Pricing and Hedging in the Black-Scholes Framework

Dynamic Hedging

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The Black-Scholes Model Made Easy
- The Key Concepts of Black-Scholes

Dynamic Hedging of Options
- Self-Financing Replicating Portfolio
- Sources of Hedging Error

Historical Perspective

- Bachelier (1900) models stock prices as a random walk, and derives the first option pricing formula.
- James Boness (1964) derives almost the same option valuation formula as Black-Scholes, but uses the expected stock return as opposed to the risk-free rate
- Paul Samuelson (1965) also derives and option pricing formula similar to the Black-Scholes model, but it involves both the expected return of the stock and the expected return of the option.
- Ed Thorp (1967) reports to have hedged an option portfolio using Boness' formula, but replacing the expected return by the risk-free rate.

The Black Scholes Model

A formal justification of the Black-Scholes model requires many assumptions:

- There is no arbitrage opportunity;
- It is possible to borrow and lend cash at a known constant risk-free interest rate r;
- It is possible to buy and sell any amount of stock (this includes short selling);
- Transactions do not incur any costs;
- The stock price follows a log-normal distribution with constant and known drift μ and volatility σ.

The Black-Scholes Key Concept

"It is possible to create a hedged position, consisting of a long position in the stock and a short position in the option, whose value will not depend on the price of the stock."

Fischer Black and Myron Scholes. "The Pricing of Options and Corporate Liabilities.". In: *Journal of Political Economy* 81 (1973), pp. 637–654.

The Black-Scholes Formula

S_0	Price of underlying asset
T	Option expiry
K	Strike
r	Interest rate
σ	Volatility

Call price:

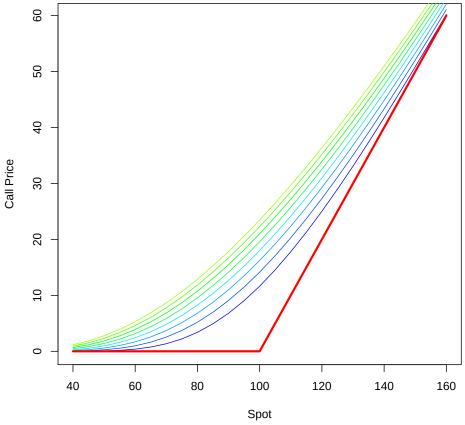
$$C_0 = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

with:

$$\begin{aligned} d_1 &= \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned}$$

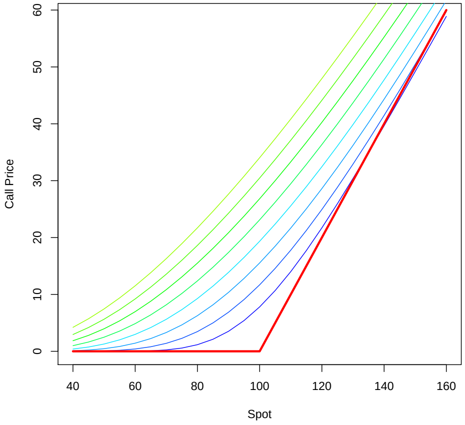
Call Price as a function of Spot and T

Strike = 100, Time to maturity 100 to 900 days. $\sigma = .3$.



Call Price as a function of Spot and Volatility

Strike = 100, Time to maturity = 1 Yr, $\sigma = .1, \ldots, .9$.



The Replicating Portfolio

The Black-Scholes formula can be directly interpreted as the description of the replicating portfolio:

$$C = S_0 N(d_1) - e^{-rT} K N(d_2)$$

The replicating portfolio has:

- $N(d_1)$ stock
- $KN(d_2)$ € of nominal of a zero-coupon bond expiring at T.

Review from Binomial Model

An option in a binomial model is equivalent to a portfolio:

- long Δ units of stock
- funded by borrowing B € at the risk-less rate

Such that, for a call worth C:

$$C = S_0\Delta - B$$

The same principle applies with the Black-Scholes model:

$$C = S_0 N(d_1) - e^{-rT} K N(d_2)$$

Black-Scholes Delta

The delta is the change in option price for a change of one e in the price of the underlying asset.

$$\text{Delta} = \frac{\text{Change in Option Value}}{\text{Change in Underlying Value}}$$

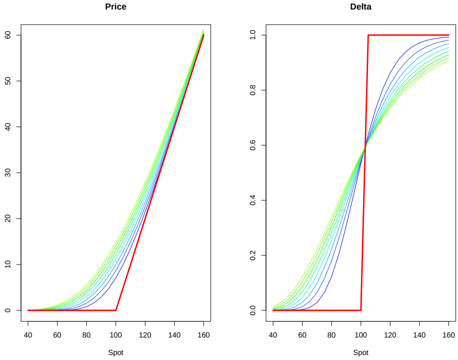
For a European call:

$$\Delta_C = N(d_1)$$

For a European put:

$$\Delta_p = N(d_1) - 1$$

Price and Delta of a Call as a function of Maturity



Construction of a hedge portfolio

Notation:

