Chapter 12: Linear Maps

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Contents

1 General approach

1.1 Definition

Let E, F two $\mathbb{K} - VS$, and f a mapping from E to F. We say that f is a linear (or f is a linear map) if:

$$\forall(\alpha, X, Y) \in \mathbb{K} \times E \times E, f(\alpha \cdot X + Y) = \alpha \cdot f(X) + f(Y)$$

$$\iff$$

$$\forall(\alpha, \beta, X, Y) \in \mathbb{K} \times \mathbb{K} \times E \times E, f(\alpha \cdot X + \beta \cdot Y) = \alpha \cdot f(X) + \beta \cdot f(Y)$$

1.2 Notation

We denote $L(E, F)$ the set of all linear maps from E to F.

1.3 Specific Linear Maps

1.3.1 Definition

1. Let $f \in \mathcal{L}(E, F)$: we say f is an endomorphism if $E = F$ we then denote $\mathcal{L}(E)$ the set of all endomorphism of E.
2. Let $f \in \mathcal{L}(E, F)$: we say f is an isomorphism if f is bijective.
3. Let $f \in \mathcal{L}(E, F)$: we say f is an automorphism if f is an endomorphism and an isomorphism. ($E = F$ and bijective)

1.4 Necessary Condition

$$f \in \mathcal{L}(E, F) \Longrightarrow f(0_E) = 0_F$$

1.4.1 Proof

Let $X \in E$	Let $X \in E$
$f(0_E) = f(0_E \times X)$	$f(0_E) = f(X - X)$
$f(0_E) = 0_F \times f(X)$	$f(0_E) = f(X) - f(X)$
$f(0_E) = 0_F$	$f(0_E) = 0_F$

2 Kernel and Images

2.1 Definition

Let E and F two $\mathbb{K}-VS$ and $f \in \mathcal{L}(E,F)$. Then:

1. We call kernel of f and denote $Ker(f)$ the subset of E defined as follows:

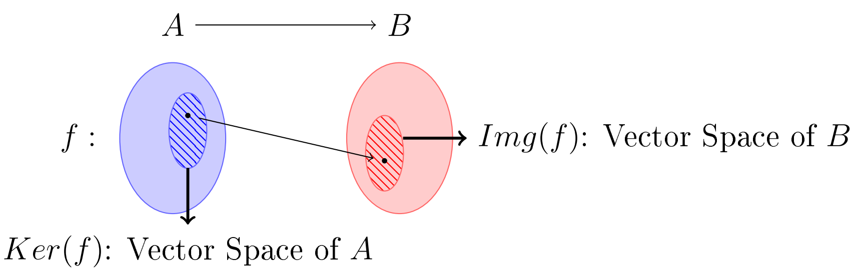
$$Ker(f) = \{X \in E, f(X) = 0_F\} = f^{-1}(\{0_F\})$$

Note: $f^{-1}()$ is NOT the inverse of f beacause f is not necessarily bijective.

2. We call image of f and denote $Im(f)$ the subset of F defined as follows:

$$Im(f) = \{f(X), X \in E\} = \{Y \in F, \exists X \in E, f(X) = Y\}$$

2.2 Graphics representation



2.3 Example

$$f\colon \quad R^2 \longrightarrow R^3$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x \\ 0 \\ y \end{pmatrix}$$

1 $f \in \mathcal{L}(R^2, R^3)$?

2 Kerf = ?

3 Imf = ?

1 Necessary condition: $f(0_E) = 0_F$: $f(0_{R^2}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$?

$$\forall (\alpha, X, Y) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2, X = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } Y = \begin{pmatrix} x' \\ y' \end{pmatrix}, x, y, x', y' \in \mathbb{R}$$

$$f(\alpha \cdot X + Y) = \begin{pmatrix} \alpha \cdot x + x' \\ \alpha \cdot y + y' \end{pmatrix} = \begin{pmatrix} \alpha \cdot x + x' \\ 0 \\ \alpha \cdot y + y' \end{pmatrix} = \alpha \cdot \begin{pmatrix} x \\ 0 \\ y \end{pmatrix} + \begin{pmatrix} x' \\ 0 \\ y' \end{pmatrix} = \alpha \cdot f(X) + f(Y)$$

so 1 $f \in \mathcal{L}(R^2, R^3)$ ✓

2 $Ker(f) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in R^2, \begin{pmatrix} x \\ 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

$$③ \; Im(f) = \left\{ \begin{pmatrix} x \\ 0 \\ y \end{pmatrix}, (x,y) \in R^2 \right\} = \left\{ x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, (x,y) \in R^2 \right\}$$

$$Im(f) = span \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

2.4 Proposition

1. Let $f \in \mathcal{L}(E,F)$ and $g \in \mathcal{L}(F,G)$. Then:

$$g \circ f \in \mathcal{L}(E,G)$$

2. If f is bijectif then f^{-1} is bijective and $f^{-1} \in \mathcal{L}(F,E)$

3. $\mathcal{L}(E,F)$ is a $\mathbb{K} - VS$:

$$\mathcal{L}(E,F): \quad \begin{aligned} E &\longrightarrow F \\ X &\longmapsto B_F \in \mathcal{L}(F,E) \end{aligned}$$

$$\forall(\alpha,f,g) \in \mathbb{K} \times \mathcal{L}^2(E,F) \quad \alpha \cdot f + g \in \mathcal{L}(E,F)$$

