Finance Quantitative
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From Cox-Ross-Rubinstein to Black-Scholes

This note sketches the relationship between the binomial model of Cox, Ross and Rubinstein and the Black-Scholes formula.

1 Distribution of Stock Price in CRR

Consider a binomial tree with n steps. There are $(n+1)$ possible values for the stock price at the end of the tree, numbered S_0 to S_n, with S_n being the largest value. A two-step tree is pictured in Figure 1.

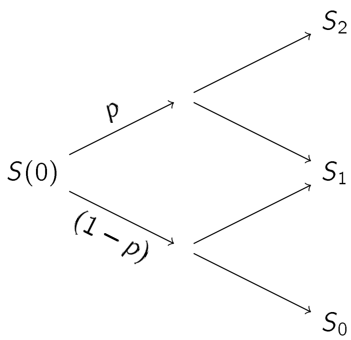

Figure 1: A two-step binomial tree

S_n is reached after n upward moves, S_{n-1} is reached after $(n-1)$ upward moves and 1 downward move, etc. There is only one path to S_n, but there are n paths to S_{n-1}.
To evaluate the number of paths leading to S_{n-k}, one needs to count how many ways we have of making k downward moves in n steps. This number is called the number of combinaisons of k among n. For example, there are 10 ways of making 2 down moves in 5 steps. This is illustrated in Figure 2, where white squares represent down moves and red squares stand for the up moves.

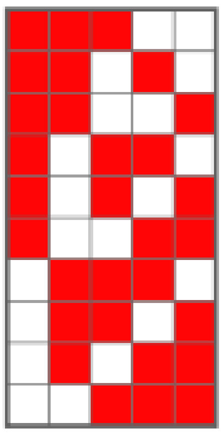

Figure 2: 10 ways of combining 2 white and 3 red squares (Source: www.wikipedia.org)

This number of ways of combining k items among n is written $\binom{n}{k}$. All paths leading to S_{n-k} have the same probability, since they all involve k down moves and $n-k$ up moves. The probability of one such path is the probability of making $(n-k)$ up moves (each one having probability p) and k down moves. This can be written:

$$p^{n-k}(1-p)^k$$

Let $S(T)$ be the stock price at expiry, and $P(S(T) = S_{n-k})$ the probability of $S(T)$ reaching node S_{n-k} in the binomial tree:

$$Pr\left(S(T) = S_{n-k}\right) = \binom{n}{k} p^{n-k}(1-p)^k$$

Figure 3 illustrates these calculations on a tree with tree steps.

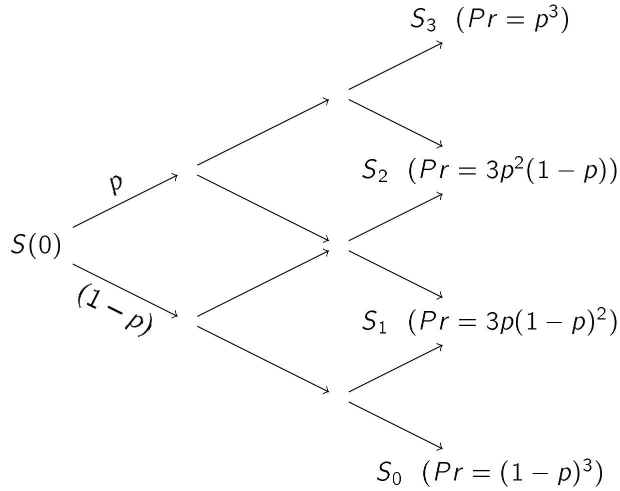

Figure 3: A three-step binomial tree, with probabilities of reaching each terminal node

The probability that $S(T)$ is greater that S_j is the sum of the probabilities of reaching at time T a node that is higher than S_j:

$$P(S(T) > S_j) = \sum_{i=j+1}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$

