

Analysis II

Notes made by Finley Cooper

23rd October 2025

Contents

1	Uniform Convergence	3
1.1	Differentiation and uniform convergence	6
2	Series of functions	8
2.1	Power series	10
3	Uniform continuity and Riemann integrability	12
3.1	Uniform continuity	12

1 Uniform Convergence

For a subset $E \subseteq \mathbb{R}$, have a sequence $f_n : E \rightarrow \mathbb{R}$. What does it mean for the sequence (f_n) to converge? The most basic notion for any $x \in E$ require that the sequence of real numbers $f_n(x)$ to converge in \mathbb{R} . If this holds we can defined a new function $f : E \rightarrow \mathbb{R}$ by setting each value to the limit of the function.

Definition. (Pointwise limit) We say that (f_n) converges *pointwise* if for all x in its domain we have that

$$f(x) = \lim_{n \rightarrow \infty} f_n(x)$$

converges. We write that $f_n \rightarrow f$ pointwise.

Are properties such as continuity, differentiability integrability, preserved in the limit? We'll use an example to show that continuity is not preserved.

We can see this by taking a sequence of functions which converge to a step function by taking tighter and tighter curvers which get steeper and steeper. For example take,

$$f_n : [-1, 1] \rightarrow \mathbb{R}, \quad f_n(x) = x^{\frac{1}{2n+1}}.$$

So in the limit we get that

$$f_n(x) \rightarrow f(x) = \begin{cases} 1 & 0 < x \leq 1 \\ 0 & x = 0 \\ -1 & -1 \leq x < 0 \end{cases}$$

which is not continious.

For an example where integability is not preserved, let q_1, q_2, q_3, \dots be an enumeration of $\mathbb{Q} \cap [0, 1]$ and define

$$f_n(x) = \begin{cases} 1 & x \in \{q_1, \dots, q_n\} \\ 0 & \text{otherwise} \end{cases}$$

so we get $f_n(x)$ continious everywhere on $[0, 1]$ apart from a finite number of points, then f_n is integrable on $[0, 1]$ (IA Analysis I). But,

$$\lim_{n \rightarrow \infty} f_n(x) = \mathbf{1}_{\mathbb{Q}}(x)$$

which we know is not integrable.

If $f_n \rightarrow f$ pointwise, f_n integrable, f integrable, does it follow that $\int f_n \rightarrow \int f$? (Spoiler: No) For example take f_n to be a 'spike' with height n and width $\frac{2}{n}$, concretely,

$$f_n(x) = \begin{cases} n^2 x & 0 \leq x \leq \frac{1}{n} \\ n^2(\frac{2}{n} - x) & \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \text{otherwise} \end{cases}$$

So the integral of f_n over $[0, 1]$ is 1, but we can see that f_n converges pointwise to zero. So $\int_0^1 f_n \rightarrow 1$ but $\int_0^1 f \rightarrow 0$.

So we need a better (stronger) notion for the convergence of a sequence of functions. We can't use something too strong, such as $f_n \rightarrow f$ if f_n is eventually f for large enough n . We've got to find something inbetween. This is uniform convergence.

Definition. (Uniform convergence) Let $f_n, f : E \rightarrow \mathbb{R}$, for $n \in \mathbb{N}$. We say that (f_n) converges *uniformly* on E if the following holds. For all $\varepsilon > 0$, $\exists N = N(\varepsilon)$ such that for every $n \geq N$ and for every $x \in E$ we have that $|f_n(x) - f(x)| < \varepsilon$.

Remark. This statement is equivalent to the following,

$$\forall \varepsilon > 0, \exists N = N(\varepsilon), \text{ s.t. } \forall n \geq N, \sup_{x \in E} |f_n(x) - f(x)| < \varepsilon.$$

Comparing this to pointwise convergence, $\forall x \in E$ and $\forall \varepsilon > 0$, $\exists N = N(\varepsilon, x)$ such that $n \geq N \implies |f_n(x) - f(x)| < \varepsilon$. So we can change our N value for each individual x . However we can't in uniform convergence, which makes this a stronger statement.

