

# Methods

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# 1 Fourier Series

## 1.1 Motivation

In 1807 J. Fourier was studying heat conduction along a metal rod. This lead him to study  $2\pi$ -periodic functions i.e. functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  was such that  $f(\theta + 2\pi) = f(\theta)$  for all  $\theta \in \mathbb{R}$  then he found that if

$$f(\theta) = \sum_{n \in \mathbb{Z}} \hat{f}_n e^{in\theta}$$

then you can write down the coefficients  $\{\hat{f}_n\}$  via the formula

$$\hat{f}_n = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-in\theta} d\theta, \quad n \in \mathbb{Z}.$$

And Fourier believed that this worked for any  $2\pi$ -periodic function  $f$ . So computing each  $\{\hat{f}_n\}$  and constructed the sum as above, then it would return the original function. He was wrong.

## 1.2 Modern Treatment

Introduce a vector space  $V$  of  $L$ -periodic functions. Hence

$$V = \{f : \mathbb{R} \rightarrow \mathbb{C} : \text{with } f \text{ a "nice" function, } f(\theta + L) = f(\theta), \forall \theta \in \mathbb{R}\}.$$

Note for  $f \in V$  need only to consider values of  $f$  taken in an interval of length  $L$ , i.e.  $[0, L)$  or  $(-\frac{L}{2}, \frac{L}{2}]$  since periodicity covers elsewhere.

We can introduce an inner product on  $V$  with

$$\langle f, g \rangle = \int_0^L f(\theta) \overline{g(\theta)} d\theta.$$

This gives the associated norm,

$$\|f\| = \sqrt{\langle f, f \rangle}.$$

For  $n \in \mathbb{Z}$  consider  $e_n \in V$  defined by  $e_n(\theta) = e^{2\pi i n \theta / L}$ .

$$\langle e_n, e_m \rangle = \int_0^L e^{2\pi i (n-m)\theta / L} d\theta = L \delta_{nm}.$$

So  $\{e_n\}$  are orthogonal and  $\|e_n\|^2 = L$  for each  $n \in \mathbb{Z}$ . This looks like IA Vectors and Matrices.

Recall that if  $v_N$  is  $N$ -dim vector space equipped with usual inner product and  $\{e_n\}_{n=1}^N$  are orthogonal with  $\|e_n\| = L$ , then for each  $x \in V$  we can write  $x = \sum_{n=1}^N \hat{x}_n e_n$  for some  $\{\hat{x}_n\}$ . To find  $\{\hat{x}_n\}$  take the inner product of both sides with  $e_m$ . So

$$(x, e_m) = \sum_{n=1}^N \hat{x}_n (e_n \cdot e_m) = L \hat{x}_m$$

i.e

$$\hat{x}_n = \frac{1}{L} (x \cdot e_n).$$

Now could this work on  $V$ ?  $V$  is not finite dimensional so it's not obvious. Every subset of  $\{e_n\}$  is linearly independent. Ignoring this for now we assume that for all  $f \in V$  we can write  $f$  in our basis  $\{e_n\}$ . Then

$$f(\theta) = \sum_n \hat{f}_n e_n(\theta),$$

So taking the inner product as before

$$\langle f, e_m \rangle = \sum_n \hat{f}_n \langle e_n, e_m \rangle$$

so using the delta as before

$$= L \hat{f}_m$$

i.e.

$$\hat{f}_n = \frac{1}{L} \langle f, e_n \rangle = \frac{1}{L} \int_0^1 f(\theta) e^{-2\pi i n \theta / L} d\theta$$

**Definition.** (Complex Fourier series) For an  $L$ -periodic  $f : \mathbb{R} \rightarrow \mathbb{C}$  define its *complex Fourier series* by

$$\sum_n \hat{f}_n e^{2\pi i n \theta / L}$$

where

$$\hat{f}_n = \frac{1}{L} \int_0^1 f(\theta) e^{-2\pi i n \theta / L} d\theta$$

are called the complex Fourier coefficients. We will write for  $f \in V$

$$f(\theta) \sim \sum_n \hat{f}_n e^{2\pi i n \theta / L}$$

to mean the series on the right corresponds to complex Fourier series for the function on the left.

We'd like to replace the  $\sim$  symbol with equality, but we require a bit more than that.

If we split the complex Fourier series into the parts  $\{n = 0\} \cup \{n > 0\} \cup \{n < 0\}$  we get

$$\sum_n \hat{f}_n e^{2\pi i n \theta / L} = \hat{f}_0 + \sum_{n=1}^{\infty} \hat{f}_n \left[ \cos\left(\frac{2\pi n \theta}{L}\right) + i \sin\left(\frac{2\pi n \theta}{L}\right) \right] + \sum_{n=1}^{\infty} \hat{f}_{-n} \left[ \cos\left(\frac{2\pi n \theta}{L}\right) - i \sin\left(\frac{2\pi n \theta}{L}\right) \right].$$

**Definition.** (Fourier series) For  $f : \mathbb{R} \rightarrow \mathbb{C}$  an  $L$ -periodic function define its *Fourier series* by

$$\frac{1}{L} a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{2\pi n \theta}{L}\right) + b_n \sin\left(\frac{2\pi n \theta}{L}\right) \right]$$

where

$$a_n = \frac{2}{L} \int_0^L f(\theta) \cos\left(\frac{2\pi n \theta}{L}\right) d\theta$$

and

$$b_n = \frac{2}{L} \int_0^L f(\theta) \sin\left(\frac{2\pi n\theta}{L}\right) d\theta$$

are called the Fourier coefficients for  $f$ .

If we set

$$\begin{aligned} c_n(\theta) &= \cos\left(\frac{2\pi n\theta}{L}\right), \\ s_n(\theta) &= \sin\left(\frac{2\pi n\theta}{L}\right), \end{aligned}$$

then we can show, for  $m, n \geq 1$  that  $\langle c_n, c_m \rangle = \langle s_n, s_m \rangle = \frac{L}{2} \delta_{mn}$  and

$$\langle c_n, 1 \rangle = \langle s_m, 1 \rangle = \langle c_n, s_m \rangle = 0.$$

So we have that  $\{1, c_n, s_n\}$  is orthogonal set in  $V$ .

For an example take  $f : \mathbb{R} \rightarrow \mathbb{R}$ , 1-periodic, such that  $f(\theta) = \theta(1 - \theta)$  on  $[0, 1)$ . For  $n \neq 0$  we have

$$\hat{f}_n = \int_0^1 \theta(1 - \theta) e^{-2\pi i n \theta} d\theta.$$

Integrating by parts (or using a standard Fourier integral computation) yields

$$\hat{f}_n = -\frac{1}{2(\pi n)^2}, \quad n \neq 0,$$

and

$$\hat{f}_0 = \int_0^1 (\theta - \theta^2) d\theta = \frac{1}{6}.$$

Hence

$$f(\theta) \sim \frac{1}{6} - \sum_{n \neq 0} \frac{e^{2\pi i n \theta}}{2(\pi n)^2}.$$

so the sine terms cancel in the sum giving just cosine terms as we expect since our  $f$  function is even.