Promised shortcuts for formulas:

Remove outside matched parentheses

Remove matched parentheses surrounding negated wff

$$(A \wedge B) \Rightarrow A \wedge B$$

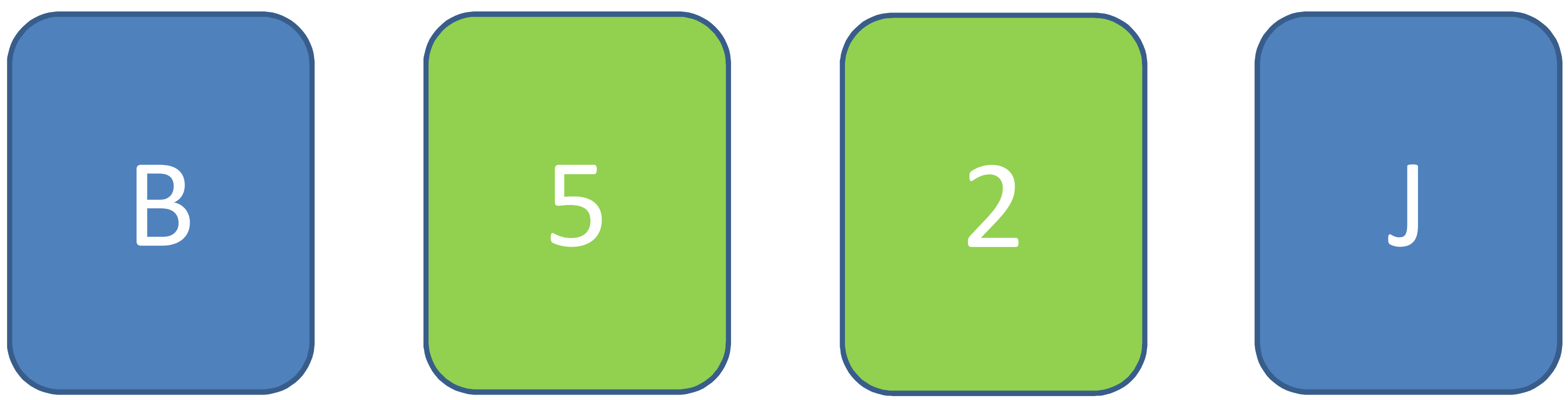
$$((\neg A) \wedge B) \Rightarrow (\neg A \wedge B) \Rightarrow \neg A \wedge B$$

$$(B \wedge (\neg A)) \Rightarrow (B \wedge \neg A) \Rightarrow B \wedge \neg A$$

Let's look at a puzzle
that can be represented
with our propositional logic.

The fun game

- You see the following cards. Each has a letter on one side and a number on the other.

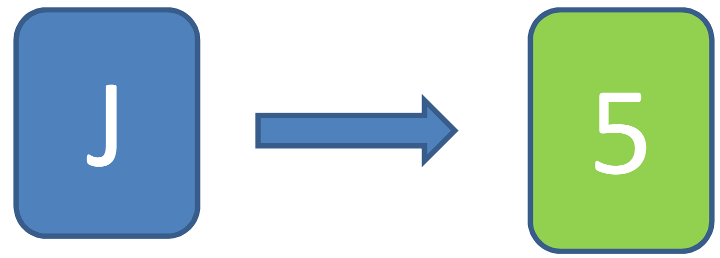


- Which cards do you need to turn to check that **“if a card has a J on it then it has a 5 on the other side”?**

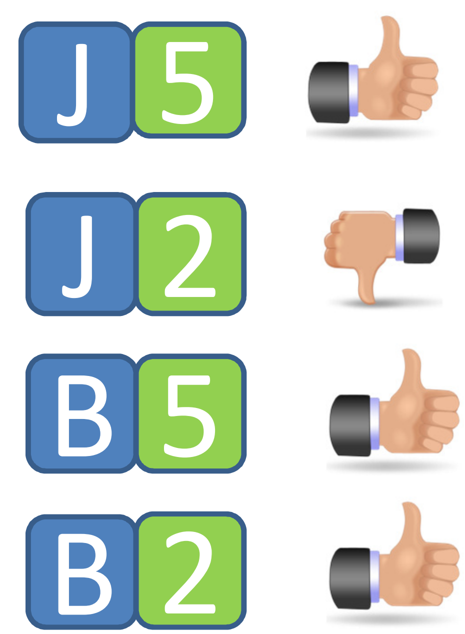
“if … then” in logic

- This puzzle has a logical structure:

“if A then B”



- What circumstances make this true?
  - A is true and B is true
  - A is true and B is false
  - A is false and B is true
  - A is false and B is false



A new binary connective

$\longleftrightarrow$

We want the meaning of this connective to be

$$((\alpha \to \beta) \wedge (\beta \to \alpha))$$

Truth table to define the meaning:

α	β	$(\alpha \to \beta)$	$(\beta \to \alpha)$	$((\alpha \to \beta) \wedge (\beta \to \alpha))$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

So, if we look at the truth table

$(\alpha \leftrightarrow \beta)$ has the truth value T

if α and β have the same truth

values and F if they differ

Also termed "if and only if" - abbreviated "iff"

if: if α then β

only if: α only if β (or if β then α)

Or if you like $\alpha \rightarrow \beta$ $\alpha \leftarrow \beta$

Introduce some new symbols:

2 object language symbols

$\top$ (this is not the letter 'T')

in logic: verum

in graph theory: top

This is a special proposition — it can only be True

$\bot$ in logic: falsum

in graph theory: bottom

This is a special proposition — it can only be False

$\equiv$ a meta-languge symbol — equivalent to

$\alpha \equiv \beta$ the wff α is equivalent to the wff β

What does it mean for two wffs to be equivalent?