Hence we see Uniform convergence \implies Pointwise convergence. This gives a nice way to compute uniform limits. If a function doesn't converge pointwise then we know it doesn't converge uniformly. If we know a sequence of functions converges pointwise to some limit function, then this function must be the limit of the uniform limit, if it exists.

Definition. (Uniformly Cauchy) Let $f_n : E \rightarrow \mathbb{R}$ be a sequence of functions. We say that (f_n) is *uniformly Cauchy* on E if

$$\forall \varepsilon > 0, \exists N = N(\varepsilon) \text{ s.t. } n, m \geq N \implies \sup_{x \in E} |f_n(x) - f_m(x)| < \varepsilon.$$

Theorem. (Cauchy criterion for uniform convergence) Let (f_n) be a sequence of functions with $f_n : E \rightarrow \mathbb{R}$. The (f_n) converges uniformly on E if and only if (f_n) is uniformly Cauchy on E .

Proof. Suppose that (f_n) is a sequence converging uniformly in E to some function f . Given some $\varepsilon > 0$, there is a N such that $\sup_{x \in E} |f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$. By the triangle inequality $\forall x \in E$, picking $n, m \geq N$,

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |f_n(x) - f(x)| + |f_m(x) - f(x)| \\ &\leq \sup_E |f_n - f| + \sup_E |f_m - f| \\ &< \varepsilon + \varepsilon \\ &< 2\varepsilon \end{aligned}$$

hence (f_n) is uniformly Cauchy.

For the converse, suppose that (f_n) is a sequence uniformly Cauchy in E . Then the sequence of real numbers $(f_n(x))$ is Cauchy so by IA Analysis I, this sequence has a limit, call it $f(x)$. So (f_n) converges pointwise to f . Now we check that $f_n \rightarrow f$ uniformly on E . Pick any $\varepsilon > 0$ and note that by the hypothesis that (f_n) is uniformly Cauchy, there exists a number N such that for all $n, m \geq N$ we have $|f_n(x) - f_m(x)| < \varepsilon$. Fix $n \geq N$ and let $m \rightarrow \infty$ in this. So since $f_m(x)$ converges to $f(x)$ pointwise, we get that

$$|f_n(x) - f(x)| \leq \varepsilon$$

hence (f_n) converges uniformly in E . □

For an example consider $f_n : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f_n(x) = \frac{x}{n}$. So $f_n \rightarrow 0$ pointwise on \mathbb{R} . But $|f_n - 0|$ is unbounded so the supremum doesn't exist so f_n does not converge uniformly on \mathbb{R} . However if we restrict the domain of f_n to $[-a, a]$ then we get uniform convergence.

Theorem. (Continuity is preserved under uniform limits) Let $f_n, f : [a, b] \rightarrow \mathbb{R}$. Suppose that (f_n) converges to f uniformly on $[a, b]$. If $x \in [a, b]$ is such that f_n is continuous at x for all $n \in \mathbb{N}$, then f is continuous at x .

Proof. Let $\varepsilon > 0$ by uniform convergence of $f_n \rightarrow f$ we have some $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\sup_{y \in [a, b]} |f_n(y) - f(y)| < \varepsilon$$

. By continuity of f_N at x we have $\delta = \delta(N, x, \varepsilon) > 0$ s.t. $y \in [a, b], |x - y| < \delta \implies |f_N(y) - f_N(x)| < \varepsilon$.

Then $y \in [a, b], |x - y| < \delta$ we have

$$\begin{aligned} |f(y) - f(x)| &\leq |f(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)| \\ &< \varepsilon + \varepsilon + \varepsilon \\ &< 3\varepsilon \end{aligned}$$

Hence f is continuous at x . □

It is instructive to see where this proof goes wrong if we only assume that (f_n) converges to f pointwise.

