CSC336

Numerical Methods

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CONTENTS

Part I

Part 1

CHAPTER

1 Introduction

1.1 Motivation

$$\text{Math Textbook} + \text{Laptop} + \text{Coding} \stackrel{?}{\Longrightarrow} \text{Compute Accurate Solution}$$

Consider the McLaurin series expansion of the function $f(x) = e^x$:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \end{aligned}$$

The issue is that we cannot compute to infinity. We need to introduce partial sums

$$S_n = \sum_{i=0}^{n} \frac{x^i}{i!}$$

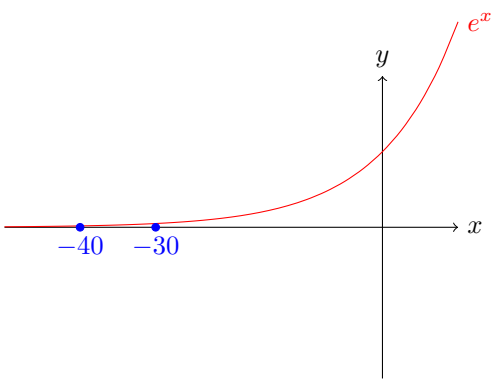
We could iterate over n until $|S_n - S_{n-1}| <$ tolerance.

x	0	1	10	20	40
Num Terms to "Converge"	2	13	42	69	104

We observe that the running time is dependent on the value of x. We need to find a better way to compute the sum – with more consistent running time.

Using the python program,

- When $x = -30$, convergence happened after 97 terms, to -6.0×10^{-5}.
- When $x = -40$, convergence happened after 124 terms, to approximately -5.9×10^{0}.



Clearly, we have inaccuracy when $x = -40$, as $0 < e^x < 1$ for all $x < 0$. The math textbooks' s techniques does not always provide good computational algorithms.

Course goal:

Show computational algorithms and discuss why they are good.

Example (e^x Better Algorithm)**.** A better algorithm is as follows

- Find k such that $r = \frac{x}{k}$ exactly with $\|r\| < 1$.
- Compute $e^r = e^{x/k}$ using the McLaurin series.
- Then, $e^x = (e^r)^k$.

♢

Remark Error due to Catasrophic Cancellation

When we subtract two numbers that are very close to each other, we lose precision.

1.2 Topics

- Computer Arithmetic and Computational Errors (Chap. 1)
  - Floating Point Arithmetic
  - Two Concepts
    - The conditioning of a math problem
    - the numerical stability of an algorithm
- Solving Systems of Linear Equations (Chap. 2)
  - Solve $Ax = b$ for x
- Solving Non-linear Equations (Chap. 5)

  Fine x s.t. $f(x) = 0$ or $g(x) = 0$ or $f(x) = g(x)$.
- Interpolation (Chap. 7)
  - Given the set of data

    $$\{(t_i, y_i)\}_{i=0}^n \qquad \text{or} \qquad \{(t_i, f(t_i))\}_{i=0}^n$$

    come up with a function $g(t)$ that approximates the data.

CHAPTER

Computer Arithmetic and Computational Errors 2

2.1 Numerical Stability

There is only finite space in computer. How would we store π, an irrational number? We can't. We can only store an approximation of π. How does introduction of approximations affect the accuracy of our computations?

Example. Suppose we want to compute the value for the sequence of integrals

$$y_n = \int_0^1 \frac{x^n}{x+5}\,dx$$

for $n = 0, 1, 2, \ldots, 8$, with 3 decimal digits of accuracy.

There are several properties that I can claim:

- $y_n > 0$ for all n, since the integrand $\frac{x^n}{x+5} > 0$ for all $x \in (0,1)$.
- $y_{n+1} < y_n$ for all n, since the integrand $\frac{x^{n+1}}{x+5} = x \cdot \frac{x^n}{x+5} < \frac{x^n}{x+5}$ for all $x \in (0,1)$.

