$(\;)$ is not a wff (none of Rules 1, 2, 3 can be used to create this formula)

$(A($ is not a wff "

(A) is not a wff "

We will look at some commonly accepted shortcuts for acceptable wffs — but we'll do this later.

Unique readability

For any wff, there is only one way to apply Rules 1, 2, 3 to generate it.

[Take this as being true. You would need structural induction to prove it. See CS 2214.]

Eg. $$((\underbrace{p}_{\text{wff}} \wedge \underbrace{q}_{\text{wff}}) \to \underbrace{r}_{\text{wff}})$$

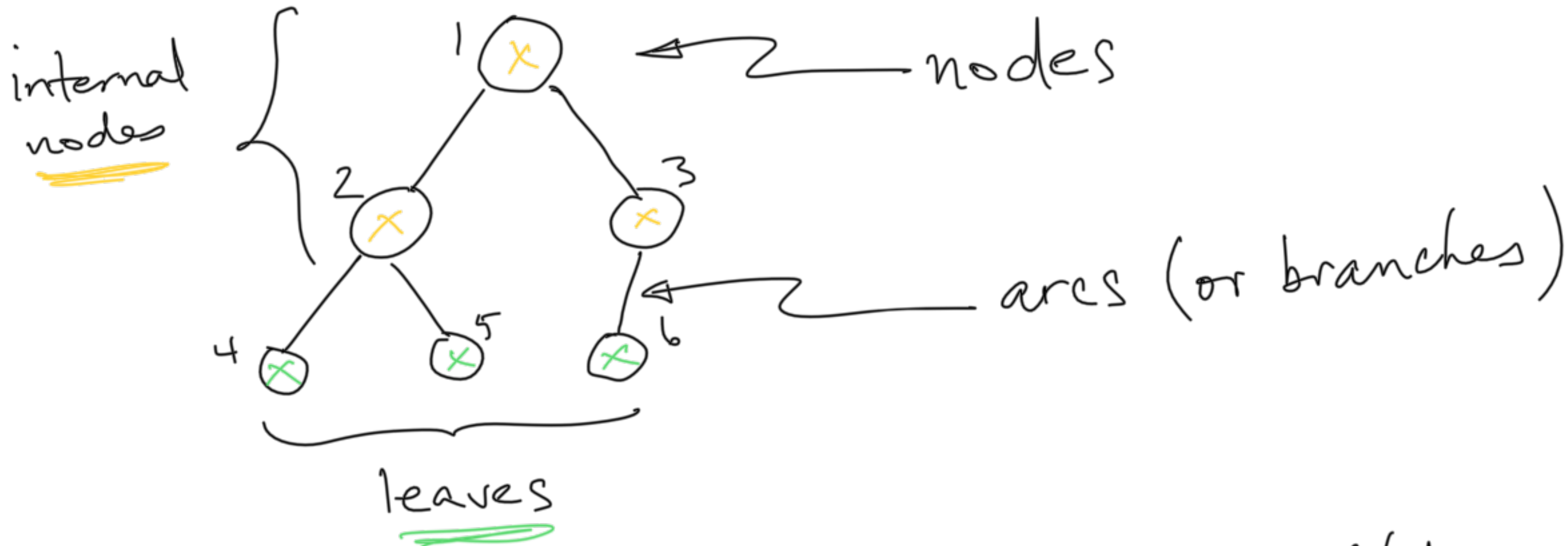
$$\underbrace{(\alpha \wedge \beta)}_{\text{wff}}$$

$$\underbrace{(\alpha \to \beta)}_{\text{wff}}$$

Try $\quad \Big(\underbrace{(p}_{\text{not a wff}} \wedge \underbrace{q) \to r}_{\text{not a wff}}\Big)$

So, we can represent this unique interpretation as a tree.