Corollary. (Uniform limits of continuous functions are continuous) If $f_n, f : [a, b] \rightarrow \mathbb{R}$, and $f_n \rightarrow f$ uniformly on $[a, b]$ and if f_n is continuous on $[a, b]$ for every n then f is continuous on $[a, b]$.

Proof. Immediate from the previous theorem. □

From now on we will denote $C([a, b]) = \{f : [a, b] \rightarrow \mathbb{R} : f \text{ is continuous on } [a, b]\}$.

Theorem. Let (f_n) be a uniformly Cauchy sequence of functions in $C([a, b])$ then it converges to a function in $C([a, b])$.

Proof. Trivial from our theorems earlier proved. □

Theorem. (Uniform convergence implies convergence of integrals) For $f_n, f : [a, b] \rightarrow \mathbb{R}$ be such that f_n, f are bounded and integrable on $[a, b]$. If $f_n \rightarrow f$ uniformly on $[a, b]$ then

$$\int_a^b f_n(x) dx \rightarrow \int_a^b f(x) dx$$

Remark. The assumption that f is integrable is redundant. We will see later that integrability of f_n implies that f is integrable if $f_n \rightarrow f$ uniformly

Proof.

$$\begin{aligned} \left| \int_a^b f_n(x) dx - \int_a^b f(x) dx \right| &= \left| \int_a^b f_n(x) - f(x) dx \right| \\ &\leq \int_a^b |f_n(x) - f(x)| dx \\ &\leq \sup_{x \in [a, b]} |f_n(x) - f(x)| (b - a) \rightarrow 0 \end{aligned}$$

by assumption.

1.1 Differentiation and uniform convergence

This is more subtle if $f_n \rightarrow f$ uniformly on some interval and if f_n are differentiable it does not follow that

- (i) That f is differentiable.
- (ii) Even if f is differentiable that $f'_n(x) \rightarrow f'(x)$.

We can view this in the example of $f_n : [-1, 1] \rightarrow \mathbb{R}$ with $f_n(x) = |x|^{1+\frac{1}{n}}$. Hence we have that

$$\lim_{x \rightarrow 0} \frac{f_n(x) - f_n(0)}{x} = \lim_{x \rightarrow 0} \operatorname{sgn}(x^{\frac{1}{n}}) = 0$$

So f_n is differentiable at 0 with $f_n(0) = 0$ and clearly f_n is differentiable everywhere where $x \neq 0$ too. We can check that $f_n \rightarrow |x|$ uniformly. But $|x|$ is not differentiable at $x = 0$.

Now consider the example $f_n : \mathbb{R} \rightarrow \mathbb{R}$ with

$$f_n(x) = \frac{\sin(nx)}{\sqrt{n}}.$$

So $f_n \rightarrow 0$ uniformly on \mathbb{R} . So we have a differentiable limit but $f'_n(x) = \sqrt{n} \cos(nx)$ which is not convergent as $n \rightarrow \infty$. So we don't have $f'_n(x) \rightarrow f'(x)$ pointwise on \mathbb{R} .

Theorem. Let $f_n : [a, b] \rightarrow \mathbb{R}$ be a sequence of differentiable functions (at the end points this means that the one-sided derivative exists). Suppose that:

- (i) $f'_n \rightarrow g$ uniformly for some function $g : [a, b] \rightarrow \mathbb{R}$.
- (ii) For some $c \in [a, b]$ the sequence $(f_n(c))$ converges.

Then (f_n) converges uniformly to some function $f : [a, b] \rightarrow \mathbb{R}$ where f is differentiable everywhere on $[a, b]$ and $f'(x) = g(x)$ for all $x \in [a, b]$.

This proves that

$$\left(\lim_{n \rightarrow \infty} f_n \right)' = \lim_{n \rightarrow \infty} f'_n$$

i.e. we can exchange the derivative and limit in this case.

