Fluid Dynamics

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Contents

1 Kinematics

1.1 Streamlines and pathlines

There are two natural ways to think of flow.

(i) A stationary observer watching flow go past. This is the Eulerian perspective. This is the approach used through this course. We define a velocity field (continuum field) $\mathbf{u}(\mathbf{x}, t)$.

(ii) A moving observing, travelling along with the flow. This is the Lagrangian perspective.

Definition. (Streamlines) These are curves that are everywhere parallel to the flow at a given instant.

Remark. The streamline that goes through $\mathbf{x}_0$ at time t_0 is given parametrically as $\mathbf{x} = \mathbf{x}(s, \mathbf{x}_0, t_0)$ and

$$\frac{d\mathbf{x}}{ds} = \mathbf{u}(\mathbf{x}, t_0)$$

(with $\mathbf{x} = \mathbf{x}_0$ at $s = 0$).

The set of streamlines shows the direction of flow a *given* instant a time (all fluid particle at one given time). Take the example $\mathbf{u} = (1, t)$. So at $t = 0$ we have $\mathbf{u} = (1,)$ so the streamlines are horizontal lines. At $t = 1$ we have $\mathbf{u} = (1, 1)$, so the streamlines are diagonal.

Definition. (Pathlines) A *pathline* is the trajectory of a fluid particle (a very small bit of fluid). The pathline $\mathbf{x} = \mathbf{x}(t, \mathbf{x}_0)$ of a fluid which is at $\mathbf{x}_0$ at $t = 0$ is such that

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t)$$

with $\mathbf{x}(0, \mathbf{x}_0) = \mathbf{x}_0$.

Again if we take $\mathbf{u} = (1, t)$ we get

$$\begin{cases} \frac{dx}{dt} = 1 \\ \frac{dy}{dt} = t \end{cases} \rightarrow \begin{cases} x = x_0 + t \\ y = y_0 + \frac{t^2}{2} \end{cases}$$

which describes the path $y - y_0 = \frac{1}{2}(x - x_0)^2$.

Remark. Pathlines are often called "Lagrangian trajectories". The applications are very useful to characterise transport (infecious diseases and pollution simulations).

If the flow is *steady* (so $\mathbf{u}$ does not depend on time). Then pathlines and streamlines are the same.

1.2 The material derivative

We will characterise the rate of change of "stuff" moving with a fluid. Consider a quantity $F(\mathbf{x}, t)$ in a fluid flow (intution is F is temperature). We want to measure how the temperature chnages as we move through the field F along the flow. Let compute the rate of change of (in time) seen by a moving observer. We will call this $\frac{DF}{Dt}$. Take a small time interval δt. Then

$$\begin{aligned}\delta F &= F(\mathbf{x}+\delta\mathbf{x}, t+\delta t) - F(\mathbf{x}, t)\\ &= \delta t\frac{\partial F}{\partial t} + (\delta\mathbf{x}\cdot\nabla)F + (\text{higher order terms}).\end{aligned}$$

We have that $\delta\mathbf{x} = \mathbf{u}\delta t$, so

$$\frac{\delta F}{\delta t} = \frac{DF}{Dt} = \frac{\partial F}{\partial t} + (\mathbf{u}\cdot\nabla)F.$$

We have the derivative and the convected derivative. This should be thought of as moving along gradients of a field.

1.3 Conservation of mass

Consider the flow through a straight rigid pipe with constant cross section. Suppose we have a $\mathbf{u}_{\text{in}}$ and a $\mathbf{u}_{\text{out}}$. Can we have $\mathbf{u}_{\text{in}} \neq \mathbf{u}_{\text{out}}$? For a gas, yes we can since they can be compressed. For a fluid, we cannot, since they are incompressible.

Define $\rho(\mathbf{x}, t)$ as the mass density with $[\rho] = \frac{\mathrm{M}}{\mathrm{L}^3}$. We want a relation between ρ and $\mathbf{u}$. Consider a fixed volume V and compute the rate of change of its mass, M.

$$M = \int_V \rho \mathrm{d}V$$

Assume that mass can only change due to the flow of mass across the boundary surface ∂V. Take a small surface element δA with normal $\mathbf{n}$. The volume out of V during δt is $(\mathbf{u}\cdot\mathbf{n})\delta A\delta t$. Hence the mass out is $\rho(\mathbf{u}\cdot\mathbf{n})\delta A\delta t$, so we get that

$$\frac{dM}{dt} = -\int_{\partial V}\rho(\mathbf{u}\cdot\mathbf{n})\mathrm{d}A.$$

The divergence theorem will allow us to rewrite this as

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0.$$

We know from IA Vector Calculus that $\nabla\cdot(\rho\mathbf{u}) = \rho\nabla\cdot\mathbf{u} + \mathbf{u}\cdot\nabla\rho$, so we can write that

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\mathbf{u}.$$

Definition. (Incompressible) A fluid flow is *incompressible* if $\frac{D\rho}{Dt} = 0$.

This is then equivalent to $\nabla\cdot\mathbf{u} = 0$ which is the equivalent condition we'll use for the course.

For this course we will assume that ρ is constant. This means as a consequence that $\nabla\cdot\mathbf{u} = 0$.