- C_t Value of derivative
- V_t Value of hedge portfolio
- B_t Amount borrowed/lent at the risk-free rate
- Δ_t Delta of derivative

At $t = 0$, the derivative is sold at price C_0 and the proceeds are used to purchase a hedge portfolio. The initial hedge is $\Delta_0 S_0 - B_0$, where B_0 is computed from the accounting identity:

$$C_0 = \Delta_0 S_0 - B_0$$

Example

A financial institution writes (sells) an at-the-money option on a stock worth €100. The option expires in two months, the hedge will be rebalanced every week (for illustration). Interest rate is 2% and volatility 30%.

Question: Compute the option price and the hedge portfolio.

Initial Hedge Portfolio

- Call price (Black-Scholes): $C_0 = 5.04$
- Delta: $\Delta_0 = 0.5352$
- Amount borrowed:

$$\begin{aligned} B_0 &= C_0 - \Delta_0 S_0 \\ &= -48.48 \end{aligned}$$

Initial hedge portfolio

	Hedge Portfolio		
Call Price	Stock	Bond	Total
C_0	$\Delta_0 \times S_0$	B_0	
5.04	53.52	-48.48	5.04

Rebalancing of hedge portfolio

The hedge must be adjusted periodically. At each step i, the decision rule is as follows:

1. Compute the value of the hedge portfolio formed at the previous time step:
$$V_i = -B_{i-1} e^{r\Delta t} + \Delta_{i-1} S_i$$
2. Compute the amount of stock to hold:
$$\Delta_i = \frac{\partial C_i}{\partial S_i}$$
3. The new hedge portfolio is $\Delta_i S_i + B_i$, with borrowing B_i determined by:
$$-B_i = V_i - \Delta_i S_i$$

At expiry of the derivative, the residual wealth is:

$$-C_T + \Delta_{T-1} S_T - B_{T-1} e^{r\Delta t}$$

Hedge Effectiveness

The quality of a model is ultimately measured by the residual error:

$$\begin{aligned} E_T &= -B_{T-1}e^{r\Delta t} + \Delta_{T-1}S_T - C_T \\ &= V_T - C_T \end{aligned}$$

Simulation 1: Call expiring out of the money

Week	stock price	Δ	call	bond	hedge port.
1	100.00	0.54	5.05	-48.9	5.05
2	98.16	0.47	3.79	-42.0	4.05
3	90.05	0.18	0.90	-16.0	0.23
4	88.01	0.11	0.42	-9.8	-0.15
5	90.28	0.13	0.50	-11.6	0.09
6	94.67	0.25	1.02	-23.0	0.66
7	94.17	0.17	0.53	-15.5	0.53
8	95.65	0.16	0.34	-14.5	0.78
9	94.67	0.00	0.00	+.6	0.62

Hedging discrepancy: 0.62 €.

Example 2: Call expiring in the money

Week	stock price	Δ	call	bond	hedge port.
1	100.00	0.54	5.05	-48.9	5.05
2	95.38	0.37	2.63	-32.7	2.55
3	93.58	0.29	1.73	-25.26	1.87
4	102.46	0.63	5.39	-60.1	4.45
5	101.23	0.58	4.22	-55.0	3.66
6	103.78	0.71	5.39	-68.6	5.12
7	103.34	0.72	4.56	-69.6	4.77
8	109.01	0.98	9.09	-98.0	8.82
9	103.94	1.00	3.94	-100.	4.06

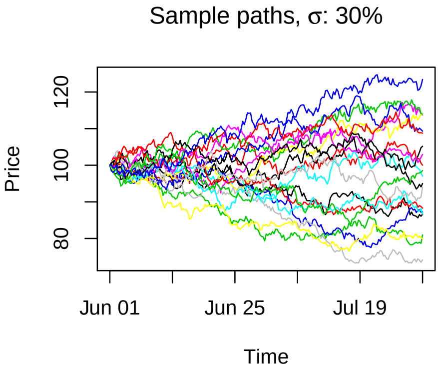
Hedging discrepancy: 4.06 - 3.94 = .12 €.

