Chapter 19
The Greek Letters

Example

- A bank has sold for \$300,000 a European call option on 100,000 shares of a non-dividend paying stock
- $S_0 = 49$, $K = 50$, $r = 5\%$, $\sigma = 20\%$,

  $T = 20$ weeks, $\mu = 13\%$
- The Black-Scholes-Merton value of the option is \$240,000
- How does the bank hedge its risk to lock in a \$60,000 profit?

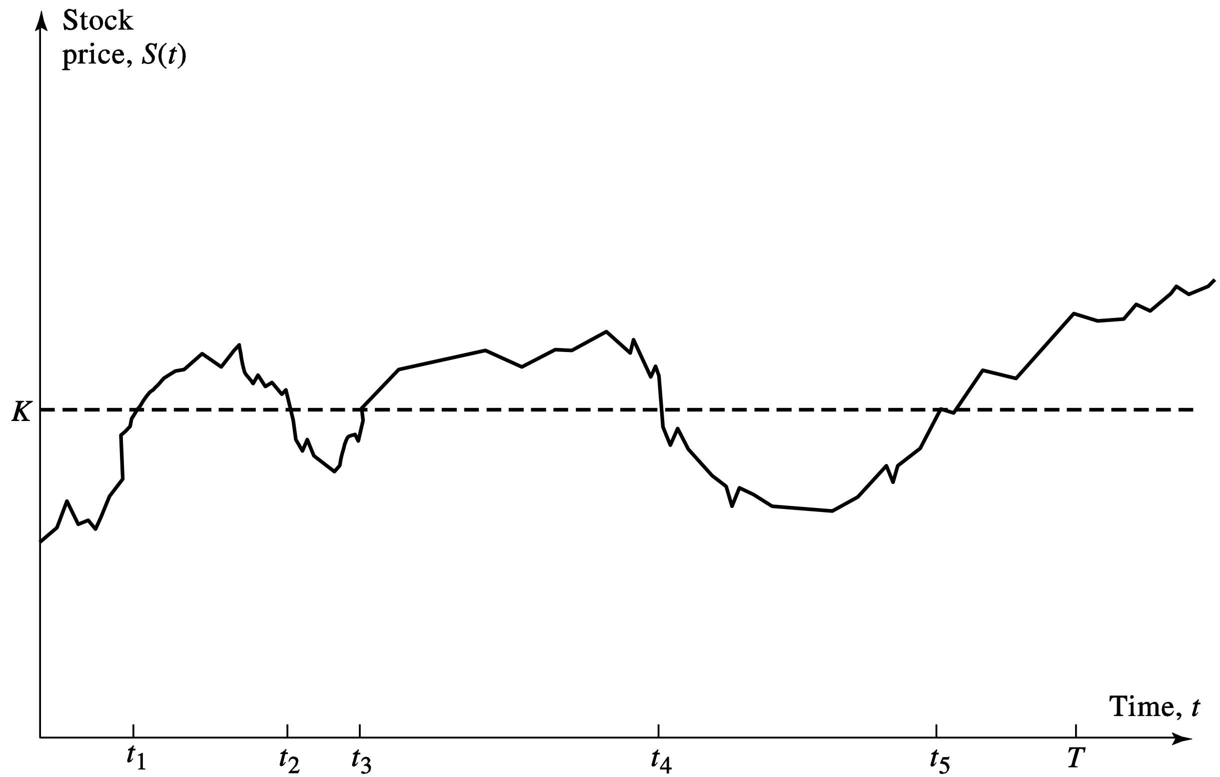
Naked & Covered Positions

- Naked position
  - Take no action
- Covered position
  - Buy 100,000 shares today
- What are the risks associated with these strategies?

Stop-Loss Strategy

- This involves:
  - Buying 100,000 shares as soon as price reaches $50
  - Selling 100,000 shares as soon as price falls below $50

Stop-Loss Strategy *continued*



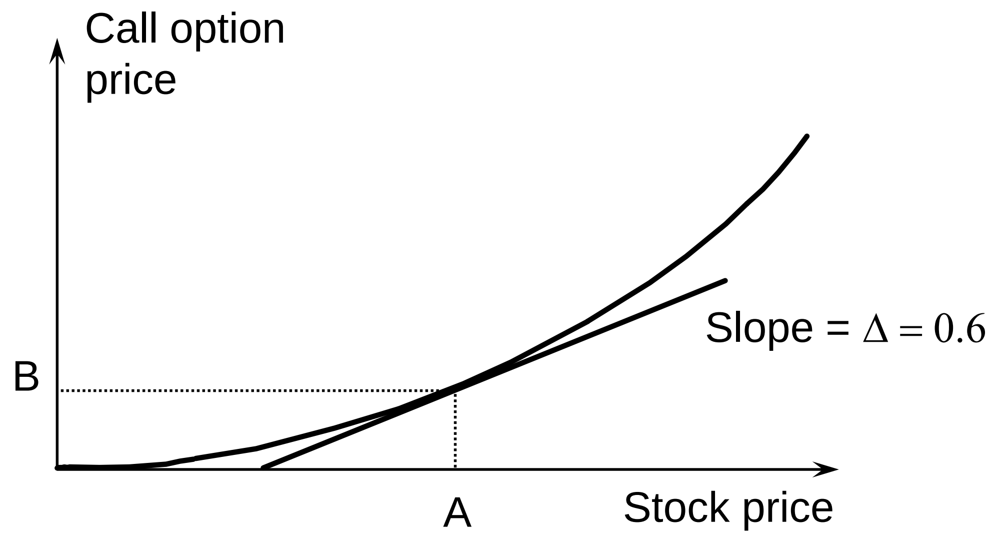
Ignoring discounting, the cost of writing and hedging the option appears to be max(S_0−K, 0). What are we overlooking?