1.4 Kinematic boundary condition

Consider the material boundary, with unit norm $\mathbf{n}$, of a body of fluid has a given velocity $\mathbf{U}(\mathbf{x}, t)$. At a point $\mathbf{x}$ on the boundary, the fluid velocity relative to the surface is $\mathbf{u} - \mathbf{U}$. Applying mass conversation on the interface over a small surface element δA in time δt. So

$$\rho(\mathbf{u}-\mathbf{U})\cdot\mathbf{n}\delta A\delta t = 0.$$

Hence we require $\mathbf{u}\cdot n = \mathbf{U}\cdot\mathbf{n}$ at the interface. This is the kinematic boundary condition.

Remark. $\mathbf{n}$ occurs on both sides, hence we don't need $\mathbf{n}$ to be a unit vector.

We have some consequences of this condition.

(i) If the boundary is fixed, $\mathbf{U} = 0$ implies that $\mathbf{u} \cdot \mathbf{n} = 0$. This is called the no penetration condition.

(ii) Consider an air/water interface (free surface). Suppose the surface is defined by $z = \xi(x, y, t)$. Then can think of the free space as $F(x, y, z, t) = 0$ where $F(x, y, z, t) = z - (x, y, t)$. So $\mathbf{n}$ is perp to $\nabla F = (-\xi_x, -\xi_y, 1)$. Then if $\mathbf{u} = (u, v, w)$ so $\mathbf{U} = (0, 0, \xi_t)$. Then the kinematic boundary condition becomes $-u\xi_x - c\xi_y + w = \xi_t$, so $w = \xi_t + u\xi_x + v\xi_y = \frac{D\xi}{Dt}$. This is equivalent to $\frac{DF}{Dt} = 0$.

1.5 Streamfunction for 2D incompressible flow

We know that $\nabla \cdot \mathbf{u} = 0$ which is equivalent to there existing a vector potential $\mathbf{A}$ such that $\mathbf{u} = \nabla \times \mathbf{A}$. In 2D if $\mathbf{u} = (u, v, 0)$ then $\mathbf{A} = (0, 0, \psi(x, y))$. So

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}.$$

We call ψ a *streamfunction*. Looking at dimensions we have that $[\psi] = \mathrm{L}^2\ \mathrm{T}^{-1}$. Now we'll see an example.

Let $\mathbf{u} = (y, x)$ (which we can see is incompressible) so

$$\frac{\partial \psi}{\partial y} = u = y$$

, hence $\psi = \frac{1}{2}y^2 + f(x)$. We also have that $-\frac{\partial \psi}{\partial x} = -f'(x) = x$, so $\psi = \frac{1}{2}(y^2 - x^2) + C$.

We have some properties about the streamfunction,

(i) Streamlines are given by $\psi =$ constant.

(ii) $|\mathbf{u}| = |\nabla \psi|$, so the flow is faster if the streamlines are closer together.

(iii) If we take two points $\mathbf{x}_0, \mathbf{x}_1$, then then the volume flux crossing the line between $\mathbf{x}_0$ and $\mathbf{x}_1$ is

$$\int_{\mathbf{x}_0}^{\mathbf{x}_1} \mathbf{u} \cdot \mathbf{n} \mathrm{d}\ell = \psi(\mathbf{x}_1) - \psi(\mathbf{x}_0).$$

(iv) ψ is constant at rigid boundaries.

We can do the same in polar coordinates. So $\mathbf{u} = (u_r(r, \theta), u_\theta(r, \theta), 0)$. We have that

$$\mathbf{u} = \nabla \times \mathbf{A} = \left(\frac{1}{r} \frac{\partial \psi}{\partial \theta}, -\frac{\partial \psi}{\partial r}, 0 \right),$$

so we can check that $\nabla \cdot \mathbf{u} = \frac{1}{r} \frac{\partial}{\partial r}(r u_r) + \frac{1}{r} \frac{\partial}{\partial \theta} = 0$

2 Dynamics of inviscid flow

2.1 Surface and volume forces

There a two types of forces exerted on a fluid.

(i) Forces proportional to the volume (gravity);

(ii) Forces proportional to the surface area (pressure, viscous stresses).

We'll first look at the first type, called volume forces. We'll denote $F(\mathbf{x}, t)\delta V$ as the force acting on a small volume element ∂V. Let's take gravity as an example, so $\mathbf{F} = \rho \mathbf{g}$. Often we have that $\mathbf{F}$ is conservative, so $\mathbf{F} = -\nabla \chi$ for some function χ (we know gravity is $\chi = \rho g z$).

Now for surface forces. Consider a small element of area of $\mathbf{n}\delta A$. Denote the surface force exerted by the positive side on the negative side by $\boldsymbol{\tau}(\mathbf{x}, t, \mathbf{n})\delta A$. We say that $\boldsymbol{\tau}$ is "stress" acting on a surface element. Note that $\boldsymbol{\tau}$ depends on orientation. By Newton's third law, we have that $\boldsymbol{\tau}(\mathbf{x}, t, -\mathbf{n}) = -\boldsymbol{\tau}(\mathbf{x}, t, \mathbf{n})$.

There are many phenomona where friction inside a fluid (viscous stress) is negligible. For example a 10cm box of water, it takes hours for viscosity to bring the fluid to rest once disturbed.

Definition. (Inviscid) A fluid is said to be *inviscid* if we can neglect viscosity.

