Sketches of Solutions to some of the Final Practice Problems

Important Note: In these sketches I did not include simpler things such as the base cases of the recursions and other similar things, focusing only on the harder parts of the solutions; **DO NOT** assume that this is enough for your answers at the exam - **for the exam follow the instructions for the final I had sent you** and which you will be given at the exam as well.

(A) Dynamic Programming

1. Consider a 2-D map with a horizontal river passing through its center. There are n cities on the southern bank with x-coordinates $a_1 \dots a_n$ and n cities on the northern bank with x-coordinates $b_1 \dots b_n$. You want to connect as many north-south pairs of cities as possible, with bridges such that no two bridges cross. When connecting cities, you are only allowed to connect the the i^{th} city on the northern bank to the i^{th} city on the southern bank.

   Solution:

   We sort in an increasing order the cities on the south bank according to their x-coordinates $a_1 \dots a_n$; thus we can now assume that indexing of the cities is such that $a_{i+1} > a_i$. Let the corresponding cities on the north bank will have x-coordinates $b_1 \dots b_n$. Let $i < j$ be arbitrary; now observe that the bridge between a_i and b_i does not cross the bridge between a_j and b_j just in case $b_i < b_j$. Thus, to solve the problem, we just have to find the longest monotonically increasing subsequence of the sequence $b_1 b_2 \dots b_n$, which is a problem solved in class, see the slides on Dynamic Programming.

2. Some people think that the bigger an elephant is, the smarter it is. To disprove this you want to analyze a collection of elephants and place as large a subset of elephants as possible into a sequence whose weights are increasing but their IQs are decreasing. Design an algorithm which given the weights and IQs of n elephants, will find a longest sequence of elephants such that their weights are increasing but IQs are decreasing.

   Solution:

   We again reduce this problem to the problem of finding the longest increasing subsequence, Thus, we sort all the elephants in a decreasing order according to their IQ; we can now assume that indexing of the elephants is chosen so that the IQ(i) of elephant i is lover than the IQ($i-1$) of elephant $i-1$, for all i such that $2 \leq i \leq n$. Let the weights of elephant i be denoted by W(i). Clearly now the problem reduces to finding the longest increasing subsequence of the sequence W(1),. . . ,W(n).

3. You have an amount of money M and you are in a candy store. There are n kinds of candies and for each candy you know how much pleasure you get by eating it, which

is a number between 1 and 100, as well as the price of each candy. Your task is to chose which candies you are going to buy to maximise the total pleasure you will get by gobbling them all.

Solution: All it takes is to notice that this is the classical knapsack problem, with duplicates allowed: the capacity of the knapsack is the amount of money M you got, the size of each item (candy) is its price and the value of each item is its pleasure score.

4. Your shipping company has just received N shipping requests (jobs). For each request i, you know it will require t_i trucks to complete, paying you d_i dollars. You have T trucks in total. Devise an algorithm to select jobs which will bring you the largest possible amount of money.

   Solution This is again the knapsack problem with the number of tracks T as the capacity of the knapsack

5. Again your shipping company has just received N shipping requests (jobs). This time, for each request i, you know it will require e_i employees and t_i trucks to complete, paying you d_i dollars. You have E employees and T trucks in total. Devise an algorithm to select jobs which will bring you the largest possible amount of money.

   Solution This is a version of the knapsack; now your knapsack has two constraint on its capacity namely the total number T of trucks available and the total number of employees E (you can think of it as a real knapsack having a limited volume capacity and a limited weight capacity). The problem is three dimensional because we cannot take multiple copies of the same job. Thus we fill a three dimensional array of size $T \times S \times N$ containing the solutions to the following subproblems $P(t, e, i)$ for all $1 \leq t \leq T$, all $1 \leq e \leq E$ and all $1 \leq i \leq N$:
   "Select among the first i jobs those that can be done with at most t trucks in total and at most e employees in total and which bring you the largest possible amount of money".
   Recursion: $opt(t, e, i) = \max\left(opt(t, e, i-1), opt(t - t_i, e - e_i, i-1) + d_i\right)$.
   Note that the first argument to the max function corresponds to the case when job i is not chosen and the second when you take job i and are left with $t - t_i$ trucks and $e - e_i$ employees for the rest of the jobs to be taken.

6. You are given a set of n types of rectangular boxes, where the i^{th} box has height h_i, width w_i and depth d_i. You want to create a stack of boxes which is as tall as possible, but you can only stack a box on top of another box if the dimensions of the 2-D base of the lower box are each strictly larger than those of the 2-D base of the higher box. Of course, you can rotate a box so that any side functions as its base. It

is also allowable to use multiple instances of the same type of box.

