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Gujarati Appendix D Questions

D1 ~ D3

population (μ, σ^2) → sample ($x_1 \cdots x_n$)

sample → population: statistical inference ← $\boxed{D2(a)}$

~~observations~~

$f(\underbrace{x_1, \cdots, x_n})$

① estimation of parameters (eg. μ)

② testing of hypotheses (eg. $\mu_1 = 10$)

estimator

$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$ → point estimator of μ ← $\boxed{D1(a)}$

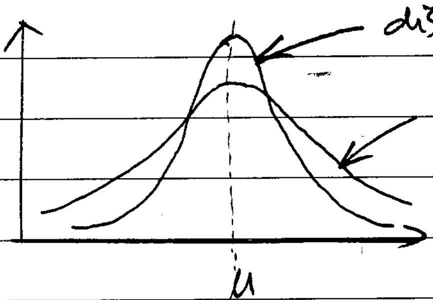
μ: constant (unknown)

as a r.v.

its distribution

distribution of $\bar{X}$ → sampling distribution ← $\boxed{D2(b)}$

distribution of x



estimate
if it represents
a figure

μ: $\begin{cases} \bar{X} : \text{point estimator} \\ [L, U] : \text{interval estimator} \end{cases}$

$P(L \le \mu \le U) = 0.95$ $\boxed{D1(a)}$

95% confidence interval

$E(\bar{X}) = \mu$: unbiased ← $\boxed{D3(a)}$

$\bar{X}$ is a linear function of $x_1 \cdots x_n$ ← $\boxed{D3(d)}$

$Var(\bar{X})$ is the smallest ← $\boxed{D3(b)}$

↓

$\boxed{D3(d)}$ (best = most efficient) $\boxed{D3(e)}$ BLUE

Per-Duet

$\boxed{D1(b)}$

(yes/no question)

null hypothesis — alternative hypothesis

$H_0 : \mu = 10 \longleftrightarrow H_1 : \mu_1 \neq 10$ (two-sided)

(H_a)

$H_1 : \mu_1 > 10$ (one-sided)

$\downarrow$

estimator (of μ) : $\bar{x} \approx \mu$

$\downarrow$

test statistic (of H_0)

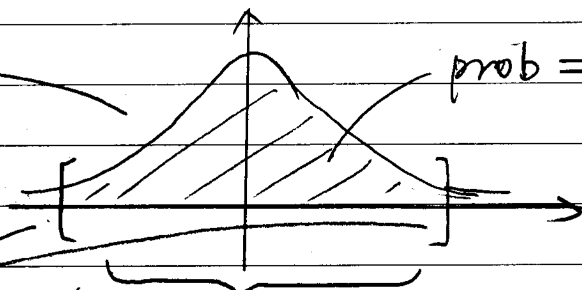
$$t = \frac{\bar{x} - 10}{se(\bar{x})}$$

hypothesized value

if $\bar{x} \approx 10$, t should be small

$\boxed{D2(d)}$

distribution of test statistic when H_0 is true



prob $= 0.95$ $= 1 - \alpha$

if t is small we tend to believe H_0 is true

critical value(s)

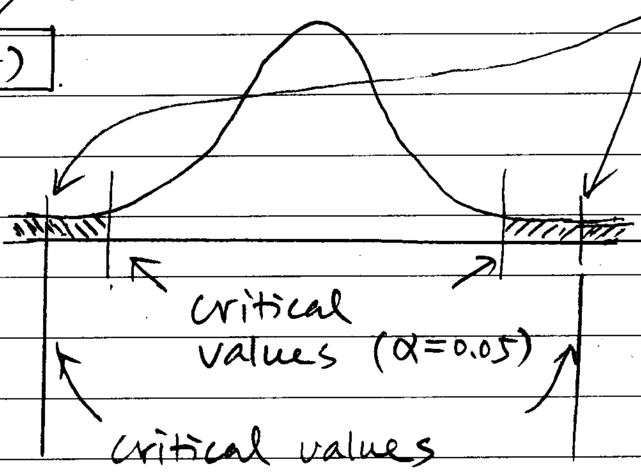
$\boxed{D2(e)}$

acceptance region ← $\boxed{D2(c)}$

depend on

level of significance $\alpha = P(\text{type I error})$
$= P(\text{reject } H_0 \mid H_0 \text{ is true})$