For inviscid flow, $\boldsymbol{\tau}$ has no tangential component and its magnitude is independent of orientation. So $\boldsymbol{\tau}(\mathbf{x}, t, \mathbf{n}) = -p(\mathbf{x}, t)\mathbf{n}$, where p is the pressure. Note we have the minus sign because the positive side pushes with pressure p towards the negative side when $p > 0$.

2.2 The Euler Momentum equation

The idea here is that we do a similar calculation for mass conservation, but now for momentum instead. Consider an arbitrary fixed volume, V with boundary ∂V. Hence the momentum inside V is

$$\int_V \rho \mathbf{u} \, \mathrm{d}V.$$

The momentum inside V can change due to

(i) Flux of momentum across ∂V;

(ii) Force acting on V or ∂V.

The volume out of δA in δt is $(\mathbf{u} \cdot \mathbf{n})\delta A \, \delta t$. So the momentum out of δA in time δt is $\rho \mathbf{u}(\mathbf{u} \cdot \mathbf{n})\delta A \, \delta t$. Hence we get the following.

Theorem. (Euler momentum integral equation)

$$\frac{d}{dt} \int_V \rho \mathbf{u} \, \mathrm{d}V = -\int_{\partial V} \rho \mathbf{u}(\mathbf{u} \cdot \mathbf{n}) \, \mathrm{d}A + \underbrace{\int_V \mathbf{f} \mathrm{d}V}_{\text{volume force}} + \underbrace{\int_{\partial V} -p\mathbf{n} \, \mathrm{d}A}_{\text{surface force}}.$$

In components,

$$\int_V \frac{\partial}{\partial t}(\rho u_i)\ \mathrm{d}V = -\int_{\partial V} \rho u_i u_j n_j\ \mathrm{d}A + \int_{\partial V} -p n_i\ \mathrm{d}A + \int_V f_i\ \mathrm{d}V.$$

Sometimes books call $\rho u_i u_j$ the momentum flux tensor. We can apply the divergence theorem to the first two integrals which gives that those two integrals become

$$\int_V \left[-\frac{\partial}{\partial x_j}(\rho u_i u_j) - \frac{\partial p}{\partial x_i} \right] \mathrm{d}V.$$

Given that this true for any fixed volume V, the integrand must be zero, hence we have that

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial p}{\partial x_i} + f_i.$$

This becomes

$$u_i \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) \right] + \rho \left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial}{\partial x_j} u_i \right] = -\frac{\partial p}{\partial x_i} + f_i.$$

So using mass conservation on the first part we get some simplification, so going back to vector form we have the Euler momentum equation,

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mathbf{f}.$$

This is the equation of motion for inviscid fluid flow.

Fluid particles accelerate under differences in pressure and volume (body) forces (inviscid flows only).

Remark. Note that at the surface, the stress exerted by the fluid at the surface is $p\mathbf{n}$.

Let's see an application of the momentum equation. Consider a 90° bent pipe with a flow U. What is the force exerted by flow on the pipe for a steady flow without gravity? We will use the Euler momentum integral equation. Since the flow is steady, the LHS term is zero and since there is no gravity, the volume force term is zero. Hence we have that

$$\int_{\text{walls}} + \int_{\text{end}} [\rho \mathbf{u}(\mathbf{u} \cdot n) p\mathbf{n}] \mathrm{d}A = 0$$

and because of our kinetmatic boundary condition, $\mathbf{u} \cdot \mathbf{n} = 0$, so our integral across the walls becomes

$$\int_{\text{walls}} p\mathbf{n}\ \mathrm{d}A = \mathbf{F} = \text{Force exerted by fluid flow on the pipe.}$$

Across the ends,

$$\int_{\text{in+ out}} [\rho \mathbf{u}(\mathbf{u} \cdot \mathbf{n}) + p][\mathrm{d}A = \rho(-U)(-U\mathbf{n}_1)A_1 + p_1\mathbf{n}_1 + \rho U(U\mathbf{n}_2)A_2 + p_2\mathbf{n}_2 A_2.$$

W ehave that $p = p_1 = p_2$ and $A_1 = A_2 = A$ so the integral becomes

$$= A[(p + \rho U^2)(\mathbf{n}_1 + \mathbf{n}_2)],$$

hence

$$\mathbf{F} = -A(p + \rho U^2)(\mathbf{n}_1 + \mathbf{n}_2).$$

2.3 Bernoulli equation for steady flow with potential forces

There are *two* Bernoullli equations in the course. We'll look at the first one for steady flow here.

Recall from the Euler equation that

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right)=-\nabla p+\mathbf{F}.$$

We will assume that

(i) We have steady flow so $\frac{\partial}{\partial t}=0$.

(ii) ρ is constant (as always in IB Fluid Dynamics).

(iii) $\mathbf{F}=-\nabla\chi$ (conservative force).

