

# Chapter 13: Matrix

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# 1 General approach

## 1.1 Definition

### 1.1.1 Definition of a matrix

We call matrix of  $n$  rows and  $p$  columns any mapping in the following form:

$$\begin{array}{ccc} \llbracket 1, n \rrbracket \times \llbracket 1, p \rrbracket & \rightarrow & \mathbb{K} \\ i, j & & a_{ij} \end{array}$$

We denote such maps as tables of  $n$  rows and  $p$  columns, and we write:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix}$$

$\forall (i, j) \in \llbracket 1, n \rrbracket \times \llbracket 1, p \rrbracket$ , we call  $a_{ij}$  a coefficient of the matrix. In this case coefficient if  $i$ -th row and  $j$ -th column.

### 1.1.2 Notation

We denote  $M_{np}(\mathbb{K})$  the set of matrix of  $n$  rows and  $p$  columns with coefficient from  $\mathbb{K}$ .

### 1.1.3 Examples

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \in M_{32}(\mathbb{R})$$

$$B = \begin{pmatrix} i \\ 1+i \\ 3 \end{pmatrix} \in M_{31}(\mathbb{C})$$

## 1.2 Particular matrices

Let  $A \in M_{np}(\mathbb{K})$  then:

### 1.2.1 Null matrix

1.  $[\forall (i, j) \in \llbracket 1, n \rrbracket \times \llbracket 1, p \rrbracket, a_{ij} = 0] \Rightarrow [A = 0_{np}]$  We say  $A$  is the null matrix  $0_{np}$ .

#### 1.2.1.1 Example

$$A' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \in M_{32}(\mathbb{R})$$

### 1.2.2 Column matrix

2.  $B \in M_{np}(\mathbb{K})$  and  $p = 1 \Rightarrow B$  is a column matrix of  $n$  rows

#### 1.2.2.1 Example

$$B' = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in M_{31}(\mathbb{R})$$

### 1.2.3 Row matrix

3.  $B \in M_{np}(\mathbb{K})$  and  $n = 1 \Rightarrow C$  is a row matrix of  $p$  columns

#### 1.2.3.1 Example

$$C' = (1 \ 2 \ 3) \in M_{13}(\mathbb{R})$$

### 1.2.4 Square matrix

We call square matrix any matrix with same number of rows and columns. We denote  $M_n(\mathbb{K})$  the set of square matrix of  $n$  rows and columns with coefficient from  $\mathbb{K}$ .

4.  $D \in M_{np}(\mathbb{K})$  and  $n = p \Rightarrow D$  is a square matrix denote  $M_n(\mathbb{K})$

#### 1.2.4.1 Example

$$D' = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \in M_3(\mathbb{R})$$

### 1.2.5 Diagonal matrix

5.  $\forall E \in M_n(\mathbb{R})$ , if  $\forall (i, j) \in \llbracket 1, n \rrbracket^2, i \neq j \Rightarrow a_{ij} = 0$  then we say  $E$  is a diagonal matrix

#### 1.2.5.1 Example

$$E' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \in M_2(\mathbb{R})$$

$$E'' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \in M_3(\mathbb{R})$$

### 1.2.6 Identity matrix

6.  $\forall I_n \in M_n(\mathbb{R})$ , if  $\forall (i, j) \in \llbracket 1, n \rrbracket^2, i \neq j \Rightarrow a_{ij} = 0$  and  $i = j \Rightarrow a_{ij} = 1$  then we say  $I_n$  is a identity matrix

### 1.2.6.1 Example

$$I'_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{R})$$

$$I''_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in M_3(\mathbb{R})$$

### 1.2.7 Triangular matrix

6.  $\forall F \in M_n(\mathbb{R})$ , if  $\forall (i, j) \in \llbracket 1, n \rrbracket^2, i > j \Rightarrow a_{ij} = 0$  then we say F is a lower triangular matrix
7.  $\forall G \in M_n(\mathbb{R})$ , if  $\forall (i, j) \in \llbracket 1, n \rrbracket^2, i < j \Rightarrow a_{ij} = 0$  then we say G is a upper triangular matrix