Greek Letters

- Greek letters are the partial derivatives with respect to the model parameters that are liable to change
- Usually traders use the Black-Scholes-Merton model when calculating partial derivatives
- The volatility parameter in BSM is set equal to the implied volatility when Greek letters are calculated. This is referred to as using the "practitioner Black-Scholes" model

Delta *(See Figure 19.2, page 401)*

- Delta (Δ) is the rate of change of the option price with respect to the underlying asset price



Hedge

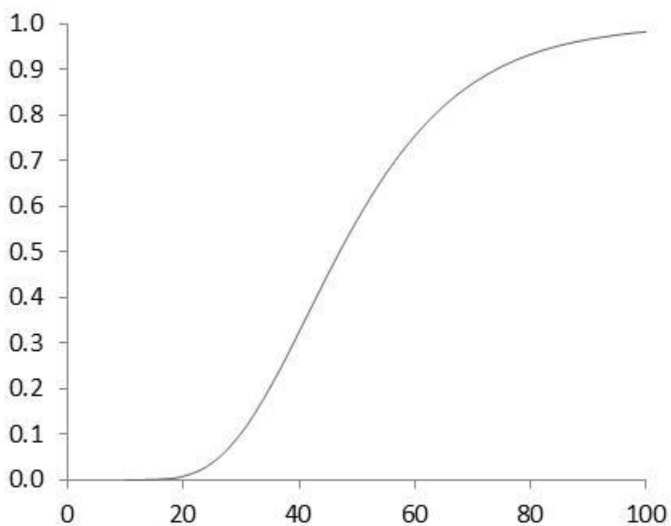
- Trader would be hedged with the position:
  - short 1000 options
  - buy 600 shares
- Gain/loss on the option position is offset by loss/gain on stock position
- Delta changes as stock price changes and time passes
- Hedge position must therefore be rebalanced

Delta Hedging

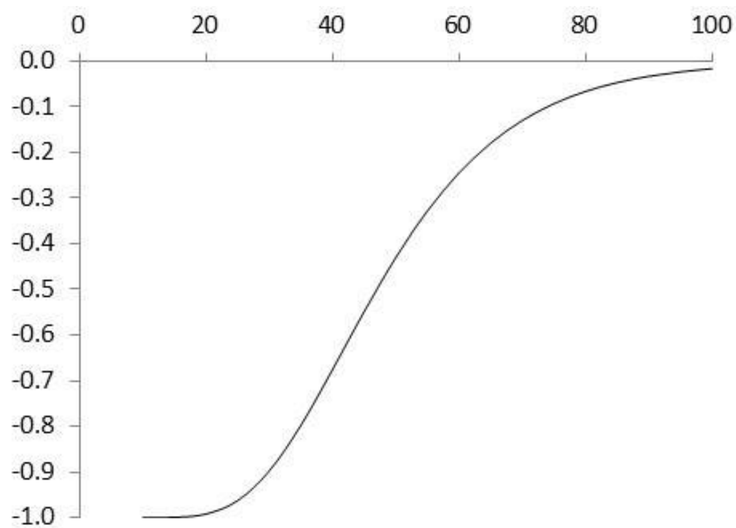
- This involves maintaining a delta neutral portfolio
- The delta of a European call on a non-dividend paying stock is $\boxed{N(d_1)}$
- The delta of a European put on the stock is

