Gujarati Appendix C
Notes and Questions

NO: 1.a
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- **Relations between Z, T, χ^2, F**

$$X \sim N(\mu, \sigma^2)$$

$\downarrow$ standardize

$$Z = \frac{X-\mu}{\sigma} \sim N(0,1) \longrightarrow Z^2 \sim \chi^2(1) \quad \Gamma\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

$n \to \infty$
$T \to Z$

$$T = \frac{Z}{\sqrt{\frac{Y}{k}}} \sim t(k) \longleftarrow Y = \sum_{i=1}^{k} Z_i^2 \sim \chi^2(k) \quad \Gamma\left(\tfrac{k}{2}, \tfrac{1}{2}\right)$$

df $\qquad$ df

$$\begin{cases} E(T) = 0 \\ Var(T) = \dfrac{k}{k-2} \end{cases} \qquad \begin{cases} E(Y) = k \\ Var(Y) = 2k \end{cases}$$

$Y_1 \sim \chi^2(k_1)$

$Y_2 \sim \chi^2(k_2)$

$$G = \frac{\frac{Y_1}{k_1}}{\frac{Y_2}{k_2}} \sim F(k_1, k_2)$$

- **sampling distributions**

$$\begin{cases} \bar{X} = \dfrac{\sum_{i=1}^{n} X_i}{n} \longrightarrow \text{estimator of } \mu \\ S^2 = \dfrac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1} \rightarrow \text{estimator of } \sigma^2 \end{cases}$$

if $X_i \sim N(\mu, \sigma^2)$

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} = Z \sim N(0,1) \Longrightarrow \left(\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)^2 = Z^2 \sim \chi^2(1)$$

if σ unknown

$$\frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} = T \sim t(n-1) \quad \text{why ?}$$

$(S = \hat{\sigma})$. $\sqrt{n}$ not $(n-1)$ $\qquad$ not. n.

To see why, we check the equality

$$\sum_{i=1}^{n}(X_i-\overline{X})^2 = \sum_{i=1}^{n}(X_i-\mu)^2 - n(\overline{X}-\mu)^2$$

$$\left(\sum_{i=1}^{n}\left[(X_i-\mu)-(\overline{X}-\mu)\right]^2\right)$$

$$\Rightarrow \underbrace{\sum_{i=1}^{n}\left(\frac{X_i-\overline{X}}{\sigma}\right)^2}_{\downarrow\ \chi^2(n-1)} = \underbrace{\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2}_{\downarrow\ \chi^2(n)} - \underbrace{\left(\frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)^2}_{\downarrow\ \chi^2(1)}$$

intuitive (not rigorous)

$$\Rightarrow S^2 = \frac{\sum_{i=1}^{n}(X_i-\overline{X})^2}{n-1}$$

$$Y \leftarrow (n-1)\frac{S^2}{\sigma^2} = \sum_{i=1}^{n}\left(\frac{X_i-\overline{X}}{\sigma}\right)^2 \sim \chi^2(n-1)$$

$$\Rightarrow T \leftarrow \frac{\overline{X}-\mu}{\frac{S}{\sqrt{n}}} = \frac{\frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}}}{\frac{\frac{S}{\sqrt{n}}}{\frac{\sigma}{\sqrt{n}}}} = \frac{Z}{\frac{S}{\sigma}}$$

$$= \frac{Z}{\sqrt{\frac{S^2}{\sigma^2}}} = \frac{Z}{\sqrt{\frac{Y}{n-1}}} \sim t(n-1)$$

$Y \rightarrow \chi^2(n-1)$

Note:
$$\begin{cases} sd(\overline{X}) = \dfrac{\sigma}{\sqrt{n}} & \leftarrow \text{population sd} \\ se(\overline{X}) = \dfrac{s}{\sqrt{n}} \left(\text{or } \dfrac{\hat{\sigma}}{\sqrt{n}}\right) & \leftarrow \text{sample sd} \end{cases}$$

- Question: $Var(S^2) = ?$

$$Var(S^2) = Var\Big(\frac{\sigma^2}{n-1}\underbrace{(n-1)\frac{S^2}{\sigma^2}}_{Y \sim \chi^2(n-1)}\Big)$$