Solution:

Let us take 6 instances of each box, and rotate them in all possible ways to obtain the 6 possible bases. Note that each box has (in general) three different rectangular faces and each rectangular face can be rotated in two possible ways, depending which of the two different edges of its base is "horizontal". For simplicity, after that, we no longer allow boxes to be rotated. Note that now we have in total $6n$ many boxes.

We now note that if a box B_2 (with a fixed orientation) can be placed on top of a box B_1 then, since both sides of the base of B_2 must be smaller than the corresponding sides of the base of B_1, the surface area of the base of B_2 must be strictly smaller than the surface area of the base of B_1.

• **Producing the right ordering for recursion:** We order all boxes in a decreasing order of the surface area of their bases (putting boxes with the same surface area of their bases in an arbitrary order); then in every possible legitimate stack of boxes if a box $B_{i_{k+1}}$ is placed on top of the box B_{i_k}, then $i_{k+1} > i_k$. From now on we will assume that all the boxes are ordered in such a way.

• **Subproblem(k):** "Find a stack built from boxes $B_1, B_2, \ldots B_k$, $1 \leq k \leq 6n$, *which must end with box B_k* and which is of the largest possible total height".

• **Recursion:** Let us denote by MaxHeight(k) the height of the stack which solves Subproblem(k). Let also length(m), width(m) and height(m) denote the length, width and height of box B_m, respectively. Then

$$\text{MaxHeight}(k) = \max_{1 \leq m \leq k-1} \{\text{MaxHeight}(m) + \text{height}(k) \ : \ \text{length}(k) < \text{length}(m) \ \&\& \ \text{width}(k) < \text{width}(m)\}$$

We also define PreviousBox(k) to be the argument m for which the above maximum is achieved.

• **Solution of the original problem:** After all of these subproblems are solved for all $1 \leq k \leq n$, we pick the largest of MaxHeight(k) to obtain the height of the tallest stack of boxes and can reconstruct such a stack using pointers PreviousBox(k).

7. Given a sequence of n real numbers $A_1 \ldots A_n$, determine *in linear time* a contiguous subsequence $A_i \ldots A_j$ for which the sum of elements in the subsequence is maximised.

Solution:

• **Subproblem(k):** "find $m \leq k$ such that suffix $A_m A_{m+1} \ldots A_k$ of the subsequence $A_1 A_2 \ldots A_k$ has a maximal possible sum."

Note that such a suffix might consist of a single element A_k.

• **Recursion:** let OptSuffix(k) denote $m \leq k$ such that suffix $A_m A_{m+1} \dots A_k$ of the subsequence $A_1 A_2 \dots A_k$ has the largest possible sum and let this sum be denoted by OptSum(k). Then OptSuffix(1) = 1 and OptSum(1) = A_1 and for $k > 1$

$$\text{OptSuffix}(k) = \begin{cases} \text{OptSuffix}(k-1) & \text{if } \text{OptSum}(k-1) > 0; \\ k & \text{otherwise} \end{cases}$$

$$\text{OptSum}(k) = \begin{cases} \text{OptSum}(k-1) + A_k & \text{if } \text{OptSum}(k-1) > 0; \\ A_k & \text{otherwise} \end{cases}$$

Thus, if the sum OptSum($k-1$) of the optimal suffix $A_m A_{m+1} \dots A_{k-1}$ (where $m = \text{OptSuffix}(k-1)$) for the subsequence $A_1 A_2 \dots A_{k-1}$ is positive, then we extend such a suffix with A_k to obtain optimal suffix $A_m A_{m+1} \dots A_{k-1} A_k$ for the subsequence $A_1 A_2 \dots A_k$ and consequently OptSuffix(k) is still equal to m and OptSum(k) = OptSum($k-1$) + A_k; otherwise, if OptSum($k-1$) is negative we discard such optimal suffix and start a new one consisting of a single number A_k; thus, in this case OptSuffix(k) = k and OptSum(k) = A_k.

• **Solution of the original problem:** After we obtain OptSum(k) and OptSuffix(k) for all $1 \leq k \leq n$ we find $1 \leq q \leq n$ such that the corresponding OptSum(q) is the largest among all OptSum(k) for $1 \leq k \leq n$; let the corresponding OptSuffix(q) be equal to p; then the solution for the initial problem is the subsequence $A_p \dots A_q$.

8. You are traveling by a canoe down a river and there are n trading posts along the way. Before starting your journey, you are given for each $1 \leq i < j \leq n$ the fee $F(i, j)$ for renting a canoe from post i to post j. These fees are arbitrary. For example it is possible that $F(1, 3) = 10$ and $F(1, 4) = 5$. You begin at trading post 1 and must end at trading post n (using rented canoes). Your goal is to design an efficient algorithms which produces the sequence of trading posts where you change your canoe which minimizes the total rental cost.