If they have the same propositional variables we can say two wffs are equivalent if they have the same true tables.

Eg. $(\alpha \to \beta) \equiv ((\neg\alpha) \vee \beta)$

α	β	$(\alpha \to \beta)$
T	T	T
T	F	F
F	T	T
F	F	T

α	β	$(\neg\alpha)$	$((\neg\alpha) \vee \beta)$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

To have the same truth tables means that the two wffs have the same truth values (meaning) in all worlds.

So, what about tautologies and contradictions? They have the same truth values (always true or always false) in every world.

We saw that $(((p \to q) \wedge (q \to r)) \to (p \to r))$ is a tautology and you can easily show that $(s \vee (\neg s))$ is also a tautology but the truth tables for these two wffs are very different.

Let's use our knowledge what the biconditional means: $(\alpha \leftrightarrow \beta)$

If the biconditional is a tautology, this means that for every truth valuation, α and β have the same truth values. This is a reasonable idea of equivalence. It also captures both the idea of "same truth tables" and the equivalence of all tautologies and the equivalence of all contradictions.

Let's see for two examples:

$$(\alpha \to \beta) \equiv ((\neg\alpha) \vee \beta)$$

$$(\alpha \to \beta) \leftrightarrow ((\neg\alpha) \vee \beta)$$

I will just copy the appropriate columns from the previous truth tables.

α	β			$(\alpha\to\beta)\leftrightarrow((\neg\alpha)\vee\beta)$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

tautology

$$(((p \to q) \wedge (q \to r)) \to (p \to r)) \leftrightarrow (s \vee (\neg s)))$$

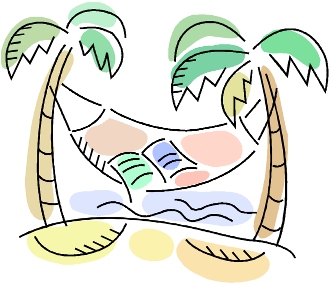
p	q	r	s			
T	T	T	T	T	T	T
T	T	T	F	T	T	T
T	T	F	T	T	T	T
T	T	F	F	T	T	T
T	F	T	T	T	T	T
T	F	T	F	T	T	T
T	F	F	T	T	T	T
T	F	F	F	T	T	T
F	T	T	T	T	T	T
F	T	T	F	T	T	T
F	T	F	T	T	T	T
F	T	F	F	T	T	T
F	F	T	T	T	T	T
F	F	T	F	T	T	T
F	F	F	T	T	T	T
F	F	F	F	T	T	T

tautology

similarly for contradictions (the two wffs will have F's in their respective columns, so the $\leftrightarrow$ column will be all T's)

Knights and knaves



- On a mystical island, there are two kinds of people: knights and knaves. Knights always tell the truth. Knaves always lie.

- Puzzle 1: You meet two people on the island, Arnold and Bob. Arnold says "Either I am a knave, or Bob is a knight". Is Arnold a knight or a knave? What about Bob?

Knights and knaves

- Puzzle 1: Arnold says "Either I am a knave, or Bob is a knight".
  Is Arnold a knight or a knave? What about Bob?
  To solve:
  - ***A***: Arnold is a knight
  - ***B***: Bob is a knight
  - Formula: $\neg \boldsymbol{A} \vee \boldsymbol{B}$: "Either Arnold is a knave, or Bob is a knight"
  - Want: scenarios where either both **A is a knight** and **the formula is true**, or **A is a knave** and **the formula is false**.
  - Use "if and only if" notation: $(\neg \boldsymbol{A} \vee \boldsymbol{B}) \leftrightarrow \boldsymbol{A}$
    True if both formulas have same value.

A	B	$\neg A$	$\neg A \vee B$	$(\neg A \vee B) \leftrightarrow A$
True	*True*	False	True	**True**
True	*False*	False	False	False
False	*True*	True	True	**False**
False	*False*	True	True	False

On the next slide

the symbol $\Longleftrightarrow$ is used instead of $\equiv$

and T is used instead of $\top$

and F is used instead of $\bot$

Logical identities

Name	$\wedge$-version	$\vee$-version
Double negation	$\neg\neg p \iff p$	
DeMorgan's laws	$\neg(p \wedge q) \iff (\neg p \vee \neg q)$	$\neg(p \vee q) \iff (\neg p \wedge \neg q)$
Commutativity	$(p \wedge q) \iff (q \wedge p)$	$(p \vee q) \iff (q \vee p)$
Associativity	$(p \wedge (q \wedge r)) \iff ((p \wedge q) \wedge r)$	$(p \vee (q \vee r)) \iff ((p \vee q) \vee r)$
Distributivity	$p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r)$
Identity	$p \wedge T \iff p$	$p \vee F \iff p$
	$p \wedge F \iff F$	$p \vee T \iff T$
Idempotence	$p \wedge p \iff p$	$p \vee p \iff p$
Absorption	$p \wedge (p \vee q) \iff p$	$p \vee (p \wedge q) \iff p$