#### 1.2.7.1 Example

$$F' = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \in M_2(\mathbb{R})$$

$$G' = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \in M_2(\mathbb{R})$$

## 1.3 Transposed matrix

### 1.3.1 Definition

Let  $A \in M_{np}(\mathbb{K})$ . We call transposed matrix of A (or A transpose) a matrix B from  $M_{pn}(\mathbb{K})$  such as:

$$\forall (i, j) \in \llbracket 1, n \rrbracket \times \llbracket 1, p \rrbracket, a_{ij} = b_{ji}$$

### 1.3.2 Notation

We denote B as  ${}^tA$

### 1.3.3 Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \in M_{23}(\mathbb{R})$$

$${}^tA = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \in M_{32}(\mathbb{R})$$

## 1.4 Symmetric matrix

### 1.4.1 Symmetric

If  ${}^tA = A$  then we say A is symmetric

#### 1.4.1.1 Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} = {}^tA = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} \in M_3(\mathbb{R})$$

### 1.4.2 Anti-Symmetric

If  ${}^tA = -A$  then we say A is Anti-symmetric

#### 1.4.2.1 Example

$$A = \begin{pmatrix} 0 & -2 & 3 \\ 2 & 0 & -5 \\ -3 & 5 & 0 \end{pmatrix} = -{}^tA = \begin{pmatrix} 0 & 2 & -3 \\ -2 & 0 & 5 \\ 3 & -5 & 0 \end{pmatrix} \in M_3(\mathbb{R})$$

## 2 Operations on matrices

### 2.1 Addition and external product

#### 2.1.1 Definition

1. We call internal operation in  $M_{np}(\mathbb{K})$  denoted  $\oplus$  "internal addition" the one defined as follows:

$$\forall A, B \in M_{np}^2(\mathbb{K}), A + B = (a_{ij} + b_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq p}}$$

$$\text{Where } A = a_{ij}_{\substack{1 \leq i \leq n \\ 1 \leq j \leq p}} \text{ and } B = b_{ij}_{\substack{1 \leq i \leq n \\ 1 \leq j \leq p}}$$

2. We call "external multiplication" or "multiplication by a scalar" the one defined as follows:

$$\forall A \in M_{np}(\mathbb{K}), \forall \alpha \in \mathbb{K}, \alpha A = (\alpha a_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq p}}$$

#### 2.1.1.1 Example

$$(A, B) \in M_{2,3}(\mathbb{R})^2 \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \text{ and } B = \begin{pmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 1+7 & 2+8 & 3+9 \\ 4+10 & 5+11 & 6+12 \end{pmatrix} = \begin{pmatrix} 8 & 10 & 12 \\ 14 & 16 & 18 \end{pmatrix}$$

$$\alpha = 3, \quad \alpha A = \begin{pmatrix} 3 \times 1 & 3 \times 2 & 3 \times 3 \\ 3 \times 4 & 3 \times 5 & 3 \times 6 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{pmatrix}$$

### 2.1.1.2 Proposition

$(M_{np}, \oplus, \cdot)$  is a vector space over  $\mathbb{K}$

### 2.1.2 Elementary matrix

For  $(n, p) \in \mathbb{N}^2, (i, j) \in \llbracket 1, n \rrbracket \times \llbracket 1, p \rrbracket$ ; We denote  $E_{ij}$  the matrix from  $M_{np}(\mathbb{K})$  such that the  $ij$ -th coefficient is 1 and all other coefficient are 0.

