Assignment 4: Risk Management
Financial Engineering course

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Introduction

In the following, we are going to compare different Risk Measure techniques to compute Value at Risk and Expected Shortfall. Lastly, we price a Cliquet exotic option in presence of counterparty risk.

Main assumptions are frozen portfolios and independent returns in time.

Data

Data in *EUROSTOXX50Dataset.csv* report daily prices over several years :

Date	ABI.BR	AD.AS		VNAn.DE	VOWG_p.DE
2013-01-02	67.35	10.344663		NaN	179.00
2013-01-03	66.54	10.344663		NaN	178.90
2013-01-04	67.62	10.465124	...	NaN	178.85
2013-01-07	66.93	10.530375		NaN	175.95
...					

In the dataset are present some missing values, whose imputation was treated differently according to the case. In fact, missing values could be due to different trading days, in which case an *ffill()* procedure was used, or to different starting trading day. In this last case, especially in Point 2, the entire column was dropped.

Variance-covariance method for VaR and ES in linear portfolio

We are tasked with computing **daily VaR and ES at 99% significance level** for a given portfolio by exploitation of a **parametric approach**. Indeed, we suppose to know that losses have a peaked and heavier tails distribution with respect to a normal one, meaning a **t-Student** distribution, information of which we are going to make use in order to derive the necessary quantiles for VaR and ES.

We consider the 20^{th} *of February 2020* as valuation date and go back in time of 5y to obtain a complete structure of prices and, thus, returns.

Finally portfolio is equally weighted with *Adidas, Allianz, Munich Re and L'Oréal* shares and invests a *Notional = 1.5e7* Euros.

Here we present a brief description of the specific **log-returns** concerning the just mentioned firms:

count = 1281	ADSGn.DE	ALVG.DE	MUVGn.DE	OREP.PA
mean	0.001120	0.000343	0.000324	0.000410
std	0.015926	0.012607	0.011145	0.012429
min	-0.070476	-0.109689	-0.075783	-0.053523
max	0.106011	0.044669	0.055446	0.072910

Once computed both the mean μ and standard deviation σ for the entire portfolio, we take advantage of the peculiar formulas for VaR and ES calculation proposed by *McNeil end al.* to write:

$$VaR_\alpha = \mu + \sigma\tau_\nu^{-1}(\alpha)$$
$$ES_\alpha^{std} = \frac{(\nu + (\tau_\nu^{-1}(\alpha))^2}{(\nu-1)}\frac{\phi_\nu(\tau_\nu^{-1}(\alpha)}{(1-\alpha)}$$

It is now sufficient to integrate the data, obtaining the following results:

VaR	579488.97 Eur
ES	804155.47 Eur

The VaR represents approximately 3.86 % of the portfolio's total value, and the ES represents about 5.36 %. We observe the portfolio is heavily concentrated in specific sectors in the consumer good and insurance sectors, thus we believe further diversification could help reduce risk, beyond considering weights balancing to align with desired risk factors and reallocating investments into assets with lower risk according to the descriptive table introduced above.

HS & WHS Simulation, Bootstrap and PCA for VaR & ES in a linear portfolio

Taking as reference date the 20^{th} *of March 2019*, we move to compute risk measures at **significance level** $\alpha = 95\%$ using **non-parametric approaches**. Any assumption on the distribution of the losses is made, but we use data historically recorded in the previous 5 years. We will take into account three different portfolios adopting different approaches and checking results' order of magnitude via a **Plausibility Check**.

Historical Simulation and Bootstrap Approaches

We consider an equity portfolio containing several firms with relative shares: *Total (25K shares)*, *AXA (20K Shares)*, *Sanofi (20K Shares)*, *Volkswagen (10K Shares)*.

