Assignment 6: Structured Products
Financial Engineering course

A.y. 2023-2024, Lecturer: Professor R. Baviera

Group 15: Ferrari Irene, Fioravanti Mattia,
Serva Giorgio Enrico Maria

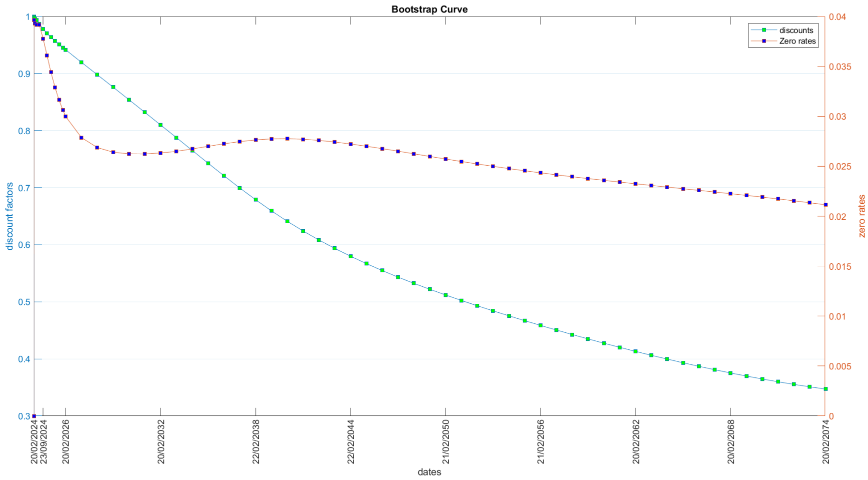
Introduction

In this assignment, we cope with a **structured bond** issued by *Bank XX* on the 16-feb-24 at 10:45 C.E.T.
We want to leverage a series of financial engineering techniques to assess and manage the risks associated with the product of interest, set within a single-curve interest rate modeling environment and without considering counterparty risk.
The terms of the contract are provided in its **termsheet**:

General Terms		Party A and B	
Principal Amount (N)	50 MIO €	Party A pays:	Euribor 3m + 2.00 %
Party A	Bank XX		Quaterly
Party B	I.B.	Party B pays:	
Trade date	today	@ Start Date	X% of N
Start Date	20-Feb-24	@ Payment dates	Coupons, Quaterly
Maturity Date (t)	15 y	First Quarter Coupon	3%
		Next Quarter Coupons	[0-5y] 3m € + 1.10% capped at 4.30%
			[5-10y] 3m € + 1.10% capped at 4.60%
			[10y-] 3m € + 1.10% capped at 5.10%

Bootstrap the market discounts

In order to proceed with the computations, we calculate the market discounts for the 16-feb-24 from various financial instruments.
Given the *Euro Interbank Mkt Zero Coupon Curve*, we start calculating mean rates for deposits, futures, and swaps by averaging bid and ask rates. For each selected date, we operate with a *modified time convention.* The control is performed inside *readExcelData-bootstrap* function which corrects holidays to the following business date.
We then use spline interpolation to fill in the gaps, creating a complete set of swap rates for up to 50 years.
Finally, we calculate discount factors handling the different types of market instruments and obtain the following shape:



Bootstrap curve

Pricing

We are provided with **flat volatilities**, which assume a constant volatility accross all maturities. This choice fails to reflect the true market conditions, thus we want to pass to **spot volatilities** aligning more closely with the market's varying expectations of risk and movement over time.

In order to bootstrap spot volatilities from flat ones, we need to derive *Cap market prices* from *Caplet prices.*

Caps Pricing

We calculate the prices of interest rate caps using a flat volatility assumption over the cap's entire term, i.e. the flat volatility stays the same for all the caplets within the specific cap. The pricing model is based on the **Bachelier** normal model to compute the value of each caplet.
Indeed, a cap consists of a series of caplets, each one providing the holder the right to receive a payment based on the difference between the reference rate and the strike rate.

$$Cap(T) = \sum_{i:T_i<=T} caplet(i, \sigma^F)$$

Here, σ^F refers, indeed, to the flat volatility.

We extract the European 3-month flat-vol structure from the market, which includes flat volatility values across various annual maturities. We exclude the row corresponding to the 18-month maturity and cover 13 different strikes.
Spanning the first 15 years, we calculate the prices of the caplets using the Bachelier formula.

