

# Pricing and Hedging in the Black-Scholes Framework

## Vanna- Volga Pricing and Hedging

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## Vana-Volga Pricing and Hedging

### Illustrations

- Interpolation of a Volatility Curve

- Pricing Exotic Derivatives

# Vanna-Volga Dynamic Hedging

- ▶ A practical method for pricing and hedging derivatives, taking into account an uncertain volatility.
- ▶ Popular for Foreign Exchange derivatives
- ▶ Relates the price of a complex derivative to the known price of simpler, liquid instruments.

## Second order Greeks

- ▶ Vega:  $\nu = \frac{\partial O}{\partial \sigma}$
- ▶ Vanna:  $\frac{\partial \nu}{\partial S}$
- ▶ Volga:  $\frac{\partial \nu}{\partial \sigma}$

# Vanna-Volga Dynamic Hedging

An extended version of Ito's lemma with random volatility:

$$\begin{aligned} dO(t, K) = & \frac{\partial O}{\partial t} dt + \frac{\partial O}{\partial S} dS_t + \frac{\partial O}{\partial \sigma} d\sigma_t \\ & + \frac{1}{2} \frac{\partial^2 O}{\partial S^2} (dS_t)^2 + \frac{1}{2} \frac{\partial^2 O}{\partial \sigma^2} (d\sigma_t)^2 + \frac{\partial^2 O}{\partial S \partial \sigma} dS_t d\sigma_t \end{aligned}$$

# Vanna-Volga Dynamic Hedging

$$\begin{aligned}
 dO(t, K) - \Delta_t dS_t - \sum_{i=1}^3 x_i dC_i(t, K_i) = & \\
 & \left[ \frac{\partial O}{\partial t} - \sum_i x_i \frac{\partial C_i}{\partial t} \right] dt + \left[ \frac{\partial O}{\partial S} - \Delta_t - \sum_i x_i \frac{\partial C_i}{\partial S} \right] dS_t \\
 & + \left[ \frac{\partial O}{\partial \sigma} - \sum_i x_i \frac{\partial C_i}{\partial \sigma} \right] d\sigma_t + \left[ \frac{\partial^2 O}{\partial S^2} - \sum_i x_i \frac{\partial^2 C_i}{\partial S^2} \right] (dS_t)^2 \\
 & + \left[ \frac{\partial^2 O}{\partial \sigma^2} - \sum_i x_i \frac{\partial^2 C_i}{\partial \sigma^2} \right] (d\sigma_t)^2 + \left[ \frac{\partial^2 O}{\partial S \partial \sigma} - \sum_i x_i \frac{\partial^2 C_i}{\partial S \partial \sigma} \right] dS_t d\sigma_t
 \end{aligned}$$

# Vanna-Volga Hedging

$$dO(t, K) - \Delta_t dS_t - \sum_{i=1}^3 x_i dC_i(t, K_i) =$$
$$r \left[ O(t, K) - \Delta_t S_t - \sum_i x_i C_i(t, K_i) \right] dt$$

## Vanna-Volga Dynamic Hedging

The weights  $x_i$  are obtained by solving the system of linear equations:

$$\frac{\partial O}{\partial \sigma} = \sum_i x_i \frac{\partial C_i}{\partial \sigma}$$

$$\frac{\partial^2 O}{\partial \sigma^2} = \sum_i x_i \frac{\partial^2 C_i}{\partial \sigma^2}$$

$$\frac{\partial^2 O}{\partial S \partial \sigma} = \sum_i x_i \frac{\partial^2 C_i}{\partial S \partial \sigma}$$

or,

$$b = Ax$$



## Vanna-Volga Dynamic Hedging

Choice of hedge instruments

- ▶ An at-the-money straddle:

$$C_1 = C(S) + P(S)$$

- ▶ A “risk reversal”, traditionally defined as

$$C_2 = P(K_1) - C(K_2)$$

with  $K_1$  and  $K_2$  chosen so that the options have a delta of .25.

- ▶ A “butterfly”, defined as

$$C_3 = \beta(P(K_1) + C(K_2)) - (P(S) + C(S))$$

with  $\beta$  determined to set the vega of the butterfly to 0.

