1) a) $\ell_1(u) = \vec{p}_1 + u\vec{d}$

$\ell_2(u) = \vec{p}_2 + u\vec{d}$

These lines are parallel, and so only intersect at ∞.

A point $\begin{bmatrix} x \\ y \\ z \end{bmatrix} + u \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} = \begin{bmatrix} x + ud_x \\ y + ud_y \\ z + ud_z \end{bmatrix}$

will project to (in homogeneous coordinates), by applying the projection matrix

$$\begin{bmatrix} \frac{(x+ud_x)f}{z+ud_z} \\ \frac{(y+ud_y)f}{z+ud_z} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{(\frac{x}{u}+d_x)f}{\frac{z}{u}+d_z} \\ \frac{(\frac{y}{u}+d_y)f}{\frac{z}{u}+d_z} \\ 1 \end{bmatrix}$$

In the limit as $u \to \infty$ this intersection will therefore project to $\begin{bmatrix} d_x f/d_z \\ d_y f/d_z \\ 1 \end{bmatrix}$ and this is the vanishing point. (Note that $\vec{p}_1$ and $\vec{p}_2$ do not actually matter)

b) If $d_z = 0$, the lines will continue to project to parallel lines, since the projection of the intersection at ∞ will be (in homogeneous coordinates) $\begin{bmatrix} d_x \\ d_y \\ 0 \end{bmatrix}$ where the 0 in the third element means the intersection is still at ∞ ($\Rightarrow$ still parallel)

c) $\pi_1(u,v) = \vec{p}_1 + u\vec{d} + v\vec{d}'$

$\pi_2(u,v) = \vec{p}_2 + u\vec{d} + v\vec{d}'$

Setting $v=0$ gives the same vanishing point as in a)

Setting $u=0$ gives a new vanishing point at $\begin{bmatrix} \frac{d_x' f}{d_z'} \\ \frac{d_y' f}{d_z'} \\ 1 \end{bmatrix}$

Taking the cross product of these points gives:

$$\vec{\ell} = \begin{bmatrix} \frac{d_x f}{d_z} \\ \frac{d_y f}{d_z} \\ 1 \end{bmatrix} \times \begin{bmatrix} \frac{d_x' f}{d_z'} \\ \frac{d_y' f}{d_z'} \\ 1 \end{bmatrix} = \begin{bmatrix} f\left(\frac{d_y}{d_z} - \frac{d_y'}{d_z'}\right) \\ f\left(\frac{d_x'}{d_z'} - \frac{d_x}{d_z}\right) \\ f^2\left(\frac{d_x d_y' - d_y d_x'}{d_z d_z'}\right) \end{bmatrix} \simeq \begin{bmatrix} \frac{1}{f}\left(\frac{d_y d_z' - d_y' d_z}{d_x d_y' - d_y d_x'}\right) \\ \frac{1}{f}\left(\frac{d_x d_z' - d_x' d_z}{d_x d_y' - d_y d_x'}\right) \\ 1 \end{bmatrix}$$

The vanishing line is given by $\vec{\ell}^T \cdot \vec{p} = 0$

2) $\vec{p}(t) = \left(a + t^2 \cos t,\ 0,\ \frac{t}{b}\right) \quad 0 \le t \le 2\pi$

a) $\vec{p}(u,v) = \left((a+u^2\cos u)\cos v,\ (a+u^2\cos u)\sin v,\ \frac{u}{b} \right)$

$0 \le u \le 2\pi$

$0 \le v \le 2\pi$

b) $$\frac{\partial \vec{p}(u,v)}{\partial u} = \left((2u\cos u - u^2 \sin u)\cos v,\ (2u\cos u - u^2\sin u)\sin v,\ \frac{1}{b} \right)$$

$$\frac{\partial \vec{p}(u,v)}{\partial v} = \left(-(a+u^2\cos u)\sin v,\ (a+u^2\cos u)\cos v,\ 0 \right)$$

$$\vec{n} = \frac{\partial \vec{p}(u,v)}{\partial u} \times \frac{\partial \vec{p}(u,v)}{\partial v}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ (2u\cos u - u^2\sin u)\cos v & (2u\cos u - u^2 \sin u)\sin v & \frac{1}{b} \\ -(a+u^2\cos u)\sin v & (a+u^2\cos u)\cos v & 0 \end{vmatrix}$$

$$= \hat{i}\left(-\frac{1}{b}(a+u^2\cos u)\cos v \right) - \hat{j}\left(\frac{1}{b}(a+u^2\cos u)\sin v \right) + \hat{k}\left((2u\cos u - u^2\sin u)(a+u^2\cos u)\cos^2 v + (2u\cos u - u^2\sin u)(a+u^2\cos u)\sin^2 v \right)$$

$$= \left(-\frac{1}{b}(a+u^2\cos u)\cos v,\ -\frac{1}{b}(a+u^2\cos u)\sin v,\ (2u\cos u - u^2\sin u)(a+u^2\cos u) \right)$$

