Gestion Obligataire

Courbe Zero-Coupon

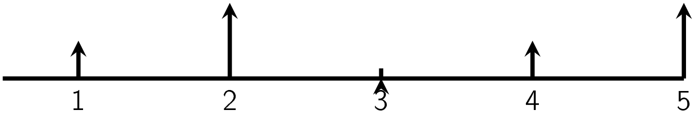
Patrick Hénaff

February 21, 2023

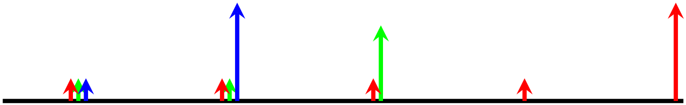
Problem Statement

How to compute the present value of an arbitrary cash flow stream when we only know the price of some bonds?

The cash flow:



The bonds:



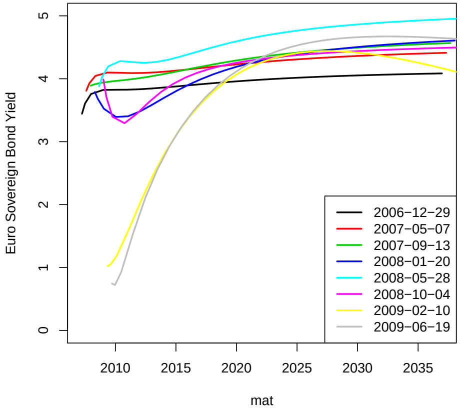
A Generic Approach to Cash Flow Discounting

- Compute the present value of € 1 paid in T years, let B_T be this value.
- The PV of any cash flow $\{F_1, \ldots, F_n\}$ is:

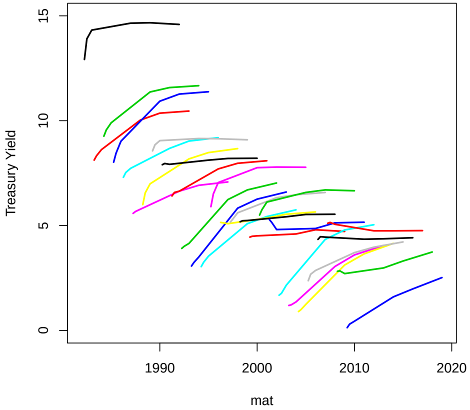
$$PV = \sum_{i=1}^{n} B_i F_i$$

- B_T is called the **discount factor** for maturity T.

Yield Curves of Euro AAA Gov't Bonds



Yield Curves of US Gov't Bonds



Observations about Term Structures

- Over time, we observe flat, upward-sloping and downward-sloping curves, upward-sloping curves are the most common.
- The short end of the curve is more volatile than the long end
- One can identify three types of shifts in the term structures:
  - parallel shifts
  - slope changes
  - convexity changes

Calculation of a ZC Curve

In practice, one imposes a functional form ("a shape") to the zero-coupon curve in order to smooth it.
This "shape" may be defined for:

1. The spot curve: $z(t)$
2. The discount factors B_t

$$z(t) = -\frac{1}{t} \ln (B_t)$$

Zero-Coupon Curve Computed by ECB

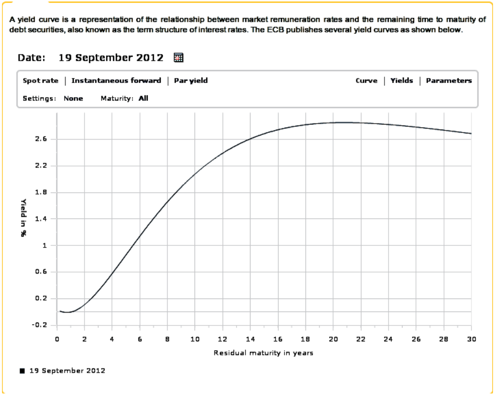


Figure: Source:European Central Bank (ecb.europa.eu)

How to get a smooth zero-coupon curve

Model of Nelson and Siegel:
Spot rates:

$$
\begin{aligned}
z(t) &= \beta_0 + \beta_1 \frac{1 - e^{-\frac{t}{\tau_1}}}{\frac{t}{\tau_1}} \\
&+ \beta_2 \left(\frac{1 - e^{-\frac{t}{\tau_1}}}{\frac{t}{\tau_1}} - e^{-\frac{t}{\tau_1}} \right)
\end{aligned}
$$

