Fixed Income Risk Management

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The Fixed Income Risk Management Problem

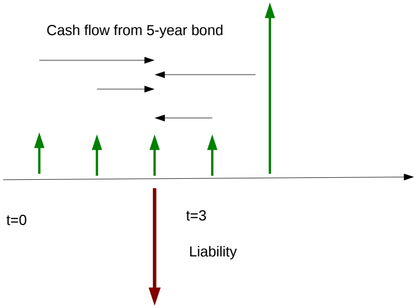
An insurance company sells Guaranteed Investment Contracts that will return $r = 5\%$ interest, compounded annually for 3 years. To receive 1000$ at maturity, you invest today:

$$\frac{1000}{(1+r)^3} = 863.83$$

How should the insurance company invest your payment?

The Fixed Income Risk Management Problem

Mismatch between asset and liability cash flows.



Assume Fixed Interest Rate

Fundamental Valuation Principle:
A cash flow stream is equivalent to its net present value (NPV).
Consequence:
If interest rates are known and fixed, any portfolio with present value L_0 will provide value L_T at the investment horizon T.
Illustration:

- Investment horizon $T = 3$ years
- $V_T = 1000$
- $r = 5\%$
- Present value of liability:

$$\begin{aligned} V_0 &= 1000\frac{1}{(1+r)^3} \\ &= 863.83 \end{aligned}$$

Funding with Fixed Interest Rate (Example 1)

Buy a 5 year bond, coupon: 3%, yield: 5%
Price (for 100€ nominal):

$$
\begin{aligned}
P &= 100\left[\frac{.03}{.05}\left(1-\frac{1}{(1+.05)^5}\right)+\frac{1}{(1+.05)^5}\right] \\
&= 91.34
\end{aligned}
$$

With a budget of 863.83, buy $N = \frac{863.83}{91.34} = 9.457$ units of bond.

Funding with Fixed Interest Rate (Example 1)

Total wealth in 3 years:

- Capitalized value of 3 coupons:

$$
\begin{aligned}
CI &= N \times 100 \times .03\left[(1+r)^2 + (1+r) + 1)\right] \\
&= 89.44
\end{aligned}
$$

- Present value of last 2 cash flows:

$$
\begin{aligned}
PV &= N \times 100 \left[\frac{.03}{.05}\left(1 - \frac{1}{(1+.05)^2}\right) + \frac{1}{(1+.05)^2}\right] \\
&= 910.56
\end{aligned}
$$

$$CI + PV = 1000$$

Funding with Fixed Interest Rate (Example 2)

Buy a 10 year bond, coupon 6%, Price: 107.72. With a budget of 863.83, buy $N = \frac{863.83}{107.72} = 8.01$ units of bond.
Value at maturity:

- Capitalized interest:

$$
\begin{aligned}
CI &= 8.01 \times 100 \times 0.06 \left[(1+r)^2 + (1+r) + 1)\right] \\
&= 151.68
\end{aligned}
$$

- Present value:

$$
\begin{aligned}
PV &= 8.01 \times 100 \left[\frac{.06}{.05}\left(1 - \frac{1}{(1+.05)^7}\right) + \frac{1}{(1+.05)^7}\right] \\
&= 848.32
\end{aligned}
$$

$$CI + PV = 1000$$

What Happens if Yield Changes (Example 1)?

Immediately after purchasing the bond, yield increases to $r = 7\%$

...

Total wealth in 3 years:

- Capitalized interest:

$$\begin{aligned} CI &= 9.45 \times 100 \times 0.03 \left[(1+r)^2 + (1+r) + 1)\right] \\ &= 91.21 \end{aligned}$$

- Present value:

$$\begin{aligned} PV &= 9.45 \times 100 \left[\frac{.03}{.07}\left(1 - \frac{1}{(1+.07)^2}\right) + \frac{1}{(1+.07)^2}\right] \\ &= 877.33 \end{aligned}$$

$$CI + PV = 968.54$$

What Happens if Yield Changes (Example 1)?

New Yield	Comp. Int.	Disc. Future CF	Value at Horizon (3Yr)
3%	87.69	945.72	1033.42
7%	91.21	877.33	968.54
5%	89.44	910.55	1000

Compounded interest and the present value of future cash flows move in opposite directions. The key to control risk will be to balance reinvestment gain and changes in PV.

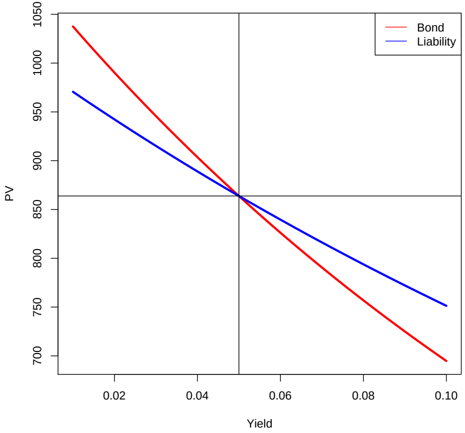
What Happens if Yield Changes (Example 1)?