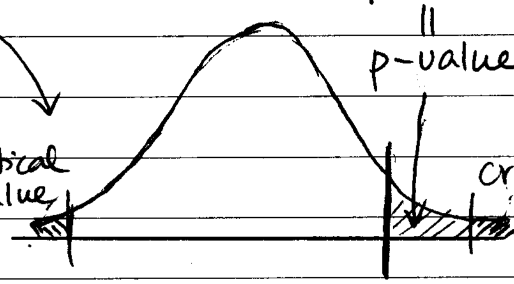
$\boxed{D2(f)}$



critical values ($\alpha = 0.05$)

critical values ($\alpha = 0.01$)

$\boxed{D2(g)}$ → $p(t_n > t_{ob})$ $=$ p-value



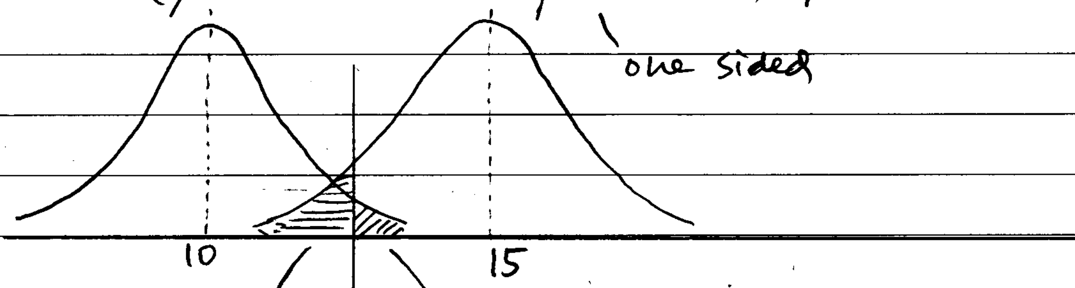
critical value

critical value

observed t statistic

H_0 is true ($\mu = 10$) H_0 is false ($\mu > 10$, say $\mu = 15$)

one sided



Critical value

β = P(type II error)

$\Downarrow$

fail to reject H_0 | H_0 is false

$\alpha = P$(type I error)

$\Downarrow$

reject H_0 | H_0 is true

$\boxed{\text{D1(c)}}$

$1-\beta$: power of the test

$\boxed{\text{D1.(e)}}$

α : significance level

$1-\alpha$: confidence coefficient

$\boxed{\text{D1(d)}}$

D4 (a) $\overline{X}$: r.v. μ : unknown constant, fixed

(b) unbiased : $E(\overline{X}) = \mu$

not : $\overline{x} = \mu$

(c) a trivial example

$$\overline{X} = 8 \quad (\mu = 10)$$

$$E(\overline{X}) = 8 \neq 10$$

$$Var(\overline{X}) = 0 \longrightarrow \text{minimal}$$

comparing variances is meaningless without unbiasedness. (Wooldridge. App. C.2.)

(d) efficient $\rightarrow$ unbiased, min variance

(e) normality $\rightarrow$ not required

(f) rigiously speaking

$P(L \le \bar{X} \le U) = 0.95$ $[L, U]$: CI

$P(L \le t \le U) = 0.95$ $[L, U]$: acceptance region

estimator

test statistic

(g) type I error $\rightarrow$ reject Ho | Ho is true

(h) type II error $\rightarrow$ fail to reject Ho | Ho is false

(i) $t_n \xrightarrow{n \to \infty} Z \sim N(0,1)$

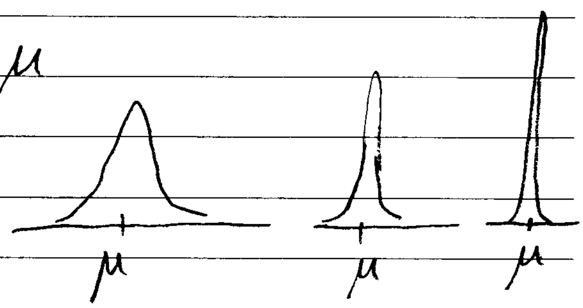
(j) $\bar{X} \xrightarrow{n \to \infty} N(\mu, \frac{\sigma^2}{n})$

asymptotically

only if $n \to \infty$

LLN : $\bar{X} \xrightarrow{n \to \infty} \mu$

write $\text{plim}(\bar{X}) = \mu$



(k)

