

# Topological Spaces

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# 1 Topologies

## 1.1 Definitions

We denote  $\mathcal{P}(X)$  as the power set of  $X$ .

**Definition.** (Topology) Let  $X$  be a set. A *topology* on  $X$  is a collection of sets  $T \subseteq \mathcal{P}(X)$  such that

- (i)  $\emptyset, X \in T$ ,
- (ii)  $T$  is closed under (possibly uncountable) unions.
- (iii)  $T$  is closed under finite intersections.

A set  $X$  with a topology  $T$  is called a *topological space* of  $X$ . An element of  $X$  is called a *point* and elements of  $T$  are called *open sets*. If  $x \in U \in T$  we say  $U$  is an open neighbourhood of  $x$ . Strictly we should always denote  $(X, T)$  for a topological space, but when  $T$  is clear, we just write  $X$  for the topological space.

**Definition.** (Continuity) If  $(X, T_X)$  and  $(Y, T_Y)$  are topological spaces then a function  $f : X \rightarrow Y$  is called *continuous* if for  $U \in T_Y$ ,  $f^{-1}(U) \in T_X$ .

**Definition.** (Homeomorphism) A function  $f : (X, T_X) \rightarrow (Y, T_Y)$  is a *homeomorphism* if it is continuous and has a continuous inverse.

**Definition.** If  $T \subseteq T'$  are topologies on  $X$  then we say that  $T$  is *coarser* and  $T'$  is *finer*. The identity function  $d : (X, T) \rightarrow (X, T')$  is continuous.

## 1.2 Topologies from metrics

If  $(X, d)$  is a metric space, recall that a subset  $U \subseteq X$  is called *open* if for every point  $x \in U$  there exists a  $\varepsilon > 0$  such that  $B_\varepsilon(x) \subseteq U$ .

**Proposition.** If  $T_d$  is the subset of  $X$  which are open under the metric  $d$ , then  $(X, T_d)$  is a topological space. We will call this the topology on  $X$  induced by the metric  $d$ .

*Proof.* Tautologically we have that  $\emptyset \in T_d$ . Clearly we have that  $X \in T_d$  too. Let  $\{U_\alpha\}_{\alpha \in I}$  be a collection of open sets in  $T_d$  with a (possibly uncountable) index set  $I$ . Let

$$x \in \bigcup_{\alpha \in I} U_\alpha.$$

Then  $x \in U_\beta$  for some  $\beta \in I$ , so  $U_\beta$  is open hence there exists a  $\varepsilon > 0$  such that  $B_\varepsilon \subseteq U_\beta \subseteq \bigcup_{\alpha \in I} U_\alpha$ , hence  $\bigcup_{\alpha \in I} U_\alpha$  is open.

Now suppose that  $I$  is finite, and  $x \in \bigcap_{\alpha \in I} U_\alpha$ . For each  $\alpha$  there exists a  $\varepsilon_\alpha > 0$  such that  $B_{\varepsilon_\alpha}(x) \subseteq U_\alpha$ . Take  $\varepsilon = \inf_{\alpha \in I} \varepsilon_\alpha$ , so  $B_\varepsilon(x) \subseteq B_{\varepsilon_\alpha}(x) \subseteq U_\alpha$  for all  $\alpha$ , hence we have that  $B_\varepsilon(x) \subseteq \bigcap_{\alpha \in I} U_\alpha$  so it's open. Hence  $T$  is a topology.  $\square$

Now we have lots of examples we can use for topological spaces. For example we have that topology induced by the Euclidean metric on  $\mathbb{R}^d$  which we will call the Euclidean topology. For any  $X \subseteq \mathbb{R}^d$  we can have a topology induced by the Euclidean metric too, like  $\mathbb{Q}$ ,  $[0, 1]$ ,  $(0, 1)$ .

**Proposition.** If we have two metric spaces  $(X, d_X)$ ,  $(Y, d_Y)$  and we have  $f : X \rightarrow Y$ , the  $f$  is continuous in the metric space sense if and only if it is continuous in the topological space sense (with the topologies induced by the metric  $d_X$  and  $d_Y$  respectively).

