Notes for Lecture 2: Induction

1 Proof by Contradiction

To prove a proposition P is true, we assume that P is false (i.e. $\neg P$ is true) and then use this hypothesis to derive a falsehood or contradiction. We can show that P is true if we can show that $\neg P \implies F$, F being a falsehood.

Example 1. Prove that $\sqrt{2}$ is irrational.

An irrational number is a number that can't be expressed as a ratio of integers.

Proof: (by contradiction)

Assume for the purpose of contradiction that $\sqrt{2}$ is rational.

$$\begin{aligned}
&\to \sqrt{2} = \frac{a}{b} \quad \text{(fraction in lowest terms)}\\
&\to 2 = \frac{a^2}{b^2}\\
&\to 2b^2 = a^2\\
&\to a \text{ is even} \quad (2 \mid a)\\
&\to 4 \mid a^2\\
&\to 4 \mid 2b^2\\
&\to 2 \mid b^2\\
&\to b \text{ is even}\\
&\to a \text{ and } b \text{ are not in lowest terms (common factor of 2)}\\
&\to \text{contradiction}\\
&\to \sqrt{2} \text{ is irrational}
\end{aligned}$$

This proof was first discovered by the Pythagoreans in ancient Greece, a religious society founded by Pythagoras. In ancient Greece, math was intertwined with religion. The Pythagoreans believed in two key gods: Aperon, the god of infinity and chaos, and Peros, the god of the finite and order. They held that all numbers were rational, as irrational numbers represented the infinite and were considered bad. One of their axioms was that the length of every line was finite and rational. According to the Pythagorean theorem, a triangle with two sides of length 1 has a hypotenuse of $\sqrt{2}$, which by their axioms, should be rational. However, they eventually discovered a proof that $\sqrt{2}$ was irrational, causing a major crisis in their society. Their axioms were inconsistent, casting doubt on all their theorems. They tried to keep this proof a secret, but a whistleblower revealed it, and legend has it that they killed the person who exposed the proof.

2 Induction Axiom

- Let $P(n)$ be a predicate. If $P(0)$ is true and $\forall\ n \in \mathbb{N}, P(n) \implies P(n+1)$ is true, then $\forall\ n \in \mathbb{N}, P(n)$ is true.
- If $P(0)$, $P(0) \implies P(1)$, $P(1) \implies P(2)$, $\ldots$ are true.

You can view this like dominoes. If you can prove that the first domino falls, and that if any domino falls, the next one will fall, then all the dominoes will fall.

Theorem: $\forall n \geq 0\ 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

Aside: Here are different ways to express the sum of the first n natural numbers:

- $1 + 2 + 3 + \ldots + n$
- $\sum_{i=1}^{n} i$
- $\sum_{1 \leq i \leq n} i$

If $n = 1$ $\quad 1 + \cancel{2} + \cancel{3} + \cdots + n = 1$
If $n \leq 0$ $\quad \cancel{1} + \cancel{2} + \cancel{3} + \cdots + n = 0$
If $n = 4$ $\quad 1 + 2 + 3 + 4 = 10$

Proof: (by induction)

Let $P(n)$ be the proposition that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

- Base case: $P(0)$ is true because $\sum_{i=1}^{0} i = 0 = \frac{0(0+1)}{2}$.
- Inductive step: For $n \geq 0$, show $P(n) \implies P(n+1)$.
- Assume $P(n)$ is true for purposes of induction (we assumed $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$).

- We need to show that $1 + 2 + \cdots + n + (n+1) = \frac{(n+1)(n+2)}{2}$.

$$\begin{aligned}
\sum_{i=1}^{n+1} i &= \sum_{i=1}^{n} i + (n+1) \\
&= \frac{n(n+1)}{2} + (n+1) \\
&= \frac{n(n+1) + 2(n+1)}{2} \\
&= \frac{(n+1)(n+2)}{2} \quad \checkmark
\end{aligned}$$

Therefore, $P(n+1)$ is true.