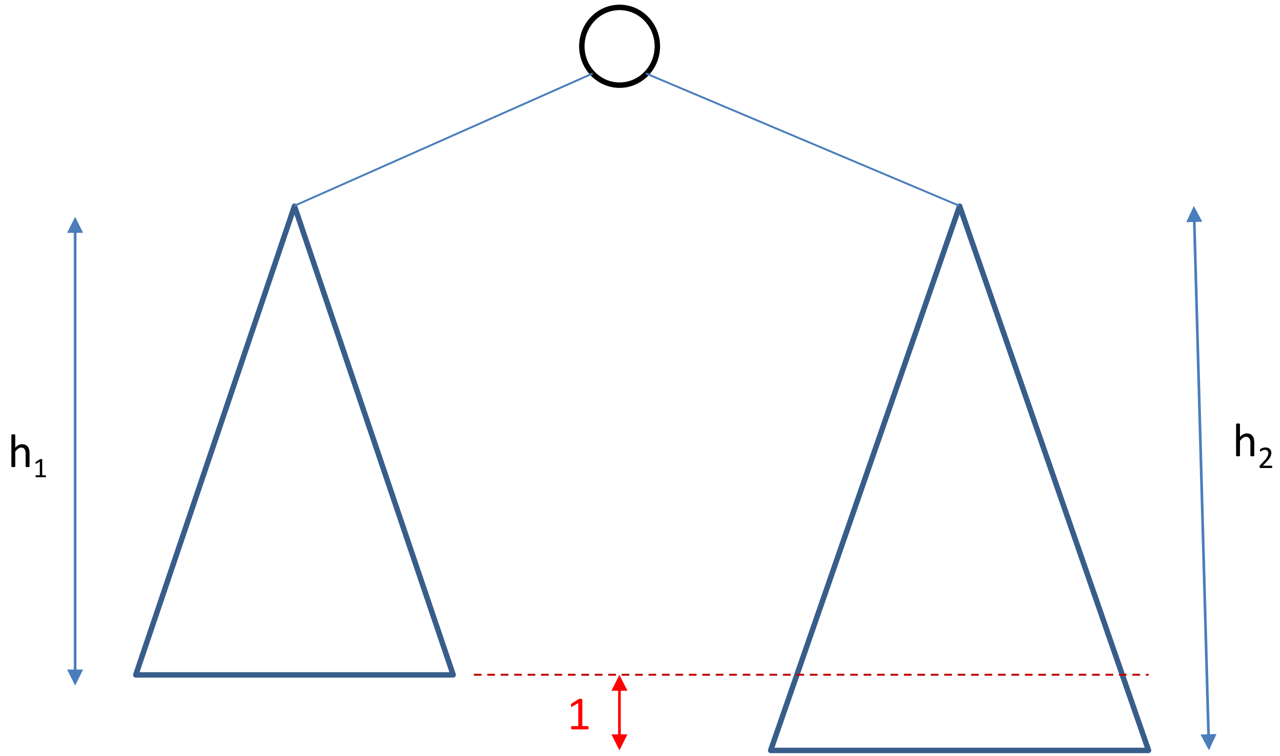
AVL Trees

An AVL tree is a binary search tree in which for every internal node the heights of its two subtrees differ by at most 1.



