Universal instantiation (UI)

Every instantiation of a universally quantified sentence is entailed by it:

$$\frac{\forall v \alpha}{\textsc{Subst}(\{v/g\}, \alpha)}$$

for any variable v and ground term g
E.g., $\forall x King(x) \land Greedy(x) \implies Evil(x)$ yields

$King(John) \land Greedy(John) \implies Evil(John)$
$King(Richard) \land Greedy(Richard) \implies Evil(Richard)$
$King(Father(John)) \land Greedy(Father(John)) \implies Evil(Father(John))$
$\vdots$

Existential instantiation (EI)

For any sentence α, variable v, and constant symbol k *that does not appear elsewhere in the knowledge base*:

$$\frac{\exists v \alpha}{\textsc{Subst}(\{v/k\}, \alpha)}$$

E.g., $\exists x Crown(x) \land OnHead(x, John)$ yields

$$Crown(C_1) \land OnHead(C_1, John)$$

provided C_1 is a new constant symbol, called a **Skolem constant**

Existential instantiation contd.

UI can be applied several times to *add* new sentences;
the new KB is logically equivalent to the old

EI can be applied once to *replace* the existential sentence;
the new KB is *not* equivalent to the old,
but is satisfiable iff the old KB was satisfiable

Existential instantiation contd.

UI can be applied several times to *add* new sentences;
the new KB is logically equivalent to the old

EI can be applied once to *replace* the existential sentence;
the new KB is *not* equivalent to the old,
but is satisfiable iff the old KB was satisfiable

Reduction to propositional inference

Suppose the KB contains just the following:

$$\forall x King(x) \land Greedy(x) \implies Evil(x)$$
$$King(John)$$
$$Greedy(John)$$
$$Brother(Richard, John)$$

Instantiating the universal sentence in *all possible* ways, we have

$$King(John) \land Greedy(John) \implies Evil(John)$$
$$King(Richard) \land Greedy(Richard) \implies Evil(Richard)$$
$$King(John)$$
$$Greedy(John)$$
$$Brother(Richard, John)$$

The new KB is **propositionalized**: proposition symbols are

$$King(John),\ Greedy(John),\ Evil(John), King(Richard)\ \mathrm{etc.}$$

Reduction contd.

Claim: a ground sentence* is entailed by new KB iff entailed by original KB
Claim: every FOL KB can be propositionalized so as to preserve entailment
Idea: propositionalize KB and query, apply resolution, return result
Problem: with function symbols, there are infinitely many ground terms,
e.g., $Father(Father(Father(John)))$
Theorem: Herbrand (1930). If a sentence α is entailed by an FOL KB,
it is entailed by a *finite* subset of the propositional KB
Idea: For $n = 0$ to ∞ do
create a propositional KB by instantiating with depth-n terms
see if α is entailed by this KB
Problem: works if α is entailed, loops if α is not entailed
Theorem: Turing (1936), Church (1936), entailment in FOL is **semidecidable** ✓

Problems with propositionalization

Propositionalization seems to generate lots of irrelevant sentences.
E.g., from

$$\forall x King(x) \land Greedy(x) \implies Evil(x)$$
$$King(John)$$
$$\forall y Greedy(y)$$
$$Brother(Richard, John)$$

it seems obvious that $Evil(John)$, but propositionalization produces lots of facts such as $Greedy(Richard)$ that are irrelevant

With p k-ary predicates and n constants, there are $p \cdot n^k$ instantiations

With function symbols, it gets much much worse!