4. Let $f \in \mathcal{L}(A,B)$:
   - f is injective $\iff Ker(f) = \{0_A\}$
   - f is surjective $\iff Im(f) = B$

3 Projects and Symmetries

3.1 Reminder:

3.1.1 Definition of supllementary subspaces

Let E a $\mathbb{K}$ - VS. Let F and G two supllementary $\mathbb{K}$ - VSS of E:

$$E = F \oplus G \iff \forall X \in E, \exists!(X_F, X_G) \in F \times G, X = X_F + X_G$$

3.2 Proposition

Let us consider:

$$p: \quad \begin{aligned} E &\longrightarrow \underbrace{F \;\subset E}_{E} \\ X &\xrightarrow{\;p\;} X_F \xrightarrow{\;p\;} X_F \end{aligned}$$

1. $p \in \mathcal{L}(E)$
2. $p \circ p \; (= p^2) = p$
3. - $Ker(p) = G$
   - $Im(p) = F$

3.3 Definition

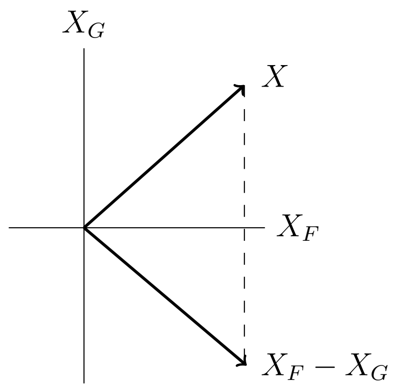
We call projector from $\mathbb{K}$ - VS E over $\mathbb{K}$ - VS F, any endomorphism p of E such that $p \circ p = p$. (We often say that p is a projector over F parallel of/alongside G.)

$$Im(p) \oplus Ker(p) = E$$

Let us consider:

$$S : \begin{array}{ccc} E & \longrightarrow & E \\ X & \longmapsto & (2P - Id_E)(X) = 2p(x) - X \end{array}$$

1. $S \in \mathcal{L}(E)$
2. $\forall X \in E, S(X) = X_E - X_G$
3. $S \circ S = Id_E$



4 Rank Nullity Theorem (RNT)

4.1 Definition of Rank

Let E and F two finite dimensional vector spaces. We call rank of any mapping f from $\mathcal{L}(E, F)$ and denote $rank(f)$ the dimension of $Im(f)$:

$$rank(f) = dim(Im(f))$$

4.2 Proposition

Let E a finite dimensional $\mathbb{K}$ - VS such that $B = (U_1, U_2, \ldots, U_n)$ a basis of E. Let F a $\mathbb{K}$ - VS. Then let $f \in \mathcal{L}(E, F)$. We have:

$$Im(f) = sp(\{f(U_1), f(U_2), \ldots, f(U_n)\})$$

We then deduce the following theorem:

4.3 Theorem (Rank Nullity Theorem)

Let E a finite dimensional VS and $f \in \mathcal{L}(E, F)$. Then:

$$dim(E) = dim(Ker(f)) + \underbrace{rank(f)}_{dim(Im(f))}$$

4.4 Corollary

Let $f \in \mathcal{L}(E, F)$ where E and F two finite dimensional vector spaces.

$$\text{If } dim(E) = dim(F)(\Longleftrightarrow dim(Ker(f)) = 0 \text{ or } dim(Im(f)) = 0)$$

$$\Longrightarrow$$

$$\text{f injective} \Longleftrightarrow \text{f surjective} \Longleftrightarrow \text{f bijective}$$

4.5 Involvement

$$\text{f injective} \Longleftrightarrow Ker(f) = \{0_E\}$$
$$[\text{f injective} \Longrightarrow dim(E) \leq dim(F)] \Longleftrightarrow [dim(E) > dim(F) \Longrightarrow \text{f not injective}]$$
$$\text{f surjective} \Longleftrightarrow Im(f) = F$$
$$[\text{f surjective} \Longrightarrow dim(E) \geq dim(F)] \Longleftrightarrow [dim(E) < dim(F) \Longrightarrow \text{f not surjective}]$$

4.6 Proof of corollary

Hypothesis: $dim(E) = dim(F)$

$$\begin{aligned}
\text{f injective} &\Longleftrightarrow Ker(f) = \{0_E\} \\
&\Longleftrightarrow dim(Ker(f)) = 0 \\
&\Longleftrightarrow dim(E) = dim(Ker(f)) + dim(Im(f)) \\
&\Longleftrightarrow dim(E) = dim(Im(f)) \\
&\Longleftrightarrow dim(F) = dim(Im(f)) \\
\text{f surjective} &\Longleftrightarrow Im(f) = F
\end{aligned}$$