Trees (an aside) – binary trees in particular



leaves have no children

internal nodes have 1 or 2 children

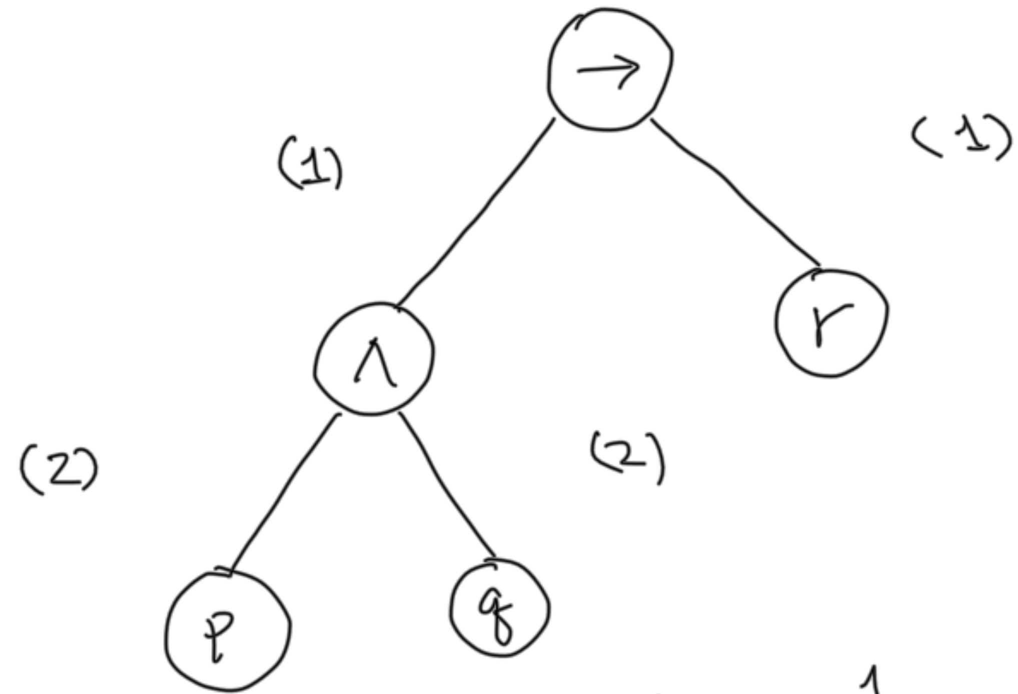
node 1 is the root of the tree

nodes 2 + 3 are children of node 1

nodes 4 + 5 are children of node 2

node 6 is a child of node 3

$((p \wedge q) \rightarrow r)$



inorder traversal :

$(lhs^1 \longrightarrow rhs^1)$

$((lhs^2 \wedge rhs^2) \longrightarrow rhs^1)$

$((p \wedge q) \longrightarrow r)$

Note:

- all leaves are propositional variables (and propositional variables can only be leaves)
- all internal nodes are connectives (with the root node being the last application of Rules 1, 2, 3)
- the parentheses (matching) are represented by subtrees

Remember one of goals:

- how to decide whether a proposition is <u>true</u> or <u>false</u>

Because we have moved to an abstract language this can be stated:

- how to decide whether a <u>wff</u> is <u>true</u> or <u>false</u>

Another way to state this goal
is to give a meaning — semantics —
to a wff
— remember the only meaning
that a wff can have is
"true" or "false"

The meaning of a wff will depend on two things:

1. the meaning of each propositional variable that it is composed of
2. the meaning of the connectives

The meaning of a propositional variable depends on the (possible) world that we are looking at.

The meaning of the connectives is fixed (and as we said before it is based on our notion of the natural language connectives)

Let's look at the connectives first.

Let's first look at "and"; i.e., "$\wedge$".

We said that it made sense to give the meaning of the proposition based on the meaning of the wffs that it connects.

— if the two wffs that it connects are both true then the compound wff is true otherwise the wff is false

Let's introduce a representation that captures this notion.

It is called a **truth table**

α	β	$\alpha \wedge \beta$
True	True	T
T	False	F
F	T	F
F	F	F

We can provide truth tables for the other connectives as well:

— not ; i.e., $\neg$

meaning: the opposite truth value

α	$\neg\alpha$
T	F
F	T

or : meaning at least one of

α	β	$\alpha \vee \beta$
T	T	T
T	F	T
F	T	T
F	F	F

if ... then meaning

"preserve truth"

