# 数学基础-概率-高斯分布

## 数学基础-概率-高斯分布

目录：



#### 正态分布性质

**if** $p(x) \sim \mathcal{N}(u, \sum)\{$
$p(x) = \{p(x1), p(x2)\} \quad p(x1), p(x2) \sim \mathcal{N}$ **即边缘分布服从正态分布**
$p(x1|x2) \sim \mathcal{N}$ **即条件分布服从正态分布**
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TODO，证明：

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#### 正态分布极大似然估计

Data: $X = (x_1, x_2, x_3..., x_n)^T$
$x_i \in \mathbb{R}^p$, $x_i \sim \mathcal{N}(\mu, \sum)$ 假设$x_i$是p维独立同分布的
$\theta = (\mu, \sum)$

**正态分布公式：**
一维正态分布：$p(x) = \frac{1}{\sqrt{(2\pi)}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$
高维正态分布：$p(x) = \frac{1}{(2\pi)^{\frac{p}{2}} |\sum|^{\frac{1}{2}}} \exp\left(-\frac{(x-\mu)^T \sum^{-1} (x-\mu)}{2}\right)$

TODO，似然函数及其意义：

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#### 一维MLE求解

考虑x为一维时，则p=1，$\theta = (\mu, \sigma^2)$，$p(x) = \frac{1}{\sqrt{(2\pi)}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$

- **对数似然：**

$$\begin{aligned}
logP(X|\theta) = log\prod_{i=1}^{N} P(x_i|\theta) &= \sum_{i=1}^{N} logP(x_i|\theta) \\
&= \sum_{i=1}^{N} log[\frac{1}{\sqrt{(2\pi)}\sigma}\exp(-\frac{(x_i-\mu)^2}{2\sigma^2})] \\
&= \sum_{i=1}^{N} log(\frac{1}{\sqrt{(2\pi)}\sigma}) + \sum_{i=1}^{N} log[\exp(-\frac{(x_i-\mu)^2}{2\sigma^2})] \\
&= \sum_{i=1}^{N} log(\frac{1}{\sqrt{(2\pi)}}) + \sum_{i=1}^{N} log\frac{1}{\sigma} - \sum_{i=1}^{N} \frac{(x_i-\mu)^2}{2\sigma^2}
\end{aligned}$$

- **求解$\mu_{MLE}$：**

求$\mu_{MLE}$第一项常数项和第二项关于$\sigma$的项无关可被忽略。

$$\begin{aligned}
\mu_{MLE} &= \arg\max_{\mu} logP(X|\theta) \\
&= \arg\max_{\mu} \sum_{i=1}^{N} -\frac{(x_i-\mu)^2}{2\sigma^2} \\
&= \arg\min_{\mu} \sum_{i=1}^{N} (x_i-\mu)^2
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \sum_{i=1}^{N}(x_i-\mu)^2}{\partial\mu} = \sum_{i=1}^{N} -2(x_i-\mu) &= 0 \\
\sum_{i=1}^{N}(x_i-\mu) &= 0 \\
\sum_{i=1}^{N} x_i - N\mu &= 0 \\
\mu_{MLE} &= \frac{1}{N}\sum_{i=1}^{N} x_i
\end{aligned}$$

$\mu$无偏估计证明：

$$\mathbb{E}[\mu_{MLE}] = \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[x_i] = \frac{1}{N}\sum_{i=1}^{N}\mu = \mu$$

- **求解$\sigma^2_{MLE}$：**

$$\sigma_{MLE}^2 = \arg\max_{\sigma} P(X|\theta) = \arg\max_{\sigma} \sum_{i=1}^{N} (-log\sigma - \frac{1}{2\sigma^2}(x_i - \mu)^2)$$

记$(-log\sigma - \frac{1}{2\sigma^2}(x_i - \mu)^2$为$\oint$

$$\frac{\partial \oint}{\partial \sigma} = \sum_{i=1}^{N} [-\frac{1}{\sigma} - \frac{1}{2}(x_i - \mu)^2(-2)\sigma^{-3}] = 0$$

$$\sum_{i=1}^{N} [-\frac{1}{\sigma} + (x_i - \mu)^2 . \sigma^{-3}] = 0$$

$$\sum_{i=1}^{N} [-\sigma^2 + (x_i - \mu)^2] = 0$$

$$-\sum_{i=1}^{N} \sigma^2 + \sum_{i=1}^{N} (x_i - \mu)^2 = 0$$

$$\sum_{i=1}^{N} \sigma^2 = \sum_{i=1}^{N} (x_i - \mu)^2$$

$$\sigma_{MLE}^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \mu)^2$$

TODO:$\sigma$有偏估计证明：

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