How to get a smooth zero-coupon curve

Svensson's model:
Spot rates:

$$
\begin{aligned}
z(t) &= \beta_0 + \beta_1 \frac{1 - e^{-\frac{t}{\tau_1}}}{\frac{t}{\tau_1}} \\
&\quad + \beta_2 \left(\frac{1 - e^{-\frac{t}{\tau_1}}}{\frac{t}{\tau_1}} - e^{-\frac{t}{\tau_1}} \right) \\
&\quad + \beta_3 \left(\frac{1 - e^{-\frac{t}{\tau_2}}}{\frac{t}{\tau_2}} - e^{-\frac{t}{\tau_2}} \right)
\end{aligned}
$$

Fitting Svensson's and Nelson-Siegel's Models

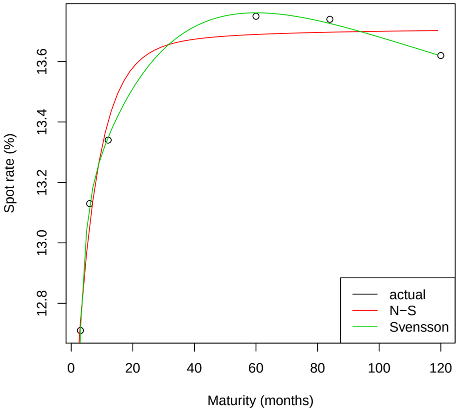


Figure: Actual vs. Fitted Treasury zero-coupon rates, with Svensson model and Nelson-Siegel model

Components of the Nelson-Siegel Model

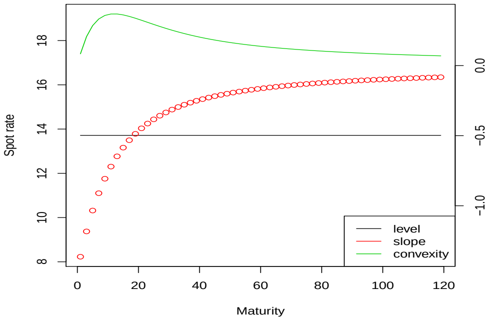
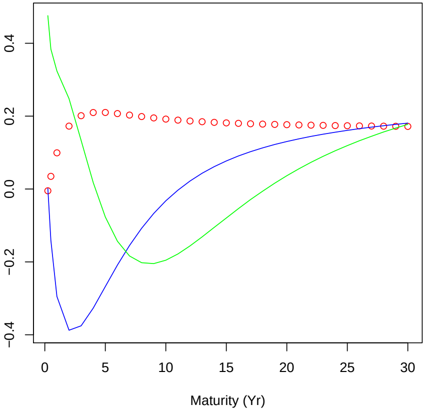


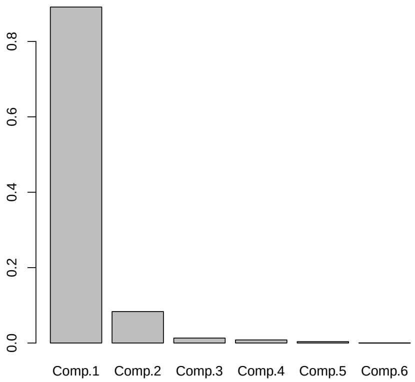
Figure: Components of Fitted Treasury zero-coupon spot curves, with Nelson-Siegel model

Empirical Risk Factors in USD Libor market



Empirical Risk Factors in USD Libor market

Fraction of total risk explained by the factors



Bond Pricing with a Zero-Coupon Curve

The ZC curve does not price each bond exactly:

$$P_k = \sum_i F_{k,i} e^{-(z(t_i)+\alpha_k)t_i}$$

P_k Price of bond k

$F_{k,i}$ Cash flow schedule for bond k

α_k zero-coupon spread specific to bond k: a measure of relative value.

- $\alpha_k > 0$ the bond is cheap with respect to the curve
- $\alpha_k < 0$ the bond is dear

Bond Pricing with a Zero-Coupon Curve

Pricing with a term structure of credit spread, specific to a market segment:

$$P_k = \sum_i F_{k,i} e^{(z(t_i)+\alpha(t_i))t_i}$$

$\alpha(t_i)$ term structure of spread, specific to the market segment of bond k.

We can compute the implied default risk from $\alpha(t)$.

