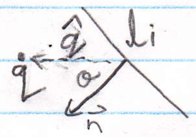
1) a) The dir from $\bar{p}_i$ to $\bar{p}_{i+1}$: $\bar{p}_{i+1} - \bar{p}_i = (x_{i+1} - x_i, y_{i+1} - y_i)$

The inward facing normal's direction is a $90°$ CCW rotation of this direction

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_{i+1} - x_i \\ y_{i+1} - y_i \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{i+1} - x_i \\ y_{i+1} - y_i \end{bmatrix} = \begin{bmatrix} y_i - y_{i+1} \\ x_{i+1} - x_i \end{bmatrix}$$

The midpoint of $\overline{p_i p_{i+1}} = \left(\frac{x_{i+1} + x_i}{2}, \frac{y_{i+1} + y_i}{2}\right)$

so the normal is: $\left(\frac{x_{i+1} + x_i}{2}, \frac{y_{i+1} + y_i}{2}\right) + t\left(\frac{y_i - y_{i+1}}{\sqrt{(y_i - y_{i+1})^2 + (x_{i+1} - x_i)^2}}, \frac{x_{i+1} - x_i}{\sqrt{(y_i - y_{i+1})^2 + (x_{i+1} - x_i)^2}}\right), \quad t \geq 0$

b) 

Take the dot product of the unit normal, $\hat{n}$, and a unit vector with direction from the midpoint of $\overline{p_i p_{i+1}}$ to $\bar{q}$ as shown in the drawing, $\hat{q}$. $\hat{n} \cdot \hat{q} = \cos\theta$.
If $\theta < 90°$, then q is on the same side of l_i as the normal. If $\theta > 90°$, it is not. If $\theta = 90°$ q is on l_i.

c) Starting with $\overline{p_0 p_1}$ continuing with $\overline{p_1 p_2}$, ..., to $\overline{p_{n-1} p_n}$ and finishing with $\overline{p_0 p_n}$, find the inward facing normal at the midpoint of each line as shown in a) and perform the test in b) using this normal and the point. If the point is on the same side of the line as the inward facing normal of each line, the point is inside p, otherwise it is not. Now perform the exact same calculations with the vertices from r instead of p. If the point is outside of r and inside of p, then it must be in the shaded region, otherwise it is not.

2) a) Translation

$$T = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Uniform Scaling

$$S = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$TS = \begin{bmatrix} a & 0 & t_x \\ 0 & a & t_y \\ 0 & 0 & 1 \end{bmatrix} \quad ST = \begin{bmatrix} a & 0 & at_x \\ 0 & a & at_y \\ 0 & 0 & 1 \end{bmatrix}$$

since $TS \neq ST$, the operations do not commute

b) Rotation

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

uniform scaling

$$S = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

$$RS = \begin{bmatrix} a\cos\theta & -a\sin\theta \\ a\sin\theta & a\cos\theta \end{bmatrix} \quad SR = \begin{bmatrix} a\cos\theta & -a\sin\theta \\ a\sin\theta & a\cos\theta \end{bmatrix}$$

Since $RS = SR$ the operations commute

c) Rotation

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Non-Uniform Scaling

$$S = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$$RS = \begin{bmatrix} a\cos\theta & -b\sin\theta \\ a\sin\theta & b\cos\theta \end{bmatrix} \quad SR = \begin{bmatrix} a\cos\theta & -a\sin\theta \\ b\sin\theta & b\cos\theta \end{bmatrix}$$

since $RS \neq SR$ the operations do not commute

d) Shear wrt x-axis

$$S_x = \begin{bmatrix} 1 & g_x \\ 0 & 1 \end{bmatrix}$$

Shear wrt y-axis

$$S_y = \begin{bmatrix} 1 & 0 \\ g_y & 1 \end{bmatrix}$$

$$S_x S_y = \begin{bmatrix} 1 + g_x g_y & g_x \\ g_y & 1 \end{bmatrix} \quad S_y S_x = \begin{bmatrix} 1 & g_x \\ g_y & g_x g_y + 1 \end{bmatrix}$$

since $S_x S_y \neq S_y S_x$ the operations do not commute

3) a) Shear parallel to y-axis

$$S_y = \begin{bmatrix} 1 & 0 \\ g_y & 1 \end{bmatrix}$$

Rotation

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Shear parallel to x-axis

$$S_x = \begin{bmatrix} 1 & g_x \\ 0 & 1 \end{bmatrix}$$

A shear parallel to the y-axis can be achieved by a rotation, a shear parallel to the x-axis, and a final rotation

$$R_\phi S_x R_\theta = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} 1 & g_x \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\theta + g_x\sin\theta & -\sin\theta + g_x\cos\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Let $\theta = 90°$, $\phi = -90°$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} g_x & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -g_x & 1 \end{bmatrix}$$