So the Euler equation gives that

$$\rho(\mathbf{u}\cdot\nabla\mathbf{u})=-\nabla(p+\chi).$$

Now we have the identity

$$(\mathbf{u}\cdot\nabla)\mathbf{u}=\nabla\left(\frac{1}{2}u^2\right)-\mathbf{u}\times(\nabla\times\mathbf{u}),$$

which gives that

$$\rho\left[\nabla\left(\frac{1}{2}u^2\right)-\mathbf{u}\times(\nabla\times\mathbf{u})\right]=-\nabla(p+\chi).$$

Now since ρ is constant we can move it inside the ∇. We'll now define $\boldsymbol{\omega}=\nabla\times\mathbf{u}$, the *vorticity* of the fluid, giving,

$$\nabla\left[\frac{1}{2}\rho u^2+p+\chi\right]=\rho\mathbf{u}\times\boldsymbol{\omega}.$$

We like to dot this with $\mathbf{u}$ to get the Steady Bernoulli equation,

$$\mathbf{u}\cdot\nabla\left[\frac{1}{2}\rho u^2+p+\chi\right]=0.$$

So this value $H=\frac{1}{2}\rho u^2+p+\chi$ is constant on streamlines for steady flow. The physical consequence of this is that when u increases, p decreases (ignoring gravity). Let's now see a simple application.

2.3.1 Tank of fluid with small drain

Take a tank full of fluid with a small hole at the bottom. The water height has height h and the fluid exists the tank with speed u. We assume that the size of the hole is small enough so the flow is steady. At the top of the tank, the pressure is atmospherical pressure and at the exit it is the same. Taking a streamline from the top on the tank to the exist we'll use the steady flow Bernoulli equation. We have gravity so $\chi=-\rho gh$. At the top we have H_1 and at the bottom we have H_2, from Bernoulli, $H_1=H_2$,

$$\frac{1}{2}\rho u^2=\rho gh,$$

so

$$u=\sqrt{2gh}.$$

Remark. We take the hole very small, so $u=0$ (approximately) at the top of the tank.

2.3.2 Venturi meter

This is a device to measure flow rates with no moving parts. We ignore gravity and assume the flow is steady and uniform across any cross section (gentle variations in the cross section A). The device is a pipe which is pinched in the middle to a much smaller cross section area. Before the pinch we have A_1, u_1, p_1 and in the pinched area we have A_2, u_2, p_2. By conservation of mass, $A_1u_1 = A_2u_2$. Now we attach two linked tubes of fluid which have a difference in height of fluid h one at the first point, and the other at the second point. From the Steady Bernoulli equation, $\frac{1}{2}\rho u_1^2 + p_1 = \frac{\rho}{u}{}^{2}_{2} + p_2$. So using our mass conservation,

$$p_1 - p_2 = \frac{1}{2}\rho u_1^2 \left(\frac{A_1^2}{A_2^2} - 1\right).$$

If we measure h using hydrostatic balance,

$$\rho g h = \frac{1}{2}\rho u_1^2 \left(\frac{A_1^2}{A_2^2} - 1\right)$$

hence

$$A_1u_1 = \sqrt{2gh}\frac{A_1A_2}{\sqrt{A_1^2 - A_2^2}}.$$

Now for the final example, let's see a water jet on a 2D oblique plane. Let the angle under the jet and the angled plane be β. We will ignore gravity. We let the incoming fluid flow rate be U and the flow going up be a_2 and the flow going down be a_1. We also want to know what the force exerted on the plane.

First we apply Bernoulli on the surface streamline. So since p is atmospherical everywhere hence $u = U$ everywhere. By mass conservation, $aU = a_1U + a_2U$, hence

$$a = a_1 + a_2.$$

Now for more information, we apply the integral form of the Euler momentum equation. We'll take the volume V to be the entire volume of fluid.

$$\frac{d}{dt}\int_V \rho\mathbf{u}\mathrm{d}t = \int_{\partial V} -\rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\mathrm{d}A + \int_{\partial V} -p\mathbf{n}\mathrm{d}A.$$

We've assumed the flow is steady, so the LHS is zero. p is atmospherical pressure everywhere except from the surface. On the surface streamlines due to the kinematic boundary condition, $\mathbf{u}\cdot\mathbf{u} = 0$. Components the Euler momentum equation on the surface gives

$$\rho a U^2 \cos\beta = \rho a_2 U^2 - \rho a_1 U^2$$

hence

$$a_2 = a_1 + a\cos\beta$$

so solving gives that

$$a_1 = \frac{a}{2}(1 - \cos\beta) \qquad a_2 = \frac{a}{2}(1 + \cos\beta).$$

Now instead if we calculate the component perpendicular to the surface,

$$F = \int p\mathbf{n}\mathrm{d}A = \rho a U^2 \sin\beta.$$

2.4 Hydrostatic pressure and Archimedes

What if we set $\mathbf{u} = 0$ in the Euler equation? Then $0 = -\nabla p + \mathbf{f}$. However there is a non-trivial distribution of pressure in the fluid. For example if we take gravity, so $\mathbf{f} = \rho\mathbf{g}$, hence

$$0 = -\nabla p + \rho\mathbf{g} = -\nabla(p + \chi).$$

So $p + \chi$ is constant, so $p = p_0 - \rho g z$. If we take $z = 0$ to the top of the water, then we have p_0 as atmospherical pressure. This is called hydrostatic pressure. We can use this to mathematically derive Archimedes principle.

