Some more inference rules

$$\frac{\alpha \qquad \neg\alpha}{\bot}$$ Contradiction

$$\frac{\bot}{\alpha}$$ $\bot$-Elimination

Derived Rules

We have our basic rules. And we have some small proofs. Let's use these small proofs without having to recreate them every time.

1.	α	Assumption
2.	$\vdots$	some proof sequence
$\vdots$	$\vdots$	
n.	$\bot$	
n+1.	$(\alpha \to \bot)$	$\to$-Introduction: 1–n
n+2.	$\neg\alpha \vee \top$	Equivalence: n+1
n+3.	$\neg\alpha$	Equivalence: n+2

$$\boxed{\begin{array}{c}\alpha \\ \vdots \\ \bot\end{array}}$$

$$\neg\alpha$$

$\neg$-Introduction

a derived rule

1. $\neg(\alpha \vee \neg\alpha)$ Assumption

2. α Assumption

3. $(\alpha \vee \neg\alpha)$ $\vee$-Introduction: 2

4. $\bot$ Contradiction: 3, 1

5. $\neg\alpha$ $\neg$-Introduction, 2–4

6. $(\neg\alpha \vee \alpha)$ $\vee$-Introduction: 5

7. $(\alpha \vee \neg\alpha)$ Equivalence: 6

8. $\bot$ Contradiction: 7, 1

9. $\neg\neg(\alpha \vee \neg\alpha)$ $\neg$-Introduction: 1–8

10. $(\alpha \vee \neg\alpha)$ Equivalence: 9

$$\frac{\qquad}{\alpha \vee \neg\alpha}$$

Law of the Excluded Middle

Note: proof by contradiction

Resolution - Another Proof System

This proof system has only one inference rule:

$$\frac{p_1 \vee p_2 \vee \dots \vee p_n \vee X \qquad q_1 \vee q_2 \vee \dots \vee q_m \vee \neg X}{p_1 \vee p_2 \vee \dots \vee p_n \vee q_1 \vee q_2 \vee \dots \vee q_m}$$

where the p_i and q_j, $i = 1, \dots, n$, $j = 1, \dots m$, and X and $\neg X$ are literals and the wff below the line is called the <u>resolvent</u>.

This is a derived rule from our Natural Deduction proof system.

Let's derive it.

$\{(p_1 \vee \dots \vee p_n \vee \chi),\ (q_1 \vee \dots \vee q_m \vee \neg\chi)\} \vdash (p_1 \vee \dots \vee p_n \vee q_1 \vee \dots \vee q_m)$

1. $(p_1 \vee \dots \vee p_n \vee \chi)$ Premise
2. $(q_1 \vee \dots \vee q_m \vee \neg\chi)$ Premise
3. $\chi \vee \neg\chi$ Law of the Excluded Middle

4. $\neg\chi$ Assumption
5. $(p_1 \vee \dots \vee p_n)$ $\vee$-Elimination: 1, 4
6. $(p_1 \vee \dots \vee p_n) \vee (q_1 \vee \dots \vee q_m)$ $\vee$-Introduction: 5

7. χ Assumption
8. $(q_1 \vee \dots \vee q_m)$ $\vee$-Elimination: 2, 7
9. $(p_1 \vee \dots \vee p_n) \vee (q_1 \vee \dots \vee q_m)$ $\vee$-Introduction: 8 + Equivalence

10. $(p_1 \vee \dots \vee p_n) \vee (q_1 \vee \dots \vee q_m)$ $\vee$-Elimination: 3, 4–6, 7–9

To make this proof system work (sound and complete) the propositional language that it works with must be in a certain form (note the wffs above the line in the inference rule are both disjunctions and the result below the line is also a disjunction).

But we have a previous result that provides this form:

Every wff can be converted to CNF.

Note:

$$\underbrace{\Gamma \vdash \alpha}$$

This is a set of wffs.

This is a wff.

Each wff can be converted to CNF.

We will now go one step further. Each wff in its CNF will be broken into it clauses. Why can we do this?

Now, the next step is to negate α.

We now use proof by contradiction to prove α.

In natural deduction form:

1.		Γ in CNF as premises
$\vdots$		
k.		
$k+1$		appropriate use of $\wedge$-Elimination to get all of the clauses
$\vdots$		
m		
$m+1$	$\neg\alpha$	Assumption
$\vdots$		These proof steps only use the resolution inference rule plus the clauses from lines $k+1$ to m
$m+n$	$\perp$	
	α	$\neg$-Introduction + Equivalence ($\neg\neg\alpha$)

This is the Resolution proof system

($\neg\alpha$ has to be converted to CNF you cannot convert α to CNF and then simply put a $\neg$ in front (would not be CNF))

Resolution

So,

1) take all of the wffs that are premises and generate the clauses
2) take $\neg\alpha$ and generate the clauses
3) take the union of the clauses from steps 1) and 2) and apply the resolution rule to the appropriate clauses, over and over, each time adding the resolvent to the set of clauses until a contradiction appears

