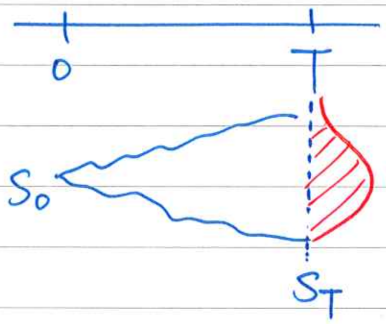
Unit 3 BS model and Greek letters

BS model $\longrightarrow$ BS formulas



$$\begin{cases} C_0 = S_0 N(d_1) - Ke^{-rT} N(d_2) \\ P_0 = Ke^{-rT} N(-d_2) - S_0 N(-d_1) \end{cases}$$

① $S_T = S_0 e^X$

$X \sim N\left((r - \frac{\sigma^2}{2})T, \sigma^2 T\right)$ under Q measure

$E^Q[S_T] = S_0 e^{rT}$

② $dS_t = rS_t dt + \sigma S_t dW_t \longleftarrow$ SDE stochastic differential equation (stochastic calculus)

$S_t = S_0 e^{\boxed{(r - \frac{\sigma^2}{2})T + \sigma W_T}} \rightarrow X$ $\longleftarrow$ solution of the SDE

W_t : Wiener process

$E^Q[S_t] = S_0 e^{rT}$

$\uparrow$ stock return under the Q measure

Note: $W_T \sim N(0, T)$

$\sigma W_T \sim N(0, \sigma^2 T)$

$X = (r - \frac{\sigma^2}{2})T + \sigma W_T \sim N\left((r - \frac{\sigma^2}{2})T, \sigma^2 T\right)$

$\parallel$

$\ln S_T - \ln S_0$ $\quad \therefore \underline{\ln S_T \sim N\left(\ln S_0 + (r - \frac{\sigma^2}{2})T, \sigma^2 T\right)}$

Delta (Δ)

$$\Delta_c = \frac{\partial c}{\partial S} = N(d_1)$$

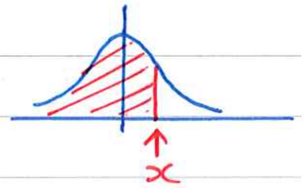
$$\Delta_p = \frac{\partial p}{\partial S} = N(d_1) - 1$$

ⓐ Proof of Δ_c

$$c = S\,N(d_1) - Ke^{-rT}N(d_2)$$

$$\frac{\partial c}{\partial S} = \underbrace{N(d_1)}_{①} + \underbrace{S\,n(d_1)\frac{\partial d_1}{\partial S}}_{②} \underbrace{- Ke^{-rT}n(d_2)\frac{\partial d_2}{\partial S}}_{③}$$

Note: $N(x) = \int_{-\infty}^{x} n(y)\,dy$



$$\frac{\partial}{\partial x}: \quad n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$n(d_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}}, \quad n(d_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}}$$

$$d_1 = \frac{\ln\frac{S}{K} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

$$\frac{\partial d_1}{\partial S} = \frac{\partial}{\partial S}\left[\frac{1}{\sigma\sqrt{T}}\ln S\right] = \frac{1}{S\sigma\sqrt{T}} = \frac{\partial d_2}{\partial S}$$

$$d_2^2 = (d_1 - \sigma\sqrt{T})^2 = d_1^2 - 2d_1\sigma\sqrt{T} + \sigma^2 T$$

$$-\frac{d_2^2}{2} = -\frac{d_1^2}{2} + d_1\sigma\sqrt{T} - \frac{\sigma^2 T}{2}$$

② − ③

$$= S\,n(d_1)\frac{\partial d_1}{\partial S_1} - Ke^{-rT}n(d_2)\frac{\partial d_2}{\partial S}$$

$$\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}} \qquad \frac{1}{S\sigma\sqrt{T}} \qquad \frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}+d_1\sigma\sqrt{T}-\frac{\sigma^2 T}{2}} \qquad \frac{1}{S\sigma\sqrt{T}}$$

$$= S\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\frac{1}{S\sigma\sqrt{T}}\underbrace{\left[1-\frac{Ke^{-rT}}{S}e^{d_1\sigma\sqrt{T}-\frac{\sigma^2T}{2}}\right]} = 0$$

$$1-\frac{Ke^{-rT}}{S}e^{\ln\frac{S}{K}+(r+\frac{\sigma^2}{2})T-\frac{\sigma^2T}{2}} = 1-\frac{Ke^{-rT}}{S}\left(\frac{S}{K}\times e^{rT}\right) = 0$$