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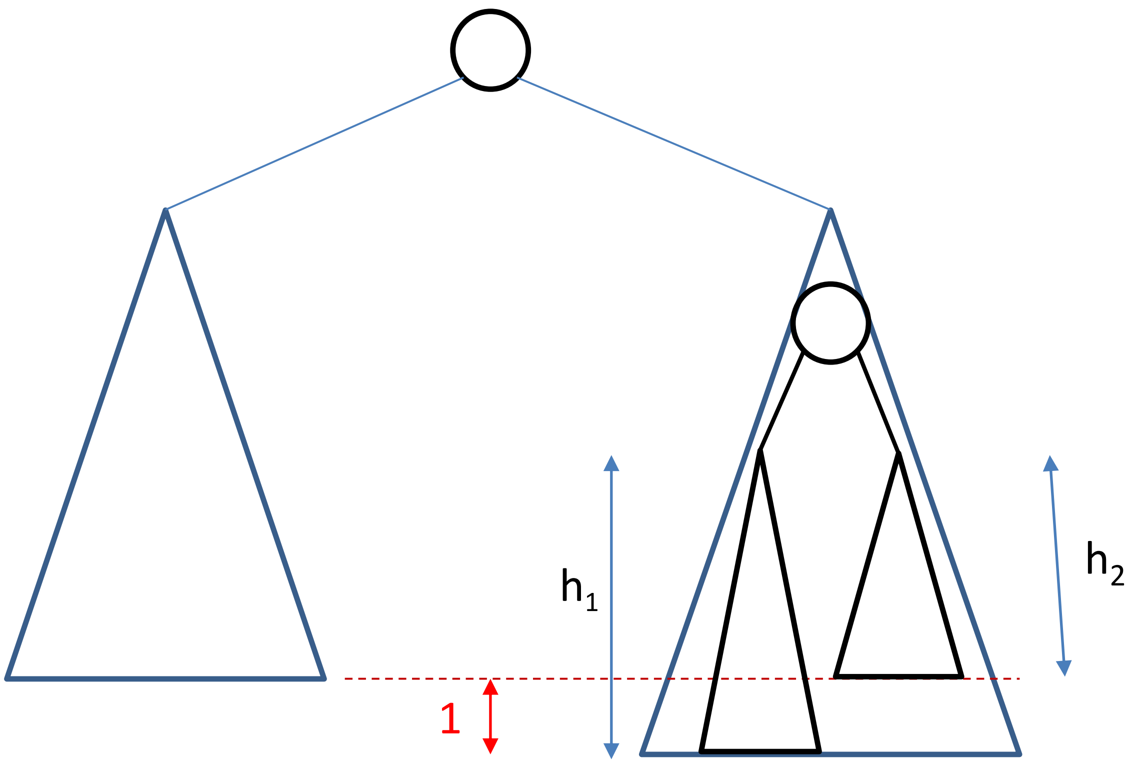
