Applying the change-of-variable formula (B.41), we have

$$\begin{aligned} h(y) &= f[w(y)] \cdot \left|\frac{dw(y)}{dy}\right| \\ &= 2\left(\tfrac{1}{2}y^{1/3}\right) \cdot \left|\tfrac{1}{6}y^{-2/3}\right| \\ &= \tfrac{1}{6}y^{-1/3},\ 0 < y < 8 \end{aligned}$$

The change-of-variable technique can be modified for the case of several random variables, $X_1,\ X_2$ being transformed into $Y_1,\ Y_2$. For a description of the method, which requires matrix algebra, see William Greene (2008) *Econometric Analysis*, 6th edition, Pearson Prentice Hall, pp. 1004–1005.

B.3 Some Important Probability Distributions

In this section we give brief descriptions and summarize the properties of the probability distributions used in this book.

B.3.1 The Bernoulli Distribution

Let the random variable X denote an experimental outcome with only two possible outcomes, A or B. Let $X = 1$ if the outcome is A and let $X = 0$ if the outcome is B. Let the probabilities of the outcomes be $P(X = 1) = p$ and $P(X = 0) = 1 - p$ where $0 \le p \le 1$. X is said to have a **Bernouilli distribution**. The *pdf* of this Bernoulli random variable is

$$f(x|p) = \begin{cases} p^x(1-p)^{1-x} & x = 0, 1 \\ 0 & \text{otherwise} \end{cases} \tag{B.42}$$

The expected value of X is $E(X) = p$, and its variance is $\text{var}(X) = p(1 - p)$. This random variable arises in **choice models**, such as the linear probability model (Chapters 7, 8, and 16) and in binary and multinomial choice models (Chapter 16).

B.3.2 The Binomial Distribution

If $X_1,\ X_2, \ldots,\ X_n$ are independent random variables, each having a Bernouilli distribution with parameter p, then $X = X_1 + X_2 + \cdots + X_n$ is a discrete random variable that is the number of successes (i.e., Bernoulli experiments with outcome $X_i = 1$) in n trials of the experiment. The random variable X is said to have a **binomial distribution**. The *pdf* of this random variable is

$$P(X = x|n, p) = f(x|n, p) = \binom{n}{x} p^x(1-p)^{n-x} \text{ for } x = 0,\ 1,\ \ldots,\ n \tag{B.43}$$

where

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

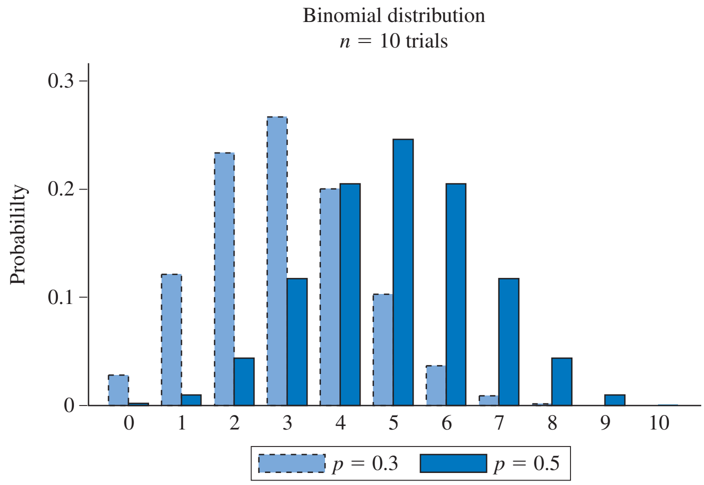

FIGURE B.3 Binomial distributions for n = 10.

is the number of combinations of n things taken x at a time. This distribution has two parameters, n and p, where n is a positive integer indicating the number of experimental trials and $0 \leq p \leq 1$. These probabilities are tedious to compute by hand, but econometric software has functions to carry out the calculations. The discrete probabilities are illustrated in Figure B.3.

The expected value and variance of X are

$$E(X) = \Sigma_{i=1}^{n} E(X_i) = np$$

$$\text{var}(X) = \Sigma_{i=1}^{n} \text{var}(X_i) = np(1 - p)$$

A related random variable is $Y = X/n$, which is the proportion of successes in n trials of an experiment. Its mean and variance are $E(Y) = p$ and $\text{var}(Y) = p(1 - p)/n$.

B.3.3 The Poisson Distribution

Whereas a binomial random variable is the number of event occurrences in a given number of experimental trials, n, the Poisson random variable is the number of event occurrences in a given interval of time or space. The probability density function for the discrete random variable X is

$$P(X = x|\mu) = f(x|\mu) = \frac{e^{-\mu}\mu^x}{x!} \quad \text{for } x = 0, 1, 2, 3, \ldots \tag{B.44}$$

Probabilities depend on the parameter μ, and $e \cong 2.71828$ is the base of natural logarithms. The expected value and variance of X are $E(X) = \text{var}(X) = \mu$. The Poisson distribution is used in models involving count variables (Chapter 16), such as the number of visits a person makes to a physician during a year. Probabilities for $x = 0$ to 10 for distributions with $\mu = 3$ and $\mu = 4$ are shown in Figure B.4.

