More preliminaries

The order of dislike quantifiers is **very important**

$$\underbrace{\exists x \forall y}_{\text{there exists an } x \text{ such that every } y} Q(x,y) \Longrightarrow$$

there is a member of $\mathcal{D}$ such that every member of $\mathcal{D}$ is in the Q relation with it

$$\underbrace{\forall x \exists y}_{\text{for every } x \text{ there exists a } y} Q(x,y) \Longrightarrow$$

for each member of $\mathcal{D}$ there is a member of $\mathcal{D}$ that is in the Q relation with it

Even more Preliminaries

When we have multiple quantifiers of the same sort we normally collapse the variables into the single quantifier (simply a notational convenience)

$$\forall x \forall y \forall z \,(P(x) \vee Q(y,z))$$

$$\forall xyz \,(P(x) \vee Q(y,z))$$

$\forall y \forall x \forall z$
$\forall z \forall x \forall y$ etc.

$$\exists y \exists z \, Q(y,z)$$

$\exists z \exists y$

$$\exists yz \, Q(y,z)$$

in any order

Also, the quantifiers can be written in any order

Entailment

Just like we had an entailment relation in propositional logic we have an entailment relation in FOL.

In FOL the notion of entailment in based on models (i.e., interpretations that make a sentence or set of sentences true)

Written: $\underbrace{\Gamma}_{\text{set of sentences}} \models \underbrace{\alpha}_{\text{sentence}}$

to be interpreted: in all models of Γ, α is true (or all models of Γ are models of α)

Validity can be written as

$\models \alpha$

this is true

in all models in which this is true (i.e., every interpretation)

i.e., α is true in all interpretations

Meta-syntax for substitution

In the inference rules for the quantifiers we will want to make substitutions for quantified variables.

We will use the following notation:

$\{v/t\}$ where v is a variable and t is a term

if t is a variable-free term

it is also called a **ground term**

To show the substitution in a formula α

$\alpha\{v/t\} \implies$ substitute t for v throughout α

System of natural deduction

In addition to the inference rules that we have for propositional logic we have 4 rules for quantifiers

1) Universal instantiation (also called $\forall$-elimination)

$$\frac{\forall v\ \alpha}{\alpha\{v/t\}}$$

2) Existential generalization (also called $\exists$-introduction)

$$\frac{\alpha\{v/t\}}{\exists v\ \alpha}$$

These two rules are quite straight forward.

3) Existential instantiation

$$\frac{\exists v\ \alpha}{\alpha\{v/t\}}$$ where t has not occured earlier in the proof

4) Universal generalization

$$\frac{\alpha \text{ (which contains } t \text{ that does not occur in premises)}}{\forall v\{t/v\}}$$

Soundness and Completeness of FOL

We have been provided with the propositional rules of inference and the 4 rules of inference for the quantifiers.

These rules of inference plus how they can be used (very informal presentation) provides a Natural Deduction system for FOL

This natural deduction system defines $\vdash$ for FOL.

We also have a definition of the semantic entailment operator $\models$

Soundness/Completion (continued)

We now have the tools to talk about soundness and completeness.

$$\Gamma \vdash \alpha \implies \Gamma \models \alpha \qquad \text{(soundness)}$$

I have tried to give an informal argument for the soundness of $\vdash$ (a formal proof would require structural induction

$$\Gamma \models \alpha \implies \Gamma \vdash \alpha \qquad \text{(completeness)}$$

This was proved by Kurt Gödel in his PhD thesis (would require a large amount of background knowledge)

Undecidability of $\models$

The completeness of FOL, i.e., $\Gamma \models \alpha \Rightarrow \Gamma \vdash \alpha$, is an important statement about the logic

That is, there is a mechanical procedure (i.e., a procedure that can be carried out by a computer) that can demonstrate (i.e., a proof of) any sentence that is entailed by a set of sentences

However, given a sentence that is not entailed by a set of sentences, i.e., $\Gamma \not\models \alpha$, the procedure may not terminate with an answer.

Undecidability ...

So, this means that the question "Is α entailed by Γ?" for FOL is undecidable — there is no mechanical procedure that will answer the question for all α.

However, since there is a procedure that can answer the question for a subset of all α's, i.e., those that are entailed, we say that the problem (ie, give an an answer to the question) is semidecidable (or, recursively enumerable).

Deduction Theorem

Just as for propositional logic we have a deduction theorem:

If $\Gamma \cup \{\alpha\} \vdash \beta$ ⊛ then $\Gamma \vdash (\alpha \to \beta)$

⊛ the proper deduction theorem requires α to be without any free variables — i.e., is a sentence. But since in our discussions we are only considering sentences, I haven't stated this in the antecedent.