Chapter 13: Matrix

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1 General approach

1.1 Definition

1.1.1 Definition of a matrix

We call matrix of n rows and p columns any mapping in the following form:

$$\begin{array}{cc} [\![1,n]\!] \times [\![1,p]\!] & \to \mathbb{K} \\ i,j & a_{ij} \end{array}$$

We denote such maps as tables of n rows and p columns, and we write:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix}$$

$\forall (i,j) \in [\![1,n]\!] \times [\![1,p]\!]$, we call a_{ij} a coefficient of the matrix. In this case coefficient if i-th row and j-th column.

1.1.2 Notation

We denote $M_{np}(\mathbb{K})$ the set of matrix of n rows and p columns with coefficient from $\mathbb{K}$.

1.1.3 Examples

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \in M_{32}(\mathbb{R})$$

$$B = \begin{pmatrix} i \\ 1+i \\ 3 \end{pmatrix} \in M_{31}(\mathbb{C})$$

1.2 Particular matrices

Let $A \in M_{np}(\mathbb{K})$ then:

1.2.1 Null matrix

1. $[\forall (i,j) \in [\![1,n]\!] \times [\![1,p]\!], a_{ij} = 0] \Rightarrow [A = 0_{np}]$ We say A is the null matrix $M_{np}(\mathbb{K})$.

1.2.1.1 Example

$$A' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \in M_{32}(\mathbb{R})$$

1.2.2 Column matrix

2. $B \in M_{np}(\mathbb{K})$ and $p = 1 \Rightarrow$ B is a column matrix of n rows

1.2.2.1 Example

$$B' = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in M_{31}(\mathbb{R})$$

1.2.3 Row matrix

3. $B \in M_{np}(\mathbb{K})$ and $n = 1 \Rightarrow$ C is a row matrix of p columns

1.2.3.1 Example

$$C' = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \in M_{13}(\mathbb{R})$$

1.2.4 Square matrix

We call square matrix any matrix with same number of rows and columns. We denote $M_n(\mathbb{K})$ the set of square matrix of n rows and columns with coefficient from $\mathbb{K}$.

4. $D \in M_{np}(\mathbb{K})$ and $n = p \Rightarrow$ D is a square matrix denote $M_n(\mathbb{K})$

1.2.4.1 Example

$$D' = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \in M_3(\mathbb{R})$$

1.2.5 Diagonal matrix

5. $\forall E \in M_n(\mathbb{R})$, if $\forall (i,j) \in [\![1,n]\!]^2, i \neq j \Rightarrow a_{ij} = 0$ then we say E is a diagonal matrix

1.2.5.1 Example

$$E' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \in M_2(\mathbb{R})$$

$$E'' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \in M_3(\mathbb{R})$$

1.2.6 Identity matrix

6. $\forall I_n \in M_n(\mathbb{R})$, if $\forall (i,j) \in [\![1,n]\!]^2, i \neq j \Rightarrow a_{ij} = 0$ and $i = j \Rightarrow a_{ij} = 1$ then we say I_n is a identity matrix

1.2.6.1 Example

$$I'_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{R})$$

$$I''_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in M_3(\mathbb{R})$$

1.2.7 Triangular matrix

6. $\forall F \in M_n(\mathbb{R})$, if $\forall (i,j) \in [\![1,n]\!]^2, i > j \Rightarrow a_{ij} = 0$ then we say F is a lower triangular matrix

7. $\forall G \in M_n(\mathbb{R})$, if $\forall (i,j) \in [\![1,n]\!]^2, i < j \Rightarrow a_{ij} = 0$ then we say G is a upper triangular matrix

1.2.7.1 Example

$$F' = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \in M_2(\mathbb{R})$$

$$G' = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \in M_2(\mathbb{R})$$

1.3 Transposed matrix

1.3.1 Definition

Let $A \in M_{np}(\mathbb{K})$. We call transposed matrix of A (or A transpose) a matrix B from $M_{pn}(\mathbb{K})$ such as:

$$\forall (i,j) \in [\![1,n]\!] \times [\![1,p]\!], a_{ij} = b_{ji}$$

1.3.2 Notation

We denote B as ${}^tA$

1.3.3 Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \in M_{23}(\mathbb{R})$$

$${}^tA = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \in M_{32}(\mathbb{R})$$

1.4 Symmetric matrix

1.4.1 Symmetric

If ${}^tA = A$ then we say A is symmetric

1.4.1.1 Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} = {}^tA = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} \in M_3(\mathbb{R})$$

1.4.2 Anti-Symmetric

If ${}^tA = -A$ then we say A is Anti-symmetric

1.4.2.1 Example

$$A = \begin{pmatrix} 0 & -2 & 3 \\ 2 & 0 & -5 \\ -3 & 5 & 0 \end{pmatrix} = {}^tA = \begin{pmatrix} 0 & 2 & -3 \\ -2 & 0 & 5 \\ 3 & -5 & 0 \end{pmatrix} \in M_3(\mathbb{R})$$

