Graph Search

Computing 2 COMP1927 17x1

PROBLEMS ON GRAPHS

- What kinds of problems do we want to solve on/via graphs?
  - Is there a simple path from A to B
  - Is the graph fully-connected?
  - Can we remove an edge and keep it fully connected?
  - Which vertices are reachable from v? (transitive closure)
  - What is the cheapest cost path from v to w?
  - Is there a cycle that passes through all V? (tour)
  - Is there a tree that links all vertices (spanning tree)
  - What is the minimum spanning tree?
  - Can a graph be drawn in a place with no crossing edges?
  - Are two graphs "equivalent"? (isomorphism)

Graph Search

- In this part of the course, we examine algorithms for:
  - Connectivity (simple graphs)
  - path finding   (simple/directed graphs)
  - minimum spanning trees   (weighted graphs)
  - shortest path   (weighted graphs)
- And look at generic algorithms for traversing (searching) graphs that can be used to solve a wide range of graph problems which involves:
  - walking along edges and visiting vertices
  - recording e.g. path taken, vertices visited, etc.
- We begin with one of the simplest graph traversals …

Simple Path Search

- Problem: is there a path from vertex v to vertex w ?
- As a function:

```
int isPath(Graph g, Vertex v, Vertex w);
```

- Approach to solving problem:
  - examine vertices adjacent to v
  - if any of them is w, then we are done
  - otherwise check try vertices two edges from v
  - repeat looking further and further from v

Simple Path Search

- Algorithm for path-finding:

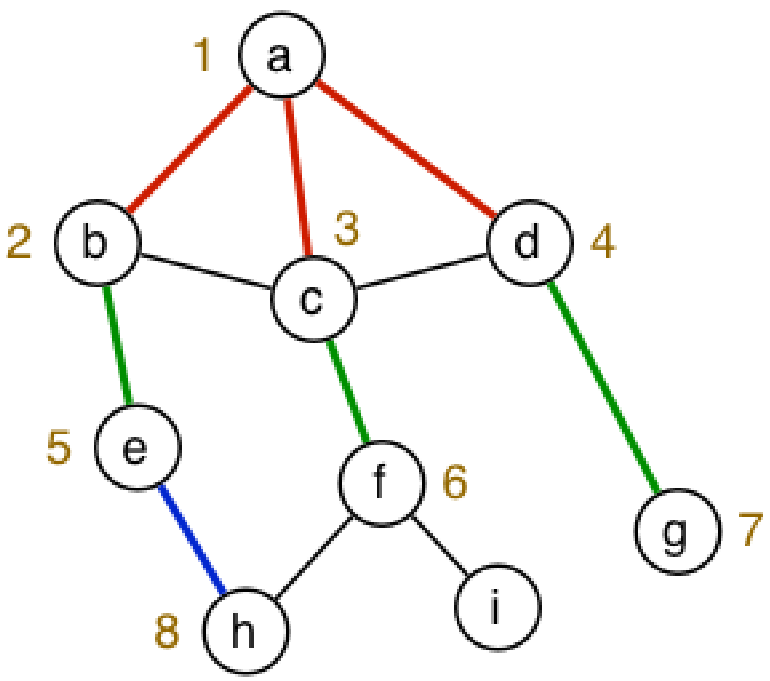
```
// is there a path from V to W?
ToVisit = { V }    // vertices to be checked
Visited = { }     // previously checked vertices
while (still some vertices in ToVisit)  {
    X = remove a vertex from ToVisit
    Visited += X
    foreach (Y in ajacentVertices(X)) {
        if (Y == W) return TRUE
        if (Y not in Visited) ToVisit += Y
    }
}
return FALSE
```

Simple Path Search

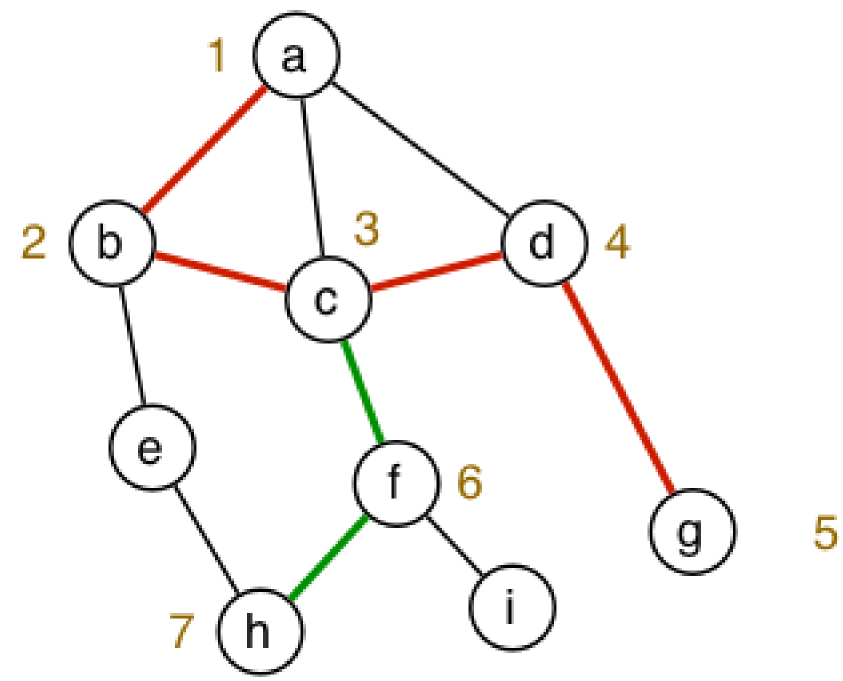
- *Graph traversal* also commonly referred to as *Graph search*
- Two different approaches to order of searching: breadth-first search (BFS), depth-first search (DFS)
  - DFS follows one path to completion before considering others
  - DFS uses recursion or a stack, and backtracking
  - BFS "fans-out" from the starting vertex ("spreading" subgraph)
  - BFS maintains a queue of to-be-visited vertices

Comparison of BFS/DFS path finding

- Is there a path from a to h?



