

# Le Modèle de Trenor-Black

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In this short note, we summarize the mathematical elements of the classical portfolio theory of Trenor-Black (Trenor & Black, 1973).

Assets excess return is modeled by a single factor model:

$$R_i = \alpha_i + \beta_i R_M + e_i$$

where  $\alpha_i$  is the idiosyncratic excess return of asset  $i$ , and  $e_i \sim N(0, \sigma_i^2)$  is the specific risk.

Recall two results from Markowitz and Sharpe:

- The expression for the tangency portfolio:

$$w^* = \frac{\Sigma^{-1} \tilde{R}}{\mathbf{1}^T \Sigma^{-1} \tilde{R}} \quad (1)$$

- Given two assets, A and M, the allocation that maximizes the Sharpe ratio is given by:

$$w_A = \frac{R_A \sigma_M^2 - R_M \sigma_A \sigma_M \rho_{AM}}{R_A \sigma_M^2 + R_M \sigma_A^2 - (R_A + R_M) \sigma_A \sigma_M \rho_{AM}} \quad (2)$$

## Calculation of the active portfolio

The active portfolio is determined by the idiosyncratic excess return and the specific risk of each asset.

The specific risks are assumed to be independent:

$$\Sigma_A = \begin{bmatrix} \sigma^2(e_1) & & \\ & \ddots & \\ & & \sigma^2(e_n) \end{bmatrix}$$

Using equation (1), we get:

$$w_{Ai} = \frac{\alpha_i / \sigma_i^2}{\sum \alpha_i / \sigma_i^2}$$

So that the active portfolio has an excess return and variance given by:

$$R_A = \alpha_A + \beta_A R_M$$

$$\sigma_A^2 = \beta_A^2 \sigma_M^2 + \sigma^2(e_A)$$

with

$$\alpha_A = \sum w_{Ai} \alpha_i$$

$$\beta_A = \sum w_{Ai} \beta_i$$

$$\sigma^2(e_A) = \sum w_{Ai}^2 \sigma^2(e_i)$$

### Allocation of wealth between the active portfolio and the market portfolio

A fraction  $w_A$  of wealth is allocated to the active portfolio, and the balance  $(1 - w_A)$  to the market portfolio so as to maximize the Sharpe ratio of the global portfolio  $x_A + (1 - x)M$ .

Using equation (2) we get after some algebra:

$$w_A = \frac{\alpha_A \sigma_M^2}{\alpha_A \sigma_M^2 (1 - \beta_A) + R_M \sigma^2(e_A)}$$

### Separability of the Sharpe ratio in the active portfolio

The first order condition for the optimal active portfolio is:

$$w_A = \lambda_A \Sigma^{-1} \alpha \tag{3}$$

Substitute in the expression

$$\alpha_A = w_A^T \alpha$$

to get:

$$\frac{\alpha_A}{\lambda_A} = \alpha^T \Sigma^{-1} \alpha \tag{4}$$

We next get an expression for  $\lambda_A$  in terms of known quantities:

$$\begin{aligned} \sigma^2(e_A) &= w_A^T \Sigma w_A \\ &= \lambda_A^2 \alpha^T \Sigma^{-1} \Sigma \Sigma^{-1} \alpha \\ &= \lambda_A^2 \alpha^T \Sigma^{-1} \alpha \end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\sigma^2(e_A)}{\lambda_A^2} &= \alpha^T \Sigma^{-1} \alpha \\ &= \frac{\alpha_A}{\lambda_A}\end{aligned}$$

Which yields:

$$\lambda_A = \frac{\sigma^2(e_A)}{\alpha_A}$$

Use this result in equation (4) to get:

$$\begin{aligned}\frac{\alpha_A^2}{\sigma^2(e_A)} &= \alpha^T \Sigma^{-1} \alpha \\ &= \sum_i \frac{\alpha_i^2}{\sigma^2(e_i)}\end{aligned}$$

which shows that the square of the Sharpe ratio of the active portfolio is the sum of the squares of the Sharpe ratios of the assets forming that portfolio.

## The Treynor-Black model in the notation of the 1973 paper and separability of the Sharpe ratio between the active and market portfolios

The investment universe is composed of  $n$  assets with asset-specific excess return:

$$r_i = \alpha_i + \beta_i r_M + e_i \quad i = 1, \dots, n \quad (5)$$

$$E(r_i) = \alpha_i + \beta_i E(r_M) = \mu_i \quad (6)$$

and of the market asset itself. Let  $w_i, i = 1, \dots, n$  be the investment in the assets with asset-specific excess returns, and  $w_M$  the investment in the market asset.