(when the consequent is <u>true</u>

truth has been preserved

when the antecedent is <u>true</u> truth has been preserved only when the consequent is true

when the antecedent is <u>false</u> there is no truth to be preserved

α	β	$\alpha \to \beta$
T	T	T
T	F	F
F	T	T
F	F	T

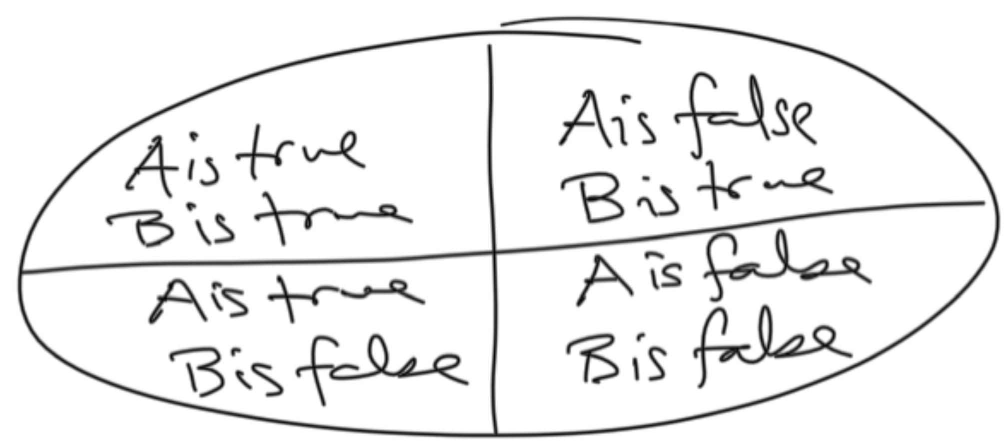
Let's return to the propositional variables

- Every propositional variable is either <u>true</u> or <u>false</u> depending on the possible world that we are considering
- Every possible world represents an infinite number of propositions

e.g. A: It is hot.
B: It is cold.
C_1: The number following 1 is 2.
C_2: The number following 1 is 3.
C_3: The number following 1 is 4.
⋮
C_{100} The number following 1 is 101.
⋮
forever

Some worlds have A true and some have A false.

Of each of these worlds B is true and some have B false.



To know the truth of propositions that are formed from these two propositions using the connectives, we don't need to know the truth of any other proposition

(e.g. the truth or falsity of "It is hot and I am sweating" depends only on the truth and falsity of "It is hot" and "I am sweating" and not whether "Today is Thursday" or "Today is Friday")

So this allows us to consider only the propositional variables that are contained in the proposition that we are interested in determining the truth value of.

The assignment of truth values to the propositional variables is called truth valuations

We will use the truth table representation to represent the truth valuations

E.G. We have 2 propositional variables, A and B

these represent the 4 sets of possible worlds

A	B	the proposition composed of A and B
T	T	the truth value for A true and B true
T	F	" A true and B false
F	T	" A false and B true
F	F	" A false and B false

we will use the meaning of the connectives to determine these truth values

The meaning (truth value) of a proposition is composed from the meaning of it subparts ("compositional semantics")

I.E., $wff = \underbrace{wff_1}_{\text{this has a meaning (truth value)}} * \underbrace{wff_2}_{\text{this has a meaning (truth value)}}$ (where $*$ could be $\wedge$, $\vee$, or $\rightarrow$)

$\underbrace{wff_1 * wff_2}_{\text{this has a meaning (truth value)}}$

this is either a prop var (has a truth value)
or is $\neg wff_{11}$
or is $\underbrace{wff_{11}}_{\text{has a t.v.}} * \underbrace{wff_{12}}_{\text{has a t.v.}}$

⋮ and so on

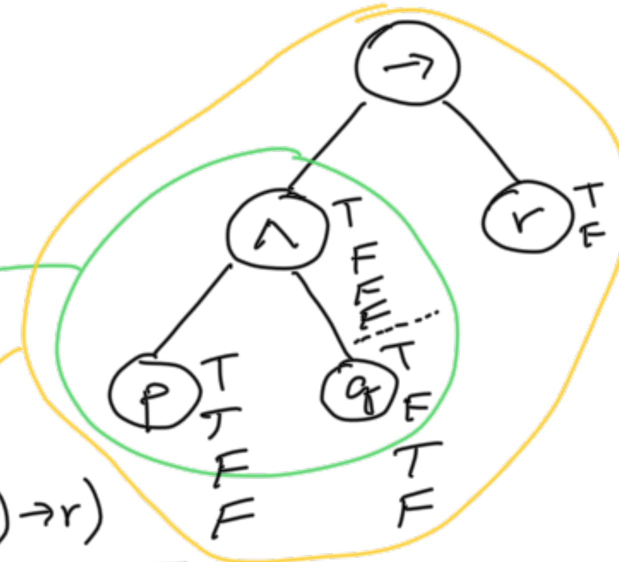
similarly

[of course $wff = \neg wff_1$ but the idea follows immediately]

Note that the structure of the wff is given by the tree structure discussed earlier.