We want to calculate the total pressure force on a body exerted due to a fluid with no flow. Let $\mathbf{n}$ be a normal to the body surface ∂V. $\mathbf{F}$ is the force exerted by the hydrostatic pressure field on the body. Hence

$$\begin{aligned}
\mathbf{F} &= \int_{\partial V} -p\mathbf{n}\mathrm{d}S \\
&= \int_{\partial V} -(p_0 - \rho g z)\mathbf{n}\mathrm{d}S \\
&= -\int_V \nabla(p_0 - \rho g z)\mathrm{d}V \\
&= \int_V \rho g \hat{z}\mathrm{d}V \\
&= \rho g V \hat{z}.
\end{aligned}$$

Where we've extended the fluid pressure field into the body. $\rho g V$ is the weight of the displaced fluid which is the buoyancy force.

Remark. Suppose we have the Euler equation

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \rho\mathbf{g}.$$

We will write $p = p_{\text{static}} + p'$, where p' is called the dynamic pressure, so our Euler equation becomes

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p'$$

hence p' denotes the pressure in fluid to motion only.

2.5 Vorticity

Recall we defined $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ as the *vorticity* of the flow. Let's see some examples of vorticity.

- Let $\mathbf{u} = \mathbf{\Omega}\times$ (solid body rotation). Then using IA Vector Calculus identities, we have that $\boldsymbol{\omega} = 2\mathbf{\Omega}$.
- Let $\mathbf{u}$ be the *shear flow* $\mathbf{u} = (0, \gamma x, 0)$, so $\boldsymbol{\omega} = (0, 0\gamma)$.
- Let $\mathbf{u} = \frac{k}{2\pi r}\mathbf{e}_\phi$ in cylindrical coordinates, representing a line singular vortex. Then $\boldsymbol{\omega} = 0$ everywhere except at $r = 0$, where

  $$\oint_r \mathbf{u} \cdot d\boldsymbol{\rho} = k = \iint \boldsymbol{\omega} \cdot \mathbf{n}\mathrm{d}\mathbf{S}$$

  by Stokes' theorem, giving that $\boldsymbol{\omega} = (0, 0, k\delta(r))$

2.5.1 Interpretation of $\boldsymbol{\omega}$

Here we will show $\boldsymbol{\omega}$ is equivalent to twice the local rotation rate of fluid particles.

Consider a material line $\delta\boldsymbol{\ell}$. How does $\boldsymbol{\ell}$ change from t to $t+\delta t$.

$$\mathbf{x}+\delta\boldsymbol{\ell} \to \mathbf{x}+\delta\boldsymbol{\ell}+\mathbf{u}(\mathbf{x}+\delta\boldsymbol{\ell},t)\delta t$$

and

$$\mathbf{x} \to \mathbf{x}+\mathbf{u}(\mathbf{x},t)\delta t.$$

Hence our difference $\delta\boldsymbol{\ell}$ goes to

$$\begin{aligned}\delta\boldsymbol{\ell} &\to \mathbf{x}+\delta\boldsymbol{\ell}+\mathbf{u}(\mathbf{x},\delta\boldsymbol{\ell},t)\delta t-[\mathbf{x}+\mathbf{u}(\mathbf{x},t)\delta t]\\ &= \delta\boldsymbol{\ell}+(\delta\boldsymbol{\ell}\cdot\nabla)\mathbf{u}\delta t.\end{aligned}$$

Hence

$$\frac{D}{Dt}\delta\boldsymbol{\ell}=(\delta\boldsymbol{\ell}\cdot\nabla)\mathbf{u}.$$

Hence the tensor $\frac{\partial u_i}{\partial x_j}$ determines changes in material lines (local rate of deformation). Write

$$\begin{aligned}\frac{\partial u_i}{\partial x_j} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)+\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right)\\ &= e_{ij}+\frac{1}{2}\varepsilon_{ijk}w_k,\end{aligned}$$

where $\boldsymbol{\omega}=\nabla\times\mathbf{u}$ by the standard decomposition of the antisymmetric tensor as a vector. The contribution of the second term to $\frac{D}{Dt}\partial\ell_i$ is

$$\begin{aligned}\frac{1}{2}\varepsilon_{jik}w_k\delta\ell_j &= \frac{1}{2}\varepsilon_{ikj}w_k\delta\ell_j\\ &= \frac{1}{2}(\boldsymbol{\omega}\times\delta\boldsymbol{\ell})_i.\end{aligned}$$

This is the rigid body rotation with angular velocity $\boldsymbol{\Omega}=\frac{1}{2}\boldsymbol{\omega}$. So $\frac{1}{2}\boldsymbol{\omega}(\mathbf{x},t)$ represents the average rotation rate of fluid particles. This is *not* the same as having flow with circular streamline.

2.5.2 Vorticity equation

Let's start with the Euler momentum equation,

$$\rho\left(\frac{\partial\mathbf{u}}{dt}+(\mathbf{u}\cdot\nabla)\mathbf{u}\right)=-\nabla p+\mathbf{f}.$$

Assume that ρ is constant and $\mathbf{f}=-\nabla\chi$. We know that $\mathbf{u}\cdot\nabla)\mathbf{u}=\nabla(\frac{1}{2}u^2)-\mathbf{u}\times\boldsymbol{\omega}$, so taking the curl of the Euler equation we get that

$$\frac{\partial\boldsymbol{\omega}}{\partial t}-\nabla\times[\mathbf{u}\times\boldsymbol{\omega}]=0.$$