There is not closed-form solution to this problem.

$$\begin{aligned}
x^n &= x^n \cdot \frac{x+5}{x+5} && \text{for } x \in (0,1)\\
x^n &= \frac{x^{n+1}}{x+5} + \frac{5x^n}{x+5}\\
\int_0^1 x^n\,dx &= \int_0^1 \frac{x^{n+1}}{x+5}\,dx + 5\int_0^1 \frac{x^n}{x+5}\,dx\\
\frac{1}{n+1}x^{n+1}\Big|_0^1 &= y_{n+1} + 5y_n\\
y_{n+1} &= \frac{1}{n+1} - 5y_n
\end{aligned}$$

Fortunately,

$$\begin{aligned}
y_0 &= \int_0^1 \frac{1}{x+5}\,dx\\
&= \ln(x+5)\Big|_0^1\\
&= \ln 6 - \ln 5\\
&= \ln\frac{6}{5} \doteq 0.182
\end{aligned}$$

By the recurrence,

$$\begin{aligned} y_1 &= \frac{1}{1} - 5y_0 \doteq 1 - 5(0.182) = 0.0900 \\ y_2 &= \frac{1}{2} - 5y_1 \doteq 0.5 - 5(0.0900) = 0.0500 \\ y_3 &= \frac{1}{3} - 5y_2 \doteq 0.333 - 5(0.0500) = 0.0830 \\ y_4 &= \frac{1}{4} - 5y_3 \doteq 0.25 - 5(0.0830) = -0.165 \end{aligned}$$

Clearly, something went wrong. We have a negative value for y_4, which is impossible. We also have $y_3 > y_2$. The problem is that we are using floating point arithmetic, which is not exact. We are losing precision in our calculations.

What if we leave y_0 as an unevaluated term?

$$\begin{aligned} y_1 &= 1 - 5y_0 \\ y_2 &= \frac{1}{2} - 5y_1 \\ &= -\frac{9}{2} + 25y_0 \\ y_3 &= \frac{1}{3} - 5y_2 \\ &= \frac{137}{6} - 125y_0 \\ y_4 &= \frac{1}{4} - 5y_3 \\ &= -\frac{1367}{12} + 625y_0 \end{aligned}$$

We approximated $y_0 = \ln \frac{6}{5} \approx 0.182$. We know that the true value of $y_0 \in [0.1815, 0.1825]$. Another way to express y_0 is $y_0 = 0.182 + E$, where $|E| \leq 0.0005 = 5 \times 10^{-4}$ is the error in our approximation.

Substituting this into the formula for y_4, we get

$$\begin{aligned} y_4 &= -\frac{1367}{12} + 625(0.182 + E) \\ &= -113.91\dot{6} + 113.75 + 625E \\ &= -0.1\dot{6} + 625E \end{aligned}$$

where

$$625E \leq 625 \times 5 \times 10^{-4} = 0.3125$$

and

$$y_4 < y_0 \doteq 0.182$$

so our propagated error is greater than the quantity to compute. ♢

A lesson learned from the previous example is that the math textbook algorithms does not necessarily produce good computational algorithms. This algorithms for computing y_n is said to be an **numerically unstable algorithm**, since a small error was magnified by the algorithm. We want the algorithms to be **numerically stable**.

Definition 2.1.1 Numerically Unstable

An algorithm is said to be **numerically unstable** if the error in the output is not bounded by the error in the input.

Remark

In the previous example, a small error E is magnified by 5 each step.

Example (Cont.). We can re-arrange the recurrent relation

$$\begin{aligned} y_{n+1} &= \frac{1}{n-1} - 5y_n \\ 5y_n &= \frac{1}{n+1} - y_{n+1} \\ y_{n+1} &= \frac{1}{5}\left(\frac{1}{n+1} - y_{n+1}\right) \end{aligned}$$

We have bounded the error in the output by the error in the input, but we are at a disadvantage since we are computing backwards. How can we start this recurrent relation?

