In [1]:
```python
import sys
import rootpath
sys.path.append(rootpath.detect())
import os
```

In [47]:
```python
import numpy as np
import matplotlib.pyplot as plt
import matplotlib
from matplotlib.patches import Patch
from testsuite.utilities import dominates
from testsuite.utilities import saf
from pymoo.factory import get_performance_indicator
from tqdm import tqdm
import time
import matplotlib.image as mpimg
```

In [3]:
```python
matplotlib.rcParams['mathtext.fontset'] = 'stix';
matplotlib.rcParams['font.family'] = 'STIXGeneral';
matplotlib.rcParams['font.size'] = 15 ;
matplotlib.rcParams['legend.fontsize'] = 15
matplotlib.rcParams['pdf.fonttype'] = 42
matplotlib.rcParams['ps.fonttype'] = 42

COLORS = {
    "a": '#ffa500',
    "b": '#008000',
    "v_b": '#a1d0a1',
    "v_br": '#e6f2e6',
    "v_a": '#ffdda1',
    "v_ar": '#fff6e6'
}
CMAP_VA = matplotlib.colors.ListedColormap(['None', COLORS['a']])
CMAP_VB = matplotlib.colors.ListedColormap(['None', COLORS['b']])
```

In [4]:
```python
def save_fig(fig, filename):
    d = "./figures/"
    path = os.path.join(d, filename)
    fig.savefig(path+".png", dpi=300, facecolor=None, edgecolor=None, orientation='portrait',
                pad_inches=0.12)
```

In [5]:
```python
def volume_a(y, targets, p, rp):
    """
    do elements in y belong to volume a, True if
        - are dominated by any of targets
        - dominate the reference point
        - are dominated by any of p
    """
    va = np.asarray([dominates(targets, yi) and dominates(xyi, rp) and dominates(p, yi)  for yi in y.
flat])
    return va

def volume_b(y, targets, p, rp):
    """
    do elements in y belong to volume b, True if
        - dominate any of targets
        - are dominated by any of p
    """
    vb = np.asarray([np.any([dominates(yi, t) for t in targets]) and dominates(p, xyi) for yi in y])
    return vb
```

In [26]:
```python
def image_case(case, ref_volume=False, resolution=200, ax=None):
    """
    returns figure showing volumes Va and Vb from the case desribed
    by case.

    :param case: dict
        dictionary containing case definition
    :param ref_volume: bool
        If true show the outer volumes of which Va and Vb should be
        calculated as a fraction. Defaults to false as calculating
        these is slow (due to many dominantion relation checks in
        the Pareto front)
    :param resolution: int
        image resoution, keep low (~50) for speed increase.
    """

    if ax is None:
        fig = plt.figure(figsize=[6, 6])
        ax = fig.gca()
        return_fig = Tue
    else:
        return_fig = False

    ax.grid('on')
    ax.axis("scaled")
    ax.set_xticks(range(0,12))
    ax.set_yticks(range(0,12))
    ax.legend(loc="lower left")

    x_lim = ax.get_xlim()
    x = np.linspace(0,x_lim[1], resolution)
    y_lim = ax.get_ylim()
    y = np.linspace(*y_lim, resolution)
    xx, yy = np.meshgrid(x, y)
    xy = np.vstack((xx.flat, yy.flat)).T
    zz = saf(xy, case['p']).reshape(resolution, resolution)

    P_x = x
    P_y = 5/x
    t_ideal = case['target'].min(axis=0)
    t_az = case['target'].max(axis=0)

    # generate boolean masks for volumes of interest: Va and Vb
    va = np.asarray([dominates(case['target'], xyi) and dominates(xyi, case['ref_point']) and dominat
es(case['p'], xyi)  for xyi in xy])
    vb = np.asarray([np.any([dominates(xyi, t) for t in case['target']]) and dominates(case['p'], xyi
) for xyi in xy])

    if ref_volume:
        va_r = np.asarray([dominates(case['target'], xyi) and dominates(xyi, case['ref_point']) for x
yi in xy])
        vb_r = np.asarray([np.any([dominates(pi, xyi) for pi in np.vstack((P_x, P_y)).T]) and np.any
([dominates(xyi, t) for t in case['target']]) for xyi in xy])



    a = ax.get_xlim()
    b = ax.get_ylim()
    vol_a = ax.imshow(va.reshape(resolution,resolution)[::-1, :], alpha=0.3, extent=[0, 11, 0, 11], c
map=CMAP_VA)
    vol_b = ax.imshow(vb.reshape(resolution,resolution)[::-1, :], alpha=0.3, extent=[*ax.get_xlim(),
*ax.get_ylim()], cmap=CMAP_VB)

    if ref_volume:
        vol_ar = ax.imshow(va_r.reshape(resolution,resolution)[::-1, :], alpha=0.1, extent=[0, 11, 0,
 11], cmap=CMAP_VA)
        vol_br = ax.imshow(vb_r.reshape(resolution,resolution)[::-1, :], alpha=0.1, extent=[0, 11, 0,
 11], cmap=CMAP_VB)

    ax.contour(xx, yy, zz, levels=[0.], colors="C0", alpha=0.5)


    ax.scatter(*case['p'].T, c="C0", label=r"$\tilde{P}$")
    ax.scatter(*case['target'].T, c="magenta", label=r"$T$")
    ax.scatter(*case['ref_point'], c="C3", label=r"$r$")

    ax.scatter(*t_ideal, c="magenta", label=r"$\tilde{t}$", marker="*")
    ax.scatter(*t_az, c="magenta", label=r"$\check{t}$", marker="v")

    ax.hlines(t_ideal[1], t_ideal[0], t_az[0], colors="magenta", linestyle="--", alpha=0.2)
    ax.hlines(t_az[1], t_ideal[0], t_az[0], colors="magenta", linestyle="--", alpha=0.2)
    ax.vlines(t_ideal[0], t_ideal[1], t_az[1], colors="magenta", linestyle="--", alpha=0.2)
    ax.vlines(t_az[0], t_ideal[1], t_az[1], colors="magenta", linestyle="--", alpha=0.2)


    ax.plot(P_x, P_y, c="k", label=r"$P$")
    handles, labels = ax.get_legend_handles_labels()
    handles += [plt.Line2D([0], [0], c="C0", alpha=0.5)]
    handles += [Patch(facecolor="orange", alpha=0.4), Patch(facecolor="green", alpha=0.3)]
    labels += [r"$A$", r"$V_a$", r"$V_b$"]
    nc = 3

    if ref_volume:
        handles += [Patch(facecolor="orange", alpha=0.1), Patch(facecolor="green", alpha=0.1)]
        labels += [r"$V_{ar}$", r"$V_{br}$"]

        nc = 4


    ax.legend(handles, labels, loc="lower left", ncol=nc)

    if ref_volume:
        va_frac = (np.sum(va)/np.sum(va_r)).round(3)
        vb_frac = (np.sum(vb)/np.sum(vb_r)).round(3)
    else:
        va_frac = (np.sum(va)/len(va)).round(3)
        vb_frac = (np.sum(vb)/len(vb)).round(3)

    pos_a = np.vstack(np.where(va.reshape(resolution, resolution))).T.mean(axis=0)/resolution*(np.arr
ay([x_lim[1]-x_lim[0], y_lim[1]-y_lim[0]]))
    pos_b = np.vstack(np.where(vb.reshape(resolution, resolution))).T.mean(axis=0)/resolution*(np.arr
ay([x_lim[1]-x_lim[0], y_lim[1]-y_lim[0]]))

    if ref_volume:
        ax.text(pos_a[1], pos_a[0], fr"$M_a={va_frac}$", c=CMAP_VA.colors[1])
        ax.text(pos_b[1], pos_b[0], fr"$M_b={vb_frac}$", c=CMAP_VB.colors[1])
    else:
        ax.text(pos_a[1], pos_a[0], fr"$V_a={va_frac}$", c=CMAP_VA.colors[1])
        ax.text(pos_b[1], pos_b[0], fr"$V_b={vb_frac}$", c=CMAP_VB.colors[1])

    ax.set_xlabel(r"$f_1$")
    ax.set_ylabel(r"$f_2$")

    if return_fig:
        return fig
```