$$\boxed{N(d_1) - 1}$$

Delta of a Stock Option *(K=50, r=0, σ = 25%, T=2, Figure 19.3, page 402)*

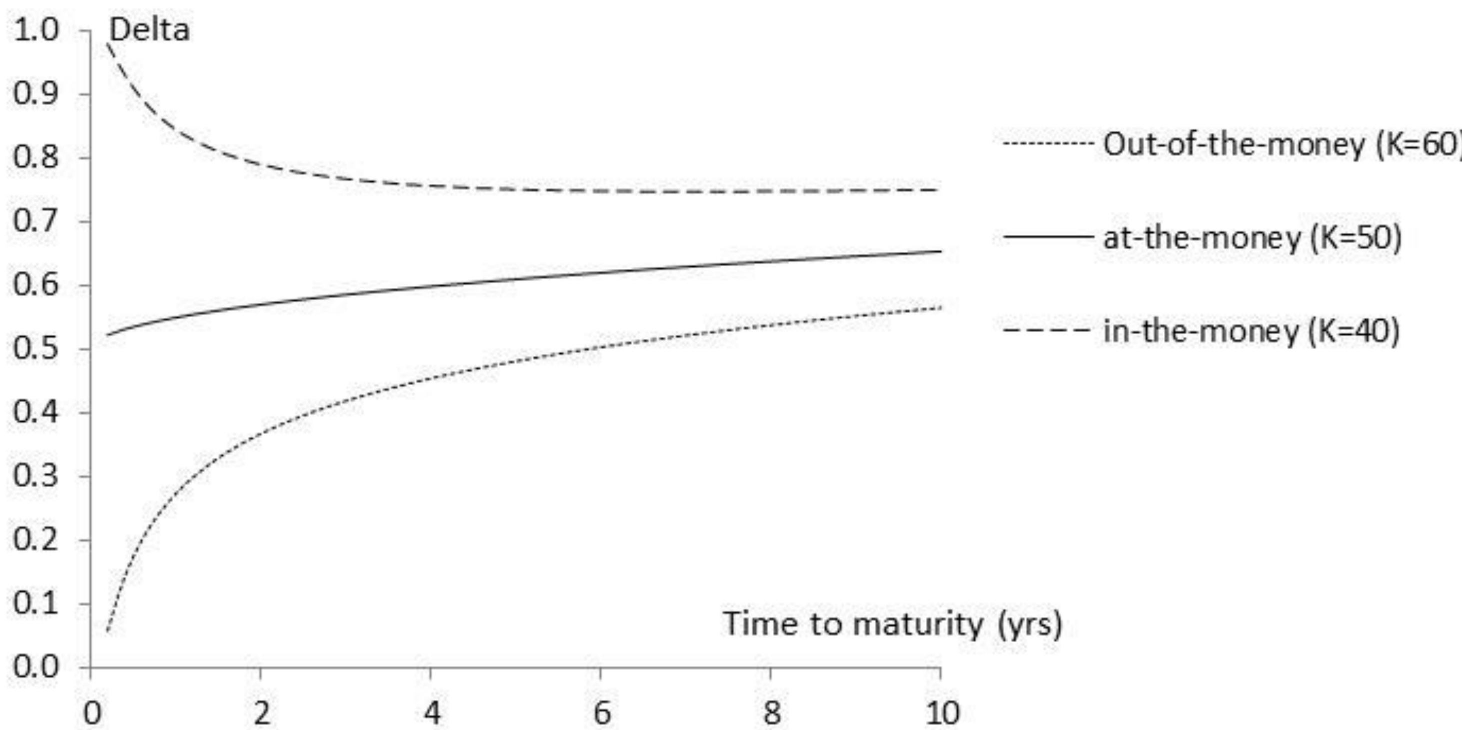


Call



Put

Variation of Delta with Time to Maturity *(S_0=50, r=0, σ=25%, Figure 19.4, page 403)*



The Costs in Delta Hedging continued

- Delta hedging a written option involves a "buy high, sell low" trading rule

First Scenario for the Example:

Table 19.2 page 404

Week	Stock price	Delta	Shares purchased	Cost (‘$000)	Cumulative Cost ($000)	Interest
0	49.00	0.522	52,200	2,557.8	2,557.8	2.5
1	48.12	0.458	(6,400)	(308.0)	2,252.3	2.2
2	47.37	0.400	(5,800)	(274.7)	1,979.8	1.9
....	....	....	....	....	....	....
19	55.87	1.000	1,000	55.9	5,258.2	5.1
20	57.25	1.000	0	0	5263.3	

