Particle Physics Problem Set

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Problem 1 Try to understand the eigenfunction expansion of the Green's function for the Helmholtz equation in a bounded region:

$$\nabla^2 y - k^2 y - \rho \tag{1}$$

The Green's function is defined by

$$\nabla^2 G(r' - r'_0) \,|\, k^2 G(r' - r'_0) = \delta(r' - r'_0) \tag{2}$$

The eigenvalue equation

$$\nabla^2 y \,|\, k^2 y = 0 \tag{3}$$

only has solutions for certain discrete values of $k - k_n$ because of the boundary conditions.

Consider a rectangular domain bounded by $x, y, z = 0, L$. You can take $L = 1$. The eigenfunctions are then

$$\alpha_n(x) = \sin(n_x \pi x) \tag{4}$$

with eigenvalues $k_n = n\pi/L$. In three dimensions the unbounded Green's function is proportional to $1/R$, in two dimensions $\ln(R)$. In the bounded case it should show this behavior close to the source at $\vec{r}_0$.

Write the Green's function in terms of the eigenfunctions as in class for one two and three dimensions.

Then take $x_o, y_o, z_o = L/2$ and explore the behavior of the Green's function using a computer mathematics program like Mathematica or Maple or MathCad. Try different number of terms in the sums over n_x, n_y, n_z. The closer you get to $\vec{r}_o$ the more terms you will need.

Produce reasonable plots for the Green's function moving towards the source in say the x direction for $1, 2, 3$ dimensions.

Solution

The Green's function in terms of eigenfunctions (1D) $\alpha_n(x)$ is written

$$G(x, x_0) = \sum_{n=1}^{\infty} \frac{\alpha_n(x)\alpha_n(x_0)}{k_n^2 - k^2} \tag{5}$$

For 1D we let $\alpha_n(x) = \sin\left(\frac{n\pi x}{L}\right)$, $k_n = n\pi/L$. Then the 1D case rewrites to

$$G(x, x_0) = \sum_{n=1}^{\infty} \frac{\sin(n\pi x)\sin(n\pi x_0)}{n^2\pi^2 - k^2} \tag{6}$$

For 2D the Green's function is written

$$G(x,y,x_0,y_0) = \sum_{n_x=1}^{\infty}\sum_{n_y=1}^{\infty} \frac{\alpha_{n_x}(x)\alpha_{n_y}(y)\alpha_{n_x}(x_0)\alpha_{n_y}(y_0)}{k_{n_x}^2 + k_{n_y}^2 - k^2} \tag{7}$$

For $\alpha_{n_x}(x) = \sin\left(\frac{n_x \pi x}{L}\right)$, $\alpha_{n_y}(y) = \sin\left(\frac{n_y \pi y}{L}\right)$, $k_{n_x} = n_x\pi/L$, $k_{n_y} = n_y\pi/L$, the 2D case is rewritten to

$$G(x,y,x_0,y_0) = \sum_{n_x=1}^{\infty}\sum_{n_y=1}^{\infty} \frac{\sin(n_x\pi x)\sin(n_y\pi y)\sin(n_x\pi x_0)\sin(n_y\pi y_0)}{n_x^2\pi^2 + n_y^2\pi^2 - k^2} \tag{8}$$