Choose $g_x = -g_y$

$$= \begin{bmatrix} 1 & 0 \\ g_y & 1 \end{bmatrix}$$

$= S_y$

b) $$S_x S_y S_x = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}\begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$$

To rotate by $0°$, $m=n=0$

$$= \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & m \\ n & 1+nm \end{bmatrix}$$

$$= \begin{bmatrix} 1+mn & 2m+nm^2 \\ n & 1+nm \end{bmatrix}$$

To make this matrix resemble a rotation matrix:

Let $n = \sin\theta$

Let $1+mn = \cos\theta \Rightarrow m = \dfrac{\cos\theta - 1}{\sin\theta}$, $\sin\theta \neq 0$

Show that $2m+nm^2 = -\sin\theta$:

$$2\left(\frac{\cos\theta - 1}{\sin\theta}\right) + \sin\theta\left(\frac{\cos\theta-1}{\sin\theta}\right)\left(\frac{\cos\theta-1}{\sin\theta}\right)$$

$$= \left(\frac{\cos\theta - 1}{\sin\theta}\right)(\cos\theta + 1)$$

$$= \frac{\cos^2\theta - 1}{\sin\theta} = -\sin\theta$$

4) Tangent:

$x(t) = \text{sgn}(\cos(2\pi t))\, a\cos^2(2\pi t)$

Considering the function over a single period, $t \in [0,1)$

For $0 \le t < \frac{1}{4}$ and $3/4 < t < 1$ $\quad \text{sgn}(\cos(2\pi t)) = 1$

$x(t) = a\cos^2(2\pi t)$, $\quad x'(t) = 2a\cos(2\pi t)(-\sin(2\pi t))(2\pi)$

$= -2\pi a \sin(4\pi t)$

For $1/4 < t < 3/4$ $\quad \text{sgn}(\cos(2\pi t)) = -1$ so $x'(t) = 2\pi a \sin(4\pi t)$

At $t = \frac{1}{4}$, $t = 3/4$ $x(t)=0$, and the limits from either side of these piecewise functions are the same, for example $\lim_{t \to \frac{1}{4}^-} a\cos^2(2\pi t) = \lim_{t \to \frac{1}{4}^+} -a\cos^2(2\pi t) = 0$

since $x(t)$ is continuous,

can therefore write: $x'(t) = -2\pi a\, \text{sgn}(\cos(2\pi t))\sin(4\pi t)$

similarly $y(t) = \text{sgn}(\sin(2\pi t))\, b\sin^2(2\pi t)$ can split an interval into $0 < t < 0.5$ and $0.5 < t < 1$ and $t = 0$, $t = 0.5$ and calculate

$y'(t) = 2\pi b\, \text{sgn}(\sin(2\pi t))\sin(4\pi t)$

Tangent $= \langle x'(t), y'(t)\rangle = \langle -2\pi a\, \text{sgn}(\cos(2\pi t))\sin(4\pi t),\ 2\pi b\, \text{sgn}(\sin(2\pi t))\sin(4\pi t)\rangle$

Normal $= \langle -y'(t), x'(t)\rangle = \langle -2\pi b\, \text{sgn}(\sin(2\pi t))\sin(4\pi t),\ -2\pi a\, \text{sgn}(\cos(2\pi t))\sin(4\pi t)\rangle$

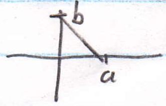
The curve is traced out once on $t \in [0, 1)$

At $t=0$, $x(0) = a$, $y(0) = 0$. At $t = \frac{1}{4}$, $x(\frac{1}{4}) = 0$, $y(\frac{1}{4}) = b$

