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CS 2214 Assignment #2
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Problem 1: Proving Properties About Integers

1. Prove: **For every integer n we have $n \leq n^2$**

This property states $\forall n\ (P(n) \rightarrow Q(n))$ where P(n) is "n is an integer" and Q(n) is "$n \leq n^2$". To begin this direct proof of this theorem, we want to prove true implies true. Therefore, we will begin with assuming P(n) is true, namely, that P(n) is an integer. We will choose an arbitrary number like **2**.

For all numbers, if 2 is an integer (**true**), THEN Q(n). Here, in order to prove this statement to be true, we must prove that Q(n) is also true. Therefore, we solve:

$$= \quad 2 \leq 2^2$$

$$= \quad 2 \leq 4$$

$$= \quad \textbf{True}$$

Consequently, we have proven that Q(n) is true, meaning true implies true, meaning $\forall n\ (P(n) \rightarrow Q(n))$ is true. Finally, the property "**For every integer n we have $n \leq n^2$**" holds true.

2. Prove: **For every integer n, the integer $n^2 + n + 1$ is odd.**

Similar to above, we will perform a direct proof utilizing $\forall n\ (P(n) \rightarrow Q(n))$ where P(n) is "n is an integer" and Q(n) is "$n^2 + n + 1$ is odd". In order to prove this, we assume the hypothesis of the conditional statement is true, namely that P(n) is an integer n and is true. To follow the definition of an odd integer:

$n = 2k + 1$ where k is some integer. Following the theorem above, we want to show that $\mathbf{n^2 + n + 1}$ is odd and true. To do this, we use the definition of an odd integer and solve:

=	$n^2 + n + 1$ where $n = 2k + 1$	//definition
=	$(2k+1)^2 + (2k + 1) + 1$	//substitution
=	$4k^2 + 4k + 1 + 2k + 1 + 1$	//expanding
=	$4k^2 + 6k + 3$	//simplifying
=	$2(2k^2 + 3k) + 3$	//simplifying

Therefore, our theorem holds. By doubling the $(2k^2 + 3k)$ and adding 3, we are always going to have an odd integer. Consequently, P(n) which is true, implies Q(n) which is also true, meaning the property "**For every integer n, the integer $n^2 + n + 1$ is odd"** is also true.

Problem 2: Proving Properties About Real Numbers

1. **Prove that:** For every real number x, if $x \leq 0$ or $1 \leq x$ holds, then $x \leq x^2$ holds as well.

Let x = 0. If the first property holds, then the second property must hold as well (proving the theorem) Therefore P $\rightarrow$ Q where P = ($x \leq 0$) and Q = $x \leq x^2$. Therefore $0 \leq x^2 = 0 \leq 0$ which means this holds. To try with a negative number, if P = -1, then P is true so we can test the second argument which is:

Q = $x \leq x^2 = -1 \leq (-1)^2 = -1 \leq 1$ which is true. Therefore, the property is true as the implication P $\rightarrow$ Q is always true implies true which is true.

2. **Prove that:** For all real number x we have $\lfloor 2x \rfloor = 2\lfloor x \rfloor$.

To disprove this theory, we look at the definition of floor function: *The floor function takes as input a real number x and gives as output the greater integer less than or equal to x.*

Therefore, if we assume x = 3.7, then $\lfloor 2(3.7) \rfloor = \lfloor 7.4 \rfloor = 7$.

Again, if we assume x = 3.7, then $2\lfloor 3.7 \rfloor = 2(3) = 6$.

Therefore, because 7 is not equal to 6, this property cannot hold for ALL real numbers, however it works for some. For example, take x = 2.2.

$\lfloor 2(2.2) \rfloor = 2\lfloor 2.2 \rfloor$

$\lfloor 4.4) \rfloor = 2\lfloor 2.2 \rfloor$

$4 = 2\lfloor 2.2 \rfloor$

$4 = 2(2)$

$4 = 4.$

Therefore, the property is true for x = 2.2, however for **<u>ALL</u>** values it does not hold as shown when x = 3.7.

Problem 3: Properties of Preimage Sets

1. Prove: $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$

In order to prove $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$, we must prove that A is a subset of B and that B is also a subset of A.

Therefore, we let x in A be an element of $f^{-1}(S \cup T)$. If this is true, which it is, then f(x) = element of $(S \cup T)$. By the definition, it means that f(x) belongs to either $f^{-1}(S)$ or $f^{-1}(T)$.