Second Scenario for the Example

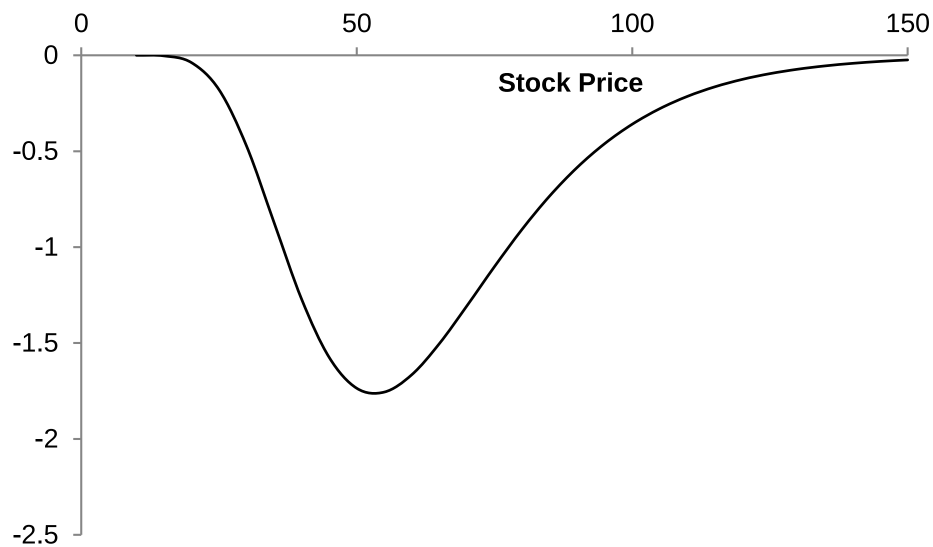
Table 19.3, page 405

Week	Stock price	Delta	Shares purchased	Cost ('$000)	Cumulative Cost ($000)	Interest
0	49.00	0.522	52,200	2,557.8	2,557.8	2.5
1	49.75	0.568	4,600	228.9	2,789.2	2.7
2	52.00	0.705	13,700	712.4	3,504.3	3.4
....	....	....	....	....	....	....
19	46.63	0.007	(17,600)	(820.7)	290.0	0.3
20	48.12	0.000	(700)	(33.7)	256.6	

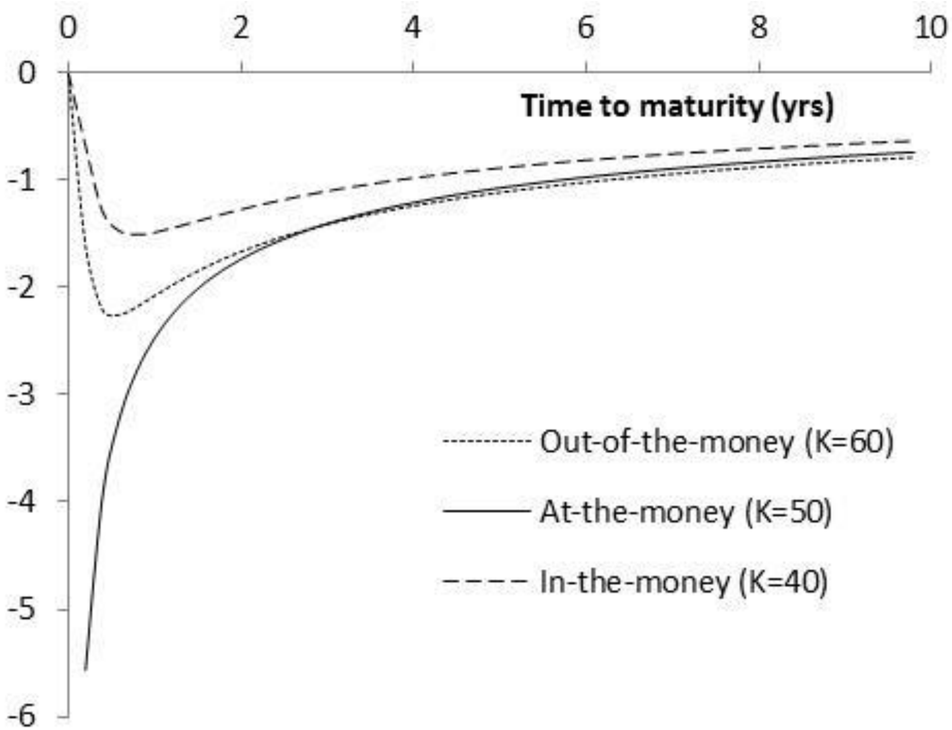
Theta

- Theta (Θ) of a derivative (or portfolio of derivatives) is the rate of change of the value with respect to the passage of time
- The theta of a call or put is usually negative. This means that, if time passes with the price of the underlying asset and its volatility remaining the same, the value of a long call or put option declines

