Some important equivalences

$$(\alpha \to \beta) \equiv (\neg\alpha \vee \beta)$$

$$(\alpha \vee \neg\alpha) \equiv \top$$

Contrapositive

- Let $\boldsymbol{A} \rightarrow \boldsymbol{B}$ be an **implication** (if A then B).
  - If a card has a J on one side then it has 5 on the other.
- Its **contrapositive** is $\neg\boldsymbol{B} \rightarrow \neg\boldsymbol{A}$.
  - If a card does not have 5 on one side then it cannot have J on the other.
- Contrapositive is equivalent to the original implication: $(\boldsymbol{A} \rightarrow \boldsymbol{B}) \equiv (\neg\boldsymbol{B} \rightarrow \neg\boldsymbol{A})$
  - This is why we need to check cards with numbers other than 5
  - Proof: $\neg B \rightarrow \neg A \equiv \neg\neg B \vee \neg A \equiv B \vee \neg A$
    $\equiv \neg A \vee B \equiv A \rightarrow B$

Converse and Inverse

- Let $A \to B$ be an **implication** (if A then B).
  - If a card has a J on one side then it has 5 on the other.
- Its **converse** is $B \to A$
  - If a card has 5 on one side, then it has J on the other.
- Its **inverse** is $\neg A \to \neg B$
  - If a card does not have J on one side, it cannot have 5 on the other.
- Converse is **not equivalent** to the original implication!
  - **For A=true, B=false, $A \to B$ is false, but $B \to A$ is true.**
- Converse is **not equivalent** to the negation of $A \to B$
  - **For A=true, B=true, $B \to A$ is true, but $\neg(A \to B)$ is false.**
- **Converse** is **equivalent** to the **inverse**. **Why?**
  - $(\neg A \to \neg B)$ is the **contrapositive** of $(B \to A)$

Example

$$(p \vee (p \wedge q)) \equiv p$$

(without using the absorption identity)

$$((p \wedge T) \vee (p \wedge q)) \qquad \text{identity}$$

$$(p \wedge (T \wedge q)) \qquad \left\{\begin{array}{l}\text{commutativity}\\ \text{distribution}\end{array}\right.$$

$$(p \wedge T) \qquad \text{identity}$$

$$p \qquad \text{identity}$$

Simplification of propositional wffs

Remove $\rightarrow$

Move $\neg$ to propositional variables

and cancel $\neg\neg$

Reduce number of propositional variables

Motivation

- Logic in computer programs

e.g., `if (x > 0 || (x <= 0) && y > 100)`

⋮

Even further, we can analyze code (see Huth & Ryan)

- Logic gates (and, or, not) on a chip

Steps in simplification

1. Change $(\alpha \rightarrow \beta)$ to $((\neg\alpha) \vee \beta)$
   (remember $(\alpha \leftrightarrow \beta) \equiv (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$)

2. Move $\neg$s to propositional variables and reduce number of propositional variables using logical equivalences (and other tautologies)

De Morgan's Laws are very important because they move $\neg$ from outside formula to inside formula

Simplifying formulas

- Start with the **outermost** connective and keep applying **de Morgan's laws** and **double negation**. Stop when all negations are on variables.

- **Example 1**: $(A \land C) \to (\neg B \lor C)$
  - By $(F \to G) \equiv (\neg F \lor G)$ (*let (A ∧ C) be F and $(\neg B \lor C)$ be G)
    - $(A \land C) \to (\neg B \lor C) \equiv \neg(A \land C) \lor (\neg B \lor C)$
  - De Morgan's law
    - $\neg(A \land C)$ is equivalent to $(\neg A \lor \neg C)$
  - So the whole formula becomes
    - $\neg A \lor \neg C \lor \neg B \lor C$
    - $\equiv \neg A \lor \neg B \lor \neg C \lor C$ //commutativity
    - but $\neg C \lor C$ is always **true**! Now we get $\neg A \lor \neg B \lor$ *True* ← use two applications of $\alpha \lor T \equiv T$
    - So the whole formula is ***True***, a **tautology**.

