Chapter 1: Numerical Series

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Contents

1 Preamble

1.1 Vocabulary

In this chapter, we will use CVG for Convergence and DVG for Divergence. We will also use GT for General Term.

1.2 Remark

Be careful, the series $\sum U_n$ is not the same as the sequence $(U_n)_{n\in\mathbb{N}}$. $\sum U_n$ is the series of general term U_n and $(U_n)_{n\in\mathbb{N}}$ is the sequence U_n.

2 General approach Convergence and Divergence

2.1 Definition

Let $(\mathbf{U_n})_{\mathbf{n}\in\mathbb{N}}$ a sequence of real numbers, we call series of general term $\mathbf{U_k}$ and denote $\sum \mathbf{U_k}$ the sequence of partial sums $(\mathbf{S_n})_{\mathbf{n}\in\mathbb{N}}$ where for any integer $\mathbf{n}\in\mathbb{N}$, $\mathbf{S_n} = \sum\limits_{\mathbf{k=0}}^{\mathbf{n}} \mathbf{U_k}$. We say $\sum \mathbf{U_k}$ is convergent if and only if $(\mathbf{S_n})_{\mathbf{n}\in\mathbb{N}}$ is convergent.

2.1.1 Example: the geometric series

Let $\mathbf{q}\in\mathbb{R}^\star$ and let us consider the series $\sum \mathbf{q^k}$. We have:

$$\forall n\in\mathbb{N}, S_n = \sum_{k=0}^{n} q^k = \left| \begin{array}{lll} \frac{1-q^{n+1}}{1-q} & \text{if } q\neq 1 & \Longrightarrow \left| \begin{array}{l} \text{if } -1<q<1, \sum\limits_{k=0}^{+\infty} q^k = \frac{1}{1-q} \sum U_k\text{: CVG} \\ \text{if } q>1 \text{ or } q<-1, \sum U_k\text{: DVG} \end{array} \right. \\ (n+1) & \text{if } q=1 & \Longrightarrow \sum U_k\text{: DVG} \end{array} \right.$$

2.2 Propositions

Let $\sum \mathbf{U_k}$ and $\sum \mathbf{V_k}$ two series of general terms and $\lambda\in\mathbb{R}$. We have:

- If $[\sum \mathbf{U_k}$ CVG and $\sum \mathbf{V_k}$ CVG$]$, then $\sum(\mathbf{U_k}+\mathbf{V_k})$ CVG
- If $[\sum \mathbf{U_k}$ CVG$]$, then $\sum \lambda\mathbf{U_k}$ CVG
- If $[\sum \mathbf{U_k}$ CVG and $\sum \mathbf{V_k}$ DVG$]$, then $\sum(\mathbf{U_k}+\mathbf{V_k})$ DVG
- $\sum \mathbf{U_k}$ DVG and $\sum \mathbf{V_k}$ DVG <u>does not imply</u> $\sum(\mathbf{U_k}+\mathbf{V_k})$ DVG

2.3 Sum and Remainder of a convergent series

Let $\sum \mathbf{U_k}$ a <u>convergent series</u>. We call sum of the series $\sum \mathbf{U_k}$ the following real number: $\sum\limits_{\mathbf{k=0}}^{+\infty} \mathbf{U_k} = \lim\limits_{\mathbf{n}\to+\infty} \mathbf{S_n}$ where $\mathbf{S_n} = \sum\limits_{\mathbf{k=0}}^{\mathbf{n}} \mathbf{U_k}$. And we call remainder of the series

$\sum \mathbf{U_k}$ sequence $(\mathbf{R_n})$ defined as follows:

$$\forall n \in \mathbb{N}, R_n = \sum_{k=n+1}^{+\infty}$$

2.3.1 Example

$\sum \mathbf{q^k}$ CVG $\Leftrightarrow -1 < q < 1$: $\mathbf{S} = \lim\limits_{\mathbf{n}\to+\infty} \mathbf{S_n} = \frac{\mathbf{1}}{\mathbf{1-q}}$

2.4 Convergence necessary condition

2.4.1 Proposition

Let $\sum(\mathbf{U_k})_{\mathbf{k}\in\mathbb{N}}$ a sequence. We have:

$$\sum U_k \text{ CVG } \underset{\not\Leftarrow}{\Longrightarrow} \left(U_k \xrightarrow[k\to+\infty]{} 0\right)$$