2 Operations on matrices

2.1 Addition and external product

2.1.1 Definition

1. We call internal operation in $M_{np}(\mathbb{K})$ denoted $\oplus$ "internal addition" the one definde as follows:
$$\forall A, B \in M_{np}^2(\mathbb{K}), A + B = (a_{ij} + b_{ij})_{\substack{1\leq i\leq n \\ 1\leq j\leq p}}$$
$$\text{Where } A = a_{ij\,\substack{1\leq i\leq n \\ 1\leq j\leq p}} \text{ and } B = b_{ij\,\substack{1\leq i\leq n \\ 1\leq j\leq p}}$$

2. We call "external multiplication" or "multiplication by a scalar" the one defined as follows:
$$\forall A \in M_{np}(\mathbb{K}), \forall \alpha \in \mathbb{K}, \alpha A = (\alpha a_{ij})_{\substack{1\leq i\leq n \\ 1\leq j\leq p}}$$

2.1.1.1 Example

$$(A, B) \in M_{2,3}(\mathbb{R})^2 \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \text{ and } B = \begin{pmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix}$$
$$A + B = \begin{pmatrix} 1+7 & 2+8 & 3+9 \\ 4+10 & 5+11 & 6+12 \end{pmatrix} = \begin{pmatrix} 8 & 10 & 12 \\ 14 & 16 & 18 \end{pmatrix}$$
$$\alpha = 3, \quad \alpha A = \begin{pmatrix} 3\times 1 & 3\times 2 & 3\times 3 \\ 3\times 4 & 3\times 5 & 3\times 6 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{pmatrix}$$

2.1.1.2 Proposition

$$(M_{np}, \oplus, \cdot) \text{ is a vector space over } \mathbb{K}$$

2.1.2 Elementary matrix

For $(n,p) \in \mathbb{N}^2, (i,j) \in [\![1,n]\!] \times [\![1,p]\!]$; We denote E_{ij} the matrix from $M_{np}(\mathbb{K})$ such that the ij-th coefficient is 1 and all other coefficient are 0.

$$E_{ij} \text{ are called elementary matrix}$$

2.1.2.1 Example

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
$$E_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$E_{33} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.1.3 Proposition

1. $(E_{ij})_{\substack{1\leq i\leq n \\ 1\leq j\leq p}}$ is a basis of $M_{np}(\mathbb{K})$
2. $dim((E_{ij})_{\substack{1\leq i\leq n \\ 1\leq j\leq p}}) = np$
   Ex: $M_2(\mathbb{R})$ a ($\mathbb{K}$)-VS: B $= \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$
   B is a Standard basis of $M_2(\mathbb{R})$, $dim(M_2(\mathbb{R})) = 2^2 = 4$

2.2 Internal product

2.2.1 Definition

Let $(n,p,q) \in \mathbb{N}^3$ and $A = a_{ij\,\substack{1\leq i\leq n \\ 1\leq j\leq p}} \in M_{np}(\mathbb{K}), B = b_{ij\,\substack{1\leq i\leq p \\ 1\leq j\leq q}} \in M_{pq}(\mathbb{K})$. We call product of A and B the matrix C form $M_{nq}(\mathbb{K})$ such that:

$$\forall (i,j) \in [\![1,n]\!] \times [\![1,q]\!], c_{ij} = \sum_{k=1}^{p} a_{ik}b_{kj}$$

2.2.1.1 Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \in M_{2,3}(\mathbb{R}) \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in M_{3,4}(\mathbb{R})$$

$$C = A \cdot B = \begin{pmatrix} 1 & 0 & 8 & 11 \\ 4 & 0 & 20 & 32 \end{pmatrix} \in M_{2,4}(\mathbb{R})$$

$$C_{2,3} = 4 \times 1 + 5 \times 2 + 6 \times 1 = 20$$

2.2.2 Remarks

R1 If A, B two matrices: we only can multiply A by B if the number of column of A is equal to the number of row of B.

R2 AB can exists but BA not or the other way around.

2.2.2.1 Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \Rightarrow M_{2,1}(\mathbb{R}), \text{ and exists but } BA \text{ does not exists}$$

R3 In the $\boxed{\text{General case}}$, where AB and BA exists:
$\boxed{\text{AB} \neq \text{BA}}$ (multiplication of matrix is not commutative)
When $AB = BA$ we say A and B commute.

2.3 Properties of matrix calculus

2.3.1 Properties

1. Let A, B two matrices such that AB exists.
   We can have $AB = 0$ and $(A \neq 0$ or $B \neq 0)$

$$\text{If } A = 0 \text{ or } B = 0 \text{ then } AB = 0$$

$$AB = 0 \nRightarrow A = 0 \text{ or } B = 0$$

2.3.1.1 Example

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

2. $(n,p,q,r) \in \mathbb{N}^4$ and let $(A,B,C) \in M_{np}(\mathbb{K}) \times M_{pq}(\mathbb{K}) \times M_{qr}(\mathbb{K})$

$$A \cdot (B \cdot C) = (A \cdot B) \cdot C \otimes \text{ is commutative}$$

3. $(A,B,C) \in M_{np}(\mathbb{K}) \times M_{pq}^2(\mathbb{K})$

$$A \cdot (B + C) = A \cdot B + A \cdot C$$

   $\otimes$ (Matrices multiplication) is distributive over matrix addition ($\oplus$).