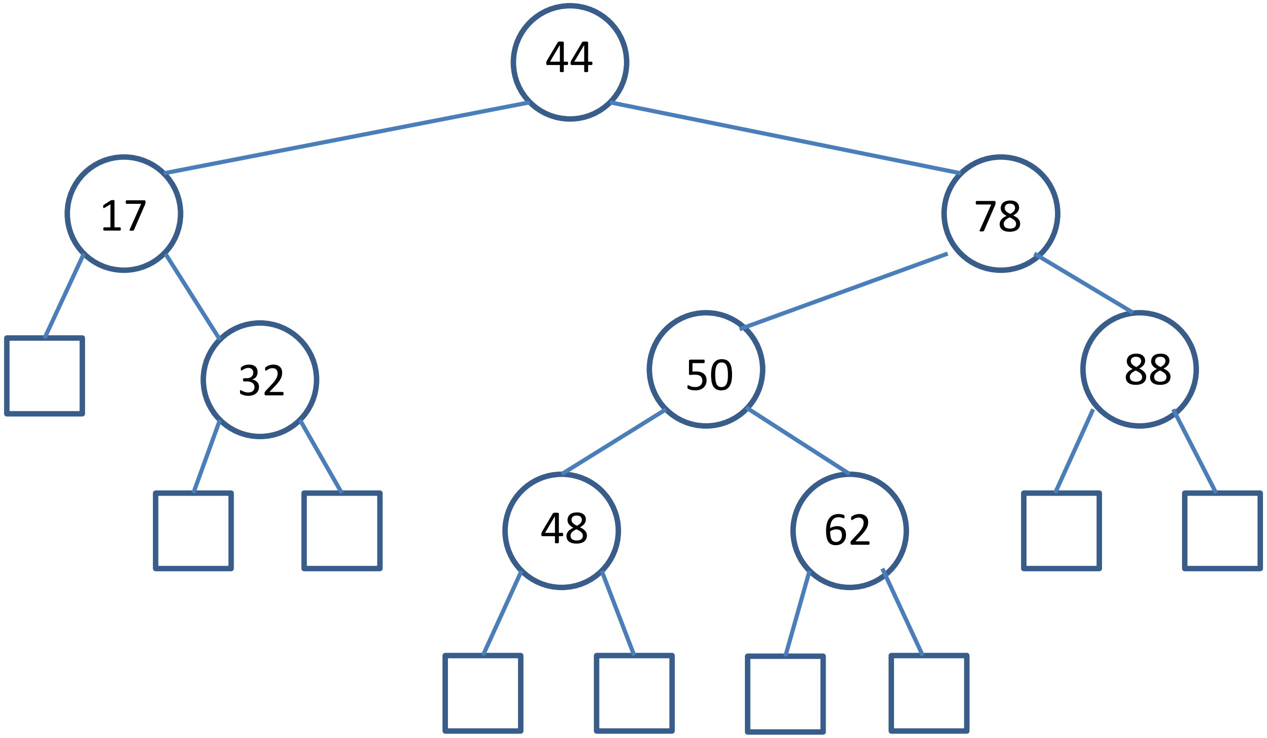
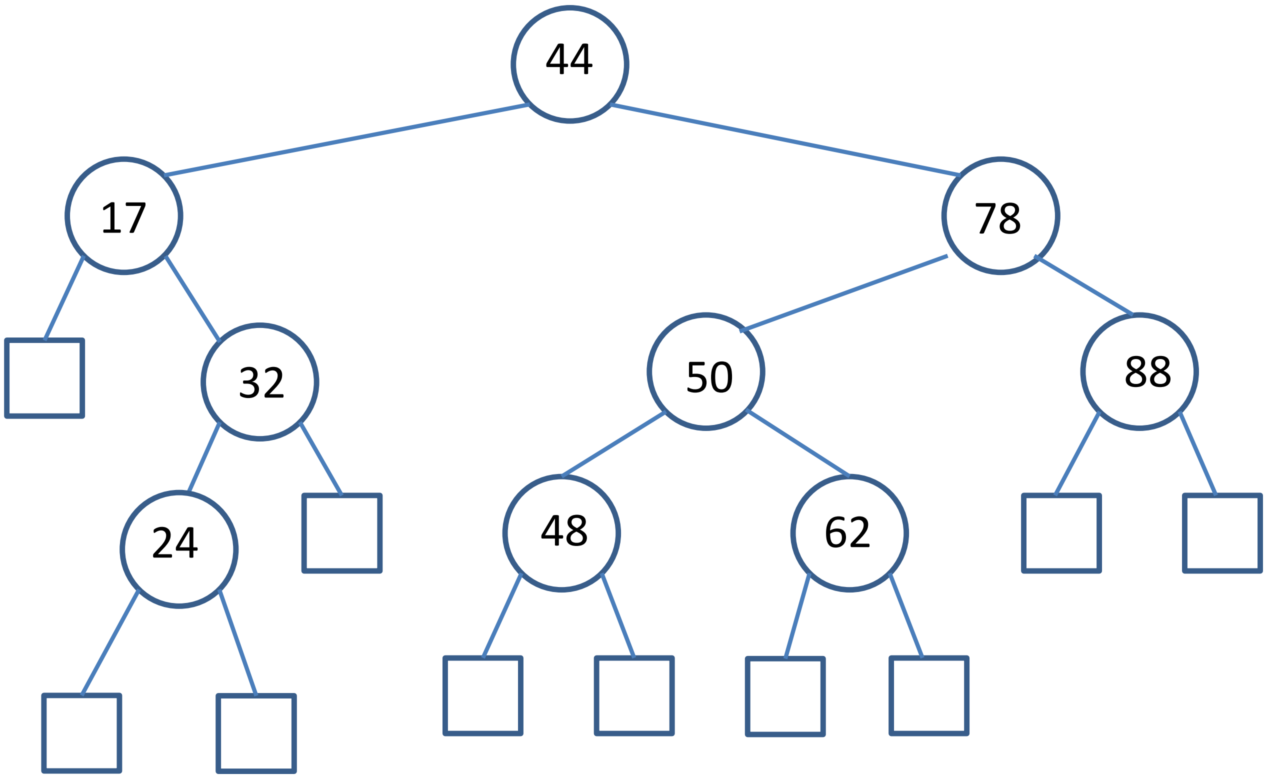