$E_{ij}$  are called elementary matrix

#### 2.1.2.1 Example

$$\begin{aligned} E_{11} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ E_{22} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ E_{33} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

### 2.1.3 Proposition

1.  $(E_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq p}}$  is a basis of  $M_{np}(\mathbb{K})$
2.  $\dim((E_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq p}}) = np$

Ex:  $M_2(\mathbb{R})$  a  $(\mathbb{K})$ -VS:  $B = \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$

$B$  is a Standard basis of  $M_2(\mathbb{R})$ ,  $\dim(M_2(\mathbb{R})) = 2^2 = 4$

## 2.2 Internal product

### 2.2.1 Definition

Let  $(n, p, q) \in \mathbb{N}^3$  and  $A = a_{ij} \substack{1 \leq i \leq n \\ 1 \leq j \leq p} \in M_{np}(\mathbb{K}), B = b_{ij} \substack{1 \leq i \leq p \\ 1 \leq j \leq q} \in M_{pq}(\mathbb{K})$ . We call product of  $A$  and  $B$  the matrix  $C$  form  $M_{nq}(\mathbb{K})$  such that:

$$\forall (i, j) \in \llbracket 1, n \rrbracket \times \llbracket 1, q \rrbracket, c_{ij} = \sum_{k=1}^p a_{ik} b_{kj}$$

### 2.2.1.1 Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \in M_{2,3}(\mathbb{R}) \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in M_{3,4}(\mathbb{R})$$

$$C = A \cdot B = \begin{pmatrix} 1 & 0 & 8 & 11 \\ 4 & 0 & 20 & 32 \end{pmatrix} \in M_{2,4}(\mathbb{R})$$

$$C_{2,3} = 4 \times 1 + 5 \times 2 + 6 \times 1 = 20$$

### 2.2.2 Remarks

- (R1) If  $A, B$  two matrices: we only can multiply  $A$  by  $B$  if the number of column of  $A$  is equal to the number of row of  $B$ .
- (R2)  $AB$  can exists but  $BA$  not or the other way around.

#### 2.2.2.1 Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \Rightarrow M_{2,1}(\mathbb{R}), \text{ and exists but } BA \text{ does not exists}$$

- (R3) In the General case, where  $AB$  and  $BA$  exists:
- $AB \neq BA$  (multiplication of matrix is not commutative)  
When  $AB = BA$  we say  $A$  and  $B$  commute.

## 2.3 Properties of matrix calculus

### 2.3.1 Properties

- Let  $A, B$  two matrices such that  $AB$  exists.  
We can have  $AB = 0$  and ( $A \neq 0$  or  $B \neq 0$ )

$$\text{If } A = 0 \text{ or } B = 0 \text{ then } AB = 0$$

$$AB = 0 \not\Rightarrow A = 0 \text{ or } B = 0$$

#### 2.3.1.1 Example

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$



2.  $(n, p, q, r) \in \mathbb{N}^4$  and let  $(A, B, C) \in M_{np}(\mathbb{K}) \times M_{pq}(\mathbb{K}) \times M_{qr}(\mathbb{K})$

$$A \cdot (B \cdot C) = (A \cdot B) \cdot C \otimes \text{ is commutative}$$

3.  $(A, B, C) \in M_{np}(\mathbb{K}) \times M_{pq}^2(\mathbb{K})$

$$A \cdot (B + C) = A \cdot B + A \cdot C$$

$\otimes$  (Matrices multiplication) is distributive over matrix addition ( $\oplus$ ).

4.  $A \in M_{np}(\mathbb{K})$  and  $B \in M_{pq}(\mathbb{K})$  and  $\lambda \in \mathbb{K}$

$$\lambda \cdot (A \cdot B) = A \cdot \lambda \cdot B = A \cdot B \cdot \lambda$$

### 2.3.2 Case of Square Matrices

1.

$$\forall A \in M_n(\mathbb{K}), A \cdot I_n = I_n \cdot A = A$$

2. Let  $(A, B) \in M_n(\mathbb{K})^2$ , such that  $AB = BA$  Then:

$$(A + B)^n = \sum_{k=0}^n \binom{n}{k} A^k B^{n-k}$$

By convention,  $A^0 = B^0 = I_n$

3.