4. $A \in M_{np}(\mathbb{K})$ and $B \in M_{pq}(\mathbb{K})$ and $\lambda \in \mathbb{K}$

$$\lambda \cdot (A \cdot B) = A \cdot \lambda \cdot B = A \cdot B \cdot \lambda$$

2.3.2 Case of Square Matrices

1. $$\forall A \in M_n(\mathbb{K}), A \cdot I_n = I_n \cdot A = A$$

2. Let $(A,B) \in M_n(\mathbb{K})^2$, such that $\boxed{AB = BA}$ Then:

$$(A+B)^n \boxed{=} \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}$$

   By convention, $A^0 = B^0 = I_n$

3. $$\forall (A,B) \in M_n(\mathbb{K})^2, {}^t(A \cdot B) = {}^tA \cdot {}^tB$$

2.4 Inverse of a matrix

2.4.1 Definition

Let $A \in M_n(\mathbb{K})$ we say that A is invertible if:

$$\exists B \in M_n(\mathbb{K}), AB = BA = I_n$$

Then we say that B is the inverse of A and denote $B = A^{-1}$ (B is unique) Hence we have (in case of A invertible): $A \cdot A^{-1} = A^{-1} \cdot A = I_n$

2.4.2 Notation

The set of invertible matrices of $M_n(\mathbb{K})$ is denoted $GL_n(\mathbb{K})$

2.4.3 How to find the inverse of a matrix

We will use the following system: Where $A \in M_n(\mathbb{K}), (U,V) \in M_{n,1}(K)^2$:

$$A \cdot U = V$$

By solving this system (Gauss elimination algorithm) when A is invertible, we will have:

$$U = A^{-1} \cdot V$$

2.4.3.1 Example

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \; A \cdot X = U \Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z-x \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z-x-y \end{pmatrix}$$

$$\iff \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} x-y+z \\ -x+y+z \\ z-x-y \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{x-y+z}{2} \\ \frac{-x+y+z}{2} \\ \frac{x+y-z}{2} \end{pmatrix}$$

$$\iff \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

3 Matrices of Linear Maps

3.1 Definition and examples

3.1.1 Definition

Let $f \in \mathcal{L}(E, F)$, E and F finite dimensional $\mathbb{K}$-vector spaces such that: $dim(E) = p$ and $dim(F) = n$ where $(p, n) \in \mathbb{N}^2$ and $B = (e_1, e_2, \ldots, e_n)$ basis of E and $B' = (e'_1, e'_2, \ldots, e'_n)$ basis of F

$$\forall U \in E, \exists!(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{K}^p, \quad U = \sum_{i=1}^{p} \lambda_i e'_i$$

We say $\begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{pmatrix}$ is the column matrix of coordinates $A \cdot U$ in B.

3.1.2 Example

Let $E = \mathbb{R}^2$, $U = \begin{pmatrix} 1 \\ 2 \end{pmatrix}_B$ with $B = (\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix})$ and $B' = (\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix})$Then:

$$U = \begin{pmatrix} 2 \\ 1 \end{pmatrix}_B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}_B + \begin{pmatrix} 1 \\ 0 \end{pmatrix}_B \Rightarrow U = \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{B'}$$

3.1.3 Definition

We call matrix of $f \in \mathcal{L}(E, F)$ with respect to basis B and B' denoted $Mat_{BB'}(f)$ the matrix whose $j - th$ column is composed of the coordinates of $f(e_j)$ in B', for all j from $[\![1, p]\!]$. This is a matrix of p columns and n rows:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix}, \forall j \in [\![1, p]\!], f(e_j) = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}_B^{\prime} = \sum_{i=1}^{n} a_{ij} e_i$$

3.1.4 Example

$$f: \begin{array}{ccc} R^2 & \longrightarrow & R^3 \\ \begin{pmatrix} x \\ y \end{pmatrix} & \longmapsto & \begin{pmatrix} x + y \\ 2x + 4y \\ -3y \end{pmatrix} \end{array}$$

1 basis for the domain (R^2): B $= (\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix})$

basis for the codomain (R^3): $B' = (\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix})$

2 $$\begin{pmatrix} x + y \\ 2x + 4y \\ -3y \end{pmatrix} \Rightarrow \begin{cases} f(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}_{B'} \\ f(\begin{pmatrix} 0 \\ 1 \end{pmatrix}) = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}_{B'} \end{cases}$$

3

$$\forall U \in R^2, U = x \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow f(U) = x \cdot f\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \cdot f\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 0 & -3 \end{pmatrix}$$