Unification

We can get the inference immediately if we can find a substitution θ such that $King(x)$ and $Greedy(x)$ match $King(John)$ and $Greedy(y)$

$\theta = \{x/John, y/John\}$ works

$\textsc{Unify}(\alpha, \beta) = \theta$ if $\alpha\theta = \beta\theta$

p	q	θ
$Knows(John, x)$	$Knows(John, Jane)$	
$Knows(John, x)$	$Knows(y, OJ)$	
$Knows(John, x)$	$Knows(y, Mother(y))$	
$Knows(John, x)$	$Knows(x, OJ)$	

Unification

We can get the inference immediately if we can find a substitution θ
such that $King(x)$ and $Greedy(x)$ match $King(John)$ and $Greedy(y)$
$\theta = \{x/John, y/John\}$ works
$\textsc{Unify}(\alpha, \beta) = \theta$ if $\alpha\theta = \beta\theta$

p	q	θ
$Knows(John, x)$	$Knows(John, Jane)$	{x/Jane}
$Knows(John, x)$	$Knows(y, OJ)$	
$Knows(John, x)$	$Knows(y, Mother(y))$	
$Knows(John, x)$	$Knows(x, OJ)$	

Unification

We can get the inference immediately if we can find a substitution θ
such that $King(x)$ and $Greedy(x)$ match $King(John)$ and $Greedy(y)$
$\theta = \{x/John, y/John\}$ works
$\textsc{Unify}(\alpha, \beta) = \theta$ if $\alpha\theta = \beta\theta$

p	q	θ
$Knows(John, x)$	$Knows(John, Jane)$	{x/Jane}
$Knows(John, x)$	$Knows(y, OJ)$	{x/OJ,y/John}
$Knows(John, x)$	$Knows(y, Mother(y))$	
$Knows(John, x)$	$Knows(x, OJ)$	

Unification

We can get the inference immediately if we can find a substitution
θ
such that $King(x)$ and $Greedy(x)$ match $King(John)$ and
$Greedy(y)$
$\theta = \{x/John, y/John\}$ works
$\textsc{Unify}(\alpha, \beta) = \theta$ if $\alpha\theta = \beta\theta$

p	q	θ
$Knows(John, x)$	$Knows(John, Jane)$	{x/Jane}
$Knows(John, x)$	$Knows(y, OJ)$	{x/OJ,y/John}
$Knows(John, x)$	$Knows(y, Mother(y))$	{y/John,x/Mother(John)}
$Knows(John, x)$	$Knows(x, OJ)$	

Unification

We can get the inference immediately if we can find a substitution θ
such that $King(x)$ and $Greedy(x)$ match $King(John)$ and $Greedy(y)$
$\theta = \{x/John, y/John\}$ works
$\textsc{Unify}(\alpha, \beta) = \theta$ if $\alpha\theta = \beta\theta$

p	q	θ
$Knows(John, x)$	$Knows(John, Jane)$	{x/Jane}
$Knows(John, x)$	$Knows(y, OJ)$	{x/OJ,y/John}
$Knows(John, x)$	$Knows(y, Mother(y))$	{y/John,x/Mother(John)
$Knows(John, x)$	$Knows(x, OJ)$	fail

Standardizing apart eliminates overlap of variables, e.g., $Knows(z_{17}, OJ)$

Unifiers

- Substitution σ is a unifier of e_1 and e_2 if $e_1\sigma = e_2\sigma$.
- Substitution σ is a most general unifier (mgu) of e_1 and e_2 if
  - σ is a unifier of e_1 and e_2; and
  - if substitution σ' also unifies e_1 and e_2, then $e\sigma'$ is an instance of $e\sigma$ for all atoms e.
- If two atoms have a unifier, they have a most general unifier.

most general unifier is unique (up to variable renaming)

Unification Example

(*) In this and the next slide predicate names and constants are in lowercase and variables are in uppercase.

Which of the following are unifiers of $p(A, b, C, D)$ and $p(X, Y, Z, e)$:

- $\sigma_1 = \{X/A, Y/b, Z/C, D/e\}$
- $\sigma_2 = \{Y/b, D/e\}$
- $\sigma_3 = \{X/A, Y/b, Z/C, D/e, W/a\}$
- $\sigma_4 = \{A/X, Y/b, C/Z, D/e\}$
- $\sigma_5 = \{X/a, Y/b, Z/c, D/e\}$
- $\sigma_6 = \{A/a, X/a, Y/b, C/c, Z/c, D/e\}$
- $\sigma_7 = \{A/V, X/V, Y/b, C/W, Z/W, D/e\}$
- $\sigma_8 = \{X/A, Y/b, Z/A, C/A, D/e\}$

Which are most general unifiers?