2.4.2 Example

- Harmonic series: $\sum \frac{\mathbf{1}}{\mathbf{n}}$, $(\frac{\mathbf{1}}{\mathbf{n}}) \xrightarrow[\mathbf{n}\to+\infty]{} \mathbf{0}$ but $\sum \frac{\mathbf{1}}{\mathbf{n}}$ DVG
- $\sum \frac{\mathbf{e^n}}{\mathbf{n^{2023}}}$, $\frac{e^n}{n^{2023}} \xrightarrow[n\to+\infty]{} +\infty \implies \sum \frac{e^n}{n^{2023}}$ DVG

3 Positive Term Series (P.T.S.)

3.1 Definition

Let $\sum \mathbf{U_k}$ a series. We say $\sum \mathbf{U_k}$ is a P.T.S., if and only if $\forall \mathbf{k} \in \mathbb{N}, \mathbf{U_k} \geq \mathbf{0}$.
We say $\sum \mathbf{U_k}$ is a P.T.S. from $\mathbf{p} \in \mathbb{N}$ onwards, if and only if $\forall \mathbf{k} \in \mathbb{N}, \mathbf{k} \geq \mathbf{p} \implies \mathbf{U_k} \geq \mathbf{0}$.

3.2 Propositions

- Let $\sum \mathbf{U_k}$ a P.T.S. and $(\mathbf{S_n})_{\mathbf{n}\in\mathbb{N}}$ the associated partial sum sequence. Then:

$$\sum U_k \text{ CVG } \Leftrightarrow (S_n)_{n\in\mathbb{N}} \text{ is upper-bounded}$$

- Let $\sum \mathbf{U_k}$ and $\sum \mathbf{V_k}$ two series such that:
  $\forall \mathbf{k} \in \mathbb{N}, \mathbf{0} \leq \mathbf{U_k} \leq \mathbf{V_k}$. Then:
  1. If $\sum \mathbf{V_k}$ CVG, then $\sum \mathbf{U_k}$ CVG
  2. If $\sum \mathbf{U_k}$ DVG, then $\sum \mathbf{V_k}$ DVG

3.2.1 Example

What's the nature of $\sum \frac{\mathbf{1}}{|\mathbf{n}\cdot\sin(\mathbf{n})|}$?

$\forall n \in \mathbb{N}^\star, 0 < |\sin(n)| \leq 1 \implies 0 < \frac{1}{n} \leq \frac{1}{|n\cdot\sin(n)|}$
$\sum \frac{\mathbf{1}}{\mathbf{n}}$ (Harmonic) DVG $\implies \sum \frac{\mathbf{1}}{|\mathbf{n}\cdot\sin(\mathbf{n})|}$ DVG

3.3 Riemann's series

3.3.1 Definition

We call Riemann's series any series of General Terms (GT) $\sum \frac{\mathbf{1}}{\mathbf{n}^{\alpha}}$ where $\alpha \in \mathbb{R}$.

3.3.2 Theorem (Riemann)

Let $\alpha \in \mathbb{R}$. Then:

$$\sum \frac{1}{n^{\alpha}} \text{ CVG } \iff \alpha > 1$$

3.3.2.1 Example

- $\sum \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} = \sum \frac{\mathbf{1}}{\mathbf{2}^{\frac{1}{2}}} \implies$ DVG
- $\sum \frac{\mathbf{1+cos(n)}}{\mathbf{n^4}}$: $\forall n \in \mathbb{N}^{\star}, 0 \leq 1 + cos(n) \leq 2 \implies 0 \leq \frac{1+cos(n)}{n^4} \leq \frac{2}{n^4}$
  And $\sum \frac{\mathbf{2}}{\mathbf{n^4}}$ of same nature as $\sum \frac{\mathbf{1}}{\mathbf{n^4}}$ (Riemann's series) CVG $\implies \sum \frac{\mathbf{1+cos(n)}}{\mathbf{n^4}}$ CVG

3.4 Comparison criteria

3.4.1 Proposition

Let $\sum \mathbf{U_n}$ and $\sum \mathbf{V_n}$ two P.T.S.

1. If $\mathbf{U_n} \underset{+\infty}{\sim} \mathbf{V_n}$ then $\sum \mathbf{U_n}$ and $\sum \mathbf{V_n}$ are of same nature

2. If $\mathbf{U_n} = \mathbf{o}(\mathbf{V_n})$ then [If $\sum \mathbf{V_n}$ CVG then $\sum \mathbf{U_n}$ CVG]

3.4.1.1 Example

What's the nature of $\sum \mathbf{U_n}$?