Solution:

• **Subproblem(k):** "Find the sequence of trading posts which provides the cheapest way of getting from post 1 to post k".

• **Recursion:** Let OptCost(k) denote the minimal possible cost of getting from post 1 to post k, and let OptSeq(k) denote post m, ($m \leq k$) which is the last intermediate

post in the corresponding optimal sequence $i_1, i_2, \ldots, i_{p-1}, i_p$ i.e., such that $i_1 = 1$, $i_{p-1} = m$ and $i_p = k$. Then OptCost(1) = 0 and for $k > 1$

$$\text{OptCost}(k) = \min_{1 \le m \le k-1} \{\text{OptCost}(m) + F(m, k)\}$$

and let OptSeq(k) be equal to m for which such a minimum is achieved (if there are several such m, pick one arbitrarily, say the smallest one), usually denoted by

$$\text{OptSeq}(k) = \arg\left(\min_{1 \le m \le k-1} \{\text{OptCost}(m) + F(m, k)\}\right)$$

• **Solution of the original problem:** After all of these subproblems are solved for all $1 \le k \le n$, we can backtrack from n to 1 using OptSeq(n) = p, OptSeq(p) = $q, \ldots$ to obtain a solution of the problem with an overall minimal cost equal to OptCost(n).

9. You are given an $n \times n$ chessboard with an integer $n(i, j)$ in each square (i, j) of its n^2 squares. You start from the top left corner of the board; at each move you can go either to the square immediately below or to the square immediately to the right of the square you are at the moment; you can never move diagonally. The goal is to reach the right bottom corner so that the sum of integers at all squares visited is minimal.

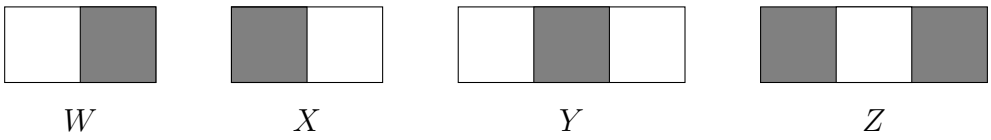
   (a) Describe an algorithm which always correctly finds a minimal sum path and runs in time $O(n^2)$.

   (b) Describe an algorithm which computes the number of such minimal paths.

Solution:

(a) We solve the following subproblems $P(i, j)$ for all $1 \le i \le n$:
"Find the minimal possible sum along a path that ends at cell (i, j) and if such a path came to cell (i, j) from cell $(i-1, j)$ or cell $(i, j-1)$"
Note we are at this moment not looking for such a path; we will reconstruct it later.
Let $opt(i, j)$ be the minimal possible sum of a path ending at (i, j) and $pred(i, j) = (i-1, j)$ if there exists such a path arriving from cell $(i-1, j)$ and $pred(i, j) = (i, j-1)$ otherwise.
Recursion: $opt(i, j) = \min(opt(i-1, j), opt(i, j-1)) + n(i, j)$;
$pred(i, j) = (i-1, j)$ if $opt(i-1, j) \le opt(i, j-)$ else $pred(i, j) = (i, j-1)$
We can now reconstruct the path by bactracking using $pred$ function and starting from cell (n, n).
(b) Having found $opt(i, j)$ and $pred(i, j)$ in (a) we now define by recursion the function $num(i, j)$ as
$num(i, j) = num(pred(i, j))$ if $opt(i-1, j) \ne opt(i, j-1)$ else $num(i, j) = num(i-1, j) + num(i, j-1)$.

10. You are given an array $A[1..n]$ of n integers. You would like to cover the array with tiles. You have an **unlimited number** of each of the following tiles available to you:



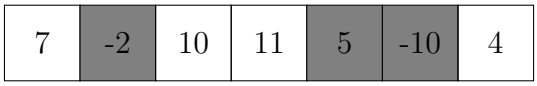
W X Y Z

Tiles W and X can be used to cover two adjacent elements of the array, and tiles Y and Z can be used to cover three consecutive (adjacent) elements of the array. Tiles may not overlap, hang off the end of the array, and each element of the array **must be completely covered by precisely one tile**.

The white parts of each tile are see-through, whereas the grey parts hide the array element beneath it.

Your *score* for a tiling is the sum of the numbers that can be seen (i.e. those that are covered by white parts of tiles). Design an $O(n)$ algorithm that determines the **maximum** possible score you can obtain.