Breadth-first Search



Depth-first Search

DFS vs BFS Approaches

- DFS and BFS are closely related.
- Implementation differs only their use of a stack or a queue

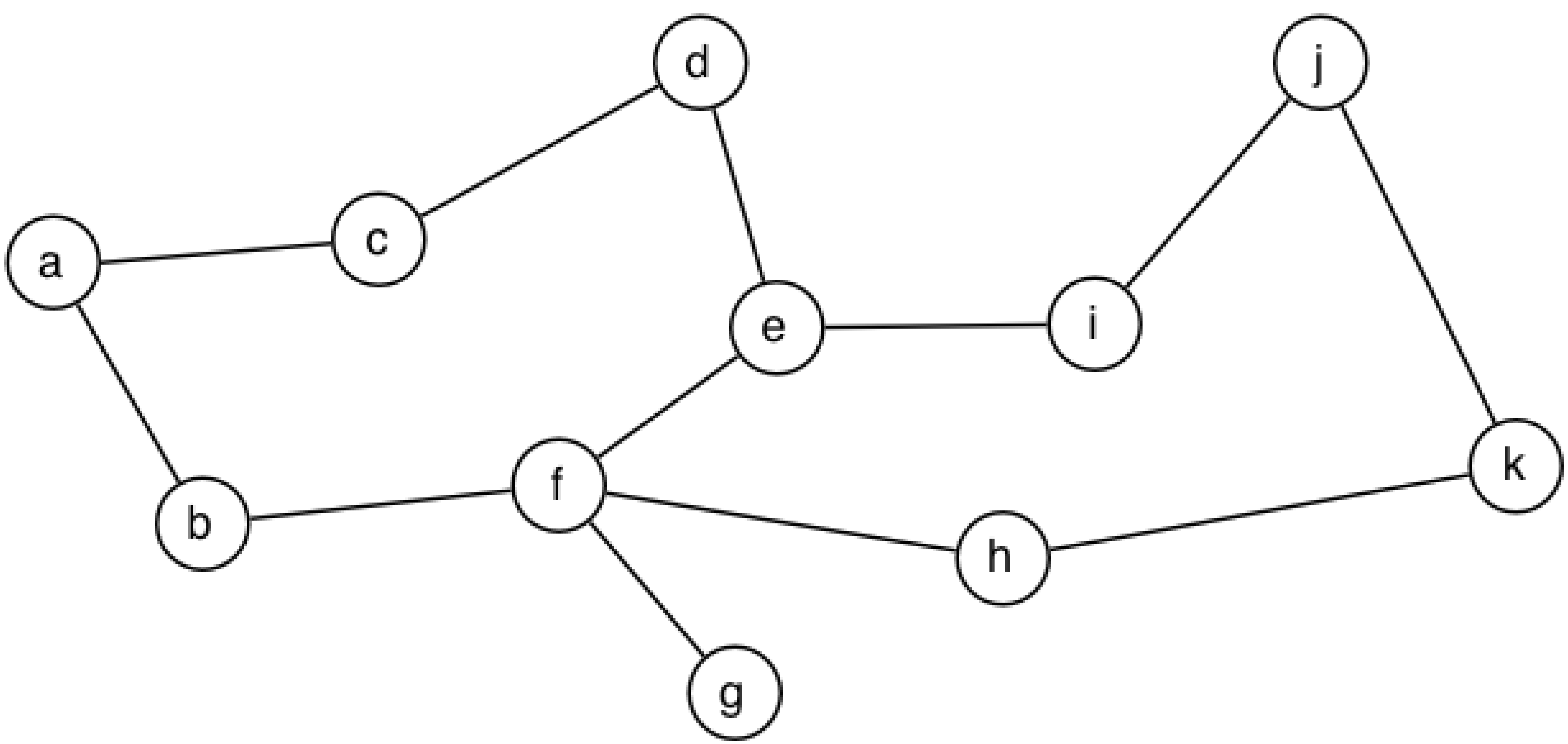
  - BFS implemented via a queue of to-be-visited vertices

  - DFS implemented via a stack of to-be-visited vertices (or recursion)

- Both approaches ignore some edges and avoid cycles by remembering previously visited vertices.

Exercise: DFS and BFS Traversal

- Show the DFS order we visit to determine isPath(a,k)
- Show the BFS order we visit to determine isPath(a,k)
- Assume neighbours are chosen in alphabetical order



Depth First Search

Depth first traversal can be described recursively as

```
depth-first(G, V)
{
   mark V as visited
   for each neighbour W of V {
      if (W already visited) continue
      depth-first(G, W)
   }
}
```

Depth First Search

Notes about DFS:

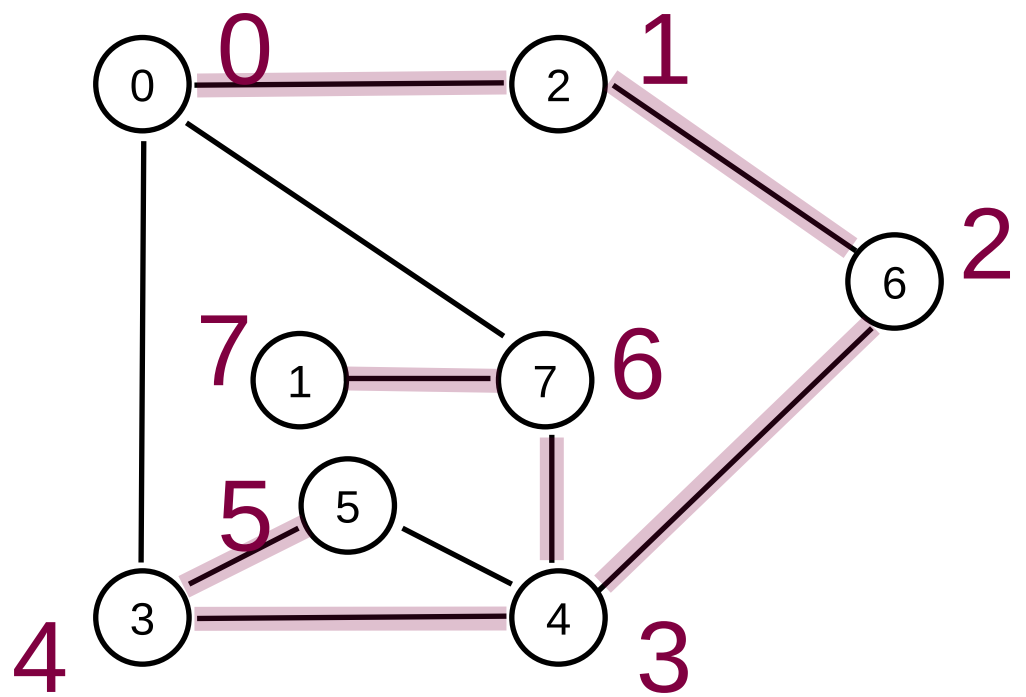
- One approach: consider neighbours in ascending order
- Recursion induces back-tracking
- need a mechanism for "marking" vertices … in fact, we number them as we visit them
(so that we could later trace path through graph)