- $\mathbf{U_n} = \mathbf{e}^{-\sqrt{\mathbf{n}}}$: Step 1: $n^2 \times U_n = \frac{n^2}{e^{\sqrt{n}}} = \frac{(\sqrt{n})^4}{e^{\sqrt{n}}} \xrightarrow[n \to +\infty]{} 0 \implies U_n = o(\frac{1}{n^2})$
  Step 2: $\sum \frac{1}{n^2}$ CVG (Riemann's series $\alpha = 2 > 1$) $\implies \sum U_n$ CVG

$$\forall n \in \mathbb{N}^{\star}, \frac{n+1}{n} = 1 + \frac{1}{n} \implies \ln(1 + \frac{1}{n}) \underset{+\infty}{=} \frac{1}{n} + o(\frac{1}{n})$$

- $\mathbf{U_n} = \ln(\frac{\mathbf{n+1}}{\mathbf{n}})$: ⚠ $\implies$ 1. $\forall n \in \mathbb{N}, U_n > 0$ since $1 + \frac{1}{n} > 1$ 2. $U_n \underset{+\infty}{=} \frac{1}{n}$
  $\implies \sum U_n$ and $\sum \frac{1}{n}$ of same nature
  and $\sum \frac{1}{n}$ DVG (Harmonic series)

3.4.2 Proposition

Let $\sum \mathbf{U_n}$ a numerical sequence. We have:

$$\sum \overbrace{(U_{n+1} - U_n)}^{W_n} \text{ CVG} \iff (U_n) \text{ CVG}$$

3.4.2.1 Example

1. General Example, limit calculation:

$$\begin{aligned}
\mathbf{S_n} = \sum_{\mathbf{k=0}}^{\mathbf{n}} \mathbf{W_k} = \sum_{\mathbf{k=0}}^{\mathbf{n}} (\mathbf{U_{k+1}} - \mathbf{U_k}) &= \sum_{k=0}^{n} U_{k+1} - \sum_{k=0}^{n} U_k \\
&= \sum_{k=1}^{n+1} U_k - \sum_{k=0}^{n} U_k \\
&= \left(\sum_{k=1}^{n} U_k + U_{n+1} \right) - \left(U_0 + \sum_{k=1}^{n} U_k \right) \\
Sn &= \sum_{k=0}^{n} (U_{k+1} - U_k) = U_{n+1} - U_0
\end{aligned}$$

2. $$\sum \overbrace{\left(\tfrac{1}{n+1} - \tfrac{1}{n}\right)}^{W_n} : \left| \begin{array}{l} \sum W_n \text{ of same nature as } \sum \left(\frac{1}{n}_{n\in\mathbb{N}^\star}\right) \left(\begin{array}{l} \frac{1}{n} \xrightarrow[n\to+\infty]{} 0 \text{ CVG} \\ \text{So: } \sum W_n \text{ CVG} \end{array} \right) \\ \text{calculation of the limit:} \\ S = \lim\limits_{n\to+\infty} S_n = \sum\limits_{k=1}^{+\infty} W_k = \lim\limits_{n\to+\infty} \left(\frac{1}{n+1} - 1\right) = -1 \end{array} \right.$$

3.5 Riemann's Rule

Let $\sum \mathbf{U_n}$ a <u>Positive</u> numerical series. If $\exists\alpha > \mathbf{1}, \mathbf{n}^\alpha \times \mathbf{U_n} \underset{+\infty}{\sim} \mathbf{0}$ then $\sum \mathbf{U_n}$ CVG

3.5.1 Proof

$$\exists\alpha > \mathbf{1}, \mathbf{n}^\alpha \times \mathbf{U_n} \xrightarrow[\mathbf{n}\to+\infty]{} \mathbf{0} \implies \frac{\mathbf{U_n}}{\frac{\mathbf{1}}{\mathbf{n}^\alpha}} \xrightarrow[\mathbf{n}\to+\infty]{} \mathbf{0}$$

$$\implies \left\{ \begin{array}{r} U_n = o(\frac{1}{n^\alpha}) \\ \text{and} \\ \alpha > 1 \\ \text{and} \\ \sum U_n \text{ P.T.S.} \end{array} \right| \left[\sum \frac{1}{n^\alpha} \text{ CVG (Riemann's series)} \implies \sum U_n \text{ CVG}\right]$$