So, if we give a truth valuation to the leaves, give the truth value to each subtree, when we reach the root of the tree, we will have the truth value for the wff for the truth valuation

$((p \wedge q) \to r)$



p	q	r	$(p \wedge q)$	$((p \wedge q) \to r)$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

Note that each row (truth valuation)
represents the set of possible worlds
in which the atomic propositions
(represented by propositional
variables) has those truth values

- note the truth value of every other
atomic proposition is irrelevant

So, because each proposition can contain only a finite number of symbols we can have a finite procedure to compute the truth value of a proposition.
(i.e., in all worlds (an infinite number) in which p is true, q is true, r is true, $((p \wedge q) \rightarrow r)$ is also true ; likewise for the other truth valuations)

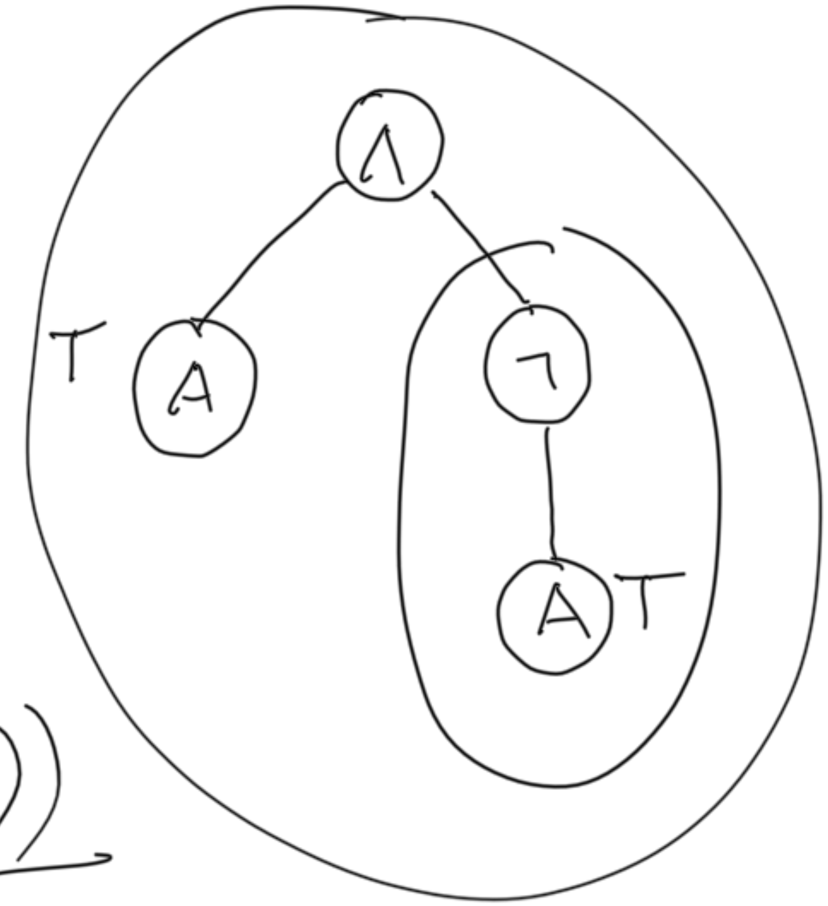
So, if p: It is hot
q: I am sweaty
r: I eat ice cream

in all worlds where these are true the proposition $((p \wedge q) \rightarrow r)$ is also true
{If it is hot and I am sweaty then I eat ice cream}

in all worlds where p and q are true but r is false the implication is false

similarly, p: pigs fly
q: the moon is made greencheese
r: the cow jumps over the moon