Present value of asset (the bond) and liability after the yield change.

New Yield	PV Bond	PV Liability
3%	945.72	915.14
7%	790.62	816.29
5%	863.83	863.83

After a change in yield, the present values of the bond and of the liability are no longer identical. The key to control risk will be select a bond such that the changes in PV are identical for asset and liability.

Present value of asset and liability as a function of yield



PV of 5Yr Bond (red) and 3Yr Liability (blue) as a function of yield.

Conclusion on How to Fund a Single Liability

Given a liability L_T to be paid at time T, with present value

$$L_0 = L_T \frac{1}{(1+r)^T}$$

Fund it with a bond portfolio (asset) such that:

- PV of asset = PV of liability
- when yield changes, the change in PV of the asset is identical to the change in PV of the liability.

Then, by the **Fudamental Principle of Valuation**, the asset will provide wealth L_T at maturity.

Measurement of Interest Rate Risk

- Consider a 4% bond maturing on 31dec2030. Current yield as of 01jan2012 is 3%.
- Price of the bond:

$$
\begin{aligned}
P &= 100\left(\frac{.04}{.03}(1-\frac{1}{(1+0.03)^18}) + \frac{1}{(1+0.03)^18}\right) \\
&= 113.7535
\end{aligned}
$$

PV01 (Present value of 1 BP):
$P(r = 2.99\%) - P(r = 3\%) = 113.9028 - 113.7535 = 0.149$

Variation

Definition

Variation is the change in price due a unit change in yield.

- $P(r)$ Bond price, function of yield
- r_1 Initial yield
- r_2 Shifted yield ($r_2 = r1 + \epsilon$)

$$P(r_2) - P(r_1) \approx -V(r_2 - r_1)$$

or

$$V = -\frac{P(r_1 + \epsilon) - P(r_1)}{\epsilon}$$

Variation or Dollar Duration

The Variation V is the negative of the derivative of price with respect to yield

$$V = -\frac{\partial P(r)}{\partial r}$$

Let

$$\begin{aligned} P(r) &= \sum_{i=1}^{n} \frac{F_i}{(1+r)^{t_i}} \\ \frac{\partial P(r)}{\partial r} &= \sum_{i=1}^{n} \frac{-t_i F_i}{(1+r)^{t_i+1}} \end{aligned}$$

Computing PV01

Two ways of computing PV01

- By difference:

$$PV01 = Price(r - 0.001) - Price(r)$$

- Analytically:

$$\begin{aligned} PV01 &= V \times 10^{-4} \\ &= -\frac{\partial P}{\partial r} \times 10^{-4} \end{aligned}$$

In practice, the calculation by difference is preferred, since it is much simpler to implement in decision-support systems.

Duration

Duration D is a measure of risk related to variation, with a more intuitive interpretation:

$$D = (1+r)\frac{V}{P}$$

or,

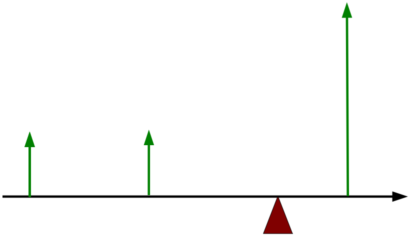
$$\begin{aligned} D &= (1+r)\frac{1}{P}\sum_{i=1}^{n} t_i \frac{F_i}{(1+r)^{t_i+1}} \\ D &= \sum_{i=1}^{n} t_i \frac{\frac{F_i}{(1+r)^{t_i}}}{P} \end{aligned}$$

Duration

$$D = \sum_{i=1}^{n} t_i \frac{\frac{F_i}{(1+r)^{t_i}}}{P}$$

- The Duration D of a bond is the average maturity of the cash-flows, weighted by the PV of each cash flow.
- It is the investment horizon at which reinvestment risk exactly balances the capital gain risk.
- The duration of a ZC bond is equal to its maturity.

Duration as Fulcrum Point



Duration: Example

An insurance company sells an investment contract that will pay 1000 € in 8 years. Yield is $r = 5\%$. The client invests

$$1000 \frac{1}{(1+r)^8} = 676.83$$

In which coupon-bearing bond should the insurance company invest to be able to make the 1000 € payment in 8 years?

Duration: Example

The duration of the liability is 8 years. We need to invest in a bond that also has a duration of 8 years.
Total wealth at horizon:

- all the coupons received before the horizon, reinvested at the prevailing yield
- market value of the bond at horizon

Duration: Example

A 10 year 5.5% bond is almost a perfect match. NPV = 103.86. Buy $\frac{676.83}{103.86} = 6.51$ units of bonds. Total value at horizon as a function of reinvestment yield:

Yield	CI	Horizon PV	Total
2%	307.63	695.96	1003.59
5%	342.26	657.73	1000.00
8%	381.24	622.62	1003.86

The duration of this bond in 7.99.

Properties of Duration

- For coupon bonds, duration is always less than maturity
- For zero-coupon bonds, duration is equal to maturity
- Duration increases with maturity, but reaches a limit.

Duration as a function of maturity

There is a limit to the duration of coupon bonds!