PY 509 HW 23-25

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Problem 1 A simple model star with equation of state $\rho = \rho_0$ (a constant) can be constructed analytically. The equation of state is not physically realistic but, nevertheless, this model can be fun and instructive to examine.

- Solve the stellar structure equations for $m(r)$ and $P(r)$, assuming constant density $\rho = \rho_0$. Express your results in terms of the total mass M and radius R. You can assume $2M/R < 1$.
- Argue that the pressure $P(r)$ is a monotonically decreasing function of r. Thus, the maximum pressure is at the center, $r = 0$.
- Show that the central pressure $P(0)$ is finite and positive only if $2M/R > 8/9$.

Solution

Part 1 For a spherically symmetric star in hydrostatic equilibrium, the stellar structure equations for $m(r)$ and $P(r)$ are given by the hydrostatic equilibrium equation and mass continuity equation

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}, \quad \frac{dm}{dr} = 4\pi r^2 \rho \tag{1}$$

For constant density $\rho = \rho_0$ the mass continuity equation has a solution

$$m(r) = \frac{4\pi}{3}\rho_0 r \tag{2}$$

Then

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2} = -\frac{4\pi G}{3}\rho_0 r \tag{3}$$

$$\int dP = -\int \frac{4\pi G}{3}\rho_0 r dr \tag{4}$$

$$P(r) = -\frac{2\pi G}{3}\rho_0 r^2 + P(0) \tag{5}$$

At the star's surface, $P(r)|_{r=R} = 0$ and so

$$0 = -\frac{2\pi G}{3}\rho_0 R^2 + P(0) \tag{6}$$

$$P(0) = \frac{2\pi G}{3}\rho_0 R^2 \tag{7}$$

Thus the pressure is finally expressed as

$$P(r) = \frac{2\pi G}{3}(R^2 - r^2) \tag{8}$$

To express in terms of the total mass M and radius R, the total mass and radius is

$$M = m(R) = \frac{4\pi}{3}\rho_0 R^3, \quad R^3 = \frac{3M}{4\pi\rho_0} \tag{9}$$

Part 2 Looking back at the hydrostatic equilibrium equation

$$\frac{dP}{dr} = -\frac{4\pi G}{3}\rho_0 r \tag{10}$$

ρ_0, G, and r are all positive, and if dP/dr is always negative, then $P(r)$ must be a monotonically decreasing function of r. And consequently, the maximum pressure is at the center ($r = 0$).

Part 3 The pressure at $r = 0$ can be further written

$$P(0) = \frac{2\pi G}{3}\rho_0 \left(\frac{3M}{4\pi\rho_0}\right) = \frac{GM^2}{2R^2} \tag{11}$$

Thus $P(0)$ is finite and positive only if $2M/R > 8/9$.

Problem 2 Consider a particle that falls freely from rest, radially, through the horizon of a static, spherically symmetry black hole of mass M.

- Using Schwarzschild or Kerr-Schild coordinates, write down the first integrals of the timelike geodesic equations.
- Use these equations to compute $r(\tau)$, the areal radius r as a function of proper time τ, for a radial geodesic. Assume the particle begins at rest at $r = \infty$.
- Find the proper time for a particle to pass through the horizon $r = 2M$ to the singularity at $r = 0$.
- Insert the appropriate factors of G and c and evaluate your result (in seconds) for various values of the black hole mass M.

Solution

Part 1 The line element for a particle moving in a radial geodesic in Schwarzschild sptm is

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 + \frac{1}{1 - \frac{2GM}{c^2 r}}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{12}$$

The proper time interval for the path between two events A and B is

$$\delta\tau = \int_A^B \sqrt{-g_{\mu\nu}u^\mu u^\nu}\, dt \tag{13}$$

But also

$$S = \int \mathcal{L} d\tau, \quad d\tau = \sqrt{-g_{\mu\nu}dx^\mu dx^\nu}, \quad \mathcal{L} = -\frac{c^2}{2}g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \tag{14}$$

When substituting the Schwarzschild metric components into this Lagrangian we get

$$\mathcal{L} = -\frac{c^2}{2}\left(1 - \frac{2GM}{c^2 r}\right)\dot{t}^2 + \frac{1}{2\left(1 - \frac{2GM}{c^2 r}\right)}\dot{r}^2 + \frac{r^2\dot{\theta}^2}{2} + \frac{r^2\sin^2\theta\dot{\phi}^2}{2} \tag{15}$$

