

Fluid Dynamics

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23rd January 2026

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1 Kinematics

1.1 Streamlines and pathlines

There are two natural ways to think of flow.

- (i) A stationary observer watching flow go past. This is the Eulerian perspective. This is the approach used through this course. We define a velocity field (continuum field) $\mathbf{u}(\mathbf{x}, t)$.
- (ii) A moving observing, travelling along with the flow. This is the Lagrangian perspective.

Definition. (Streamlines) These are curves that are everywhere parallel to the flow at a given instant.

Remark. The streamline that goes through \mathbf{x}_0 at time t_0 is given parametrically as $\mathbf{x} = \mathbf{x}(s, \mathbf{x}_0, t_0)$ and

$$\frac{d\mathbf{x}}{ds} = \mathbf{u}(\mathbf{x}, t_0)$$

(with $\mathbf{x} = \mathbf{x}_0$ at $s = 0$).

The set of streamlines shows the direction of flow a *given* instant a time (all fluid particle at one given time). Take the example $\mathbf{u} = (1, t)$. So at $t = 0$ we have $\mathbf{u} = (1, 0)$ so the streamlines are horizontal lines. At $t = 1$ we have $\mathbf{u} = (1, 1)$, so the streamlines are diagonal.

Definition. (Pathlines) A *pathline* is the trajectory of a fluid particle (a very small bit of fluid). The pathline $\mathbf{x} = \mathbf{x}(t, \mathbf{x}_0)$ of a fluid which is at \mathbf{x}_0 at $t = 0$ is such that

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t)$$

with $\mathbf{x}(0, \mathbf{x}_0) = \mathbf{x}_0$.

Again if we take $\mathbf{u} = (1, t)$ we get

$$\begin{cases} \frac{dx}{dt} = 1 \\ \frac{dy}{dt} = t \end{cases} \rightarrow \begin{cases} x = x_0 + t \\ y = y_0 + \frac{t^2}{2} \end{cases}$$

which describes the path $y - y_0 = \frac{1}{2}(x - x_0)^2$.

Remark. Pathlines are often called "Lagrangian trajectories". The applications are very useful to characterise transport (infectious diseases and pollution simulations).

If the flow is *steady* (so \mathbf{u} does not depend on time). Then pathlines and streamlines are the same.

1.2 The material derivative

We will characterise the rate of change of "stuff" moving with a fluid. Consider a quantity $F(\mathbf{x}, t)$ in a fluid flow (intuition is F is temperature). We want to measure how the temperature changes as we move through the field F along the flow. Let compute the rate of change of (in time) seen

by a moving observer. We will call this $\frac{DF}{Dt}$. Take a small time interval δt . Then

$$\begin{aligned}\delta F &= F(\mathbf{x} + \delta \mathbf{x}, t + \delta t) - F(\mathbf{x}, t) \\ &= \delta t \frac{\partial F}{\partial t} + (\delta \mathbf{x} \cdot \nabla) F + (\text{higher order terms}).\end{aligned}$$

We have that $\delta \mathbf{x} = \mathbf{u} \delta t$, so

$$\frac{\delta F}{\delta t} = \frac{DF}{Dt} = \frac{\partial F}{\partial t} + (\mathbf{u} \cdot \nabla) F.$$

We have the derivative and the convected derivative. This should be thought of as moving along gradients of a field.

1.3 Conservation of mass

Consider the flow through a straight rigid pipe with constant cross section. Suppose we have a \mathbf{u}_{in} and a \mathbf{u}_{out} . Can we have $\mathbf{u}_{\text{in}} \neq \mathbf{u}_{\text{out}}$? For a gas, yes we can since they can be compressed. For a fluid, we cannot, since they are incompressible.

Define $\rho(\mathbf{x}, t)$ as the mass density with $[\rho] = \frac{M}{L^3}$. We want a relation between ρ and \mathbf{u} . Consider a fixed volume V and compute the rate of change of its mass, M .

$$M = \int_V \rho dV$$

Assume that mass can only change due to the flow of mass across the boundary surface ∂V . Take a small surface element δA with normal \mathbf{n} . The volume out of V during δt is $(\mathbf{u} \cdot \mathbf{n}) \delta A \delta t$. Hence the mass out is $\rho(\mathbf{u} \cdot \mathbf{n}) \delta A \delta t$, so we get that

$$\frac{dM}{dt} = - \int_{\partial V} \rho(\mathbf{u} \cdot \mathbf{n}) dA.$$