$\alpha = P(t > c)$ determined by α

r.v. critical value

p-value $= P(t > t)$

r.v. observed

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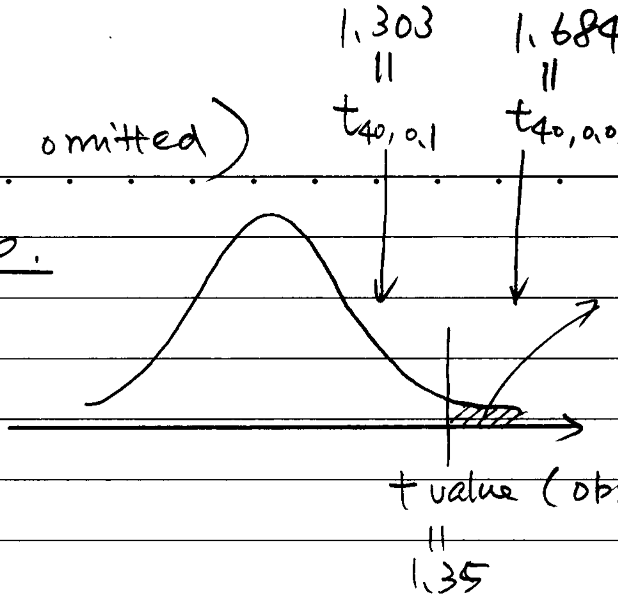
(D5 omitted)

D6.



one tailed

$$p\text{-value} = P(t_{r.v.} > t_{obs}) = 0.05 \sim 0.10$$

note: two-tailed

$$\begin{aligned} p\text{-value} &= P(|t_{r.v.}| > |t_{ob}|) \\ &= P(t_{r.v} > |t_{obs}| \text{ or } t_{r.v.} < -|t_{obs}|) \\ &= 2 \cdot P(t_{r.v.} > |t_{obs}|) \end{aligned}$$

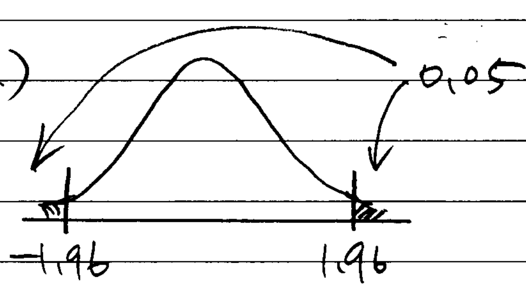
if $\alpha = 0.1$, p-value $< \alpha$, reject H_0

(t value $> t_{40,\alpha}$)

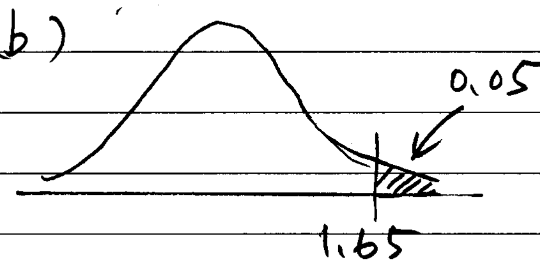
if $\alpha = 0.05$, p-value $> \alpha$, fail to reject H_0

(t value $< t_{40,\alpha}$)

D7 (a)



(b)



(b)(c), D8, (a)–(f) omitted

look up the relevant tables by yourselves

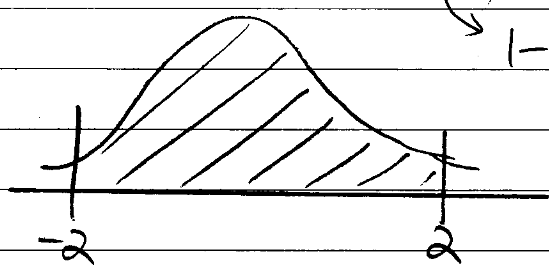
$$X \sim N(\underset{1000}{\mu}, \underset{10000}{\sigma^2})$$

D9 (a) $P(800 \leq X \leq 1200)$

$$= P\left(\frac{800-\mu}{\sigma} \leq \underbrace{\frac{X-\mu}{\sigma}}_{Z} \leq \frac{1200-\mu}{\sigma}\right)$$

$$= P(-2 \leq Z \leq 2)$$

$$= \Phi(2) - \underbrace{\Phi(-2)}_{1-\Phi(2)} = 2\underbrace{\Phi(2)}_{0.9772} - 1 = 0.9544$$



