cases}$$

where

$$\alpha = \int_{-1}^{1} \exp\left[-\frac{1}{1-y^2}\right] \mathrm{d}y.$$

Then we have that

(i) $x \to \delta_n(x)$ is a smooth function for each n.

(ii) $\delta_n(x) = 0$ on $|x| \geq \frac{1}{n}$.

(iii) $\forall \varepsilon > 0,\ \exists N > 0,\ \forall n > N$ we have that

$$\int_{-\varepsilon}^{\varepsilon} \delta_n(x)\mathrm{d}x = 1$$

We'll prove (iii) since (i) and (ii) are obvious. Given some $\varepsilon > 0$ take $N = \frac{1}{\varepsilon}$, so if $n > N$ we have that $\frac{1}{n} < \varepsilon$, hence

$$\begin{aligned} \int_{-\varepsilon}^{\varepsilon} \delta_n(x)\mathrm{d}x &= \int_{-\frac{1}{n}}^{\frac{1}{n}} \delta_n(x)\mathrm{d}x \\ &= \frac{1}{\alpha}\int_{-1}^{1} \exp\left[-\frac{1}{1-y^2}\right] \mathrm{d}y = 1 \end{aligned}$$

We can see that

$$\lim_{n\to\infty} \delta_n(x) = 0 \quad \text{if } x \neq 0.$$

And for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \int_{-\varepsilon}^{\varepsilon} \delta_n(x)\mathrm{d}x = 1$$

So it almost looks like $\delta(x) = \lim_{n\to\infty} \delta_n(x)$. But the problem is that the pointwise limit doesn't converge at 0. But the limit does exist in a weaker sense. Note that $\delta(x)$ rarely appears in isolation (we usually see it inside an integral or as part of a forcing term in a differential equation). We can interpret these as a sequence of statements that we will eventually take the limit of once the limiting behaviour becomes well-defined using the surrounding terms which makes the limit behave nicely. For example if we have the differential equation

$$\ddot{y} + \omega^2 y = \delta(t)$$

we can solve for y_n, the solution to $\ddot{y} + \omega^2 y = \delta_n(t)$, and then assert that $y = \lim_{n\to\infty} y_n(t)$. We could even look at the derivative of $\delta(x)$ via

$$\begin{aligned} \int_{-\infty}^{\infty} \delta'(x) f(x)\,\mathrm{d}x &= \lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n'(x) f(x)\,\mathrm{d}x \\ &= -\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) f'(x)\,\mathrm{d}x \\ &= -\int_{-\infty}^{\infty} \delta(x) f'(x)\,\mathrm{d}x = -f'(0) \end{aligned}$$

4.1.1 Periodic delta functions

Let $\delta^L(x)$ denote the L-periodic extension of $\delta(x)$ outside $[-\frac{L}{2}, \frac{L}{2}]$. It has Fourier coefficients given by

$$\frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \delta(x) e^{-2\pi i n x/L} \mathrm{d}x = \frac{1}{L}$$

So we have that

$$\delta^L(x) \sim \frac{1}{L}\sum_n e^{2\pi i n x/L}$$

which also doesn't converge, which is good since now neither side makes sense. Let's try and chuck our Fourier series into where we would usually use a delta function.

$$\begin{aligned}
f(0) &= \int_{-\frac{L}{2}}^{\frac{L}{2}} \delta^L(x) f(x) \mathrm{d}x \\
&= \sum_n \left[\frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} e^{2\pi i n x/L} f(x) \mathrm{d}x \right] \\
&= \sum_n \hat{f}_{-n} = \sum_n \hat{f}_n = \sum_n \hat{f}_n e^{2\pi i n(0)/L} \\
&= f(0)
\end{aligned}$$

Note that the Fourier series for each $\delta_n^L(x)$ each converge (rapidly) since $\delta_n^L \in C^\infty(\mathbb{R})$.

4.1.2 Eigenfucntion expansion of $\delta(x)$

Let L be a Sturm-Liouville operator on

$$V = \Big\{ y \in C^2[a,b] : \text{ y satisfies real homogeneous B.Cs at each non-singular endpoint} \Big\}$$

Let $\{Y_k\}_{k=1}^\infty$ be normalised eigenfunctions and fix $\xi \in (a,b)$ and consider functions $x \to \delta_n(x-\xi)$. For n sufficiently large $\delta_n(x-\xi) \in V$. By completeness of the space V we should have

$$\begin{aligned}
\delta_n(x-\xi) &= \sum_{k=1}^\infty \langle \delta_n(\cdot - \xi), Y_k \rangle_w Y_k(x) \\
&= \sum_{k=1}^\infty \left[\int_a^b \delta_n(t-\xi) Y_k(t) w(t) \mathrm{d}t \right] Y_k(x).
\end{aligned}$$

Formally, letting $n \to \infty$,

$$\delta(x-\xi) = \sum_{k=1}^\infty w(\xi) Y_k(\xi) Y_k(x).$$

Let's do a sanity check for this result. For $f \in V$ we should have

$$\begin{aligned} f(x) &= \int_a^b \delta(x-\xi) f(\xi) \mathrm{d}\xi \\ &= \sum_{k=1}^{\infty} \left[\int_a^b f(\xi) Y_k(\xi) w(\xi) \mathrm{d}\xi \right] Y_k(x) \\ &= \sum_{k=1}^{\infty} \hat{f}_k Y_k(x), \quad \hat{f}_k = \langle f, Y_k \rangle_w \end{aligned}$$

4.2 Green's Functions

We want to solve

$$(\dagger) \begin{cases} Ly = f(x) & a < x < b \\ y(a) = 0 \\ y(b) = 0 \end{cases} .$$

Where $L = \alpha(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \beta(x)\frac{\mathrm{d}}{\mathrm{d}x} + \gamma(x)$, with $\alpha(x) \neq 0$ on (a, b). Fix $\xi \in (a, b)$. Suppose we can find a function $G = G(x, \xi)$ such that

$$\begin{cases} L_x[G(x,\xi)] = \delta(x-\xi) & a < x < b \\ G(a,\xi) = 0 \\ G(b,\xi) = 0 \end{cases} .$$

Consider

$$y(x) = \int_a^b G(x,\xi) f(\xi) \mathrm{d}\xi.$$

Clearly we have that $y(a) = y(b) = 0$ and

$$\begin{aligned} L_x[y(x)] &= \int_a^b L_x[G(x,\xi)] f(\xi) \mathrm{d}\xi \\ &= \int_a^b \delta(x-\xi) f(\xi) \mathrm{d}\xi \\ &= f(x). \end{aligned}$$

So y solves $(\dagger)$. We call G the Green's function for L with Dirichlet boundary conditions.

Let's look at some properties of $G = G(x, \xi)$. Note on $a < x < \xi$, $\xi < x < b$, G satisfied $L_x[G(x,\xi)] = 0$ which is a nice second order homogeneous ODE so we expect $x \to G(x, \xi)$ to be well-behaved. In the neighbourhood of $x = \xi$ the worst term on the LHS is

$$\alpha(x)\frac{\mathrm{d}^2 G}{\mathrm{d}x^2}$$

and on the RHS the worst term is $\delta(x - \xi)$. So we expect

$$\alpha(x)\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}G}{\mathrm{d}x}\right] = \delta(x-\xi) + (\text{more regular terms})$$

What do we need for

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\cdots\right] \sim \delta(x-\xi)?$$

Recall the sequence $\{\delta_n\}$ were such that

$$\lim_{n\to\infty} \delta_n(x) = 0 \; x \neq 0, \quad \forall \varepsilon > 0 \; \lim_{n\to\infty} \int_{-\varepsilon}^{\varepsilon} \delta_n(x)\mathrm{d}x = 1.$$

So it's natural to look at

$$H_n(x) = \int_{-\infty}^{x} \delta_n(t)\mathrm{d}t,$$

since $H_n'(x) = \delta_n(x)$.

If we fix $x > 0$ then for n sufficiently large so $\frac{1}{n} < x$, hence

$$H_n(x) = \int_{-\frac{1}{n}}^{-\frac{1}{n}} \delta_n(t)\mathrm{d}t = 1$$

Similarly if we have $x < 0$ fixed, then for n sufficiently large we get that

$$H_n(x) = 0.$$

If we take

$$\lim_{n\to\infty} H_n(x) = H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

we get the Heaviside function. i.e. the unique function such that $H'(x) = \delta(x)$ and $H(x) = 0$ for $x < 0$. We expect to need $\frac{\mathrm{d}G}{\mathrm{d}x}$ to behave like $H(x-\xi)$ near $x = \xi$, i.e. we expect a jump discontinuity at $x = \xi$. Since

$$G(x,\xi) = \int_{a}^{x} \frac{\mathrm{d}G(t,\xi)}{\mathrm{d}t}\mathrm{d}t$$

and

$$\int_{a}^{x} H(t-\xi)\mathrm{d}t = \begin{cases} x-\xi & x \geq \xi \\ 0 & x < \xi \end{cases}$$

which is a continuious function. To conclude, expect

- $x \to G(x,\xi)$ to be continuous at $x = \xi$;
- $x \to \frac{\mathrm{d}G(x,\xi)}{\mathrm{d}x}$ to have a jump disconuinuity at $x = \xi$.

To fix the jump disconuinuity, integrate

$$\frac{\mathrm{d}^2G}{\mathrm{d}x^2} + \frac{\beta(x)}{\alpha(x)}\frac{\mathrm{d}G}{\mathrm{d}x} + \frac{\gamma(x)}{\alpha(x)}G = \frac{\delta(x-\xi)}{\alpha(x)}$$

over $(\xi-\varepsilon, \xi+\varepsilon$ with $\varepsilon > 0$,

$$\frac{\mathrm{d}G}{\mathrm{d}x}\Big|_{x=\xi-\varepsilon}^{x=\xi+\varepsilon} + \int_{\xi-\varepsilon}^{\xi+\varepsilon} \left[\frac{\beta}{\alpha}\frac{\mathrm{d}G}{\mathrm{d}x} + \frac{\gamma}{\alpha}G\right]\mathrm{d}x = \int_{\xi-\varepsilon}^{\xi+\varepsilon} \frac{\delta(x-\xi)}{\alpha(x)}\mathrm{d}x = \frac{1}{\alpha(\xi)}.$$

Taking $\varepsilon \to 0$ we get that

$$\left[\frac{\mathrm{d}G}{\mathrm{d}x}\right]_{x=\xi^-}^{x=\xi^+} = \frac{1}{\alpha(\xi)}.$$

In summary, G satisfies

(i) $G(\xi^+, \xi) = G(\xi^-, \xi)$;

(ii) $\left[\frac{\mathrm{d}G}{\mathrm{d}x}\right]_{x=\xi^-}^{x=\xi^+} = \frac{1}{\alpha(\xi)}$.