$$\forall (A, B) \in M_n(\mathbb{K})^2, {}^t(A \cdot B) = {}^tA \cdot {}^tB$$

## 2.4 Inverse of a matrix

### 2.4.1 Definition

Let  $A \in M_n(\mathbb{K})$  we say that  $A$  is invertible if:

$$\exists B \in M_n(\mathbb{K}), AB = BA = I_n$$

Then we say that  $B$  is the inverse of  $A$  and denote  $B = A^{-1}$  (B is unique) Hence we have (in case of A invertible):  $A \cdot A^{-1} = A^{-1} \cdot A = I_n$

### 2.4.2 Notation

The set of invertible matrices of  $M_n(\mathbb{K})$  is denoted  $GL_n(\mathbb{K})$

### 2.4.3 How to find the inverse of a matrix

We will use the following system: Where  $A \in M_n(\mathbb{K}), (U, V) \in M_{n,1}(K)^2$ :

$$A \cdot U = V$$

By solving this system (Gauss elimination algorithm) when  $A$  is invertible, we will have:

$$U = A^{-1} \cdot V$$

### 2.4.3.1 Example

$$\begin{aligned}
 A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} A \cdot X = U &\Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
 &\Leftrightarrow \begin{pmatrix} \textcircled{1} & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z - x \end{pmatrix} \\
 &\Leftrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z - x - y \end{pmatrix} \\
 &\Leftrightarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} x - y + z \\ -x + y + z \\ z - x - y \end{pmatrix} \\
 &\Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{x-y+z}{2} \\ \frac{-x+y+z}{2} \\ \frac{x+y-z}{2} \end{pmatrix} \\
 &\Leftrightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
 A^{-1} &= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}
 \end{aligned}$$

## 3 Matrices of Linear Maps

### 3.1 Definition and examples

#### 3.1.1 Definition

Let  $f \in \mathcal{L}(E, F)$ ,  $E$  and  $F$  finite dimensional  $\mathbb{K}$ -vector spaces such that:  $\dim(E) = p$  and  $\dim(F) = n$  where  $(p, n) \in \mathbb{N}^2$  and  $B = (e_1, e_2, \dots, e_n)$  basis of  $E$  and  $B' = (e'_1, e'_2, \dots, e'_n)$  basis of  $F$

$$\forall U \in E, \exists! (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{K}^p, \quad U = \sum_{i=1}^p \lambda_i e'_i$$

We say  $\begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{pmatrix}$  is the column matrix of coordinates  $A \cdot U$  in  $B$ .

### 3.1.2 Example

Let  $E = \mathbb{R}^2$ ,  $U = \begin{pmatrix} 1 \\ 2 \end{pmatrix}_B$  with  $B = (\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix})$  and  $B' = (\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix})$  Then:

$$U = \begin{pmatrix} 2 \\ 1 \end{pmatrix}_B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}_B + \begin{pmatrix} 1 \\ 0 \end{pmatrix}_B \Rightarrow U = \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{B'}$$

### 3.1.3 Definition

We call matrix of  $f \in \mathcal{L}(E, F)$  with respect to basis  $B$  and  $B'$  denoted  $Mat_{BB'}(f)$  the matrix whose  $j$ -th column is composed of the coordinates of  $f(e_j)$  in  $B'$ , for all  $j$  from  $\llbracket 1, p \rrbracket$ . This is a matrix of  $p$  columns and  $n$  rows:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix}, \forall j \in \llbracket 1, p \rrbracket, f(e_j) = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}_{B'} = \sum_{i=1}^n a_{ij} e_i$$

### 3.1.4 Example

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y \\ 2x + 4y \\ -3y \end{pmatrix}$$