In [27]:
```python
def montecarlo_case(case, samples=1e6, ax=None):
    """
    :param case: dict
        dictionary containing case definition
    :param ref_volume: bool
        If true show the outer volumes of which Va and Vb should be
        calculated as a fraction. Defaults to false as calculating
        these is slow (due to many dominantion relation checks in
        the Pareto front)
    :param resolution: int
        image resoution, keep low (~50) for speed increase.
    """
    samples = int(samples)
    resolution = int(np.sqrt(samples)/5)

    if ax is None:
        fig = plt.figure(figsize=[6, 6])
        ax = fig.gca()
        return_fig = Tue
    else:
        return_fig = False

    ax.grid('on')
    ax.axis("scaled")
    ax.set_xticks(range(0,12))
    ax.set_yticks(range(0,12))
    ax.legend(loc="lower left")

    x_lim = ax.get_xlim()
    x = np.linspace(0,x_lim[1], resolution)
    y_lim = ax.get_ylim()
    y = np.linspace(*y_lim, resolution)
    xx, yy = np.meshgrid(x, y)
    xy = np.vstack((xx.flat, yy.flat)).T
    zz = saf(xy, case['p']).reshape(resolution, resolution)

    xy = np.vstack([np.random.uniform(0, i, samples) for i in case['ref_point']]).T

    P_x = x
    P_y = 5/x
    t_ideal = case['target'].min(axis=0)
    t_az = case['target'].max(axis=0)

    va_r = xy[[dominates(case['target'], xyi) and dominates(xyi, case['ref_point']) for xyi in xy]]
    vb_r = xy[[np.any([dominates(pi, xyi) for pi in np.vstack((P_x, P_y)).T]) and np.any([dominates(x
yi, t) for t in case['target']]) for xyi in xy]]
    va = va_r[[dominates(case['target'], xyi) and dominates(xyi, case['ref_point']) and dominates(cas
e['p'], xyi)  for xyi in va_r]]
    vb = vb_r[[np.any([dominates(xyi, t) for t in case['target']]) and dominates(case['p'], xyi) for
xyi in vb_r]]

    ax.scatter(*va_r.T, c=COLORS['v_ar'], label=r"$V_{ar}$", s=5)
    ax.scatter(*vb_r.T, c=COLORS['v_br'], label=r"$V_{br}$", s=5)
    ax.scatter(*va.T, c=COLORS['v_a'], label=r"$V_{a}$", s=5)
    ax.scatter(*vb.T, c=COLORS['v_b'], label=r"$V_{b}$", s=5)


    t_ideal = case['target'].min(axis=0)
    t_az = case['target'].max(axis=0)

    ax.contour(xx, yy, zz, levels=[0.], colors="C0", alpha=0.5)


    ax.scatter(*case['p'].T, c="C0", label=r"$\tilde{P}$")
    ax.scatter(*case['target'].T, c="magenta", label=r"$T$")
    ax.scatter(*case['ref_point'], c="C3", label=r"$r$")

    ax.scatter(*t_ideal, c="magenta", label=r"$\tilde{t}$", marker="*")
    ax.scatter(*t_az, c="magenta", label=r"$\check{t}$", marker="v")

    ax.hlines(t_ideal[1], t_ideal[0], t_az[0], colors="magenta", linestyle="--", alpha=0.2)
    ax.hlines(t_az[1], t_ideal[0], t_az[0], colors="magenta", linestyle="--", alpha=0.2)
    ax.vlines(t_ideal[0], t_ideal[1], t_az[1], colors="magenta", linestyle="--", alpha=0.2)
    ax.vlines(t_az[0], t_ideal[1], t_az[1], colors="magenta", linestyle="--", alpha=0.2)

    ax.plot(P_x, P_y, c="k", label=r"$P$")
    nc =  int(np.ceil(len(ax.get_legend_handles_labels()[0])/3))
    handles, labels = ax.get_legend_handles_labels()
    handles += [plt.Line2D([0], [0], c="C0", alpha=0.5)]
    labels += [r"$A$"]
    ax.legend(handles, labels, ncol=nc, loc="lower left")

    va_frac = np.round(len(va)/len(va_r), 3)
    vb_frac = np.round(len(vb)/len(vb_r), 3)

    print(va_frac)
    print(vb_frac)
    pos_a = va.mean(axis=0)
    pos_b = vb.mean(axis=0)

    ax.text(pos_a[0], pos_a[1], fr"$M_a={va_frac}$", c="k")
    ax.text(pos_b[0], pos_b[1], fr"$M_b={vb_frac}$", c="k")

    ax.set_xlabel(r"$f_1$")
    ax.set_ylabel(r"$f_2$")


    if return_fig:
        return fig
```