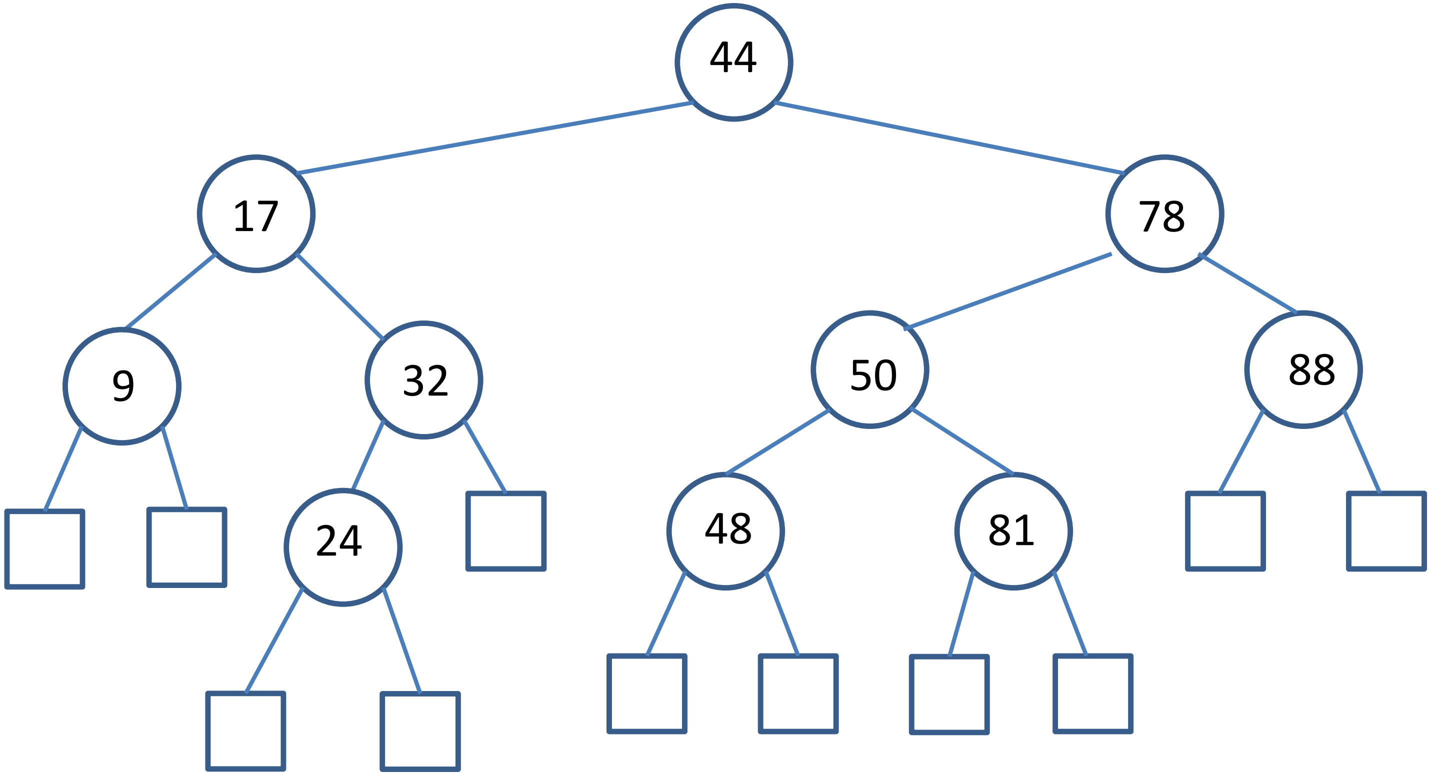
AVL Trees



AVL Trees



AVL Trees



What is the Maximum Height of an AVL Tree?

Let $n(h)$ = minimum number of nodes in an AVL tree of height h.

What is the Maximum Height of an AVL Tree?

Let $n(h)$ = minimum number of nodes in an AVL tree of height h.

$n(0) = 1$, $n(1) = 3$, $n(2) = 5$, $n(3) = 9$, $n(4) = 15$, ...