∴ ② − ③ = 0

$$\Delta_c = \frac{\partial C}{\partial S} = N(d_1)$$

ⓑ Proof of Δ_p is left to you as an exercise

Note: put-call parity

$$c + Ke^{-rT} = p + S$$

$\frac{\partial}{\partial S}$

$$\frac{\partial c}{\partial S} + 0 = \frac{\partial p}{\partial S} + 1$$

$$\Delta_c + 0 = \Delta_p + 1$$

$\Delta_c = N(d_1)$

$$\therefore \Delta_p = \Delta_c - 1 = N(d_1) - 1$$

Gamma (Γ)

$$\Gamma_c = \frac{\partial^2 C}{\partial S^2} = \frac{\partial \Delta_c}{\partial S} = \frac{n(d_1)}{S\sigma\sqrt{T}}$$

$$\Gamma_p = \frac{\partial^2 P}{\partial S^2} = \frac{\partial \Delta_p}{\partial S} = \frac{n(d_1)}{S\sigma\sqrt{T}}$$

ⓐ Proof of Γ_c

$$\Gamma_c = \frac{\partial \overbrace{\Delta_c}^{N(d_1)}}{\partial S} = n(d_1)\underbrace{\frac{\partial d_1}{\partial S}}_{=\frac{1}{S\sigma\sqrt{T}}} = \frac{n(d_1)}{S\sigma\sqrt{T}}$$

ⓑ Proof of Γ_p

$$\Gamma_p = \frac{\partial \overbrace{\Delta_p}^{N(d_1)-1}}{\partial S} = n(d_1)\frac{\partial d_1}{\partial S} = \frac{n(d_1)}{S\sigma\sqrt{T}}$$

Note: put-call parity

$$C + Ke^{-rT} = P + S$$

$$\xrightarrow{\frac{\partial}{\partial S}} \quad \Delta_c + 0 = \Delta_p + 1$$

$$\xrightarrow{\frac{\partial}{\partial S}} \quad \Gamma_c + 0 = \Gamma_p + 0$$

Vega (ν)

$$\nu_c = \frac{\partial C}{\partial \sigma} = S\sqrt{T}\, n(d_1)$$

$$\nu_p = \frac{\partial P}{\partial \sigma} = S\sqrt{T}\, n(d_1)$$

(a) Proof of ν_c

$$\nu_c = \frac{\partial C}{\partial \sigma} = S\,n(d_1)\frac{\partial d_1}{\partial \sigma} - Ke^{-rT} n(d_2)\frac{\partial d_2}{\partial \sigma}$$

$$d_1 = d_2 + \sigma\sqrt{T}$$

$$\frac{\partial d_1}{\partial \sigma} = \frac{\partial d_2}{\partial \sigma} + \sqrt{T}$$

$$= \underbrace{\left[S\,n(d_1) - Ke^{-rT} n(d_2)\right]}_{=0}\frac{\partial d_2}{\partial \sigma} + \underbrace{S\sqrt{T}\,n(d_1)}_{\text{answer!}}$$

Why?

See p. 2a: $S\,n(d_1) - Ke^{-rT} n(d_2)$

$$= S\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}\underbrace{\left[1 - \frac{Ke^{-rT}}{S}e^{d_1\sigma\sqrt{T} - \frac{\sigma^2 T}{2}}\right]}_{=0} = 0$$

$$= S\sqrt{T}\,n(d_1)$$

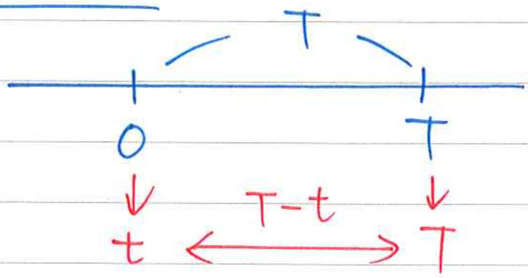
(b) Proof of ν_p : your exercise

Theta (Θ)

$$\Theta_c = -\frac{S\sigma\,n(d_1)}{2\sqrt{T}} - rKe^{-rT}N(d_2)$$

$$\Theta_p = -\frac{S\sigma\,n(d_1)}{2\sqrt{T}} + rKe^{-rT}N(-d_2)$$

ⓐ Proof of Θ_c



$t \to t + \Delta t$ current time moves forward

$T \to T - \Delta T$ maturity time decreases!