Nomanclature

- P: Pareto front
- $\tilde{P}$: Approximation to Pareto front: $\tilde{P} = \{y_i\}_{i=1}^{n}$
- A: attainment front
- r: reference point
- T: targets: $T = \{t_i\}_{i=1}^{n_t}$
- $\tilde{t}$: target ideal $\tilde{t} = \{\tilde{t}_j\}_{j=1}^{n}$ where:
$$\tilde{t}_j = \min_{u \in T}\{u_j\}$$
- $\check{t}$: target zenith $\check{t} = \{\check{t}_j\}_{j=1}^{n}$ where:
$$\check{t}_j = \max_{u \in T}\{u_j\}$$
- $H(Y, r)$: Volume dominated by Y referred to r

Region that is dominated by a set Z in objective space:
$$\mathrm{dom}(Z) = \{u \mid \exists z \in Z \; s.t. \; z \prec u\}$$

Volume definitions

$$V_a = H(\mathrm{dom}(T) \cap \mathrm{dom}(\tilde{P}), r)$$
$$V_b = H(\mathrm{dom}(\tilde{P}) \setminus \mathrm{dom}(\check{t}), r)$$

Reference Volume definitions

$$V_{ar} = H(\mathrm{dom}(T) \cap \mathrm{dom}(P), r)$$
$$V_{br} = H(\mathrm{dom}(P) \setminus \mathrm{dom}(\check{t}), r)$$

Relative Hypervolume Scores

$$M_a = \frac{V_a}{V_{ar}} = \frac{H(\mathrm{dom}(T) \cap \mathrm{dom}(\tilde{P}), r)}{H(\mathrm{dom}(T) \cap \mathrm{dom}(P), r)}$$
$$M_b = \frac{V_b}{V_{br}} = \frac{H(\mathrm{dom}(\tilde{P}) \setminus \mathrm{dom}(\check{t}), r)}{H(\mathrm{dom}(P) \setminus \mathrm{dom}(\check{t}), r)}$$

