Analysis II - Example Sheet 2

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Question 1

Part (a)

Suppose the norms $||\cdot||$ and $||\cdot||'$ are Lipschitz equivalent. Then there exists real numbers A, B positive such that

$$A||x|| \leq ||x||' \leq B||x|| \quad \text{for all } x \in V. \tag{†}$$

Let $r = \frac{1}{A}$ and $R = \frac{1}{B}$. Supose that $x \in B_1'$. Hence we have that $||x||' < 1$, so by (†) we get $A||x|| < 1$ Hence it follows that $||x|| < r$ so $x \in B_r$. This gives $B_r \subseteq B_1'$. Now take some $x \in B_R$. Hence $||x|| < R = \frac{1}{B}$. So again by (†) $||x||' < 1$ so $x \in B_1'$. So $B_r \subseteq B_1' \subseteq B_R$.

Conversely suppose that there exists real numbers r, R such that

$$B_r \subseteq B_1' \subseteq B_R.$$

Fix a $x \in V$. We have that

$$\begin{aligned}
\frac{x}{||x||'} \in B_1' &\implies \frac{x}{||x||'} \in B_r \\
&\implies \left|\left|\frac{x}{||x||'}\right|\right| < r \\
&\implies \frac{1}{r}||x|| < ||x||'.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{Rx}{||x||} \in B_R &\implies \frac{Rx}{||x||} \in B_1' \\
&\implies \left|\left|\frac{Rx}{||x||}\right|\right|' < 1 \\
&\implies ||x||' < \frac{1}{R}||x||.
\end{aligned}$$

Hence $||\cdot||$ and $||\cdot||'$ are Lipschitz equivalent.

Part (b)

Suppose the norms $||\cdot||$ and $||\cdot||'$ are Lipschitz equivalent.

$$A||x|| \leq ||x||' \leq B||x|| \quad \text{for all } x \in V.$$

Suppose now (x_n) is a sequence in V converging to x with respect to $||\cdot||$. Then we have that

$$||x_n - x||' \leq B||x_n - x|| \to B \cdot 0 = 0$$

as $n \to \infty$. Symmetrically we have the converse statement.

Now suppose that $x_n \to x$ with respect to $||\cdot|| \iff x_n \to x$ with respect to $||\cdot||'$. Take α as the identity map from $(V, ||\cdot||)$ to $(V, ||\cdot||')$. By the hypothesis it is continuous, hence there is some $\delta > 0$ such that

$$||x|| < \delta \implies ||x||' < 1$$

and taking the inverse map there is some ε such that

$$||x||' < 1 \implies ||x|| < \varepsilon.$$

Hence by part (a) the norms are Lipschitz equivalent.

Part (c)

Let's show that $||\cdot|| + |\varphi(\cdot)|$ defines a norm on V. Firstly,

$$||0|| + |\varphi(0)| = 0 + 0 = 0$$

since φ is linear. The norm is clearly positive definite since $||x|| = 0 \iff x = 0$. φ is linear, so the norm has linearity in scalar multiplication. We're just left to prove the triangle inequality.

$$\begin{aligned} ||x+y|| + |\varphi(x+y)| &\leq ||x|| + ||y|| + |\varphi(x) + \varphi(y)| \\ &\leq ||x|| + |\varphi(x)| + ||y|| + |\varphi(y)|. \end{aligned}$$

So this does define a norm on V.

Suppose that φ is not continuous. So there exists a point $x \in V$ and a sequence such that $x_n \to x$ and $\varphi(x_n) \to y$ with $y \neq \varphi(x)$ (with the first limit being taken with respect to the $||\cdot||$ norm). We'll show the norms are not Lipschitz equivalent using (c). Call our newly define norm $||\cdot||'$. Then

$$||x_n - x||' = ||x_n - x|| + |\varphi(x_n - x)|. \qquad = ||x_n - x|| + |\varphi(x_n) - \varphi(x)|$$

But as $n \to \infty$ the first term vanishes and the second term doesn't go to zero since $\varphi(x) \neq \varphi(y)$ as limits are unique. Hence x_n doesn't converge to x in this new norm, so by (c) the norms are not Lipschitz equivalent.

Part (d)

For this part we'll assume the Axiom of Choice so we can construct a basis for any vector space through Zorn's lemma. Let V be an infinite dimensional vector space. Let B be a basis for V, take (e_n) to be some countable subset of the basis vectors and define a $\varphi : V \to \mathbb{R}$ on this vector space by

$$\varphi(e_n) = n, \quad \varphi = 0 \text{ on other basis}$$

so we can then extend φ to make it linear. φ is not bounded on the unit ball so not continuous. Then by (c) we have at least two non-Lipschitz equivalent norms, hence if we have a vector space with exactly one norm up to Lipschitz equivalence, V *must* be finite-dimensional.

Question 2

Suppose that $f : X \to X'$ is continuous. Let V be some open set in X' and let's show that $X \setminus f^{-1}(V)$ is closed instead. Take some sequence (x_n) in $X \setminus f^{-1}(V)$ with $x_n \to x$. So it is sufficient to show that $x \notin f^{-1}(V)$.

$x_n \notin f^{-1}(V)$, so $f(x_n) \in X' \setminus V$. Since f is continuous, $f(x_n) \to f(x)$ and since V is open, $X' \setminus V$ is closed, therefore $f(x) \in X' \setminus V$. So $f(x) \notin V$ so $x \notin f^{-1}(V)$, hence $f^{-1}(V)$ is open.

Question 3

Since we're working over the vector space $\mathbb{R}^n$ it is enough to show that X is closed and bounded. If we let f be the function describing the Euclidean metric (which is continuous), then it has a bounded image, so X is bounded. Now we're left to show X is closed. Suppose not. Then there is a point x which is a limit point of X but not in X, so we can make $||y-x||_2$ as small as we like for $y \in X$. Hence the function

$$\begin{aligned} f : & X \to \mathbb{R} \\ & y \to \frac{1}{||y-x||_2} \end{aligned}$$

is clearly continuous but unbounded. So X must be closed, so it is compact.

For a general metric space, (X, d).

Question 4

Part (a)

Let (x_k) be a sequence in X, then since the open balls cover X and there are finitely many of then, there must be a ball centred at x_i that contains infinitely many terms of the sequence.

Question 5

(i) Not topological. We can relate the open sets $(0,1)$ and $(1,\infty)$ by a the transformation $\frac{1}{x}$, which preserves the topology, but not the bo undedness.

(ii) Topological. A set is closed $\iff$ it's complement is open, so the closed-ness of a set is completely determined by not being in the collection of open sets. Cons

(iii)