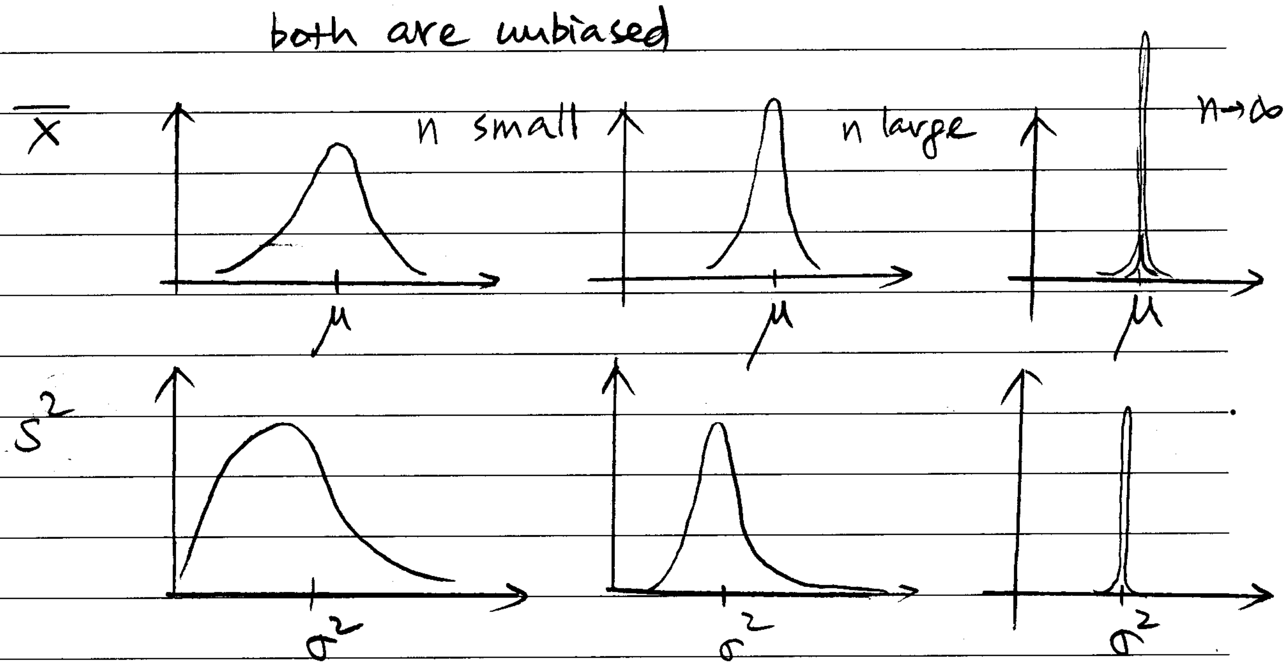
$$= \Big(\frac{\sigma^2}{n-1}\Big)^2 \underbrace{Var(Y)}_{2(n-1)} = \frac{2\sigma^4}{n-1}.$$

<u>Summary</u>

	mean	variance
$\bar{X}$	$E(\bar{X}) = \mu$	$Var(\bar{X}) = \frac{\sigma^2}{n} \xrightarrow{n\to\infty} 0$
S^2	$E(S^2) = \sigma^2$	$Var(S^2) = \frac{2\sigma^4}{n-1} \longrightarrow 0$

↓

both are unbiased



we write $\text{plim}(\underbrace{\bar{X}}_{\text{r.v.}}) = \underbrace{\mu}_{\text{const}}$, $\text{plim}(\underbrace{S^2}_{\text{r.v.}}) = \underbrace{\sigma^2}_{\text{const}}$

X_i: any distribution, iid

① LLN: $\text{plim}(\bar{X}) = \mu$, or $\bar{X} \xrightarrow{n\to\infty} \mu$

② CLT: $\bar{X} \xrightarrow[n\to\infty]{} N(\mu, \frac{\sigma^2}{n})$, or $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \to N(0,1)$

- **Test statistic (r.v)**

1 population	2 populations (omitted)
① $H_0 : \mu = 10$ (μ_0)	③ $H_0 : \mu_1 = \mu_2$
② $H_0 : \sigma^2 = 20$ (σ_0^2)	④ $H_0 : \sigma_1^2 = \sigma_2^2$

① if $\mu = 10$ is true, $\bar{X} \approx 10$

test statistic $T = \dfrac{\bar{X} - \mu_0}{se(\bar{X})} \sim t(n-1)$, $se(\bar{X}) \to \dfrac{s}{\sqrt{n}}$

② if $\sigma^2 = 20$ is true, $s^2 \approx 20$

test statistic $\underset{(\chi^2)}{Y} = \dfrac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1)$