*Proof.* Let  $f : X \rightarrow Y$  be continuous in the metric space sense. Let  $U$  be an open set in  $T_{d_Y}$  so we need to show that  $f^{-1}(U)$  is open. Let  $x \in f^{-1}(U)$ , so  $f(x) \in U$ . Hence there exists an  $\varepsilon > 0$  such that  $B_\varepsilon(f(x)) \subseteq U$ . So since  $f$  is continuous there exists a  $\delta > 0$  such that if  $d_X(x, x') < \delta$ , then  $d_Y(f(x), f(x')) < \varepsilon$ . Hence  $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$ . So  $B_\delta(x) \subseteq f^{-1}(U)$ , hence  $f^{-1}(U)$  is open.

Now let's do the converse and suppose that  $f : X \rightarrow Y$  is continuous in the topological sense. Fix some  $x \in X$  and  $\varepsilon > 0$ . Consider  $B_\varepsilon(f(x))$  which is open in  $Y$ . Then  $f^{-1}(B_\varepsilon(f(x)))$  is in  $T_{d_X}$ . It contains  $x$  so there exists a  $\delta > 0$  such that  $x \in B_\delta(x)$ , so

$$d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \varepsilon.$$

So  $f$  is continuous in the metric sense. □

**Definition.** Let  $(X, T)$  be a topological space and  $x_1, x_2, \dots \in X$  say. We say that  $x_n$  converges to  $x$  if for every open neighbourhood  $U$  of  $x$  there exists a  $N$  such that  $x_n \in U$  for all  $n \geq N$ .

**Proposition.** If  $(X, d)$  is a metric space with topology  $T_d$  then a sequence  $(x_n)$  converges in the metric sense if and only if it converges in the topological sense.

*Proof.* Suppose it converges in the metric sense to  $x$ . Then for all  $\varepsilon > 0$  there exists a  $N$  such that for all  $n \geq N$  we have that  $x_n \in B_\varepsilon(x)$ . If  $U$  is a neighbourhood of  $x$  then there is some  $\varepsilon$  such that the ball of radius  $\varepsilon$  centred at  $x$  is contained in  $U$ . Conversely if  $(x_n)$  converges in the topological sense to  $x$ , let  $\varepsilon > 0$  and consider the open ball centred at  $x$  with radius  $\varepsilon$ . Now  $B_\varepsilon(x)$  is an open neighbourhood of  $x$  so there exists an integer  $N$  such that  $x_n \in B_\varepsilon(x)$  for all  $n > N$ . Hence  $(x_n)$  converges to  $x$  in the metric sense. □

Consider  $\mathbb{R}$  and  $(0, 1)$  with the Euclidean metric and topology. Then the two spaces are related, by the function  $(0, 1) \rightarrow \mathbb{R}$  by  $\tan^{-1} x$  which is invertible. Hence we say the two spaces are homeomorphic, and  $\mathbb{R} \cong (0, 1)$ . However the two spaces are not isometric since  $\mathbb{R}$  is not complete under the Euclidean metric and  $(0, 1)$  is not. Hence the property of completeness is not a topological property: it is a property induced by the metric.

**Definition.** (Discrete topology) Let  $X$  be a set. The *discrete* topology is the topology  $T_{\text{discrete}} = \mathcal{P}(X)$  (so every set is open).

*Remark.* Any function from  $(X, T_{\text{discrete}})$  to any space is continuous. This topology can be induced by the discrete metric, where  $d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$ . So  $B_{\frac{1}{2}}(x) = \{x\}$  so  $\{x\}$  is open, hence all

the sets are open.

**Definition.** (Indiscrete topology) Let  $X$  be a set. The *indiscrete* topology  $T_{\text{indiscrete}} = \{\emptyset, X\}$  (as little as possible sets are open).