- Make use of three global variables:
  - count … counter to remember how many vertices traversed so far
  - pre[] … array saying order in which each vertex was visited (pre stands for preorder)
  - st[] … array storing the predecessor of each vertex

Depth First Search Tree

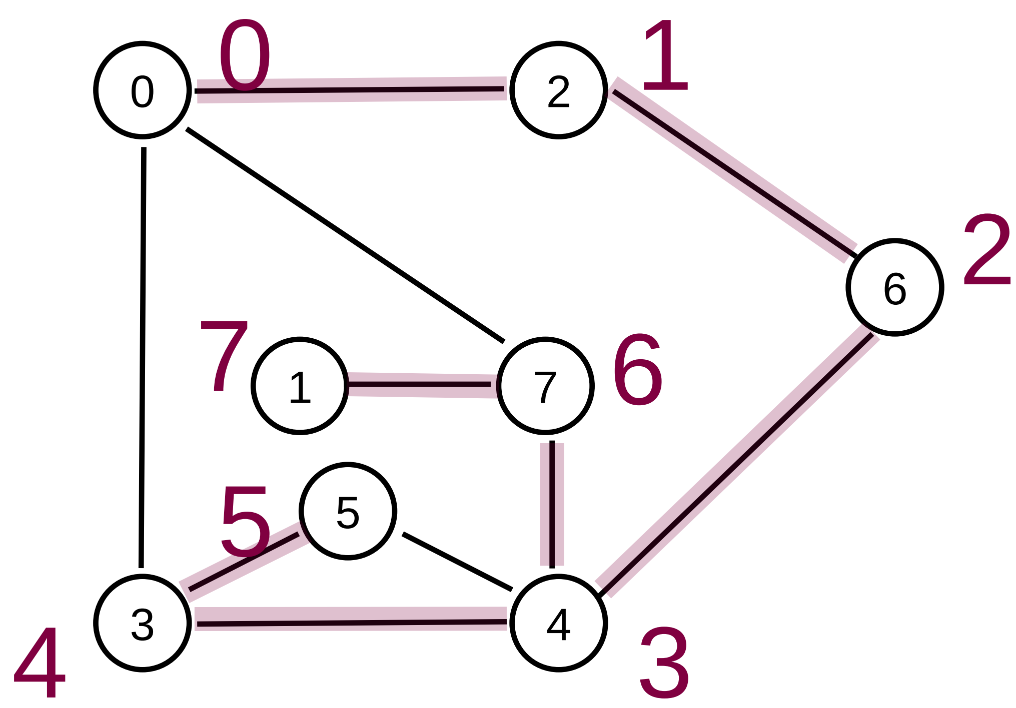
- The edges traversed in a graph walk form a tree
- It corresponds to the call tree of the recursive *dfs* function
- Represents the original graph minus any cycles or alternate paths
- We can use a tree to encode the whole search process
- Each time we visit a vertex we record the previous vertex we came from - if the graph is connected this forms a spanning tree
    - We store this in the *st* array

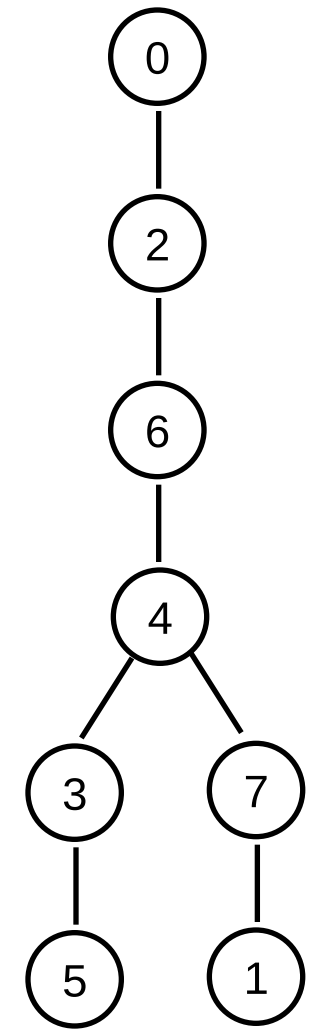
Depth First Search (DFS)




```
// Assume we start with dummy Edge {0,0}
// assume we start with count = 0
// pre[v] = -1 for all v
// st[v] = -1 for all v (stores the predecessor)
// assume adjacency matrix representation
void dfsR (Graph g, Edge e) {
     Vertex i, w = e.w;
     pre[w] = count++;
     st[w] = e.v;
     for (i=0; i < g->V; i++){
         if ((g->edges[w][i] == 1) &&  (pre[i] == -1)
             dfsR (g, mkEdge(g,w,i));
         }
     }
}
```

DEPTH FIRST SEARCH (DFS)