Remark. If we assume that f'_n are continuous, then the proof is more straightforward and can be based on the fundamental theorem of calculus.

Proof. By the mean value theorem applied to the difference $(f_n - f_m)$ we have that for any $x \in [a, b]$

$$\begin{aligned} f_n(x) - f_m(x) &= f_n(c) - f_m(c) + (x - c)(f_n - f_m)'(x_{n,m}) \\ \implies |f_n(x) - f_m(x)| &\leq |f_n(c) - f_m(c)| + (b - a)|f_n'(x_{n,m}) - f_m'(x_{n,m})| \\ \implies \sup |f_n - f_m| &< |f_n(c) - f_m(c)| + (b - a) \sup |f_n' - f_m'| \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. So (f_n) is uniformly Cauchy and hence there is an $f : [a, b] \rightarrow \mathbb{R}$ s.t. $f_n \rightarrow f$ uniformly.

For the next part fix some $y \in [a, b]$. Define

$$h(x) = \begin{cases} \frac{f(x) - f(y)}{x - y} & x \neq y \\ g(y) & x = y \end{cases}$$

Now we only have to establish that h is continuous at y to show that f is differentiable at y with $f'(y) = g(y)$. Let

$$h_n(x) = \begin{cases} \frac{f_n(x) - f_n(y)}{x - y} & x \neq y \\ f_n'(y) & x = y \end{cases}$$

then since f_n is differentiable at y we see that h_n is continuous on $[a, b]$. The pointwise limit of (h_n) is h almost by definition since $f_n' \rightarrow g$ at $x = y$. Since the uniform limit of sequence of continuous functions is continuous, we just need to show that (h_n) is uniformly Cauchy on $[a, b]$ since the limit must be h since it converges pointwise to h .

$$h_n(x) - h_m(x) = \begin{cases} \frac{(f_n - f_m)(x) - (f_n - f_m)(y)}{x - y} & x \neq y \\ (f_n' - f_m')(y) & x = y \end{cases}.$$

By the mean value theorem,

$$\begin{aligned} h_n(x) - h_m(x) &= \begin{cases} (f_n - f_m)'(x_{n,m}) \text{ for some } x_{n,m} \text{ between } x \text{ and } y & x \neq y \\ (f_n - f_m)'(y) & x = y \end{cases} \\ \sup_{[a,b]} |h_n - h_m| &\leq \sup_{[a,b]} |f_n' - f_m'| \rightarrow 0 \end{aligned}$$

as $n, m \rightarrow \infty$. So (h_n) is uniformly Cauchy so we're done. \square

Remark. f_n' need not be continuous consider

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

the f is differentiable on $[-1, 1]$ with $f'(x)$ not continuous at $x = 0$ and we can take $f_n(x) = f(x)$ for all n (or $f_n(x) = f(x) + \frac{x}{n}$).

We have a shorter proof of the above theorem, assuming that (f_n') are continuous in addition to the hypothesis. For any $x \in [a, b]$ we can write

$$f_n(x) = f_n(c) + \int_c^x f_n'(t) dt$$

by FTC. Then

$$\begin{aligned} |f_n(x) - f_m(x)| &= \left| f_n(c) - f_m(c) + \int_c^x (f'_n(t) - f'_m(t)) dt \right| \\ &\leq |f_n(c) - f_m(c)| + \sup_{t \in [a, b]} |f'_n(t) - f'_m(t)| (b - a) \rightarrow 0 \end{aligned}$$

as $n, m \rightarrow \infty$. So (f_n) is uniformly Cauchy, hence converges uniformly.