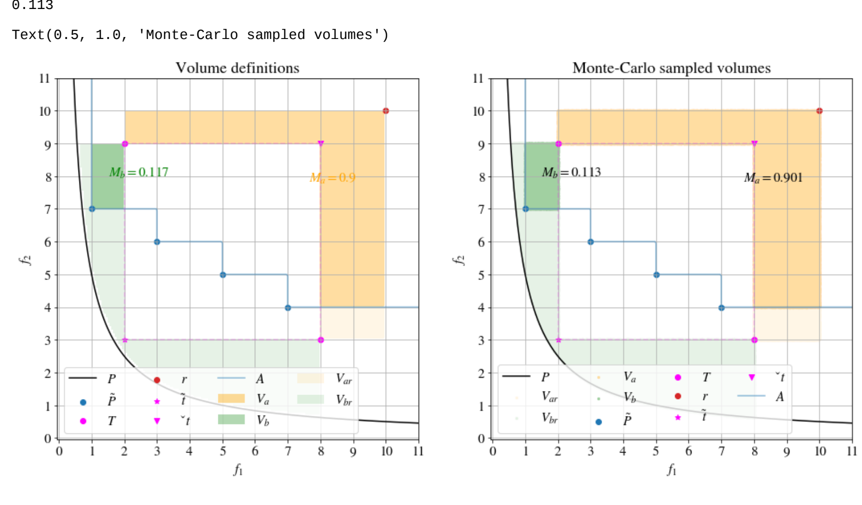
In [8]:
```python
# define test cases

# single target attained
case_00 = {'ref_point': np.array([10., 10.]),
           'name': 'case_00',
          'target' : np.array([[6., 7.]]),
          'p' : np.array([[1., 7.],
                          [3., 6.],
                          [5., 5.],
                          [7., 4.]]),
          'doh': (12., 4.)
          }

# single target unattained
case_01 = {'ref_point': np.array([10., 10.]),
           'name': 'case_01',
           'target' : np.array([[2., 4.]]),
          'p' : np.array([[1., 7.],
                          [3., 6.],
                          [5., 5.],
                          [7., 4.]]),
          'doh': (39., 0.)
          }

# dual targets, both attained
case_02 = {'ref_point': np.array([10., 10.]),
           'name': 'case_02',
           'target' : np.array([[6., 7.],
                                [8., 5.]
                                ]),
           'p' : np.array([[1., 7.],
                           [3., 6.],
                           [5., 5.],
                           [7., 4.]]),
           'doh': (12., 4.)
          }

# dual targets, both unattained
case_03 = {'ref_point': np.array([10., 10.]),
           'name': 'case_03',
           'target' : np.array([[2., 5.],
                                [5., 3.]
                                ]),
           'p' : np.array([[1., 7.],
                           [3., 6.],
                           [5., 5.],
                           [7., 4.]]),
           'doh': (12., 4.)
          }

# dual targets, one unattained, one unattained
case_04 = {'ref_point': np.array([10., 10.]),
           'name': 'case_04',
           'target' : np.array([[2., 5.],
                                [8., 5.]
                                ]),
           'p' : np.array([[1., 7.],
                           [3., 6.],
                           [5., 5.],
                           [7., 4.]]),
           'doh': (12., 4.)
          }

# dual targets, one unattained, one unattained, attained beyond span of t
case_05 = {'ref_point': np.array([10., 10.]),
           'name': 'case_05',
           'target' : np.array([[8., 3.],
                                [2., 9.]
                               ]),
           'p' : np.array([[1., 7.],
                           [3., 6.],
                           [5., 5.],
                           [7., 4.]]),
           'doh': (12., 4.)
          }

# dual targets, both unattained, one beyond span of p
case_06 = {'ref_point': np.array([10., 10.]),
           'name': 'case_06',
           'target' : np.array([[2., 5.],
                                [0.8, 9.]
                               ]),
           'p' : np.array([[1., 7.],
                           [3., 6.],
                           [5., 5.],
                           [7., 4.]]),
           'doh': (12., 4.)
          }

# dual targets, both unattained, overlapping volumes
case_07 = {'ref_point': np.array([10., 10.]),
           'name': 'case_07',
           'target' : np.array([[2., 4.],
                                [3., 3.]
                               ]),
           'p' : np.array([[1., 7.],
                           [3., 6.],
                           [5., 5.],
                           [7., 4.]]),
           'doh': (12., 4.)
          }

# dual targets, both attained, one beyond span of p
case_08 = {'ref_point': np.array([10., 10.]),
           'name': 'case_08',
           'target' : np.array([[6., 7.],
                                [9, 5.]
                               ]),
           'p' : np.array([[1., 7.],
                           [3., 6.],
                           [5., 5.],
                           [7., 4.]]),
           'doh': (12., 4.)
          }

# dual targets, both attained, overlapping volumes
case_09 = {'ref_point': np.array([10., 10.]),
           'name': 'case_09',
           'target' : np.array([[5., 8.],
                                [6, 7.]
                               ]),
           'p' : np.array([[1., 7.],
                           [3., 6.],
                           [5., 5.],
                           [7., 4.]]),
           'doh': (12., 4.)
          }

# three targets, one unattained, two unattained, attained beyond span of t
case_10 = {'ref_point': np.array([10., 10.]),
           'name': 'case_10',
           'target' : np.array([[8., 3.],
                                [5., 7.],
                                [2., 9.]
                               ]),
           'p' : np.array([[1., 7.],
                           [3., 6.],
                           [5., 5.],
                           [7., 4.]]),
           'doh': (12., 4.)
          }

case_11 = {'ref_point': np.array([10., 10.]),
           'name': 'case_10',
           'target' : np.array([[4., 9],
                                [6, 8],
                                [8, 7]
                               ]),
           'p' : np.array([[1., 7.],
                           [3., 6.],
                           [5., 5.],
                           [7., 4.]]),
           'doh': (12., 4.)
          }


cases = [case_00,
         case_01,
         case_02,
         case_03,
         case_04,
         case_05,
         case_06,
         case_07,
         case_08,
         case_09,
         case_10,
         case_11]
```

In [29]:
```python
# compute any single case
n_samples = int(2e5)
resolution = int(np.sqrt(n_samples)/5)

case = case_05
tic = time.time()
fig, axes = plt.subplots(1,2, figsize=[16, 8])
image_case(case, True, 200, axes[0])
toc = time.time()
montecarlo_case(case, n_samples, axes[1])

tac = time.time()

axes[0].set_title("Volume definitions")
axes[1].set_title(fr"Monte-Carlo sampled volumes")
```

```
No handles with labels found to put in legend.
<ipython-input-26-c71c711372c0>:39: RuntimeWarning: divide by zero encountered in true_divide
  P_y = 5/x
No handles with labels found to put in legend.
<ipython-input-27-781db8fff530>:40: RuntimeWarning: divide by zero encountered in true_divide
  P_y = 5/x
```