Theta for Call Option *(K=50, σ = 25%, r = 0, T = 2, Figure 19.5, page 408)*



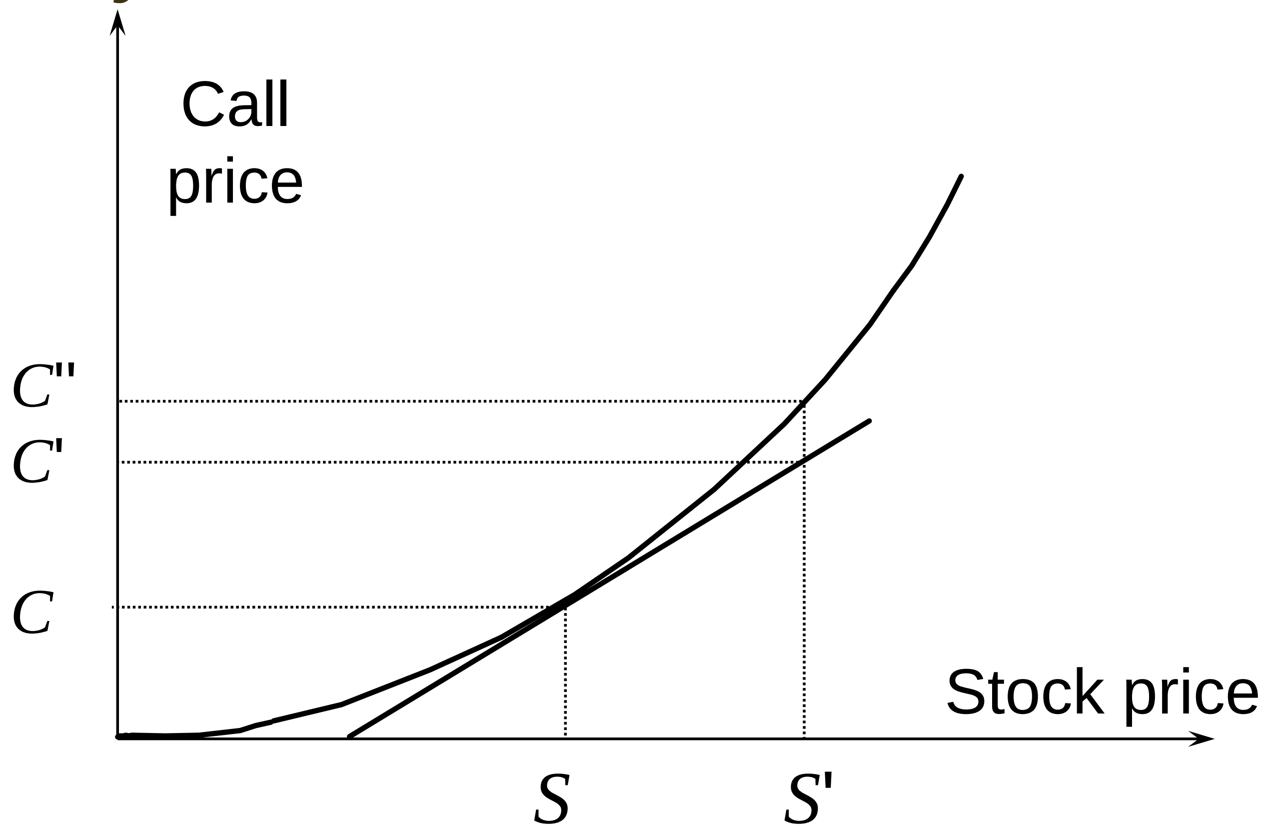
Variation of Theta with Time to Maturity *(S_0=50, r=0, σ=25%, Figure 19.6, page 409)*



Gamma

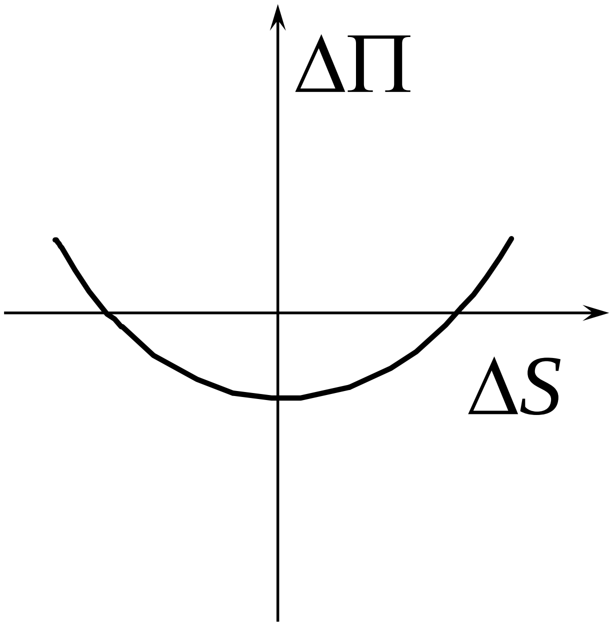
- Gamma (Γ) is the rate of change of delta (Δ) with respect to the price of the underlying asset
- Gamma is greatest for options that are close to the money

Gamma Addresses Delta Hedging Errors Caused By Curvature *(Figure 19.7, page 411)*

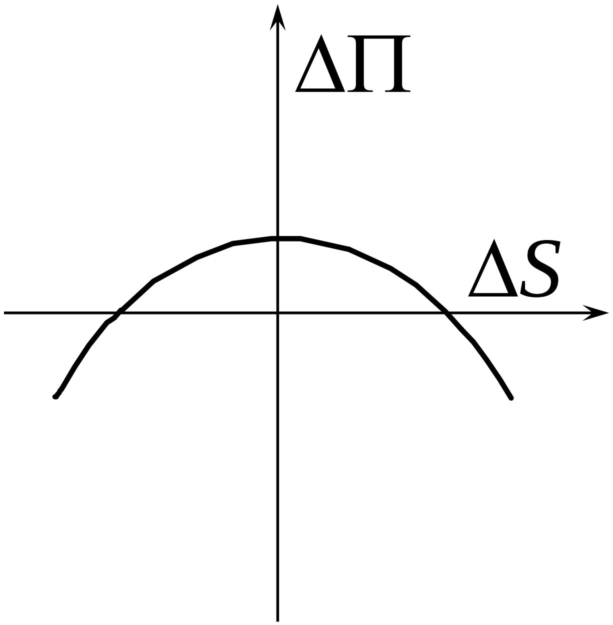


Interpretation of Gamma

For a delta neutral portfolio, $\boxed{\Delta\Pi \approx \Theta\,\Delta t + \tfrac{1}{2}\Gamma\Delta S^{2}}$