① basis for the domain ( $\mathbb{R}^2$ ):  $B = (\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix})$

basis for the codomain ( $\mathbb{R}^3$ ):  $B' = (\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix})$

$$\textcircled{2} \quad \begin{pmatrix} x + y \\ 2x + 4y \\ -3y \end{pmatrix} \Rightarrow \begin{cases} f(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}_{B'} \\ f(\begin{pmatrix} 0 \\ 1 \end{pmatrix}) = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}_{B'} \end{cases}$$

③

$$\forall U \in \mathbb{R}^2, U = x \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow f(U) = x \cdot f\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + y \cdot f\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 0 & -3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 0 & -3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$$

$$\text{so } f\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$$

## 3.2 Matrix interpretation of Linear Transformation

### 3.2.1 Proposition

Let  $E$  and  $F$  two finite dimensional  $\mathbb{K}$ -VS,  $B$  and  $B'$  bases of respectively  $E$  and  $F$ . Let  $U \in E$  and  $f \in \mathcal{L}(E, F)$ . Then:

$$\text{Mat}_{B'}(f(u)) = \text{Mat}_{BB'}(f) \cdot \text{Mat}_B(u)$$

### 3.2.2 Example

With the same function as before:

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y \\ 2x + 4y \\ -3y \end{pmatrix}$$

And with the same basis as before:  $B = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$  and  $B' = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$

And with  $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$  we have:

$$\begin{aligned} \text{Mat}_{B'}(f(u)) &= \text{Mat}_{BB'}(f) \cdot \text{Mat}_B(u) \\ &= \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 0 & -3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \\ -6 \end{pmatrix} = f(u) \end{aligned}$$

## 3.3 Matrix of $g \circ f$

### 3.3.1 Proposition

Let  $E, F$  and  $G$  three finite dimensional  $\mathbb{K}$ -VS, and  $B, B', B''$  bases of respectively  $E, F$  and  $G$ . Considering  $f \in \mathcal{L}(E, F)$  and  $g \in \mathcal{L}(F, G)$ , we have  $g \circ f \in \mathcal{L}(E, G)$  and:

$$\text{Mat}_{BB''}(g \circ f) = \text{Mat}_{B'B''}(g) \cdot \text{Mat}_{BB'}(f)$$

### 3.3.2 Example

$$\begin{aligned}
 f : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 & g : \mathbb{R}^2 &\longrightarrow \mathbb{R}^3 \\
 \begin{pmatrix} x \\ y \end{pmatrix} &\longmapsto \begin{pmatrix} x+y \\ x-y \end{pmatrix} & \begin{pmatrix} x \\ y \end{pmatrix} &\longmapsto \begin{pmatrix} x+2y \\ x \\ -x+y \end{pmatrix}
 \end{aligned}$$

And with the following basis:  $B = \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$  and  $B' = \left( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$

$$\text{Mat}_B(f) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \text{Mat}_{BB'}(g) = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$\forall X = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2, \quad g \circ f(X) = g(f(X)) = g \left( \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \right)$$

$$g \circ f(X) = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & 1 \\ 0 & -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

## 3.4 Matrix of a bijection

### 3.4.1 Proposition

Let  $E$  and  $F$  two  $\mathbb{K}$ -VS of same dimension.  $B$  a bases of  $E$  and  $B'$  a bases of  $F$ . Let  $f \in \mathcal{L}(E, F)$  Then:

$$f \text{ bijective} \iff \text{Mat}_{BB'}(f) \text{ is invertible}$$

### 3.4.2 Example

$$\begin{aligned}
 f : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\
 \begin{pmatrix} x \\ y \end{pmatrix} &\longmapsto \begin{pmatrix} 2x+y \\ x-4y \end{pmatrix}
 \end{aligned}$$

To find if  $f$  is bijective, we have to show that  $f$  is surjective or injective, because  $f$  is a endomorphism.

$$\text{with: } B = \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \text{ and } \text{Mat}(f) = \begin{pmatrix} 2 & 1 \\ 1 & -4 \end{pmatrix}$$