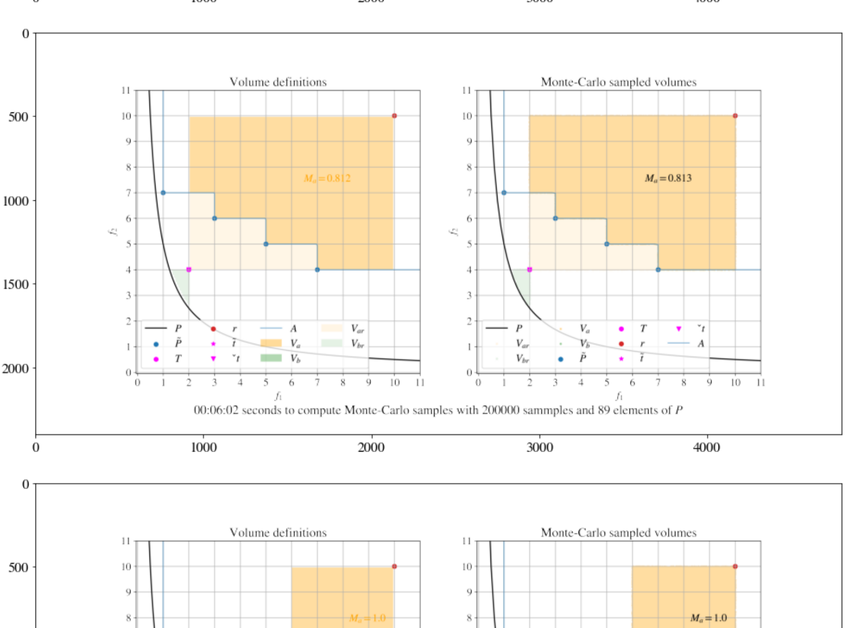
```
0.901
0.113
```

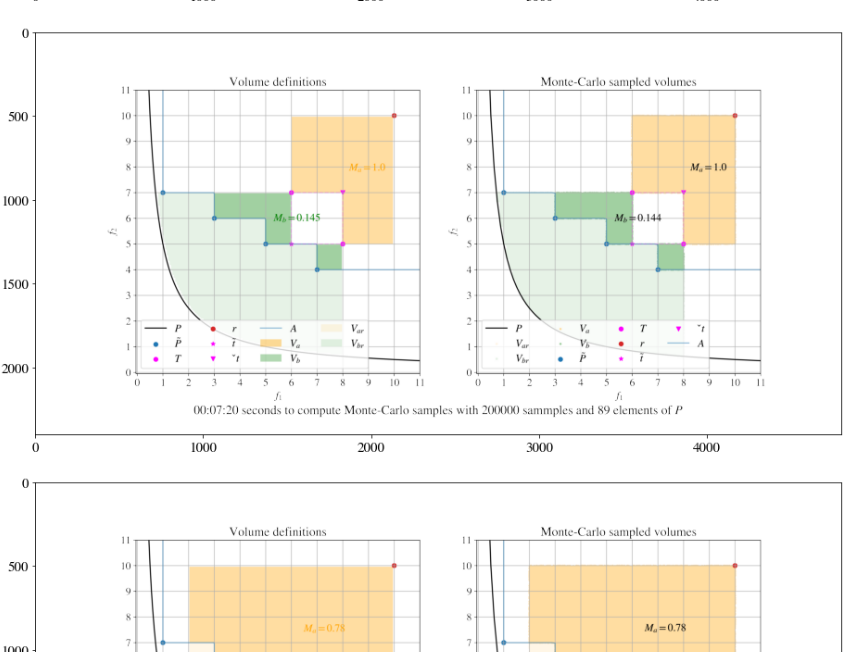
Out[29]:
```
Text(0.5, 1.0, 'Monte-Carlo sampled volumes')
```