$(p \vee s) \quad (q \vee \neg s)$

$(p \vee q)$

$(p \vee q \vee r) \quad (\neg r \vee s)$

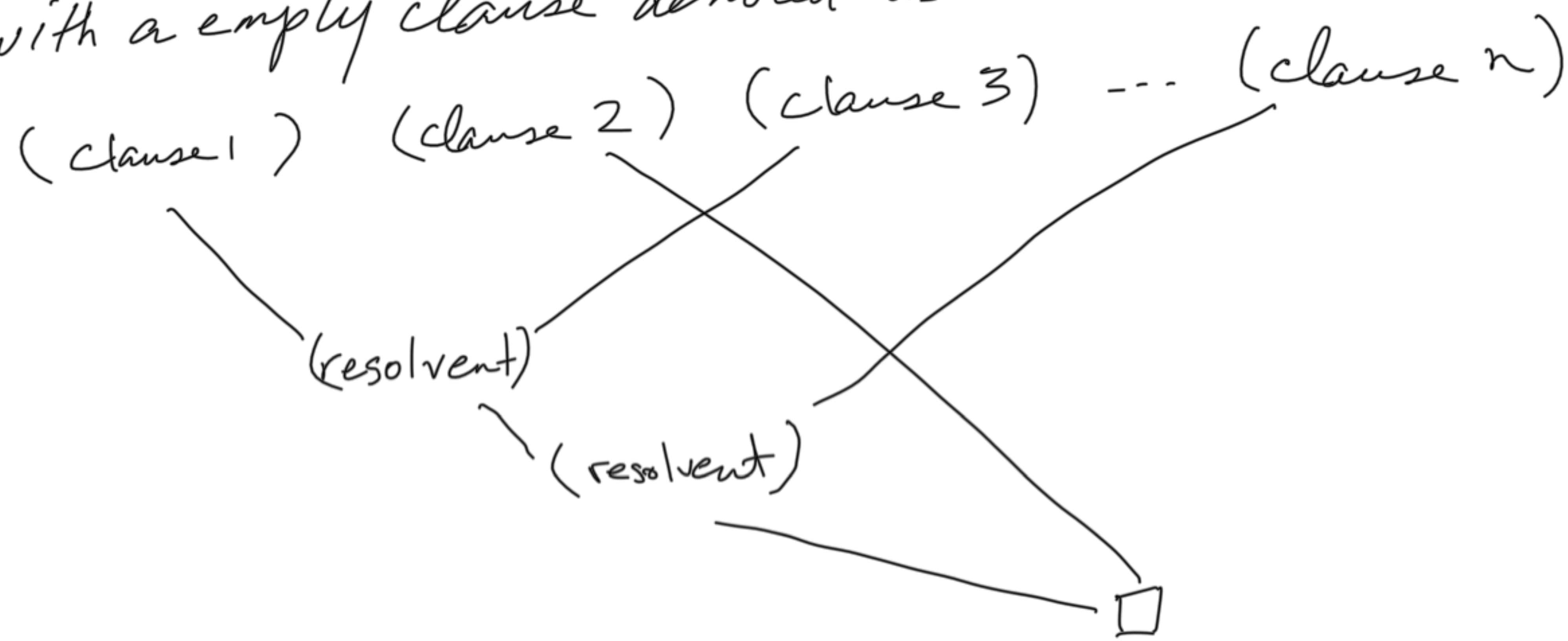
$(p \vee q \vee s)$

$(p \vee q \vee \neg r) \quad (\neg q \vee \neg r)$

$(p \vee \neg r)$

(note: remove duplicates)

The normal way of presenting a resolution proof is to list the clauses, then draw combinations reflecting the resolution rule, incorporating the new clause into the set of clauses, ending with a empty clause denoted as □.



Also, some presentations show the clauses as lists of literals rather than as disjunctions

eg $(p, \neg q, \neg r, s)$ $\quad\quad$ $(p \vee \neg q \vee \neg r \vee s)$

Prove Modus Ponens by resolution

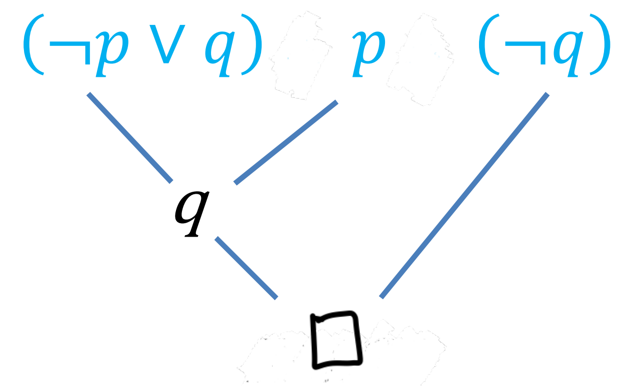
- **If** $\boldsymbol{p}$ **then** $\boldsymbol{q}$
- $\boldsymbol{p}$

$\therefore$ $\boldsymbol{q}$

- Prove by resolution:

$$(p \to q) \quad p \quad (\neg q)$$

$$(\neg \boldsymbol{p} \lor \boldsymbol{q}) \quad \boldsymbol{p} \quad \neg \boldsymbol{q}$$



Prove Hypothetical Syllogism by resolution

- If $\boldsymbol{p}$ then $\boldsymbol{q}$
- If $\boldsymbol{q}$ then $\boldsymbol{r}$

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$\therefore$ If $\boldsymbol{p}$ then $\boldsymbol{r}$

➢ Prove by resolution:

$(p \rightarrow q) \quad (q \rightarrow r) \qquad (\neg(p \rightarrow r))$

$(\neg \boldsymbol{p} \vee \boldsymbol{q}) \quad (\neg \boldsymbol{q} \vee \boldsymbol{r}) \quad \neg(\neg \boldsymbol{p} \vee \boldsymbol{r})$

$(\neg \boldsymbol{p} \vee \boldsymbol{q}) \quad (\neg \boldsymbol{q} \vee \boldsymbol{r}) \quad \boldsymbol{p} \quad \neg \boldsymbol{r}$ //De Morgan, double negation