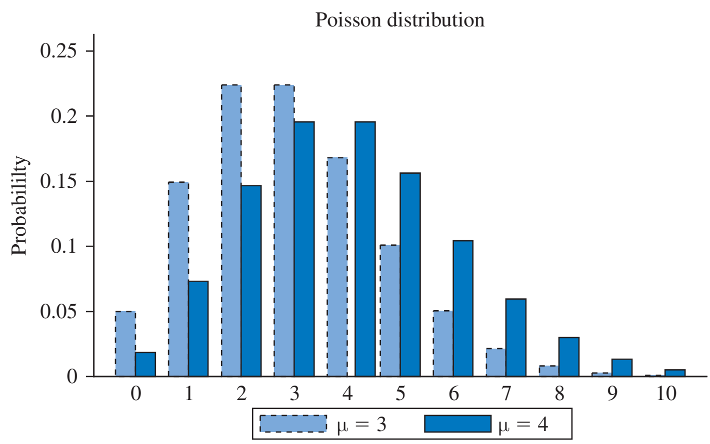

FIGURE **B.4** Poisson distribution.

B.3.4 The Uniform Distribution

A continuous distribution that is vastly important for theoretical purposes is the **uniform distribution**. The random variable X with values $a \le x \le b$ has a uniform distribution if its *pdf* is given by

$$f(x|a,b) = \frac{1}{b-a} \text{ for } a \le x \le b \tag{B.45}$$

The plot of the density function is given in Figure B.5

The area under $f(x)$ between a and b is one which is required of any probability density function for a continuous random variable. The expected value of X is the midpoint of the interval $[a, b]$, $E(X) = (a+b)/2$. This can be deduced from the symmetry of the distribution. The variance of X is $\text{var}(X) = E(X^2) - \mu^2 = (b-a)^2/12$.

An interesting special case occurs when $a = 0$ and $b = 1$, so that $f(x) = 1$ for $0 \le x \le 1$. The distribution, shown in Figure B.6, describes one common meaning of "a random number between zero and one."

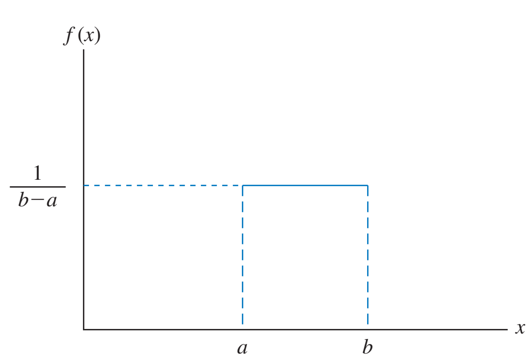

FIGURE **B.5** A uniform distribution.

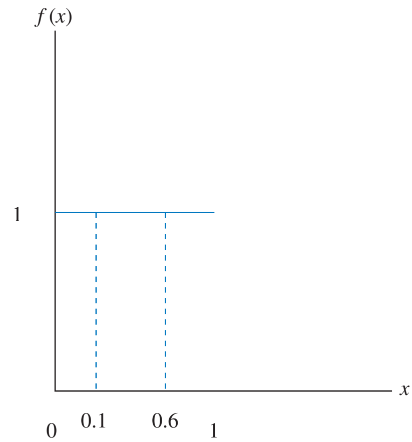

FIGURE **B.6** A uniform distribution on [0, 1] interval.

The uniform distribution has the property that any two intervals of equal width have the same probability of occurring. That is

$$P(0.1 \le X \le 0.6) = P(0.3 \le X \le 0.8) = P(0.21131 \le X \le 0.71131) = 0.5$$

Picking a number randomly between zero and one is conceptually complicated by the fact that the interval has an uncountably infinite number of values, and the probability of any one of them occurring is zero. What is more likely meant by such a statement is that each interval of equal width has the same probability of occurring, no matter how narrow. This is exactly the nature of the uniform distribution.

B.3.5 The Normal Distribution

The normal distribution was described in the Probability Primer, Section P.6. A point not stressed at that time was why we must consult tables, like Table 1 at the end of the book, to calculate normal probabilities. For example, we now know that for the continuous and normally distributed random variable X, with mean μ and variance σ^2, the probability that X falls in the interval $[a, b]$ is

$$\int_a^b f(x)dx = \int_a^b \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-(x-\mu)^2 \Big/ 2\sigma^2\right] dx$$

Unfortunately this integral does not have a closed-form, algebraic solution. Consequently, we wind up working with tabled values containing numerical approximations to areas under the standard normal distribution, or we use computer software functions in a similar manner.

