Chapter 1: Numerical Series

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Contents

1   Preamble

1.1   Vocabulary

In this chapter, we will use CVG for Convergence and DVG for Divergence. We will also use GT for General Term.

1.2   Remark

⚠ Be careful, the series $\sum U_n$ is not the same as the sequence $(U_n)_{n\in\mathbb{N}}$. $\sum U_n$ is the series of general term U_n and $(U_n)_{n\in\mathbb{N}}$ is the sequence U_n.

2   General approach Convergence and Divergence

2.1   Definition

Let $(\mathbf{U_n})_{\mathbf{n}\in\mathbb{N}}$ a sequence of real numbers, we call series of general term $\mathbf{U_k}$ and denote $\sum \mathbf{U_k}$ the sequence of partial sums $(\mathbf{S_n})_{\mathbf{n}\in\mathbb{N}}$ where for any integer $\mathbf{n}\in\mathbb{N}$, $\mathbf{S_n} = \sum_{\mathbf{k=0}}^{\mathbf{n}} \mathbf{U_k}$. We say $\sum \mathbf{U_k}$ is convergent if and only if $(\mathbf{S_n})_{\mathbf{n}\in\mathbb{N}}$ is convergent.

2.1.1   Example: the geometric series

Let $\mathbf{q}\in\mathbb{R}^\star$ and let us consider the series $\sum \mathbf{q^k}$. We have:

$$\forall n\in\mathbb{N}, S_n = \sum_{k=0}^{n} q^k = \begin{vmatrix} \frac{1-q^{n+1}}{1-q} & \text{if } q\neq 1 \implies \begin{vmatrix} \text{if } -1 < q < 1, \sum_{k=0}^{+\infty} q^k = \frac{1}{1-q}\sum U_k \text{: CVG} \\ \text{if } q > 1 \text{ or } q < -1, \sum U_k \text{: DVG} \end{vmatrix} \\ (n+1) & \text{if } q = 1 \implies \sum U_k \text{: DVG} \end{vmatrix}$$

2.2   Propositions

Let $\sum \mathbf{U_k}$ and $\sum \mathbf{V_k}$ two series of general terms and $\lambda\in\mathbb{R}$. We have:

- If [$\sum \mathbf{U_k}$ CVG and $\sum \mathbf{V_k}$ CVG], then $\sum(\mathbf{U_k} + \mathbf{V_k})$ CVG

- If [$\sum \mathbf{U_k}$ CVG], then $\sum \lambda\mathbf{U_k}$ CVG

- If [$\sum \mathbf{U_k}$ CVG and $\sum \mathbf{V_k}$ DVG], then $\sum(\mathbf{U_k} + \mathbf{V_k})$ DVG

- ⚠ $\sum \mathbf{U_k}$ DVG and $\sum \mathbf{V_k}$ DVG <u>does not imply</u> $\sum(\mathbf{U_k} + \mathbf{V_k})$ DVG

2.3   Sum and Remainder of a convergent series

Let $\sum \mathbf{U_k}$ a <u>convergent series</u>. We call sum of the series $\sum \mathbf{U_k}$ the following real number: $\sum_{\mathbf{k=0}}^{+\infty} \mathbf{U_k} = \lim_{\mathbf{n}\to+\infty} \mathbf{S_n}$ where $\mathbf{S_n} = \sum_{\mathbf{k=0}}^{\mathbf{n}} \mathbf{U_k}$. And we call remainder of the series

$\sum \mathbf{U_k}$ sequence $(\mathbf{R_n})$ defined as follows:

$$\forall n \in \mathbb{N}, R_n = \sum_{k=n+1}^{+\infty}$$

2.3.1   Example

$\sum \mathbf{q^k}$ CVG $\Leftrightarrow -1 < q < 1 : \mathbf{S} = \lim_{\mathbf{n \to +\infty}} \mathbf{S_n} = \frac{\mathbf{1}}{\mathbf{1-q}}$

2.4   Convergence necessary condition

2.4.1   Proposition

Let $\sum (\mathbf{U_k})_{\mathbf{k} \in \mathbb{N}}$ a sequence. We have:

$$\sum U_k \text{ CVG } \overset{\Longrightarrow}{\Longleftarrow} \left(U_k \xrightarrow[k \to +\infty]{} 0 \right)$$