Recall that

$$y_{100} < y_{99} < \cdots < y_0 \doteq 0.182$$

We start by approximating $y_{100} \doteq 0$. We know the exact value is $y_{100} = 0 + \varepsilon$, where $0 < \epsilon < 0.182$. Then,

$$\begin{aligned} y_{99} &= \frac{1}{5}\left(\frac{1}{100} - y_{100}\right) \\ &= \frac{1}{5}\left(\frac{1}{100} - (0+\varepsilon)\right) \\ &= \frac{1}{500} - \frac{\varepsilon}{5} \end{aligned} \qquad \begin{aligned} y_{98} &= \frac{1}{5}\left(\frac{1}{100} - y_{99}\right) \\ &= \frac{1}{5}\left(\frac{1}{100} - \left(\frac{1}{500} + \frac{\varepsilon}{5}\right)\right) \\ &= \cdots + \frac{\varepsilon}{25} \end{aligned}$$

We observe that the effect of the error is ε is diminished by a factor of $\frac{1}{5}$ each step. This is a numerically stable algorithm. By the time we get to y_8, we can expect an accurate result. ♢

Definition 2.1.2 Numerical Stability

An algorithm is said to be **numerically stable** if the error in the output is bounded by the error in the input.

2.2 Floating Point Arithmetic

2.2.1 Floating Point Representation

We only have finite space in the computer. This means we cannot store all real numbers, only an approximation of them. We use the **floating point representation** to store real numbers.

Definition 2.2.1 Floating Point Representation

A real number x is represented in the form

$$fl(x) = \pm d_0.d_1d_2\ldots d_{p-1} \times \beta^e \qquad d_0 \neq 0$$

where m is the **significant** or **mantissa**, β is the **base**, and e is the **exponent**.

Note:

- d_i's are bounded by
$$0 \leq d_i < \beta.$$
- p is the precision of the accuracy.
- E is also bounded by
$$L \leq E \leq U$$

With the floating point representation, we have a finite set of floating point numbers

$$F(\beta, p, L, U) = \{\pm d_0.d_1 d_2 \dots d_{p-1} \times \beta^E : 0 \leq d_i < \beta, L \leq E \leq U\}.$$

Remark $d_0 \neq 0$

We observe that this representation is not unique. Consider a system

$$F(\beta = 10, p = 5, L = -10, U = 10)$$

and the number 1.23, we have the representation

$$+1.2300 \times 10^0 \in F(10, 5, -10, 10)$$

but also

$$+0.1230 \times 10^1 \in F(10, 5, -10, 10)$$

and

$$+0.0123 \times 10^2 \in F(10, 5, -10, 10)$$

Thus, we need to choose a unique representation for each number. We can add a rule that **the first digit is not zero**, which normalizes the floating point representation, and makes representations unique.

Remark Representating Zero

Since $d_0 \neq 0$, how can we represent zero? We define another rule

$$fl(0) = 0.000 \dots 0 \times \beta^{L-1}$$

where **an exponent of $L - 1$ is used to indicate the value is denormalized**.

In the early days of computing, different manufacturers used different floating point representations. This made it difficult to write portable code. The IEEE 754 standard [2] was introduced to standardize floating point representations.

	Single Precision	Double Precision
β	2	2
p	24	53
L	-126	-1022
U	127	1023

Note:

For single precision, we have
$$E \in \{-126, -125, \dots, 127\}$$
with a cardinality of 254. If we use 8 bits to represent the exponent, we have $2^8 = 256$ possible

values. We have 2 special values: $E = L - 1 = -127$ for denormalized numbers, and $E = U + 1 = 128$ for infinity and NaN (Not a Number).

Remark Memory Requirements

For single precision, we need $1 + 24 + 8 = 33$ bits to represent a number. This is a very weird number of bits. We don't know any computer that uses 33 bits to represent a number.