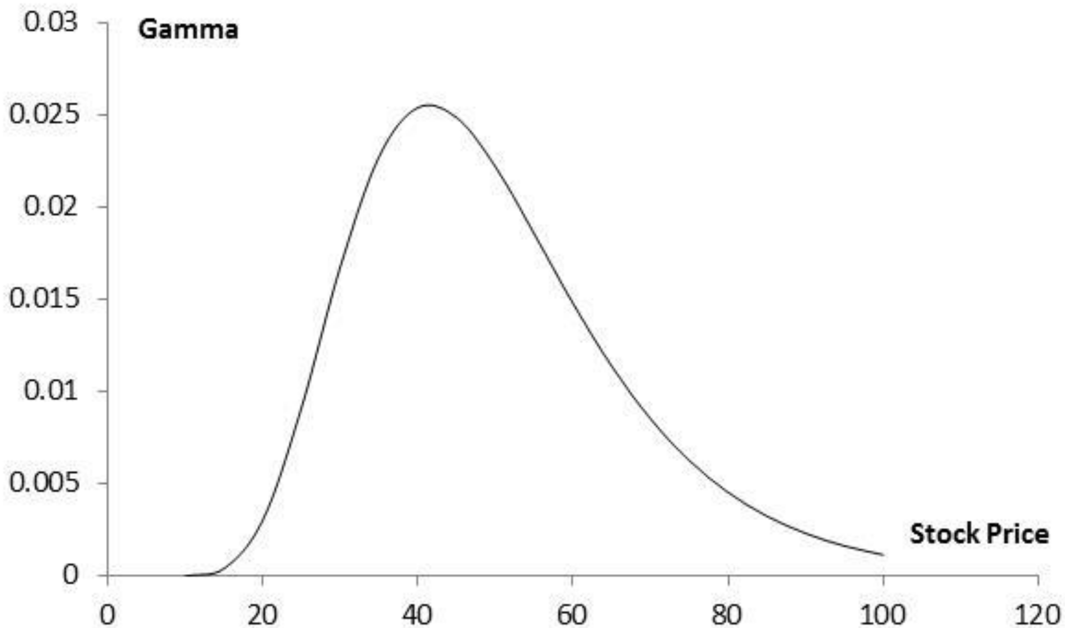
Positive Gamma



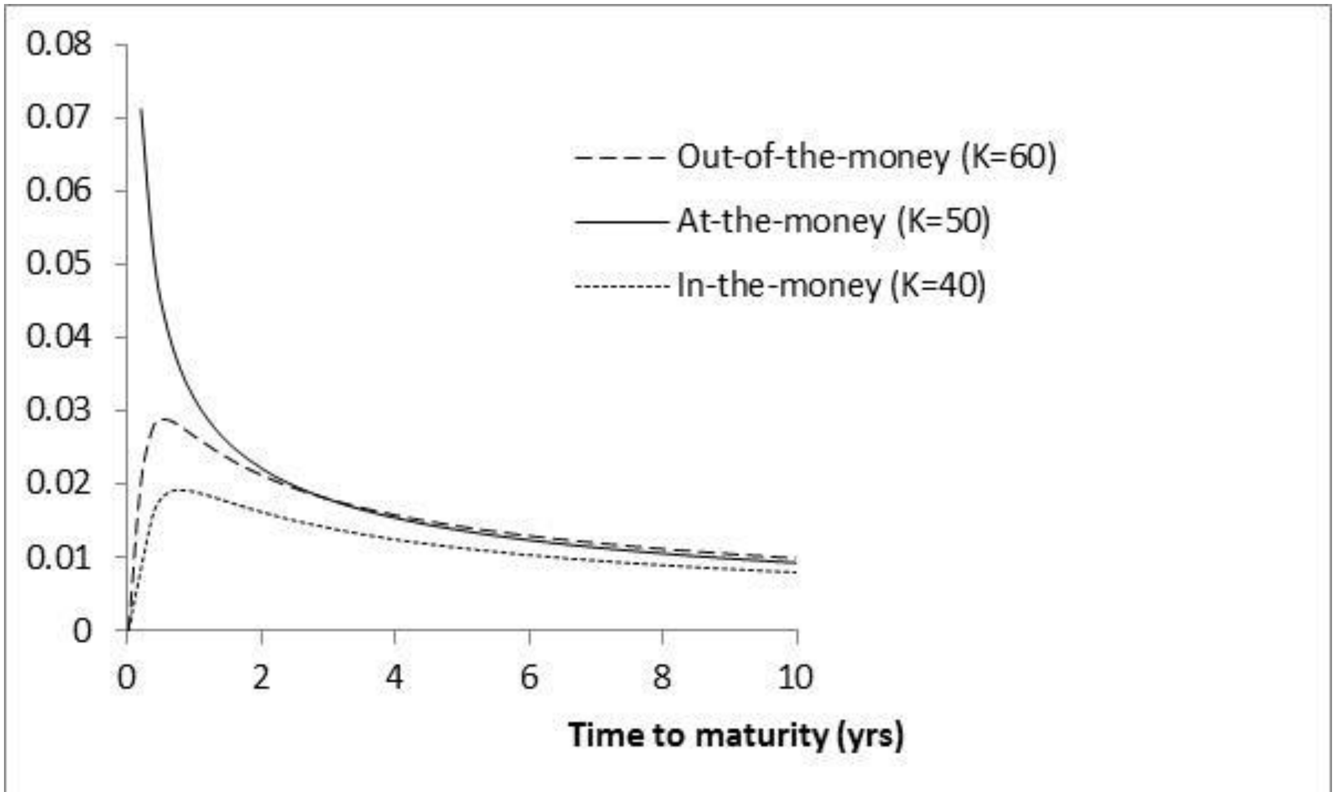
Negative Gamma

Gamma for Call or Put Option:

(K=50, σ = 25%, r = 0%, T = 2, Figure 19.9, page 412)



Variation of Gamma with Time to Maturity *(S_0=50, r=0, σ=25%, Figure 19.10, page 413)*



Relationship Between Delta, Gamma, and Theta *(page 415)*

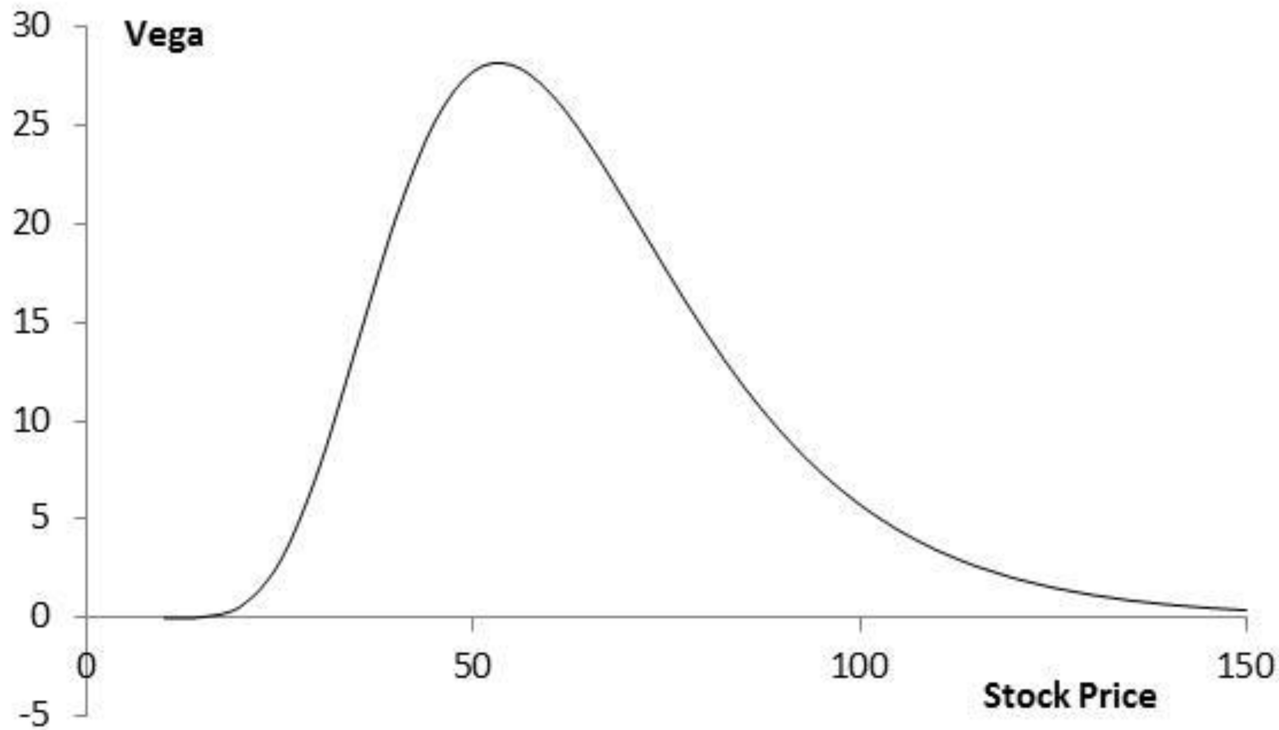
For a portfolio of derivatives on a stock paying a continuous dividend yield at rate q it follows from the Black-Scholes-Merton differential equation that

$$\boxed{\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi}$$

Vega

- Vega ($\mathcal{V}$) is the rate of change of the value of a derivatives portfolio with respect to volatility
- If vega is calculated for a portfolio as a weighted average of the vegas for the individual transactions comprising the portfolio, the result shows the effect of all implied volatilities changing by the same small amount