*Remark.* A function from any space to  $(X, T_{\text{indiscrete}})$  is continuous. This topology does not come from a metric unless  $X$  is a singleton set. This is because if  $x \neq y$  then  $d(x, y) = \varepsilon > 0$ , so  $y \notin B_\varepsilon(x)$  and since  $y$  is arbitrary, then  $B_\varepsilon(x) = \{x\} = X$ .

Let  $X = \{o, c\}$ . Then let  $T = \{\emptyset, \{o, c\}, \{o\}\}$  be a topology of  $X$ . This is called the Sierpinski space. It has the property that every sequence converges to  $c$ . A continuous function  $f : T \rightarrow (X, T_{\text{Sierpinski}})$  is exactly an open subset of  $Y$ .

Let  $X = \mathbb{R}$  we'll define the right order topology on  $X$  as

$$T_{\text{ord}} = \{(a, \infty) \mid -\infty \leq a \leq \infty\}.$$

Let  $\{(a, \infty)\}_{a \in I}$  be a collect of elements of  $T_{\text{ord}}$ . Then

$$\bigcup_{a \in I} (a, \infty) = (\inf_{a \in I} a, \infty) \in T_{\text{ord}}.$$

Similarly for finite  $I$ ,

$$\bigcap_{a \in I} (a, \infty) = (\max_{a \in I} a, \infty) \in T_{\text{ord}}$$

### 1.3 Bases and subbases

**Definition.** (Basis) Let  $T$  be a topology of  $X$ . A *basis*,  $B \subseteq T$  for  $T$  is a subcollection such that every element of  $T$  is a union of elements in  $B$ .

**Definition.** (Subbasis) Let  $T$  be a topology of  $X$ . A *subbasis*,  $S \subseteq T$  for  $T$  is a subcollection such that every element of  $T$  is a union of sets which are finite intersections of elements of  $S$ .

**Lemma.** Let  $f : (X, T_X) \rightarrow (Y, T_Y)$  and  $S \subseteq T_Y$  is a subbasis. If  $f^{-1}(U)$  is open for all  $U \in S$  then  $f$  is continuous.

*Proof.* If  $V \subseteq T_Y$ , then  $V = \bigcup_{a \in I} V_a$  where  $V_a \in \bigcap_{b \in J_a} U_{a,b}$  with  $U_{a,b} \in S$  and  $J_a$  finite. Then

$$f^{-1}(V) = \bigcup_{a \in I} V_a = \bigcup_{a \in I} \left( \bigcap_{b \in J_a} f^{-1}(U_{a,b}) \right) \in T_X,$$

by the axioms of the topology. □

Consider the Euclidean topology on  $\mathbb{R}^n$ . The collection  $B = \{B_r(x) \mid x \in \mathbb{R}^n, r > 0\}$  is a basis. Likewise the collection of  $n$ -cubes everywhere are also a basis. Interestingly the set  $QB \subseteq B$  with balls at rational points with rational radii is also a basis. This is interesting since  $QB$  is countable while  $B$  is uncountable and  $\mathcal{P}(\mathbb{R}^n)$  is  $\aleph_2$ .

**Definition.** (Closed set) Let  $(X, T)$  be a topological space. A subset  $C \subseteq X$  is *closed* if  $X \setminus C \in T$ .

**Proposition.** Let  $(X, T)$  be a topological space and  $\mathcal{F} = \{C \subseteq X \mid C \text{ closed}\}$ . Then

- (i)  $\emptyset, X \in \mathcal{F}$ ;
- (ii)  $\mathcal{F}$  is closed under (possibly uncountable) intersections;
- (iii)  $\mathcal{F}$  is closed under finite unions.

**Proposition.** A function  $f : X \rightarrow Y$  between topological spaces is continuous if and only if the preimage of every closed set is closed.