Let's look at an example. Take,

$$L = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \omega^2, \quad (a, b) = (0, 1).$$

We want to solve

$$\begin{cases} \frac{\mathrm{d}^2 G}{\mathrm{d}x^2} + \omega^2 G = \delta(x - \xi) & 0 < x < 1 \\ G(0, \xi) = G(1, \xi) = 0 \end{cases}.$$

Solve the ODE on either side of $x = \xi$.

$$G(x, \xi) = \begin{cases} A(\xi)\sin(\omega x) + B(\xi)\cos(\omega x), & 0 < x < \xi \\ C(\xi)\sin[\omega(x-1)] + D(\xi)\cos[\omega(x-1)] & \xi < x < 1 \end{cases}.$$

By the boundary conditions, $G(0, \xi) = G(1, \xi) = 0$, hence

$$\implies B(\xi) = 0,\ D(\xi) = 0.$$

From continuity at $x = \xi$ we get that

$$A(\xi) = C(\xi)\frac{\sin(\omega(\xi - 1))}{\sin(\omega\xi)}.$$

From jump of $\frac{\mathrm{d}G}{\mathrm{d}x}$ at $x = \xi$ we get that

$$\frac{C(\xi)}{\sin(\omega\xi)}\omega\left[\cos[\omega(\xi-1)]\sin(\omega\xi) - \cos(\omega\xi)\sin[\omega(\xi-1)]\right] = 1$$

So the inside expression becomes $\sin[\omega\xi - \omega(\xi - 1] = \sin\omega$. Hence

$$C(\xi) = \frac{\sin(\omega\xi)}{\omega\sin(\omega)}$$

and

$$A(\xi) = \frac{\sin(\omega(\xi-1))}{\omega\sin\omega}.$$

Finially we get,

$$\begin{aligned} G(x, \xi) &= \frac{1}{\omega\sin\omega} \times \begin{cases} \sin(\omega x)\sin(\omega(\xi-1)) & 0 < x < \xi \\ \sin(\omega\xi)\sin(\omega(x-1)) & \xi < x < 1 \end{cases} \\ &\equiv \text{const} \times \begin{cases} y_1(x)y_2(\xi) & 0 < x < \xi \\ y_1(\xi)y_1(x) & \xi < x < 1 \end{cases} \end{aligned}$$

Note that y_1, y_2 are L.I solutions to $Ly = 0$ with $y_1(0) = 0$ and $y_2(1) = 0$.

4.3 A General Result

We want to solve

$$(\dagger)\begin{cases} Ly = f(x) & a < x < b \\ y(a) = y(b) = 0 \end{cases}$$

where $L = \alpha(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \beta(x)\frac{\mathrm{d}}{(\mathrm{d}x} + \gamma(x)$.

Proposition. Let y_1, y_2 be solutions to $Ly = 0$ with $y_1(a) = 0$ and $y_2(b) = 0$ linearly independent. Then,

$$G(x,\xi) = \frac{1}{\alpha(xi)W(y_1,y_2)(\xi)} \times \begin{cases} y_1(x)y_2(\xi) & a < x < \xi \\ y_1(\xi)y_2(x) & \xi < x < b \end{cases}$$

satisfies

$$L_x[G(x,\xi)] = \delta(x-\xi), \quad G(a,\xi) = G(b,\xi) = 0.$$

Proof. We have that

$$L_x[G(x,\xi)] = 0$$

for $x \neq \xi$. Clearly $G(\xi^+,\xi) = G(\xi^-,\xi)$ and

$$\begin{aligned} \frac{\mathrm{d}G}{\mathrm{d}x}|^{x=\xi^+} - \frac{\mathrm{d}G}{\mathrm{d}x}|^{x=\xi^-} &= \frac{1}{\alpha(\xi)W(y_1,y_2)(\xi)}\left[y_1(\xi)y_2'(\xi) - y_1'(\xi)y_2(\xi)\right] \\ &= \frac{1}{\alpha(\xi)}. \end{aligned}$$

So $G(x,\xi)$ has all desired properties for the Dirichlet Green's function. □

So the solution to $(\dagger)$ is

$$\begin{aligned} y(x) &= \int_a^b G(x,\xi)f(\xi)\mathrm{d}\xi \\ &= \left[\int_a^x + \int_x^b\right] G(x,\xi)\mathrm{d}\xi \\ &= y_2(x)\int_a^x \frac{y_1(\xi)f(\xi)}{\alpha(\xi)W(y_1,y_2)(\xi)}\mathrm{d}\xi + y_1(x)\int_a^b \frac{y_2(\xi)f(\xi)}{\alpha(\xi)W(y_1,y_2)(\xi)}\mathrm{d}\xi \end{aligned}$$

Remark. Take care with the definition of $G(x,\xi)$ on $x > \xi$, $x < \xi$.

Take the example

$$L = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \omega^2, \quad (a,b) = (0,1).$$

Use $y_1(x) = \sin(\omega x)$ and $y_2 = \sin(\omega(x-1))$, and $\alpha(\xi) = 1$. We can compute

$$\begin{aligned} W(y_1,y_2)(\xi) &= \omega\sin(\omega\xi)\cos(\omega(\xi-1)) - \omega\cos(\omega\xi)\sin(\omega(\xi-1)) \\ &= \omega\sin(\omega\xi - \omega(\xi-1)) \\ &= \omega\sin(\omega). \end{aligned}$$

which gives

$$G(x,\xi) = \frac{1}{\omega\sin\omega} \times \begin{cases} \sin(\omega x)\sin(\omega(\xi-1)) & 0<x<\xi \\ \sin(\omega\xi)\sin(\omega(x-1)) & \xi<x<1 \end{cases}$$

same as before.

4.4 Green's functions for Sturm-Liouville operators

Suppose L has form

$$L = \frac{1}{w(x)}\left[-\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) + q(x)\right].$$

Since we're interested in solutions to $Ly = f$, we can take $w = 1$ *wlog.* (by just changing f to wf). Note for Sturm-Liouville operators we have that $\alpha(x) \equiv -p(x)$.

Proposition. If L is a Sturm-Lioville operator and y_1, y_2 satisfy $Ly_1 = Ly_2 = 0$, then

$$p(x)W(y_1,y_2)(x)$$

is constant.

Proof. We have that

$$\begin{aligned} y_2Ly_1 - y_1Ly_2 &= y_2(-(py_1')' + qy_1) - y_1(-(py_2')' + qy_2) \\ &= y_1(py_2')' - y_2(py_1')' \\ &= (p(x)[y_1y_2' - y_1'y_2])' \end{aligned}$$

So if $Ly_1 = Ly_2 = 0$ we get that $p(x) = W(y_1,y_2)(x)$ is constant. □

So for L, a Sturm Liouville operator, we get that

$$G(x,\xi) = \text{const.} \times \begin{cases} y_1(x)y_2(\xi) & a<x<\xi \\ y_1(\xi)y_2(x) & \xi<x<b \end{cases}.$$

And the solution to $Ly = f$ with $y(a) = y(b) = 0$ is

$$y(x) = \left[-\frac{1}{p(c)W(y_1,y_2)(c)}\right] \times \left[y_2(x)\int_a^x y_1(\xi)f(\xi)\mathrm{d}\xi + y_1(x)\int_x^b y_2(\xi)f(\xi)\mathrm{d}\xi\right]$$

where $c \in (a,b)$ is constant.

4.5 Eigenfunction Expansions Revisited

Recall that if L is a Sturm-Liouville operator on $V = \{y \in C^2([a,b]) : y(a) = y(b) = 0\}$, then there exists *normalised* eigenfunctions $\{Y_k\}_{k=1}^\infty$ such that $LY_k = \lambda_k Y_k$ and $\langle Y_k, Y_\ell\rangle = \delta_{k\ell}$. And for any $f \in V$ with have that

$$f(x) = \sum_{k=1}^\infty \hat{f}_k Y_k(x), \quad \hat{f}_k = \langle f, Y_k\rangle = \int_a^b f(x)Y_k(x)\mathrm{d}x. \qquad (\star)$$

Note that we have that $G(a,\xi) = G(b,\xi) = 0$, but $G(x,\xi) \notin V$ since it has a jump discontinuity so it's not twice continuously differentiable. Instead, consder the sequence $\{G_n(x,\xi)\}_{n=1}^{\infty}$ such that

$$\begin{aligned} L_x[G_n(x,\xi)] &= \delta_n(x-\xi) \\ G_n(a,\xi) &= G_n(b,\xi) = 0. \end{aligned}$$

Note that each $G_n(\cdot,\xi)\xi V$ for any $\xi \in (a,b)$ so $\star$ holds for each $G_n(x,\xi)$ with $f(x) \equiv G_n(x,\xi)$. Then take the limit to get that

$$G(x,\xi) = \sum_{k=1}^{\infty} \hat{G}_k(x) Y_k(x).$$

Instead pretend that $G(x,\xi) \in V$.

Proposition. If $\{Y_k\}$ are as above then the Dirichlet Green's function for the Sturm Liouville operator L satisfies,

$$G(x,\xi) = \sum_{k=1}^{\infty} \frac{Y_k(x)Y_K(\xi)}{\lambda_k}$$

where $LY_k = \lambda_k Y_k$.

By completeness of V we know that

$$G(x,\xi) = \sum_{k=1}^{\infty} \hat{G}_k(\xi) Y_k(x)$$

with

$$\begin{aligned} \hat{G}_k(\xi) &= \langle G(\cdot,\xi), Y_k \rangle \\ &= \frac{1}{\lambda_k} \langle G(\cdot,\xi), LY_k \rangle \\ &= \frac{1}{\lambda_k} \langle LG(\cdot,\xi), Y_k \rangle \quad \text{since } L \text{ is self-adjoint} \\ &= \frac{1}{\lambda_k} \langle \delta(\cdot - \xi), Y_k \rangle \\ &= \frac{1}{\lambda_k} \int_a^b \delta(x-a) Y_k(x) \mathrm{d}x \\ &= \frac{Y_k(\xi)}{\lambda_k}. \end{aligned}$$

i.e. we have that

$$G(x,\xi) = \sum_{k=1}^{\infty} \frac{Y_k(x)Y_k(\xi)}{\lambda_k} \quad \square$$

Remark. If we have $\lambda_k = 0$ for some k, suppose $\tilde{y} \in V$ with $L\tilde{y} = 0$. We will have $y = (L^{-1}y)(x)$ so if $y \in V$ satisfies $Ly = f$ so does $L(y + \tilde{y}) = f$ so L is not invertiable, hence we can't find a Green's function.

Let's look an example now. Take $L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with $(a, b) = (0, 1)$. So $V = \{y \in C^2([0, 1]) : y(0) = y(1) = 0\}$. We have that

$$Y_k(x) = \sqrt{2}\sin(k\pi x), \quad \lambda_k = (k\pi)^2.$$

So

$$G(x, \xi) = \sum_{k=1}^{\infty} \frac{2\sin(k\pi x)\sin(k\pi\xi)}{(k\pi)^2}$$

and we can check all properties.

We have

$$G(x, \xi) = \sum_{k=1}^{\infty} \frac{Y_k(x)Y_k(\xi)}{\lambda_k}$$

and we apply L to both sides, we get that

$$\begin{aligned}\delta(x - \xi) &= L[G(x, \xi)] \\ &= \sum_{k=1}^{\infty} \frac{Y_k(\xi)L_x[Y_k(x)]}{\lambda_k} \\ &= \sum_{k=1}^{\infty} Y_k(\xi)Y_k(x)\end{aligned}$$

so

$$\delta(x - \xi) = \sum_{k=1}^{\infty} Y_k(x)Y_k(\xi).$$

This is a similar form to the Kronecker delta expansion as the outer product of two vectors.