$$\Theta_c = -\frac{\partial C}{\partial T}$$

$$= -S\,n(d_1)\frac{\partial d_1}{\partial T} - rKe^{-rT}N(d_2) + Ke^{-rT}n(d_2)\frac{\partial d_2}{\partial T}$$

$d_1 = d_2 + \sigma\sqrt{T}$

$$\frac{\partial d_1}{\partial T} = \frac{\partial d_2}{\partial T} + \frac{\sigma}{2\sqrt{T}}$$

cancel out

$$= \frac{S\sigma\, n(d_1)}{2\sqrt{T}} - rKe^{-rT}N(d_2)$$

ⓑ Proof of Θ_p : your exercise

Rho (ρ)

r usually doesn't change much!

$\frac{\partial \boxed{}}{\partial r} \Rightarrow$ this is not very useful!

$$\rho_c = \frac{\partial c}{\partial r} = KTe^{-rT}N(d_2)$$

$$\rho_p = \frac{\partial p}{\partial r} = -KTe^{-rT}N(-d_2)$$

ⓐ <u>Proof of ρ_c</u>

Answer

$$\rho_c = \frac{\partial c}{\partial r} = Sn(d_1)\frac{\partial d_1}{\partial r} + \underline{KTe^{-rT}N(d_2)}$$

$$- Ke^{-rT}n(d_2)\frac{\partial d_2}{\partial r}$$

$$d_1 = d_2 + \sigma\sqrt{T}$$

$$\frac{\partial d_1}{\partial r} = \frac{\partial d_2}{\partial r}$$

$$= KTe^{-rT}N(d_2)$$

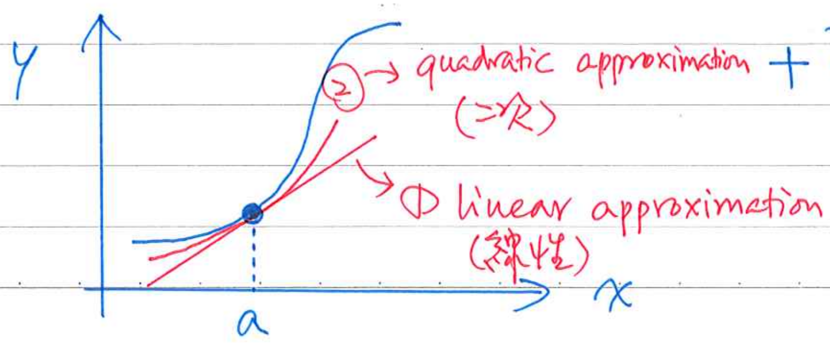
ⓑ <u>Proof of ρ_p</u> : your exercise

● Taylor series expansion

$$y = f(x)$$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots\cdots$$

$$+ \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots\cdots$$



① linear approximation

$$f(x) \cong f(a) + f'(a)(x-a)$$

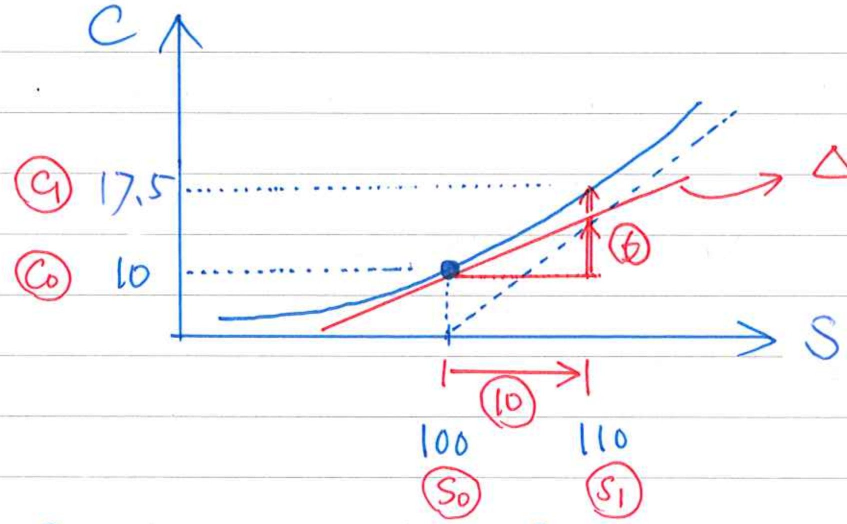
② quadratic approximation

$$f(x) \cong f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

When we see C as a function of S

$$C = f(S)$$

$$C = \underbrace{f(S_0)}_{=\,C_0} + \underbrace{f'(S_0)}_{\frac{\partial C}{\partial S} = \Delta}(S-S_0) + \frac{\overbrace{f''(S_0)}^{\frac{\partial^2 C}{\partial S^2} = \Gamma}}{2!}(S-S_0)^2$$



