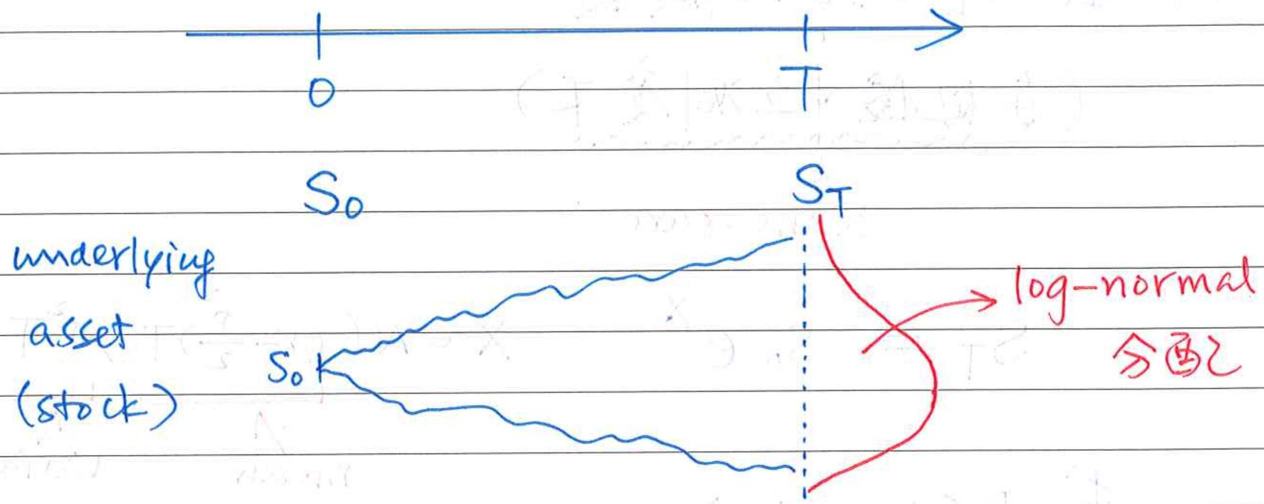
BS 公式推導



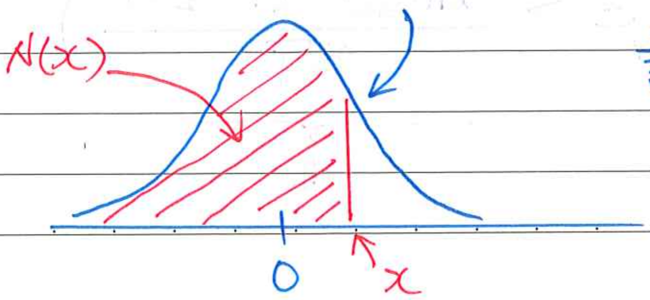
options $\begin{cases} C_0 \longleftarrow\text{-------------}\ C_T = (S_T - K)^+ \\ P_0 \longleftarrow\text{-------------}\ P_T = (K - S_T)^+ \end{cases}$

① 公式 $\begin{cases} C_0 = S_0 N(d_1) - Ke^{-rT} N(d_2) \\ P_0 = Ke^{-rT} N(-d_2) - S_0 N(-d_1) \end{cases}$

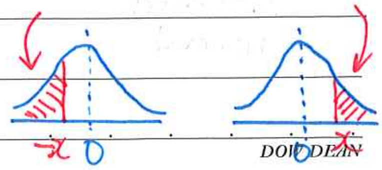
其中 $\begin{cases} d_1 = \dfrac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \\ d_2 = \dfrac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \end{cases}$