(b) – (d) omitted

D10 $n = 1000, \quad \overline{X} = 900, \quad \mu = 1000$

(a) $P(\overline{X} \geq 900) = P\left(\frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \geq \frac{900-\mu}{\frac{\sigma}{\sqrt{n}}}\right)$

$\overline{X} \sim N(\underset{1000}{\mu}, \underset{10}{\frac{\sigma^2}{n}})$

$$= P(Z \geq 31.6228)$$

$$= \Phi(-31.6228) \approx 0$$

(b) $$P\left(\underbrace{\overline{X} - 1.96\frac{\sigma}{\sqrt{n}}}_{893.8019} < \mu < \underbrace{\overline{X} + 1.96\frac{\sigma}{\sqrt{n}}}_{906.1981}\right) = 0.95$$

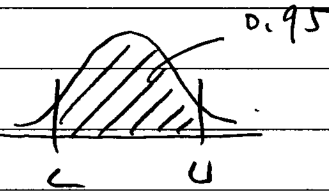
(this is from

$$P\left(-1.96 \leq \frac{\overset{900}{\overline{X}}-\mu}{\frac{\sigma}{\sqrt{n}}} \leq 1.96\right) = 0.95$$)

(c) $\overbrace{H_0 : \mu = 1000}$

$\because \mu \notin \overbrace{[\underset{L}{893.8019}, \underset{U}{906.1981}]}^{CI,\ \text{acceptance region}} \Rightarrow$ reject H_0

$P(\mu \notin [L, U]) = \underset{\uparrow\, 1-\alpha}{0.95}$



P11 $X \sim N(\mu, \sigma^2)$

$P(X < 6.5) = 0.05$, $P(X > 6.8) = 0.10$

(a) $P\left(\frac{X-\mu}{\sigma} < \frac{6.5-\mu}{\sigma}\right) = 0.05$ $\quad P\left(\frac{X-\mu}{\sigma} > \frac{6.8-\mu}{\sigma}\right) = 0.10$

$P(Z < a) = 0.05$ $\quad P(Z > b) = 0.10$

$\Phi(a) \downarrow -1.65$ $\quad 1 - \Phi(b) \downarrow 1.28$

$\frac{6.5-\mu}{\sigma} = -1.65$ $\quad \frac{6.8-\mu}{\sigma} = 1.28$

$\mu = 6.6689$, $\sigma = 0.1024$

(b) $P(X > 7) = P\left(\frac{X-\mu}{\sigma} > \frac{7-\mu}{\sigma}\right)$

$= P(Z > 3.2333)$

$= 0.00016$ (very small)

$\Phi(3.23) = 0.9994 \leftarrow$ from table

$P(Z > 3.23) = 0.0006$

D12 $X \sim N(\mu, \sigma^2)$ $\sigma^2 \to 2$

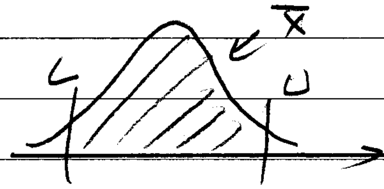
$\overline{X} = 9$ ← estimate

r.v. ← estimator

z-statistic → $Z = \dfrac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} = 8.9445$ (not t because σ is known)

$\overline{X} \sim N\left(\mu, \dfrac{\sigma^2}{n}\right)$, $\mu = 5$, $\dfrac{\sigma^2}{n} = \dfrac{2}{10}$

(a) 95% CI for $\overline{X}$ (r.v)



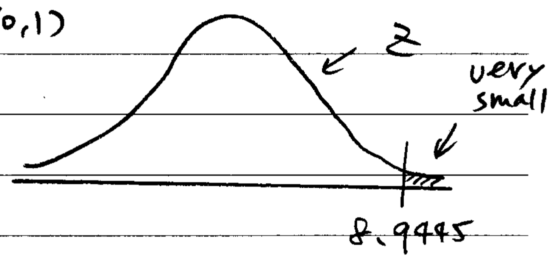
$$\left(\underbrace{\mu - 1.96\frac{\sigma}{\sqrt{n}}}_{L},\ \underbrace{\mu + 1.96\frac{\sigma}{\sqrt{n}}}_{U}\right) = (4.1235, 5.8765)$$