For $0 < t < \frac{1}{4}$, $\operatorname{sgn}(\sin(2\pi t)) = 1$, $\operatorname{sgn}(\cos(2\pi t)) = 1$ and $\sin(4\pi t) \neq 0$

$$\frac{y'(t)}{x'(t)} = \frac{2\pi b \operatorname{sgn}(\sin(2\pi t)) \sin(4\pi t)}{-2\pi a \operatorname{sgn}(\cos(2\pi t)) \sin(4\pi t)} = -\frac{b}{a} = \text{constant}$$

Therefore, the curve traces a line with endpoints $(a, 0)$ and $(0, b)$ over this interval,

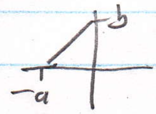


with area $\frac{ab}{2}$

Over $\frac{1}{4} < t < \frac{1}{2}$

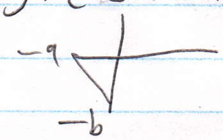
Similarly, at $t = \frac{1}{2}$, $x(\frac{1}{2}) = -a$, $y(\frac{1}{2}) = 0$. $\operatorname{sgn}(\sin(2\pi t)) = 1$
$\operatorname{sgn}(\cos(2\pi t)) = -1$, $\sin(4\pi t) \neq 0$

and again $\frac{y'(t)}{x'(t)} = \text{const}$



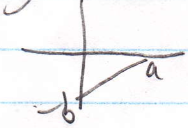
with area $\frac{ab}{2}$

At $t = 3/4$, $x(3/4) = 0$, $y(3/4) = -b$. Over $\frac{1}{2} < t < 3/4$,
$\operatorname{sgn}(\sin(2\pi t))$, $\operatorname{sgn}(\cos(2\pi t))$ are both constant, and $\sin(4\pi t) \neq 0$
so $\frac{y'(t)}{x'(t)} = \text{const}$



with area $\frac{ab}{2}$

Finally at $t = 1$ $x(1) = a$, $y(1) = 0$. Over $3/4 < t < 1$
$\operatorname{sgn}(\sin(2\pi t))$, $\operatorname{sgn}(\cos(2\pi t))$ are both constant and $\sin(4\pi t) \neq 0$
so $\frac{y'(t)}{x'(t)} = \text{const}$



with area $\frac{ab}{2}$

So the total area is $4 \cdot \frac{ab}{2} = 2ab$

The arc length of the curve is the sum of the length of the four line segments,

$$\text{Length} = 4 \cdot \sqrt{(a-0)^2 + (0-b)^2} = 4\sqrt{a^2+b^2}$$

When there are no closed form formullae, it is still true that $A = \int_a^b y(t)x'(t)\,dt$ and arc length $= \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt$ and these values can be computed using numerical integration, taking care to split the integral based on its parametric plot to take care of sign changes and the direction of integration

5) $x(t) = at$
$y(t) = -\frac{1}{2}gt^2 + bt + h$

a) Tangent $= \langle x'(t), y'(t) \rangle$
$x'(t) = a$
$y'(t) = -gt + b$
Normal $= \langle -y'(t), x'(t) \rangle$
$= \langle gt - b, a \rangle$

b) $y(t_i) = 0$
$0 = -\frac{1}{2}gt_i^2 + bt_i + h$
$gt_i^2 - 2bt_i - 2h = 0$
Using quadratic formula
$t_i = \dfrac{b \pm \sqrt{b^2+2hg}}{g}$

$t_i = \dfrac{b + \sqrt{b^2+2hg}}{g}$

(Discard the negative value of t_i since we are looking for $t_i > 0$)
$x(t_i) = a\left(\dfrac{b + \sqrt{b^2+2hg}}{g}\right)$

Position $(t_i) = \left\langle a\left(\dfrac{b+\sqrt{b^2+hg}}{g}\right), 0 \right\rangle$

$y'(t_i) = -gt_i + b$
$= -\cancel{g}\left(\dfrac{b+\sqrt{b^2+2hg}}{\cancel{g}}\right) + b$
$= -\sqrt{b^2+2hg}$

Velocity $(t_i) = \langle x'(t_i), y'(t_i) \rangle = \langle a, -\sqrt{b^2+2hg} \rangle$