To prove  $f$  is injective, we have to show that  $\ker(f) = \{0_{\mathbb{R}^2}\}$

$$\begin{aligned}
 f(X) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} &\iff \begin{pmatrix} 2 & 1 \\ 1 & -4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
 &\iff \begin{pmatrix} 2 & 1 \\ 0 & -9 \end{pmatrix} : \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
 &\iff x = y = 0 \Rightarrow S = \{0_{\mathbb{R}^2}\} \\
 &\iff f \text{ is injective so } f \text{ is bijective}
 \end{aligned}$$

Lets find the inverse of  $f$ , using two methods:

### 3.4.2.1 Method 1: Gauss elimination algorithm

$$\begin{aligned}
 \begin{pmatrix} 2 & 1 \\ 1 & -4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} a \\ b \end{pmatrix} \\
 \iff \begin{pmatrix} 2 & 1 \\ 0 & -9 \end{pmatrix} : \begin{pmatrix} a \\ 2b - a \end{pmatrix} \\
 \iff \begin{pmatrix} 18 & 0 \\ 0 & -9 \end{pmatrix} : \begin{pmatrix} 8a + 2b \\ 2b - a \end{pmatrix} \\
 \iff \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \frac{4a+b}{9} \\ \frac{a-2b}{9} \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 4a + b \\ a - 2b \end{pmatrix} \\
 \iff \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{9} \begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}
 \end{aligned}$$

### 3.4.2.2 Method 2: System, Gauss-Jordan

Operations on left must be done also on right:

$$\begin{aligned}
 \left( \begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \end{array} \right) &\iff \left( \begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & -9 & -1 & 2 \end{array} \right) \\
 &\iff \left( \begin{array}{cc|cc} 18 & 0 & 8 & 2 \\ 0 & -9 & -1 & 2 \end{array} \right) \\
 &\iff \left( \begin{array}{cc|cc} 1 & 0 & \frac{4}{9} & \frac{1}{9} \\ 0 & 1 & -\frac{1}{9} & -\frac{2}{9} \end{array} \right)
 \end{aligned}$$

With both methods we have:

$$[Mat_B(f)]^{-1} = \frac{1}{9} \begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix}$$

### 3.4.3 Proposition

Let  $E$  and  $F$  two  $\mathbb{K}$ -VS of same dimension.  $B$  a bases of  $E$  and  $B'$  a bases of  $F$ . Let  $f \in \mathcal{L}(E, F)$  Then:

$$f \text{ bijective} \iff Mat_{BB'}(f) \text{ is invertible}$$

And in this case we have:

$$[Mat_{BB'}(f)]^{-1} = Mat_{BB'}(f^{-1})$$

### 3.4.4 Examples

#### 3.4.4.1 Example 1

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

① Is  $A$  invertible, compute the inverse of  $A$

$$\begin{aligned} & \left( \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array} \right) \\ \Leftrightarrow & \left( \begin{array}{ccc|ccc} \textcircled{1} & 1 & 0 & 1 & 0 & 0 \\ 0 & \textcircled{-1} & 1 & -1 & 1 & 0 \\ 0 & 0 & \textcircled{1} & 0 & 0 & 1 \end{array} \right) \quad \text{There are pivot (circled) so } A \text{ is invertible} \\ \Leftrightarrow & \left( \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array} \right) \\ \Leftrightarrow & \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array} \right) \\ \Leftrightarrow & A^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \end{aligned}$$