**Definition.** Let  $(X, T)$  be a topological space. Let  $A \subseteq X$  be a subset of  $X$ . Then

- (i) The closure  $\bar{A}$  is the smallest (by inclusion) closed set containing  $A$  so

$$\bar{A} = \bigcap_{S \text{ closed}, A \subseteq S} S.$$

- (ii) We say that  $A$  is dense in  $X$  if  $A = \bar{A}$ .
- (iii) The interior  $\overset{\circ}{A}$  is the largest open set contained in  $A$  so

$$\overset{\circ}{A} = \bigcup_{S \text{ open}, S \subseteq A} S.$$

**Definition.** (Limit point) Let  $X$  be a topological space and  $A \subseteq X$ . A *limit point* of  $A$  is a point in  $X$  which is a limit of a sequence in  $A$ .

**Proposition.** If  $C$  is a closed subset of  $(X, T)$ , then the limit points of  $C$  lie in  $C$ .

*Proof.* Let  $\{x_n\}$  be a sequence in  $C$  with limit  $x_\infty$ . If  $x_\infty \notin C$ , then  $x_\infty \in X \setminus C$  which is open. Then if  $x_n \rightarrow x_\infty$  then we should have that  $x_n \in X \setminus C$  for  $n \geq N$  but  $x_n \in C$  so  $x_n \notin X \setminus C$  which is a contradiction.  $\square$

**Corollary.** A limit point of a  $A$  lies in  $\bar{A}$ .

For an example  $\overline{\mathbb{Q}} = \mathbb{R}$  since any real number is a limit of a sequence of rational numbers. We have that  $\overline{(0, 1)} = [0, 1]$  too. The cocountable topology on  $\mathbb{R}$  is the topology  $T_{\text{countable}} = \{\emptyset\} \cup \{\mathbb{R} \setminus C \mid C \text{ countable}\}$ . Let  $\{x_n\}$  be a sequence in  $\mathbb{R}$ , for  $x \in \mathbb{R}$  consider  $\{x\} \cup \{\mathbb{R} - \{x_n\}\}$  is open and contains  $x$ . If  $x_n \rightarrow x$ , then  $x_n$  must be in a  $U$  for all  $n \geq N$  so  $x_n = x$  for all  $n \geq N$ . Hence the convergent sequences are exactly the eventually constant sequences with the limits being the value they are eventually constant to. So the limit points of a set  $A$  are  $A$  under this topology. However almost all  $A$  is not closed. For example  $(0, 1)$  is not closed since  $\mathbb{R} \setminus (0, 1)$  is not countable. But the closure of  $(0, 1)$  must be closed, so it must be  $\mathbb{R}$  hence the sense of limit points and closure are actually two very different properties in topology instead of metric spaces.

## 1.4 Hausdorff spaces

**Definition.** (Hausdorff) A space  $(X, T)$  is *Hausdorff* if for  $x \neq y \in X$  there are open neighbourhoods  $x \in U, y \in V$  with  $U \cap V = \emptyset$ .

*Remark.* This is the notion that points are separated by open sets.

**Lemma.** If the topology  $T$  is induced by a metric then it is Hausdorff.

*Proof.* If  $x \neq y$  then  $d(x, y) = s > 0$ . So consider  $U = B_{s/2}(x)$  and  $V = B_{s/2}(y)$ . The triangle inequality shows that  $U \cap V = \emptyset$  and we know all balls are open.  $\square$

**Proposition.** If a space is Hausdorff then a sequence in  $X$  has at most 1 limit.

*Proof.* Let  $(x_n)$  be a sequence in  $X$ . Suppose it has limits  $y \neq z \in X$ . Let  $U$  and  $V$  be disjoint local neighbourhoods for  $y$  and  $z$  respectively. Then  $x_n \in U$  for all  $n \geq N_1$  and  $x_n \in V$  for all  $n \geq N_2$ . So if we take that  $N = \max\{N_1, N_2\}$  then for all  $n \geq N$ , we have that  $x_n \in U \cap V$  which is empty, hence we have a contradiction.  $\square$