For example, if $n = 7$ and $A = [7, -2, 10, 11, 5, -10, 4]$, then tiling with tile Y followed by tile tile W then tile X:



achieves a score of $7 + 10 + 11 + 4 = 32$, which is the maximum possible in this case.

Hint: This is a straightforward dynamic programming, by solving recursively subproblems for subarrays $A[1..i]$ for all $2 \leq i \leq n$, taking the max over the 4 possible choices for the ending tile.

11. You have to cut a wood stick into several pieces at the marks on the stick. The most affordable company, Analog Cutting Machinery (ACM), charges money according to the length of the stick being cut. Their cutting saw allows them to make only one cut at a time. It is easy to see that different cutting orders can lead to different prices. For example, consider a stick of length 10 m that has to be cut at 2, 4, and 7 m from one end. There are several choices. One can cut first at 2, then at 4, then at 7. This leads to a price of $10 + 8 + 6 = 24$ because the first stick was of 10 m, the resulting stick of 8 m, and the last one of 6 m. Another choice could cut at 4, then at 2, then at 7. This would lead to a price of $10 + 4 + 6 = 20$, which is better for us. Your boss demands that you design an algorithm to find the minimum possible cutting cost for

any given stick.

Solution:

Let us index left edge of the stick by 0, then consecutive marks where the stick is to be cut by 1 to n and finally the right edge by $n+1$. For every i such that $0 \leq i \leq n$ and for every j such that $i < j \leq n+1$ we consider the following subproblems:

• **Subproblem(i,j):** "Find the minimum total cost of cutting into pieces a stick with the left end at point i and the right end at point j."

• **Ordering of Subproblems:** Even though the subproblems are indexed with two variables it is enough to order subproblems by their corresponding values of $j - i$. Thus, we first solve subproblems with smaller values of $j - i$, solving the problems with the same value of $j - i$ in an arbitrary order.

Let also $l(i, j)$ denote the length of the part of the stick between marks i and j; we can then take that the cost of making a single cut on the piece of the stick with ends at marks i and j is equal to $l(i, j)$.

• **Recursion:** Let MinCost(i, j) be the minimal cost solution for Subproblem(i, j); then

$$\text{MinCost}(i,j) = \begin{cases} \min_{i<k<j}\{\text{MinCost}(i,k) + \text{MinCost}(k,j) + l(i,j)\} & \text{if} \quad j \geq i+2 \\ 0 & \text{otherwise} \end{cases}$$

Clearly, the above algorithm runs in time $O(|V|)$, because during the recursion steps each entry MinCost(w_i) is computed only once and looked at only once (when computing MinCost(v) for its parent v).

12. You are given a rooted tree. Each edge of the tree has a cost for removing it. Devise an algorithm to compute the minimum total cost of removing edges to disconnect the root from all the leaves of the tree.

Solution 1: Make all the edges directed pointing towards the leaves, and then treat this tree as a flow network with the root of the tree as the source and with the leaves as sinks and with the capacity of each edge equal to the cost of removing that edge. Add a super-sink and connect it with all leaves with edges with infinite capacity. Now run your favorite max flow algorithm until it converged. Take the minimal cut defined as all the vertices in the last residual flow network which are accessible from the source. The edges to be removed are now the edges crossing the minimal cut.

Note that the fastest max flow algorithm to date runs in time $O(|V|^3)$. We now present a faster dynamic programming solution to the above problem.

Solution 2:

Let v be any vertex of the given tree T and let T_v denote the subtree of T with root at v. Let also the cost of removing an edge (u, v) be denoted by $\text{cost}(u, v)$.

• **Subproblem(v):** "Find the minimum total cost of removing some of the edges of T_v to disconnect the root v from all the leaves of T_v."

• **Ordering of Subproblems:** Subproblem(v) precedes Subproblem(u) if v belongs to subtree T_u.

• **Recursion:** Let MinCost(v) be the minimal cost solution for Subproblem(v) and the children of v be $w_1, w_2, \ldots, w_k$; then

$$\text{MinCost}(v) = \sum_{i=1}^{k} \min\{\text{cost}(v, w_i), \text{MinCost}(w_i)\}$$

Clearly, the above algorithm runs in time $O(|V|)$, because during the recursion steps each entry MinCost(w_i) is computed only once and looked at only once (when computing MinCost(v) for its parent v).

13. A company is organizing a party for its employees. The organizers of the party want it to be a fun party, and so have assigned a fun rating to every employee. The employees are organized into a strict hierarchy, i.e. a tree rooted at the president. There is one restriction, though, on the guest list to the party: an employee and their immediate supervisor (parent in the tree) cannot both attend the party (because that would be no fun at all). Give an algorithm that makes a guest list for the party that maximizes the sum of the fun ratings of the guests.