Finally for 3D the Green's function is written

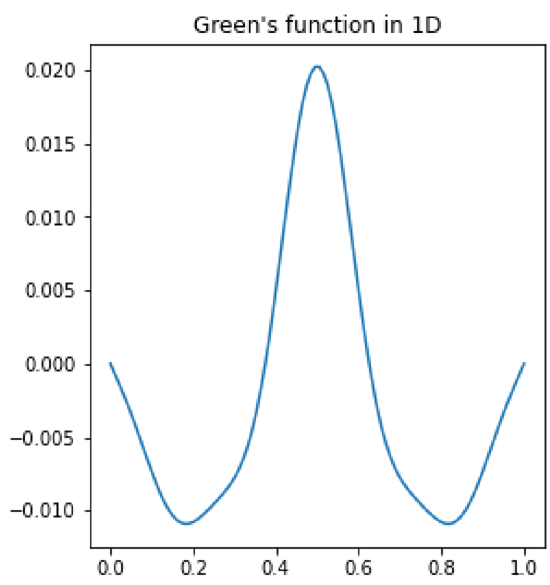
$$G(x,y,z,x_0,y_0,z_0) = \sum_{n_x=1}^{\infty}\sum_{n_y=1}^{\infty}\sum_{n_z=1}^{\infty} \frac{\alpha_{n_x}(x)\alpha_{n_y}(y)\alpha_{n_z}(z)\alpha_{n_x}(x_0)\alpha_{n_y}(y_0)\alpha_{n_z}(z_0)}{k_{n_x}^2 + k_{n_y}^2 + k_{n_z}^2 - k^2} \tag{9}$$

For $\alpha_{n_x}(x) = \sin\left(\frac{n_x \pi x}{L}\right)$, $\alpha_{n_y}(y) = \sin\left(\frac{n_y \pi y}{L}\right)$, $\alpha_{n_z}(z) = \sin\left(\frac{n_z \pi z}{L}\right)$, $k_{n_x} = n_x\pi/L$, $k_{n_y} = n_y\pi/L$, $k_{n_z} = n_z\pi/L$, the 3D case is rewritten to

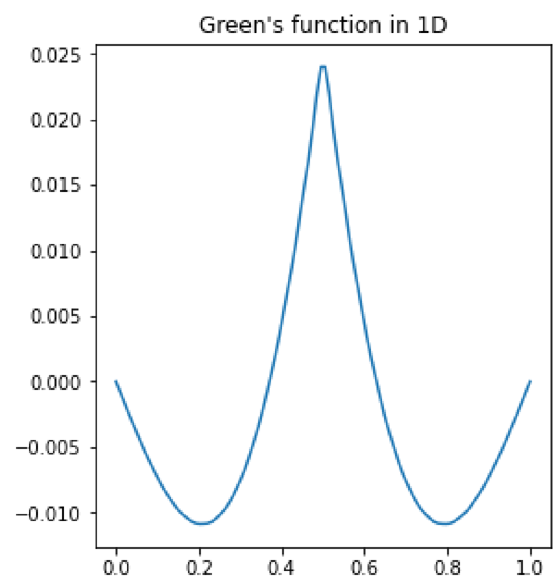
$$G(x,y,z,x_0,y_0,z_0) = \sum_{n_x=1}^{\infty}\sum_{n_y=1}^{\infty}\sum_{n_z=1}^{\infty} \frac{\sin(n_x\pi x)\sin(n_y\pi y)\sin(n_z\pi z)\sin(n_x\pi x_0)\sin(n_y\pi y_0)\sin(n_z\pi z_0)}{n_x^2\pi^2 + n_y^2\pi^2 + n_z^2\pi^2 - k^2} \tag{10}$$

Using Python as our computer mathematics program, we try a different number of terms in the sums over n_x, n_y, and n_z.

First is a comparison of images for the 1D Green's function when having 10 terms compared to 50 terms:



(a) 1D Green's function (10 terms)

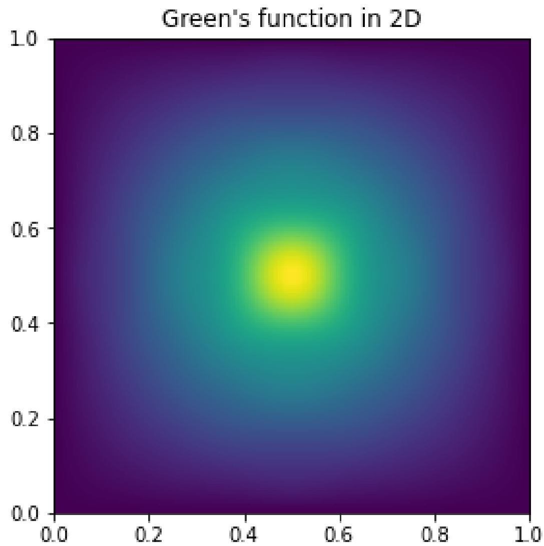


(b) 1D Green's function (50 terms)