Term structure of credit spread

$\Phi(t)$ Cumulative default probability from now until time t

q_i Unconditional probability of default between cash flow dates $[t_{i-1}, t_i]$.

$$\Phi(t_n) = \sum_{i=1}^{n} q_i$$

Pricing a Risky Zero-Coupon Bond

- P Price of zero-coupon of maturity T
- β Recovery in case of default
- $\Phi(T)$ Cumulative default probability from now until maturity
- $\alpha(T)$ credit spread, function of maturity

$$
\begin{aligned}
P &= (1-\Phi(T))e^{-r_T T} + \Phi(T)\beta e^{-r_T T} \\
&= e^{-r_T T}(1-\Phi(T)(1-\beta)) \\
&= e^{-(r_T+\alpha(T))T}
\end{aligned}
$$

Pricing a Risky Zero-Coupon Bond

$$
\begin{aligned}
e^{-\alpha(T)T} &= 1 - \Phi(T)(1-\beta) \\
-\alpha(T)T &= \ln(1 - \Phi(T)(1-\beta)) \\
-\alpha(T)T &\approx -\Phi(T)(1-\beta) \\
\alpha(T) &\approx \frac{1}{T}\Phi(T)(1-\beta)
\end{aligned}
$$

Example: Implied Default Probability

Example $\beta = 50\%$, $\alpha(T)$ typical of a A3 bond:

	1Y	2Y	3Y	5Y
$\alpha(T)$ (BP)	50	54	57	72
$\Phi(T)$ (%)	1.00	2.16	3.42	7.20
q_i (%)	1.00	1.16	1.26	3.78

A 5Yr zero-coupon spread of 72 bp implies a cumulative probability of default of 7.2% within 5 years. The probability of default in the 5th year is 3.78%.

Measuring Zero-Coupon Interest Rate Risk

Recall: PV01 is the change in bond price for a 1 b.p. decrease in bond YTM.

How to adapt the concept when discounting is done with a zero-coupon curve?

PV01 with zero-coupon bond pricing

Define PV01 as the change in bond price for a 1 b.p. shift of the entire zero-coupon yield curve.

1. Price the bond with the zero-coupon yield curve

$$P = F_1 e^{-z_1 t_1} + F_2 e^{-z_2 t_2} + \ldots + F_n e^{-z_n t_n}$$

2. Subtract 1 b.p. to the zero-coupon curve and re-price the bond:

$$P^* = F_1 e^{-(z_1 - 1b.p.)t_1} + F_2 e^{-(z_2 - 1b.p.)t_2} + \ldots + F_n e^{-(z_n - 1b.p.)t_n}$$

3. the PV01 is the change in price:

$$PV01 = P^* - P$$

Bucket Hedging

- Divide maturities into time buckets
- Hedge separately each time bucket
- Report exposure by time bucket

Cash Flow Projection onto Maturity Pillars

Given a cash flow $F(t)$, and two maturities $T_1 < t \leq T_2$, find the quantities q_1 and q_2 of zero-coupon bonds maturing at T_1 and T_2, such that cash flow $F(t)$ is "equivalent" to the portfolio of two zero-coupon bonds.

Cash Flow Projection onto Maturity Pillars

Instrument	Maturity	Price	PV01
Cash Flow	t	P_t	$PV01_t$
ZC-1	T_1	P_1	$PV01_1$
ZC-2	T_2	P_2	$PV01_2$

Cash Flow Projection onto Maturity Pillars

Solve for q_1 and q_2:

$$\begin{aligned} P_t &= q_1 P_1 + q_2 P_2 \\ PV01_t &= q_1 PV01_1 + q_2 PV01_2 \end{aligned}$$

- Do this for each cash flow in the portfolio to get a *shadow portfolio*.
- Hedge with instruments that are "close" to the pillars

Factor Hedging

A statistical approach to hedging.

- Identify independent risk factors
- hedge exposure to each risk factor

Risk Factors

A risk factor: a type of yield curve shift.

- Defined *a-priori*
  - parallel shift
  - twist of the 2-10 Yrs maturity, rotating around a 3 Yr pivot
- Identified by a statistical analysis of actual curve movements

Risk Factors

$$\Delta z(t) = \alpha_1 F_1(t) + \alpha_2 F_2(t) + \alpha_3 F_3(t)$$

with:

$\Delta z(t)$	Change in zero-coupon maturity t
α_1	Movement along the first risk factor
$F_1(t)$	Exposure of $z(t)$ to the first risk factor

Exposure of Bond to Risk Factors

$$\Delta P = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3$$

with:

ΔP	Change in bond price
α_1	Movement along the first risk factor
P_1	Exposure of bond to the first risk factor

P_1 is PV01 if the 1st factor is a parallel shift.