$\underset{\text{(observed)}}{\overline{X}} = 9 \notin [L, U] \Rightarrow$ reject H_0 (two-sided)

(b) $H_0: \mu = 5$ against $\mu > 5$

one-sided

$$P(\underset{\substack{\uparrow \\ \text{test stat} \\ \text{r.v.}}}{Z} > \underset{\substack{\uparrow \\ \text{observed} \\ \| \\ 8.9445}}{z}) = P(\underset{\sim N(0,1)}{Z} > 8.9445) < 0.0001$$



p-value < 0.05

$\Rightarrow$ reject H_0

(c) p-value $= P(Z > 8.9445) < 0.0001$

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D13 $X \sim N(\mu, \sigma^2)$

$n = 10, \quad \bar{X} = 8, \quad S = 4$

$$T = \frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} \sim t_9$$

$se(\bar{X})$

$t_{9, 0.025} = 2.262 \leftarrow$ critical value

$$P(-2.262 \le T \le 2.262) = 0.95$$

given $\frac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}}$ unknown

$$P\left(\underbrace{\bar{X} - 2.262 \frac{S}{\sqrt{n}}}_{5.1388} \le \mu \le \underbrace{\bar{X} + 2.262 \frac{S}{\sqrt{n}}}_{10.8612}\right) = 0.95$$

D14 $X \sim N(\underbrace{\mu}_{8}, \underbrace{\sigma^2}_{36})$ $\quad \bar{X} = 7.5, \quad n = 25$

(a) $\bar{X} \sim N(\mu, \frac{\sigma^2}{n}) = N(8, \frac{36}{25})$

(b) $P(\bar{X} \le 7.5)$

$$= P\left(\frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \le \frac{7.5 - \mu}{\frac{\sigma}{\sqrt{n}}}\right)$$

$$= P(Z \le -0.4167) = 0.3372$$

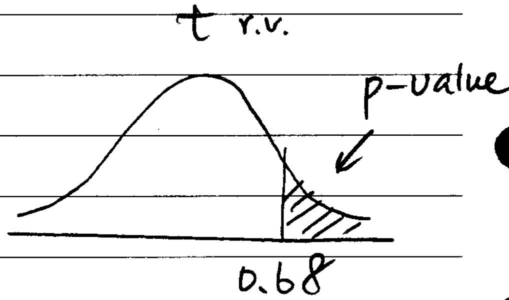
(c) $P(-1.96 \le Z \le 1.96) = 0.95$

$$\Rightarrow P\left(\underbrace{\mu - 1.96 \frac{\sigma}{\sqrt{n}}}_{L = 5.6480} \le \bar{X} \le \underbrace{\mu + 1.96 \frac{\sigma}{\sqrt{n}}}_{10.3520 = U}\right) = 0.95$$

$7.5 \in [L, U]$
It's likely!

(D.15 omitted)

D.16 $t = 0.68$, $df = 30$



$p(t_{30} > 0.68) = 0.25$

t_{30}: r.v.; $p(t_{30} > 0.68)$ = p-value

if $\alpha > 0.25$, p-value $< \alpha \Rightarrow$ reject H_0

if $\alpha > 0.25$ p-value $> \alpha \Rightarrow$ not reject H_0
$(0.1, 0.05, \cdots)$

D17 (a) both are unbiased, because

$$E(\hat{\mu}_1) = \frac{E(X_1) + E(X_2) + E(X_3)}{3} = \mu$$

$$E(\hat{\mu}_2) = \frac{E(X_1)}{6} + \frac{E(X_2)}{3} + \frac{E(X_3)}{2} = \mu$$

(b) $\hat{\mu}_1$ is relatively more efficient, because

$$Var(\hat{\mu}_1) = \frac{1}{3^2}\left[Var(X_1) + Var(X_2) + Var(X_3)\right] = \frac{\sigma^2}{3}$$

$$Var(\hat{\mu}_2) = \frac{1}{6^2}Var(X_1) + \frac{1}{3^2}Var(X_2) + \frac{1}{2^2}Var(X_3)$$

$$= \frac{7}{18}\sigma^2 > \frac{\sigma^2}{3} = Var(\hat{\mu}_1)$$

D18 $X \sim N(\mu, \sigma^2)$

$n = 10$, $\bar{X} = 900000$, $S = 100000$

(a) $T = \dfrac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} \sim t_9$

$$P(-2.262 \le T \le 2.262) = 0.95$$

critical value $t_{9,0.025}$

$$P\Big(\underbrace{\bar{X}-2.262\frac{S}{\sqrt{n}}}_{828.469} \le \mu \le \underbrace{\bar{X}+2.262\frac{S}{\sqrt{n}}}_{971.531}\Big) = 0.95$$