Also known as **Normal Libor Market Model**, it supposes that the Libor rate evolves according to:

$$dL_i(t) = \underline{\sigma}_i \cdot dW_t^{(i+1)}$$

Collateral to that, the price of a caplet with expiry in T_{i+1} could be derived as:

$$caplet_i(t_0) = B(t_0, T_{i+1})\delta_i\{[L_i[t_0] - K]\mathcal{N}(d^N) + \sigma_i\sqrt{T_i - t_0}\phi(d^N)\}$$

where,

$$d^N = \frac{L_i(t_0) - K}{\sigma_i\sqrt{T_i - t_0}}$$

and σ_i is suppose to be the flat volatility relative to the maturity of the cap we are about to price.

When summing up the caplets prices as previously presented, it is crucial to remember that for each maturity, we should include all previous quarterly maturity caplets, except for the one with a 3-month maturity, which is excluded.

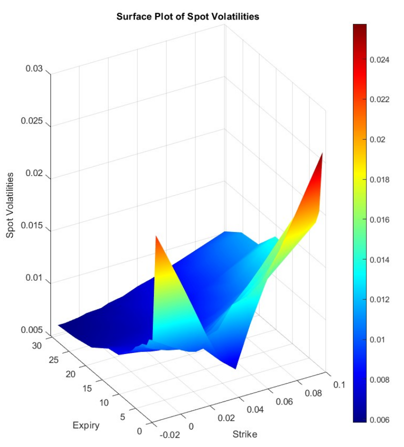
Bootstrap volatility

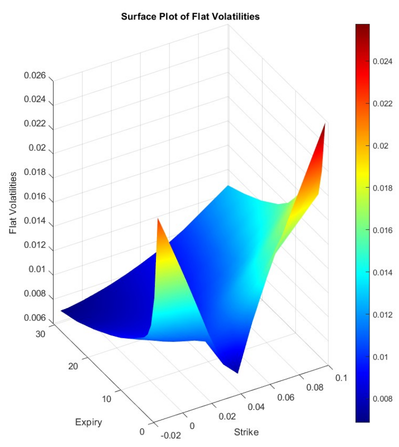
We proceed with the estimation of spot volatility structure from market data on cap volatilities. Firstly, we compute for each year the differences in cap prices to determine the annual increment on prices. Since in general,

$$\Delta C_{spot\ vol} = \Delta C_{flat\ vol}$$

Imposing this equality we are able to retrieve the spot volatilities. Indeed, we start by setting the initial spot volatility (related to the first year) values equal to the ones related to flat volatilities. For each year beyond, we solve the equation that balances the Bachelier formula-derived caplet price difference with the market observed price difference finding the roots of a set of non-linear equations employing *fsolve* routine.
For the intemid volatilities value, we exploit linear interpolation.

We are now able to compare the surfaces of flat and spot volatilities:





While the overall trends in volatility changes might be similar, they offer specific values and sensitivities, with spot volatilities reflectig a more dynamic marlet view. The flat volatilities, in contrast, offer a smoothed-out perspective that averages out sensitivities over time. Nonetheless, the curves exhibit a tendency to converge and overlay each other.

Pricing the structured bond

Ultimately, we calculate the upfront cost of the contract. The contract's termsheet reveals that it consists of two counterparties, each with their respective payment cashflows.

We delve with the different parties separately.

As regards *Party A*, it pays a floating leg and a spread of 2% quaterly. Thus, the relative net present value of its payments would turn out to be:

$$NPV_A = \sum_i E_0[\delta(t_{i-1}, t_i) D(t_0, t_i)(s^{spol_A} + L_{i-1}(t_0))]$$

which could be simplified as:

$$NPV_A = 1 - B(t_0, 15y) + s^{spol_A} * BPV_A$$

Respectively, $B(t_0, 15y)$ indicates the discount factor at maturity, s^{spol_A} is the aforementioned spread at 2% and BPV_A is the basis point value relative to its payments.