3.6 D'Alembert's Rule (Ratio Test)

Let $(\mathbf{U_n})$ be a strictly positive sequence such that:

$$\frac{U_{n+1}}{U_n} \xrightarrow[n\to+\infty]{} \ell \in \mathbb{R}_+ \cup \{+\infty\}$$

$$\text{Then: } \begin{array}{l} \ell < 1 \implies \sum U_n \text{ CVG} \\ \ell > 1 \implies \sum U_n \text{ DVG} \\ \ell = 1 \implies \text{ no conclusion} \end{array}$$

3.6.1 Example

$\sum \frac{\mathbf{1}}{\mathbf{n!}}$: $\forall n \in \mathbb{N}, \frac{1}{n!} > 0$ (P.T.S.) and $\frac{U_{n+1}}{U_n} = \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{1}{n+1} \xrightarrow[n\to+\infty]{} 0 < 1 \implies \sum \frac{1}{n!}$ CVG

3.7 Cauchy's Rule

Let $(\mathbf{U_n})$ be a strictly positive sequence such that:

$$\sqrt[n]{U_n} \xrightarrow[n\to+\infty]{} \ell \in \mathbb{R}_+ \cup \{+\infty\}$$

$$\text{Then: } \begin{array}{l} \ell < 1 \implies \sum U_n \text{ CVG} \\ \ell > 1 \implies \sum U_n \text{ DVG} \\ \ell = 1 \implies \text{ no conclusion} \end{array}$$

3.7.1 Example

$\sum (\frac{\mathbf{n}}{\mathbf{n+1}})^{\mathbf{n^2}}$: $\forall n \in \mathbb{N}, (\frac{n}{n+1})^{n^2} > 0$ (P.T.S.), $\sqrt[n]{U_n} = \left((\frac{n}{n+1})^{n^2}\right)^{\frac{1}{n}} = (\frac{n}{n+1})^n = e^{n\ln(1-\frac{n}{n+1})}$
$\ln(1-\frac{n}{n+1}) \sim n\times(-\frac{n}{n+1}) \xrightarrow[n\to+\infty]{} -1 < 0 \implies \sqrt[n]{U_n} \xrightarrow[n\to+\infty]{} e^{-1} = \frac{1}{e} < 1 \underset{\text{Cauchy}}{\implies} \sum U_n$ CVG

3.8 Examples

1 $\sum(1+\frac{1}{n})^n$: $1+\frac{1}{n} \not\xrightarrow[n\to+\infty]{} 0$ (don't have the necessary condition) $\implies \sum(1+\frac{1}{n})^n$ DVG

2 $\sum\left((1+\frac{1}{n})^n - e\right)$:

(a)

$$\begin{aligned}\left((1+\frac{1}{n})^n - e\right) = e^{n\times\ln(1+\frac{1}{n})} - e &= e^{n\times(\frac{1}{n}-\frac{1}{2n^2}+o(\frac{1}{n^2}))} - e \\ &= e^{1-\frac{1}{2n}+o(\frac{1}{n})} - e \\ &= e \times e^{-\frac{1}{2n}+o(\frac{1}{n})} - e \\ &= e \times (1 - \frac{1}{2n} + o(\frac{1}{n})) - e \\ &= -\frac{e}{2n} + o(\frac{1}{n})\end{aligned}$$

So $\left((1+\frac{1}{n})^n - e\right) \underset{+\infty}{\sim} -\frac{e}{2n}$ (Can't use P.T.S. property)

(b) $\sum -\frac{e}{2n} < 0$ for $n \in \mathbb{N}^\star$

(c) $\exists p \in \mathbb{N}^\star, (n \geq p) \implies \left(\left((1+\frac{1}{n})^n - e\right) \leq 0\right)$ (Same sign as $\sum -\frac{e}{2n}$)
$\implies \sum\left((1+\frac{1}{n})^n - e\right)$ has the same nature as $\sum -\frac{e}{2n}$ wich is of same nature as $\sum \frac{1}{n}$ DVG

③ $\sum n^{2023} \times e^{-n} = \sum \frac{n^{2023}}{e^n}$: $n^{2023} = o(e^n)$ (growth comparison) $n^{2025} \times e^{-n} = \frac{\frac{n^{2023}}{e^n}}{\frac{1}{n^2}} \xrightarrow[n \to +\infty]{} 0 \implies U_n = o(\frac{1}{n^2}) \underset{\text{Riemann}(\alpha=2>1)}{\implies} \sum U_n$ CVG