**Proposition.** If  $(X, T)$  is Hausdorff then points are closed.

*Proof.* Let  $x \in X$ . We want to show that  $\{x\} = \overline{\{x\}}$ . Let  $y \neq x$ . Let  $U, V$  be disjoint neighbourhoods of  $x$  and  $y$  respectively. We know that  $x \in X \setminus V$  which is closed. Hence  $\overline{\{x\}} \subseteq X \setminus V$ . But  $y \notin V$ , so  $y$  is not in the closure of  $\{x\}$  hence the closure of  $\{x\}$  is just  $\{x\}$ , so  $\{x\}$  is closed.  $\square$

Let's see an example. Let  $X$  be an infinite set and consider the cofinite topology on  $X$ . Take two non-empty open sets, so

$$(X \setminus F) \cap (X \setminus F') = X \setminus (F \cup F')$$

which is non-empty since  $F \cup F'$  is finite and  $X$  is infinite so the set on the RHS is non-empty hence the space is not Hausdorff.

## 1.5 Defining new topologies on existing ones

We have three main ways to define new topologies when given a topology already.

### 1.5.1 The subspace topology

**Definition.** (Subset topology) Let  $(X, T_X)$  be a topological space. Let  $Y \subseteq X$  a subset. The *subset topology* on  $Y$  is

$$T|_Y = \{Y \cap U \mid U \in T\}.$$

**Definition.** (Subspace) A subspace of  $(X, T)$  is a subset equipped with the subspace topology.

**Proposition.** The subset topology is a topology.

*Proof.* Simple exercise of the axioms.  $\square$

**Proposition.** The inclusion map  $\iota : (Y, T|_Y) \rightarrow (X, T)$  is continuous. In fact  $T|_Y$  is the constant topology on  $Y$  such that the inclusion map is continuous.

*Proof.* Let  $U \in T$  then  $\iota^{-1}(U) = U \cap Y \in T|_Y$  by definition. So it is continuous. Suppose  $\iota : (Y, T') \rightarrow (X, T)$  is continuous. For  $U \in T$ ,  $\iota^{-1}(U) \in T'$  so  $T|_Y \subseteq T'$ .  $\square$

A further point of view, a function  $f : (Z, T_Z) \rightarrow (Y, T|_Y)$  is continuous if and only if  $\iota \circ f$  is continuous.

**Lemma.** (Gluing Lemma) Let  $f : X \rightarrow Y$  be a function between topological spaces.

- (i) If  $\{U_\alpha\}_{\alpha \in I}$  are open subsets which cover  $X$  and each  $f|_{U_\alpha} : U_\alpha \rightarrow Y$  are continuous (where  $U_\alpha$  is given the subspace topology) then  $f$  is continuous.
- (ii) If  $\{C_\alpha\}_{\alpha \in I}$  is a finite collection of closed sets containing  $X$  and  $f|_{C_\alpha} : C_\alpha \rightarrow Y$  is continuous for each  $\alpha \in I$  then  $f$  is continuous.

*Proof.* Let  $V \subseteq Y$  be open. We want to show that  $f^{-1}(V)$  is open. We know that

$$\begin{aligned} f^{-1}(V) &= (f^{-1}(V)) \cap X = f^{-1}(V) \cap \left( \bigcup_{\alpha \in I} U_\alpha \right) \\ &= \bigcup_{\alpha \in I} f^{-1}(V) \cap U_\alpha \end{aligned}$$

Since  $f|_{U_\alpha}$  are continuous, we have that  $f^{-1}|_{U_\alpha}(V)$  is open in  $U_\alpha$  in the subspace topology. So there exists a  $W$  open in  $X$  such that  $f^{-1}|_{U_\alpha}(V) = U_\alpha \cap W$  hence this is the intersection on open subsets of  $X$  so is open in  $X$ , hence since the union of open subsets is open  $f^{-1}(V)$  is open, so  $f$  continuous.

The second part can be proved the same using the closed set definition of continuity.  $\square$

If  $(X, d)$  is a metric space with topology  $T_d$  and  $Y \subseteq X$  then  $T_d|_Y$  is the topology induced by  $d|_Y$ .