The first integrals of the timelike geodesic equations can be found in the Lagrangian. The energy per unit mass E is

$$\mathcal{L} = -\frac{c^2}{2}\left(1 - \frac{2GM}{c^2 r}\right)\dot{t}^2 \implies E = -\left(1 - \frac{2GM}{c^2 r}\right)\dot{t} \tag{16}$$

Where the angular momentum per unit mass L is

$$\mathcal{L} = \frac{r^2 \sin^2 \theta \dot{\phi}^2}{2} \implies L = r^2 \sin^2 \theta \dot{\phi} \tag{17}$$

In a radial geodesic L is zero ($\dot{\theta} = 0, \theta = \pi/2$) and thus

$$E = -\left(1 - \frac{2GM}{c^2 r}\right) \dot{t} \implies \dot{t} = \frac{E}{1 - \frac{2GM}{c^2 r}} \tag{18}$$

Then

$$\frac{dr}{d\tau} = \pm \sqrt{\frac{E^2}{c^2} - \left(1 - \frac{2GM}{c^2 r}\right)\left(1 + \frac{L^2}{r^2}\right)} \tag{19}$$

If for a radial geodesic (when $L = 0$) we get

$$\frac{dr}{d\tau} = \pm \sqrt{\frac{E^2}{c^2} - \left(1 - \frac{2GM}{c^2 r}\right)} \implies \int d\tau = \pm \int \frac{dr}{\sqrt{\frac{E^2}{c^2} - \left(1 - \frac{2GM}{c^2 r}\right)}} \tag{20}$$

Part 2 Assuming the particle begins at rest at $r = \infty$, the areal radius as a function of proper time can be computed numerically. Implementing an RK4 method to iterate τ for steps in r, we use 1000 steps, $M = E = c = 1$ with $\tau_{final} = 10.0$ and $r_{initial} = 100.0$. Here we get a linear relation of the proper time to the areal radius shown below:

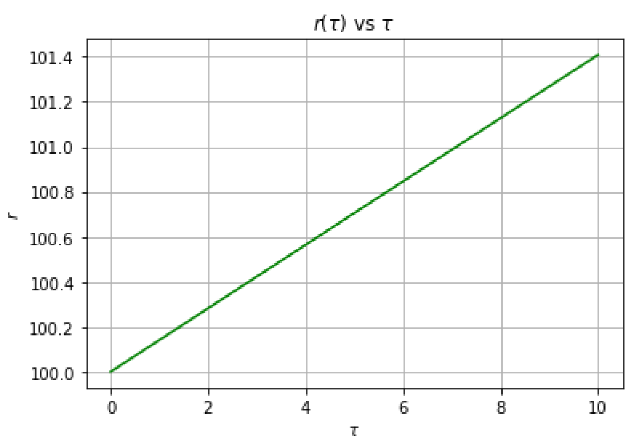

Figure 1: Caption

Part 3 To find the proper time for a particle to pass through the horizon $r = 2M$ to the singularity at $r = 0$, we have (still using the assumption that the particle begins at $r = \infty$)

$$\tau = \int_{\infty}^{2M} \frac{dr}{\sqrt{\frac{E^2}{c^2} - \left(1 - \frac{2GM}{c^2 r}\right)}} \tag{21}$$

Where $E = -\left(1 - \frac{2GM}{c^2 r}\right) \dot{t}$. Into python using quadrature in the *mpmath* package with *numpy*, the proper time comes out to be roughly $6.83 * 10^{19}$ seconds (when the black hole mass is 6 solar masses).

Part 4 We finally insert the appropriate factors of G and c to evaluate our result (in seconds) for various values of the black hole mass M. When the black hole mass is 8 solar masses, the proper time is 100 times more than when the black hole mass is 6 solar masses. Further, the proper time for a black hole with a mass of 10 solar masses is 100 times greater than when the black hole mass is 8 solar masses. Thus, the proper time increases proportionately to the change in black hole mass.

Problem 3 For the collapse of an Oppenheimer-Snyder dust star:

- Compute the proper time from the beginning of the collapse to the formation of the horizon, as measured by an observer on the surface of the star.
- Compute the proper time from the formation of the horizon to the singularity, as measured by an observer on the surface of the star.
- Compute the proper time from the beginning of the collapse to the formation of the singularity, as measured by an observer at the center of the star.

Insert the appropriate factors of G and c and find numerical values for a star of mass $M = M_{sun}$ and initial radius $R_0 = R_{sun}$.