As regards part B, instead, the computation of the NPV is slightly more complex and needs to take into account the detailed coupon payment structure presented in the termsheet of the contract.

$$NPV_B = X\% + 3\% B(t_0, 3m)\delta(t_0, 3m) + s^{spol_B} * BPV_B([6m : 3m : 15y]) + B(t_0, 3m)$$

$$\text{- B}(\mathrm{t}_0, 15y) - Cap(0, 5y, k_1) - Cap(5y, 10y, k_2) - Cap(10y, 15y, k_3)$$

Delving into the problem, we know party B pays X % of the notional as upfront, then it pays 3 % in the first quarter and the next coupons are all given by the sum of the Euribor 3m plus 1.1 % capped to different rates. Specifically, these rates are:

$$q_1 = 4.3\% \quad q_2 = 4.6\% \quad q_3 = 5.1\%$$

Then, it can be seen that, for every i:

$$min(Eur_{3m} + 1.1\%, q_i) = Eur_{3m} + 1.1\% - max(Eur_{3m} - (q_i - 1.1\%), 0)$$

We recognize it to be equivalent to the Libor plus a fixed spread minus a caplet, allowing us to factorize the $s^{spol_B} = 1.1\%$ payments and simplify the calculation. Finally, we distinguish three buckets of caplets, each with strike $k_1 = 3.2\%$,
$k_2 = 3.5\% \;\; and \;\; k_3 = 4\%$. The values of spot volatilites corresponding to the first two

strikes are not yet available, thus we need to interpolate them. The interpolation will be linear over time and spline over strikes values.

Finally, we can retrieve the upfront by equating the two net present values and obtain that:

$$X = NPV_A - NPV_B = 18.56\%$$

Risk measures computation

Delta-bucket sensitivities

In the final section, following the practices of financial engineers, we calculate risk measures and mitigate the associated risks through hedging. Thus, our process begins with computing the delta-bucket sensitivities. Considering 4 deposits, 7 futures, and 18 swap rates for bootstrapping the discount curve, we derive a total of 29 delta-bucket sensitivities. Each sensitivity is determined by incrementally adjusting one of the aforementioned rates by 1 basis point, then iteratively performing the discount curve bootstrap, recalibrating spot volatilities, recomputing the "shifted" upfront value X, and finally calculating the delta-bucket sensitivity as the difference between the shifted and original values of X.

Below are the tables presenting the results obtained for each instrument we manipulated:

Depo Date	$\Delta - sensitivity$
21-feb-24	0
27-feb-24	0
20-mar-24	6.1285e-06
22-apr-24	1.2809e-06

Tabella 1: varying Depo rates

Future Expiry Date	$\Delta - sensitivity$
24-giu-24	1.9668e-05
23-set-24	1.4859e-05
20-dic-24	7.0648e-06
20-mar-25	2.7932e-06
23-giu-25	1.1287e-06
22-set-25	4.9101e-07
19-dic-25	2.1607e-07

Tabella 2: varying Future rates

Swap Date	$\Delta - sensitivity$
20-Feb-25	1.2332e-11
20-Feb-26	-6.4848e-06
22-Feb-27	-1.1777e-05
21-Feb-28	-1.3142e-05
20-Feb-29	7.2447e-06
20-Feb-30	-1.2696e-05
20-Feb-31	-1.4931e-05
20-Feb-32	-1.0836e-05
21-Feb-33	-1.5686e-05
20-Feb-34	4.4537e-05
20-Feb-35	1.7052e-05
20-Feb-36	-3.4925e-05
21-Feb-39	4.1153e-04
22-Feb-44	1.4326e-06
22-Feb-49	-3.5847e-07
20-Feb-54	7.6449e-08
20-Feb-64	-6.6957e-09
20-Feb-74	8.3677e-10

Tabella 3: varying Swap rates

From these tables we can make several **observations**: first of all, the **first 2 depo rates** do not contribute to the Δ since they do not affect the discounts we use in the pricing.
Moreover, also the **first swap** has a very low Δ since in the bootstrap we derive that discount from the futures and do not use that swap rate directly.
Finally we can see that the **highest** $\Delta - sensitivity$, in absolute value, is the one of the swap with **maturity 15 years**, which is the same of the certificate. In fact the next swaps, i.e the ones with maturity **longer than 15 years**, have a very low $\Delta - sensitivity$, but not exactly 0. We believe this is due to the fact that they affect the **spline** interpolation to obtain the complete set of swaps.

A **criticality** occurs when we scrutinize the accuracy of our model. Ideally, we expect the sum of the delta-bucket sensitivities to be equal to the total delta. However, upon using an increment of *1bp* for the rates, we observe a discrepancy: the sum of the delta-bucket sensitivities amounts to 4.1466e-04, while the total delta measures 3.164e-04. This single basis point difference could prove crucial, especially in the context of hedging the total delta of the portfolio.