We infer:

Portfolio Value: 4731416.99 Eur

	TTEF.PA	AXAF.PA	SASY.PA	VOWG_p.DE
weights	0.27	0.10	0.33	0.30

While briefly describing the single **log-returns** distributions:

count = 1282	TTEF.PA	AXAF.PA	SASY.PA	VOWG_p.DE
mean	0.000049	0.000167	0.000063	-0.000182
std	0.014484	0.016272	0.013681	0.020569
min	-0.082287	-0.168196	-0.112447	-0.220877
max	0.072995	0.064341	0.056800	0.068753

Let us start considering **HS Simulation**: the procedure developed in the corresponding function consists in value portfolio losses, ordering them in decreasing order and, finally, apply the appropriate formulas à la *Ferguson (1996)*:

$$VaR_\alpha = L^{(\lfloor n(1-\alpha)\rfloor, n)}$$
$$ES_\alpha^{std} = mean(L^{(i,n)}, i = \lfloor n(1-\alpha)\rfloor, ..., 1)$$

where n corresponds to the number of losses observation taken into account. Here, n = 1282.

On the other hand, since it is not guaranteed that all the historical series are available, we explore **Statistical Bootstrap** approach with 200 simulations. Random sampling with replacement from the original dataset is performed and, then, risk measures are retrieved via HS Simulation approach on this reduced sample.

Finally, we can confront values obtained with the aforementioned approaches and check the order or magnitude via **Plausibility Check**, obtaining in Euro:

	HS	Bootstrap	Plausibility
VaR	95569.56	105439.80	91526.47
ES	143630.83	155425.75	

Results are similar using the three methods and VaR order of magnitude is plausible.

Weighted Historical Simulation Approach

An equally weighted equity portfolio containing *Adidas, Airbus, BBVA, BMW and Deutsche Telekom* assets is considered, having unitary notional and log-returns such that:

	ADSGn.DE	AIR.PA	BBVA.MC	BMWG.DE	DTEGn.DE
mean	0.000772	0.000648	-0.000305	-0.000144	0.000240
std	0.016233	0.017212	0.017691	0.015505	0.013548
min	-0.166886	-0.109994	-0.176490	-0.078271	-0.059115
max	0.106011	0.097824	0.070598	0.061626	0.062132

It is not said that all events in the past have the same importance. This eventuality is explored in case of **Weighted Historical Simulation**, where, letting $\lambda = 0.95$, we associate a new set of weights to losses decreasing in the past. We construct a sequence of weights $\{w_s\}_{s=t-n+1,\dots,t}$ such that:

$$w_s = C\lambda^{t-s}$$

where,

$$C = \frac{1-\lambda}{1-\lambda^n}$$

Being n = 1282 still.

We look for the minimum index $i^\star$ that satisfies the inequality:

$$\sum_{i=1}^{i^\star} w_i \leq 1-\alpha$$

Finally, we compute risk measurements as:

$$VaR_\alpha = L^{(i^\star,n)}$$

$$ES_\alpha = \frac{\sum_{i=1}^{i^\star} w_i L^{(i,n)}}{\sum_{i=1}^{i^\star} w_i}$$

Daily VaR and ES result in the following values, whose order of magnitude is checked via **Plausibility Check**:

	WHS	Plausibility
VaR	1.59%	1.93%
ES	2.15%	

As it can be deduced, the extent of potential extreme losses, given by the difference between VaR and ES (0.56%) is limited and the fact that these values are relatively close might indicate that the portfolio is subject to regular, not extremely volatile market conditions.

Principal Component Analysis

In this case, we have to compute **10days VaR and ES** for an equally weighted equity portfolio over the first 18 companies in EUROSTOXX50 with unitary Notional.

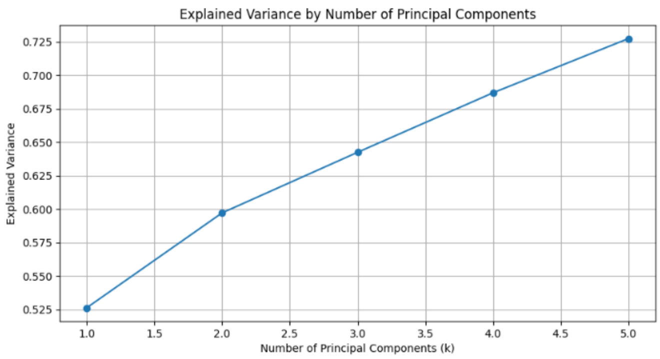
We undergo an approach based on **Principal Component Analysis**. The aim is to reduce the problem dimensionality considering **k components** at the time, with k going **from 1 to 5**.