$n(h) = 1 + n(h-1) + n(h-2) > 2n(h-2)$

Solve the recurrence equation for h even

$n(0) = 1$

$n(h) > 2n(h-2)$

$2n(h-2) > 2^2 n(h-2\times 2)$

$2^2 n(h-2\times 2) > 2^3\, n(h-2\times 3)$

…

$$2^i n(h-2\times i) > 2^{i+1}\, n(\underbrace{h-2\times(i+1)}_{=\,0})$$

Then, $n(h) > 2^{i+1} n(0) = 2^{i+1}$

Solve the recurrence equation for h even

Since $h-2\times(i+1) = 0$, then $i+1 = h/2$ and so

$n(h) = n > 2^{i+1} = 2^{h/2}$

Therefore, taking logarithms on both sides we get

$h/2 \le \log_2 n$

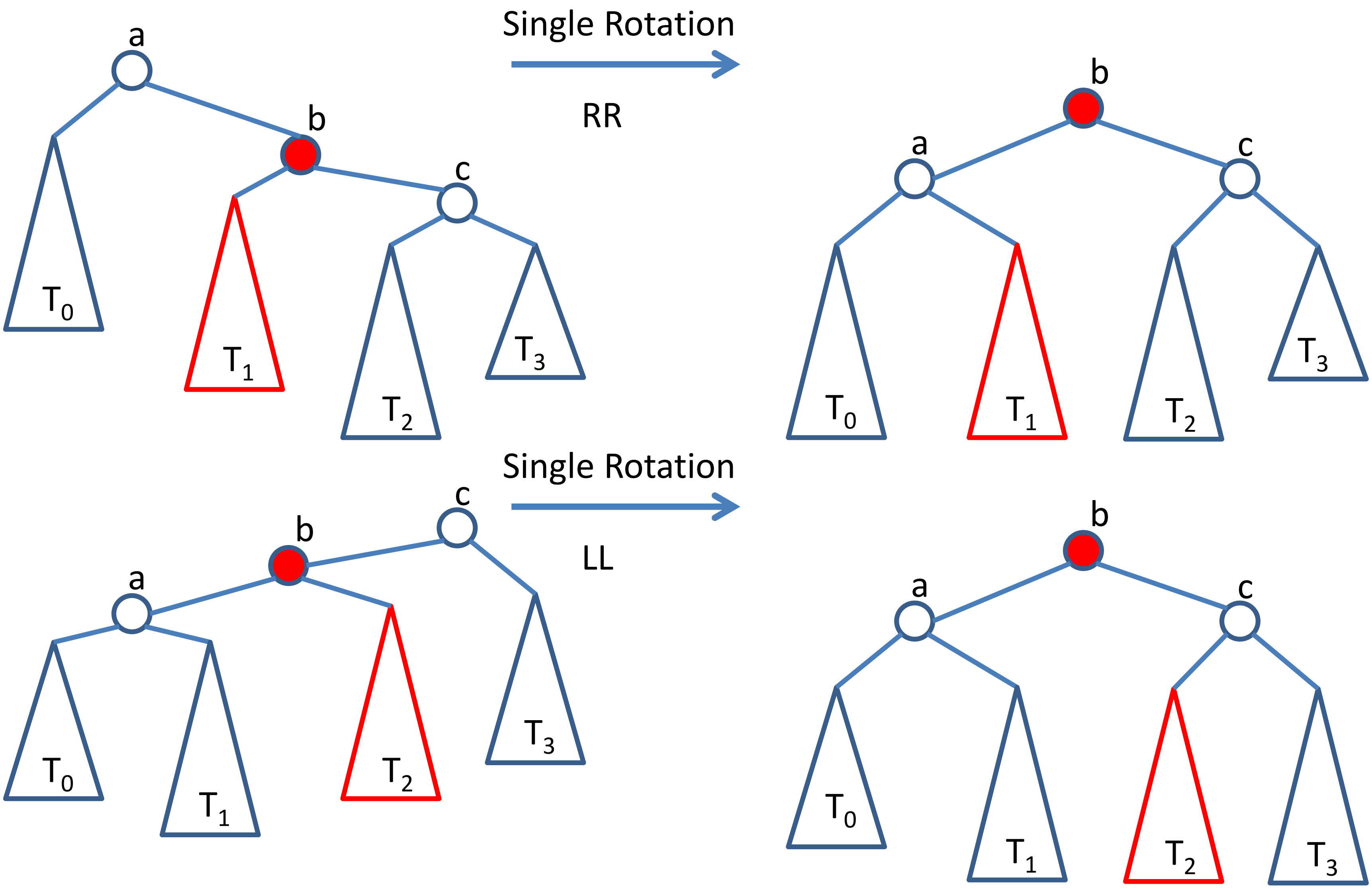
and so

height = $h < 2 \log_2 n$, so height is $O(\log n)$

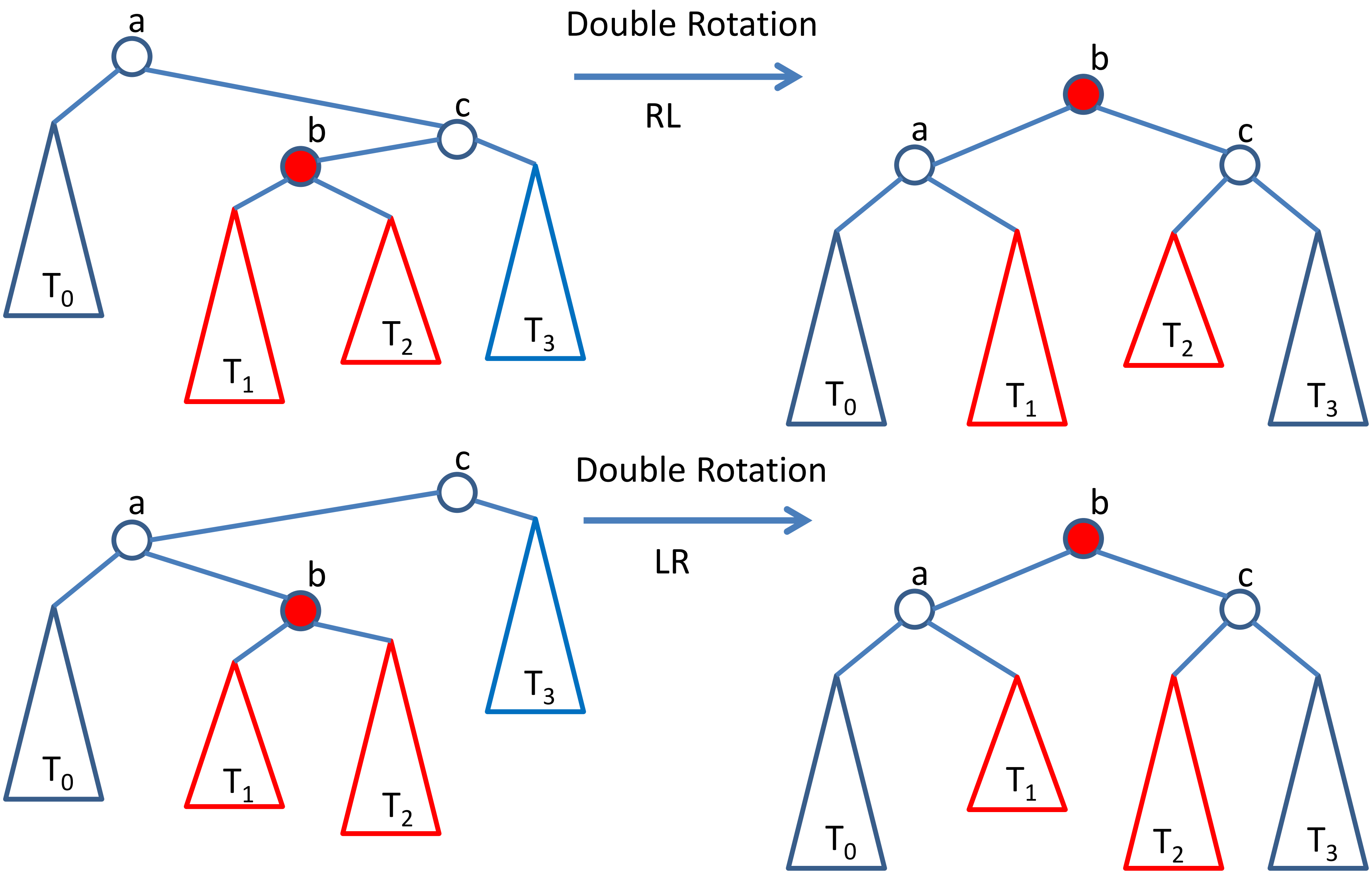
Re-Balancing AVL Trees

To re-balance an AVL tree we always rebalance the smallest un-balanced subtree.

Single Rotations



Double Rotations

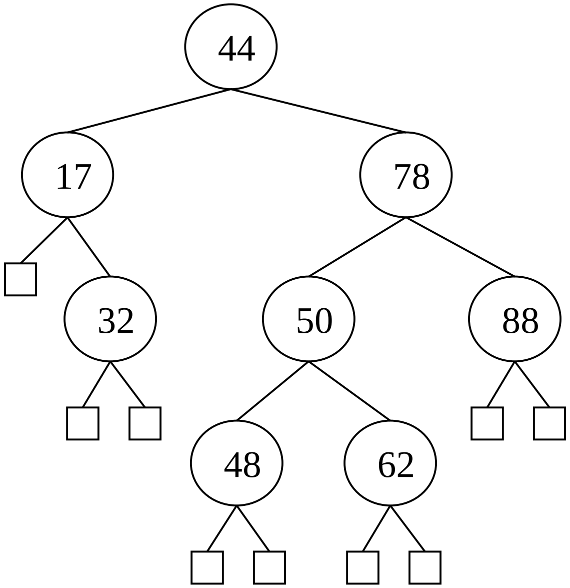


Re-Balancing AVL Trees

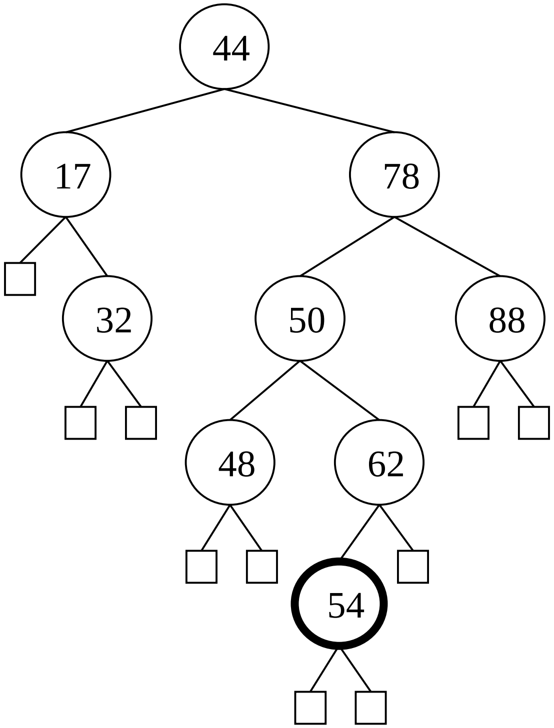
If the tree becomes unbalanced due to an insertion ONE rotation will re-balance the tree.