2.4.2   Example

- Harmonic series: $\sum \frac{\mathbf{1}}{\mathbf{n}}$, $\left(\frac{\mathbf{1}}{\mathbf{n}}\right) \xrightarrow[\mathbf{n \to +\infty}]{} \mathbf{0}$ but $\sum \frac{\mathbf{1}}{\mathbf{n}}$ DVG

- $\sum \frac{\mathbf{e^n}}{\mathbf{n^{2023}}}$, $\frac{e^n}{n^{2023}} \xrightarrow[n \to +\infty]{} +\infty \implies \sum \frac{e^n}{n^{2023}}$ DVG

3   Positive Term Series (P.T.S.)

3.1   Definition

Let $\sum \mathbf{U_k}$ a series. We say $\sum \mathbf{U_k}$ is a P.T.S., if and only if $\forall \mathbf{k} \in \mathbb{N}, \mathbf{U_k} \geq \mathbf{0}$.
We say $\sum \mathbf{U_k}$ is a P.T.S. from $\mathbf{p} \in \mathbb{N}$ onwards, if and only if $\forall \mathbf{k} \in \mathbb{N}, \mathbf{k} \geq \mathbf{p} \implies \mathbf{U_k} \geq \mathbf{0}$.

3.2   Propositions

- Let $\sum \mathbf{U_k}$ a P.T.S. and $(\mathbf{S_n})_{\mathbf{n} \in \mathbb{N}}$ the associated partial sum sequence. Then:

$$\sum U_k \text{ CVG } \Leftrightarrow (S_n)_{n \in \mathbb{N}} \text{ is upper-bounded}$$

- Let $\sum \mathbf{U_k}$ and $\sum \mathbf{V_k}$ two series such that:
  $\forall \mathbf{k} \in \mathbb{N}, \mathbf{0} \leq \mathbf{U_k} \leq \mathbf{V_k}$. Then:

  1. If $\sum \mathbf{V_k}$ CVG, then $\sum \mathbf{U_k}$ CVG
  2. If $\sum \mathbf{U_k}$ DVG, then $\sum \mathbf{V_k}$ DVG

3.2.1   Example

What's the nature of $\sum \frac{\mathbf{1}}{\mathbf{|n \cdot \sin(n)|}}$?

$\forall n \in \mathbb{N}^\star, 0 < |\sin(n)| \leq 1 \implies 0 < \frac{1}{n} \leq \frac{1}{|n \cdot \sin(n)|}$
$\sum \frac{\mathbf{1}}{\mathbf{n}}$ (Harmonic) DVG $\implies \sum \frac{\mathbf{1}}{\mathbf{|n \cdot \sin(n)|}}$ DVG

3.3   Riemann's series

3.3.1   Definition

We call Riemann's series any series of General Terms (GT) $\sum \frac{1}{n^\alpha}$ where $\alpha \in \mathbb{R}$.

3.3.2   Theorem (Riemann)

Let $\alpha \in \mathbb{R}$. Then:
$$\sum \frac{1}{n^\alpha} \text{ CVG} \iff \alpha > 1$$

3.3.2.1   Example

- $\sum \frac{1}{\sqrt{2}} = \sum \frac{1}{2^{\frac{1}{2}}} \implies$ DVG

- $\sum \frac{1+\cos(n)}{n^4}$: $\forall n \in \mathbb{N}^\star, 0 \leq 1 + \cos(n) \leq 2 \implies 0 \leq \frac{1+\cos(n)}{n^4} \leq \frac{2}{n^4}$
  And $\sum \frac{2}{n^4}$ of same nature as $\sum \frac{1}{n^4}$ (Riemann's series) CVG $\implies \sum \frac{1+\cos(n)}{n^4}$ CVG

3.4   Comparison criteria

3.4.1   Proposition

Let $\sum U_n$ and $\sum V_n$ two P.T.S.

① If $U_n \underset{+\infty}{\sim} V_n$ then $\sum U_n$ and $\sum V_n$ are of same nature

② If $U_n = o(V_n)$ then [If $\sum V_n$ CVG then $\sum U_n$ CVG]

3.4.1.1   Example

What's the nature of $\sum U_n$?