Note that

$$\int_c^x f'_n(t) dt \rightarrow \int_c^x g(t) dt$$

by uniform convergence of $f'_n \rightarrow g$ which implies g is continuous and hence also integrable. We can let $n \rightarrow \infty$ the first equation for $f_n(x)$ which gives that

$$f(x) = f(c) + \int_c^x g(t) dt$$

So we can take the derivative of both sides giving that $f'(x) = g(x) = \lim f'_n(x)$. \square

Proposition. If $f_n, g_n : E \rightarrow \mathbb{R}$ with $f_n \rightarrow f$ uniformly on E and $g_n \rightarrow g$ uniformly on E then $f_n + g_n$ converges uniformly to $f + g$ on E , and if $h : E \rightarrow \mathbb{R}$ is a bounded function then $hf_n \rightarrow hf$ uniformly on E also.

Proof. On the example sheet.

2 Series of functions

Definition. (Convergence of a series of functions) Let $g_n : E \rightarrow \mathbb{R}$ for $n \in \mathbb{N}$ then write

$$f_n = \sum_{j=1}^n g_j$$

defined pointwise. Then we say that that,

- (i) The series of functions $\sum_{n=1}^{\infty} g_n$ is convergent at a point $x \in E$ if the sequence of partial sums $(f_n(x))$ converges.
- (ii) The series of functions $\sum_{n=1}^{\infty} g_n$ uniformly on E if the sequence (f_n) converges uniformly on E .
- (iii) $\sum_{n=1}^{\infty} g_n$ converges absolutely at $x \in E$ if the series $\sum_{n=1}^{\infty} |g_n(x)|$ converges.
- (iv) $\sum_{n=1}^{\infty} g_n$ converges absolutely uniformly on E if $\sum_{n=1}^{\infty} |g_n|$ converges uniformly on E .

We know from IA Analysis I that absolute convergence \implies convergence for a sequences in \mathbb{R} . From this we have that if $\sum_{n=1}^{\infty} g_n$ converges absolutely at a point $x \in E$ then $\sum_{n=1}^{\infty} g_n$ converges at x . Similiar to this we have the following proposition relating absolute uniform convergence and uniform convergence.

Proposition. (Absolute uniform convergence implies uniform convergence) If $g_n : E \rightarrow \mathbb{R}$ and if $\sum_{n=1}^{\infty} g_n$ converges absolutely uniformly on E then $\sum_{n=1}^{\infty} g_n$ converges uniformly on E .

Proof. Let $f_n = \sum_{i=1}^n g_i$ Then

$$\begin{aligned} |f_n(x) - f_m(x)| &= \left| \sum_{i=m+1}^n g_i(x) \right| \\ &= \sum_{i=m+1}^n |g_i(x)| = h_n(x) - h_m(x), \quad \text{where } h_n(x) = \sum_{i=1}^n |g_i(x)| \\ \sup_{x \in E} |f_n(x) - f_m(x)| &\leq \sup_{x \in E} |h_n(x) - h_m(x)| \rightarrow 0 \end{aligned}$$

as $n, m \rightarrow \infty$ so (f_n) converges uniformly on E . □

Remark. Uniform convergence and absolute pointwise convergence aren't enough to conclude that the series convergence absolutely uniformly.

Theorem. (Weierstrass M-test) Let $g_n : E \rightarrow \mathbb{R}$ be a sequence of functions and suppose that $\exists M_n$ such that

$$\sup_{x \in E} |g_n(x)| \leq M_n$$

and that

$$\sum_{n=1}^{\infty} M_n$$

converges. Then

$$\sum_{n=1}^{\infty} g_n$$

converges absolutely uniformly on E .

Proof. Let

$$h_n(x) = \sum_{j=1}^n |g_j(x)|$$

for $n > m$,

$$\begin{aligned} h_n(x) - h_m(x) &= \sum_{j=m+1}^n |g_j(x)| \leq \sum_{j=m+1}^n M_j = \sum_{j=1}^n M_j - \sum_{j=1}^m M_j \\ \implies \sup_{x \in E} |h_n(x) - h_m(x)| &\leq \left| \sum_{j=1}^n M_j - \sum_{j=1}^m M_j \right| \quad \forall n, m \end{aligned}$$

by assumption the right hand side $\rightarrow 0$ since $\sum_{j=1}^{\infty} M_j$ is convergent, hence (h_n) is uniformly Cauchy hence converges uniformly.