Turns out we only need 23 bits for the significant, since the first digit in a normalized system is always 1. We can drop the first digit, and use 23 bits for the significant, 8 bits for the exponent, and 1 bit for the sign. This gives us 32 bits, which is a more common number of bits.

Note: Types in Different Languages

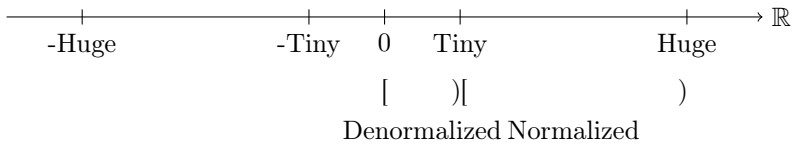
	Single Precision	Double Precision
C	`float`	`double`
Python		`float`
Rust	`f32`	`f64`

Note: Extended Precision and Reduced Precision

For accurate engineering, we may need more precision, called **extended precision** (Quad Precision). This is not part of the IEEE 754 standard, but is available in some systems.

In machine learning systems, we want the opposite – we want to use less precision to save memory and computation time. This is called **reduced precision**, and we often use 16 or 8 bits. This is a topic of active discussion, and the IEEE committee is working on a standard for reduced precision for ML.

Note that the distribution of floating point numbers is not uniform. There are more floating point numbers near zero than near $\pm\infty$. This is because the exponent is distributed uniformly, but the significant is not.



	Single	Double
HUGE	$+1.11\ldots1 \times 2^{127} \approx 3.4 \times 10^{38}$	$+1.11\ldots1 \times 2^{1023} \approx 1.8 \times 10^{308}$
TINY	$+1.00\ldots0 \times 2^{-126} \approx 1.2 \times 10^{-38}$	$+1.00\ldots0 \times 2^{-1022} \approx 2.2 \times 10^{-308}$

Remark

What about $x \in \mathbb{R}$, $x >$ HUGE? In this case, we have an overflow, and the computer will return $fl(x) = +\infty = 1.00\ldots0 \times \beta^{u+1}$.

Remark

What happens if we have $(+\infty) - (+\infty)$? In this case, we have an indeterminate form, and the

computer will return NaN (Not a Number),

$$1.xx\ldots x \times \beta^{u+1}$$

where at least one of the x's is not zero.

Remark

For $0 \leq x \leq$ TINY, take

$$fl(x) = \begin{cases} 0.00d_i d_{i+1} \ldots d_{p-1} \times \beta^{L-1} \\ 0 \qquad\qquad\qquad\qquad\qquad \text{eventually} \end{cases}$$

Example. Suppose we have

$$F(\beta, p, L, U) = F(10, 3, -10, 10)$$

and

$$x = 3.00 \times 10^5, y = 3.00 \times 10^5, \qquad x, y \in \mathbb{R}$$

Computing

$$z = x^2 + y^2$$

would give

$$z = 1.60 \times 10^{11} + 9.00 \times 10^{10} \notin F(10, 3, -10, 10)$$

and hence $fl(z) = +\text{Inf}$.

What if we instead want to compute

$$z = \sqrt{x^2 + y^2}?$$

Using the standard algorithm, we get +Inf inside the square root, and the computer will return NaN. We need to repair our algorithm.

$$h = \max(|x|, |y|) \times \sqrt{1 + \left(\frac{\min(|x|, |y|)}{\max(|x|, |y|)}\right)^2}$$

♢

Remark

The `libm` library in C and the `math` module in python both have a function called `hypot` that computes the Euclidean distance. This is a numerically stable algorithm that avoids overflow and underflow.

Note: Inf Could Have Meanings

Suppose you are computing the slope of a line, and you get infinity. This would indicate that the line is parallel to the y-axis. This is a meaningful result, and not an error.