The following table reports the obtained results:

PCA principal components k	VaR	ES
1	5.7879%	7.2152%
2	5.8221%	7.2510%
3	5.8203%	7.2492%
4	5.8155%	7.2447%
5	5.8172%	7.2464%
Plausibility Check	5.4471%	

The order of magnitude seems reasonable.

We can directly observe how much of the losses variance is explained according to different values of k:



Each component represents a pattern of variance across the portfolio's assets. The first component explains alone nearly 52%, while each subsequent component captures progressively less, up to nearly 72% of it considering their combination.

The changes in VaR and ES are relatively small as k increases, as the first few components capture the most significant variance patterns, adding more components may only slightly adjust risk measures.

We have efficiently simplified the problem we have taken into account.

Full Monte-Carlo and Delta normal VaR

In this exercise, we are in presence of a non linear derivative portfolio, which generates the urge to consider adapted approaches.

We take as reference date the 16^{th} *of January 2017* and a portfolio formed by stocks of *BMW* for **1,186,680€** and short the same number of call options (i.e **13714**).

Their expiry is set on the 18^{th} *of April 2017*, strike **K=25€**, volatility σ = 15.4%, dividend yield **d=3.1%** and a fixed interest rate **r=0.5%** for the period.

We compute VaR with $\Delta = 10$ **days and significance level** $\alpha = 95\%$. In order to simulate the underlying we use a 2y lapse in the past.

Full Monte-Carlo

It is known that for an equity derivative:

$$S_{t+\Delta} = S_t e^{X_{t+\Delta}}$$

Since we need to consider a 10days loss, in each MC simulation, we take randomly one value of the logreturn in the past and multiply by a factor of 10, assuming for simplicity that the logreturn remains constant in that time lapse. An alternative way is to simulate 10 values in the past and sum them, as it'll be done in the next approach. We then evaluate losses related to the option according to:

$$L^{der}(X_t; \Delta) = -\sum_i (C_i(t + \Delta) - C_i(t))$$

where i is the i-th derivative in our portfolio. In evaluating the call losses we recall having a short position and exploit black formula in order to compute the derivative price, including dividends. When computing the Call price in the 10days lapse we have to update the time to maturity, which has become 10 days shorter.

Then we sum up the loss of the shorted calls and the ones of the shares to get the losses of the portfolio.

Finally, we compute VaR via WHS with $\lambda = 0.95$ and it results being:

$$VaR = 4550.36 Eur$$

Full Monte-Carlo can be numerical intensive for an exotic derivative that cannot be priced via a closed formula since we would have to **simulate the entire path of the underlying rather than just the terminal price**. The complexity becomes even computationally worse if considering exotic derivatives, such as basket ones, where multiple paths have to be simulated.

Delta-Normal

In conclusion, we compute VaR via a Delta-Normal approach where we consider a first order expansion of derivatives losses:

$$L^{der}(X_t) = -\sum_{i=1}^{d} sens_i(t) * X_{t,i}$$

where $sens_i$ is the first order portfolio sensitivity.

In this specific case, the Δ_c of a call is given by:

$$\Delta_c = e^{-d*TTM} N(d_1)$$

Of course $\Delta_s = 1$.

Losses have to be considered in a span of 10 years. Then, here, we select randomly 10 values in the past of the logreturns, sum them and, finally, compute the value of the underlying ten days in the future by the aforementioned formula in the Full Monte Carlo point. We derive the losses and adjust with respect to the number of contracts held.

Summing up the losses of the shorted calls and of the stock we obtain the total loss of the portfolio and finally we computed the VaR via WHS with $\lambda = 0.95$.

It results being:

$$VaR = 771.47 Eur$$

We are not totally surprised by the underestimation of Delta Normal method. Indeed, Delta Normal Method assumes changes in portfolio value are normally distributed and linearly related to changes in risk factors, underestimating tail risk and lacking of computational complexity.

Moreover the portfolio is nearly **Δ-hedged** (not perfectly since the delta of the single call is not 1 but $\Delta = 0.993$).

As we can see the Delta-Normal can be improved, especially if the regulator wants a better estimation of the VaR, in this case we may consider a **Delta-Gamma** approach, considering also the cross-gamma terms in the computation of the losses.