Solution: Let the fun rating of an employee v be denoted by $\rho(v)$ We again do recursion on rooted subtrees, solving the following two subproblems by a simultaneous recursion:

• **Subproblem(v):** "Find the maximal rating score INCLUDE(v) of guests belonging to the subtree T_v rooted at a node v which must include v and the maximal rating score EXCLUDE(v) of guests belonging to the subtree T_v rooted at a node v which must exclude v ."

• **Recursion:**

$$\text{INCLUDE}(v) = \rho(v) + \sum_{w} \{\text{EXCLUDE}(w) \ : \ w \text{ is a child of } \ v\}$$

$$\text{EXCLUDE}(v) = \sum_{w} \{\max\big(\text{INCLUDE}(w), \text{EXCLUDE}(w)\big) \ : \ w \text{ is a child of } \ v\}$$

14. You are given a boolean expression consisting of a string of the symbols *true* and *false* and with exactly one operation *and, or, xor* between any two consecutive truth

values. Count the number of ways to place brackets in the expression such that it will evaluate to true. For example, there is only 1 way to place parentheses in the expression *true* and *false* xor *true* such that it evaluates to true.

Solution:

Let us enumerate all *true* and *false* symbols by indices $1 \dots n$.

• **Subproblem(i,j):** "For every substring $S(i,j)$ of the symbols $\sigma_k \in \{true, false\}$, $(i \leq k \leq j)$, between the i^{th} and j^{th} symbol inclusively compute the number $T(i,j)$ of ways to place brackets in the corresponding expression such that it will evaluate to true **and** the number $F(i,j)$ of ways to place brackets in the corresponding expression such that it will evaluate to false."

• **Ordering of subproblems:** Subproblems are solved in the order of increasing values of $j-i$, breaking evens arbitrarily.

• **Recursion:** Let us denote the k^{th} operand by $\otimes_k$; thus $\otimes_k \in \{and, or, xor\}$. If $j-i \leq 2$ then the truth value of the substring $S(i,j)$ is uniquely determined; for $j-i \geq 3$ we have

$$T(i,j) = \sum_{i<k<j} \begin{cases} T(i,k)T(k+1,j) & \text{if } \otimes_k = and \\ T(i,k)T(k+1,j) + T(i,k)F(k+1,j) + F(i,k)T(k+1,j) & \text{if } \otimes_k = or \\ T(i,k)F(k+1,j) + F(i,k)T(k+1,j) & \text{if } \otimes_k = xor \end{cases}$$

$$F(i,j) = \sum_{i<k<j} \begin{cases} T(i,k)F(k+1,j) + F(i,k)T(k+1,j) + F(i,k)F(k+1,j) & \text{if } \otimes_k = and \\ F(i,k)F(k+1,j) & \text{if } \otimes_k = or \\ T(i,k)T(k+1,j) + F(i,k)F(k+1,j) & \text{if } \otimes_k = xor \end{cases}$$

The solution to our problem is given $T(1,n)$.

15. You are given a sequence of numbers with operations $+, -, \times$ in between, for example

$$1 + 2 - 3 \times 6 - 1 - 2 \times 3 - 5 \times 7 + 2 - 8 \times 9$$

Your task is to place brackets in a way that the resulting expression has the largest possible value.

Solution:

Similar to the previous one; this time for every substring consisting of the numbers between the i^{th} and the j^{th} number (inclusively) we compute the minimal value

$\mathrm{MIN}(i,j)$ of the corresponding subexpression as well as the maximal such value $\mathrm{MAX}(i,j)$.

• **Recursion:** Let us denote the k^{th} operand by $\otimes_k$; thus $\otimes_k \in \{+,-,\times\}$. If $j-i \leq 2$ then the value of the substring $S(i,j)$ is uniquely determined; for $j-i \geq 3$ we have

$$\mathrm{MIN}(i,j) = \begin{cases} \mathrm{MIN}(i,k) + \mathrm{MIN}(k+1,j) & \text{if } \otimes_k = + \\ \mathrm{MIN}(i,k) - \mathrm{MAX}(k+1,j) & \text{if } \otimes_k = - \\ \min \begin{cases} \mathrm{MIN}(i,k) \times \mathrm{MIN}(k+1,j), \\ \mathrm{MIN}(i,k) \times \mathrm{MAX}(k+1,j) \\ \mathrm{MAX}(i,k) \times \mathrm{MIN}(k+1,j) \\ \mathrm{MAX}(i,k) \times \mathrm{MAX}(k+1,j) \end{cases} & \text{if } \otimes_k = \times \end{cases}$$

$$\mathrm{MAX}(i,j) = \begin{cases} \mathrm{MAX}(i,k) + \mathrm{MAX}(k+1,j) & \text{if } \otimes_k = + \\ \mathrm{MAX}(i,k) - \mathrm{MIN}(k+1,j) & \text{if } \otimes_k = - \\ \max \begin{cases} \mathrm{MIN}(i,k) \times \mathrm{MIN}(k+1,j), \\ \mathrm{MIN}(i,k) \times \mathrm{MAX}(k+1,j) \\ \mathrm{MAX}(i,k) \times \mathrm{MIN}(k+1,j) \\ \mathrm{MAX}(i,k) \times \mathrm{MAX}(k+1,j) \end{cases} & \text{if } \otimes_k = \times \end{cases}$$