Simplifying formulas

- **Example 2:** $\neg\,(\,(\boldsymbol{A} \vee \neg \boldsymbol{B}) \rightarrow (\neg \boldsymbol{A} \wedge \boldsymbol{C})\,)$
  - $\equiv \neg\,(\,\neg(A \vee \neg B) \vee (\neg A \wedge C)\,)$ // →
  - $\equiv \neg\neg(A \vee \neg B) \wedge \neg(\neg A \wedge C)$ // de Morgan to ∨
  - $\equiv (A \vee \neg B) \wedge \neg(\neg A \wedge C)$ // double negation
  - $\equiv (A \vee \neg B) \wedge (\neg\neg A \vee \neg C)$ //de Morgan to ∧
  - $\equiv (A \vee \neg B) \wedge (A \vee \neg C)$ //double negation

  - $\equiv A \vee (\neg B \wedge \neg C)$ //distributivity, taking A outside the parentheses

Simplifying formulas

- **Example 3:** $(\boldsymbol{A} \wedge \neg \boldsymbol{B}) \rightarrow (\boldsymbol{A} \vee \boldsymbol{B} \rightarrow \neg \boldsymbol{B})$
  - $\equiv \neg(A \wedge \neg B) \vee (A \vee B \rightarrow \neg B)$ //→
  - $\equiv \neg(A \wedge \neg B) \vee (\neg(A \vee B) \vee \neg B)$ //→
  - $\equiv (\neg A \vee \neg\neg B) \vee (\neg(A \vee B) \vee \neg B)$ //De Morgan to ∧
  - $\equiv (\neg A \vee B) \vee (\neg(A \vee B) \vee \neg B)$ //double negation
  - $\equiv \neg A \vee B \vee \neg B \vee (\neg(A \vee B))$ //associativity & commutativity
  - $\equiv \neg A \vee True \vee (\neg(A \vee B))$ //law of the excluded middle
  - $\equiv True$ //identity

Normal Forms

Conjunctive Normal Form (CNF)

- conjunction of disjuncts

$$(p \vee \neg q \vee r) \wedge (\neg p \vee \neg q) \wedge (\neg q \vee \neg r)$$

Disjunctive Normal Form (DNF)

- disjunction of conjuncts

$$(p \wedge q) \vee (p \wedge \neg q \wedge \neg r) \vee (q \wedge \neg r)$$

Terminology

(propositional) variable – p

literal – $p, \neg p$ (a variable or a negated variable)

clause (or maxterm) – $(p \vee \neg q \vee r)$

term (or min term) – $(p \wedge \neg r)$

So, CNF is a conjunction of clauses that contain literals.

We only look at CNF.

Every wff can be converted to its CNF and its DNF (i.e., the CNF and the DNF are logically equivalent to the original wff)

Convert to CNF

1. Remove $\leftrightarrow$ $\quad (\alpha \leftrightarrow \beta) \equiv (\alpha \to \beta) \wedge (\beta \to \alpha)$
2. Remove $\to$ $\quad (\alpha \to \beta) \equiv ((\neg\alpha) \vee \beta)$
3. Use de Morgan's laws to make all ~~negations~~ literal negations or double negation
4. Remove double negations
5. Use distributivity to get the correct form

Convert to CNF

- **Example 2:** $\neg\,(\,(\boldsymbol{A} \vee \neg \boldsymbol{B}) \rightarrow (\neg \boldsymbol{A} \wedge \boldsymbol{C})\,)$
  - $\equiv \neg\,(\,\neg(A \vee \neg B) \vee (\neg A \wedge C)\,)$ // $\rightarrow$
  - $\equiv \neg\neg(A \vee \neg B) \wedge \neg(\neg A \wedge C)$ // de Morgan to $\vee$
  - $\equiv (A \vee \neg B) \wedge \neg(\neg A \wedge C)$ // double negation
  - $\equiv (A \vee \neg B) \wedge (\neg\neg A \vee \neg C)$ //de Morgan to $\wedge$
  - $\equiv (A \vee \neg B) \wedge (A \vee \neg C)$ //double negation