Unification Example

$p(A, b, C, D)$ and $p(X, Y, Z, e)$ have as unifiers:

- $\sigma_1 = \{X/A, Y/b, Z/C, D/e\}$
- $\sigma_4 = \{A/X, Y/b, C/Z, D/e\}$
- $\sigma_7 = \{A/V, X/V, Y/b, C/W, Z/W, D/e\}$
- $\sigma_6 = \{A/a, X/a, Y/b, C/c, Z/c, D/e\}$
- $\sigma_8 = \{X/A, Y/b, Z/A, C/A, D/e\}$
- $\sigma_3 = \{X/A, Y/b, Z/C, D/e, W/a\}$

The first three are most general unifiers.
The following substitutions are not unifiers:

- $\sigma_2 = \{Y/b, D/e\}$
- $\sigma_5 = \{X/a, Y/b, Z/c, D/e\}$

Unification

```
Procedure Unify(t1,t2)
        Inputs
                t1,t2: atoms Output
                most general unifier of t1 and t2 if it exists or ⊥ otherwise
        Local
                E: a set of equality statements
                S: substitution
         E ←{t1 = t2}
         S={}
          while (E≠{})
                select and remove x = y from E
                if (y is not identical to x) then
                        if (x is a variable) then
                                replace x with y everywhere in E and S
                                S←x/y∪S
                        else if (y is a variable) then
                                replace y with x everywhere in E and S
                                S←y/x∪S
                        else if (x is f(x1, ..., xn) and y is f(y1, ..., yn)) then
                                E←E∪{x1 = y1,...,xn = yn}
                        else
                                return ⊥
          return S
```

Unification example

x, y, z variables
a, b constants

$\text{Unify}\left(P(x,y,y),\ P(a,z,b)\right)$

$E \leftarrow \{ P(x,y,y) = P(a,z,b) \}$ select + remove so $E \leftarrow \{\}$

$E \leftarrow \{\} \cup \{ x=a,\ y=z,\ y=b \}$ select + remove $x=a$ so $E \leftarrow \{y=z,\ y=b\}$

$S \leftarrow \{x/a\}$ select + remove $y=z$ so $E \leftarrow \{y=b\}$

y is replaced by z in S and E

$E \leftarrow \{ z=b \}$ select + remove so $E \leftarrow \{\}$

$S \leftarrow \{x/a,\ \underline{y/z}\}$

z is replaced by b in S and E

$S \leftarrow \{x/a,\ y/b,\ z/b\}$ $\leftarrow$ mgu (σ) $\left\{\begin{array}{l} P(x,y,y)\sigma = P(a,b,b) \\ P(a,z,b)\sigma = P(a,b,b) \end{array}\right.$

Another unification example

y, z variables
a, b constants

$$\text{Unify}\left(P(a,y,y),\ P(z,z,b)\right)$$

$E \leftarrow \{P(a,y,y) = P(z,z,b)\}$

select + remove so $E \leftarrow \{\}$

$E \leftarrow \{a=z,\ y=z,\ y=b\}$

select + remove $a=z$
so $E \leftarrow \{y=z,\ y=b\}$

z is replaced by a in E+S

$E \leftarrow \{y=a,\ y=b\}$

select + remove $y=a$
$E \leftarrow \{y=b\}$

$S \leftarrow \{z/a\}$

y is replaced by a in E+S

so $E \leftarrow \{a=b\}$

$S \leftarrow \{z/a,\ y/a\}$

next time through while loop the if…else ends with return failure