Cliquet option

Our aim is to price a 7 year cliquet option with yearly payoff

$$[L \cdot S(t_i) - S(t_{i-1})]^+$$

where $L = 0.99$ is a participation coefficient, $S(0) = 1$, $Notion = 30MIO$ and volatility $\sigma = 0.20$, with no dividends paid.

Bank XX is buying this option, exposing itself to the counterparty risk of not receiving the yearly payoff due ISP default. Conversely, Bank ISP has no counterparty risk since Bank XX commits to pay the option in the settlement date, before ISP pays any of the payoff. In the computation of the price we are going to present results for the case we neglect the probability that ISP defaults and the case we consider the default probability of ISP.

For the Risk Neutral Valuation Formula, we have that

$$\Pi_0 = E^Q[L \cdot S(t_i) - S(t_{i-1})]$$

where Q is the risk neutral probability measure.

From the computation made in Appendix A, we obtain the following closed formula for the price of the option:

$$\Pi_0 = \sum_{i=1}^{6} \tau_i \frac{e^{-r(t_{i-1})(t_{i-1}-t_i)}}{B(0,t_{i-1})} C^{BS}(\frac{LB(0,t_{i-1})}{B(0,t_i)} e^{-r(t_i-t_{i-1})}, 1, \sigma)$$

where C^{BS} is the price of an european call option with expiry 1y computed with Black and Scholes model. It is important to notice that we approximated the price using a constant expiry of 1y for all the calls. For further information, check the attached code.

Substituting the values, we obtain the following results:

	no default	default
price	17606124.32 Eur	17378473.89 Eur

We notice that the price in default case is lower than the one in which we neglect the default scenario: this is reasonable since bank XX will pay less to enter in the contract knowing that ISP has the possibility to default.

Finally, ISP should try to sell the product at 17492299.11 Eur, that is the mid value between the two, as we can interpret the reported values respectively as the bid and ask in the market.

APPENDIX

APPENDIX A: Cliquet option pricing

We want to compute the price of the option

$$\Pi_0 = E^Q[L \cdot S(t_i) - S(t_{i-1})]^+$$

We notice that the form of the payoff recalls the one of an european call option. Following this idea, we collect $S(t_{i-1})$ in the expectation to get a constant "strike price"

$$E^Q[L \cdot S(t_i) - S(t_{i-1})]^+ = E^Q[S(t_{i-1}) \cdot (L \cdot \frac{S(t_i)}{S(t_{i-1})} - 1)]^+$$

We now want to take off $S(t_{i-1})$ from the expectation, but the two factors are not independent under Q. Hence we need to change the numeraire in order to have an expectation under a probability measure in which the two factors are independent. For this purpose, we choose the forward measure t_{i-1}. We now can bring out the factor since it is independent of the other one and it is $\mathcal{F}_{t_{i-1}}$ measurable and we get

$$E^Q[S(t_{i-1}) \cdot (L \cdot \frac{S(t_i)}{S(t_{i-1})} - 1)]^+ = S(t_{i-1})E^{t_{i-1}}[L \cdot \frac{S(t_i)}{S(t_{i-1})} - 1]^+$$

The expectation is now in the form of

$$E^{t_{i-1}}[S(t) - K]^+$$

where in our case $S(t) = L \cdot \frac{S(t_i)}{S(t_{i-1})}$ and $K = 1$.
Exploiting inter-bank market data, we write the price function with respect to discount factors, i.e.

$$S(t_{i-1})E^{t_{i-1}}[L \cdot \frac{S(t_i)}{S(t_{i-1})} - 1]^+ = \frac{1}{B(t_{i-1})}E^{t_{i-1}}[L \cdot \frac{B(t_{i-1})}{B(t_i)} - 1]^+$$

By these last remarks, computing the expectation, we finally get

$$\Pi_0 = \sum_{i=1}^{6} \tau_i \frac{e^{-r(t_{i-1})(t_{i-1}-t_i)}}{B(0,t_{i-1})} C^{BS}(\frac{LB(0,t_{i-1})}{B(0,t_i)} e^{-r(t_i - t_{i-1})}, 1, \sigma)$$

where $\tau_i = P(t_0, t_i) + \pi(1 - P(t_0, t_i))$, with P being survival probabilities and π being the recovery rate, is a term we use to weight the payoff with respect to the probability for ISP to default.