Treynor and Black restate this portfolio as an investment in  $n + 1$  assets, asset 1 to  $n$  being only exposed to the specific risk, and the  $n + 1$  asset being only exposed to the market risk:

$$w_{n+1} = w_M + \sum_{i=1}^n \beta_i w_i$$

Note that these  $n + 1$  assets are independent. The mean and variance of the portfolio are:

$$E(r_P) = \sum_{i=1}^{n+1} w_i E(r_i) = \mu_P \quad (7)$$

$$\sigma_P^2 = \sum_{i=1}^{n+1} w_i^2 \sigma_i^2 \quad (8)$$

As usual, maximize the Sharpe ratio by solving:

$$\begin{aligned} \min \quad & \frac{1}{2} w^T \Sigma w \\ \text{s.t.} \quad & \\ & \mu^T w = \mu_P \end{aligned}$$

Keeping in mind that the assets are independent, the Lagrangian is:

$$L(w, \lambda) = \sum_{i=1}^{n+1} w_i^2 \sigma_i^2 - 2\lambda \left( \sum_{i=1}^{n+1} w_i \mu_i - \mu_P \right)$$

First order conditions for optimality yield:

$$2w_i \sigma_i^2 - 2\lambda \mu_i = 0 \quad i = 1, \dots, n+1$$

or,

$$w_i = \lambda \frac{\mu_i}{\sigma_i^2} \quad (9)$$

Substitute in (6) to get:

$$\mu_P = \lambda \sum_{i=1}^{n+1} \mu_i^2 / \sigma_i^2 \quad (10)$$

$$\sigma_P^2 = \lambda^2 \sum_{i=1}^{n+1} \mu_i^2 \sigma_i^2 \quad (11)$$

so that,

$$\lambda = \frac{\sigma_P^2}{\mu_P}$$

To summarize, the weights of the assets in the active portfolio are:

$$w_i = \frac{\mu_i}{\mu_P} \frac{\sigma_P^2}{\sigma_i^2} \quad i = 1, \dots, n$$

To determine the investment in the market asset,  $w_M$ , recall that,

$$\mu_{n+1} = E(r_M) = \mu_M \quad (12)$$

$$\sigma_{n+1}^2 = \sigma_M^2 \quad (13)$$

Thus,

$$w_{n+1} = \sum_{i=1}^n w_i \beta_i + w_M \quad (14)$$

$$= \lambda \frac{\mu_M}{\sigma_M^2} \quad (15)$$

From equation (9), we have:

$$\sum_{i=1}^n w_i \beta_i = \lambda \sum_{i=1}^n \frac{\beta_i \mu_i}{\sigma_i^2}$$

So that the investment in the market asset can be written as

$$w_M = \lambda \left[ \frac{\mu_M}{\sigma_M^2} - \sum_{i=1}^n \frac{\beta_i \mu_i}{\sigma_i^2} \right]$$

To establish the separability of the Sharpe ratio between the active and the market portfolios, combine equations (10) and (11) to get:

$$\frac{\mu_P^2}{\sigma_P^2} = \sum_{i=1}^{n+1} \frac{\mu_i^2}{\sigma_i^2}$$

Denoting  $S_A, S_M, S_P$  the Sharpe ratios of, respectively, the active, market and overall portfolios, we can restate the previous equation as:

$$S_P^2 = \sum_{i=1}^n \frac{\mu_i^2}{\sigma_i^2} + S_M^2 \quad (16)$$

$$= \frac{\alpha_A^2}{\sigma_A^2 + S_M^2} \quad (17)$$

$$S_A^2 + S_M^2 \quad (18)$$

Treynor and Black call  $\alpha_A = \sum_{i=1}^n w_i \alpha_i$  the “appraisal premium” and  $\sigma_A^2 = \sum_{i=1}^n w_i^2 \sigma_i^2$  the “appraisal risk”.

## **Bibliography**

Treynor, J. L., & Black, F. (1973). How to Use Security Analysis to Improve Portfolio Selection. *The Journal of Business*, 46(1), 66–86. <http://www.jstor.org/stable/2351280>