Evaluating the ith component of this equation,

$$\begin{aligned}[\nabla\times(\mathbf{u}\times\boldsymbol{\omega})]_i &= \varepsilon_{ijk}\frac{\partial}{\partial x_j}[\varepsilon_{kmn}u_m\omega_n]\\ &= (\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm})\left(\frac{\partial u_m}{\partial x_j}\omega_n+u_m\frac{\partial\omega_n}{\partial x_j}\right)\\ &= \omega_j\frac{\partial u_i}{\partial x_j}-\omega_i\frac{\partial u_j}{\partial x_j}+u_i\frac{\partial\omega_j}{\partial x_j}-u_j\frac{\partial\omega_i}{\partial x_j}.\end{aligned}$$

Mass is conserved and the divergence of the curl is zero, so the second and third terms are zero, hence this is equal to

$$(\boldsymbol{\omega} \cdot \nabla)u_i - (\mathbf{u} \cdot \nabla)\omega_i.$$

So the vorticity equation is

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} = (\cdot \nabla)\mathbf{u}.$$

This is equivalent to

$$\frac{D}{Dt}\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u}.$$

The material derivative recall is the rate of change of the field while moving with the flow.

2.5.3 Vortex stretching

Let's compare the two equations

$$\frac{D}{Dt}\delta\boldsymbol{\ell} = (\delta\boldsymbol{\ell} \cdot \nabla)\mathbf{u} \qquad \frac{D}{Dt}\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u}.$$

We can see these are the same equation. So moving with the fluid, $\boldsymbol{\omega}$ changes just like a material line element initially parallel to $\boldsymbol{\omega}$. In particular, if $\delta\boldsymbol{\ell}$ gets longer (stretching) then $\boldsymbol{\omega}$ gets larger. This is conservation of angular momentum. When the fluid is stretched in the direction of $\boldsymbol{\omega}$, we get an increase in vorticity.

Definition. (Circulation) Given a closed curve Γ, the *circulation* around Γ is defined as

$$C(t) = \int_\Gamma \mathbf{u} \cdot \mathrm{d}\boldsymbol{\ell}.$$

By Stokes' theorem we have that

$$C(t) = \iint_S \boldsymbol{\omega} \cdot S$$

where S is an open surface with $\partial S = \Gamma$.

Theorem. *Non-examinable* (Kelvine circulation theorem) For an inviscid fluid with constant density and conservative body forces then if we take Γ to be a material line then

$$\frac{d}{dt}C = 0$$

Remark. If the flow is irrotatoinal at $t = 0$ then it is irrotional at all times.

3 Viscosity

The previous chapter only covered flows which were inviscid and we neglected any friction in fluid flow. We had that

(i) $\boldsymbol{\tau} = -p\mathbf{n}$

(ii) In our momentum balance equation (Euler) we had $\rho \frac{D\mathbf{u}}{Dt} \sim (\nabla p;\ \mathbf{F})$.

Now we will include viscosity; we will find

(i) New *tangential* stress between fluid layers or between fluid and the boundary.

(ii) New term in the momentum balance equation.

Here in IB Fluid Dynamics we will only focus on 2D parallel viscious flows. So we will have that

$$\mathbf{u} = (u(y,t), 0, 0).$$

We instantly get that $\nabla \cdot \mathbf{u} = 0$.

3.1 Plane Couette flow

Consider a a steady flow between two parallel plates, driven only by the motion of the top plate parallel to the bottom one with speed U. Experiments report that for the wide variety of fluids (Newtonian fluids) the find that

(i) The fluid velocity at the top is U and at the bottom it is zero.

(ii) The fluid flow velocity varies linearly between 0 and V, so $u(y) = U\frac{y}{h}$.

(iii) The tangential force τ required to move the top plane is linear in U and inversely proportional to h.

So we have that

$$\tau \propto \frac{U}{h}$$

Definition. (Dynamic viscosity) The *dynamic viscosity*, μ, of a fluid is the proportionality constant from above, so

$$\tau = \mu\frac{U}{h}.$$

We usually write this as

$$\tau = \mu\frac{\partial u}{\partial n},$$

the viscous tangential stress exerted by the positive side to the negative side, where the normal is pointing from the interface to the positive side.

Remark. The viscosity is a material property of the fluid.

We call $\frac{U}{h}$ the shear rate of the fluid.

3.2 2D parallel viscous flow

3.2.1 Steady case with $\mathbf{F} = 0$

Consider an infinitesimal volume of fluid $\delta x \delta y$. Consider a flow $u(y,t)$ moving left to right in the volume. On the left and right side we only have the pressure force and no tangential forces. On the top and bottom we have the tangetial frictional forces equal to

$$\pm\mu\frac{\partial u}{\partial y}.$$

Using the balance of forces in the x-direction we have that

$$p(x)\delta y - p(x+\delta x)\delta y + \left(\mu\frac{\partial y}{\partial y}(y+\delta y) - \mu\frac{\partial u}{\partial y}(y)\right)\delta x = 0.$$

Expanding the $\delta x \delta y$ terms we get that

$$\delta x \delta y\left(-\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2}\right) = 0$$

which gives that

$$-\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} = 0.$$

In the y-direction we get that

$$\frac{\partial p}{\partial y} = 0$$

3.2.2 General case with $u(y,t)$ and $\mathbf{F}$

On Example Sheet 2 we will see that the general case solution is

$$\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} + F_x$$
$$0 = -\frac{\partial p}{\partial y} + F_y.$$

Remark. For flows of the form $\mathbf{u} = (u(y,t),0,0)$ we have that $(\mathbf{u}\cdot\nabla)\mathbf{u} = 0$ so this explains why this result is much simplier than it seems it would be.