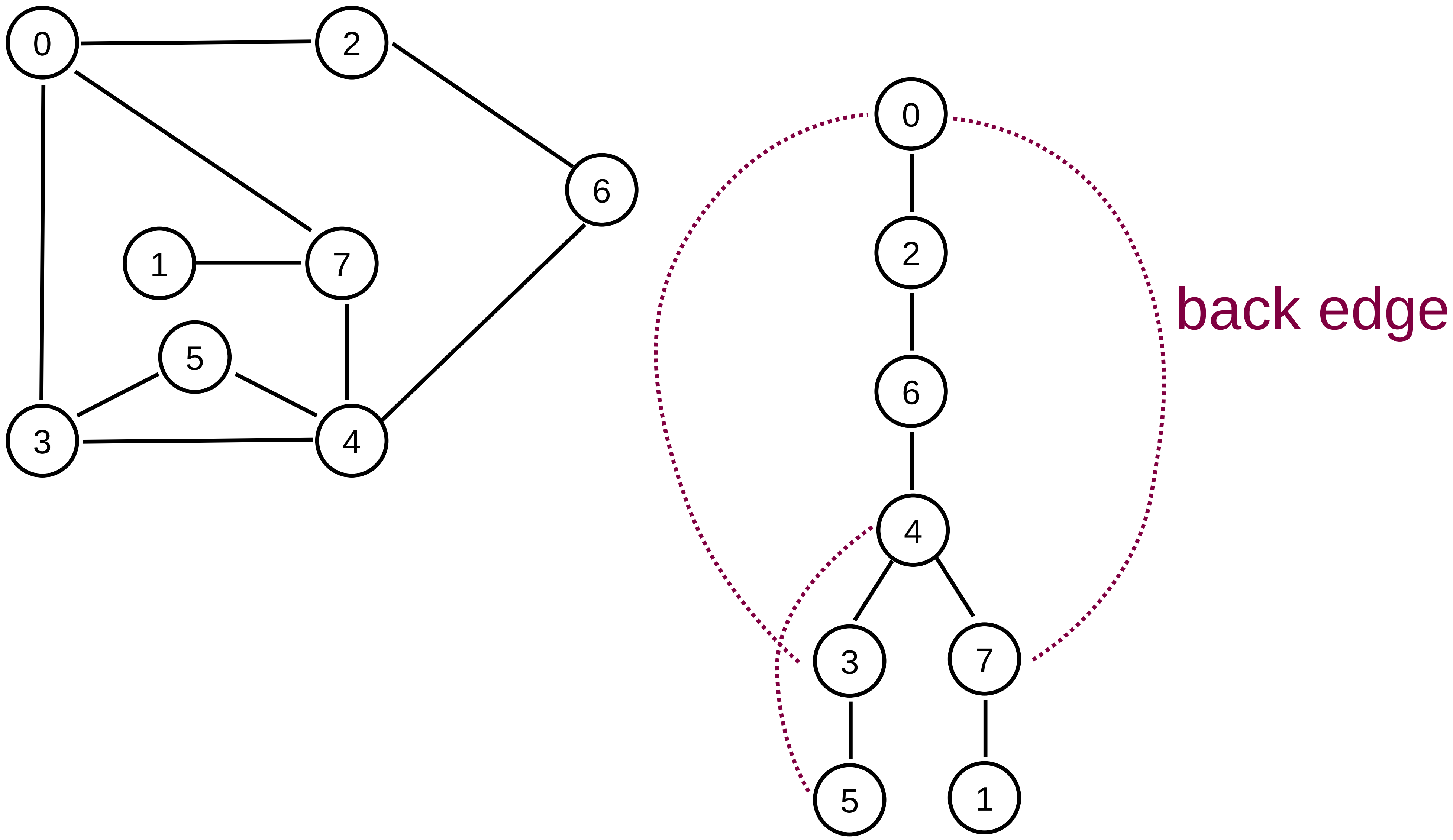
	0	1	2	3	4	5	6	7
pre	0	7	1	4	3	5	2	6
st	0	7	0	4	6	3	2	4

- the edges traversed in the graph walk form a tree
- the tree corresponds to the call tree of the depth first search
- and contents of the *st* array

  and to the contents of the st array - spanning tree
- pre contains the pre-ordering of the vertices

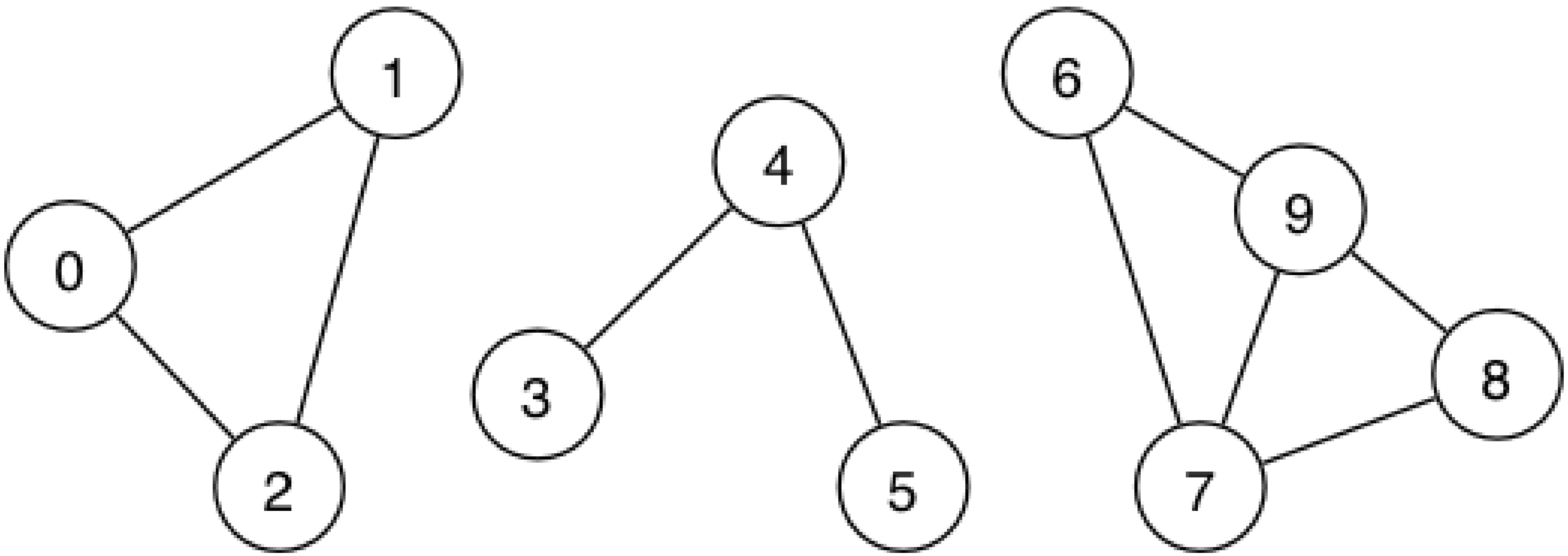
Properties of DFS forests

- If a graph is not connected it will produce a spanning forest
  - If it is connected it will form a spanning tree
- we call an edge connecting a vertex with an ancestor in the DFS tree that is not its parent a back edge



Exercise: DFS Traversal

- Which vertices will be visited during dfs(g):



- How can we ensure that *all* vertices are visited?