4.6 Initial Value Problems

We want to solve

$$(\dagger) \begin{cases} Ly = f(t) & t > 0 \\ y(0) = \dot{y}(0) = 0 \end{cases}.$$

where $L = \alpha(t)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \beta(t)\frac{\mathrm{d}}{\mathrm{d}x} + \gamma(t)$, with $\alpha(t) \neq 0$.

For each $\tau \in (0, \infty)$, suppose we can find $G = G(t, \tau)$ such that

$$L_t[G(t, \tau)] = \delta(t - \tau) \quad t > 0$$

and also that

$$G(0, \tau) = \frac{\mathrm{d}G}{\mathrm{d}t}(0, \tau) = 0.$$

Then if we set

$$y(t) = \int_0^{\infty} G(t, \tau)f(\tau)\mathrm{d}\tau,$$

we get the solution to $(\dagger)$.

Proposition. The Green's function for (†) is characterised by
 (i) $G(t,\tau) = 0$ on $t < \tau$;
 (ii) $L_t[G(t,\tau)] = 0$ on $t > \tau$ with $G(t^+,\tau) = 0$ and $\frac{\mathrm{d}G}{\mathrm{d}t}(\tau^+,\tau) = \frac{1}{\alpha(\tau)}$.

Proof. We know that (i) guarantees that $L_t[G(t,\tau)] = 0$ for $t < \tau$, and $G(0,\tau) = \frac{\mathrm{d}G}{\mathrm{d}t}(0,\tau) = 0$. Part (ii) guarantees that $L_t[G(t,\tau)] = \delta(t-\tau)$, since $G(\tau^+,\tau) = G(\tau^-,\tau) = 0$ and

$$\frac{\mathrm{d}G}{\mathrm{d}t}(\tau^+,\tau) - \frac{\mathrm{d}G}{\mathrm{d}t}(\tau^-,\tau) = \frac{1}{\alpha(\tau)} \quad \square$$

Note that since $G(t,\tau) = 0$ on $\tau > t$,

$$y(t) = \int_0^\infty G(t,\tau)f(\tau)\mathrm{d}\tau) = \int_0^\tau G(t,\tau)f(\tau)\mathrm{d}\tau.$$

i.e. causality is intact.

Let's see an example. Take $L = \frac{\mathrm{d}^2}{\mathrm{d}t^2}$. Then $G(t,\tau) = 0$ when $t < \tau$ and for $t > \tau$, we have

$$\frac{\mathrm{d}^2G}{\mathrm{d}t^2}(t,\tau) = 0, \quad G(\tau^+,\tau) = 0, \quad \frac{\mathrm{d}G}{\mathrm{d}t}(\tau^+,\tau) = 1.$$

We get that $G(t,\tau) = A(\tau)(t-\tau) + B(\tau)$. This gives that $B = 0$ and differentiating with resepct to t gives that $A = 1$. So our solution to (†) gives that

$$y(t) = \int_0^\tau (t-\tau)f(\tau)\mathrm{d}\tau.$$

Note that this is consistant with Taylor's theroem.

5 The Fourier Transform

5.1 Definitions and simple properties

The Fourier transform is the $L \to \infty$ limit of the theory of Fourier series.

Definition. (Fourier Transform) For $f :\to \mathbb{R} \to \mathbb{C}$ define its *Fourier Transform* by

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x} f(x)\mathrm{d}x, \quad \lambda \in \mathbb{R}.$$

We will use $\mathcal{F}$ to denote the relevant linear map, i.e. $\hat{f} = \mathcal{F}[f]$. We will occasionally write that $\mathcal{F}_{x\to\lambda}[f(x)] = \hat{f}(\lambda)$.

Also if g is integrable on $\mathbb{R}$ then

$$\int_{-\infty}^{\infty} e^{-i\lambda x} g(x)\mathrm{d}x \to 0$$

as $|\lambda| \to \infty$, by the Riemann Lebsegue lemma.

Proposition. We'll make some statements about the arthimetic of Fourier transformations.

(i)
$$\mathcal{F}_{x\to\lambda}\left[\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k f(x)\right] = (i\lambda)^k \hat{f}(\lambda)$$
$$\mathcal{F}_{x\to\lambda}\left[x^k f(x)\right] = \left(i\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^k \hat{f}(\lambda)$$

(ii)
$$\mathcal{F}_{x\to\infty}[f(x-a)] = e^{-i\lambda a}\hat{f}(\lambda)$$
$$\mathcal{F}_{x\to\infty}[e^{-iax}f(x)] = \hat{f}(\lambda+a)$$

Proof. We'll just prove part (i) and leave (ii) as an exercise.

$$\begin{aligned}\int_{-\infty}^{\infty} f^{(k)}(x)e^{-i\lambda x}\mathrm{d}x &= \int_{-\infty}^{\infty} f(x)\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^k e^{-i\lambda x}\mathrm{d}x \\ &= (i\lambda)^k\hat{f}(\lambda).\end{aligned}$$

$$\begin{aligned}\left(i\frac{\mathrm{d}}{\mathrm{d}\lambda}\right)^k \int_{-\infty}^{\infty} e^{-i\lambda x}f(x)\mathrm{d}x &= \int_{-\infty}^{\infty} e^{-i\lambda x}x^k f(x)\mathrm{d}x \\ &= \mathcal{F}_{x\to\lambda}[x^k f(x)].\end{aligned}$$

Property (i) is important in solving ODEs.

$$P\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)y = F(x), \quad x\in\mathbb{R},$$

where P is a polynomial. Take the Fourier transform, so

$$P(i\lambda)\hat{y}(\lambda) = \hat{F}(\lambda),$$

i.e. $\hat{y}(\lambda) = \frac{\hat{F}(\lambda)}{P(i\lambda)}$.

But can we get y back from $\hat{y}$?

Proposition. We can reconstruct f from $\hat{f}$, using the inversion formula for the Fourier transformation,
$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x}\hat{f}(\lambda)\mathrm{d}\lambda.$$

Proof. Write
$$\int_{-\infty}^{\infty} e^{i\lambda x}\hat{f}(\lambda)\mathrm{d}\lambda = \lim_{n\to\infty}\int_{-n}^{n} e^{i\lambda x}\hat{f}(\lambda)\mathrm{d}\lambda.$$

So the integral is the limit of

$$\begin{aligned}\frac{1}{2\pi}\int_{-n}^{n} e^{i\lambda x}\left[\int_{-\infty}^{\infty} e^{-i\lambda y} f(y)\mathrm{d}y\right]\mathrm{d}\lambda &= \int_{-\infty}^{\infty} f(y)\left[\frac{1}{2\pi}\int_{-n}^{n} e^{i\lambda(x-y)}\mathrm{d}\lambda\right]\mathrm{d}y\\ &= \int_{-\infty}^{\infty} f(y)\frac{\sin[n(x-y)]}{\pi(x-y)}\mathrm{d}y\\ &= \int_{-\infty}^{\infty} f(x+y)\frac{\sin(ny)}{\pi y}\mathrm{d}y.\end{aligned}$$

Recall from IA Differential Equations Example Sheet 1 Question 13. We get that for all n

$$\int_{-\infty}^{\infty}\frac{\sin(ny)}{\pi y}\mathrm{d}y = 1.$$

So

$$\begin{aligned}\frac{1}{2\pi}\int_{-n}^{n} e^{i\lambda x}\hat{f}(\lambda)\mathrm{d}y - f(x) &= \int_{-\infty}^{\infty}\sin(ny)\frac{f(x+y)-f(x)}{\pi y}\mathrm{d}y\\ &= \int_{-\infty}^{\infty}\sin(ny)F(y,x)\mathrm{d}y\\ &= \frac{1}{n}\int_{-\infty}^{\infty}\cos(ny)F_y(y,x)\mathrm{d}y \to 0 \quad \square\end{aligned}$$

From previous removing limits we get that

$$f(x) = \int_{-\infty}^{\infty} f(y)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda(x-y)}\mathrm{d}\lambda\right]\mathrm{d}y$$

i.e

$$\delta(x-y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda(x-y)}\mathrm{d}\lambda.$$

We can use this to prove a proposition.

Proposition. For $f, g : \mathbb{R} \to \mathbb{C}$,

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\lambda)\overline{\hat{g}(\lambda)}\mathrm{d}\lambda$$

hence

$$\int_{-\infty}^{\infty}|f(x)|^2\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{f}(\lambda)|^2\mathrm{d}\lambda.$$

Proof.

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{-i\lambda y} f(y)\mathrm{d}y\right]\left[e^{i\lambda x\overline{g(x)}}\mathrm{d}x\right]\mathrm{d}\lambda &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y)\overline{g(x)}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda(x-y)}\mathrm{d}y\right]\mathrm{d}x\mathrm{d}y\\ &= \int_{-\infty}^{\infty} f(x)\overline{g(x)}\mathrm{d}x \quad \square\end{aligned}$$

Definition. (Convolution) For $f, g : \mathbb{R} \to \mathbb{C}$ define the *convolution* by,

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\mathrm{d}y.$$

Proposition.

$$\mathcal{F}_{x\to\lambda}[f * g(x)] = \hat{f}(\lambda)\hat{g}(\lambda).$$

Proof.

$$\begin{aligned}
\text{LHS} &= \int_{-\infty}^{\infty} e^{-i\lambda x}\left[\int_{-\infty}^{\infty} f(x-y)f(y)\mathrm{d}y\right]\mathrm{d}x \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\lambda x} f(x-y)g(y)\mathrm{d}x\mathrm{d}y \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\lambda(X+Y)} f(X)g(Y)\mathrm{d}X\mathrm{d}Y \quad \text{where } X = x-y,\ Y = y \\
&= \hat{f}(\lambda)\hat{g}(\lambda).
\end{aligned}$$

5.2 Important Examples

We'll first look at exponentials.

Let $\sigma \in \mathbb{C}$ with $\mathrm{Re}(\sigma) > 0$. Let

$$f(x) = H(x)e^{-\sigma x} = \begin{cases} e^{-\sigma x} & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

. Then

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(x)e^{-i\lambda x}\mathrm{d}x = \int_{0}^{\infty} e^{-(\sigma+i\lambda)x}\mathrm{d}x = \frac{1}{\sigma+i\lambda}.$$

Note that $|\hat{f}(\lambda)| \to 0$ as $|\lambda| \to \infty$.

