Functional Derivatives

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I have $n_e(x) = n_e(n_i(x), E(x)) = Z_i n_i(x) + b\nabla \cdot \boldsymbol{E}(x)$ and some functional $F[n_e(n_i, \boldsymbol{E})]$. The differential of F

$$\delta F = F[n_e + \delta n_e] - F[n_e] = \int dx \, \frac{\delta F[n_e]}{\delta n_e} \delta n_e$$

lets me find $\frac{\delta F[n_e]}{\delta n_e}$. Wikipedia gives chain rules

$$\begin{aligned} \frac{\delta F[n_e]}{\delta n_i} &= \frac{\delta F[n_e]}{\delta n_e} \frac{\partial n_e}{\partial n_i}, \\ \frac{\delta F[n_e]}{\delta \boldsymbol{E}} &= \frac{\delta F[n_e]}{\delta n_e} \frac{\partial n_e}{\partial \boldsymbol{E}}, \end{aligned}$$

where

$$\begin{aligned} \frac{\partial n_e}{\partial n_i} &= Z_i, \\ \frac{\partial n_e}{\partial \boldsymbol{E}} &= \boldsymbol{E} \cdot \nabla n_e = \text{just gross vector calc stuff.} \end{aligned}$$

These are chain rules for when n_e is just a function of other functions ($\nabla \cdot$ is just a differential operator between function spaces). Wikipedia has another chain rule for $\frac{\delta F[G[\rho]]}{\delta \rho}$. G is a functional that turns ρ into a number (by integrating).

Example if $N[n_e] = \int dx \, n_e(x)$:

$$\delta N = \int dx \, \frac{\delta N[n_e]}{\delta n_e} \delta n_e = N[n_e + \delta n_e] - N[n_e] = \int dx \, \delta n_e \Longrightarrow \frac{\delta N[n_e]}{\delta n_e} = 1.$$

$$\begin{aligned} \frac{\delta N[n_e]}{\delta n_i} &= Z_i, \\ \frac{\delta N[n_e]}{\delta \boldsymbol{E}} &= \boldsymbol{E} \cdot \nabla n_e. \end{aligned}$$

I think the chain rules have to be the way they are so that we get

$$\begin{aligned} \delta F &= \int dx \, \frac{\delta F[n_e]}{\delta n_e} \delta n_e \\ &= \int dx \, \frac{\delta F}{\delta n_e} \left[\frac{\partial n_e}{\partial n_i} \delta n_i + \frac{\partial n_e}{\partial \boldsymbol{E}} \cdot \delta \boldsymbol{E} \right] \\ &= \int dx \, \left[\frac{\delta F[n_e]}{\delta n_i} \delta n_i + \frac{\delta F[n_e]}{\delta \boldsymbol{E}} \cdot \delta \boldsymbol{E} \right] \\ &= F[n_e(n_i + \delta n_i, \boldsymbol{E} + \delta \boldsymbol{E})] - F[n_e(n_i, \boldsymbol{E})]. \end{aligned}$$

in analogy with

$$dn_e = \frac{\partial n_e}{\partial n_i} dn_i + \frac{\partial n_e}{\partial \boldsymbol{E}} \cdot d\boldsymbol{E}.$$

Each component of the divergence in $\frac{\partial n_e}{\partial \boldsymbol{E}}$ looks like $\frac{\partial}{\partial E_i} \partial_j E_j = d(\partial_j E_j) \left(\frac{\partial}{\partial E_i} \right)$. We have $dE_i = \partial_j E_i dx_j$ and $d(\partial_j E_j) = \partial_k \partial_j E_j dx_k$, as well as

$$\frac{\partial}{\partial E_i} = \frac{\partial x_j}{\partial E_i} \frac{\partial}{\partial x_j}.$$

Together, this should be

$$d(\partial_j E_j) \left(\frac{\partial}{\partial E_i} \right) = \partial_k \partial_j E_j \frac{\partial x_l}{\partial E_i} \delta_{lk} = \partial_k \partial_j E_j \frac{\partial x_k}{\partial E_i}.$$

If each $E_i : \mathbb{R}^3 \to \mathbb{R}$ is a bijection, then $\boldsymbol{E}_i^{-1} \left(E_i(\boldsymbol{x}) \right) = \boldsymbol{x}$, and

$$\boldsymbol{e}_y \cdot \partial_y \boldsymbol{x} = 1 = \boldsymbol{e}_y \cdot \partial_y \boldsymbol{E}_i^{-1} \left(E_i(\boldsymbol{x}) \right) = \frac{\partial y}{\partial E_i} \frac{\partial E_i}{\partial y}.$$

Idk how to generalize this to E field not being bijective. Then

$$\left[\frac{\partial}{\partial \boldsymbol{E}} \nabla \cdot \boldsymbol{E} \right]_i = \frac{\partial}{\partial E_i} \partial_j E_j = \frac{\partial_k \partial_j E_j}{\partial_k E_i} = \left[\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{E}} \cdot \nabla \left(\nabla \cdot \boldsymbol{E} \right) \right]_i.$$

Nope, actually I think this is how it should work:

$$\begin{aligned} dn_e = \frac{\partial n_e}{\partial n_i} dn_i + \frac{\partial n_e}{\partial \boldsymbol{E}} \cdot d\boldsymbol{E} &= n_e(n_i + dn_i, \boldsymbol{E} + d\boldsymbol{E}) - n_e(n_i, \boldsymbol{E}) \\ &= Z_i \left(n_i + dn_i \right) + b \nabla \cdot \left(\boldsymbol{E} + d\boldsymbol{E} \right) - Z_i n_i - b \nabla \cdot \boldsymbol{E} \\ &= Z_i dn_i + b \nabla \cdot d\boldsymbol{E}, \end{aligned}$$

so $\frac{\partial n_e}{\partial n_i} = Z_i$ and $\frac{\partial n_e}{\partial \boldsymbol{E}} = b\nabla$.