S: $S_0 \rightarrow S_1 (= S)$

100 110

C: $C_0 \rightarrow C_1 (= C)$

10 17.5

$\Delta = 0.6$

$\Gamma = 0.02$

$$\underbrace{C - C_0}_{\Delta C} = \Delta \underbrace{(S - S_0)}_{\Delta S} + \frac{\Gamma}{2} (\underbrace{S - S_0}_{\Delta S})^2$$

① $\Delta C = \Delta \cdot \Delta S$

$\Delta C = 0.6 \times 10 = 6 \quad (10 \to 16)$

② $\Delta C = \Delta \cdot \Delta S + \frac{\Gamma}{2} \cdot (\Delta S)^2$

$\Delta C = 0.6 \times 10 + \frac{0.02}{2} \times (10)^2 = 7 \quad (10 \to 17)$

③ In reality

$\Delta C = 7.5 \quad (10 \to 17.5)$

Multivariate function

$z = f(x, y)$

$$f(x,y) = f(\underbrace{x_0}_{a}, \underbrace{y_0}_{b}) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0)$$

$$+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x - x_0)^2 + \frac{\partial^2 f}{\partial x \partial y}(x - x_0)(y - y_0) + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(y - y_0)^2$$

$$\Rightarrow \Delta f = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$$

$$+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\Delta x)^2 + \frac{\partial^2 f}{\partial x \partial y}(\Delta x)(\Delta y) + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(\Delta y)^2$$

In fact, c is a fn of S, σ, T, r

don't need to consider K ($\because K =$ constant)

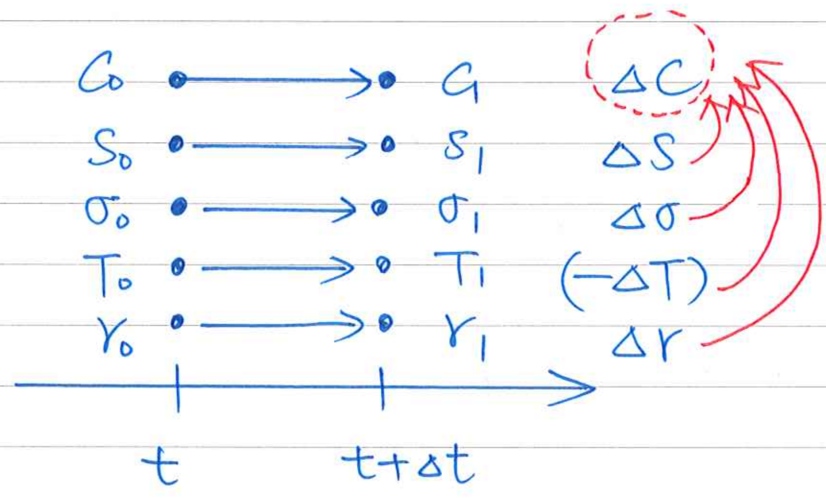
$$c = f(S, \sigma, T, r)$$

$$\Delta C = \underbrace{\frac{\partial f}{\partial S}}_{\Delta}(\Delta S) + \frac{1}{2}\underbrace{\frac{\partial^2 f}{\partial S^2}}_{\Gamma}(\Delta S)^2$$

$$+ \underbrace{\frac{\partial f}{\partial \sigma}}_{\nu}(\Delta\sigma) + \underbrace{\left(-\frac{\partial f}{\partial T}\right)}_{\Theta}(-\Delta T) + \underbrace{\frac{\partial f}{\partial r}}_{\rho}(\Delta r)$$

$+ \cdots$ (cross terms, high order terms) $\cdots$

Idea:



In a small time interval, everything can change

$$\Delta C = \Delta \cdot (\Delta S) + \frac{\Gamma}{2}(\Delta S)^2$$

$$+ \nu \cdot (\Delta\sigma) + \Theta(-\Delta T) + \rho \cdot (\Delta r)$$

(change in each factor can contribute to ΔC)

($T = 10 \to 9$
$\Delta T = -1$, $-\Delta T = 1$)