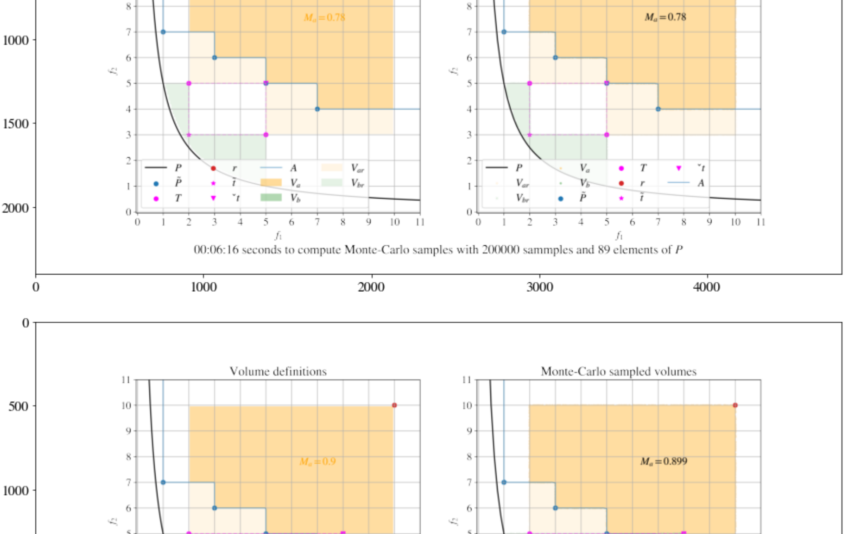


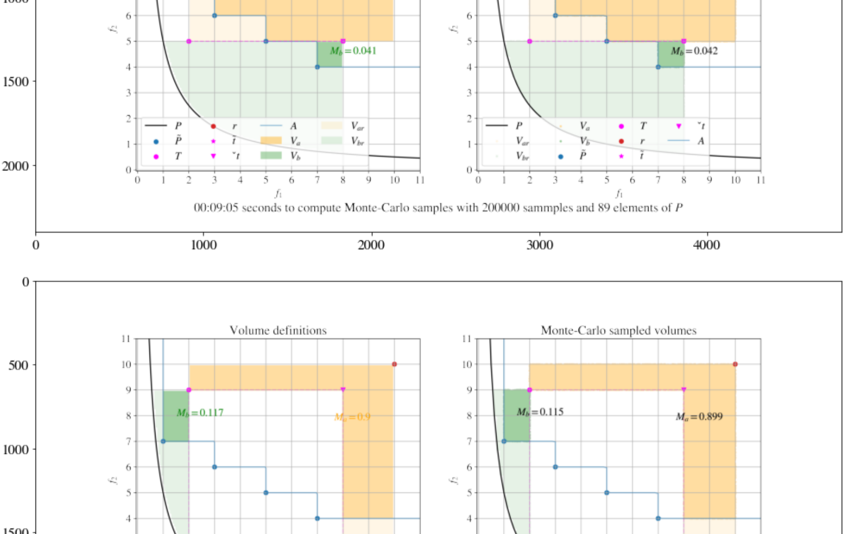
In [52]:
```python
n_samples = int(2e5)
resolution = int(np.sqrt(n_samples)/5)

for case in tqdm(cases):
    case_name = f"figures/{case['name']}.png"

    if os.path.isfile(case_name):
        fig = plt.figure(figsize=[16, 8])
        img = mpimg.imread(case_name)
        fig.gca().imshow(img)
    else:
        fig, axes = plt.subplots(1,2, figsize=[16, 8])
        tic=time.time()
        image_case(case, True, 200, axes[0])
        toc = time.time()
        montecarlo_case(case, n_samples, axes[1])
        tac = time.time()

        axes[0].set_title("Volume definitions")
        axes[1].set_title(f"Monte-Carlo sampled volumes")

        fig.text(0.5, 0.05, fr"{time.strftime('%H:%M:%S', time.gmtime(tac-toc))} seconds to compute Monte-Carlo samples with {n_samples} sammples and {resolution} elements of $P$", ha="center", fontsize=18)
    fig.show()
#    save_fig(fig, case['name'])
```

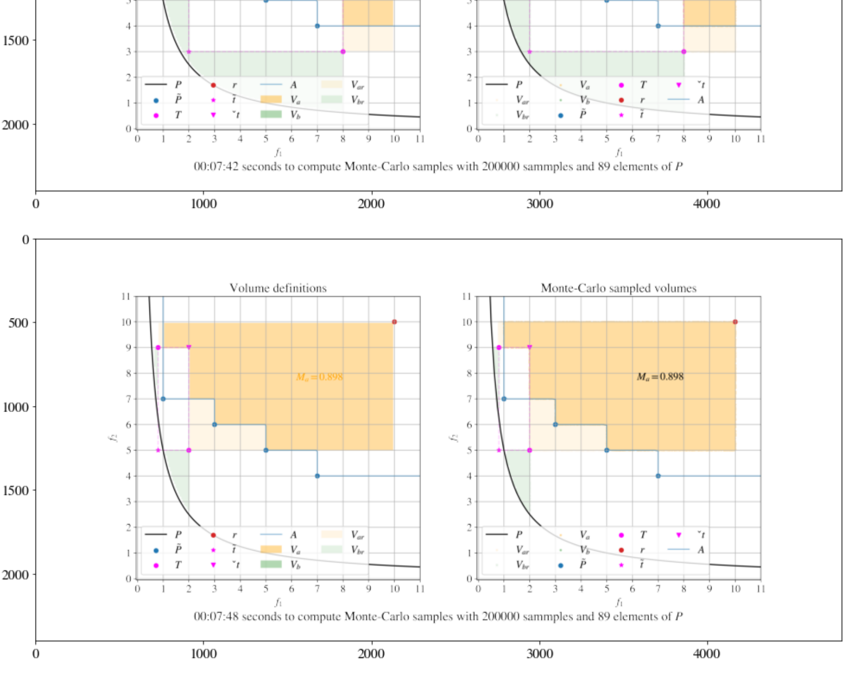
```
  0%|          | 0/12 [00:00<?, ?it/s]<ipython-input-52-65614a23e4f8>:24: UserWarning: Matplotlib is currently using module://ipykernel.pylab.backend_inline, which is a non-GUI backend, so cannot show the figure.
  fig.show()
100%|██████████| 12/12 [00:19<00:00,  1.63s/it]
```