Graph Search function

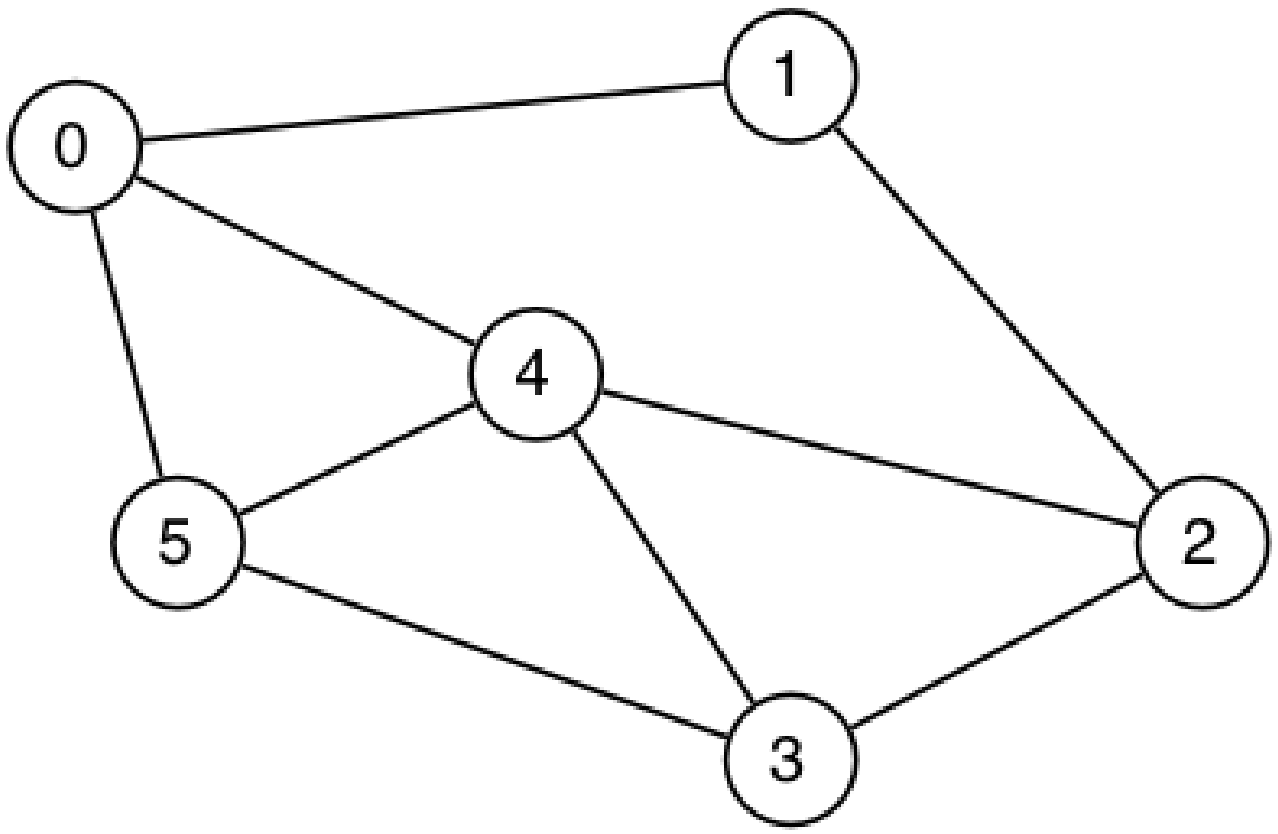
- The graph may not be connected
  - We need to make sure that we visit every connected component:

```
void  dfSearch (Graph g) {
      int v;
      count = 0;
      pre = malloc (sizeof (int) * g->nV));
      st  = malloc(sizeof (int) * g->nV));
      for (v = 0; v < g->nV; v++){
          pre[v] = -1;
          st[v] = -1;
      }
      for (v = 0; v < g->V; v++) {
          if (pre[v] == -1)
              dfsR (g, mkEdge(g,v,v));
      }
}
```

- The work complexity of the graph search algorithm is O(V^2) for adjacency matrix representation, and O($V+E$) for adjacency list representation

Exercise: DFS Traversal

- Trace the execution of dfs(g,0) on:



- What if we were using DFS to search for a path from 0..5? We would get 0-1-2-3-4-5.
- If we want the shortest (least edges/vertices) path we need to use BFS instead. See later slides for this.

