Finance Quantitative

Risque de Taux

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Cash flow matching

Suppose the price is $212 for a 2-year coupon bond with face of $200 and an annual coupon (first one is one year from now) of $40. Suppose also that the price is $150 for a 1-year coupon bond with face of $150 and an annual coupon (one remaining, one year from now) of $15.

Remaining pension benefits in a plan having two more years to go are $95,000 one year from now and $60,000 two years from now.

1. What replicating portfolio of the two coupon bonds covers the pension liabilities exactly?
2. What is the price of the replicating portfolio?

Let q_1, q_2 be the quantities of bonds 1 and 2.

$$
\begin{aligned}
95000 &= 40q_1 + 165q_2 \\
60000 &= 240q_1
\end{aligned}
$$

Therefore,

```
q1 <- 60000/240
q2 <- (95000 - 40*q1)/165
```

Solution: $q_1 = 250, q_2 = 515.152$.

```
P <- 212 * q1 + 150 * q2
```

Portfolio cost: $P = 1.3027273 \times 10^5$.

Immunisation solution:

Bond dedication

In this problem, we construct a bond portfolio that generates a cash flow stream that matches a liability.

Assume that you must pay the amounts summarized in table~1. You can invest in a portfolio of 5 bonds described in table~2.

At every period, we can re-invest excess cash flow at a rate of .02, but cannot borrow.

Let's define the following notation:

Year	Cash Flow
1	-100
2	-200
3	-150
4	-400
5	-300

Table 1: Liability cash flow stream

Bond	Maturity	Coupon	Yield
A	1	.05	.05
B	2	.07	.075
C	3	.06	.058
D	4	.05	.049
E	5	.08	.081

Table 2: Available bonds for dedication

q_i quantity of bond i

$C(t)$ cash balance at time t

$F_i(t)$ cash flow from 1 unit of bond i at time t.

The purpose of the problem is to determine $q_i, i \in A, B, C, D$ and $C(t), t = 0, \ldots, 4$. The cash balance at end of year 5, $C(5)$ should be 0.

1. Write the accounting identity defining the cash-flow balance at each period (i.e. the balance between the money received and the money paid out).

$$C(t-1) * (1+r) + \sum_i q_i F_i(t) - C(t) = L(t)$$

2. Write the constrains on the variables q_i and $C(t)$.

$$q_i >= 0 \;\; \forall i C(t) >= 0 \;\; t = 0, \ldots, 5$$

3. Your goal is to minimize the cost of this dedication strategy. Write the corresponding objective function.

$$\sum_i q_i P_i + C(0)$$

4. Use the linprog package in R to solve the problem

```
# Cash Flow matrix
cf <- matrix(
  data=c(105, 0, 0, 0, 0,
           7, 107, 0, 0, 0,
         6, 6, 106, 0, 0,
         5, 5, 5, 105, 0,
         8, 8, 8, 8, 108), byrow=FALSE, ncol=5)

# cash reinvestment
rf <- .02
```

```
cash <- matrix(
  data = c((1+rf), -1, 0, 0, 0, 0,
           0, (1+rf), -1, 0, 0, 0,
           0, 0, (1+rf), -1, 0, 0,
           0, 0, 0, (1+rf), -1, 0,
           0, 0, 0,  0, (1+rf), -1),   byrow=TRUE, ncol=6)

# discount factors
r <- c(0.05, .075, .058, .049, .081)
df <- sapply(seq(5), function(t) (1/(1+r))**t)
P <- colSums(cf * df)
A <- cbind(cf, cash)
obj <- c(P, 1, rep(0,5))
rhs = c(100,200,150,400,300)
lower.bound <- rep(0, length(obj))

res <- solveLP(cvec=obj, Amat=A, b=rhs, const.dir=rep("=", nrow(A)), lpSolve=TRUE)
```

Since there are 5 independent bonds available to match 5 liabilities, an exact solution can be found:

Table 3: Optimal portfolio

bonds	quantity
A	0.415
B	1.435
C	1.036
D	3.598
E	2.778

If bounds are sets on quantity, then cash needs to be carried from period to period in order to balance inflows and outflows. Let's limit quantities to 2.5.

```
bounds <- cbind(diag(5), zeros(5, 6))
rhs.bounds <- rep(2.5,5)
res <- solveLP(cvec=obj, Amat=rbind(A, bounds), b=c(rhs, rhs.bounds), const.dir=c(rep("=", nrow(A)), rep
```

The optimal portfolio is

Table 4: Optimal portfolio with constraints on quantities

bonds	quantity
A	0.407
B	1.427
C	2.467
D	2.500
E	2.500

and, as expected, a positive cash balance needs to be maintained in some periods.

Table 5: Cash balance in each period

Year	Cash
0	0.00
1	0.00
2	0.00
3	144.03
4	29.41
5	0.00

Reinvestment Risk and Market Risk

Consider a 3-year standard bond with a 6% YTM and a 100 € face value, which delivers a 10% coupon rate. Coupon frequency and compounding frequency are assumed to be annual. Its price is 110.69 € and its duration is equal to 2.75. We assume that YTM changes instantaneously to become 5%, 5.5%, 6.5% or 7% and stays at this level during the life of the bond. Whatever the change in this YTM, show that the sum of the bond price and the reinvested coupons after 2.75 years is always the same.

Duration Hedging

An investor holds 100,000 units of bond A whose features are summarized in the following table. He wishes to be hedged against a rise in interest rates by selling some bond H.

Bond	Maturity	Coupon rate (%)	YTM (%)	Duration	Price
A	18	9.5	8	9.5055	114.181
H	20	10	8	9.87	119.792

Coupon frequency and compounding frequency are assumed to be semiannual. YTM stands for yield to maturity. The YTM curve is flat at an 8% level.

1. What is the quantity of the hedging instrument H that the investor has to sell?
2. We suppose that the YTM curve increases instantaneously by 0.1%.
   (a) What happens if the bond portfolio has not been hedged?
   (b) And if it has been hedged?
3. Same question as the previous one when the YTM curve increases instantaneously by 2%.
4. Conclude.

What is the quantity of the hedging instrument H that the investor has to sell?

PV01 is related to duration by the formula:

$$\text{PV01} = \frac{D}{1+\frac{r}{2}}\frac{P}{100} \times \frac{1}{10000}$$

PV01 of A and H:

```r
PV01.A <- 9.5055 / (1 + .08/2) * (114.181/100) * (1/10000)
PV01.H <- 9.87 / (1 + .08/2) * (119.792/100) * (1/10000)

Q.A <- 100000
Q.H <- Q.A * PV01.A / PV01.H
```

The investor should sell $Q.H = 9.1796 \times 10^4$ units of bond H.

We suppose that the YTM curve increases instantaneously by 0.1%.

We use the following pricing function for a semi-annual bond:

```r
pv.calc.semi <- function(maturity, coupon, yield) {
100 * ((coupon/yield) * (1 - 1/(1+(yield/2))^(2*maturity)) + 1/(1+(yield/2))^(2*maturity))
}

u.pc.1 <- 100000*(pv.calc.semi(18, .095, .081) - 114.181)
h.pc.1 <- u.pc.1 - Q.H*(pv.calc.semi(20, .1, .081) - 119.792)

u.pc.2 <- 100000*(pv.calc.semi(18, .095, .1) - 114.181)
h.pc.2 <- u.pc.2 - Q.H*(pv.calc.semi(20, .1, .1) - 119.792)
```

For a .1% increase:

1. The unhedged price change for a .1% increase is -1.0363596×10^5.
2. The hedged price change for a .1% increase is -93.3.

For a 2% increase in yield:

1. The unhedged price change for a 2% increase is -1.8317713×10^6.
2. The hedged price change for a 2% increase is -1.494451×10^4.

This illustrates that the PV01 hedge is valid for small changes in yield.