Note that in case $\otimes_k = \times$ the computation of the values of $\mathrm{MIN}(i,j)$ and $\mathrm{MAX}(i,j)$ depend on the signs of the values of $\mathrm{MIN}(i,k)$, $\mathrm{MAX}(i,k)$, $\mathrm{MIN}(k+1,j)$, $\mathrm{MAX}(k+1,j)$; for example if $\mathrm{MIN}(i,k)$ and $\mathrm{MIN}(k+1,j)$ are both negative numbers whose absolute values are larger than $\mathrm{MAX}(i,k)$ and $\mathrm{MAX}(k+1,j)$ respectively, then $\mathrm{MAX}(i,j)$ is equal to the product of these two negative minima (rather than the products of the corresponding two maxima).

16. You have n_1 items of size s_1, n_2 items of size s_2, and n_3 items of size s_3. You would like to pack all of these items into bins, each of capacity C, using as few bins as possible.

Solution:

For every $i \leq n_1$, $j \leq n_2$ and $k \leq n_3$ we solve the following subproblems:

• **Subproblem(i,j,k):** "Pack i items of size s_1, j items of size s_2, and k items of size s_3 into the smallest possible number of bins, each of capacity C."

Denote such minimal number by $\mathrm{OPT}(i,j,k)$.

• **Recursion:**

$$\mathrm{OPT}(i,j,k) = 1 + \min\{\mathrm{OPT}(i-\alpha, j-\beta, k-\gamma) \ : \ \alpha s_1 + \beta s_2 + \gamma s_3 \leq C \ \&$$
$$\& \ (\alpha+1)s_1 + \beta s_2 + \gamma s_3 > C \ \& \ \alpha s_1 + (\beta+1)s_2 + \gamma s_3 > C \ \& \ \alpha s_1 + \beta s_2 + (\gamma+1)s_3 > C\}$$

Thus, we fill in all possible ways a single box up to its capacity C and then look for the optimal solutions for the remaining items. Note that "filling a box up to capacity" conditions are not necessary; $\alpha s_1 + \beta s_2 + \gamma s_3 \leq C$ suffices but there many more cases to inspect. Think how you would code this recursion efficiently with two nested loops.

17. For three bit strings $X = x_1 \dots x_m$,$Y = y_1 \dots y_n$ and $Z = z_1 \dots z_{m+n}$ we say that Z is an interleaving of X and Y if Z can be obtained by interleaving the bits in X and Y in a way that maintains the left-to-right order of the bits in X and Y. For example if $X = 101$ and $Y = 01$ then $x_1x_2y_1x_3y_2 = 10011$ is an interleaving of X and Y, whereas 11010 is not. Give an efficient algorithm to determine if Z is an interleaving of X and Y.

Solution Subproblems $P(i, j)$, for all $0 \leq i \leq m,\ 0 \leq j \leq n$: "Determine if the substrings $X[1..i]$ and $Y[1..j]$ can be interleaved to produce substring $Z[1..i + j]$."

Let $I(i, j) = True$ if the answer to $P(i, j)$ is true and $False$ otherwise; then

$$I(i,j) = \begin{cases} True & \text{if } Z[i+j] = X[i] \ \&I(i-1,j) = True \\ True & \text{if } Z[i+j] = Y[i] \ \&I(i,j-1) = True \\ False & \text{otherwise} \end{cases}$$

18. Consider a row of n coins of values $v_1 \dots v_n$, where n is even. We play a game against an opponent by alternating turns. In each turn, a player selects either the first or last coin from the row, removes it from the row permanently, and receives the value of the coin. Determine the maximum possible amount of money we can definitely win if we move first.

Solution:

Let v_i denote the value of the coin c_i. For every i and j such that $i < j$ and such that $j - (i - 1)$ is even we consider the following subproblems:

• **Subproblem(i,j):** "For the subsequence of coins $c_i, c_{i+1}, \dots c_j$ determine the maximum possible amount of money $\mathrm{OPT}(i, j)$ we can definitely win if we move first."