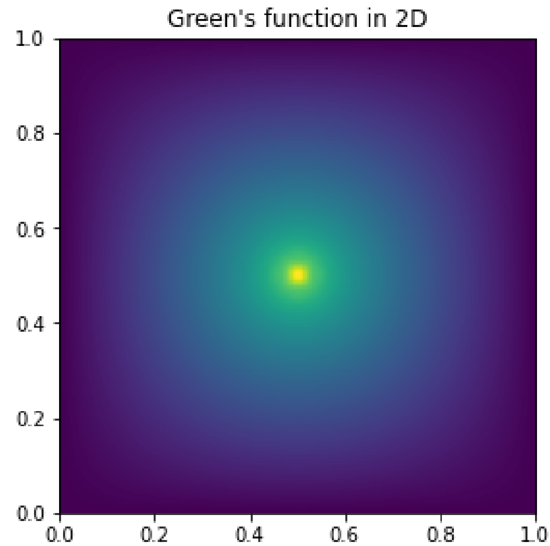
Figure 1: 1D Green's function (10 terms versus 50 terms)

As we can see, the number of terms in the Green's function makes the curve more well-defined.

Next is a comparison of images for the 2D Green's function when having 10 terms versus 50 terms:



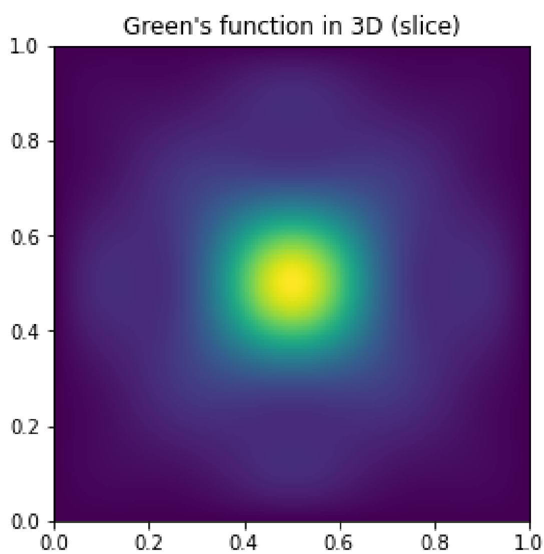
(a) 2D Green's function (10 terms)



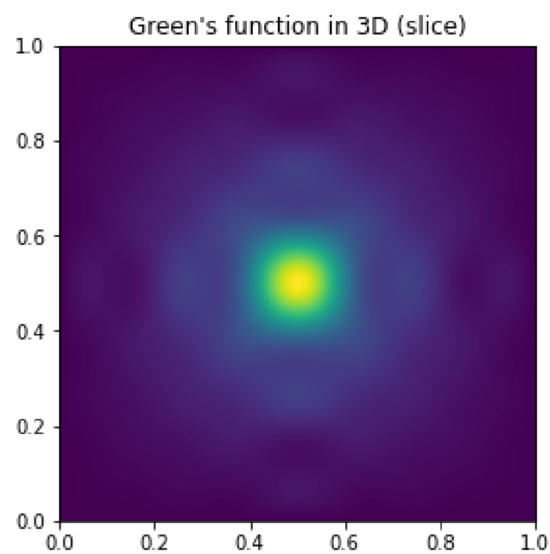
(b) 2D Green's function (50 terms)

Figure 2: 2D Green's function (10 terms versus 50 terms)

Lastly is a comparison of cross-sectional images for the 3D Green's function when having 5 terms versus 10 terms (for the sake of runtime, we decrease the number of terms for comparison):



(a) 3D Green's function (5 terms)



(b) 3D Green's function (10 terms)

Figure 3: 3D Green's function (5 terms versus 10 terms)

As we can see, the Green's functions of every dimension correctly approach $\vec{r}_o$ the more terms that are included. Unsurprisingly, the behavior in the cross-sectional image for the 3D Green's function resembles that of the image for the 2D Green's function.