④ $\sum n! \times e^{-n}$

4 Alternating Series

4.1 Definition

Let $(\mathbf{U_n}) \in \mathbb{R}^{\mathbb{N}}$, we say $(\mathbf{U_n})$ is an alternating sequence thus $\sum \mathbf{U_n}$ an alternating series, if there exists $\left|\begin{array}{l}\text{a positive}\\ \text{a negative}\end{array}\right.$ sequence $(\mathbf{a_n})$ such that:

$$\forall \mathbf{n} \in \mathbb{N}, \left|\begin{array}{l} U_n = (-1)^n \times a_n \\ U_n = (-1)^{n+1} \times a_n \end{array}\right.$$

4.1.1 Example

$\sum \frac{(-1)^{\mathbf{n}}}{\mathbf{n}}$ is an alternating series because $\forall \mathbf{n} \in \mathbb{N}, \frac{(-1)^{\mathbf{n}}}{\mathbf{n}} = (-1)^{\mathbf{n}} \times \frac{1}{\mathbf{n}}$

4.2 Alternating Series Special Criteria (A.S.S.C.)

4.2.1 Theorem

Let $(\mathbf{U_n})$ an alternating sequence, such that:

$$\left.\begin{array}{l} U_n \xrightarrow[n \to +\infty]{} 0 \\ (|U_n|)_{n \in \mathbb{N}} \text{ is decreasing} \end{array}\right] \implies \sum \mathbf{U_n} \text{ CVG}$$

4.2.2 Explanation

An alternating sequence is of the form

$$\mathbf{U_n} = (-1)^{\mathbf{n}} \times \mathbf{a_n} \quad \text{or} \quad \mathbf{U_n} = (-1)^{\mathbf{n+1}} \times \mathbf{a_n}$$

$$|\mathbf{U_n}| = |(-1)^{\mathbf{n}} \times \mathbf{a_n}| = |\mathbf{a_n}| \quad \text{or} \quad |\mathbf{U_n}| = \left|(-1)^{\mathbf{n+1}} \times \mathbf{a_n}\right| = |\mathbf{a_n}|$$

$$\text{So } (|\mathbf{U_n}|)_{\mathbf{n} \in \mathbb{N}} = (\mathbf{a_n})_{\mathbf{n} \in \mathbb{N}}$$

5 Absolute Convergence

5.1 Definition

Let $(\mathbf{U_n})$ a sequence, we say $\sum \mathbf{U_n}$ is absolutely convergent if $\sum |\mathbf{U_n}|$ is convergent.

5.1.1 Example

$\sum \frac{(-1)^n}{n^2}$ is absolutely convergent because $\sum \left|\frac{(-1)^n}{n^2}\right| = \sum \frac{1}{n^2}$ is convergent.

5.2 Proposition

Let $\sum U_n$ a series, if $\sum U_n$ is absolutely convergent then $\sum U_n$ is convergent.

$$\sum |U_n| \text{ CVG } \underset{\not\Longleftarrow}{\Longrightarrow} \sum U_n \text{ CVG.}$$

5.2.1 Counter Example

$\sum \frac{(-1)^n}{n}$ is convergent BUT $\sum \left|\frac{(-1)^n}{n}\right| = \sum \frac{1}{n}$ is divergent.

5.3 Examples

5.3.1 Example 1

- $\sum \frac{(-1)^n}{n^\alpha}, \alpha \in \mathbb{R}$:
  - case $\alpha \leq 0$: $\sum \frac{(-1)^n}{n^\alpha} \underset{n\to+\infty}{\not\longrightarrow} 0$ (necessary condition) $\Longrightarrow \sum \frac{(-1)^n}{n^\alpha}$ DVG.
  - case $\alpha > 0$: $\sum \frac{1}{n^\alpha} > 0 \Longrightarrow \sum \frac{(-1)^n}{n^\alpha}$ is an alternating series.

$$\left.\begin{array}{l} \frac{1}{n^\alpha} \xrightarrow[n\to+\infty]{} 0 \\ \left|\frac{1}{n^\alpha}\right| = \frac{1}{n^\alpha} \text{ is decreasing} \end{array}\right| \overset{\text{A.S.S.C.}}{\Longrightarrow} \sum \frac{(-1)^n}{n^\alpha} \text{ CVG.}$$

5.3.1.1 Proposition deduced from example 1

$\forall \alpha > 0, \sum \frac{(-1)^n}{n^\alpha}$ is convergent.