③ omitted

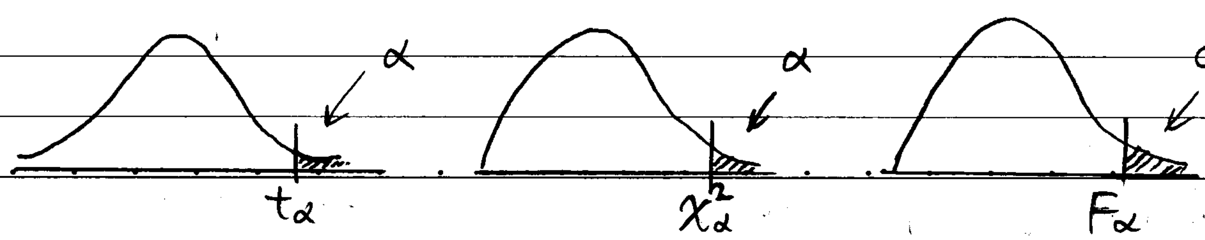
④ if $\sigma_1^2 = \sigma_2^2$ is true, $s_1^2 \approx s_2^2$

we do not know the dist of $s_1^2 - s_2^2$
but we do know the dist of $\dfrac{s_1^2}{s_2^2}$

test statistic $\underset{(F)}{G} = \dfrac{s_1^2}{s_2^2} \sim F(n_1-1, n_2-1)$

critical values (α: significance level)

$t_{\alpha, n-1}$ $\qquad$ $\chi^2_{\alpha, n-1}$ $\qquad$ F_{α, n_1-1, n_2-1}



- Relations Z, t, χ^2, F

convergence

① $Z = \dfrac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \xleftarrow{n\to\infty} T = \dfrac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}}$

$\frac{\sigma}{\sqrt{n}} \to sd(\bar{X})$ $\qquad$ $\frac{S}{\sqrt{n}} \to se(\bar{X})$

$Z \leftrightarrow T$ $\qquad Z \sim N(0,1)$ $\qquad T \sim t(n-1)$

$E(Z)=0$ $\qquad\qquad E(T)=0$

$Var(Z)=1$ $\qquad\qquad Var(T) = \dfrac{n-1}{n-3} \to 1$

<u>other way to see this</u>

$T \leftrightarrow \chi^2$ $\qquad T = \dfrac{Z}{\sqrt{\frac{\chi^2(k)}{k}}}$, if $k\to\infty$, $\chi^2(k) \to$ normal (CLT)

$\downarrow$

$N(k, 2k)$

$\dfrac{\chi^2(k)}{k} \to N(1, \frac{2}{k})$, $\frac{2}{k} \to 0$

it will converge to a constant 1

$(\ plim\ (\frac{\chi^2(k)}{k}) = 1\ )$

$T \swarrow$ Z, χ^2

$\downarrow k\to\infty$

$\dfrac{Z}{1} = Z$

② $\chi^2(k) \to Z$, why? in what sense?

$Z \leftrightarrow \chi^2$ $\qquad \chi^2(k) = \underbrace{\chi^2(1) + \cdots + \chi^2(1)}_{k}$

$$\frac{\chi^2(k) - k}{\sqrt{2k}} \longrightarrow N(0,1)$$

③ $\left\{\begin{array}{l} t^2(k) = F(1,k) \\ F(m,n) = \dfrac{\chi^2(m)}{m}, \text{ if } n\to\infty \end{array}\right\}$ useful in tests to see why, think

- $F(1,k) = \dfrac{\frac{\chi^2(1)}{1}}{\frac{\chi^2(k)}{k}} = \dfrac{Z^2}{\left(\sqrt{\frac{\chi^2(k)}{k}}\right)^2}$

$$= \left(\frac{Z}{\sqrt{\frac{\chi^2(k)}{k}}}\right)^2 = (t(k))^2$$

- $F(m,n) = \dfrac{\frac{\chi^2(m)}{m}}{\frac{\chi^2(n)}{n}} \xrightarrow{n\to\infty} \dfrac{\chi^2(m)}{m}$, $\quad \dfrac{\chi^2(n)}{n} \to 1$