Insertion

- Insertion is as in a binary search tree
- Re-balance if needed

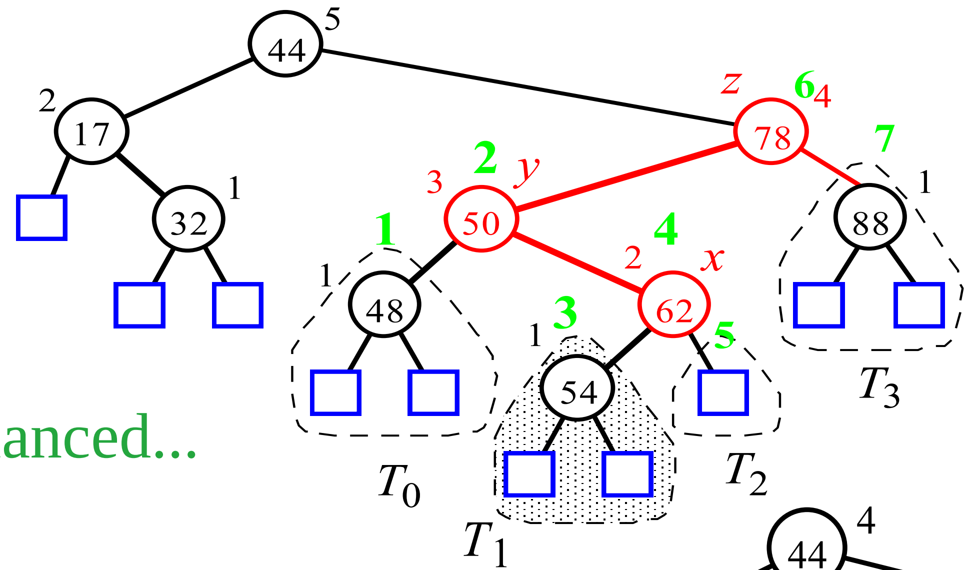


before insertion

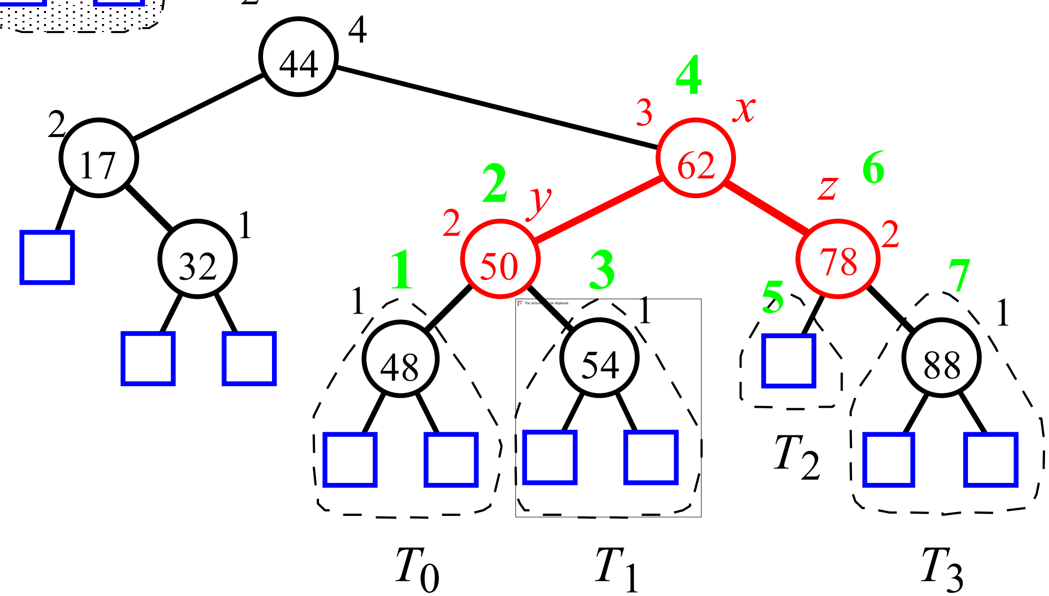


after insertion of 54

Insertion Example, continued



unbalanced...