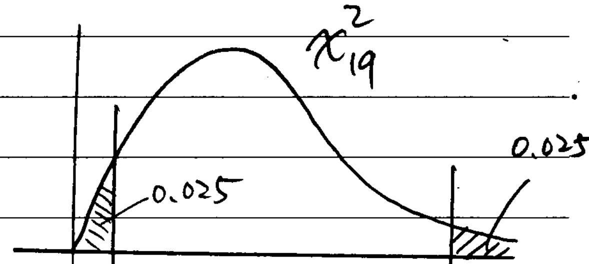
(b) critical value of t statistic

$\because$ σ is unknown, we need to use S

$\Rightarrow \dfrac{\bar{X}-\mu}{\frac{S}{\sqrt{n}}} \sim t$ distribution

D19 $Y=(n-1)\dfrac{S^2}{\sigma^2} \sim \chi^2_{(n-1)}$

$n=20, \quad S^2=16$



(a) $P(8.9065 \le Y \le 32.8523) = 0.95$, $\quad Y = (n-1)\dfrac{S^2}{\sigma^2}$

$\chi^2_{19,0.975} = 8.9065 \qquad \chi^2_{19,0.025} = 32.8523$

$$\Rightarrow P\left(\frac{(n-1)S^2}{32.8523} \le \sigma^2 \le \frac{(n-1)S^2}{8.9065}\right) = 0.95$$

$$P(\underset{L}{9.2535} \le \sigma^2 \le \underset{U}{34.1324}) = 0.95$$

(b) $\sigma^2 = 8.2 \notin [L, U] \Rightarrow$ reject H_0

D20

	$\bar{X}$	S	
X_1	1.075	0.5796	μ_1, σ_1^2
X_2	1.159	0.6134	μ_2, σ_2^2

(a) $H_0: \sigma_1^2 = \sigma_2^2 \longrightarrow$ F test $\leftarrow$ (b)

$$F = \frac{S_1^2}{S_2^2} = \frac{\dfrac{\sum_{i=1}^{n_1}(X_{1i}-\bar{X}_1)^2}{n_1-1}}{\dfrac{\sum_{i=1}^{n_2}(X_{2i}-\bar{X}_2)^2}{n_2-1}} \sim F_{n_1,n_2}$$

statistic r.v.

$$F = \frac{S_2^2}{S_1^2} = \frac{0.3762}{0.3359} = 1.12$$

greater smaller

$$p\text{-value} = P(F_{rv} > \underset{1.12}{F_{ob}}) = 0.4146 > \underset{0.1,\,0.05}{\alpha}$$

$\Rightarrow$ fail to reject H_0

D21

$$X^* = \frac{\sum_{i=1}^{n} X_i}{n+1}$$

$$E(X^*) = \frac{\sum^{n} E(X_i)}{n+1} = \frac{n}{n+1}\mu \neq \mu \quad \therefore \text{ biased}$$

$$\text{plim}(X^*) = \text{plim}\left(\frac{n}{n+1}\frac{\sum^{n} E(X_i)}{n}\right) = \text{plim}\left(\frac{n}{n+1}\right)\mu = \mu$$

$\therefore$ consistent

D22 $n=28$, $\bar{x}=23.25$, $s=9.49$

two sided CI

$n=28$

$$t_{27,0.025} = 2.502$$

$$P(-2.502 \le T \le 2.502) = 0.95$$

$$P\Big(\underbrace{\bar{X} - 2.502\frac{S}{\sqrt{n}}}_{19.57} \le \mu \le \underbrace{\bar{X} + 2.502\frac{S}{\sqrt{n}}}_{26.93}\Big) = 0.95$$

one sided CI

$$t_{27,0.05} = 1.703$$

$$P(T \le 1.703) = 0.95$$

$$P\Big(\mu \le \underbrace{\bar{X} + 1.703\frac{S}{\sqrt{n}}}_{26.31}\Big) = 0.95$$