Large Scale Dynamic Hedging Simulation

To test the effectiveness of delta-hedging with the Black-Scholes model:

1. Simulate price scenarios
2. Simulate the dynamic rebalancing of the hedge portfolio
3. For each path, observe the hedging error at expiry

Simulated paths - log-normal process $\sigma = 30\%$, $T = 2$ months



First Simulation

Simulations in a “perfect Black-Scholes world”

- The volatility is known and constant
- the interest rate is constant
- No transaction costs

Delta-hedging simulation, maturity: 2 months,
$\sigma = .3$, $r = .02$, $K = 100$, $S_0 = 100$. Option price: 5.05.
200 time steps, 1000 simulations.

Distribution of Hedging Error - ATM European Call

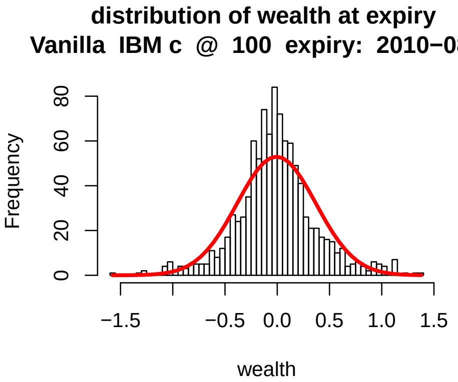


Figure: Distribution of hedging error $\sigma = 0.4$

Delta Hedging Error

Sources of hedging error:

1. Hedging frequency (Black-Scholes assumes continuous rebalancing);
2. Along one path, the volatility that is experienced may not be the theoretical volatility σ.

Delta Hedging Error vs. Hedging Frequency

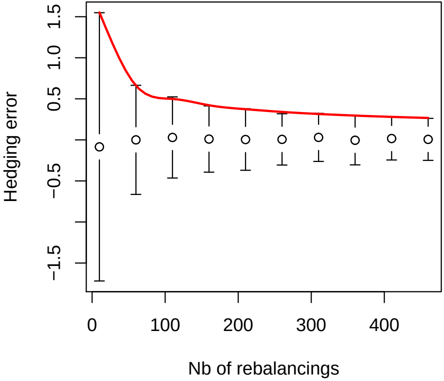


Figure: The variance of the hedging error is inversely related to the hedging frequency

Black-Scholes in Real Life

In real life, every assumption of the Black-Scholes model is invalidated:

- Interest rate is not constant
- Future volatility is unknown
- The distribution of S_T is not log-normal
- There are transaction costs and restrictions to shorting stocks

Let's measure the impact on hedging error.

Delta Hedging Error vs. Transaction Cost

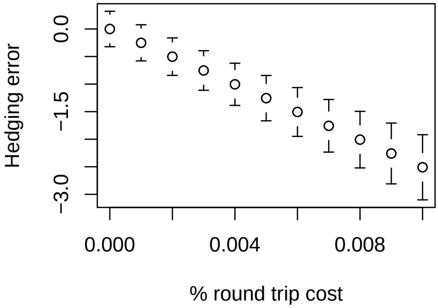


Figure: The hedging error is directly related to the magnitude of the transaction costs

Hedging Error and Unknown Volatility

Consider:

- The pricing volatility, used to determine the initial value of the derivative
- The actual volatility, the one that is experienced during the live of the option.

You sell an option and dynamically hedge your risk by trading the delta-hedging portfolio.

- If the actual volatility is lower than the pricing (expected) volatility, you will on average make money.
- If the actual volatility is higher than the pricing (expected) volatility, you will on average loose money.

More Greeks...

Hedging error is strongly dependent upon the curvature of the option price curve: the Gamma.

$$\begin{aligned}\Gamma &= \frac{\partial^2 C}{\partial S^2} \\ &\approx \frac{C(S_0 + h) - 2C(S_0) + C(S_0 - h)}{h^2}\end{aligned}$$

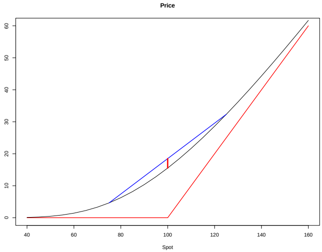


Figure: Gamma of a Call

More Greeks...