...balanced

Algorithm putAVL (r, k, data)

In: Root r of an AVL tree, record (k,data)

Out: {Insert (k,data) and re-balance if needed}

put(r,k,data) // Algorithm for binary search trees

Let p be the node where (k,data) was inserted

while ($p \neq$ null) **and** (subtrees of p differ in height ≤ 1) **do**

  p = parent of p

if $p \neq$ null **then** rebalance subtree rooted at p by performing appropriate rotation

Re-Balancing AVL Trees

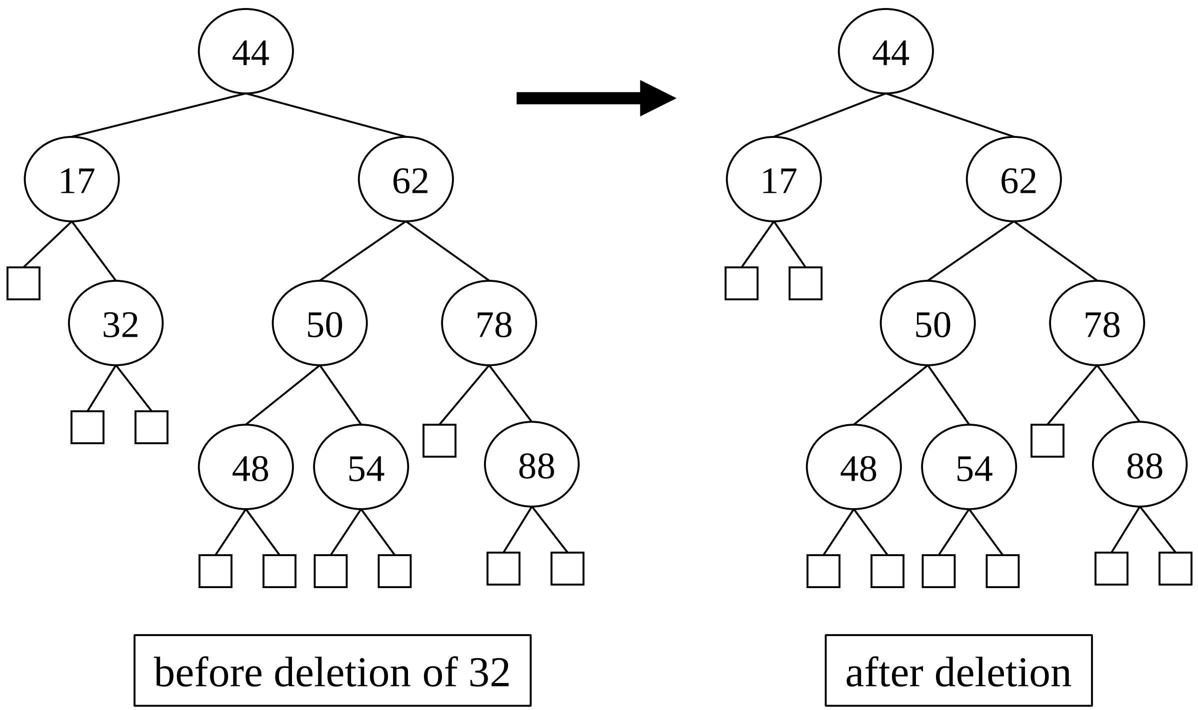
When a single and a double rotation can be applied to an un-balanced subtree the single rotation always re-balances the subtree.

Re-Balancing AVL Trees

If the tree becomes unbalanced due to a removal SEVERAL rotations might be needed to re-balance the tree.

Removal

- Removal begins as in a binary search tree, which means the node removed will become a leaf.
- Re-balance if needed.



Algorithm removeAVL (*r, k*)

In: Root *r* of an AVL tree, key *k* to remove

Out: {Remove *k* and re-balance if needed}

remove(*r*,*k*) // Algorithm for binary search trees

Let *p* be the parent of the node that was removed

while ($p \neq$ null) **do {**

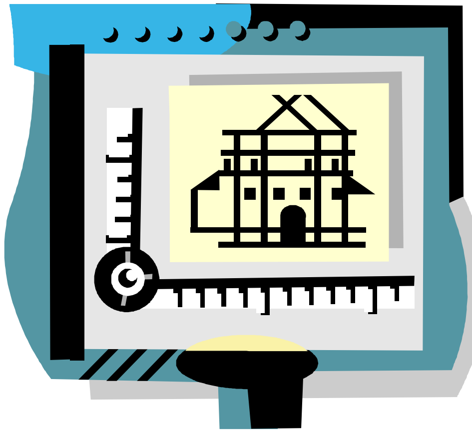
if subtrees of *p* differ in height > 1 **then**

rebalance subtree rooted at *p* by performing appropriate rotation

p = parent of *p*

}

AVL Tree Performance



- AVL tree storing n items
  - The data structure uses O(n) space
  - A single rotation takes O(1) time
    - using a linked-structure binary tree
  - Get takes O(log n) time
    - height of tree is O(log n), no re-balancing needed
  - Put takes O(log n) time
    - initial get operation takes O(log n) time
    - rebalancing the tree takes O(1) time, as at most one rebalancing operation is needed
  - Removal takes O(log n) time
    - initial get operation takes O(log n) time
    - rebalancing the tree needs O(log n) time as several rebalancing operations might be needed