Therefore, x lies in $f^{-1}(S \cup T)$ which means x also lies in $f^{-1}(S) \cup f^{-1}(T)$. Therefore, $f^{-1}(S \cup T) = f^{-1}(S) \cup f^{-1}(T)$.

2. Prove: $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$

$= \quad x \in (S \cap T)$

$= \quad f(x) \in (S \cap T)$

$= \quad f(x) \in S, f(x) \in T$

$= \quad x \in f^{-1}(S), x \in f^{-1}(T)$

$= \quad x \in f^{-1}(S) \cap f^{-1}(T)$

Problem 4: Properties of Functions

1. **This function is injective (one-to-one).**

$= \quad f(n) = 2019n + 1$

$= \quad f(n) = f(y)$

$= \quad 2019n + 1 = 2019y + 1$

$= \quad 2019n = 2019y$

$= \quad n = y$

The function is also surjective (onto):

= $y = 2019n + 1$

= $y - 1 = 2019n$

= $(y-1)/2019 = n$

= $n = (y-1)/2019$

Therefore there is some n such that 2019n + 1 = y. To get the inverse of f(n) = 2019n + 1:

= $f(n) = 2019n + 1$

= $n = 2019y + 1$

= $2019y = n - 1$

= $y = (n-1) / 2019$

= $f^{-1}(n) = (n-1) / 2019$

2. **This function is injective.**

= $f(n) = n/2 + n/2$

= $f(n) = f(y)$

= $n/2 + n/2 = y/2 + y/2$

= $2n/2 = 2y/2$

= $n = y$

The function is also surjective (onto):

= $y = n/2 + n/2$

= $y = 2n/2$

= $y = n$

Since the function is a bijection, the inverse is denoted as follows:

= $f(n) = n/2 + n/2$

= $n = 2y/2$

= $y = n$

= $f^{-1}(n) = n$

3. **This function is injective.**

= $f(x) = x - \lfloor x \rfloor$

= $f(x) = f(y)$

= $x - \lfloor x \rfloor = y - \lfloor y \rfloor$

= $x – n = y – n$ for $n \leq x < n+1$

= $x = y$

= Therefore $f(x) = f(y)$ implies that $x = y$ for all x and y in the domain of $[1,2) \rightarrow [0,1)$.

The function is NOT surjective:

= $y = x - \lfloor x \rfloor$

= $y = x – n$ for $n \leq x < n+1$

= $x = y + n$ for $n \leq x < n+1$

= This function is not onto because there cannot be an x which is equal to $y + n$ when x is less than n by the property for $n \leq x < n+1$.

4. **This function is injective.**

= $f(x) = (x - \lfloor x \rfloor)^2$

= $f(x) = f(y)$

= $(x - \lfloor x \rfloor)^2 = (y - \lfloor y \rfloor)^2$

= $(x - n)^2 = (y - n)^2$ for $n \leq x < n+1$

= $x = y$

= Therefore $f(x) = f(y)$ implies that $x = y$ for all x and y in the domain of $[1,2) \to [0,1)$. Because the domain is restricted to positive numbers, it is able to map to the same number $x = y$.

This function is surjective.

= $y = (x - \lfloor x \rfloor)^2$

= $y = (x - n)^2$ for $n \leq x < n+1$

= $\sqrt{y} = x - n$ for $n \leq x < n+1$

= $x = \sqrt{y} + n$ for $n \leq x < n+1$

= Therefore this function is not surjective because the values that restrict the function to the domain do not allow the equality to be true.

Problem 5: Properties of Functions

1. Prove: **if *g* is surjective then so is $g \circ f$**

Let $y = f(x) \in B$. If z is any element of C ($z \in C$), then we have $g(y) = z$. Therefore, there is an x in set A such that $(g \circ f)(x) = g(f(x)) = z$. Thus, $g \circ f$ is surjective if g is surjective.

2. Prove: **if *f* and *g* are both injective, then so is *g* ∘ *f***

$g \circ f$ is also the set from $A \rightarrow C$. To prove this is injective, we must show $(g \circ f)(x) = (g \circ f)(y)$. This will eventually lead us to $x = y$ which means the functions map equally.

If $(g \circ f)(x) = (g \circ f)(y) = c \in C$, then equally $g(f(x)) = g(f(y))$. Let $f(x) = x$, $f(y) = y$, so $g(x) = g(y)$.

Because g is the set from $B \rightarrow C$ which is injective and $g(a) = g(b)$, we know that $a = b$. This means that $f(x) = f(y)$. Since f is also the set from $A \rightarrow B$ is injective and $f(x) = f(y)$, we know that $x = y$. Therefore, $g \circ f$ is injective if both f and g are injective.