Note: Sometimes, Underflow is OK

Recall the McLaurin series for

$$y = 1 + x + \frac{x^2}{2}$$

Suppose we are computing in $F(10, 3, -10, 10)$.

$$y = 1.00 \times 10^0 + 1.00 \times 10^{-5} + 5.00 \times 10^{-11}$$

The last term would underflow, but that's OK. We can ignore it, since it is negligible.

2.3 Computational Errors

2.3.1 Intuition

Example. We know that $\sqrt{255} = 15.9687194227\ldots$, but how should we represent this in our floating point system $F(\beta, p, L, U) = F(10, 3, -10, 10)$?

The easiest solution is **chopping** / **truncation**, where we only keep the first p digits. This gives us

$$fl(\sqrt{255}) = 1.59 \times 10^1$$

Another approach is **round to nearest**, which gives us

$$fl(\sqrt{255}) = 1.60 \times 10^1$$

♢

Remark

Heath called $x \in \mathbb{R} \to fl(x) \in F$ **rounding**. This is why we use **round-to-nearest** to disambiguate.

Note:

We can also round to +Inf or to −Inf

Note: Banker's Rounding

Banker's rounding is a method of rounding that rounds the last digit to the nearest even number, if it is exactly halfway between two numbers. This is the default rounding method in IEEE 754. This method is also known as **round to even**.

Example (Cont.). After rounding, we have error

$$fl(\sqrt{255}) - \sqrt{255} \doteq \begin{cases} -0.0687 & \text{chop} \\ 0.0313 & \text{round-to-nearest} \end{cases}$$

♢

Definition 2.3.1 Absolute and Relative Error

If $\tilde{x}$ is an approximation to x, the **absolute error in the approximation** is

$$|\tilde{x} - x|$$

and the **relative error** is

$$\frac{|\tilde{x} - x|}{|x|} \qquad x \neq 0$$

Example. The relative error of the approximation of $\sqrt{255}$ is

$$\doteq \begin{cases} 4.3 \times 10^{-3} & \text{chop} \\ 1.96 \times 10^{-3} & \text{round-to-nearest} \end{cases}$$

♢

> **Remark**
> Absolute error is not sensitive to scale, while relative error is.

Example. Let $x, y \in \mathbb{R}$, and they are approximated by $\tilde{x}, \tilde{y} \in F$.

Consider $x = 100.001, y = 0.112, \tilde{x} = 100., \tilde{y} = 0.113$. The absolute error is

$$|\tilde{x} - x| = 0.001, \quad |\tilde{y} - y| = 0.001$$

They absolute errors are the same, but this does not meas the two approximations are equally good. The relative error is

$$\frac{|\tilde{x} - x|}{|x|} \doteq 1.0 \times 10^{-5}, \quad \frac{|\tilde{y} - y|}{|y|} \doteq 9.0$$

The relative error for y is much larger than for x, so the approximation for y is worse. ♢

2.3.2 Machine Epsilon

If $x \in \mathbb{R}$ is approximated by $fl(x) \in F$, how large can the relative error be?

Example. Consider $F(10, 3, L, U), a \in \mathbb{R}$ with $a = w.xyz \times 10^E$ with $0 < w \leq 9$, $0 \leq x, y, z \leq 9$, and $L \leq E \leq U$.

Assume we determine $fl(a)$ using round to nearest,

$$|fl(a) - a| \leq 0.005 \times 10^E$$

The maximum relative error is

$$\begin{aligned} \max \frac{|fl(a) - a|}{|a|} &\leq \frac{\max |fl(a) - a|}{\min |a|} \\ &= \frac{0.005 \times 10^E}{1.000 \times 10^E} \\ &= 0.005 = \frac{1}{2} \times 10^{-2} \end{aligned}$$

♢

In general, for $x \in \mathbb{R}$ and $fl(x) \in F(\beta, p, L, U)$ with round to nearest, we have

$$\frac{|fl(x) - x|}{|x|} \leq \frac{1}{2} \times \beta^{1-p}$$

assuming x is representable.