2 Option Price in CRR

The price of an option in the CRR framework is the discounted expected value of the payoff. Consider a call option with strike K. To simplify notation, assume that K coincide with a node in the tree at expiry, say S_k. The option is in the money for states $S_{k+1}, \ldots, S_n$. Since we know the probability associated with each terminal state, we can write the value of the call option:

$$\begin{aligned}
C &= e^{-rT} \sum_{i=k+1}^{n} \binom{n}{i} p^i (1-p)^{n-i} (S_i - K) \\
&= e^{-rT} \sum_{i=k+1}^{n} \binom{n}{i} p^i (1-p)^{n-i} S_i - K \sum_{j=k+1}^{n} \binom{n}{i} p^i (1-p)^{n-i}
\end{aligned}$$

To simplify calculations, let's focus on the first part of the equation above:

$$A = e^{-rT} \sum_{j=i+1}^{n} \binom{n}{j} p^j (1-p)^{n-j} S_j$$

To reach S_i, you need to make i up-moves and $(n-i)$ down-moves. This can be written as:

$$S_i = S(0) u^i d^{n-i}$$

The time to maturity T is split into n steps of length Δt. Use both observations to get:

$$\begin{aligned}
A &= e^{-rT} \sum_{i=k+1}^{n} \binom{n}{i} p^i (1-p)^{n-i} S u^i d^{n-i} \\
&= e^{-r(k\Delta t + (n-k)\Delta t)} \sum_{i=k+1}^{n} \binom{n}{i} (pu)^i \left((1-p)d\right)^{n-i} S(0) \\
&= S(0) \sum_{i=k+1}^{n} \binom{n}{i} (pue^{-r\Delta t})^i \left((1-p)de^{-r\Delta t}\right)^{n-i}
\end{aligned}$$

Let

$$pue^{-r\Delta t} = q$$

After some algebra, one find that

$$(1-p)de^{-r\Delta t} = 1-q$$

And therefore,

$$A = S(0) \sum_{i=k+1}^{n} \binom{n}{i} q^i (1-q)^{n-i}$$

Putting things back together, we get:

$$C = S(0) \sum_{j=k+1}^{n} \binom{n}{i} q^i (1-q)^{n-i} - Ke^{-rT} \sum_{i=k+1}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$

which is starting to look like the Black-Scholes equation.
The term

$$\sum_{i=k+1}^{n} \binom{n}{i} q^i (1-q)^{n-i}$$

is the probability that the stock at expiry, $S(T)$ is greater than S_k, if the probability of an up move is q. Similarly, the term

$$\sum_{i=k+1}^{n} \binom{n}{i} q^i (1-q)^{n-i}$$

is the probability that the stock at expiry, $S(T)$ is greater than S_k, if the probability of an up move is p. When the number of steps in the binomial tree increases, the distribution of the price at expiry, $S(T)$, becomes a log-normal distribution, and after more algebra, one gets

$$\sum_{j=i+1}^{n} \binom{n}{j} q^j (1-q)^{n-j} \approx N(d1)$$

and

$$\sum_{j=i+1}^{n} \binom{n}{j} p^j (1-p)^{n-j} \approx N(d2)$$

where $N(x)$ is the cumulative normal density, and d_1 and d_2 are the familiar terms of the Black-Scholes equation:

$$\begin{aligned} d_1 &= \frac{\ln(S(0)/K) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned}$$

which leads to the Black-Scholes formula:

$$C = S(0)N(d_1) - Ke^{-rT}N(d_2)$$

3 Interpretation

This discussion helps to motivate an informal interpretation of the Black-Scholes formula:

- $N(d2)$ is the probability that the call will be exercised (i.e. $S(T)$ will be greater than K), assuming that the stock expected return is the risk-free rate.
- $S(0)N(d1)$ is the present value, using the risk-free interest rate, of the expected asset price at expiration, $S(T)$, given that the asset price at expiration is above the strike K.

In summary, when you own a call, you will purchase the stock at a fixed price K, provided that the stock price at expiry is greater than K. The value of this contract is the discounted expected value of what you will receive ($S(0)N(d_1)$), less the discounted value of your expected payment ($Ke^{-rT}N(d_2)$).