(earlier we saw $\operatorname{plim}\left(\frac{\chi^2(n)}{n}\right) = 1$)

$$\frac{\chi^2(m)}{m} \xrightarrow[\text{CLT again}]{m\to\infty} N\left(1, \frac{2}{m}\right) \to 1$$

cb

$$t(n) \xrightarrow{n\to\infty} N(0,1)$$

$$\chi^2(n) \xrightarrow{n\to\infty} N(n, 2n)$$

$$\left(\frac{\chi^2(n)}{n} \longrightarrow N\left(1, \frac{2}{n}\right)\right)$$

$$F(m,n) \xrightarrow{n\to\infty} \frac{\chi^2(m)}{m} \xrightarrow{m\to\infty} N\left(1, \frac{2}{m}\right)$$

C17 $X \sim N(\mu, \sigma^2)$, $\sigma^2 \to 102.07$

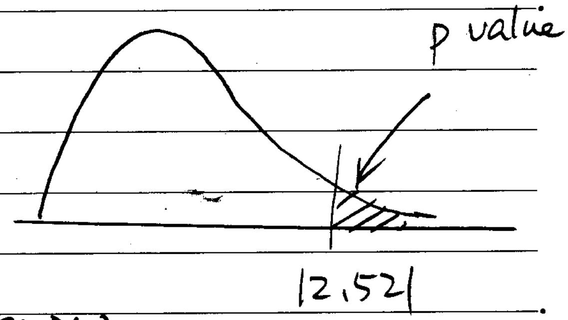
$n = 10$, $S^2 = 142$

$P(\underset{\text{r.v.}}{S^2} > \underset{\text{observed}}{142})$ $\qquad (n-1)\frac{S^2}{\sigma^2} \sim \chi^2_{(n-1)}$

$= P\left((n-1)\frac{S^2}{\sigma^2} > \underset{10}{(n-1)}\frac{142}{\underset{102.07}{\sigma^2}}\right)$

$= P(\underset{\text{r.v.}}{\chi^2_{(9)}} > 12.521)$

$= 0.1855$ (p-value)



(from $\chi^2_{(9)}$ table we only know

$P(S^2 > 142)$ is between $0.1 \sim 0.25$)

C18 (a) $P(S^2 > x \cdot \sigma^2) = 0.1$ find $x = ?$

$P(\underbrace{(n-1)\frac{S^2}{\sigma^2}}_{\chi^2_{(n-1)}} > \underbrace{(n-1)x}_{9x}) = 0.1$

from χ^2 table we find $P(\chi^2_{(9)} > 14.6837) = 0.1$

$\therefore 9x = 14.6837$, $x = 1.6315$

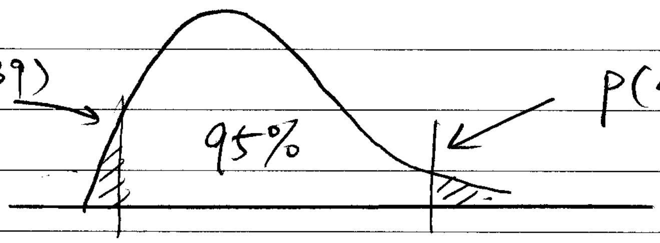
$P(S^2 - \sigma^2 > \underline{0.63}\,\sigma^2) = 0.1$

S^2 greater than σ^2 by 63%.

(b) $P(x\sigma^2 \le S^2 \le y\sigma^2) = 0.95$

$$P\Big((n-1)x \le \underbrace{(n-1)\frac{S^2}{\sigma^2}}_{\chi^2_{(n-1)}} \le (n-1)y\Big) = 0.95$$

$$P(2.70039 \le \chi^2_{(9)} \le 19.0228) = 0.95$$



$\therefore \quad x = 0.30000, \quad y = 2.1136$

C20 $\quad H_0 : \sigma_1^2 = \sigma_2^2$

r.v. $\quad F = \dfrac{S_1^2}{S_2^2} \sim F(\underset{50}{n_1 - 1}, \underset{40}{n_2 - 1})$

statistic value $\quad F = \dfrac{9}{7.2} = 1.25$

$$P(\underset{\text{r.v.}}{F} > 1.25) = 0.2371$$

C24 $\quad \underbrace{m F(m,n) = \chi^2(m)}$ if $n \to \infty$

choose $\alpha = 0.05$

critical values $\chi^2_{(10)} = 18.3070$, $F_{(10,10)} = 2.98$, $F_{(10,20)} = 2.35$, $F_{(10,60)} = 1.99$

We see $10\, F_{(10,n)} \xrightarrow[n\to\infty]{} \chi^2_{(10)}$