此處 $N(x)$ 是標準常態分配的 CDF

$X \sim N(0,1)$



註: $N(-x) = 1 - N(x)$



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② 對 S_T 分配的假設

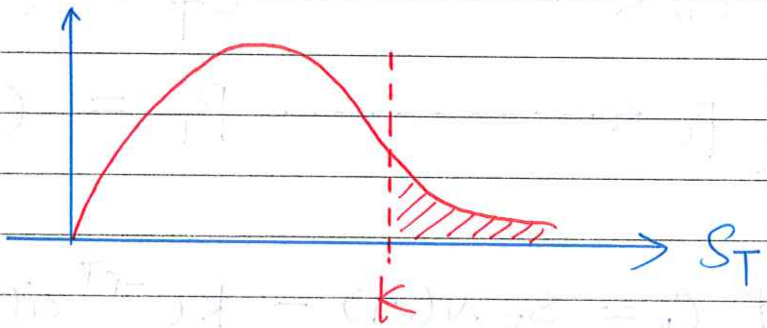
(在風險中立測度下)

$\Downarrow$

Q measure

$$S_T = S_0 e^X \qquad X \sim N\Big(\underbrace{(r-\tfrac{\sigma^2}{2})T}_{\text{mean}},\ \underbrace{\sigma^2 T}_{\text{variance}}\Big)$$

求 $P(S_T > K)$?



$$P(S_T > K)$$

$$= P(\underbrace{\ln S_T}_{= \ln S_0 + X} > \ln K)$$

$$= P\left(X > \ln\left(\frac{K}{S_0}\right)\right)$$

標準化

$X \sim N(\mu, \sigma^2)$

$\frac{X-\mu}{\sigma} \sim N(0,1)$

$\uparrow$

Z

$(r-\frac{\sigma^2}{2})T$

$$= P\left(\frac{X - \boxed{}}{\bigcirc} > \frac{\ln(\frac{K}{S_0}) - \boxed{}}{\bigcirc}\right)$$

$\sigma\sqrt{T}$

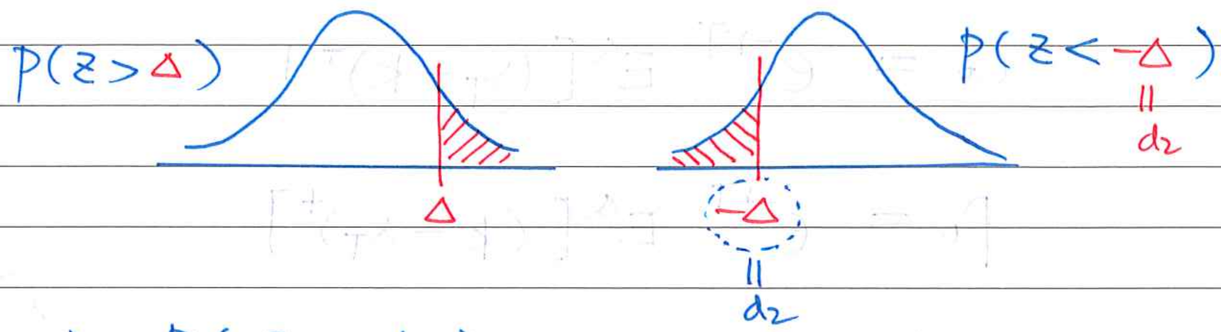
$$= P\Bigg(Z > \underbrace{\frac{\ln(\frac{K}{S_0}) - (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}}_{= \triangle}\Bigg)$$

standard normal

$$= P\left(z < \underbrace{\frac{\ln\left(\frac{S_0}{K}\right) + \left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}}_{= -\triangle = d_2}\right)$$