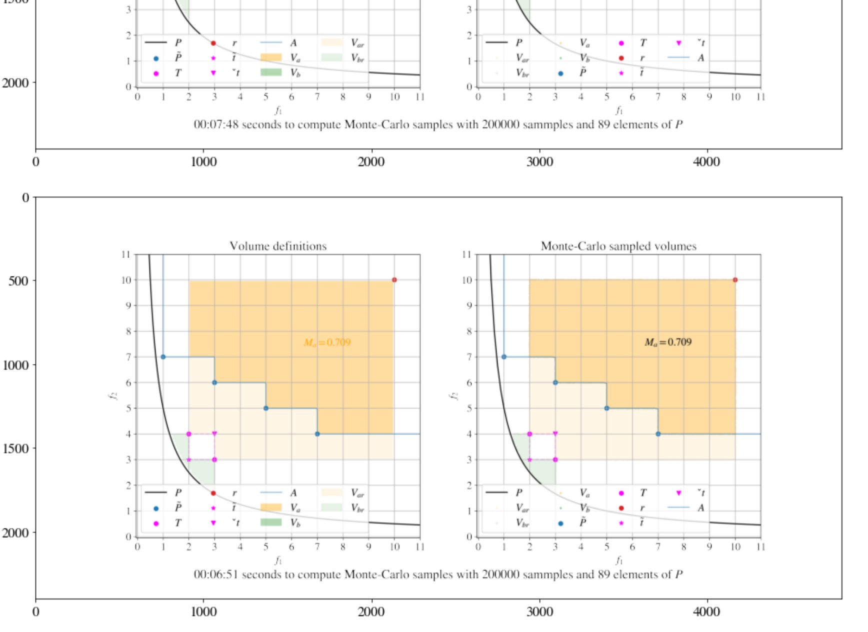


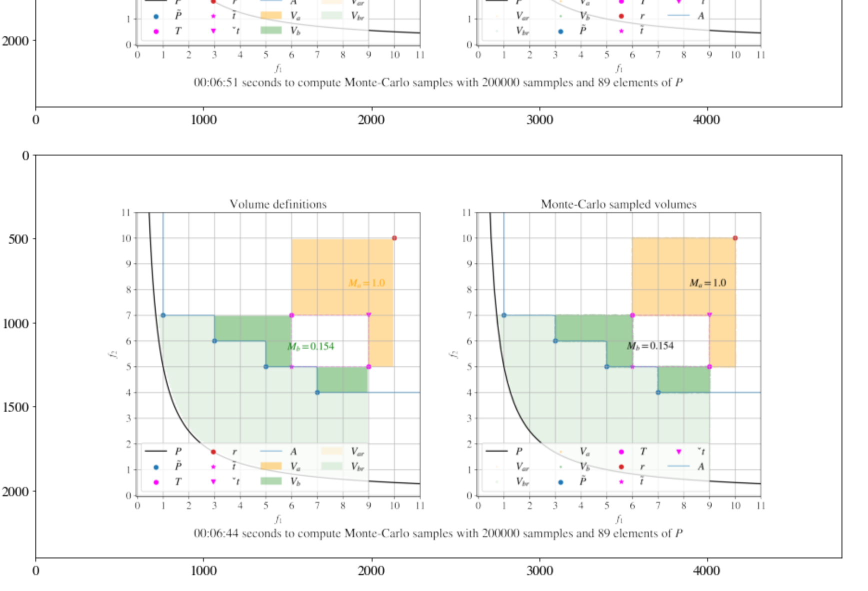


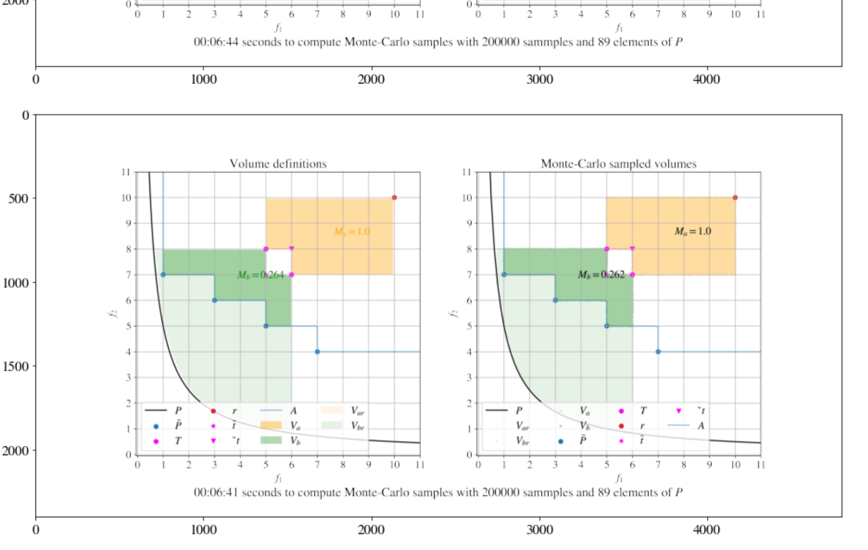


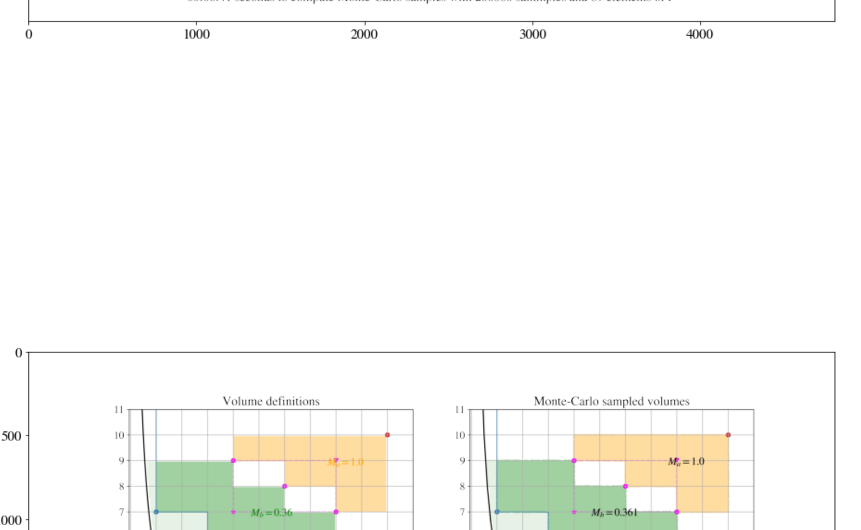


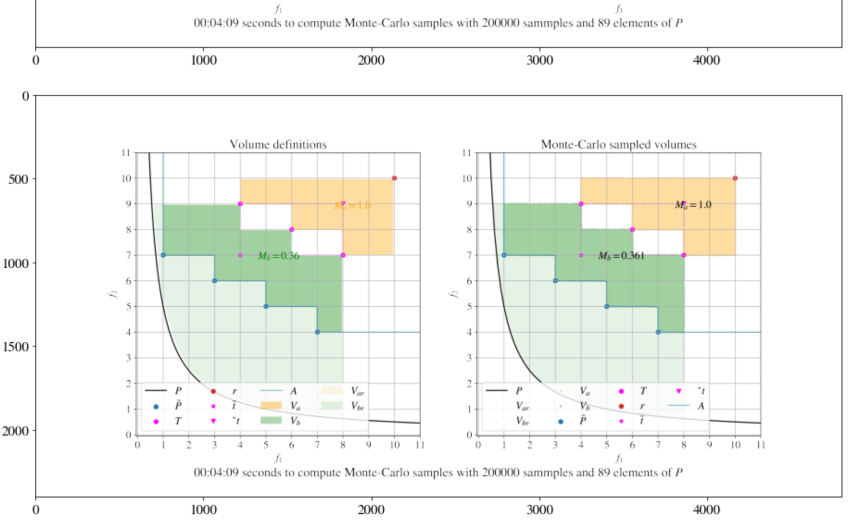


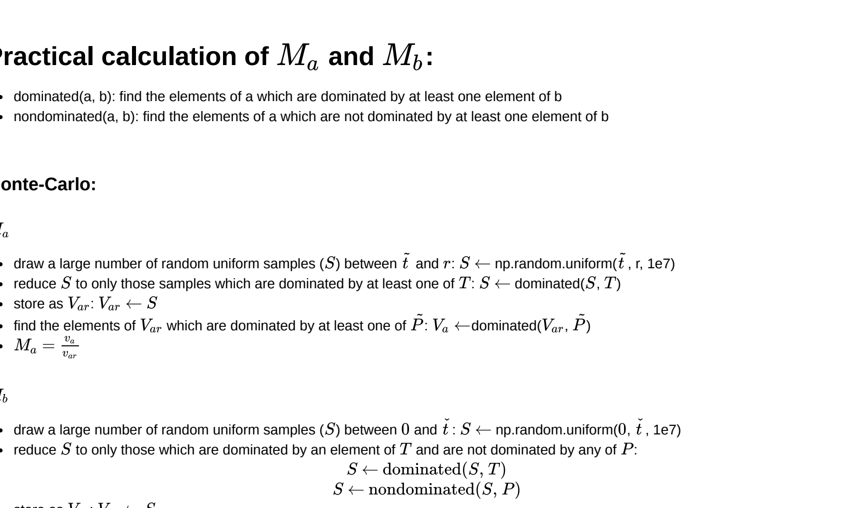












Practical calculation of M_a and M_b:

- dominated(a, b): find the elements of a which are dominated by at least one element of b
- nondominated(a, b): find the elements of a which are not dominated by at least one element of b

Monte-Carlo:

M_a

- draw a large number of random uniform samples (S) between $\tilde{t}$ and r: $S \leftarrow$ np.random.uniform($\tilde{t}$, r, 1e7)
- reduce S to only those samples which are dominated by at least one of T: $S \leftarrow$ dominated(S, T)
- store as V_{ar}: $V_{ar} \leftarrow S$
- find the elements of V_{ar} which are dominated by at least one of $\tilde{P}$: $V_a \leftarrow$dominated(V_{ar}, $\tilde{P}$)
- $M_a = \frac{v_a}{v_{ar}}$

M_b

- draw a large number of random uniform samples (S) between 0 and $\check{t}$: $S \leftarrow$ np.random.uniform(0, $\check{t}$, 1e7)
- reduce S to only those which are dominated by an element of T and are not dominated by any of P:
$$S \leftarrow \mathrm{dominated}(S, T)$$
$$S \leftarrow \mathrm{nondominated}(S, P)$$
- store as V_{br}: $V_{br} \leftarrow S$
- find the elements of V_{br} which are not dominated by any of $\tilde{P}$: $V_b \leftarrow$nondominated(V_{br}, $\tilde{P}$)
- $M_b = \frac{v_b}{v_{br}}$

note: calculating V_{br} will be expensive, but only needs to be done once for each problem configuration