2.1 Power series

We'll now specialise to the case where $g_n(x) = c_n(x - a)^n$ for $a, c_n \in \mathbb{R}$. This gives a real power series.

Theorem. (Radius of convergence) Let $\sum_{n=0}^{\infty} c_n(x - a)^n$ be a real power series then there exists a $R \in [0, \infty]$ called the *radius of convergence* of the power series such that

- (i) If $|x - a| < R$ then the power series converges absolutely.
- (ii) If $|x - a| > R$ then the power series diverges.
- (iii) R is given by

$$R = \frac{1}{\limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}}}$$

where if the limit is zero, then $R = \infty$.

- (iv) For any $r \in (0, R)$ we have the power series converges uniformly on $[a - r, a + r]$, in particular the function that the power series converges to is continuous on $(a - R, a + R)$.

Proof. The proof for (i), (ii), and (iii) are in IA Analysis I. We'll just prove (iv). Note first that the power series converges absolutely at $x = a + r$ i.e. we have that

$$\sum_{n=0}^{\infty} |c_n| r^n$$

is convergent. Since $|c_n(x - a)^n| \leq |c_n| r^n$ for any $x \in [a - r, a + r]$ we can apply the Weierstrass M -test with $M_n = |c_n| r^n$ to conclude that the series

$$\sum_{n=0}^{\infty} c_n(x - a)^n \rightarrow f$$

converges absolutely uniformly on $[a - r, a + r]$. It follows that f is continuous. at any point in $(a - R, a + R)$ by picking r small enough.

Remark. (Boundary behaviour. Let

$$f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n$$

with power series boundary R with $0 < R < \infty$. If the power series converges at one of the boundary points of the interval of convergence, say at $x = a + R$ i.e. $\sum_{n=0}^{\infty} c_n R^n$ is convergent then

$$\lim_{x \rightarrow a+R} f(x) = \sum_{n=0}^{\infty} c_n R^n$$

so f extends to $(a - R, a + R]$ as a continuous function.

Moreover, under the same conditions that $\sum_{n=0}^{\infty} c_n R^n$ converges we have that the series converges uniformly on $[a - r, a + r]$ for any $r \in (0, R)$. Same discussion applies at the endpoint $a - R$.

Theorem. (Differentiation of power series) Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a power series with radius of convergent $R > 0$. Let

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

defined on $(a-R, a+R)$. We have the following

(i) The derived series

$$\sum_{n=1}^{\infty} n c_n(x-a)^{n-1}$$

has radius of convergent R .

(ii) f is differentiable on $(a-R, a+R)$ with

$$f'(x) = \sum_{n=1}^{\infty} n c_n(x-a)^{n-1} \quad \forall x \in (a-R, a+R)$$

Proof. Before we prove the theorem let's give a definition we've seen slightly before.

Definition. If (a_n) is a sequence of reals let

$$p_n = \sup\{a_m : m \geq n\}$$

$$q_n = \inf\{a_m : m \geq n\}.$$

Then we define

$$\limsup_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} p_n$$

$$\liminf_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} q_n.$$

which exists in $\mathbb{R} \cup \{\infty\}$ since (q_n) and (p_n) are monotone.

$$\limsup_{n \rightarrow \infty} (n|c_n|)^{\frac{1}{n}} = \limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}}$$

since we have that $\lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1$. So we have (i).