From the inversion formula, we have that

$$\begin{aligned}
H(x)e^{-\sigma x} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{\sigma+i\lambda}\mathrm{d}\lambda \\
&= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{\lambda - i\sigma}\mathrm{d}\lambda
\end{aligned}$$

Differentaion with respect to σ gives that

$$H(x)x^k e^{-\sigma x} = \left(-\frac{\partial}{\partial\sigma}\right)^k \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{i\lambda+\sigma}\mathrm{d}\lambda = \frac{k!}{2}\int_{-\infty}^{\infty} \frac{e^{i\lambda x}}{(\sigma+i\lambda)^{k+1}}\mathrm{d}\lambda.$$

Let's now consider $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$. So

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\lambda x - x^2/2}\mathrm{d}\lambda.$$

This is tough without IB Complex Analysis/Methods.
Instead we'll notice the following,

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) f = -xf$$
$$\implies (i\lambda)\hat{f} = -\left(i\frac{\mathrm{d}}{\mathrm{d}x}\right)\hat{f}.$$

i.e. we have that

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\hat{f} = -\lambda\hat{f}.$$

Hence

$$\hat{f}(\lambda) = \hat{f}(0)e^{-\lambda^2/2},$$

so from the definition,

$$\hat{f}(0) = \int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1$$

i.e.

$$\hat{f}(\lambda) = e^{-\lambda^2/2}.$$

So Gaussians are eigenfunctions of the operator $\mathcal{F}$ (with eigenvalue $\sqrt{2\pi}$ in this case). We also have that

$$e^{-\lambda^2/2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\lambda x - x^2/2}\mathrm{d}x.$$

Now we look at the Dirac delta function. From the definition of $\delta(x)$ we have that

$$\hat{\delta}(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x}\delta(x)\mathrm{d}x = 1$$

From the inversion formula we get that

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x}\mathrm{d}\lambda.$$

5.3 Initial Value Problems revisited

Recall from Section 4.6 we have the initial value problem

$$(\dagger)\begin{cases} Ly = f(t) & t > 0 \\ y(0) = \dot{y}(0) = 0 \end{cases}.$$

Now assume that L has constant coefficients, so $L = \alpha\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \beta\frac{\mathrm{d}}{\mathrm{d}x} + \gamma$. Without loss of generality we can assume that $\alpha = 1$. Extend definition of $t \to y(t)$ to all of $\mathbb{R}$ by setting $y(t) = 0$ on $t < 0$. We'll do the same for f. It is customary to write

$$\mathcal{F}_{t\to\omega}[y(t)] = \hat{y}(\omega)$$

when dealing without time-like variables.

Taking the Fourier transformation we get that

$$[(i\omega)^2 + \beta(i\omega) + \gamma]\hat{y}(\omega) = \hat{f}(\omega).$$

i.e. we have that

$$\hat{y}(\omega) = \frac{\hat{f}(\omega)}{P(\omega)}, \quad P(\omega) = (i\omega)^2 + \beta(\omega) + \gamma.$$

Write $P(\omega)$ as

$$P(\omega) = (i\omega + \sigma_1)(i\omega + \sigma_2).$$

Assume that β and γ are such that $\mathrm{Re}(\sigma_i) > 0$ for $i = 1, 2$, and additionally that $\sigma_1 \neq \sigma_2$. Note that

$$\frac{1}{P(\omega)} = \frac{1}{\sigma_2 - \sigma_1}\left[\frac{1}{i\omega + \sigma_1} - \frac{1}{i\omega + \sigma_2}\right]$$

and by Section 5.2 we get that the right hand side of the equation is

$$= \frac{1}{\sigma_2 - \sigma_1}\left[\mathcal{F}_{t\to\omega}[H(t)e^{-\sigma_1 t}] - \mathcal{F}_{t\to\omega}[H(t)e^{-\sigma_2 t}]\right].$$

If we define

$$R(t) = H(t)\left[\frac{e^{-\sigma_1 t} - e^{-\sigma_2 t}}{\sigma_2 - \sigma_1}\right],$$

then

$$\hat{R}(\omega) = \frac{1}{P(\omega)}, \hat{y}(\omega) = \hat{R}(\omega)\hat{f}(\omega).$$

We call $R = R(t)$ the response function for L. If $\sigma_1 = \sigma_2$, set $\sigma_1 = \sigma$, and $\sigma_2 = \sigma + \varepsilon$, and take the limit as $\varepsilon \to 0$, to get

$$R(t) = H(t)te^{-\sigma t}.$$

In the other case, by the convolution theorem we get that

$$\begin{aligned} y(t) &= R * f(t) \\ &= \int_{-\infty}^{\infty} R(t-\tau)f(\tau)\mathrm{d}\tau \\ &= \int_0^{\tau} R(t-\tau)f(\tau)\mathrm{d}\tau. \end{aligned}$$

So $R(t-\tau) = G(t,\tau)$. If $f = \delta(\tau)$, then $y(t) = R(t)$.

However it's not clear where we used the initial conditions of the problem to get our solution. We actually used them implicitly to ensure that the boundary terms vansih when we integrated the differential equation when we first took the Fourier transformation, to ensure the function is continuous at 0 and the first derivative is continuous at 0.

Let's see an example. Consider the damped simple harmonic oscillator.

$$\begin{cases} \ddot{y} + 2\gamma\dot{y} + y = f(t) & t > 0 \\ y(0) = \dot{y}(0) = 0 \end{cases}$$

with $\gamma > 0$. By the Fourier transformation we ge that

$$\hat{y}(\omega) = \frac{\hat{f}(\omega)}{P(\omega)}, \quad P(\omega) = [i\omega + \gamma + \sqrt{\gamma^2 - 1}][i\omega + \gamma - \sqrt{\gamma^2 - 1}].$$

Note that

$$\mathrm{Re}[\gamma \pm \sqrt{\gamma^2 - 1}] \equiv \mathrm{Re}[\omega_\pm(\gamma)] > 0.$$

So in the case $\gamma > 1$,

$$R(t) = H(t)\left[\frac{e^{-\sigma_-(t)} - e^{-\sigma_+ t}}{\sigma_+ - \sigma_-}\right] = H(t)e^{-\gamma t}\frac{\sinh[t\sqrt{\gamma^2 - 1}]}{\sqrt{\gamma^2 - 1}}$$

This is *overdamping*. In the case that $0 < \gamma < 1$, we get that

$$R(t) = H(t)e^{-\gamma t}\frac{\sin[t\sqrt{1-\gamma^2}]}{\sqrt{1-\gamma^2}}$$

This is *underdamping*. When $\gamma = 1$, take $\gamma = 1 + \varepsilon$ and send $\varepsilon \to 0$, to get

$$R(t) = H(t)te^{-t}.$$

This is *critical damping*.

5.4 Some Neat Examples

5.4.1 Poisson Summation Formula

Proposition. (Poisson summation formula) For a "nice" function f we have that

$$\sum_n f(x+n) = \sum_n \hat{f}(2\pi n)e^{2\pi i n x}$$

Proof. Notice that the LHS and RHS functions are periodic with period 1. So they have a Fourier series. Say that the LHS is $F(x)$ and the RHS is $G(x)$, so $\hat{G}_n = \hat{f}(2\pi n)$. F has Fourer coefficients given by

$$\begin{aligned}\hat{F}_n &= \int_0^1 e^{-2\pi i n x}F(x)\mathrm{d}x \\ &= \sum_m \int_0^1 e^{-2\pi i n x}f(x+m)\mathrm{d}x \\ &= \sum_m \int_m^{m+1} e^{-2\pi i n x}f(x)\mathrm{d}x \\ &= \int_{-\infty}^{\infty} e^{-2\pi i n x}f(x)\mathrm{d}x \\ &= \hat{f}(2\pi n).\end{aligned}$$

So $\hat{F}_n = \hat{G}_n$, so $F(x) = G(x)$. □

Let's see an example of this in action. Try $f(x) = H(x)e^{-\sigma(x)}$ with $\mathrm{Re}(\sigma) > 0$. Then $\hat{f}(\lambda) = (\sigma + i\lambda)^{-2}$ using the Poisson summation formula with $x = 0$ we get that

$$\sum_n \frac{1}{(2\pi i n + \sigma)^2} = \sum_{n=1}^{\infty} n^{-\sigma n} = \frac{e^\sigma}{(1-e^\sigma)^2} = \frac{4}{\sinh^2(\sigma/2)}.$$

Let's take

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad \hat{f}(\lambda) = e^{-\lambda^2/2}$$

which gives that

$$\frac{1}{\sqrt{2\pi}} \sum_n e^{-(x+n)^2/2} = \sum_n e^{-(2\pi n)^2/2} e^{2\pi i n x}$$

5.4.2 Heisenberg's Uncertainty Principle

If $\int_{-\infty}^{\infty} |\psi(x)|^2 \mathrm{d}x = 1$ then we claim that

$$\left(\int_{-\infty}^{\infty} x^2 |\psi(x)|^2 \mathrm{d}x \right) \left(\int_{-\infty}^{\infty} \lambda^2 |\hat{\psi}(\lambda)|^2 \mathrm{d}\lambda \right) \geq \frac{\pi}{2}$$

Integrate by parts to get that

$$\begin{aligned} I &= -\int_{-\infty}^{\infty} x \frac{\mathrm{d}}{\mathrm{d}x} [\psi \bar{\psi}] \mathrm{d}x \\ &= -\int_{-\infty}^{\infty} \left(x\psi \bar{\psi}' + x\psi' \bar{\psi} \right) \mathrm{d}x. \end{aligned}$$

Then by the triangle inequality we get that

$$\begin{aligned} 1 &\leq 2 \int_{-\infty}^{\infty} x |\psi| |\psi'| \mathrm{d}x \\ &= 2 \left(\int_{-\infty}^{\infty} x^2 |\psi|^2 \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} |\psi'|^2 \mathrm{d}x \right)^{\frac{1}{2}}. \end{aligned}$$

Recall that $\mathcal{F}_{x \to \lambda}[f'(x)] = (i\lambda) \hat{f}(\lambda)$, and $\int_{-\infty}^{\infty} |f(x)|^2 \mathrm{d}x = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 \mathrm{d}\lambda$. So

$$\int_{-\infty}^{\infty} |f'(x)|^2 \mathrm{d}x = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lambda^2 |\hat{f}(\lambda)|^2 \mathrm{d}\lambda.$$

Hence

$$1 \leq \left(\int_{-\infty}^{\infty} x^2 |\psi|^2 \mathrm{d}x)^{\frac{1}{2}} \right) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \lambda^2 |\hat{\psi}(\lambda)|^2 \mathrm{d}\lambda)^{\frac{1}{2}} \right)$$

which gives the result after moving constants and squaring.

Replacing $\psi(x)$ with $e^{-i\lambda_0 x} \psi(x + x_0)$ we get the new Fourier transform as $e^{i\lambda x_0} \hat{\psi}(\lambda + \lambda_0)$. So

$$\left(\int_{-\infty}^{\infty} x^2 |\psi(x + x_0)|^2 \mathrm{d}x \right) \cdot \left(\int_{-\infty}^{\infty} \lambda^2 |\hat{\psi}(\lambda + \lambda_0)|^2 \mathrm{d}\lambda \right) \geq \frac{\pi}{2},$$

so we get that

$$\left(\int_{-\infty}^{\infty} x^2 |\psi(x)|^2 \mathrm{d}x \right) \cdot \left(\int_{-\infty}^{\infty} \lambda^2 |\hat{\psi}(\lambda)|^2 \mathrm{d}\lambda \right) \geq \frac{\pi}{2}.$$

In quantum mechanics the probability that a particle's momentum is in (A, B) is (up to a constant) $\int_A^B |\hat{\psi}(\lambda)|^2 \mathrm{d}\lambda$. If we choose

$$x_0 = \int_{-\infty}^{\infty} x|\psi(x)|^2 \mathrm{d}x$$
$$\lambda_0 = \int_{-\infty}^{\infty} \lambda|\hat{\psi}(\lambda)|^2 \lambda,$$

then our equation becomes a statement about variance i.e the variance of position multiplied by the variance of momentum must be greater than some constant.