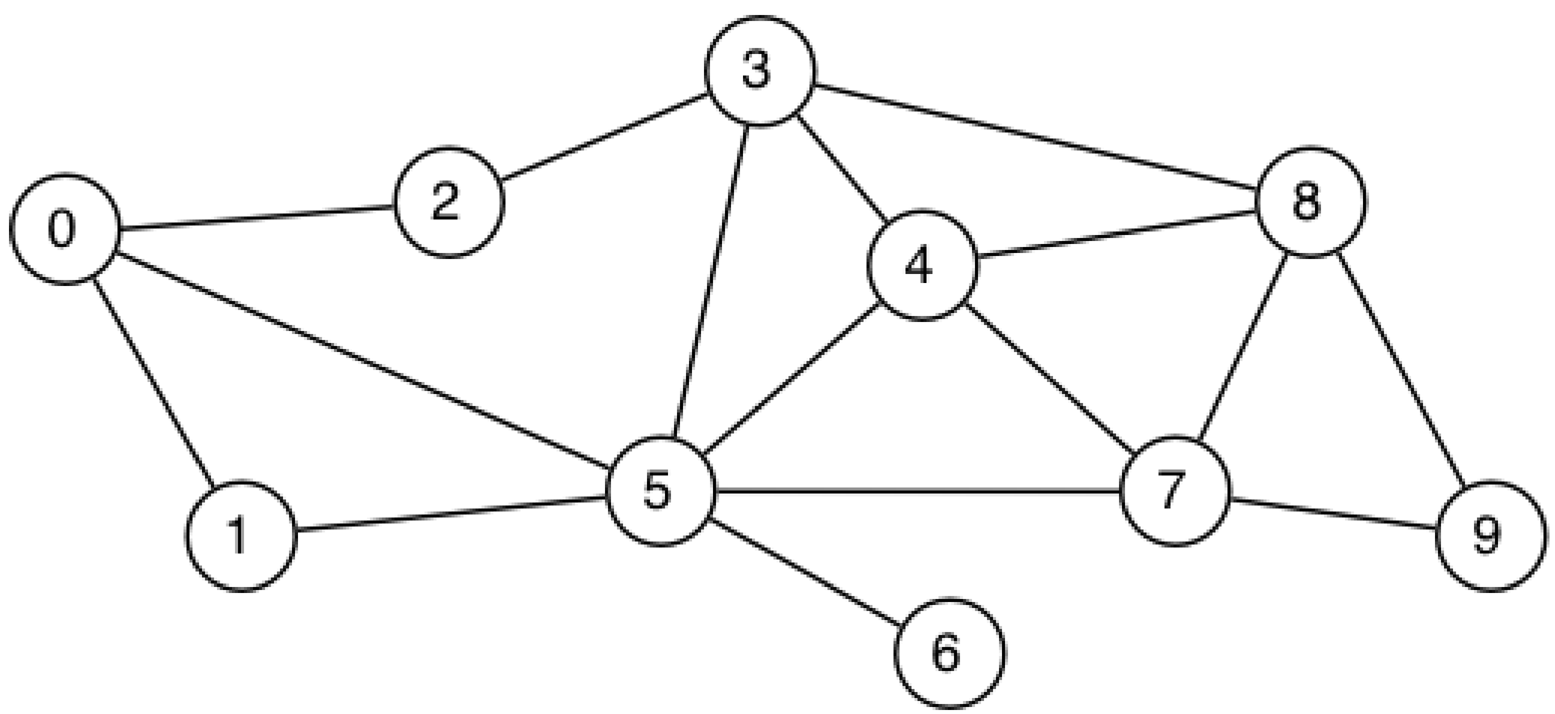
Non-Recursive Depth First Search

- We can use a stack instead of recursion:

```
void dfs (Graph g, Edge e) {
    int i;
    Stack s = newStack();
    StackPush (s,e);
    while (!StackIsEmpty(s)) {
        e = StackPop(s);
        if (pre[e.w] == -1) {
            pre[e.w] = count++;
            st[e.w] = e.v;
            for (i = 0; i < g->nV; i++) {
                if ((g->edges[e.w][i] == 1)&&
                            (pre[i] == -1)) {
                    StackPush (s,mkEdge(g,e.w,i));
                }
            }
        }
    }
}
```

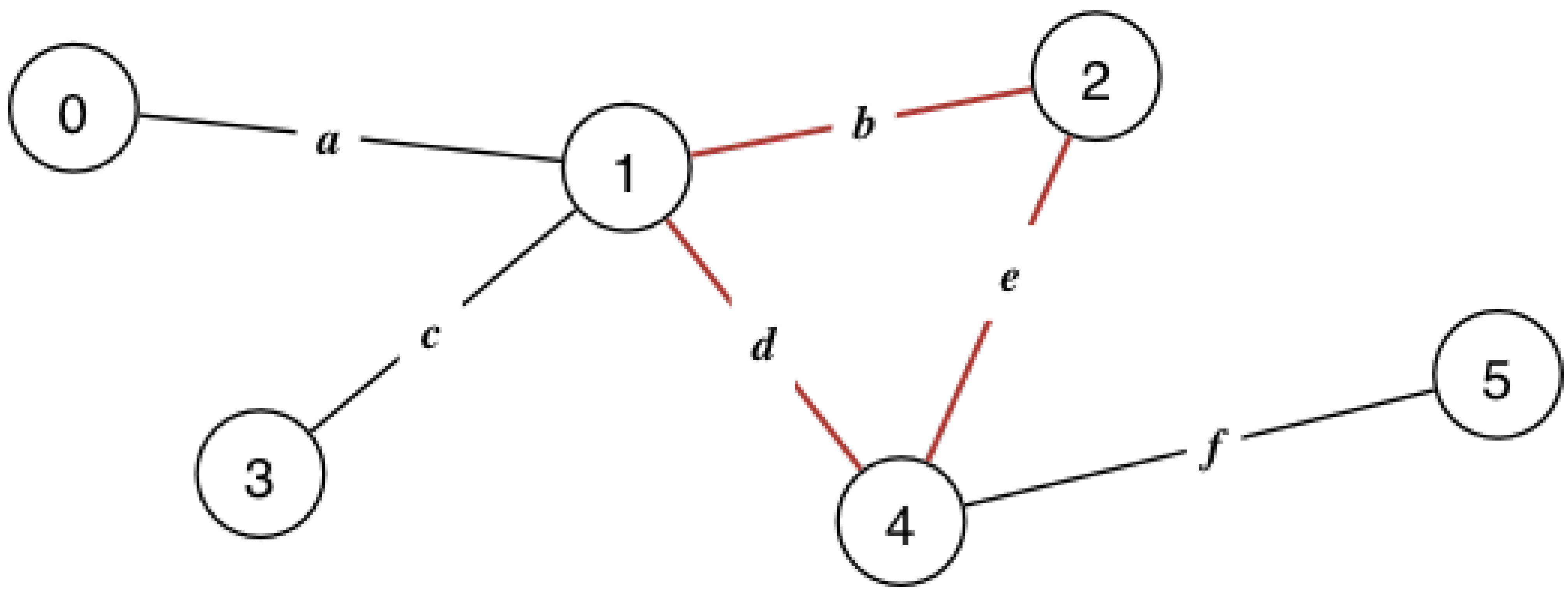
Exercise: DFS Traversal

- Show the final state of the pre and st arrays after dfs(g,0):



DFS Algorithms: Cycle Detection

- **Cycle detection**: does a given graph have any cycles?
  - if and only if the DFS graph has back edges, it contains cycles
  - we can easily detect this in the DFS search:



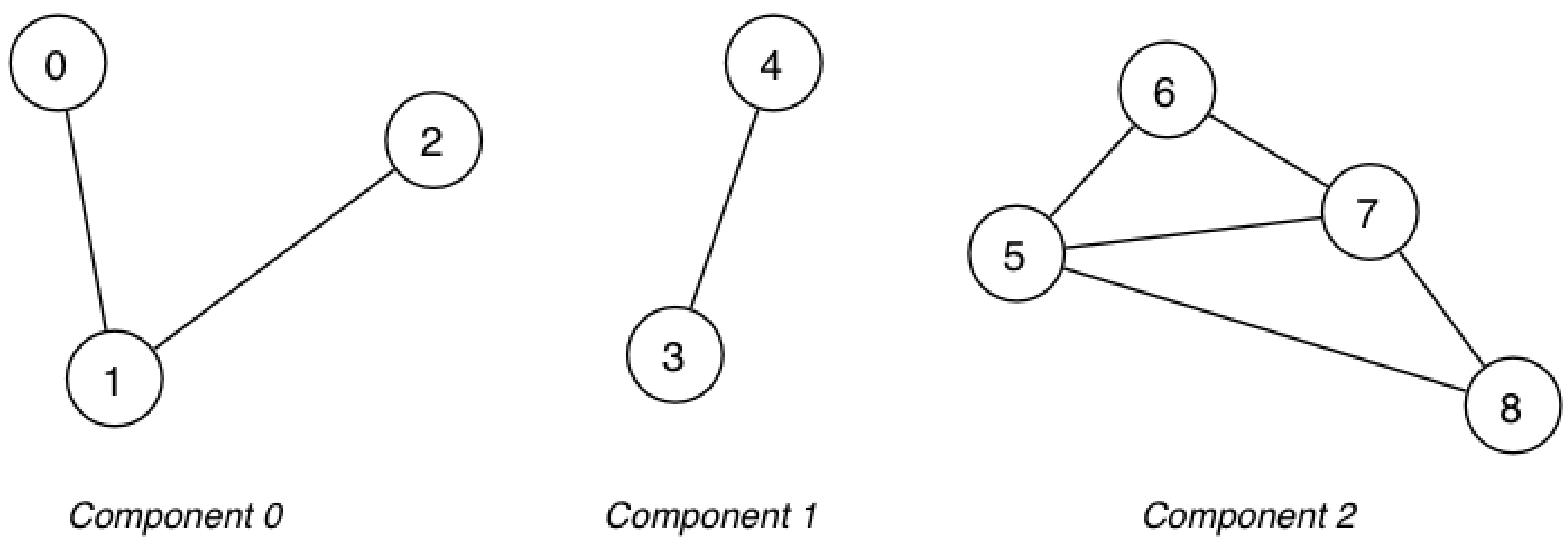
DFS Algorithms: Cycle Detection

- We are only checking for the existence of cycle, we are not returning it

```
//Return 1 if there is a cycle
int hasCycle (Graph g, Edge e) {
   int i, w = e.w;
   pre[w] = count++;
   st[w] = e.v;
   for (i=0; i < g->V; i++){
       if ((g->edges[w][i] == 1) && (pre[i] == -1)) {
           if(hasCycle (g, mkEdge(g,w,i)))
               return 1;
       } else if( (g->edges[w][i] == 1) && i != e.v){
           //if it is not the predecessor
           return 1;
       }
   }
   return 0;
}
```

DFS Algorithms: Connectivity

- Each vertex belongs to a connected component
- The function connectedComponents sets up the array cc to indicate which component contains each vertex



cc

[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]
0	0	0	1	1	2	2	2	2

DFS Algorithms

- Connectivity:
  - maintain an extra array cc for connected components

```
void connectedComponents (Graph g) {
    int v;
    count = 0;
    ccCount = 0;
    pre = malloc (g->nV *sizeof (int));
    cc  = malloc (g->nV *sizeof (int));
    st  = malloc (g->nV *sizeof (int));

    for (v = 0; v < g->nV; v++) {
        pre[v] = -1;
        st[v] = -1;
        cc[v]  = -1;
    }
    for (v = 0; v < g->V; v++) {
      if (pre[v] == -1) {
         connectedR (g, mkEdge(g,v,v));
         ccCount++;
      }
    }
}
```

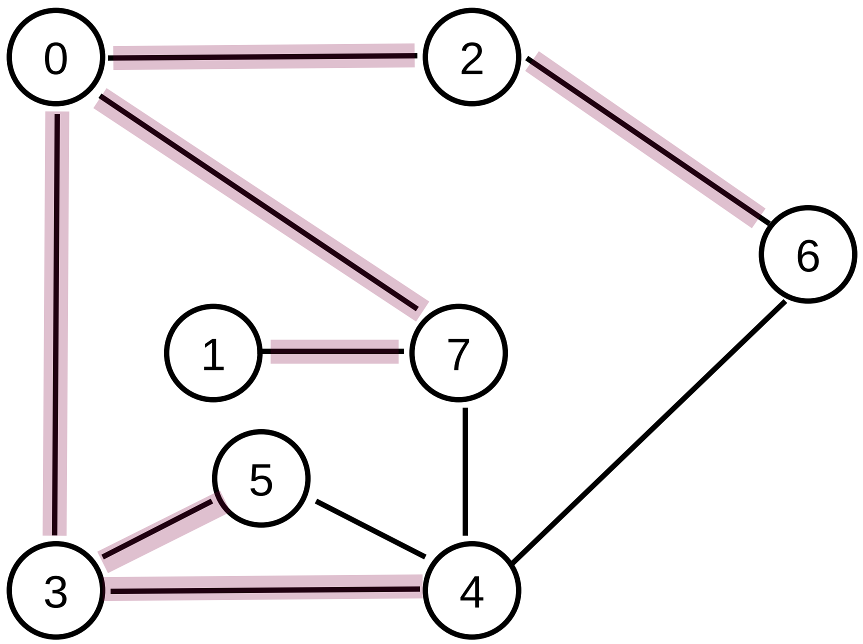
```
void connectedR (Graph g, Edge e) {
    int i, w = e.w;
    pre[w] = count++;
    st[w] = e.v;
    cc[w] = ccCount;

    for (i=0; i < g->V; i++){
        if ((g->edges[currV][i] == 1) &&
            (pre[i] == -1)) {
            dfsR (g, mkEdge(g,w,i));
        }
    }
}
```

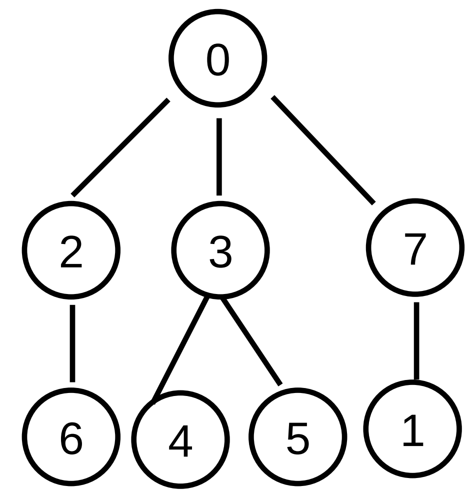
Breadth-First Search

- What if we want the shortest path between two vertices?
  - DFS doesn't help us with this problem
- To find the shortest path between v and any vertex w
  - we visit all the vertices adjacent to v (distance 1)
  - then all the vertices adjacent to those we visited in the first step (distance 2)

```
breadth-first(G, V)
{
  add V to queue
  while (queue not empty) {
    V = remove head of queue
    mark V as visited
    foreach neighbour W of V {
      if (W already visited) continue
      add W to tail of queue
    }
  }
}
```