• **Recursion:**

$$\mathrm{OPT}(i,j) = \max \begin{cases} v_i + \min\{\mathrm{OPT}(i+1, j-1), \mathrm{OPT}(i+2, j)\} \\ v_j + \min\{\mathrm{OPT}(i+1, j-1), \mathrm{OPT}(i, j-2)\} \end{cases}$$

In the first case we remove coin c_i and our opponent can then remove either coin c_j or coin c_{i+1} and we have to assume that he will leave us with the worse of the two possible outcomes and thus we take the min of the two options. In the second case we remove the last coin c_j and our opponent can then remove either coin c_i or coin c_{j-1}.

19. Find the number of partitions of n, i.e. the number of sets of integers $\{p_1, p_2, \ldots, p_k\}$ such that such that $p_1 + p_2 + \ldots + p_k = n$. *(Hint: try a relaxation with respect to the size of the largest number in such a sum.)*

    Solution:

    For every i and j such that $1 \leq j \leq i \leq n$ we solve the following subproblems:

    • **Subproblem(i,j):** "Find the number $N(i, j)$ of partitions $\{p_1, p_2, \ldots, p_k\}$ of i such that $p_m \leq j$ for all m such that $1 \leq m \leq k$."

    • **Recursion:** $N(i, j) = N(i, j-1) + N(i-j, j)$.

    Thus, the number of partitions of integer i where each part does not exceed j is equal to the number of partitions of i where each part does not exceed $j-1$ plus the number of partitions which have at least one element equal to j. But, taking out one element of size j, the number of partitions which have at least one element equal to j is equal to the number of partitions of integer $i-j$ (to account for the number j we took out) where each element of such partitions does not exceed j (because there might be more than one element of size j).

20. Devise a dynamic programming algorithm that counts the number of non-decreasing sequences of integers of length N, such that the numbers are between 0 and M inclusive.

    Solution: let $N(i, j)$ be the number of non-decreasing sequences of length i, each smaller or equal j. then
    $N(i, j) = N(i, j-1) + N(i-1, j)$
    Here $N(i, j-1)$ is the number of non-decreasing sequences of length i, such that the last element is smaller than j, and $N(i-1, j)$ is actually equal to the number of non-decreasing sequences of length i, such that the last element is j, because every sequence such that the last element is equal j can be obtained from a sequence of length $i-1$ where each element is at most j by concatenating at its end j.

21. A palindrome is a sequence of at least three letters which reads equally from left to right and from right to left.

    (a) Given a sequence of letters S, find efficiently its longest subsequence (not necessarily contiguous) which is a palindrome. Thus, we are looking for a longest palindrome which can be obtained by crossing out some of the letters of the initial sequence without permuting the remaining letters.

    (b) Find the total number of occurrences of all subsequences of S which are palindromes of any length ≥ 2.

Solution: This is a straightforward dynamic programming we did in class, by considering subsequences $S[i..j]$ and distinguishing in recursion two cases: $S[i] = S[j]$ and $S[i] \neq S[j]$.

22. You are given an ordered sequence of n cities, and the distances between every pair of cities. You must partition the cities into two subsequences (not necessarily contiguous) such that person A visits all cities in the first subsequence (in order), person B visits all cities in the second subsequence (in order), and such that the sum of the total distances travelled by A and B is minimized.

    Solution: You have a very similar problem, called *Minimizing Total Variation of a Sequence* solved in the notes available at the website, solved by an identical technique with the only difference that the distances between cities are replaced by $|x_i - x_j|$.

23. You have been handed responsibility for a business in Texas for the next N days. At the beginning of each day, you may hire an illegal worker, keep the number of illegal workers the same or fire an illegal worker. At the end of each day, there will be an inspection. The inspector on the i^{th} day will check that you have between l_i and r_i illegal workers (inclusive). If you do not, you will fail the inspection. Design an algorithm that determines the fewest number of inspections you will fail if you hire and fire illegal employees optimally.

    Solution: Solve recursively the subproblems $P(i, j)$ for all $1 \leq i \leq N$ and all $0 \leq j \leq i$: "Determine how to hire and fire illegal workers for the first i days, such that the number of illegal workers on day i is equal to j and the number of failed inspections is as small as possible".

24. You have invented a new version of hopscotch! On the ground you have drawn a line of $n + 1$ evenly spaced hopscotch squares indexed from 0 to n, and have written an integer (positive or negative) in each square. These squares can be represented by an array $A[0..n]$ where $A[i]$ is the number written in the square with index i.