3.2.3 The no-slip boundary condition

Notice that we've now increased the order of our differential equations. This means that we now need another boundary condition to solve our equations.

Experiements tell us that for viscious flow at a rigid boundary $\mathbf{u} = \mathbf{U}$, so the tagnetial components of the flow match the velocity of the rigid boundary. This is called the no-slip boundary condition and is *only* applicable to viscious flows.

Recall we still have our other boundary condition: the no penetration boundary condition

$$\mathbf{u}\cdot\mathbf{n} = \mathbf{U}\cdot\mathbf{n}.$$

3.2.4 Examples

Poiseuille flow in a channel: This is steady flow in a (2D) cannel driven by a pressure gradient. Say the channel has a length L and height h, with pressure p_0 at the input of the channel and p_1 outwards. We have steady flow, so $\frac{\partial}{\partial t} = 0$ and our no-slip boundary conditions gives that $u(0) = 0$ and $u(h) = 0$. We have that $\frac{\partial p}{\partial y} = 0$ so $p = p(x)$. Further we have that

$$\frac{\partial p}{\partial x} = \mu\frac{\partial^2 u}{\partial y^2}$$

hence both are constant functions. Solving gives that

$$u(y) = \frac{G}{2\mu} y(h-y)$$

where we define

$$G = -\left(\frac{p_1 - p_0}{2}\right).$$

So our flow is a parabola, with zero flow at the boundaries of the channel walls. We can calculate the flow rate, q, as

$$q = \int_0^h u \, \mathrm{d}y = \frac{(p_1 - p_0)h^3}{12\mu L}.$$

Remark. We can see if we balance forces that we have that

$$p_1 h - p_0 h + \tau_{\text{top}} L - \tau_{\text{bottom}} L = 0$$

which we can calculate and expand out to see using the fact that

$$\tau_{\text{top}} = \mu \frac{\partial u}{\partial \mathbf{n}} = \mu \frac{\partial u}{\partial y} \mid_{y=h} .$$

Now let's look at a viscous flow driven down an inclined plane by gravity (body force). Suppose the plane is inclined by an angle α and let the x direction be parallel to the plane and the y direction be perpendicular to the plane. Let the fluid thickness be h.

$$\mathbf{F} = \rho \mathbf{g} = (\rho g \sin\alpha, -\rho g \cos\alpha).$$

In the y direction we have that

$$0 = -\frac{\partial p}{\partial y} - \rho g \cos\alpha$$

with boundary conditions

$$p(h) = p_a$$

so we get that

$$p = p_a + \rho g (h-y)\cos\alpha$$

which resembles the hydrostatic balance equation (but now tilted). In the x direction we have that

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} + \rho g \sin\alpha.$$

Since h is constant the first term is zero. We assume there is no shear stress exerted by the fluid on the air hence we can see that

$$\mu \frac{\partial u}{\partial y} \mid_{y=h} = 0.$$

So solving

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\rho g}{\mu} \sin\alpha$$

gives that

$$u(y) = \frac{\rho g \sin\alpha}{2\mu} y(2h-y).$$

3.2.5 Boundary condition at an interface

Suppose we have two fluids. Fluid 1 has viscosity μ_1 and fluid 2 has viscosity has viscosity μ_2. What are the boundary conditions at the interface of the fluid?

(i) No slip: $u_1 = u_2$ at the interface

(ii) Continuity of stress: In the normal direction $p_1 = p_2$ and in the tangential direction

$$\mu_1 \frac{\partial u_1}{\partial y} = \mu_2 \frac{\partial u_2}{\partial y}$$

.

3.3 Unsteady parallel viscous flows and viscous diffusion

Example. Fundamental example: Rayleigh's problem (Stokes' 1st problem). This is the impulsively started plate. Consider a semi-infinite viscous fluid in $y > 0$ that is initially at rest with no applied pressure gradient, $\mathbf{f} = 0$. At $t = 0^+$ the plate starts to move at constant velocity $(U, 0)$. What are the equations of motion? We have that

$$\frac{\partial p}{\partial y} = 0$$

so $p = p_0$. In the x-direction,

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} + f_x$$

which reduces to the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\mu}{\rho} \frac{\partial^2 u}{\partial y^2}.$$

The initial condition is $u(y, t) = 0$ at $t = 0$. And the boundary conditions are

$$u(0, t) = U$$
$$u(\infty, t) = 0$$

with the first equation coming from the no-slip condition. We define $\frac{\mu}{\rho} = \nu$ as the kinematic viscosity. If $\mu = 0$ nothing happens. We see that ν is the diffusivity for momentum (and vorticity: see later). So all we have to do is solve the differential equation using a Fourier series or seperation of variables (IB Methods). Here we use dimensional analysis so $u(y, t) = Uf(t, y, \nu)$ so $f(t, y, \nu)$ is dimensionless. Hence

$$\frac{u}{U} = f\left(\frac{y}{\sqrt{\nu t}}\right)$$

so $y \sim \sqrt{\nu t}$. Define our *similarity variable* as

$$\eta = \frac{y}{\sqrt{vt}}$$

so putting this into our diffusion equation we get that

$$-\frac{1}{2}\frac{\eta}{t} f' U = \frac{\nu}{\nu t} U f''$$

and solving using $f(0) = 1$ and $f(\infty) = 0$ gives that

$$u(y,t) = U\text{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right)$$

where

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} \mathrm{d}t.$$

So the force required to move the plate at $y = 0$ is

$$\tau = \mu \frac{\partial u}{\partial y} \mid_{y=0} = -\frac{\mu U}{\sqrt{\nu t}} f'(0) = -\frac{\mu U}{\sqrt{\pi \nu t}}.$$

We have that $\mathbf{u} = (u, 0, 0)$, so $\boldsymbol{\omega} = (0, 0, -\frac{\partial u}{\partial y}$ hence the vorticity also diffuses.