Breadth-First Search





BREADTH-FIRST SEARCH

- We observed previously that we can simply replace the stack with a queue in the non-recursive implementation to get breadth -first search:

```
void bfs (Graph g, Edge e) {
    int i;
    Queue q = newQueue();
    QueueJoin(q,e);
    while (!QueueIsEmpty(q)) {
        e = QueueLeave(q);
        if(pre[e.w] == -1){
            pre[e.w] = count++;
            st[e.w] = e.v;
            for (i = 0; i < g->nV; i++){
                if ((g->edges[e.w][i] != 0)&&
                             (pre[i] == -1)) {
                    QueueJoin (q,mkEdge(g,e.w,i));
                }
            }
        }
    }
}
```

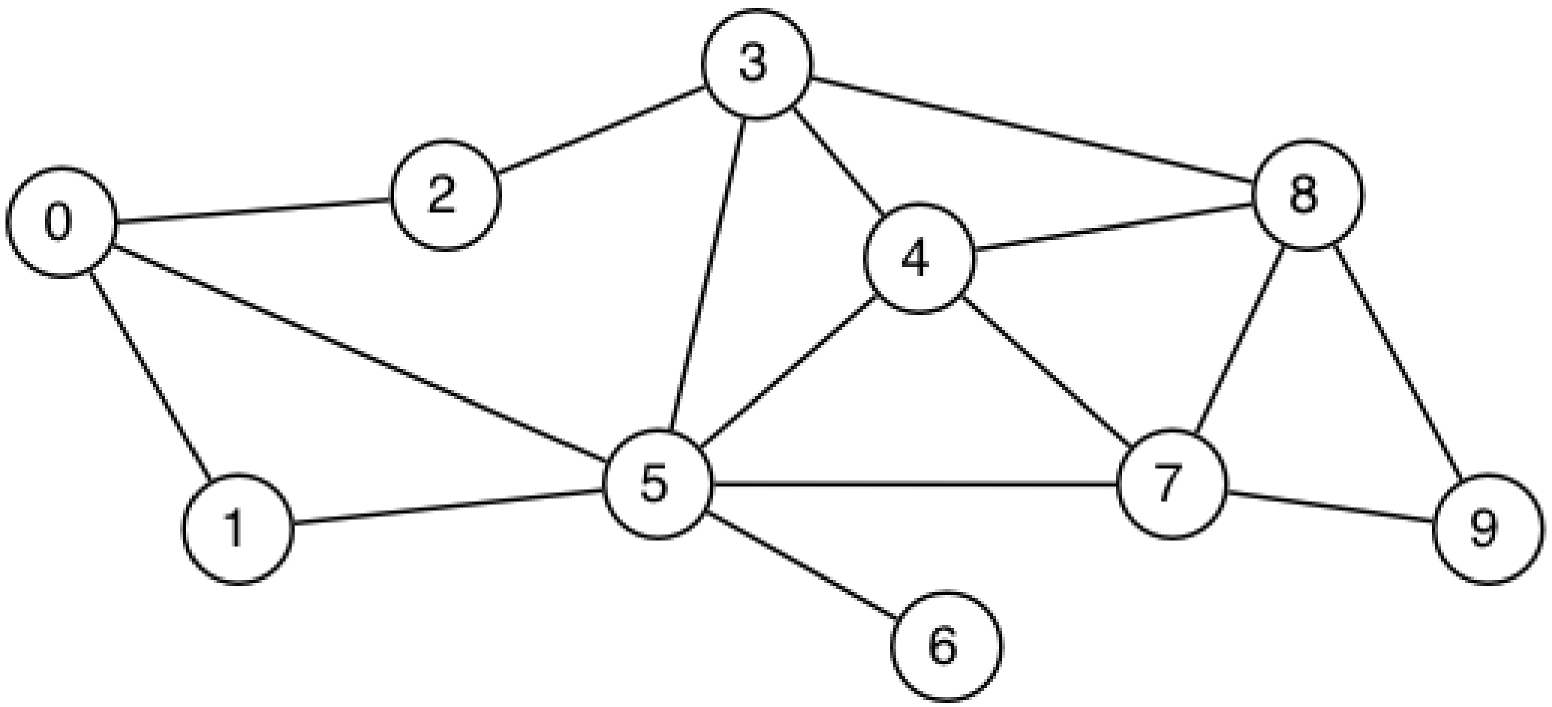
Improved Breadth-First Search

- We can mark them as visited as we put them on the queue since the queue will retain their order. Queue will have at most V entries

```
void bfs (Graph g, Edge e) {
    int i;
    Queue q = newQueue();
    QueueJoin (q,e);
    pre[e.w] = count++;
    st[e.w] = e.v;
    while (!QueueIsEmpty(q)) {
        e = QueueLeave(q);
        for (i = 0; i < g->V; i++)  {
            if ((g->edges[e.w][i] != 0)&&(pre[i] == -1)) {
                QueueJoin (q,mkEdge(g,e.w,i));
                pre[i] = count++;
                st[i] = e.w;
            }
        }
    }
}
```

Exercise: BFS Traversal

- Show the final state of the pre and st arrays after bfs(g,0):




```
breadth-first(G, V)
{
   add V to queue
   while (queue not empty) {
      V = remove head of queue
      mark V as visited
      foreach neighbour W of V {
         if (W already visited) continue
         add W to tail of queue
      }
   }
}
```

Write code to print out the shortest path from 0 to a given vertex v.

Breadth-First Search

- For one BFS: O(V^2) for adjacency matrix and O(V+E) for adjacency list
- We can do BFS for every node as root node, and store the resulting spanning trees in a V x V matrix to store all the shortest paths between any two vertices
- To store and calculate these spanning trees, we need
  - memory proportional to $V * V$
  - time proportional to $V * E$
- Then, we can
  - return path length in constant time
  - path in time proportional to the path length