5.4.3 From Fourier Series to Fourier Transforms

Consider the smooth function $f : \mathbb{R} \to \mathbb{C}$, and assume that $f(x)$ has compact support i.e. that $f(x) = 0$ for all $|x| > L$. On our interval $(-L, L)$ we have that

$$f(x) = \sum_n \hat{f}_n e^{i\pi nx/L},$$

so

$$\begin{aligned}\hat{f}_n &= \frac{1}{2L} \int_{-L}^{L} e^{-i\pi nx/L} f(x) \mathrm{d}x \\ &= \frac{1}{2L} \int_{-\infty}^{\infty} e^{-\pi nx/L} f(x) \mathrm{d}x\end{aligned}$$

since $f(x) = 0$ on $|x| > L$. Write $\lambda_n = \frac{n\pi}{L}$ and $\delta\lambda_n = \lambda_{n+1} - \lambda_n = \frac{\pi}{L}$. Then

$$\hat{f}_n = \frac{1}{2\pi} \delta\lambda_n \hat{f}(\lambda_n).$$

So for $x \in (-L, L)$ we have that

$$f(x) = \frac{1}{2\pi} \sum_n e^{i\lambda_n x} \hat{f}(\lambda_n) \delta\lambda_n.$$

Take $L \to \infty$ i.e. $\delta\lambda_n \to 0$, we get that

$$f(x) = \frac{1}{2\pi} e^{i\lambda x} \hat{f}(\lambda) \mathrm{d}\lambda.$$

This is the Fourier inversion formula we saw before!

Also from Parseval's theorem we have that

$$\begin{aligned}\int_{-L}^{L} |f(x)|^2 \mathrm{d}x &= 2L \sum_n |\hat{f}_n|^2 \\ &= \frac{1}{2\pi} \sum_n |\hat{f}(\lambda_n)|^2 \delta\lambda_n\end{aligned}$$

So as $L \to \infty$ get the that

$$\int_{-\infty}^{\infty} |f(x)|^2 \mathrm{d}x = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 \mathrm{d}\lambda.$$

This is Parseval's theorem for the Fourier transform from the equivalent theorem from the Fourier series theorem.

5.4.4 Central Limit Theorem

Let $X_1, \dots, X_n$ be i.i.d random variables with $\mathbb{E}(X_1) = \mu = 0$ and $\text{Var}\,(X_1) = \sigma^2$. Define

$$S_n = \frac{1}{\sqrt{n}}(X_1 + \cdots + X_n)$$

and denote the pdf of S_n by $f_n(x)$. The characteristic function of S_n is

$$\begin{aligned}
\hat{f}_n(\lambda) &= \mathbb{E}[e^{-i\lambda S_n}] \\
&= \mathbb{E}[e^{-i\lambda(X_1+\cdots X_n)/\sqrt{n}}] \\
&= \mathbb{E}[e^{-i\lambda X_1/\sqrt{n}}] \cdots \mathbb{E}[e^{-i\lambda X_n/\sqrt{n}}] \\
&= \mathbb{E}[e^{-i\lambda X_1/\sqrt{n}}]^n \\
&= \varphi_{X_1}(\frac{\lambda}{\sqrt{n}})^n
\end{aligned}$$

Notice

$$\begin{aligned}
\varphi_{X_1}(0) &= 1 \\
\varphi'_{X_1}(0) &= -i\mu = 0 \\
\varphi''_{X_2}(0) &= -\sigma^2.
\end{aligned}$$

Take $\sigma^2 = 1$, so

$$\begin{aligned}
\hat{f}_n(\lambda) &= \left(1 - \frac{1}{2}\frac{\lambda^2}{n} + o\left(\frac{1}{n} right)\right)^n \\
&= e^{-\lambda^2/2} + o(1) \quad \text{as } n \to \infty.
\end{aligned}$$

Recall that

$$\mathcal{F}_{x\to\lambda}\left[\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right] = e^{-\lambda^2/2}.$$

So by the inverse Fourier transform we get that

$$\begin{aligned}
\lim_{n\to\infty} f_n(x) &= \lim_{n\to\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x}\hat{f}_n(\lambda)\mathrm{d}\lambda \\
&= \frac{1}{\sqrt{2\pi}}e^{-x^2/2}
\end{aligned}$$

i.e.

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}} \to \mathcal{N}(0,1).$$

5.5 Discrete Fourier Transform

Given a sequence $\{X_n\}_{n=0}^{N-1}$, define a new sequence $\{\hat{X}_k\}_{k=0}^{N-1}$ by the discrete Fourier Transform (DFT)

$$\hat{X}_k = \sum_{n=0}^{N-1} X_n e^{-i(2\pi n/N)k}.$$

Let $f : [0, 2\pi) \to \mathbb{C}$ be given and suppose that $X_n = f(x_n)$, where $x_n = \frac{2\pi n}{N} \in [0, 2\pi)$. Set $\delta x_n = x_{n-1} - x_n = \frac{2\pi}{N}$.

$$\hat{X}_k = \frac{N}{2\pi} \sum_{n=0}^{N-1} f(x_n) e^{-ikx_n} \delta x_n.$$

If we extend $f(x) = 0$ when $x \notin [0, 2\pi)$, we get that

$$\hat{X}_K \approx \frac{N}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} f(x) \mathrm{d}x = \frac{N}{2\pi} \hat{f}(k) \quad \text{for } k = 0, 1, \cdots, N-1.$$

i.e. we have that for N large,

$$\hat{f}(k) \approx \frac{2\pi}{N} \hat{X}_k.$$

Proposition. Given $\{\hat{X}_k\}_{k=0}^{N-1}$ we can recover $\{X_n\}_{n=0}^{N-1}$ via

$$X_n = \frac{1}{N} \sum_{k=0}^{N-1} \hat{X}_k e^{i(2\pi k/N)n}.$$

Proof. By direct computation,

$$\sum_{k=0}^{N-1} \hat{X}_k e^{i(2\pi n/N)k} = \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} X_m e^{2\pi i(n-m)k/N}.$$

Write $\omega_N = e^{2\pi i/N}$, we get that

$$= \sum_{m=0}^{N-1} X_m \sum_{k=0}^{N-1} \omega_N^{(n-m)k}.$$

Note that

$$\sum_{k=0}^{N-1} \omega_N^{(n-m)k} = \begin{cases} N & n = m \\ \frac{1-\omega_N^{(n-m)N}}{1-\omega_N^{(n-m)}} = 0 & n \neq m \end{cases}.$$

So our sum becomes

$$= \sum_{m=0}^{N-1} X_m (N\delta_{nm}) = NX_n. \quad \square$$

Instead with vectors, if we have $\mathbf{x} \in \mathbb{C}^N$ we have

$$\mathbf{x} = \begin{pmatrix} X_0 \\ \vdots \\ X_{N-1} \end{pmatrix}, \text{ so } \quad \hat{\mathbf{X}} = \begin{pmatrix} \hat{X}_0 \\ \vdots \\ \hat{X}_{N-1} \end{pmatrix} = \mathcal{F}\mathbf{X}$$

where $\mathcal{F}$ is an $N \times N$ matrix, where $\mathcal{F}_{kn} = e^{-i(2\pi n/N)k}$. From previous,

$$\mathcal{F}\mathcal{F}^\dagger = \mathcal{F}^\dagger \mathcal{F} = NI_N,$$

hence if we define $U = \frac{1}{\sqrt{N}}\mathcal{F}$ we have that $UU^\dagger = U^\dagger U$ so U is unitary. i.e $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$.

For an example take $X_n = \binom{N-1}{n}$ for $n = 0, \cdots, N-1$. Then we have that

$$\hat{X}_k = \sum_{n=0}^{N_1} \binom{N-1}{n} e^{-i(2\pi n/N)k} = \left(1 + e^{-2\pi i k/N}\right)^{N-1}.$$

Proposition. (Parseval's theorem) We have that,

$$\sum_{n=0}^{N-1} X_n \overline{Y}_n = \frac{1}{N} \sum_{k=0}^{N-1} \hat{X}_k \overline{\hat{Y}}_k.$$

This also means that

$$\sum_{n=0}^{N-1} |X_n|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |\hat{X}_k|^2.$$

Proof.

$$\begin{aligned} \text{RHS} &= \frac{1}{N} (\mathcal{F}\mathbf{x}) \cdot (\mathcal{F}\mathbf{y}) \\ &= (U\mathbf{x}) \cdot (U\mathbf{y}) \\ &= \mathbf{x} \cdot \mathbf{y} \\ &= \text{LHS}. \end{aligned}$$

And we can take $\mathbf{x} = \mathbf{y}$ to get latter part. $\square$

Let's see an example. Take $X_n = \binom{N-1}{n}$ as before. So

$$\begin{aligned} \sum_{n=0}^{N-1} \binom{N-1}{n}^2 &= \frac{1}{N} = \sum_{k=0}^{N-1} (1 + e^{-2\pi i k/N})^{N-1} (1 + e^{2\pi i k/N})^{N-1} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} (e^{i\pi k/N} + e^{-i\pi k/N})^{2N-2} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} \left(2\cos\left(\frac{\pi k}{N}\right)\right)^{2N-2} \end{aligned}$$

From IA Numbers and Sets we get that

$$\sum_{n=0}^{N-1} \binom{N-1}{n} \binom{N-1}{n} = \sum_{n=0}^{N-1} \binom{N-1}{n} \binom{N-1}{N-1-n} = \binom{2N-2}{N-1}.$$

This gives the result

$$\frac{1}{2^{2M}} \binom{2M}{M} = \frac{1}{M+1} \sum_{k=0}^{M} \cos^{2M}\left(\frac{\pi k}{M+1}\right) = \mathbb{P}(\text{getting } M \text{ heads tossing a coin } 2M \text{ times}).$$

Just like with the usual Fourier transformation, we can define a conovuolution of sequences $\{X_n\}_{n=0}^{N-1}$, $\{Y_n\}_{n=0}^{N-1}$, defined as

$$(X * Y)_n = \sum_{m=0}^{N-1} X_{(n-m) \bmod N} Y_m.$$

Proposition.

$$[\mathcal{F}(X * Y)] = \hat{X}_k \hat{Y}_k.$$

Proof.

$$\begin{aligned}
\text{LHS} &= \sum_{n=0}^{N-1} \left(\sum_{m=0}^{N-1} X_{(n-m) \bmod N} Y_m \right) e^{-i\left(\frac{2\pi n}{N}\right)k} \\
&= \sum_{m=0}^{N-1} Y_m e^{-i(2\pi m/N)k} \sum_{n=0}^{N-1} X_{(n-m) \bmod N} e^{-i(2\pi(n-m)/N)k} \\
&= \sum_{m=0}^{N-1} Y_m e^{-i(2\pi m/N)k} \hat{X}_k \\
&= \hat{X}_k \hat{Y}_k. \quad \square
\end{aligned}$$

Let's look at the computation complexity of the discrete Fourier transformation. From the sum, the time complexity of the discrete Fourier transformation computed naively is $O(n^2)$. The Fast Fourier Transform (FFT) can be used to speed up if $N = 2^M$, then we can compute the Fourier transform in time complexity $O(N \log N)$.

6 PDEs on unbounded domains

6.1 Well-posedness

Hadamard declared that a problem is well-posed if

(i) A solution exists;

(ii) The solution is unique;

(iii) The solution depends continuously on the given initial conditions and boundary data.