We can also compute  $A^{-1}$  this way: (if  $A$  is invertible we have:)

$$\begin{aligned} AX = U & \Leftrightarrow X = A^{-1}U \\ \Leftrightarrow & \begin{pmatrix} \textcircled{1} & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{array}{l} R_1 \\ R_2 - R_1 \\ R_3 - R_1 \end{array} \\ \Leftrightarrow & \begin{pmatrix} \textcircled{1} & 1 & 0 \\ 0 & \textcircled{-1} & 1 \\ 0 & 0 & \textcircled{1} \end{pmatrix} : \begin{pmatrix} a \\ b-a \\ c-a \end{pmatrix} \begin{array}{l} R_1 \\ R_2 - R_3 \\ R_3 \end{array} \\ \Leftrightarrow & \begin{pmatrix} 1 & \textcircled{1} & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{array}{l} R_1 + R_2 \\ -R_2 \\ R_3 \end{array} \\ \Leftrightarrow & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} a+b-c \\ -b+c \\ c-a \end{pmatrix} \\ \Leftrightarrow & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \end{aligned}$$

② Let  $f \in \mathcal{L}(\mathbb{R}^3)$ , Show that  $f$  is an automorphism and determine  $f^{-1}$

$$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ x+z \\ x+y+z \end{pmatrix} \quad B = \left( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

$A = \text{Mat}_B(f)$  so  $A$  invertible  $\iff f$  is an automorphism  
so  $A^{-1} = \text{Mat}_B(f^{-1})$

$$f^{-1} \in \mathcal{L}(\mathbb{R}^3): \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x+y-z \\ -y+z \\ -x+z \end{pmatrix} \quad (\text{using } A^{-1})$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto A^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

### 3.4.4.2 Example 2

$$\begin{aligned} \phi: \mathbb{R}_2[X] &\longrightarrow \mathbb{R}_2[X] & B &= (1, X, X^2) \\ P &\longmapsto XP' + P(X+1) \end{aligned}$$

So we have:  $\phi(1) = 1$  and  $\phi(X) = 2X + 1$  and  $\phi(X^2) = 3X^2 + 2X + 1$

$$\text{So: } A = \text{Mat}_B(\phi) = \begin{pmatrix} \phi(1) & \phi(X) & \phi(X^2) \\ 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix} \begin{matrix} 1 \\ X \\ X^2 \end{matrix}$$

an invertible matrix (upper right triangle with non zero diagonal coefficient)

We have to find  $A^{-1}$  to find  $\phi^{-1}$

$$\begin{aligned} \left( \begin{array}{ccc|ccc} \textcircled{1} & 1 & 1 & 1 & 0 & 0 \\ 0 & \textcircled{2} & 2 & 0 & 1 & 0 \\ 0 & 0 & \textcircled{3} & 0 & 0 & 1 \end{array} \right) & \begin{matrix} 3R_1 - R_3 \\ R_2 - 2R_3 \\ R_3 \end{matrix} \iff \left( \begin{array}{ccc|ccc} 3 & 3 & 0 & 3 & 0 & -1 \\ 0 & 6 & 0 & 0 & 3 & -1 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array} \right) & \begin{matrix} 2R_1 - R_2 \\ R_2 \\ R_3 \end{matrix} \\ & \iff \left( \begin{array}{ccc|ccc} 6 & 0 & 0 & 6 & -3 & 0 \\ 0 & 6 & 0 & 0 & 3 & -2 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array} \right) & \begin{matrix} \frac{1}{6}R_1 \\ \frac{1}{6}R_2 \\ \frac{1}{3}R_3 \end{matrix} \\ & \implies [\text{Mat}_B(\phi)]^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \\ & \implies [\text{Mat}_B(\phi)]^{-1} = \frac{1}{6} \begin{pmatrix} 6 & -3 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{pmatrix} \end{aligned}$$



Now find  $Q \in \mathbb{R}_2[X]$ ,  $\phi(Q)(X) = 1 + (X - 1)^2$

$\phi^{-1}(1 + (X - 1)^2)$ ? we use the matrix to find the coordinates:  $1 + (X - 1)^2 = 2 - 2X + X^2$

$$\frac{1}{6} \begin{pmatrix} 6 & -3 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}_B = \frac{1}{6} \begin{pmatrix} 18 \\ -8 \\ 2 \end{pmatrix}_B = \begin{pmatrix} 3 \\ -\frac{4}{3} \\ \frac{1}{3} \end{pmatrix}_B$$

$$\exists! Q \in \mathbb{R}[X], \phi(Q) = \mathbb{R} \quad Q(X) = 3 - \frac{4}{3}X + \frac{1}{3}X^2$$