$$= P(z < d_2)$$

$$= N(d_2)$$



$$\therefore P(S_T < K)$$

$$= 1 - P(S_T > K)$$

$$= 1 - N(d_2) = N(-d_2)$$

用到 $N(-x) = 1 - N(x)$

<u>小結</u>

在 Q-measure 下

$P(S_T > K) = N(d_2)$ → 買權履約的機率

$P(S_T < K) = N(-d_2)$ → 賣權履約的機率

注意 $N(d_2)$, $N(-d_2)$ 在 BS 公式中的位置

並賦予其<u>直觀意義</u>!
intuitive meaning

③ BS 公式架構　　利率 = r

$0 \longrightarrow T$

$C_0 \leftsquigarrow C_T = (S_T - K)^+$

$P_0 \leftsquigarrow P_T = (K - S_T)^+$

$$C_0 = e^{-rT} E^Q[(S_T - K)^+]$$

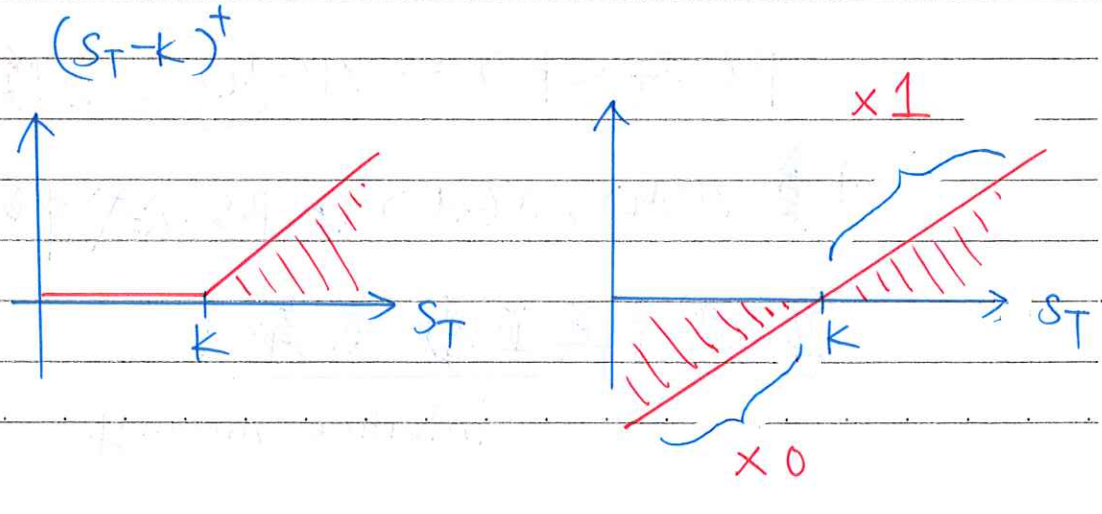
$$P_0 = e^{-rT} E^Q[(K - S_T)^+]$$

推導 C_0，定義 $\underbrace{\mathbb{1}_{(S_T>K)}}_{\text{indicator r.v.}} = \begin{cases} 1 & \text{if } S_T > K \quad \text{履約} \\ 0 & \text{if } S_T \leq K \quad \text{不履約} \end{cases}$

$$\mathbb{1}_A = \begin{cases} 1 & \text{if A happens} \\ 0 & \text{if A doesn't happen} \end{cases}$$

$$C_0 = e^{-rT} E[(S_T - K)^+]$$

$$= e^{-rT} E[(S_T - K) \cdot \mathbb{1}_{(S_T>K)}]$$



$$= e^{-rT}\underbrace{E[S_T \cdot \mathbb{1}_{(S_T>K)}]}_{S_0 e^{rT} N(d_1)} - Ke^{-rT}\underbrace{E[\mathbb{1}_{(S_T>K)}]}_{= N(d_2)}$$

ⓐ 先看後項：

$$E[\mathbb{1}_A] = p(A)\times 1 + \underbrace{p(A^c)}_{= 1-p(A)}\times 0 = p(A)$$

$$\therefore E[\mathbb{1}_{(S_T>K)}] = p(S_T>K) = N(d_2)$$

ⓑ 再求前項：

$$S_T = S_0 e^X \qquad X \sim N(\underset{(r-\frac{\sigma^2}{2})T}{a}, \underset{\sigma^2 T}{b^2})$$