(i) and (ii) are obvious, but (iii) is subtle. to maintain continuity we implicity mean we have certain norms or metrics inducing a topology we have to keep in mind. For example if we have some initial boundary value problem (IBVP) whose initial data and boundary data lie in X and time $t > 0$ the solution $u(\cdot, t)$ belongs to some Y. For each $t > 0$ we get some abstract map, $S_t : X \to Y$. Continuity depends on the metrics d_X and d_Y. Intuitively if we start close, we stay close.

For an example consider the IVP

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x, \quad x(0) = x_0.$$

Clearly (i) and (ii) are satisfied, we have a solution

$$X_0(t) = x_0 e^{-t}.$$

Solution with initial data $x(0) = x_1$ is $X_1(t) = x_1 e^{-t}$. So

$$\begin{aligned} |X_1(t) - X_0(t)| &= e^{-t}|x_1 - x_0| \\ &\leq |x_1 - x_0|. \end{aligned}$$

So problem is well-posed for the usual metric on $\mathbb{R}$. Let's look an ill-posed example. Consider the backwards heat equation IBVP on $\Omega = (0, \pi)$.

$$\begin{cases} \varphi_t + \varphi_{xx} = 0 & \Omega \times (0,\infty) \\ \varphi = 0 & \partial\Omega \times (0,\infty) \\ \varphi = f & \Omega \times \{t = 0\} \end{cases}.$$

If $f(x) = 0$ we get $\varphi(x,t) = 0$.
If $f(x) = f_n(x) = \frac{1}{n}\sin(nx)$.
Then $\varphi_n(x,t) = \frac{1}{n}e^{n^2 t} sin(nx)$, so

$$||f - f_n||_\infty = \sup_{(0,\infty)} |f(x) - f_n(x)| = \frac{1}{n} \to 0.$$

However

$$||\varphi - \varphi_n||_\infty = \frac{1}{n}e^{n^2 t} \to \infty.$$

So the problem is ill-posed even locally in time.

6.2 The Method of Characteristics

We want to solve

$$a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} = c(x,y,u).$$

This equation is quasi-linear, since the complicated part (the left-hand side) is linear. We also have that u satisfies some initial data on some curve C in $\mathbb{R}^2$, i.e we want $u(x,y) = \phi(x,y)$ where $(x,y) \in C$.

Consider curves $(x,y) = (x(t), y(t))$ defined by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a(x,y), \quad \frac{\mathrm{d}y}{\mathrm{d}x} = b(x,y)$$

and $(x(0), y(0)) \in C$. We call these the *characteristic curves* for the PDE. We get a whole family of solutions determined by starting point $(x(0), y(0))$. Consider evolution of $u(x,y)$ along a given characteristic, i.e set

$$z(t) = u(x(t), y(t)).$$

By the chain rule, we get that (writing $x = x(t)$ and $y = y(t)$),

$$\begin{aligned} \frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial y}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\partial u}{\partial y} \\ &= a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} \\ &= c(x,y,z). \end{aligned}$$

Since $u(x, y) = z$ on $x = x(t)$ and $y = y(t)$. Note that $z(0) = u(x(0), y(0)) = \phi(x(0), y(0))$ since $(x(0), y(0)) \in C$. So

$$\frac{\mathrm{d}z}{\mathrm{d}t} = c(x, y, z)$$

and $z(0) = \phi(x(0), y(0))$. We need to invert the relationship between (x, y) and (s, t) where s is the arc along C to the characteristic curve we need. Let's see an example

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u, \quad u(x, 0) = f(x), C = \{(s, 0), s \in \mathbb{R}\}.$$

The characteristic curves are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 1.$$

hence we get solutions,

$$\begin{cases} x = t + x_0 \\ y = t + y_0 \end{cases}.$$

We want $(x_0, y_0) \in C$, so take $(x_0, y_0) = (s, 0)$ for $s \in \mathbb{R}$, hence our characteristic curves are $x = t + s$ and $y = t$.

$$\frac{\mathrm{d}z}{\mathrm{d}t} = c(x, y, z) = z.$$

This has a solution $z(t) = z_0 e^t$. Where $z_0 = z(0) = u(x_0, y_0) = u(s, 0) = f(s)$. Therefore $z(t, s) = f(s)e^t$.

We invert using the characteristic curves to get that

$$\begin{cases} t = y \\ s = x - y \end{cases}.$$

Hence finially, $u(x, y) = z(t(x, y), s(x, y)) = f(x - y)e^y$.

Now let's see an harder example. Take

$$(1 + x^2)\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u + 1, \quad u(0, y) = f(y), \quad C = \{(0, s) : s \in \mathbb{R}\}.$$

The characteristic curves are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1 + x^2, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 1.$$

Solving gives $x(t) = \tan(t + \arctan x_0)$ and $y(t) = t + y_0$. We want $(x_0, y_0) \in C$ so $(x_0, y_0) = (0, s)$. Therefore

$$\begin{cases} x = \tan t \\ y = t + s = \arctan x + s \end{cases}$$

To find z, solve

$$\frac{\mathrm{d}z}{\mathrm{d}t} = z + 1$$

, which has solution $z(t) = -1 + [z_0 + 1]e^t$. Recall that $z_0 = u(x_0, y_0) = u(0, s) = f(s)$, hence

$$z(t, s) = -1 + (f(s) + 1)e^t.$$

Inverting this *flow map* we get that

$$\begin{aligned} t &= \arctan x \\ s &= y - \arctan x. \end{aligned}$$

So

$$u(x,y) = -1 + [f(y - \arctan x) + 1]e^{\arctan x}.$$

6.3 Classification of second order linear PDEs in 2 variables

Consider the generic problem $Lu = y$ where

$$Lu = a(x,y)\frac{\partial^2 u}{\partial x^2} + 2b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} + d(x,y)\frac{\partial u}{x} + e(x,y)\frac{\partial u}{\partial y} + f(x,y)u$$

. We want to introduce $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$ such that the PDE simplifies. Write $(x,y) = (x_1, x_2)$ and $(\xi, \eta) = (\xi_1, \xi_2)$. Then

$$Lu = \sum_{i,j=1}^{2} a_{ij}(x,y)\frac{\partial^2 u}{\partial x \partial y} + [\text{lower order stuff}],$$

where

$$\{a_{ij}\} = \begin{pmatrix} a(x,y) & b(x,y) \\ b(x,y) & c(x,y) \end{pmatrix}.$$

Proposition. If $(x,y) \to (\xi, \eta)$ and $U(\xi, \eta) = u(x,y)$ then Lu becomes

$$\tilde{L}U = \sum_{p,q=1}^{2} A_{pq}\frac{\partial^2 U}{\partial \xi_p \partial \xi_q} + [\text{lower order terms}]$$

where

$$A_{pq} = \sum_{i,j=1}^{2} a_{ij}\frac{\partial \xi_p}{\partial x_i}\frac{\partial \xi_q}{\partial x_j}.$$

Proof. By the chain rule and using summation convention, we get

$$\frac{\partial y}{\partial x_i} = \frac{\partial \xi_p}{\partial x_i}\frac{\partial U}{\partial \xi_p}.$$

Hence

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial \xi_p}{\partial x_i}\frac{\partial \xi_q}{\partial x_j}\frac{\partial^2 U}{\partial \xi_p \partial \xi_q} + \frac{\partial^2 \xi_p}{\partial x_i \partial x_j}\frac{\partial U}{\partial \xi_p}.$$

This means that

$$a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} = A_{pq}\frac{\partial^2 U}{\partial \xi_p \partial \xi_q} + [\text{lower order terms}]. \quad \square$$

We can read of that

$$\begin{aligned} A_{11} &= a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 \\ A_{12} &= a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y \\ A_{22} &= a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2. \end{aligned}$$

We can make $A_{11} = 0$ and $A_{22} = 0$ if $M = \frac{\xi_x}{\xi_y}$ and $N = \frac{\eta_x}{\eta_y}$ satifying $az^2 + 2bz + c = 0$. This has roots

$$z = \left[\frac{-b \pm \sqrt{b^2 - ac}}{a}\right].$$

If M and N are chosen to satisfy this quadratic, we call the curves $\xi(x, y) =$ const. and $\eta(x, y) =$ const. the *characteristic curves* of the PDE. On the characteristic curves

$$\frac{\mathrm{d}}{\mathrm{d}x}[\xi(x, y(x))] = \xi_x + \frac{\mathrm{d}y}{\mathrm{d}x}\xi_y = 0.$$

Similarly we have that

$$\eta_x + \frac{\mathrm{d}y}{\mathrm{d}x}\eta_y = 0.$$

This gives that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\xi_x}{\xi_y} = -M, \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\eta_x}{\eta_y} = -N.$$

So on characteristic curves,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\left[\frac{-b \pm \sqrt{b^2 - ac}}{a}\right]. \tag{$\star$}$$

This leads to a classification.

- If $b^2 - ac < 0$ we say the PDE is *elliptic* and has no real characteristic curves.
- If $b^2 - ac = 0$ we say the PDE is *parabolic* and has a single family of characteristic curves.
- If $b^2 - ac > 0$ we say the PDE is *hyperbolic* and has two families of characteristic curves.

In hyperbolic regions, we can integrate up equation ($\star$) to get coordinates $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ for which $A_{11} = A_{22} = 0$. Our partial differential operator becomes

$$2A_{12}\frac{\partial^2 U}{\partial\xi\partial\eta} + [\text{lower order terms}].$$

We call this the *canonical form* of the PDE¿.

Let's see an example of this on the wave equation with $c = 1$.

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0.$$

Thus, $a = 1, b = 0, c = -1$, so $b^2 - ac = 1 > 0$, so the PDE is *globally hyperbolic.* To get characteristic curves, we set

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mp 1.$$

So integrating we get curves $x \pm y =$ const. i.e we can take

$$\xi(x, y) = x + y$$
$$\eta(x, y) = x - y.$$

In these coordinates we get that

$$4\frac{\partial^2 U}{\partial\xi\partial\eta} = 0.$$

Hence $U(\xi,\eta)=A(\xi)+B(\eta)$. This gives solution

$$u(x,y)=A(x+y)+B(x-y).$$

We can solve

$$\begin{cases} u_{tt}-u_{xx}=0 & \mathbb{R}\times(0,\infty) \\ u=f & \mathbb{R}\times\{t=0\} \\ u_t=g & \mathbb{R}\times\{t=0\} \end{cases}.$$

Set $A(x)+B(x)=f(x)$ and $A'(x)-B'(x)=g(x)$. So we get the solution to the IVP as

$$u(x,t)=\frac{1}{2}[f(x+t)+f(x-t)]+\frac{1}{2}\int_{x-t}^{x+t} g(s)\mathrm{d}s.$$

Let's see the example

$$xy\frac{\partial^2 u}{\partial^2 x^2}-\frac{\partial^2 u}{\partial y^2}=0$$

so $a=xy, b=0, c=-1$. Hence $b^2-ac=xy$, so the hyperbolic region is the region $\{x,y>0\}\cup\{x,y<0\}$. On the characteristic curves we have that

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\mp\frac{1}{\sqrt{xy}}.$$

Solving this in $\{x,y>0\}$ region we get that

$$\frac{1}{3}y^{3/2}\pm x^{1/2}=\text{const.}$$

So this gives

$$\xi(x,y)=\frac{1}{3}y^{3/2}+x^{1/2}$$
$$\eta(x,y)=\frac{1}{3}y^{3/2}-x^{1/2}.$$

So the PDE becomes

$$-\frac{1}{2}y\frac{\partial^2 U}{\partial\xi\partial\eta}+[\text{lower order terms}]=0.$$

Note that

$$y=\left[\frac{3}{2}(\xi+\eta)\right]^{2/3},$$

so we can write our PDE in canonical form.