5.3.2 Example 2

- $\forall n \in \mathbb{N}, U_n = \frac{\sin(n)}{n^\alpha}$: $|U_n| = \frac{|\sin(n)|}{n^\alpha}$, $\Longrightarrow 0 \leq |U_n| \leq \frac{1}{n^\alpha}$

$$\begin{aligned} \text{If } \alpha > 1, \text{ then } & \sum \frac{1}{n^\alpha} \text{ CVG (Riemann } \alpha > 1) \\ \text{then } & \sum |U_n| \text{ CVG (Comparison test)} \\ \text{then } & \sum U_n \text{ Absolutely CVG (Proposition)} \\ \text{then } & \sum U_n \text{ CVG (Proposition)} \end{aligned}$$

6 Important Proof

6.1 Series whose general term is positive

6.1.1 Theorem (Comparison rules)

Consider two sequences $(\mathbf{U_n})$ and $(\mathbf{V_n})$.

1. If for all $n \in \mathbb{N}, u_n \leq v_n$, then
   (a) $\sum \mathbf{v_n}$ converges $\implies \sum u_n$ converges
   (b) $\sum \mathbf{u_n}$ diverges $\implies \sum v_n$ diverges

   If $\mathbf{u_n} \sim \mathbf{v_n}$ then the series $\sum \mathbf{u_n}$ and $\sum \mathbf{v_n}$ have the same nature.

6.1.1.1 Remarks

– Property 1 remains true if the relation $\mathbf{u_n} \leq \mathbf{v_n}$ satisfied only above a certain rank, instead of for all $n \in \mathbb{N}$. That is, it is true if there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N}, n \geq n_0 \implies u_n \leq v_n$$

– Property 1 includes the case $u_n = o(v_n)$. Indeed, in this case, the relation $\mathbf{u_n} \leq \mathbf{v_n}$ is satisfied above a certain rank.

6.1.1.2 Proof

1. Let $(\mathbf{S_n})$ denote the partial sums of $\sum \mathbf{u_n}$ and $(\mathbf{T_n})$ the partial sums of $\sum \mathbf{v_n}$. To start with, note that the sequences $(\mathbf{S_n})$ and $(\mathbf{T_n})$ are both increasing. Indeed, for all $n \in \mathbb{N}$,
$$S_{n+1} - S_n = u_{n+1} \geq 0 \quad \text{and} \quad T_{n+1} - T_n = v_{n+1} \geq 0$$
   Thus, we know that
$$(S_n) \text{ converges } \iff (S_n) \text{ is bounded above}$$
   Furthermore, since for all $n \in \mathbb{N}$, $u_n \leq v_n$, we can write:
$$\forall n \in \mathbb{N}, \quad S_n \leq T_n$$
   Thus, if $\sum \mathbf{v_n}$ converges, then $(\mathbf{T_n})$ is bounded. It hence admits an upper bound M. Then for all $n \in \mathbb{N}$:
$$S_n \leq T_n \leq M$$
   and M is also an upper bound of $(\mathbf{S_n})$. The sequence $(\mathbf{S_n})$ is hence bounded above and, since it is increasing, it converges. This proves the property (a).

   Proving property (b) is now straightforward: it is the contrapositive of property (a).

2. Assume that $(u_n) \sim (v_n)$. Then there exists a sequence (ϵ_n) such that
$$\forall n \in \mathbb{N}, u_n = v_n \times (1 + \epsilon_n) \quad \text{and} \quad \epsilon_n \xrightarrow[n \to +\infty]{} 0$$

Since (ϵ_n) converges to 0, it remains between $-\frac{1}{2}$ and $\frac{1}{2}$ above a certain rank: there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}\forall n \in \mathbb{N}, n \geq n_0 &\implies -\frac{1}{2} \leq \epsilon_n \leq \frac{1}{2} \\ &\implies \frac{1}{2} \leq 1+\epsilon_n \leq \frac{3}{2} \\ &\implies \frac{1}{2}v_n \leq u_n \leq \frac{3}{2}v_n\end{aligned}$$

If $\sum \mathbf{u_n}$ converges then, using property 1 and the relation $\frac{1}{2}v_n \leq u_n$, we know that $\sum \mathbf{\frac{1}{2}v_n}$ converges. Thus, $\sum \mathbf{v_n}$ converges.

If $\sum \mathbf{u_n}$ diverges then, using property 1 and the relation $u_n \leq \frac{3}{2}v_n$, we know that $\sum \mathbf{\frac{3}{2}v_n}$ diverges. Thus, $\sum \mathbf{v_n}$ diverges.