Solution

Part 1 The Oppenheimer-Snyder metric for a dust star is given by

$$ds^2 = -c^2 dt^2 + \frac{R^2}{1 - \frac{r^2}{R^2}} dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{22}$$

Where $R = R(t)$ is the collapsing star's radius and t is the proper time from a comoving observer. The proper time τ from an observer on the surface of the star occurs when $dr = d\theta = d\phi = 0$ for the metric. The event horizon occurs when the Schwarzschild radius is the proper radius of the collapsing star, or $R(t) = \frac{2GM}{c^2}$. At the horizon, $r = R(t)$ and $dr = d\theta = d\phi = 0$. Thus, $ds^2 = -c^2 dt^2$. The proper time from the beginning of the collapse of the formation of the horizon, as measured by an observer on the surface of the star, is the time for when $R(t) = \frac{2GM}{c^2}$.

The Friedmann equation for dust clouds takes the form

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} \tag{23}$$

Or differentiating and rearranging,

$$2\left(\frac{\dot{R}}{R}\right)\frac{d\dot{R}}{dt} = \frac{8\pi G}{3}\frac{d\rho}{dt} = \frac{8\pi G}{3}\frac{d\rho}{dR}\frac{dR}{dt} = \frac{8\pi G}{3}\sqrt{\frac{8\pi G\rho}{3}}\frac{dR}{dt} \tag{24}$$

$$\frac{d\dot{R}}{dt} = -\sqrt{\frac{8\pi G\rho}{3}}R^{1/2} \tag{25}$$

Where $\rho = M/\left(\frac{4}{3}\pi R_0^3\right)$. We can now solve for $R(t)$ and then t:

$$\frac{dR}{dt} = -\sqrt{\frac{8\pi G\rho}{3}}R(t) = -\sqrt{\frac{8\pi G}{3}\frac{M}{\frac{4}{3}\pi R_0^3}}R(t) = -\sqrt{\frac{2GM}{R_0^3}}R(t) \tag{26}$$

Integration:

$$\int_{R_0}^{2GM/c^2} \frac{dR}{R} = -\int_0^{\Delta t} \sqrt{\frac{2GM}{R_0^3}} dt \tag{27}$$

$$\ln\left(\frac{2GM}{R_0^3 c^2}\right) = -\sqrt{\frac{2GM}{R_0^3}} \Delta t \tag{28}$$

$$\Delta t = -\ln\left(\frac{\dots}{R_0}\right)\left(\frac{2GM}{R_0^3}\right)^{-1/2} \tag{29}$$

Using R_0 as the radius of the sun and M as one solar mass, into a calculator we find that the proper time is roughly 6.38×10^8 seconds.

Part 2 The proper time from the formation of the horizon to the singularity, as measured by an observer on the surface of the star occurs when $R(t) = 0$.

Implementing into python to find the proper time when $R(t) = 0$, we stop the numerical integration for when $R(t)$ approaches zero in a tolerance equal to $1 * 10^{-10}$. Our code takes the integration as solving an ODE system with *solve_ivp* as the integration package of *scipy*.

Doing so, the computed proper time from the formation of the horizon to the singularity, as measured by an observer on the star's surface τ is equal to roughly 28.35 seconds.

Part 3 The proper time form the Schwarzschild metric for an observer from the beginning of the collapse to the formation of the singularity is given by

$$d\tau = \sqrt{1 - \frac{2GM}{c^2 R(t)}} dt \implies \tau = \int d\tau = \int \sqrt{1 - \frac{2GM}{c^2 R(t)}} dt \tag{30}$$

From earlier

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G\rho}{3} = \frac{8\pi G}{3}\frac{3M}{4\pi R^3} = \frac{8GM}{R^3} \tag{31}$$

Letting $R = R_0$ at $t = 0$, then

$$\int_0^{t_{collapse}} dt = \int_{R_0}^0 \frac{dR}{\sqrt{\frac{8GM}{R^3}}} \tag{32}$$

$$t_{collapse} = \frac{2}{\sqrt{8GM}} \int_{R_0}^0 R^{-3/2} dR = \frac{2}{\sqrt{8GM}} \left(-2R^{-1/2}\right)_{R_0}^0 \tag{33}$$

$$= \frac{4}{\sqrt{8GM}} R_0^{-1/2} = \sqrt{\frac{R_0}{2GM}} \tag{34}$$

Plugging in for a mass of one solar mass with the radius of the sun, the collapse time is calculated to equal

$$t_{collapse} = \sqrt{\frac{(6.957 * 10^8}{2(6.674 * 10^{-11})(1.989 * 10^{30}}} \tag{35}$$

$$\approx 1.62 * 10^{-6}\ s \tag{36}$$