6.4 Fourier Transform in higher dimensions

For $f:\mathbb{R}^n\to\mathbb{C}$ define the Fourier transform pair

$$\hat{f}(\boldsymbol{\lambda})=\int_{\mathbb{R}^n} e^{-i\boldsymbol{\lambda}\cdot\mathbf{x}}f(\mathbf{x})\mathrm{d}^n\mathbf{x}$$

$$f(\mathbf{x})=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-i\lambda\cdot\mathbf{x}}\hat{f}(\boldsymbol{\lambda})\mathrm{d}^n\boldsymbol{\lambda},$$

where $\mathbf{x}, \boldsymbol{\lambda} \in \mathbb{R}^n$, and $\mathrm{d}^n\mathbf{x} = \mathrm{d}x_1 \cdots \mathrm{d}x_n$.

From now on we'll use use $\int$ to mean $\int_{\mathbb{R}^n}$. For $f, g : \mathbb{R}^n \to \mathbb{C}$ define the convolution as

$$f * g(\mathbf{x}) = (\mathbf{x} - \mathbf{y})g(\mathbf{y})\mathrm{d}^n\mathbf{y}.$$

Hence the convolution theorem is

$$\mathcal{F}_{\mathbf{x}\to\boldsymbol{\lambda}}[f * g(\mathbf{x})] = \hat{f}(\boldsymbol{\lambda})\hat{g}(\boldsymbol{\lambda})$$

and

$$\mathcal{F}_{\mathbf{x}\to\boldsymbol{\lambda}}\left[\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} f(\mathbf{x})\right] = (i\lambda_1)^{\alpha_1} \cdots (i\lambda_n)^{\alpha_n}\hat{f}(\boldsymbol{\lambda}).$$

Let's seen an example with Δ.

$$\begin{aligned}\mathcal{F}_{\mathbf{x}\to\boldsymbol{\lambda}}[\Delta f(\mathbf{x})] &= (-\lambda_1^2 - \lambda_2^2 - \cdots - \lambda_n^2)\hat{f}(\boldsymbol{\lambda}) \\ &= -|\boldsymbol{\lambda}|^2\hat{f}(\lambda).\end{aligned}$$

We have a higher dimensional version of the Dirac delta function, $\delta(\mathbf{x})$ defined by properties,

- $\delta(\mathbf{x}) = 0$ for $\mathbf{x} \neq \mathbf{0}$.
- For all $\varepsilon > 0$ we have
$$\int_{|x|<\varepsilon} \delta(\mathbf{x})\mathrm{d}^n\mathbf{x} = 1.$$

So $\int \delta(\mathbf{x} - \mathbf{y})f(\mathbf{y})\mathrm{d}^n\mathbf{y} = f(\mathbf{x})$.

We can take the Fourier transform of the delta function, giving

$$\hat{\delta}(\boldsymbol{\lambda}) = \int e^{-i\boldsymbol{\lambda}\cdot\mathbf{x}}\delta(\mathbf{x})\mathrm{d}^n\mathbf{x} = 1.$$

By the Fourier inversion formula, we find that

$$\delta(\mathbf{x}) = \frac{1}{(2\pi)^n}\int e^{i\boldsymbol{\lambda}\cdot\mathbf{x}}\mathrm{d}^n\boldsymbol{\lambda}.$$

Recall that if

$$g(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2},$$

then $\hat{g}(\lambda) = e^{-\lambda^2/2}$. We want an n-dimensional analogue of this, so take

$$g(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}}e^{-|\mathbf{x}|^2/2},$$

giving

$$\begin{aligned}\hat{g}(\boldsymbol{\lambda}) &= \frac{1}{(2\pi)^{n/2}}\int e^{-i\boldsymbol{\lambda}\cdot x}e^{-|\mathbf{x}|^2/2}\mathrm{d}^n\mathbf{x} \\ &= \int \mathrm{d}x_1 \cdots \int \mathrm{d}x_n \left[\frac{1}{\sqrt{2\pi}}e^{-i\lambda_1 x_i - x_1^2/2}\right] \cdots \left[\frac{1}{\sqrt{2\pi}}e^{-i\lambda_n x_n - x_n^2/2}\right] \\ &= e^{-\lambda_1^2/2} \cdots e^{-\lambda_n^2/2} \\ &= e^{-|\boldsymbol{\lambda}|^2/2}.\end{aligned}$$

Which is what we expect.

6.5 Green's Function for the Heat equation

We want solve

$$(\dagger) \begin{cases} u_t - \kappa \Delta u = F(\mathbf{x}, t) & (x,t) \in \mathbb{R}^n \times (0, \infty) \\ u(\mathbf{x}, 0) = f(\mathbf{x}) & \mathbf{x} \in \mathbb{R}^n \end{cases}$$

Split ($\dagger$) into

(i) Zero forcing ($F = 0$) with arbitrary initial data.

(ii) Zero inital data ($f = 0$) with arbitrary forcing.

Then we can use superposition to get our solution to ($\dagger$).

To solve, we will use the *heat kernel*,

$$K_t(\mathbf{x}) = \frac{1}{(4\pi\kappa t)^{n/2}} \exp\left[-\frac{|\mathbf{x}|^2}{4\kappa t}\right], \quad t > 0.$$

Proposition. We have that for all $t > 0$,

$$\hat{K}_t(\boldsymbol{\lambda}) = \mathcal{F}_{\mathbf{x}\to\boldsymbol{\lambda}}[K_t(\mathbf{x})] = e^{-\kappa t|\boldsymbol{\lambda}|^2}.$$

Proof.

$$\begin{aligned} \hat{K}_t(\boldsymbol{\lambda}) &= \int e^{-i\boldsymbol{\lambda}\cdot\mathbf{x}} K_t(\mathbf{x}) \mathrm{d}^n\mathbf{x} \\ &= \frac{1}{(4\pi\kappa t)^{n/2}} \int e^{-i\boldsymbol{\lambda}\cdot\mathbf{x}} e^{-\frac{|\mathbf{x}|^2}{4\kappa t}} \mathrm{d}^n\mathbf{x}. \end{aligned}$$

Use the substitution

$$\mathbf{x} = \sqrt{2\kappa t}\mathbf{x}'.$$

This implies that $\mathbf{d}^n\mathbf{x} = (2\kappa t)^{n/2}\mathbf{d}^n\mathbf{x}'$, so

$$\begin{aligned} \hat{K}_t(\boldsymbol{\lambda}) &= \frac{(2\kappa t)^{n/2}}{(4\pi\kappa t)^{n/2}} \int e^{-i\boldsymbol{\lambda}\cdot[\sqrt{2\kappa t}\mathbf{x}']} e^{-|\mathbf{x}'|^2/2} \mathrm{d}^n\mathbf{x}' \\ &= \frac{1}{(2\pi)^{n/2}} \int e^{-i(\sqrt{2\kappa t}\boldsymbol{\lambda})\cdot\mathbf{x}} e^{-|\mathbf{x}|^2/2} \mathrm{d}^n\mathbf{x} \\ &= \int e^{-i(\sqrt{2\kappa t}\boldsymbol{\lambda})\cdot\mathbf{x}} g(\mathbf{x}) \mathrm{d}^n\mathbf{x} \\ &= \hat{g}(\boldsymbol{\lambda}\sqrt{2\kappa t}) = e^{-\kappa t|\boldsymbol{\lambda}|^2}. \quad \square \end{aligned}$$

We can see that $\hat{K}_t(\boldsymbol{\lambda}) \to 1$ as $t \to 0$ so it looks like $K_t(\boldsymbol{\lambda}) \to \delta(\mathbf{x})$.

We can now solve problems (i) and (ii) using the Fourier transform.

Proposition. The solution to (i) is

$$u(\mathbf{x}, t) = K_t * f(\mathbf{x}).$$

Proof. Take Fourier transforms of $u_t - \kappa \Delta u = 0$ and $u(\mathbf{x}, 0) = f(\mathbf{x})$. This gives that

$$\frac{\partial \hat{u}}{\partial t} + \kappa |\boldsymbol{\lambda}|^2 \hat{u} = 0, \quad \hat{u}(\boldsymbol{\lambda}, 0) = \hat{f}(\boldsymbol{\lambda})$$

where $\hat{u} = \hat{u}(\boldsymbol{\lambda}, t) = \mathcal{F}_{\mathbf{x} \to \boldsymbol{\lambda}}[u(\mathbf{x}, t)]$. Solving this ODE gives

$$\hat{u}(\boldsymbol{\lambda}, t) = \hat{u}(\boldsymbol{\lambda}, 0) e^{-\kappa |\boldsymbol{\lambda}|^2 t} = \hat{f}(\boldsymbol{\lambda}) \hat{K}_t(\boldsymbol{\lambda}).$$

By the convolution theorem we get that

$$u(\mathbf{x}, t) = (K_t * f)(\mathbf{x}). \quad \square$$

Proposition. The solution to (ii) is

$$u(\mathbf{x}, t) = \int_0^t \left[\int K_{t-s}(\mathbf{x} - \mathbf{y}) F(\mathbf{y}, s) \mathrm{d}^n \mathbf{y} \right] \mathrm{d}s.$$

Proof. Taking Fourier transforms again, we get that

$$\frac{\partial \hat{u}}{\partial t} + \kappa |\boldsymbol{\lambda}|^2 \hat{u} = \hat{F}(\boldsymbol{\lambda}, t), \quad \hat{u}(\boldsymbol{\lambda}, 0) = 0.$$

Let's solve the ODE by using the integrating factor $e^{\kappa |\boldsymbol{\lambda}|^2 t}$, which gives,

$$\frac{\partial}{\partial t} \left[\hat{u} e^{\kappa |\boldsymbol{\lambda}|^2 t} \right] = \hat{F}(\boldsymbol{\lambda}, t) e^{\kappa |\boldsymbol{\lambda}|^2 t},$$

so after integrating we get

$$\hat{u}(\boldsymbol{\lambda}, t) e^{\kappa |\boldsymbol{\lambda}|^2 t} = \int_0^t \hat{F}(\boldsymbol{\lambda}, s) e^{\kappa |\boldsymbol{\lambda}|^2 s} \mathrm{d}s.$$

So

$$\begin{aligned} \hat{u}(\boldsymbol{\lambda}, t) &= \int_0^t e^{-\kappa |\boldsymbol{\lambda}|^2 (t-s)} \hat{F}(\boldsymbol{\lambda}, s) \mathrm{d}s \\ &= \int_0^t \hat{K}_{t-s}(\boldsymbol{\lambda}) \hat{F}(\boldsymbol{\lambda}, s) \mathrm{d}s. \end{aligned}$$

Hence by the convolution theorem, we get that

$$u(\mathbf{x}, t) = \int_0^t \left[\int K_{t-s}(\mathbf{x} - \mathbf{y}) F(\mathbf{y}, s) \mathrm{d}^n \mathbf{y} \right] \mathrm{d}s. \quad \square$$

So the solution to (†) is

$$u(\mathbf{x}, t) = (K_t * f)(\mathbf{x}) + \int_0^t \left[\int K_{t-s}(\mathbf{x} - \mathbf{y}) F(\mathbf{y}, s) \mathrm{d}^n \mathbf{y} \right] \mathrm{d}s.$$

Let's focus on (ii), and define the function

$$G(\mathbf{x}, t; \mathbf{y}, s) = H(t-s) K_{t-s}(\mathbf{x} - \mathbf{y}).$$

If we set $F(\mathbf{x}, t) = 0$ in $t < 0$, then the solution to (ii) is

$$u(\mathbf{x}, t) = \int_{-\infty}^{\infty} \left[\int G(\mathbf{x}, t; \mathbf{y}, s) F(\mathbf{y}, s) \mathrm{d}^n \mathbf{y} \right] \mathrm{d}s.$$

So G is a Green's function for the heat equation with zero initial data. We can show that

$$\frac{\partial G}{\partial t} - \kappa \Delta G = \delta(t - s)\delta(\mathbf{x} - \mathbf{y})$$

or equivalently,

$$\frac{\partial \hat{G}}{\partial t} + \kappa |\boldsymbol{\lambda}|^2 \hat{G} = e^{-i\boldsymbol{\lambda} \cdot y} \delta(t - s),$$

so $\hat{G}(\boldsymbol{\lambda}, t; \mathbf{y}, s) = H(t - s) e^{-i\boldsymbol{\lambda} \cdot y} \hat{K}_{t-s}(\boldsymbol{\lambda})$.