$$S_T > K \iff X > \ln\left(\frac{K}{S_0}\right)$$

$$\therefore E[S_T \cdot \mathbb{1}_{(S_T>K)}]$$

$$= S_0 \cdot \underbrace{E\left[e^X \cdot \mathbb{1}_{(X>\ln(\frac{K}{S_0}))}\right]}$$

$$X \sim N(a, b^2) \qquad \underbrace{\frac{X-a}{b}}_{= Z} \sim N(0,1)$$

$$X = a + bZ$$

$$X > \ln\left(\frac{K}{S_0}\right) \iff Z > \underbrace{\frac{\ln(\frac{K}{S_0})-a}{b}}_{\text{令為 } c}$$

$$\boxed{} = E\left[e^{a+bZ} \cdot \mathbb{1}_{(Z>c)}\right]$$

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$$= \int_{c}^{\infty} e^{a+bz} \cdot \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}}_{\text{standard normal 之 pdf}} dz$$

注意 從c開始積

若 $z \sim N(0,1)$，則 $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$

$$= e^{a+\frac{b^2}{2}} \int_{c}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-b)^2}{2}} dz$$

(註：指數部分配方
$$\frac{z^2-2bz+b^2}{2} = \frac{(z-b)^2}{2}$$
)

$$= e^{a+\frac{b^2}{2}} \int_{c-b}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z')^2}{2}} dz'$$

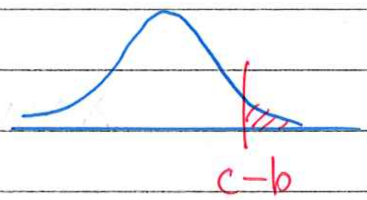
(註：令 $z' = z-b$

$dz' = dz$

z	c	∞
z'	$c-b$	∞

)

$$= e^{a+\frac{b^2}{2}} \underbrace{P(z' > c-b)}_{N(b-c)}$$



$$\Longrightarrow \boxed{} = e^{a+\frac{b^2}{2}} N(b-c)$$

$$\therefore E[S_T \mathbb{1}_{(S_T>K)}] = S_0 e^{a+\frac{b^2}{2}} N(b-c)$$

$$\because \quad a = \left(r - \frac{\sigma^2}{2}\right)T, \quad b^2 = \sigma^2 T$$

$$c = \frac{\ln\left(\frac{K}{S_0}\right) - a}{b}$$

$$= \frac{\ln\left(\frac{K}{S_0}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$\therefore \quad a + \frac{b^2}{2} = rT$$

$$b - c = \sigma\sqrt{T} - \frac{\ln\left(\frac{K}{S_0}\right) - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$= \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1$$

$$\therefore \quad E\left[S_T \mathbb{1}_{(S_T > K)}\right] = S_0 e^{rT} N(d_1)$$

ⓒ 兩項合併，可得

$$C_0 = S_0 N(d_1) - K e^{-rT} N(d_2)$$

④ 循此方法可推導出 p_0 公式

$$p_0 = e^{-rT} E\left[(K - S_T)^+\right]$$

$$= e^{-rT} E\left[(K - S_T)\mathbb{1}_{(S_T < K)}\right]$$

$$= K e^{-rT} \underbrace{E\left[\mathbb{1}_{(S_T < K)}\right]}_{N(-d_2)} - e^{-rT} \underbrace{E\left[S_T \mathbb{1}_{(S_T < K)}\right]}_{S_0 e^{rT} N(-d_1)}$$

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請自行推出：

ⓐ $E[\mathbb{1}_{(S_T<K)}] = P(S_T<K) = N(-d_2)$

ⓑ $E[S_T \mathbb{1}_{(S_T<K)}] = S_0 e^{rT} N(-d_1)$

ⓒ 合併即得

$$P_0 = Ke^{-rT}N(-d_2) - S_0 N(-d_1)$$

核心問題

為何假設 $S_T = S_0 e^X$

$X \sim N(a, b^2)$?

$a \to (r-\frac{\sigma^2}{2})T$, $b^2 \to \sigma^2 T$

這是我們後續要探討的問題！