## 4 Important Proof

### 4.1 Proposition

Let  $n$  and  $p$  be two non-zero natural numbers,  $E$  and  $F$  two finite-dimensional vector spaces over  $\mathbb{R}$  such that  $\dim(E) = p$  and  $\dim(F) = n$ .

Let  $B = (e_1, \dots, e_p)$  be a basis of  $E$  and  $B' = (\epsilon_1, \dots, \epsilon_n)$  a basis of  $F$ .

Consider  $f \in \mathcal{L}(E, F)$  and  $A = \text{Mat}_{B,B'}(f) = (a_{ij})$ .

For all  $u \in E$ , let  $X$  denote the column matrix containing the coordinates of  $u$  in basis  $B$  and  $Y$  the coordinates of  $f(u)$  in basis  $B'$ .

Then,

$$Y = AX$$

#### 4.1.1 Remarks:

1. Note that  $Y = AX \iff \text{Mat}_{B'}(f(u)) = \text{Mat}_{B,B'}(f) \cdot \text{Mat}_B(u)$
2.  $A = \text{Mat}_{B,B'}(f) = (a_{ij})$ . The columns of  $A$  contain the coordinates of the vectors  $f(e_j)$  in basis  $B'$ :

$$\begin{array}{ccc} f(e_1) & & f(e_p) \\ \downarrow & & \downarrow \\ \begin{pmatrix} a_{1,1} & \dots & a_{1,p} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,p} \end{pmatrix} & \leftarrow & \begin{array}{l} \text{coefficient of } \epsilon_1 \\ \vdots \\ \text{coefficient of } \epsilon_n \end{array} \end{array}$$

that is,  $\forall j \in \llbracket 1, p \rrbracket$ ,  $f(e_j) = a_{1,j}\epsilon_1 + \dots + a_{n,j}\epsilon_n$ .

3.  $A \in \mathcal{M}_{n,p}(\mathbb{R})$ ,  $X \in \mathcal{M}_{p,1}(\mathbb{R})$  and  $Y \in \mathcal{M}_{n,1}(\mathbb{R})$ .

#### 4.1.2 Proof:

Let  $u \in E$ .

- Since  $B$  is a basis of  $E$ :

$$\exists! (\lambda_1, \dots, \lambda_p) \in \mathbb{R}^p, u = \lambda_1 e_1 + \dots + \lambda_p e_p$$

$$\text{Then } X = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_p \end{pmatrix}.$$

- Using the linearity of  $f$ , we get:

$$\begin{aligned} f(u) &= \lambda_1 f(e_1) + \dots + \lambda_p f(e_p) \\ &= \lambda_1 (a_{1,1}\epsilon_1 + \dots + a_{n,1}\epsilon_n) + \dots + \lambda_p (a_{1,p}\epsilon_1 + \dots + a_{n,p}\epsilon_n) \\ &= (\lambda_1 a_{1,1} + \dots + \lambda_p a_{1,p})\epsilon_1 + \dots + (\lambda_1 a_{n,1} + \dots + \lambda_p a_{n,p})\epsilon_n \end{aligned}$$

$$\text{It follows that } Y = \begin{pmatrix} \lambda_1 a_{1,1} + \cdots + \lambda_p a_{1,p} \\ \vdots \\ \lambda_1 a_{n,1} + \cdots + \lambda_p a_{n,p} \end{pmatrix} = \begin{pmatrix} a_{1,1} & \cdots & a_{1,p} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,p} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_p \end{pmatrix} = AX.$$