Define $f_n(x) = \sum_{j=0}^n c_j(x-a)^j$ is clearly differentiable on \mathbb{R} with $f'_n(x) = \sum_{j=1}^n j c_j(x-a)^{j-1}$. By (i) we have that $f'_n(x)$ converges uniformly on $[a-r, a+r]$ for all $r < R$ and $f_n(a) = c_0 \forall n$ so $(f_n(a))$ converges. So the limit is differentiable in $[a-r, a+r]$, with

$$f'(x) = \lim_{n \rightarrow \infty} f'_n(x) = \lim_{n \rightarrow \infty} \sum_{j=1}^n j c_j(x-a)^{j-1}$$

□

If we have a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ we say the power series converges *locally uniformly* on the interval of convergence $(a-R, a+R)$ i.e. for all $0 < r < R$ the power series converges uniformly on $[a-r, a+r]$.

Remark. By repeatedly applying the above theorem we get that if $f(x) = \sum_{n=1}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$ then f is differentiable to any order $k \in \mathbb{N}$ in $(a-R, a+R)$ and the k th derivative is given by

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1) c_n (x-a)^{(n-k)}.$$

Plugging in $x = a$ we get that

$$c_n = \frac{f^{(k)}(a)}{k!}.$$

This says that f is uniquely determined by its values in an arbitrarily small interval around the point $x = a$ since that's all we need to capture it's derivatives and form its power series.

3 Uniform continuity and Riemann integrability

3.1 Uniform continuity

Definition. (Uniform continuity) Let $E \subseteq \mathbb{R}$ and let $f : E \rightarrow \mathbb{R}$. We say that f is *Uniformly continuous* on E if $\forall \varepsilon > 0$ there exists a $\delta > 0$ such that $\forall x, y \in E$ we have that

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

This differs from our usual definition of continuity. We require some δ to work for *any* $x, y \in E$ given some ε , rather than picking a δ for each ε and x value. Clearly uniform continuity implies continuity but the converse is not true. For an example consider $f(x) = \frac{1}{x}$ on $(0, 1)$. Clearly continuous at each x , but not uniformly continuous since it gets too steep around 0.

Not even boundedness and continuity is enough for uniform continuity, consider $\sin(\frac{1}{x})$, take $x_n = \frac{1}{2n\pi}$ and $y_n = \frac{1}{2n+\frac{1}{2}}\pi$ then $|f(x) - f(y)| = 1$, so no δ works, we can always choose an n large enough.

Theorem. Let $[a, b]$ be a closed, bounded interval and $f : [a, b] \rightarrow \mathbb{R}$ a continuous function. Then f is uniformly continuous.

Proof. Argue by contradiction. Suppose that f is not uniformly continuous, so there exists an $\varepsilon > 0$ such that for all $\delta > 0$ there is a pair of points $x, y \in [a, b]$ such that $|y - x| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$. Now let $\delta_n = \frac{1}{n}$, so we get a sequence of functions x_n and y_n satisfying the above for each δ_n . By Bolzano-Weiestrass, there exists a subsequence (x_{n_k}) that converges to a point $x \in [a, b]$.

$$|x - y_{n_k}| \leq |x - x_{n_k}| + |x_{n_k} - y_{n_k}| \leq |x - y_{n_k}| + \frac{1}{n_k} \rightarrow 0 \text{ as } n \rightarrow \infty$$

By the continuity of f at x we get $f(x_{n_k}) \rightarrow f(x)$ and $f(y_{n_k}) \rightarrow f(x)$. But this is contradiction since $f(x)$ and $f(y_{n_k})$ are always separated by some distance ε . \square

We can actually strengthen this theorem.

Theorem. Let $f : [a, b] \rightarrow \mathbb{R}$ where $-\infty < a < b < \infty$ be any function. Suppose that there is a collection \mathcal{C} of open intervals $I \subseteq \mathbb{R}$ such that if

$$F = [a, b] \setminus \bigcup_{I \in \mathcal{C}} I$$

then f is continuous at every point in F (i.e. the set of discontinuities is contained in the union). Then $\forall \varepsilon > 0 \exists \delta > 0$ s.t. $x \in F, y \in [a, b]$, with $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$.

Proof. Same as above, using the fact that F is *closed* so it contains all of its limit points.