6.1.2 Theorem (Riemann series)

Let α in $\mathbb{R}$. The series $\sum \frac{1}{n^\alpha}$ converges if and only if $\alpha > 1$.

6.1.2.1 Some explanations before the proof:

Before the explicit proof, here are the main ideas we will use:

1. We focus on the case $\alpha > 0$ (otherwise, $\frac{1}{n^\alpha}$ does not converge to 0, hence the series diverges).

2. When $0 < \alpha \leq 1$, we try to lower-bound $\frac{1}{n^\alpha}$ by a positive sequence $(\mathbf{v_n})$ such that $\sum v_n$ diverges. And when $\alpha > 1$, we try to upper-bound $\frac{1}{n^\alpha}$ by a positive sequence $(\mathbf{w_n})$ such that $\sum w_n$ converges.

3. In that purpose, we use the property that, since the function $t \mapsto \frac{1}{t^\alpha}$ decreases, we know that for all $n \geq 2$:

$$\forall t \in [n-1, n], \frac{1}{t^\alpha} \geq \frac{1}{n^\alpha} \quad \text{and} \quad \forall t \in [n, n+1], \frac{1}{n^\alpha} \geq \frac{1}{t^\alpha}$$

We can hence integrate the first inequality on $[n-1, n]$ and the second one on $[n, n+1]$:

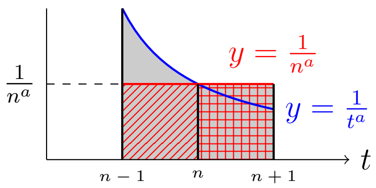
$$\int_{n-1}^{n} \frac{1}{t^\alpha}\mathrm{d}t \geq \int_{n-1}^{n} \frac{1}{n^\alpha}\mathrm{d}t \quad \text{and} \quad \int_{n}^{n+1} \frac{1}{n^\alpha}\mathrm{d}t \geq \int_{n}^{n+1} \frac{1}{t^\alpha}\mathrm{d}t$$

The function $t \mapsto \frac{1}{t^\alpha}$ is a constant function. Thus,

$$\int_{n-1}^{n} \frac{1}{n^\alpha}\mathrm{d}t = \left[\frac{t}{n^\alpha}\right]_{n-1}^{n} = \frac{1}{n^\alpha} \quad \text{and} \quad \int_{n}^{n+1} \frac{1}{n^\alpha}\mathrm{d}t = \left[\frac{t}{n^\alpha}\right]_{n}^{n+1} = \frac{1}{n^\alpha}$$

Finally, for all $n \geq 2$,

$$\int_{n-1}^{n} \frac{1}{t^\alpha}\mathrm{d}t \geq \frac{1}{n^\alpha} \geq \int_{n}^{n+1} \frac{1}{t^\alpha}\mathrm{d}t$$



4. To compute the integral of $\frac{1}{t^\alpha}$: a primitive function F is defined on $\mathbb{R}_+^*$ by:

$$F(t) = \frac{t^{-\alpha+1}}{-\alpha+1} \text{ (case } \alpha \neq 1) \quad \text{or} \quad F(t) = \ln(t) \text{ (case } \alpha = 1)$$

6.1.2.2 Theorem's proof

Let $\alpha \in \mathbb{R}$ and the series $\sum \frac{1}{n^\alpha}$.

1. If $\alpha \leq 0$, then $\frac{1}{n^\alpha} = n^{-\alpha}$ with $-\alpha \geq 0$. Thus, $\left(\frac{1}{n^\alpha}\right)$ does not converge to 0 and the series diverges.

2. If $0 < \alpha \leq 1$: since the function $t \mapsto \frac{1}{t^\alpha}$ decreases on $\mathbb{R}_+^*$, we know that for all $n \geq 1$:

$$\forall t \in [n, n+1], \frac{1}{n^\alpha} \geq \frac{1}{t^\alpha}$$

   By integrating this inequality on $[n, n+1]$, we get: $\int_n^{n+1} \frac{1}{n^\alpha}\mathrm{d}t \geq \int_n^{n+1} \frac{1}{t^\alpha}\mathrm{d}t$.

   The first integral is $\left[\frac{t}{n^\alpha}\right]_n^{n+1} = \frac{1}{n^\alpha}$.