To address this discrepancy, we reduced the increment to $1e-8$ and rescaled the difference to 1bp. We discovered that the sum of the delta-bucket sensitivities now measures 3.1558e-04, while the total delta is 3.1557e-04. These values exhibit the desired close resemblance. This seems reasonable since we are narrowing the shift, improving accuracy in delta sensitivity. We can conclude that h significantly influences the computation of sensitivities, highlighting its importance in the process.

Total Vega

Then we proceed to compute the total Vega of the certificate. We perform it by bumping up all the flat volatilities of the markets of $1e-8$ for every maturity and for every strike.

As in the previous point, we recalibrate the spot volatilities and compute the new upfront obtaining

$$\nu_{total} = (X_{shifted} - X)/h * 1\%$$

In our case, $\nu_{total} = 0.1119\%$.

We are aware of the possibility of deriving a closed formula for its computation. However, since we have exploited flat volatilities to better capture market movements, its deduction would have been over-complicated by the presence of composite derivatives. Thus, we opted for this procedure.

Vega-bucket sensitivities

We proceed with the analysis of the Vega, performing the Vega-bucket sensitivities, i.e we bump up by $1e-8$ all the flat volatilities relative to the same maturity for all the strikes, therefore at each iteration we bump up an entire row of the flat vols matrix and exploit the bucket-Vega through the shifted upfront as for the total Vega.

Caps maturity	$\nu - sensitivity$
1y	-2.152e-04
2y	-3.7793e-05
3y	1.2688e-05
4y	9.9459e-06
5y	1.0400e-03
6y	-2.7974e-04
7y	-1.6290e-04
8y	-4.7258e-04
9y	1.9250e-04
10y	2.4006e-03
12y	-2.8343e-04
15y	1.0967e-01
20y	0
25y	0
30y	0

Tabella 4: $\nu - bucket$ sensitivities varying flat vols

We observe the highest Vega-bucket sensitivity for a maturity of 15 years. However, for longer maturities up to 30 years, the sensitivity is exactly 0. This is because these longer maturities do not impact the spot volatilities used for pricing the certificate.

Hedging

Delta-bucket hedging

After computing the risk measures, we proceed to hedge the relative risks of our position in the certificate. Initially, we group the $\Delta - bucket$ sensitivities into four coarse-grained buckets: (0-2 years; 2-5 years; 5-10 years; 10-15 years). Then, we calculate the weights corresponding to each $\Delta - bucket$ sensitivity within these coarse-grained buckets and aggregate them.

The weights for the macro-buckets except the first and the last one are:

$$w_i^j = \frac{t_i - \hat{t}_{j-1}}{\hat{t}_j - \hat{t}_{j-1}} \quad when \quad k_{j-1} \leq i < k_j \quad and \quad k_j < i \leq k_{j+1}$$

$$w_i^j = 1 \quad when \quad i = k_j$$

While for the first it remains constant for the first two years and equal to 1, then decreases linearly up to 5y. Finally, for the last year, it increases linearly from 10y up to 15y, where it arrives to 1.

Here are the coarse-grained bucket deltas for our certificate:

Coarse-grained bucket	$\Delta - sensitivity$
0-2y	3.0034e-05
2y-5y	-5.3636e-05
5y-10y	-4.801e-05
10y-15y	3.8609e-04

Tabella 5: coarse-grained bucket Δ of our certificate

In order to hedge each coarse bucket we select 4 swaps with different maturities:(2y, 5y, 10y,15y), for each one we compute the 4 relative coarse-grained bucket Δ sensitivities, obtaining this 4x4 matrix:

Swap maturity:	2y	5y	10y	15y
Δ (0-2y)	-1.8227e-04	1.46398e-05	1.2933e-05	1.1648e-05
Δ (2y-5y)	0	-4.3961e-04	4.8569e-05	4.3463e-05
Δ (5y-10y)	0	0	-8.0715e-04	1.0148e-04
Δ (10y-15y)	0	0	0	-1.1239e-03

Tabella 6: coarse-grained bucket Δ of the swaps

As expected, this matrix is upper triangular because the swaps will only have a $\Delta \neq 0$ until their expiry. Consequently, they will have a $\Delta = 0$ for the other buckets.
Therefore, we can solely utilize the swap with a maturity of 15 years to hedge the 10-15 year bucket, and proceeding in reverse, we solve a 4x4 linear system to determine the positions for the other three swaps.
These positions are as follows:

Position on 2y swap:	8’918’253.42€	(0.18)
Position on 5y swap:	-4’492’220.59€	(-0.09)
Position on 10y swap:	-814’443.75€	(-0.02)
Position on 15y swap:	17’176’912.61€	(0.34)

Tabella 7: swaps positions

In the table we display both the positions we retrieve from the linear system between the brackets and also the position in €. In this case a positive position tells us that we enter in a swap as receiver of the fixed leg while a negative one as payers, this is also due to the way we computed the DV01 of the swaps We see it from the perspective of a receiver of the fixed leg, this also explains the signs in the table of the coarse-grained buckets Δ of the swaps.

