CENG 371 - Scientific Computing
Fall 2022
Homework 3

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December 25, 2022

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1. c) For the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

largest eigenpair $(\lambda_{\max}, v_{\max})$ using power method and smallest eigenpair $(\lambda_{\min}, v_{\min})$ using inverse power method with shifting (as A is not invertable without shifting) with $\alpha = 0.001$ and

$$x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

for both are

$$\lambda_{\max} = 3.7321, \qquad v_{\max} = \begin{bmatrix} 0.2887 \\ -0.5000 \\ 0.5774 \\ -0.5000 \\ 0.2887 \end{bmatrix}$$

$$\lambda_{\min} = 0.2679, \qquad v_{\min} = \begin{bmatrix} 0.2887 \\ 0.5000 \\ 0.5774 \\ 0.5000 \\ 0.2887 \end{bmatrix}$$

If we calculate relative error as $\frac{\|Av-\lambda v\|_2}{|\lambda|}$,

$$\frac{\|Av_{\max} - \lambda_{\max} v_{\max}\|_2}{|\lambda_{\max}|} = 1.3370e - 15$$

$$\frac{\|Av_{\min} - \lambda_{\min} v_{\min}\|_2}{|\lambda_{\min}|} = 2.4979e - 15$$

d) For

$$B = \begin{bmatrix} 0.2 & 0.3 & -0.5 \\ 0.6 & -0.8 & 0.2 \\ -1.0 & 0.1 & 0.9 \end{bmatrix}, x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

If we use power method by hand,

$$x_1 = Bx_0 = \begin{bmatrix} 0.2 & 0.3 & -0.5 \\ 0.6 & -0.8 & 0.2 \\ -1.0 & 0.1 & 0.9 \end{bmatrix} * \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.2 + 0.3 - 0.5 \\ 0.6 - 0.8 + 0.2 \\ -1.0 + 0.1 + 0.9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_2 = Bx_1 = \begin{bmatrix} 0.2 & 0.3 & -0.5 \\ 0.6 & -0.8 & 0.2 \\ -1.0 & 0.1 & 0.9 \end{bmatrix} * \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+0+0 \\ 0+0+0 \\ 0+0+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

As $x_1 = x_2$, x converges to 0, meaning power method by hand failed to find the eigenpair. If we use the power method we implemented in MATLAB, the eigenpair is computes as

$$(\lambda, v) = \left(1.3427, \begin{bmatrix} -0.407031 \\ -0.028761 \\ 0.912961 \end{bmatrix}\right).$$

This means

$$\begin{bmatrix} 0.2 + 0.3 - 0.5 \\ 0.6 - 0.8 + 0.2 \\ -1.0 + 0.1 + 0.9 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

probably because of cancellation errors, causing the x to converge without getting canceled out completely.

2. a) If $v_i v_i^T$ is orthogonal or at least equal to a multiple of I, most likely it is $v_i^T v_i I$, then

$$\lambda_i \frac{v_i v_i^T}{v_i^T v_i} = \lambda_i I.$$

So, if this condition holds, what we are doing is removing λ_i from the eigenvalues of A, meaning the second greatest in absolute value eigenvalue becomes the greatest eigenvalue in absolute value of $A - \lambda_1 I$ and so on.
To prove $\frac{v_i v_i^T}{v_i^T v_i} = I$, lets take v_i, λ_i, which are eigenpairs of the real symmetric matrix A. Because A is real symmetric, there exists a eigendecomposition $A = V \Lambda V^T$ where

$$\begin{aligned}
V &= \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \\
\Lambda &= \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \\
VV^T &= I \\
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} &= \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \\
\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} &= \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \\
v_i v_i^T &= I \\
v_i^T v_i &= 1
\end{aligned}$$

d)

Because we used same convergence metric, there is no significant difference between the both algorithms, their performance in term of relative error but in term of time performance, subspace iteration is significantly worse. This is because it takes a lot of time to converge, making it slower than calculating each eigenpair separately.

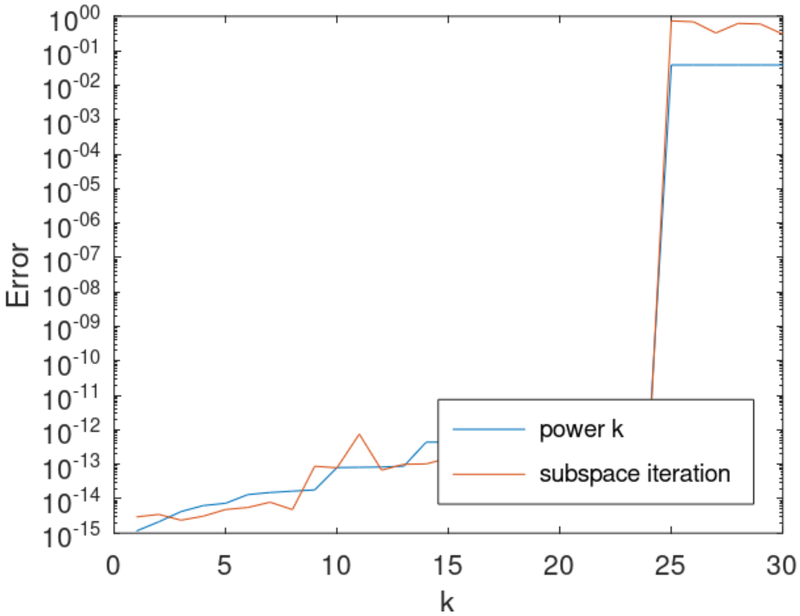

Figure 1: Relative Error Comparison

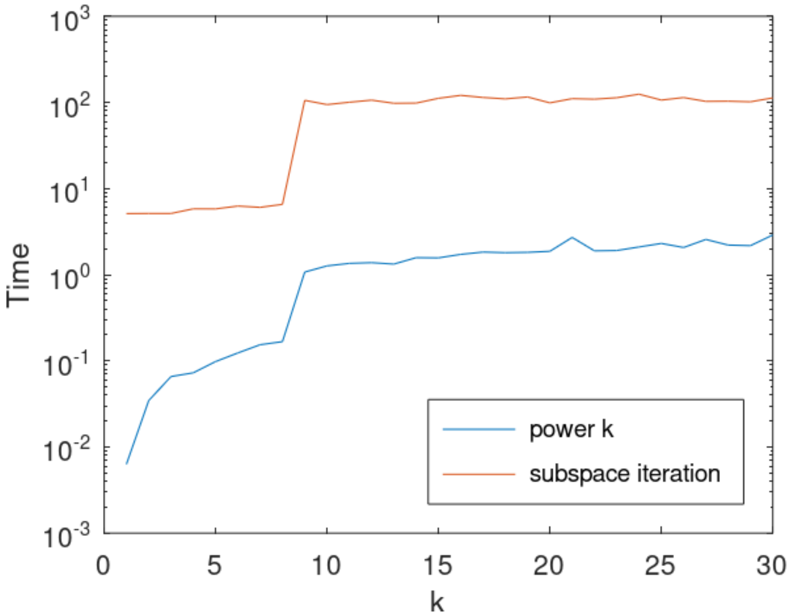

Figure 2: Time Comparison