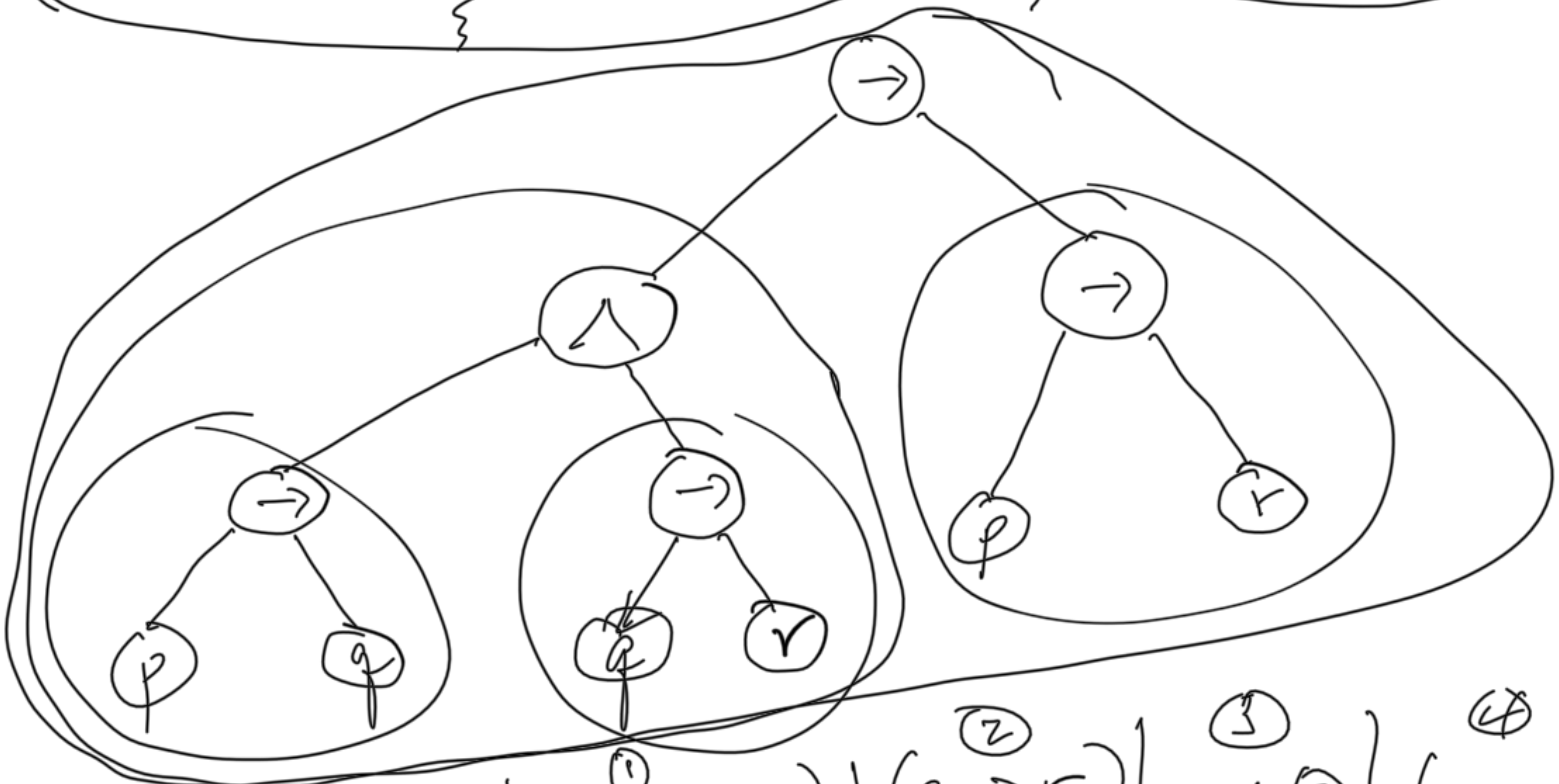
$(A \wedge (\neg A))$



A	$(\neg A)$	$(A \wedge (\neg A))$
T	F	F
F	T	F

<u>contradiction</u>

$$((p \to q) \wedge (q \to r)) \to (p \to r)$$



p	q	r	① $(p \to q)$	② $(q \to r)$	③ ① $\wedge$ ②	④ $(p \to r)$	③ $\to$ ④
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T