

Chapter 11: Vector Spaces

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1 Introduction

1.1 General approach

Studying Vector spaces will allow us to notice general theorems that can be applied to many mathematical structures.

1.2 Definition

A vector space is a set whose elements, often called vectors, may be added together and multiplied ("scaled") by numbers called scalars.

1.3 Notation

A $\mathbb{K} - VectorSpace$ is non-empty set E that has :

- An internal law, which is an map of $E \times E$ in E :

$$\begin{aligned} E \times E &\longrightarrow E \\ (u, v) &\longmapsto u + v \end{aligned}$$

- An external law, which is an map of $\mathbb{K} \times E$ in E :

$$\begin{aligned} \mathbb{K} \times E &\longrightarrow E \\ (\lambda, u) &\longmapsto \lambda \cdot u \end{aligned}$$

(\mathbb{K} is a set, often \mathbb{R})

The elements of E are called vectors

The elements of \mathbb{K} are called scalars

The neutral element 0_E is also called the null vector

The symmetrical $-u$ is also called the opposite

The internal composition law on E , denoted $+$, is the addition

The external composition law on E is the multiplication by a scalar

axioms relative to the internal law :

- 0_E is unique
- $-u$ is unique

1.4 Properties

To know if a space is a vector space, there are properties that need to match up.

- The internal law
- The external law
- Both laws together
- The neutral element
- It's symmetrical

This makes up 8 laws that need to be respected :

1. $u + v = v + u (\forall (u, v) \in E)$
2. $u + (v + w) = (u + v) + w (\forall (u, v, w) \in E)$
3. There exists a neutral element $0_E \in E$ so that $u + 0_E = u (\forall u \in E)$
4. All elements admit a symmetric u' so that $u + u' = 0_E$. This element u' is denoted $-u$
5. $1 \cdot u = u (\forall u \in E)$
6. $\lambda \cdot (\mu \cdot u) = (\lambda\mu) \cdot u (\forall \lambda, \mu \in \mathbb{K}, u \in E)$
7. $\lambda \cdot (v + u) = (\lambda \cdot v) + (\lambda \cdot u) (\forall \lambda \in \mathbb{K}, v, u \in E)$
8. $(\lambda + \mu) \cdot u = (\lambda \cdot u) + (\mu \cdot u) (\forall \lambda, \mu \in \mathbb{K}, u \in E)$

2 Vector Sub-Spaces

A sub space is very useful to prove that a set is a Vector space. We will see that a vector sub-space is vector space.

2.1 Definition

Let E be a vector space, F is a subspace if and only if

- $0_E \in F$
- $u + v \in F \quad \forall (u, v) \in F^2$
- $\lambda \cdot u \in F \quad \forall \lambda \in \mathbb{K}, v \in F$

2.2 Properties

Showing that a space is a subspace of a bigger (or equal) vector space is enough to prove that it is itself a vector space.

3 Relations between V. Space and V. SubSpace

3.1 Linear Combinations

Let v_1, v_2, \dots, v_n , n vectors from a vector space E . Then :
Any vector of the form :

$$u = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n$$

is called a Linear Combination of the vectors v_1, v_2, \dots, v_n

The scalars $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{K}$ are the coefficients of the linear combination.

Let E be a vector space. Then F is a vector sub space if and only if all linear combination of two element of F also belongs to E .

3.2 Vector SubSpace Intersection

- the intersection \cap of two vector sub spaces **is** also a sub space
- the union \cup of two vector sub spaces **is not** a sub space

3.3 Vector SubSpace Sum

Let F and G be two vectorial sub spaces of E

The sum of two vectorial sub spaces is also a vss, in fact, it is the smallest vss including both F and G

F and G are in direct sum in E if

- $F \cap G = \{0_E\}$
- $F + G = E$

We then denote $F \oplus G = E$

F and G are called additional sub-spaces

F and G are additional sub-spaces in E if and only if any element of E is uniquely written as the sum of an element of F and an element of G

$$\text{if } \begin{cases} w = u + v \\ w = u' + v' \end{cases} \quad \text{with } \begin{cases} u \in F, v \in G \\ u' \in F, v' \in G \end{cases} \quad \text{then } \begin{cases} u = u' \\ v = v' \end{cases}$$

4 Families

4.1 Free Family

A family $\{v_1, v_2, \dots, v_p\}$ of E is called a *free family* (or *linearly independent family*) if all null linear combination

$$\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_p v_p = 0$$

is such that all λ coefficients are null.

The opposite (there exists a null linear combination with at least 1 coefficient that is not null) is called a linked family or a linearly dependent family.

Let E a \mathbb{K} -vector space,
A family $F = \{v_1, v_2, \dots, v_p\}$ with $p \geq 2$ vectors of E is a linked family iff at least one vector is a linear combination of the other vectors

4.2 Spanning Family

A family $\{v_1, v_2, \dots, v_p\}$ is a generative family of E if all vector of E is a linear combination of the vectors v_1, v_2, \dots, v_p

$$\forall x \in E, \exists (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{K}^n : x = \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n$$

To prove that a family spans a vector space, we need to find the solutions $\lambda_1, \dots, \lambda_n$ of the equation given just above.

5 Basis

A family of vectors of E is a basis if it is both a free family and a spanning family. It acts as a reference for the vector space

Every basis of a finite dimension vector space E has the same number of elements. This number is called the **dimension** ($\dim E$) of the vector space.

TO COMPLETE !

6 Dimension of a Vector SubSpace

Let E be a \mathbb{K} -vector space of dimension n . Then :

- Any free family of E has at most n elements.
- Any spanning family of E has at least n elements.

Let E be a \mathbb{K} -vector space of dimension n . Then :

- A v.s.s of dimension 1 is called a **vector line**
- A v.s.s of dimension 2 is called a **vector plane**
- A v.s.s of dimension $n - 1$ is called an **hyperplan**

Let F and G be two v.s.s of E , a finite vector space.

- F is of a finite dimension too
- $\dim F \leq \dim E$

- $\dim F = \dim E \iff F = E$

$$\dim(F + G) = \dim(F) + \dim(G) - \dim(F \cap G)$$

7 Proofs

7.1 Intersection of linear subspaces

7.2 Sum of linear subspaces

7.3 Spanned linear subspaces

7.4 Basis

7.5 Existence of a basis in a finite dimension