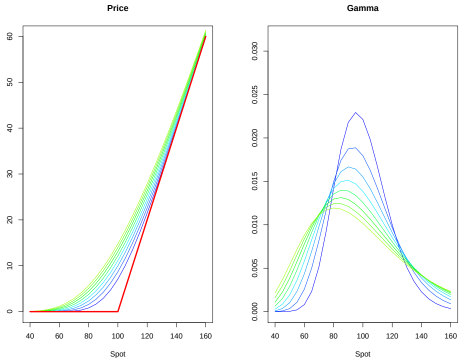


Figure: Price and Gamma of a Call as a function of maturity

Hedging Error and Unknown Volatility

In the presence of unknown volatility, it can be shown that hedging error is related to:

1. The difference between the pricing (i.e. assumed) volatility and the effective volatility
2. The gamma of the option:
    - Hedging error is small when the gamma is small and keeps the same sign
    - Hedging error is large when the gamma is large and changes sign

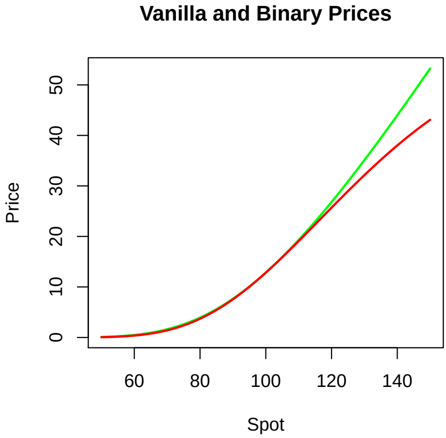
Influence of Gamma on Hedging Error

We compare the hedging error of two simple derivatives expiring in 1 year:

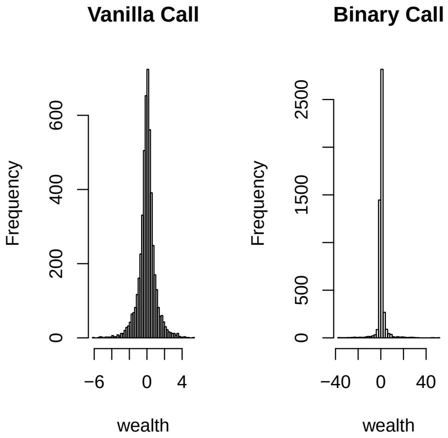
- a European call, $K = 100$,
- a binary call struck at 126.65 e, that will pay 62.09 e if the option expires in the money.

The binary call is designed to have the same premium and same initial delta as the European option.

PV of Vanilla and Binary Calls with same current price and delta



Delta Hedging Errors: Vanilla vs. Binary Calls



In identical experimental conditions, the hedging error of the binary call is about 10 times larger than the one of the European

Robustness of Black-Scholes

$$\epsilon_T = \frac{1}{2}\int_0^T \left[\Sigma^2 - \sigma_t^2\right] \frac{\partial^2 C}{\partial S^2} S^2 dt$$

N El-Karoui, M Jeanblanc-Picquè, and Steven E. Shreve. "Robustness of the Black and Scholes Formula". In: *Mathematical Finance* 8.2 (1998), pp. 93–126. ISSN: 1467-9965. DOI: 10.1111/1467-9965.00047

Hedging Error: Conclusion

In a Black-Scholes world, hedging error is determined by four factors:

1. the hedging frequency
2. the transaction costs
3. the magnitude of the option gamma
4. the difference between Σ (the BS volatility used for pricing and hedging) and the volatility that is actually experienced, σ.

Is Black-Scholes Still in Use?

- Black-Scholes in its pure form (delta hedging argument) has probably never been used in practice
- Complex derivatives have been managed with much more complex models (stochastic volatility, jumps, local volatility) since the mid 1990's.

Is Black-Scholes Still in Use?

The Black-Scholes formula is widely used in vanilla option markets, but not in the way consistent with the theory:

- The Black-Scholes formula is used to operate a change of unit of measure for vanilla options: to convert prices into volatilities (it is easier to interpret volatility differences than price differences).
- Traders recognize that the main risk is unknown volatility / non normality of returns, and therefore hedge options with other options.

Espen Gaarder Haug and Nassim Nicholas Taleb. "Option Traders Use (very) Sophisticated Heuristics, Never the Black-Scholes-Merton Formula.". In: *Economic Behavior and Organization* 77 (2011).

Is Black-Scholes Still in Use?

The Black-Scholes assumptions (normality of returns) are unfortunately still sometimes used by regulators to define risk measures.

1. Value at Risk, Pillar I of Basel II/III;
2. Solvency II risk models.

References

[1] Fischer Black and Myron Scholes. "The Pricing of Options and Corporate Liabilities.". In: *Journal of Political Economy* 81 (1973), pp. 637–654.

[2] Espen Gaarder Haug and Nassim Nicholas Taleb. "Option Traders Use (very) Sophisticated Heuristics, Never the Black-Scholes-Merton Formula.". In: *Economic Behavior and Organization* 77 (2011).

[3] N El-Karoui, M Jeanblanc-Picquè, and Steven E. Shreve. "Robustness of the Black and Scholes Formula". In: *Mathematical Finance* 8.2 (1998), pp. 93–126. ISSN: 1467-9965. DOI: 10.1111/1467-9965.00047.