Vega for Call or Put Option
(K=50, σ = 25%, r = 0, T = 2)



Taylor Series Expansion *(Appendix to Chapter 19)*

- The value of a portfolio of derivatives dependent on an asset is a function of of the asset price S, its volatility σ, and time t

$$\boxed{\begin{aligned}\Delta\Pi &= \frac{\partial\Pi}{\partial S}\Delta S + \frac{\partial\Pi}{\partial\sigma}\Delta\sigma + \frac{\partial\Pi}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2\Pi}{\partial S^2}(\Delta S)^2 + \ldots \\ &= \text{Delta}\times\Delta S + \text{Vega}\times\Delta\sigma + \text{Theta}\times\Delta t + \frac{1}{2}\text{Gamma}\times(\Delta S)^2 + \ldots\end{aligned}}$$

Managing Delta, Gamma, & Vega

- Delta can be changed by taking a position in the underlying asset
- To adjust gamma and vega it is necessary to take a position in an option or other derivative

Example

	Delta	*Gamma*	*Vega*
Portfolio	0	−5000	−8000
Option 1	0.6	0.5	2.0
Option 2	0.5	0.8	1.2

What position in option 1 and the underlying asset will make the portfolio delta and gamma neutral? Answer: Long 10,000 options, short 6000 of the asset

What position in option 1 and the underlying asset will make the portfolio delta and vega neutral? Answer: Long 4000 options, short 2400 of the asset

Example *continued*

	Delta	*Gamma*	*Vega*
Portfolio	0	−5000	−8000
Option 1	0.6	0.5	2.0
Option 2	0.5	0.8	1.2

What position in option 1, option 2, and the asset will make the portfolio delta, gamma, and vega neutral?

We solve

$$-5000+0.5w_1+0.8w_2=0$$
$$-8000+2.0w_1+1.2w_2=0$$

to get $w_1 = 400$ and $w_2 = 6000$. We require long positions of 400 and 6000 in option 1 and option 2. A short position of 3240 in the asset is then required to make the portfolio delta neutral

Rho

- Rho is the rate of change of the value of a derivative with respect to the interest rate

Hedging in Practice

- Traders usually ensure that their portfolios are delta-neutral at least once a day
- Whenever the opportunity arises, they improve gamma and vega
- There are economies of scale
  - As portfolio becomes larger hedging becomes less expensive per option in the portfolio

Scenario Analysis

A scenario analysis involves testing the effect on the value of a portfolio of different assumptions concerning asset prices and their volatilities

Greek Letters for European Options on an Asset that Provides a Yield at Rate q

Greek Letter	*Call Option*	*Put Option*
Delta	$e^{-qT}N(d_1)$	$e^{-qT}\left[N(d_1)-1\right]$
Gamma	$\dfrac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}$	$\dfrac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}$
Theta	$-S_0N'(d_1)\sigma e^{-qT}/\left(2\sqrt{T}\right)$ $+qS_0N(d_1)e^{-qT}-rKe^{-rT}N(d_2)$	$-S_0N'(d_1)\sigma e^{-qT}/\left(2\sqrt{T}\right)$ $+qS_0N(-d_1)e^{-qT}+rKe^{-rT}N(-d_2)$
Vega	$S_0\sqrt{T}N'(d_1)e^{-qT}$	$S_0\sqrt{T}N'(d_1)e^{-qT}$
Rho	$KTe^{-rT}N(d_2)$	$-KTe^{-rT}N(-d_2)$

Futures Contract Can Be Used for Hedging

- The delta of a futures contract on an asset paying a yield at rate q is $e^{(r-q)T}$ times the delta of a spot contract
- The position required in futures for delta hedging is therefore $e^{-(r-q)T}$ times the position required in the corresponding spot contract

Hedging vs Creation of an Option Synthetically

- When we are hedging we take positions that offset delta, gamma, vega, etc
- When we create an option synthetically we take positions that match delta, gamma, vega, etc

Portfolio Insurance

- In October of 1987 many portfolio managers attempted to create a put option on a portfolio synthetically
- This involves initially selling enough of the portfolio (or of index futures) to match the Δ of the put option

Portfolio Insurance continued

- As the value of the portfolio increases, the Δ of the put becomes less negative and some of the original portfolio is repurchased
- As the value of the portfolio decreases, the Δ of the put becomes more negative and more of the portfolio must be sold

Portfolio Insurance continued

The strategy did not work well on October 19, 1987...