Delta-Vega hedging

In this section, our aim is to hedge the total Vega according to our position in the certificate(we recall that $\nu_{total} = 0.1119\%$).

To achieve Vega hedging, we opt for an ATM 5-year Cap with a strike equal to the ATM 5-year Swap rate, which is 2.6908% .
Therefore, we price the cap with both the spot volatilities derived from, respectively, the shifted and non-shifted real flat cap volatility surfaces. Via linear interpolation on the expires and spline interpolation on the strikes we obtain the missing values from the bootstrap.

Subsequently, we obtain the Vega for this cap:

$$\nu = (Cap_{shifted} - Cap_{fixed})/h * 1\%$$

Where we consider an increment $h = 1e - 8$ for the forward numerical computation of Vega.

Then, we retrieve the position on this Cap easily, it is:

$$\#Cap_{5y} = \frac{-\nu_{total}}{\nu_{Cap5y}}$$

Ultimately, the position on the 5-year ATM Cap is calculated to be -222'812'512.42€ (-4.46), which aligns with our expectations.
In fact our position in the certificate is long on Caplets with various strikes and maturities ranging from 6 months to 15 years. This allows to think about and approximate it as being long on a 15-year Cap. This way, we explain the reason why we are short of -4.46 5-year ATM Caps with a notional of 50 million€.

As we're not satisfied with solely Vega hedging; we also aim to be Δ-hedged. Therefore we open a position against a 5y Swap quoted on the mkt, which we recall has $\nu = 0$ and thus does not affect the Vega hedging. We compute the total DV01 for both the Swap and the Cap, maintaining $h = 1e - 08$ for increased accuracy.

The computed DV01 values are as follows:

$$DV01_{Swap5y} \;=\; -4.2497e - 04, \quad DV01_{ATMCap5y} \;=\; 2.0683e - 04$$

We easily retrieve the swap position by imposing:

$$\Delta_{certificate} = \#Cap * DV01_{ATMCap5y} + \#Swap * DV01_{Swap5y}$$

Finally our position on 5y swap is -71313431.11€ (-1.43). Again, the sign suggests use to enter in a swap as payer of the fixed leg.

Vega-bucket hedging

Lastly we consider the course-grained buckets for the Vega (0-5y and 5y-15y), which we hedge with an ATM 5y Cap and an ATM 15 year Cap.
Here we follow an analogue procedure as in the Delta-bucket hedging, so we compute the bucket-Vega for the certificate and for the 2 Caps, i.e yearly bucket Vega and aggregate them through the weights of the 2 coarse-grained buckets and get the coarse-grained buckets Vega for the certificate and for the 2 Caps which we show here:

	Certificate	ATM 5y Cap	ATM 15 year Cap
ν bucket (0-5y)	0.00133	0.0251	-3.5103e-04
ν bucket (5y-15y)	0.11054	0	0.11958

Tabella 8: coarse-grained buckets Vega

All the values are as we expect: the 2 coarsed-grained Vega of the certificate sum up to the total Vega we computed above; the Vega(0-5y) of the ATM 5y Cap is the same of the point above and is 0 for the other bucket since the Cap expires and therefore we can use the ATM 15 year Cap to hedge the Vega(5y-15y) of the certificate.
So we have a straightforward linear system 2x2, again with an upper triangular matrix of the Vega of the Caps. By easily solving it, we retrieve the positions:

Position on 5y ATM Cap: -3290290.75 € (-0.07);
Position on 15y ATM Cap: -46218700.93 € (-0.92).

Also in this case following the same reasoning we have to short the Caps to hedge the Vega.

A **remarkable observation** here is that by dividing the risk sensitivities into more buckets, such as in the case of CGBs, the hedged portfolio becomes more diversified and necessitates lower financial leverage. Consequently, this can lead to reduced fees required to purchase all the financial instruments needed to hedge our certificate. Therefore, the approach of dividing our risk into coarse-grained buckets when hedging structured products based on interest rates appears to be significantly more efficient.