## Vanna-Volga Dynamic Hedging: Calculation Steps

- compute the risk indicators for the option to be priced:

$$b = \begin{pmatrix} \frac{\partial O}{\partial \sigma} \\ \frac{\partial^2 O}{\partial \sigma^2} \\ \frac{\partial^2 O}{\partial \sigma \partial S} \end{pmatrix} \quad (1)$$

- compute A matrix

$$A = \begin{pmatrix} \frac{\partial C_1}{\partial \sigma} & \cdots & \frac{\partial C_3}{\partial \sigma} \\ \frac{\partial^2 C_1}{\partial \sigma^2} & \cdots & \frac{\partial^2 C_3}{\partial \sigma^2} \\ \frac{\partial^2 C_1}{\partial \sigma \partial S} & \cdots & \frac{\partial^2 C_3}{\partial \sigma \partial S} \end{pmatrix} \quad (2)$$

## Vanna-Volga Dynamic Hedging: Calculation Steps

- ▶ solve for  $x$ :

$$b = Ax$$

- ▶ the corrected price for  $O$  is:

$$O^M(t, K) = O^{BS}(t, K) + \sum_{i=2}^3 x_i (C_i^M(t) - C_i^{BS}(t)) \quad (3)$$

where  $C_i^M(t)$  is the market price and  $C_i^{BS}(t)$  the Black-Scholes price (i.e. with flat volatility).

# Vanna-Volga Dynamic Hedging

Neglecting the off diagonal terms in  $A$ , a simplified procedure is to estimate  $x_i$  by:

$$x_2 = \frac{\frac{\partial^2 O}{\partial \sigma^2}}{\frac{\partial^2 C_2}{\partial \sigma^2}}$$
$$x_3 = \frac{\frac{\partial^2 O}{\partial \sigma \partial S}}{\frac{\partial^2 C_3}{\partial \sigma \partial S}}$$

# Volatility Interpolation

Given the ATM volatility and at two other strikes, determine the volatility at an arbitrary strike  $K$ .

# Vana-Volga Smile Interpolation

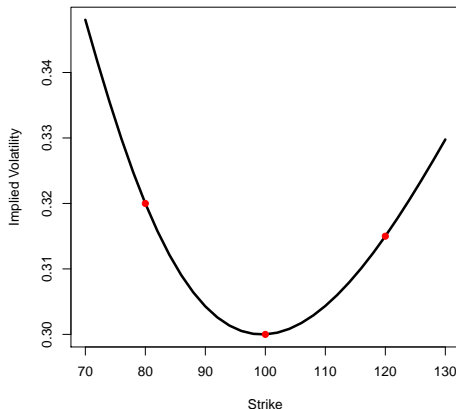


Figure: Interpolated Volatility Curve with Vanna-Volga algorithm

# Vanna-Volga Pricing of a Binary Option

Consider a one-year binary call, struck at the money. Assume that the smile is quadratic.

Use the traditional benchmark instruments of the FX market:

- ▶ Straddle,
- ▶ Risk-reversal
- ▶ Butterfly

# Vanna-Volga Pricing of a Binary Option

## Steps:

- ▶ Compute the strikes corresponding to a  $25\Delta$  call and put,
- ▶ Compute the value of each benchmark instrument. The butterfly must be vega-neutral,
- ▶ Compute the risk indicators (vega, vanna, volga) for the binary option,
- ▶ ... and for the benchmark instruments,
- ▶ Compute the smile cost of each benchmark: the price with the smile effect less price at the ATM volatility.
- ▶ Compute the price correction for the binary option.



# Pricing a Binary Option

In summary, we get:

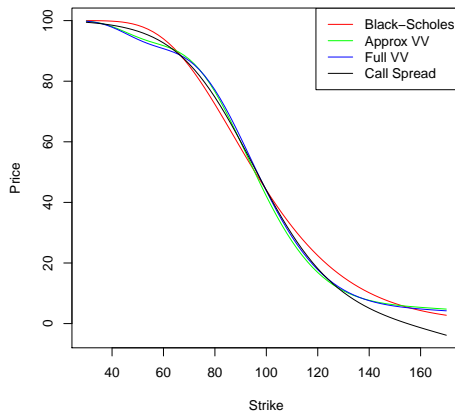
- ▶ Black-Scholes price: 32
- ▶ With approximate Vanna-Volga correction:  
 $32 + -4.86 = 27.14$
- ▶ With accurate Vanna-Volga correction:  $32 + -3.39 = 28.61$
- ▶ the approximation by a call spread is: 29.44

# Pricing Comparison

comparison of binary option value for a range of strikes, computed with four methods:

1. the regular Black-Scholes method, assuming a flat volatility
2. the Black-Scholes price with approximate Volga-Vanna correction
3. same as above, but with an accurate calculation of the Vanna-Volga correction
4. the value of a call spread centered at the strike

# Vana-Volga Pricing of a Binary Option



# References