- $U_n = e^{-\sqrt{n}}$: Step 1: $n^2 \times U_n = \frac{n^2}{e^{\sqrt{n}}} = \frac{(\sqrt{n})^4}{e^{\sqrt{n}}} \xrightarrow[n \to +\infty]{} 0 \implies U_n = o(\frac{1}{n^2})$
  Step 2: $\sum \frac{1}{n^2}$ CVG (Riemann's series $\alpha = 2 > 1$) $\implies \sum U_n$ CVG

- $U_n = \ln(\frac{n+1}{n})$: ⚠ $\implies$
  $\forall n \in \mathbb{N}^\star, \frac{n+1}{n} = 1 + \frac{1}{n} \implies \ln(1 + \frac{1}{n}) \underset{+\infty}{=} \frac{1}{n} + o(\frac{1}{n})$

  ① $\forall n \in \mathbb{N}, U_n > 0$ since $1 + \frac{1}{n} > 1$
  ② $U_n \underset{+\infty}{=} \frac{1}{n}$
  $\implies \sum U_n$ and $\sum \frac{1}{n}$ of same nature
  and $\sum \frac{1}{n}$ DVG (Harmonic series)

3.4.2   Proposition

Let $\sum U_n$ a numerical sequence. We have:
$$\sum \overbrace{(U_{n+1} - U_n)}^{w_n} \text{ CVG} \iff (U_n) \text{ CVG}$$

3.4.2.1   Example

1. ⚠ General Example, limit calculation:

$$\mathbf{S_n} = \sum_{\mathbf{k=0}}^{\mathbf{n}} \mathbf{W_k} = \sum_{\mathbf{k=0}}^{\mathbf{n}} (\mathbf{U_{k+1}} - \mathbf{U_k}) = \sum_{k=0}^{n} U_{k+1} - \sum_{k=0}^{n} U_k$$

$$= \sum_{k=1}^{n+1} U_k - \sum_{k=0}^{n} U_k$$

$$= \left(\sum_{k=1}^{n} U_k + U_{n+1} \right) - \left(U_0 + \sum_{k=1}^{n} U_k \right)$$

$$Sn = \sum_{k=0}^{n} (U_{k+1} - U_k) = U_{n+1} - U_0$$

$$\sum \overbrace{\left(\tfrac{1}{n+1} - \tfrac{1}{n} \right)}^{W_n} : \left| \begin{array}{l} \sum W_n \text{ of same nature as } \sum \left(\tfrac{1}{n}_{n \in \mathbb{N}^\star} \right) \left(\begin{array}{c} \tfrac{1}{n} \xrightarrow[n \to +\infty]{} 0 \text{ CVG} \\ \text{So:} \sum W_n \text{ CVG} \end{array} \right) \\ \text{calculation of the limit:} \\ S = \lim_{n \to +\infty} S_n = \sum_{k=1}^{+\infty} W_k = \lim_{n \to +\infty} \left(\tfrac{1}{n+1} - 1 \right) = -1 \end{array} \right.$$

2.

3.5   Riemann's Rule

Let $\sum \mathbf{U_n}$ a <u>Positive</u> numerical series. If $\exists \alpha > \mathbf{1}, \mathbf{n}^\alpha \times \mathbf{U_n} \underset{+\infty}{\sim} \mathbf{0}$ then $\sum \mathbf{U_n}$ CVG

3.5.1   Proof

$$\exists \alpha > \mathbf{1}, \mathbf{n}^\alpha \times \mathbf{U_n} \xrightarrow[\mathbf{n \to +\infty}]{} \mathbf{0} \implies \frac{\mathbf{U_n}}{\frac{1}{\mathbf{n}^\alpha}} \xrightarrow[\mathbf{n \to +\infty}]{} \mathbf{0}$$

$$\implies \left\{ \begin{array}{r} U_n = o(\tfrac{1}{n^\alpha}) \\ \text{and} \\ \alpha > 1 \\ \text{and} \\ \sum U_n \text{ P.T.S.} \end{array} \right| \left[\sum \tfrac{1}{n^\alpha} \text{ CVG (Riemann's series)} \implies \sum U_n \text{ CVG} \right]$$