The normal distribution is related to the chi-square, t, and F distributions, which we now discuss.

B.3.6 The Chi-square Distribution

Chi-square random variables arise when standard normal random variables are squared. If $Z_1, Z_2, \ldots, Z_m$ denote m *independent* $N(0, 1)$ random variables, then

$$V = Z_1^2 + Z_2^2 + \ldots + Z_m^2 \sim \chi^2_{(m)} \tag{B.46}$$

The notation $V \sim \chi^2_{(m)}$ is read as: The random variable V has a chi-square distribution with m **degrees of freedom**. The degrees of freedom parameter m indicates the number of *independent* $N(0,1)$ random variables that are squared and summed to form V. The value of m determines the entire shape of the chi-square distribution, including its mean and variance

$$\begin{aligned} E[V] &= E\left[\chi^2_{(m)}\right] = m \\ \operatorname{var}[V] &= \operatorname{var}\left[\chi^2_{(m)}\right] = 2m \end{aligned} \tag{B.47}$$

In Figure B.7 graphs of the chi-square distribution for various degrees of freedom are presented. The values of V must be nonnegative, $v \geq 0$, because V is formed by squaring and summing m standardized normal $N(0,1)$ random variables. The distribution has a long tail, or is *skewed*, to the right. As the degrees of freedom m gets larger, however, the distribution becomes more symmetric and "bell-shaped." In fact, as m gets larger, the chi-square distribution converges to, and essentially becomes, a normal distribution.

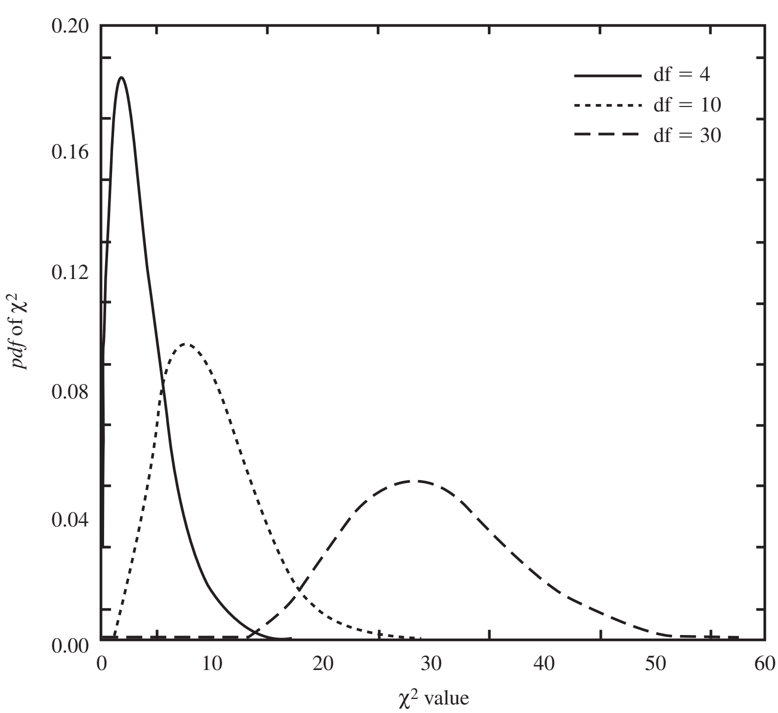

FIGURE B.7 The chi-square distribution.

The 90th, 95th, and 99th percentile values of the chi-square distribution for selected values of the degrees of freedom are given in Table 3 at the end of the book. These values are often of interest in hypothesis testing.

B.3.7 The t-distribution

A t random variable (no upper case) is formed by dividing a standard normal random variable $Z \sim N(0, 1)$ by the square root of an *independent* chi-square random variable, $V \sim \chi^2_{(m)}$, that has been divided by its degrees of freedom m. If $Z \sim N(0, 1)$ and $V \sim \chi^2_{(m)}$, and if Z and V are independent, then

$$t = \frac{Z}{\sqrt{V/m}} \sim t_{(m)} \tag{B.48}$$

The t-distribution's shape is completely determined by the degrees of freedom parameter, m, and the distribution is symbolized by $t_{(m)}$.

Figure B.8 shows a graph of the t-distribution with $m = 3$ degrees of freedom relative to the $N(0,1)$. Note that the t-distribution is less "peaked," and more spread out than the $N(0,1)$. The t-distribution is symmetric, with mean $E(t_{(m)}) = 0$ and variance $\text{var}(t_{(m)}) = m/(m-2)$. As the degrees of freedom parameter $m \to \infty$ the $t_{(m)}$ distribution approaches the standard normal $N(0,1)$.

