# Semantics of FOL

First, we need to define an interpretation.

We will do this using some mathematical tools:

- sets
- equality
- functions

An interpretation is a mapping (total, into functions) from the language to sets.

# Interpretation

Objects $\mathcal{D} = \{d1, d2, d3, \ldots, dn\}$ or

$\{d1, d2, d3, \ldots\}$

nonempty finite or infinite set

(Domain or Universe)

Examples : blocks, natural numbers

Think of it simply as a set of things.

Interpretation of constants in the language

$\longrightarrow$ a mapping (total, into function) from constants to $\mathcal{D}$

Examples

$b1 \longrightarrow d3$

$b2 \longrightarrow d8$

etc.

$1 \longrightarrow d2$

$2 \longrightarrow d5$

etc.

Note:

$b1 \rightarrow d3$

$b2 \rightarrow d3$

$1 \rightarrow d2$

$2 \rightarrow d2$

but not

$\left.\begin{array}{l} b1 \rightarrow d3 \\ b1 \rightarrow d8 \end{array}\right\}$ not a function

Properties (unary predicates) sets of domain elements (finite or infinite)

Example $\text{Blue} = \{d2, d4, d8\}$

$P = \{d1, d2\}$

So, the interpretation of the predicate Blue is the set $\{d2, d4, d8\}$

(the world that is interpreting our language, there are three "Blue" objects which are listed in the set above

→ NOTE: this is just a subset of $\mathcal{D}$)

Relations (n-ary predicates)

sets of n-tuples formed with domain elements (finite or infinite)

Example $On = \{\langle d3, d5\rangle, \langle d2, d1\rangle\}$

So, the interpretation of the predicate On is the set of pairs of domain elements $\langle d3, d5\rangle$ and $\langle d2, d1\rangle$

For ternary predicates, the interpretation would be sets of 3-tuples of domain elements.

The same for n-ary predicates:

# Functions

- another way to refer to domain elements

Examples $f(a) = b$ if $a$ is mapped to domain element $d2$
and $b$ is mapped to domain element $d4$
then in our interpretation $f(d2) = d4$

similarly for n-ary functions

$$g(a, b) = c \qquad \left.\begin{array}{l} a \rightarrow d2 \\ b \rightarrow d4 \\ c \rightarrow d5 \end{array}\right\} \; g(d2, d4) = d5$$

# Truth in an Interpretation

$\mathcal{D} = \{d1, d2, d3, d4\}$

$P = \{d2, d3\}$

$Q = \{\langle d1, d4\rangle, \langle d4, d4\rangle\}$

$a \to d1$
$b \to d2$
$c \to d3$
$d \to d4$

In this interpretation

$P(a)$ is F(alse) $d1 \notin P$

$P(b)$ is T(rue) $d2 \in P$

$Q(a, d)$ is T $\langle d1, d4\rangle \in Q$

$Q(b, c)$ is F $\langle d2, d3\rangle \notin Q$

$\forall x\, P(x)$ is F because P does not contain all members of set $\mathcal{D}$

$\exists x\, P(x)$ is T because P contains at least one member of set $\mathcal{D}$

Now let's look at more complicated formulas.

# Truth (continued)

$$\underbrace{\forall x}_{\text{for every domain element}} \underbrace{(\neg P(x) \rightarrow Q(x,d))}_{\text{what is the truth value of this formula}}$$

if $x$ is $d1$, $\neg P(x)$ is T (because $d1 \notin P$) and $Q(x,d)$ is T (because $\langle d1, d4\rangle \in Q$)

so, the implication is T

if $x$ is $d2$, $\neg P(x)$ is F (because $d2 \in P$) so the implication is T

if $x$ is $d3$, $\neg P(x)$ is F (because $d3 \in P$) so the implication is T

if $x$ is $d4$, $\neg P(x)$ is T (because $d4 \notin P$) and $Q(x,d)$ is T (because $\langle d4, d4\rangle \in Q$)

so, the implication is T

Truth (still continued)

so, since the formula in the scope of $\forall x$ is true for every domain element

$$\forall x (\neg P(x) \rightarrow Q(x,d)) \text{ is } T$$

Let's try

$$\exists x ( \underbrace{P(x)}_{\text{this is } T} \wedge \neg \underbrace{Q(a,x)}_{\text{this is } T} )$$

if $x$ is $d3$

because $d3 \in P$ because $\langle d1, d3 \rangle \notin Q$

so this is true

so this is true because there is atleast one element in the domain which makes the conjunction T

# Truth in an Interpretation - Part II

Let's keep the $\mathcal{D}$ and the interpretations of P and Q the same as in the previous example but now map the constants differently:

$a \rightarrow d2$
$b \rightarrow d2$
$c \rightarrow d3$
$d \rightarrow d3$

So, now $P(a)$, $P(b)$, $P(c)$, $P(d)$ are all true because d2 and d3 are members of P

But $\forall x\, P(x)$ is still false

[Note: the truth of the quantifier depends on $\mathcal{D}$ and the interpretation of P not on the truth value of all of the statements about P containing all of the constants]

An interpretation in which a FOL sentence (or set of FOL sentences) is true is called a <u>model</u> (of that FOL sentence or set of FOL sentences).

## Satisfiability

A FOL sentence (or set of FOL sentences) is **satisfiable** iff it has a model (i.e., it is true in at least one interpretation)

## Validity

A FOL sentence (or set of FOL sentences) is **valid** iff every interpretation is a model (i.e., it is true in every interpretation)

## Unsatisfiable

A FOL sentence (or set of FOL sentences) is <u>unsatisfiable</u> if it has no models (i.e., it is true in no interpretation)

[Compare with satisfiable, tautology, and contradiction for Propositional Logic.]

Satisfiability, Validity, Unsatisfiability

$\forall x\, P(x) \rightarrow \exists y\, P(y)$ ①

$\exists x\, (P(x) \wedge Q(x, d))$ ②

$\exists x\, (P(x) \wedge \neg P(x))$ ③

Interpretation 1

$\mathcal{D} = \{george, mary, john\}$

$d \rightarrow mary$

$P = \{george, john\}$

$Q = \{\langle george, mary \rangle, \langle john, mary \rangle\}$

① is T

② is T

③ is F

Interpretation 2

$\mathcal{D} = \{block1, block2\}$

$d \rightarrow block1$

$P \rightarrow \{block1, block2\}$

$Q \rightarrow \{\langle block1, block2 \rangle\}$

① is T

② is F

③ is F