Let's see some more examples

Example. Now add a boudary at $y = h$. Now our similarity dimensionless solution is no longer possible.

(i) At small times $(\nu t)^{\frac{1}{2}} \ll h$ this is approximately like the previous example.

(ii) At long times eventually we have Couette flow.

The characteristic time-scale is $t \sim \frac{h^2}{\nu}$ for diffusion of momentum.

Example. Now let's look a Stokes' 2nd problem. Here we have the same set up as in Stokes' 1st problem, but now the boudary is oscilatory with velocity $U \cos(\omega t)$. A full treatment of the problem is in Example Sheet 2. The imposed time scale is $\frac{1}{\omega}$ during which velocity variations can diffuse distance $\sqrt{\frac{\nu}{\omega}} = \delta$ known as Stokes layer.

Example. Again we can see this problem with the fluid bounded from above. This gives two length scales δ and h related by

$$S = \frac{\omega h^2}{\nu} = \frac{h^2}{\delta^2}$$

where S is known as the Stokes number for the fluid

Remark. The examples illustrate that the inivscid solution is *not* uniformly the same as the limit of a small viscosity.

3.4 Navier-Stokes equations (NE)

These equations derive the most generla momentum equation for a Newtonian fluid. If we have that the fluid is incompressible, so $\nabla \cdot \mathbf{u} = 0$ then the equation is

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mathbf{F} + \mu \nabla^2 \mathbf{u}.$$

This reduces to Euler in inviscid limit and the 2D parallel equations when $\mathbf{u} = (u(y,t), 0, 0)$. We also have that $\boldsymbol{\omega}$ satisfies

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + U\nabla^2 \boldsymbol{\omega}.$$

3.5 The Reynolds number (NE)

Consider a flow with characteristic length scale L and velocity U. Then we have that

$$\rho \frac{D\mathbf{u}}{Dt} \sim \rho \frac{u}{L/u} \sim \rho u^2 L$$

and

$$\mu \nabla^2 u \sim \mu \frac{u}{L^2}.$$

Taking the ratio between these terms and setting it to a new quantity **Re**. We get that

$$\mathrm{Re} = \frac{\rho U L}{\mu} = \frac{UL}{\nu}.$$

We call Re the Reynolds number and it is a dimensionless quantity useful for study what kind of flow we have.

4 Inviscid irrotational flow

This chapter will study what we call irrotatoinal flow also known as potential flow. This is flow with $\boldsymbol{\omega} = 0$.

4.1 Velocity potential

Since all the domains we care about in IB Fluid Dynamics are simply connected, since the flow is irrotational, it is conservative hence there exists a scale potential ϕ such that

$$\mathbf{u} = \nabla \phi.$$

We also have that the flow is incompressible, hence $\nabla^2 \phi = 0$, so ϕ is harmonic. ϕ also satisfies the Von Neumann boundary condition $\frac{\partial \phi}{\partial \mathbf{n}} = \mathbf{U} \cdot \mathbf{n}$ which follows from the kinematic boundary condition $\mathbf{u} \cdot \mathbf{n} = \mathbf{U} \cdot \mathbf{n}$.

Hence solving the Euler equation for irrotational incompressible flow now reduces to solving Laplace's equation with Von Neumann boundary conditions.

4.2 Examples

For simple geometries we build solutions using seperable solutions to Laplace's equations in suitable coordinate systems.

In spherical coordinates recall our general antisyemmtric solution to Laplace's equation is

$$\phi = \sum_{n \geq 0} (A_n r^n + B_n r^{-1-n}) P_n(\cos \theta)$$

where P_n is the nth Legendre polynomial. Usually we only need a small number of modes.

Example. Let's consider radial flow so $\phi = \phi(r)$. Then from Laplace's equation's solution if we don't want θ dependence we need only the $n = 0$ mode. So

$$\frac{\partial \phi}{\partial r} = \frac{A}{r^2}$$

for some constant A. Hence *wlog* $\phi = -\frac{A}{r}$. The physical significance of the factor A can be seen by applying Gauss' law like we would in IB Electrodynamics. Consider the volume flux q across a sphere with radius r.

$$\int \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}S = -\frac{A}{r^2} 4\pi r^2 = -4\pi A.$$

So

$$\phi = -\frac{q}{4\pi r}$$

. This represents a source at the origin if $q > 0$ and a sink if $q < 0$.

Example. If instead we have uniform flow (in the z direction) Then we only want the 1st mode so $\phi = Ur\cos\theta = U_z$ hence

$$\mathbf{u} = \nabla\phi = u_{\hat{\mathbf{z}}}.$$