This quantity is called the **relative round-off error bound**.

> **Definition 2.3.2** Machine Epsilon
> The **round-off error bound** or **machine epsilon** is the maximum relative error that can occur when approximating a number,
>
> $$\varepsilon_{\text{machine}} = \frac{1}{2} \times \beta^{1-p}$$

Remark

In IEEE 754, we have

$$\varepsilon_{\text{machine}} = \begin{cases} 2^{-24} \doteq 5.96 \times 10^{-8} & \text{Single Precision} \\ 2^{-53} \doteq 1.1 \times 10^{-16} & \text{Double Precision} \end{cases}$$

Example. Suppose we have $F(10, 3, -10, 10)$, $x = 1.51 \times 10^8$, and $y = 3.71 \times 10^6$, $x, y \in \mathbb{R}$, $fl(x) = x, fl(y) = y$.

We want to compute $z = x + y$. We have

$$z = 1.51 \times 10^8 + 3.71 \times 10^6 = 1.5471 \times 10^8 \notin F.$$

♢

Remark

Mathematically, addition is not closed in F.

Note:

In the CPU, there is extended precision in the floating point unit. This means that the CPU can store more digits than the standard floating point representation. This is useful for intermediate calculations, for example, the 0.0371×10^8 in the previous example, but the final result is rounded to the standard representation.

Example (Cont.).

$$fl(z) = \begin{cases} 1.54 \times 10^8 & \text{chop} \\ 1.55 \times 10^8 & \text{round-to-nearest} \end{cases}$$

We have the relative error

$$\frac{|fl(x+y) - (x+y)|}{|x+y|} = \begin{cases} 0.00459 & \text{chop} \\ 0.00187 & \text{round-to-nearest} \end{cases} \leq \varepsilon_{\text{machine}} = \begin{cases} 0.01 & \text{chop} \\ 0.005 & \text{round-to-nearest} \end{cases}$$

♢

Remark

For IEEE arithemetic, for $x, y \in F$ and operation op $\in \{+, -, *, /\}$, we must have

$$\frac{|fl(x \text{ op } y) - (x \text{ op } y)|}{|x \text{ op } y|} \leq \varepsilon_{\text{machine}}$$

and the square root of x

$$\frac{|fl(\text{sqrt}(x)) - \sqrt{x}|}{|\sqrt{x}|} \leq \varepsilon_{\text{machine}}$$

How does the error accumulates as we perform more operations?

Example.

- $\text{expr1} = \text{sqrt}(2.0) + \log(42.0)$
- $\text{expr1} = \exp(7.0) + \sin(45.0)$

We have

$$\varepsilon_{\text{expr1}} = \frac{|\text{expr1} - (\sqrt{2} + \ln(42))|}{|\sqrt{2} + \ln(42)|}$$

and

$$\varepsilon_{\text{expr2}} = \frac{|\text{expr2} - (e^7 + \sin(45))|}{|e^7 + \sin(45)|}$$

Suppose we compute

$$\text{expr1} + \text{expr2} \qquad \text{expr1} \times \text{expr2} \qquad \text{expr1} \div \text{expr2}$$

We know that

- $\varepsilon_{\text{expr1}+\text{expr2}} \leq \max(\varepsilon_{\text{expr1}}, \varepsilon_{\text{expr2}}) + |\epsilon_A|$
- $\varepsilon_{\text{expr1}\times\text{expr2}} \leq \varepsilon_{\text{expr1}} + \varepsilon_{\text{expr2}} + |\epsilon_B|$
- $\varepsilon_{\text{expr1}\div\text{expr2}} \leq \varepsilon_{\text{expr1}} + \varepsilon_{\text{expr2}} + |\epsilon_C|$

The first terms are error due to in exact operands. The last terms are due to the representing the result in floating point, and since they are representation errors, they need to be smaller than the machine epsilon,

$$|\epsilon_A|, |\epsilon_B|, |\epsilon_C| \leq \varepsilon_{\text{machine}}$$

♢

> **Remark**
> Error does not grow too quickly, but it does grow. This is why we need to be careful when performing many operations.