A tangent plane at a point $\vec{p}_0 = \vec{p}(u_0, v_0)$ is given by

$$\vec{T}(\alpha, B) = \vec{p}_0 + \alpha \left.\frac{\partial \vec{p}(u,v)}{\partial u}\right|_{\vec{p}=\vec{p}_0} + B \left.\frac{\partial \vec{p}(u,v)}{\partial v}\right|_{\vec{p}=\vec{p}_0}$$

c) The outward facing normal is the negative of what was calculated above.

$$\vec{n} = \left(\frac{1}{b}(a+u^2\cos u)\cos v,\ \frac{1}{b}(a+u^2\cos u)\sin v,\ (u^2\sin u - 2u\cos u)(a+u^2\cos u) \right)$$

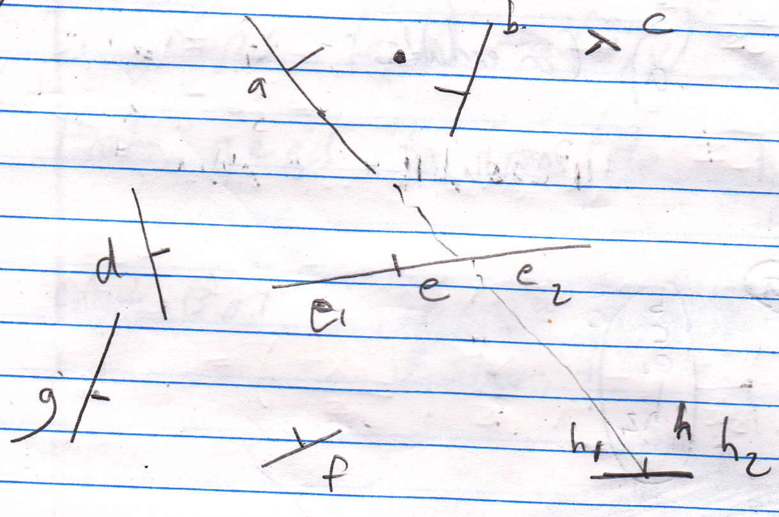
$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} \qquad |\vec{n}| = \sqrt{\left(\frac{1}{b}(a+u^2\cos u)\cos v\right)^2 + \left(\frac{1}{b}(a+u^2\cos u)\sin v\right)^2 + (u^2\sin u - 2u\cos u)^2(a+u^2\cos u)^2}$$

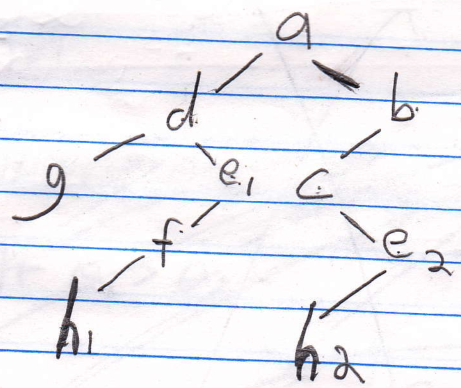
$$= (a+u^2\cos u)\sqrt{\frac{1}{b^2} + (u^2\sin v - 2u\cos u)^2}$$

$$= \frac{\left(\frac{1}{b}\cos v,\ \frac{1}{b}\sin v,\ u^2\sin u - 2u\cos u \right)}{\sqrt{\frac{1}{b^2} + (u^2\sin v - 2u\cos u)^2}}$$

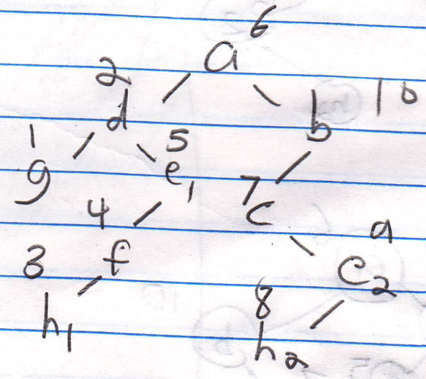
3a) It is not possible to exclude anything from rendering. None of the surfaces are backfacing. While some surfaces, like c, appear completely obfuscated from the eye's position in the 2d image, we do not know the relative depths of the surfaces – for example, if b is a short wall and c is a tall wall, then the eye will be able to see c the part of c that is higher than b. Without this information, none of the surfaces can be excluded from rendering.

3) b)





c) Draw order:



g d h_1 f e_1 a c h_2 e_2 b