Problem 2 The Green's function in eqn. 12-314 is the amplitude to go from $\vec{r} = \vec{r}_S$ at $t = t_S$ to $\vec{r}$ at t. We can also define a Green's function to go from one state to another

$$G(k_2, k_1, t_2 - t_1) = \langle 0|a_{k_2}(t_2)a^\dagger_{k_1}(t_1)|0\rangle \tag{11}$$

is the amplitude for a particle to go from the state denoted by k_1 at t_1 to the state k_2 at t_2.

Using the time dependence found in (1) find the Fourier transform of $G_{k_1k_2}(t) = G(k_2, k_1, t_2 - t_1)$ where $t = t_2 - t_1$.

Solution

For the Green's function to go from one state to another

$$G(k_2, k_1, t_2 - t_1) = \langle 0|a_{k_2}(t_2)a^\dagger_{k_1}(t_1)|0\rangle \tag{12}$$

we can express $a_{k_2}(t_2)$ and $a^\dagger_{k_1}(t_1)$ in terms of $a_{k_2}(0)$ and $a^\dagger_{k_1}(0)$ at $t = 0$. This is possible via the time evolution operator $U(t, t_0)$:

$$a_{k_2}(t_2) = U(t_2, 0)a_{k_2}(0)U^\dagger(t_2, 0) \tag{13}$$
$$a^\dagger_{k_1}(t_1) = U(t_1, 0)a^\dagger_{k_1}(0)U^\dagger(t_1, 0) \tag{14}$$

then

$$G_{k_1k_2}(t) = \langle 0|U(t_2, 0)a_{k_2}(0)U^\dagger(t_2, 0)U(t_1, 0)a^\dagger_{k_1}(0)U^\dagger(t_1, 0)|0\rangle \tag{15}$$

Using the time evolution operator $U(t, t_0) = e^{-iH(t-t_0)}$ and that there exists a complete set of states for the particle, we can write

$$G_{k_1k_2}(t) = \sum_n \langle 0|a_{k_2}(0)|n\rangle\langle n|a^\dagger_{k_1}(0)|0\rangle e^{-i(E_n-E_0)t} \tag{16}$$

where E_n, E_0 are the corresponding eigenvalues to the eigenstates $|n\rangle$, $|0\rangle$. Then the Fourier transform of $G_{k_1k_2}(t) = G(k_2, k_1, t_2 - t_1)$ is

$$\mathcal{F}[G_{k_1k_2}(t)] = G(k_2, k_1, \omega) = \int_{-\infty}^{\infty} G(k_2, k_1, t)e^{i\omega t}dt \tag{17}$$
$$= \int_{-\infty}^{\infty} \sum_n \langle 0|a_{k_2}(0)|n\rangle\langle n|a^\dagger_{k_1}(0)|n\rangle e^{-i(E_n-E_0)t}e^{i\omega t}dt \tag{18}$$
$$= \sum_n \langle 0|a_{k_2}(0)|n\rangle\langle n|a^\dagger_{k_1}(0)|0\rangle \int_{-\infty}^{\infty} e^{-i(E_n-E_0-\omega)t}dt \tag{19}$$

Since we know

$$\int_{-\infty}^{\infty} e^{-i(E_n-E_0-\omega)t}dt = 2\pi\delta(E_n - E_0 - \omega) \tag{20}$$

The Fourier transform of $G_{k_1k_2}(t)$ is thus

$$G(k_2, k_1, \omega) = 2\pi \sum_n \langle 0|a_{k_2}(0)|n\rangle\langle n|a^\dagger_{k_1}(0)|0\rangle\delta(E_n - E_0 - \omega) \tag{21}$$