Five questions, each question 20 marks for a total of 100 marks. Due: on March 12, before 2:00 PM. Note: no extensions can be given because we will go through the solutions in class on that date.

1. **[20 marks]** You're given an array A of n integers, and must answer a series of n queries, each of the form: "How many elements a of the array A satisfy $L_k \leq a \leq R_k$?", where L_k and R_k $(1 \leq k \leq n)$ are some integers such that $L_k \leq R_k$. Design an $O(n \log n)$ algorithm that answers all of these queries.
Hint: preprocess the array to allow for fast search for the largest element not exceeding L_n and a fast search for the smallest element larger or equal to R_k. The question for (L_k, R_k) can be answered using the indices of the two elements found in such a way.

2. **[20 marks, both (a) and (b) 10 marks each]** You are given an array S of n integers and another integer x.

   (a) Describe an $O(n \log n)$ algorithm (in the sense of the worst case performance) that determines whether or not there exist two elements in S whose sum is exactly x.
   Hint: how can you preprocess the array to allow a fast search for an element?

   (b) Describe an algorithm that accomplishes the same task, but runs in $O(n)$ **expected** (i.e., average) time.
   Hint: What kind of data structure allows an almost instantaneous search for an element in the expected time (i.e., on average)?

3. **[20 marks, both (a) and (b) 10 marks each; if you solve (b) you do not have to solve (a)]** You are at a party attended by n people (not including yourself), and you suspect that there might be a celebrity present. A *celebrity* is someone known by everyone, but does not know anyone except themselves. You may assume everyone knows themselves.

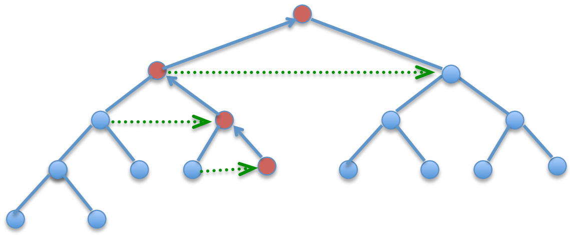
   Your task is to work out if there is a celebrity present, and if so, which of the n people present is a celebrity. To do so, you can ask a person X if they know another person Y (where you choose X and Y when asking the question).

   (a) Show that your task can always be accomplished by asking no more than $3n - 3$ such questions, even in the worst case.

   Hint: Assume you ask A if he knows B. If the answer is yes what can you conclude about A? If the answer is no what can you conclude about B? Note that, as some of you have noticed, 3n-4 questions are enough because at least one answer to one question can be used twice!

   (b) Show that your task can always be accomplished by asking no more than $3n - \lfloor \log_2 n \rfloor - 2$ such questions, even in the worst case.
   How can you chose which pair of people you ask such questions in order to maximally reuse some of the previously obtained answers? You might find useful the following kind of a binary tree where every node has 0 or exactly 2 children and all layers are filled except the last one (only the leaves represent different people). Again, it seems the bound is a bit sloppy and actually $3n - 3 - -\lfloor \log_2 n \rfloor$ questions suffice.

4. **[20 marks, each pair 4 marks]** Read the review material from the class website on asymptotic notation and basic properties of logarithms, pages 38-44 and then determine if $f(n) = \Omega(g(n))$, $f(n) = O(g(n))$ or $f(n) = \Theta(g(n))$ for the following pairs. Justify your



answers.

$f(n)$	$g(n)$
$(\log_2 n)^2$	$\log_2(n^{\log_2 n}) + 2\log_2 n$
n^{100}	$2^{n/100}$
$\sqrt{n}$	$2^{\sqrt{\log_2 n}}$
$n^{1.001}$	$n \log_2 n$
$n^{(1+\sin(\pi n/2))/2}$	$\sqrt{n}$

To compare the growth rate of $f(n)$ and $g(n)$ you might use the L'Hôpital Rule applied to the ratio $f(x)/g(x)$. Note that, if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ then clearly eventually $f(n) < g(n)$ and thus $f(n) = O(g(n))$.

5. **[20 marks, each recurrence 5 marks)** Determine the asymptotic growth rate of the solutions to the following recurrences. If possible, you can use the Master Theorem, if not, find another way of solving it.

   (a) $T(n) = 2T(n/2) + n(2 + \sin n)$
   Hint: $1 \leq 2 + \sin n \leq 3$.

   (b) $T(n) = 2T(n/2) + \sqrt{n} + \log n$
   Hint: which term grows faster, $\sqrt{n}$ *or* $\log n$*?*

   (c) $T(n) = 8T(n/2) + n^{\log n}$
   Hint: use algebra of logs.

   (d) $T(n) = T(n-1) + n$
   Does Master Theorem apply here? If not, just try unwinding the recurrence and see if you can sum up the linear overheads.