Computer programs have functions for the *cdf* of t-random variables that can be used to calculate probabilities. Since certain probabilities are widely used, Table 2 at the back of this book, also inside the front cover, contains frequently used percentiles of t-distributions, called **critical values** of the distribution. For example, the 95th percentile of a t-distribution with 20 degrees of freedom is $t_{(0.95, 20)} = 1.725$. The t-distribution is symmetric, so Table 2 shows only the right tail of the distribution.

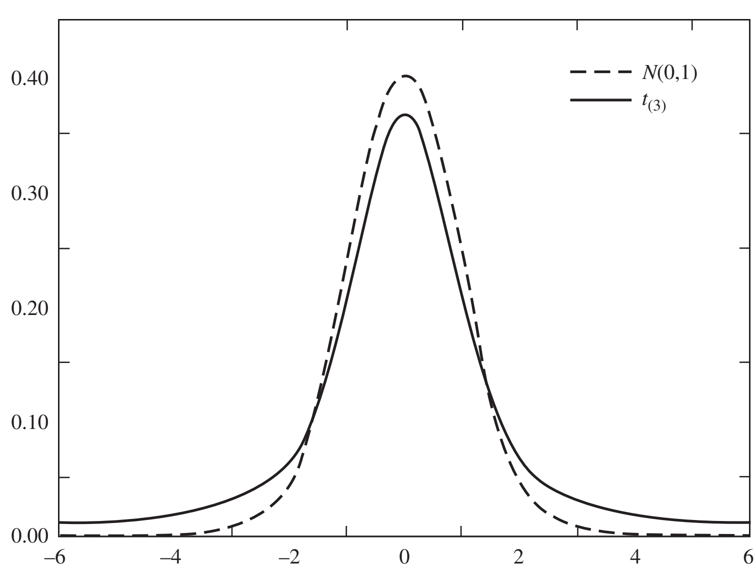

FIGURE B.8 The standard normal and $t_{(3)}$ probability density functions.

B.3.8 The F-distribution

An F random variable is formed by the ratio of two independent chi-square random variables that have been divided by their degrees of freedom. If $V_1 \sim \chi^2_{(m_1)}$ and $V_2 \sim \chi^2_{(m_2)}$, and if V_1 and V_2 are independent, then

$$F = \frac{V_1/m_1}{V_2/m_2} \sim F_{(m_1,m_2)} \tag{B.49}$$

The F-distribution is said to have m_1 *numerator degrees of freedom* and m_2 *denominator degrees of freedom*. The values of m_1 and m_2 determine the shape of the distribution, which in general looks like Figure B.9. The range of the random variable is $(0,\infty)$ and it has a long tail to the right. For example, the 95th percentile value for an F-distribution with $m_1 = 8$ numerator degrees of freedom and $m_2 = 20$ denominator degrees of freedom is $F_{(0.95,8,20)} = 2.45$. Critical values for the F-distribution are given in Table 4 (the 95th percentile) and Table 5 (the 99th percentile).

B.4 Random Numbers

In several chapters we carry out Monte Carlo simulations to illustrate the sampling properties of estimators. See, for example, Chapters 3, 4, 5, 10, and 11. To use Monte Carlo simulations we rely upon the ability to create **random numbers** from specific probability distributions, such as the uniform and the normal. The use of computer simulations is widespread in all sciences. In this section we introduce to you this aspect of computing.[1] You should first realize that the idea of creating random numbers using a computer is paradoxical, because by definition random numbers that are "created" cannot be truly random. The random numbers generated by a computer are **pseudo-random numbers** in that they "behave as if they were random." We present one method for generating pseudo-random numbers called the **inverse transformation** approach, or the

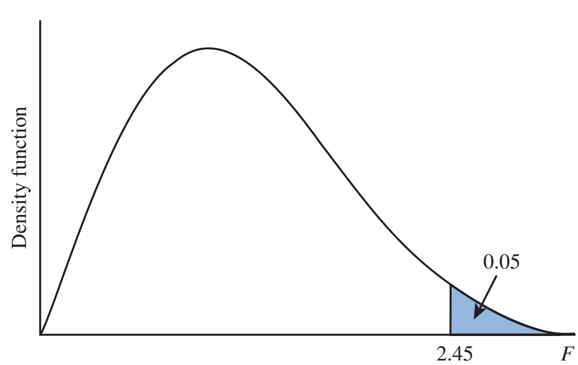

FIGURE B.9 The probability density function of an $F_{(8,20)}$ random variable.

[1] A well-written book on the subject is by James E. Gentle (2003) *Random Number Generation and Monte Carlo Methods,* New York: Springer.