    You start in square 0 and will hop until you reach square n. Each time, you will jump from the current square to a square with a strictly greater *index*. Specifically, if you are at square with index i then you will jump to a square with index $i + k$ for some $k > 0$. Since hopping is tiring, the length of each jump cannot exceed the length of the previous jump. Your first jump may be of any length.

    Your score from such a jump sequence is the sum of the scores in all squares you have hopped onto. Design an efficient algorithm that determines the maximum score you can earn.

    Solution: Solve recursively subproblems $P(i, j)$ for all $1 \leq i \leq N$ and all $0 \leq j \leq i$: "Determine how to jump, ending at square i, such that the last jump was of size j and the score achieved is maximal".

(B) Max Flow

25. Assume that you are the administrator of a network of computers; each computer is connected by unidirectional fiberoptic cables to a few other computers on the same network (so the network can be modelled by a directed graph). You noticed that computers $P_1, P_2, ...P_n$ are mounting an attack on computers $Q_1, Q_2, ..., Q_m$. The total number of computers on the network is $N > m + n$. Since it is a real emergency, you must disconnect some of the optical cables of the network so that none of computers $P_1, P_2, ...P_n$ can send packets to any of $Q_1, Q_2, ..., Q_m$. Since you must send crews to disconnect some of the fiberoptic cables, for each cable c_{ij} for traffic from a computer X_i to a computer X_j there is an associated cost c_{ij} for disconnecting it. Your task is to design an algorithm for determining which cables to disconnect to isolate computers $Q_1, Q_2, ..., Q_m$ from all of the computers $P_1, ..., P_n$ so that the total cost incurred is minimal.

    Solution: Set the capacities of edges (cables) equal to the cost of disconnecting them. Connect a super-source with all of the attacking computer and all victim computers with a super-sink with links of infinite capacity. Find Max Flow and the associated Min Cut (using the method explained on the slides). The links crossing the cut should be disconnected.

26. You are running a dating agency and have m guys and f girls as customers. Each guy and each girl have reviewed all profiles of the candidates of opposite sex and have sent you their corresponding lists of people whose profiles they liked. Your task is to organise a largest possible number of first dates so that everyone meets at most one person of opposite sex and so that both parties liked each others profile (i.e., have included the other party on their list of people whose profile they liked).

    Solution This is a typical Max Flow/Min Cut type of a problem. Construct a bipartite graph with boys on one side and girls on the other side, connecting a boy with a girl by a directed edge of capacity 1 just in case they both liked each others profiles. Add a source and connect it with all the boys with directed edges of capacity 1; connect all the girls with a sink yo also add, with edges of capacity 1. Find max flow in such a network using a feasible algorithm which guarantees all flows are integer, such as Edmonds - Karp version of the Ford - Fulkerson algorithm and find all the edges between boys and girls which ave a unit flow in them to form a maximal number of pairs.

    Note: The sam methods applies to the problems below.

27. You work for a new private university which wants to keep the sizes of classes small. Each class is assigned its maximal capacity - the largest number of students which can enrol in it. Students pay the same tuition fee for each class they get enrolled in. Students can apply to be enrolled in as many classes as they wish, but each of them will eventually be enrolled to at most 5 classes at any given semester. You are

given the wish lists of all students, containing for each student the list of all classes they would like to enrol this particular semester and you have to chose from the classes they have put on their wish lists in which classes you will enrol them, without exceeding the maximal enrolment of any of the classes and without enrolling any student into more than 5 classes. Your goal is, surprisingly, to maximise the income from the tuition fees for your university. Design an efficient algorithm for such a task.

28. Assume each student can borrow at most 10 books from the library, and the library has three copies of each title in its inventory. Each student submits a list of books he wishes to borrow. You have to assign books to students, so that a maximal number of volumes is checked out.

29. The emergency services are responding to a major earthquake that has hit a wide region, and left n people injured who need to be sent to a hospital. Let P be the set of n people and H be the set of k hospitals. Several hospitals are available to treat these people, but there are some constraints:

    (a) Each injured person needs to be sent to a hospital no further than one hour drive away. Let H_p be the set of hospitals that are within range for person p.

    (b) Each hospital h has a capacity c_h, the maximum number of people that the hospital can receive.

    Design an efficient algorithm that determines whether it is possible to assign each person to a hospital in a way that satisfies these constraints, and returns such an assignment if so.

30. There is a partially hidden image containing n rows and n columns of pixels. Each of the n^2 pixels is either black or white. You know the colour of $m \leq n^2$ of these pixels, but the colour of the remaining pixels remains a mystery.

    Additionally, you know the number of black pixels in each row, and also the number of black pixels in each column.

    Design an algorithm that produces any possible image satisfying the information you know. If there are multiple possible such images, your algorithm may produce any.