6.6 Green's Function for Laplace's Equation

We want to find a $G = G(\mathbf{x}, \mathbf{y})$ such that for each fixed $\mathbf{y} \in \mathbb{R}^n$ we have that

$$(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}), \quad G(\mathbf{x}, \mathbf{y}) \to 0$$

as $|\mathbf{x}| \to \infty$. We call this the Free Space Green's function for Laplace's equation. It is helpful to think of $G(\mathbf{x}, \mathbf{y})$ as the electric potential at $\mathbf{x}$ due to point charge at $\mathbf{y}$. By taking the Fourier transform, we get that

$$\hat{G}(\boldsymbol{\lambda} \mathbf{y}) = \frac{e^{-i\boldsymbol{\lambda} \cdot \mathbf{y})}}{|\boldsymbol{\lambda}|^2}$$

Proposition. For $\alpha > 0$, $\mathbf{x} \in \mathbb{R}^n$,

$$\mathcal{F}_{\mathbf{x} \to \boldsymbol{\lambda}}[|\mathbf{x}|^{-\alpha}] = C_{n,\alpha} |\boldsymbol{\lambda}|^{\alpha - n}$$

Proof. Write that $f_\alpha(\mathbf{x}) = |\mathbf{x}|^{-\alpha}$.

$$\begin{aligned} \hat{f}_\alpha(\boldsymbol{\lambda}) &= \int e^{-i\boldsymbol{\lambda} \cdot \mathbf{x}} |\mathbf{x}|^{-\alpha} \mathrm{d}^n \mathbf{x} \\ &= \int e^{-i\boldsymbol{\lambda} \cdot R^T \mathbf{x}'} |R^T \mathbf{x}'|^{-\alpha} \mathrm{d}^n \mathbf{x}' \\ &= \int e^{-i(R\boldsymbol{\lambda}) \cdot \mathbf{x}} |\mathbf{x}|^{-\alpha} \mathrm{d}^n \mathbf{x} \\ &= \hat{f}_\alpha(R\boldsymbol{\lambda}) \end{aligned}$$

Where $R \in SO(n)$ such that $\mathbf{x} = R^T \mathbf{x}'$. So $\hat{f}_\alpha$ is rotation invariant, i.e if $\boldsymbol{\lambda} = |\boldsymbol{\lambda}| \mathbf{m}$, then $\hat{f}_\alpha(\boldsymbol{\lambda})$ is independent of $\mathbf{m}$. If we make the substitution, $|\boldsymbol{\lambda}| \mathbf{x} = \mathbf{x}'$, then get that

$$\begin{aligned} \hat{f}_\alpha(\boldsymbol{\lambda}) &= \int e^{-i|\boldsymbol{\lambda}| \mathbf{m} \cdot \mathbf{x}} |\mathbf{x}|^{-\alpha} \mathrm{d}^n \mathbf{x} \\ &= \left(\int e^{-i\mathbf{m} \cdot \mathbf{x}'} |\mathbf{x}'|^{-\alpha} \mathrm{d}^n \mathbf{x}' \right) |\boldsymbol{\lambda}|^{\alpha - n} \\ &= C_{n,\alpha} |\boldsymbol{\lambda}|^{\alpha - n}. \quad \square \end{aligned}$$

For $n > 2$ let $\alpha = n - 2 > 0$.

$$\mathcal{F}_{\mathbf{x}\to\boldsymbol{\lambda}}[|\mathbf{x}|^{2-n}] = C_{n,n-2}|\boldsymbol{\lambda}|^{-2}$$

i.e.

$$\frac{1}{|\boldsymbol{\lambda}|^2} = \mathcal{F}_{\mathbf{x}\to\boldsymbol{\lambda}}[c_n|\mathbf{x}|^{2-n}].$$

Proposition. For $n > 2$ we have that

$$G(\mathbf{x},\mathbf{y}) = -\frac{1}{(n-2)|S^{n-1}|}\cdot\frac{1}{|\mathbf{x}-\mathbf{y}|^{n-2}}$$

where $|S^{n-1}|$ is the area of S^{n-1}. If $n = 3$ we get that

$$G(\mathbf{x},\mathbf{y}) = -\frac{1}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{y}|}.$$

Proof. Set $F(\mathbf{x}) = c_n|\mathbf{x}|^{2-n}$, so

$$\hat{G}(\boldsymbol{\lambda},\mathbf{y}) = -e^{-i\boldsymbol{\lambda}\cdot y}\hat{F}(\boldsymbol{\lambda}) = -\mathcal{F}_{\mathbf{x}\to\boldsymbol{\lambda}}(F(\mathbf{x}-\mathbf{y}))$$

This implies that

$$G(\mathbf{x},\mathbf{y}) = -\frac{c_n}{|\mathbf{x}-\mathbf{y}|^{n-2}}.$$

Since $\Delta G(\mathbf{x},0) = \delta(\mathbf{x})$, we have that

$$\frac{1}{c_n}\delta(\mathbf{x}) = -\Delta[|\mathbf{x}|^{2-n}].$$

Integrate over $|\mathbf{x}| \leq 1$, and use the divergence theorem to get that

$$\begin{aligned}\frac{1}{c_n} &= -\int_{|\mathbf{x}|\leq 1}\Delta[|\mathbf{x}|^{2-n}]\mathrm{d}^n\mathbf{x} \\ &= -\int_{|\mathbf{x}|=1}\nabla[|\mathbf{x}|^{2-n}|]\cdot\mathrm{d}\mathbf{S} \\ &= (n-2)\int_{|\mathbf{x}|=1}\mathrm{d}S = (n-2)|S^{n-1}|. \quad\square\end{aligned}$$

Proposition. If $n = 2$, a Free Space Green's function for Laplace's equation is

$$G(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\log|\mathbf{x}-\mathbf{y}|.$$

Proof. Take $\mathbf{y} = 0$ *wlog.* Note on $r = |\mathbf{x}| \neq 0$, then $\Delta[\log r] = 0$, since

$$\Delta[\log r] = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\log r\right] = \frac{1}{r}\frac{\partial}{\partial r}[1] = 0.$$

Fix $\varepsilon > 0$, and use the divergence theorem.

$$\begin{aligned}\int_{|\mathbf{x}|<\varepsilon}\Delta[\log r]\mathrm{d}^2\mathbf{x} &= \int_{|\mathbf{x}|=\varepsilon}\nabla(\log r)\cdot\mathrm{d}\mathbf{S} \\ &= \int_0^{2\pi}\frac{\mathbf{e}_r}{\varepsilon}\cdot\mathbf{e}_r\varepsilon\mathrm{d}\theta = 2\pi.\end{aligned}$$

So
$$\Delta\left[\frac{1}{2\pi}\log|\mathbf{x}|\right]=0 \quad \text{for } \mathbf{x}\neq 0.$$
and
$$\int_{|\mathbf{x}|<\varepsilon}\Delta\left[\frac{1}{2\pi}\log|\mathbf{x}|\right]\mathrm{d}^2\mathbf{x}=1 \quad \text{for all } \varepsilon>0.$$
Hence
$$\Delta\left[\frac{1}{2\pi}\log|\mathbf{x}|\right]=\delta(\mathbf{x}). \quad \square$$
We want to solve
$$(\dagger)\begin{cases}\Delta u=F & \mathbf{x}\in\Omega\\ u=0 & \mathbf{x}\in\partial\Omega\end{cases}$$
To do this, find the *Dirichlet* Green's function, defined for each $\mathbf{y}\in\Omega$, by
$$\begin{cases}\Delta\mathcal{G}(\mathbf{x},\mathbf{y})=\delta(\mathbf{x}-\mathbf{y}) & \mathbf{x}\in\Omega\\ \mathcal{G}(\mathbf{x},\mathbf{y})=0 & \mathbf{x}\in\partial\Omega\end{cases}$$
which is such that
$$u(\mathbf{x})=\int_\Omega \mathcal{G}(\mathbf{x},\mathbf{y})F(\mathbf{y})\mathrm{d}^n\mathbf{y}$$
solves $(\dagger)$.

To find $\mathcal{G}(\mathbf{x},\mathbf{y})$, we need to add some harmonic function to the free space Green's function so that $\mathcal{G}(\mathbf{x},\mathbf{y})=0$ when $\mathbf{x}\in\partial\Omega$. We do this by using the method of images. This is done by adding a series of charges in Ω^c, $\mathbf{y}_0,\mathbf{y}_1,\ldots$ so that the total potential due to $\mathbf{y}$ and the point charges vanishes on $\partial\Omega$. For an example, take
$$\Omega=\{\mathbf{x}\in\mathbb{R}^3 : x_3>0\}.$$
We guess that
$$\mathcal{G}(\mathbf{x},\mathbf{y})=G(\mathbf{x},\mathbf{y})+[-G(\mathbf{x},\mathbf{y}_0)].$$
For each $\mathbf{x}\in\Omega$,
$$\begin{aligned}\Delta\mathcal{G}(\mathbf{x},\mathbf{y})&=\delta(\mathbf{x}-\mathbf{y})-\delta(\mathbf{x}-\mathbf{y}_0)\\ &=\delta(\mathbf{x}-\mathbf{y}) \quad \text{since } \mathbf{x}\neq\mathbf{y}_0.\end{aligned}$$
So for $\mathbf{x}\in\partial\Omega$,
$$(\mathbf{x},\mathbf{y})=-\frac{1}{4\pi}[\cdots]=0$$
So the solution to
$$\begin{cases}\Delta u=F & \text{in } \Omega\\ u=0 & \text{on } \partial\Omega\end{cases}$$
is
$$u(\mathbf{x})=\int_\Omega[G(\mathbf{x},\mathbf{y})-G(\mathbf{x},\mathbf{y}_0)]F(\mathbf{y})\mathrm{d}^n\mathbf{y}.$$