What about subtraction? That is a whole different story.

2.3.3 Catastrophic Cancellation

Example. Suppose we want to find the roots of

$$x^2 + (-320x) + 16 = 0.$$

From math textbooks, we know the quadratic formula

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Using $F(10, 4, L, U)$, we have

$$\begin{aligned}
r_1 &= \frac{3.200 \times 10^2 + \sqrt{1.024 \times 10^5 - 6.400 \times 10^1}}{2.000 \times 10^0} \\
&\doteq \frac{3.200 \times 10^2 + \sqrt{1.023 \times 10^5}}{2.000 \times 10^0} && \text{1 round off error} \\
&\doteq \frac{3.200 \times 10^2 + 3.198 \times 10^2}{2.000 \times 10^0} \\
&= \frac{6.398 \times 10^2}{2.000 \times 10^0} \\
&= 3.199 \times 10^2
\end{aligned}$$

Similarly,

$$\begin{aligned} r_2 &= \frac{3.200 \times 10^2 - \sqrt{1.024 \times 10^5 - 6.400 \times 10^1}}{2.000 \times 10^0} \\ &\doteq \frac{3.200 \times 10^2 - \sqrt{1.023 \times 10^5}}{2.000 \times 10^0} && \text{1 round off error} \\ &\doteq \frac{3.200 \times 10^2 - 3.198 \times 10^2}{2.000 \times 10^0} \\ &= \frac{2.000 \times 10^{-1}}{2.000 \times 10^0} \\ &= 1.000 \times 10^{-1} \end{aligned}$$

You can show that the exact roots are

$$r_1^* \doteq 319.950 \quad r_2^* \doteq 0.050 \qquad r_2^* = 0.05001$$

So the relative errors

$$\varepsilon_{r_1} \doteq 1.6 \times 10^{-4} \leq \varepsilon_{\text{machine}} = 5 \times 10^{-4} \qquad \varepsilon_{r_2} \doteq 1.0$$

♢

> **Remark**
> A machine epsilon of 1 tells us that the result is completely wrong. There is no digits of accuracy in the result.

Why would this happen?

$$r_2 = \frac{3.200 \times 10^2 - 3.198 \times 10^2}{2.000 \times 10^0}$$

We know that

- 3.200×10^2 is exact
- 3.198×10^2 has 2 rounding errors, $fl(\sqrt{fl(102400 - 64)})$
- 2.000×10^0 is exact

We wanted to subtract

$$3.200 \times 10^2 - 3.198\texttt{xxxxxx} \times 10^2$$

We see that the leading terms cancel out, and `xxxxxx` are insignificant in the intermediate result, but they are significant in the final result. This is know as the **catastrophic cancellation**.

> **Definition 2.3.3** Catastrophic Cancellation
> **Catastrophic Cancellation** is the loss of significance in the result in subtracting two possibly inexact quantities close in value.

> **Remark**
> To avoid catastrophic cancellation, avoid subtracting nearly equal inexact quantities.

Example (Cont.). So how to determine r_2 accurately?

We know that

$$(1x^2 - 320x + 16) = 1 \cdot (x - r_1)(x - r_2)$$

so

- $1 \cdot r_1 r_2 = 16$ when $x = 0$
- $r_2 = \dfrac{16}{r_1} = \frac{1.600 \times 10^1}{3{,}199 \times 10^2} = 5.002 \times 10^{-2} < \epsilon_{\text{machine}}$

♢

Remark

A better algorithm to compute roots of

$$ax^2 + bx + c = 0$$

is

$$r_1 = \frac{-b - \text{sign}(b)\sqrt{b^2 - 4ac}}{2a} \qquad r_2 = \frac{c}{ar_1}$$

Part II

Appendices

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