   If F denotes a primitive function of $\frac{1}{t^\alpha}$, we hence get:

$$\forall n \geq 1, \frac{1}{n^\alpha} \geq F(n+1) - F(n) \geq 0$$

   Since both series $\sum \frac{1}{n^\alpha}$ and $\sum (F(n+1) - F(n))$ have positive terms, we can use comparison theorem. Let us prove that the serie $\sum (F(n+1) - F(n))$ diverges: the latter is a telescoping series, it hence has the same nature as the sequence $(F(n))$.

   If $\alpha < 1$, then for all $n \in \mathbb{N}^*$, $F(n) = \frac{n^{1-\alpha}}{1-\alpha}$ with $1 - a > 0$. The sequence $(F(n))$ hence diverges to $+\infty$, that is, $\sum (F(n+1) - F(n))$ diverges.

   If $\alpha = 1$, then for all $n \in \mathbb{N}^*$, $F(n) = \ln(n)$ and the sequence $(F(n))$ diverges to $+\infty$. Here also, $\sum (F(n+1) - F(n))$ diverges.

   Finally, for all α such that $0 < \alpha \leq 1$, the series $\sum (F(n+1) - F(n))$ diverges. Using comparison theorem, it results that $\sum \frac{1}{n^\alpha}$ diverges too.

3. If $\alpha > 1$: since the function $t \mapsto \frac{1}{t^\alpha}$ decreases on $\mathbb{R}_+^*$, we know that for all $n \geq 2$:

$$\forall t \in [n-1, n], \frac{1}{n^\alpha} \leq \frac{1}{n^\alpha}$$

By integrating this inequality on $[n-1,n]$, we get: $\int_{n-1}^{n} \frac{1}{n^\alpha}\mathrm{d}t \leq \int_{n-1}^{n} \frac{1}{t^\alpha}\mathrm{d}t$.

The first integral is $\left[\frac{t}{n^\alpha}\right]_{n-1}^{n} = \frac{1}{n^\alpha}$.

If F denotes a primitive function of $\frac{1}{t^\alpha}$, we hence get:

$$\forall n \geq 1, 0 \leq \frac{1}{n^\alpha} \leq F(n) - F(n-1)$$

Since both series $\sum \frac{1}{n^\alpha}$ and $\sum (F(n) - F(n-1))$ have positive terms, we can use comparison theorem. Let us prove that the serie $\sum (F(n) - F(n-1))$ converges: the latter is a telescoping series, it hence has the same nature as the sequence $(F(n))$.

But $F(n) = \frac{n^{1-\alpha}}{1-\alpha} = -\frac{1}{(1-\alpha)} \times \frac{1}{n^{\alpha-1}}$ with $\alpha - 1 > 0$.

Thus, the sequence $(F(n))$ converges to 0 and the telescoping series $\sum (F(n) - F(n-1))$ converges.

Using comparison theorem, it results that $\sum \frac{1}{n^\alpha}$ converges too.

6.2 Series whose general term has a non-constant sign

6.2.1 Theorem (Absolute convergence)

If a series $\sum u_n$ converges absolutely, then it converges.

Reminder: a series $\sum u_n$ converges absolutely if $\sum |u_n|$ converges.

6.2.1.1 Proof

Consider a series $\sum u_n$ converging absolutely. We hence assume that $\sum |u_n|$ converges.

Let us define the two series (u_n^+) and (u_n^-) by:

$$\forall n \in \mathbb{N}, \quad u_n^+ = \begin{cases} u_n & \text{if } u_n \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad u_n^- = \begin{cases} -u_n & \text{if } u_n \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

These sequences are both positive $u_n = u_n^+ - u_n^-$. Furthermore,

- $\forall n \in \mathbb{N}, 0 \leq u_n^+ \leq |u_n|$ and $\sum |u_n|$ converges, so $\sum u_n^+$ converges.
- $\forall n \in \mathbb{N}, 0 \leq u_n^- \leq |u_n|$ and $\sum |u_n|$ converges, so $\sum u_n^-$ converges.

Thus, $\sum u_n = \sum (u_n^+ - u_n^-)$ is the sum of two convergent series. It is hence convergent.

6.2.2 Theorem (Leibniz's rule)

Let (u_n) be an alternating sequence. If $(|u_n|)$ is decreasing and converges to 0, then:

1. $\sum u_n$ converges.
2. The remainder (R_n) of the series satisfy to: $\forall n \in \mathbb{N}, |R_n| \leq |u_{n+1}|$.

6.2.2.1 Reminders

the theorem's proof relies on the notion of adjacent sequences and on two properties seen during the chapter 5 (sequences) the previous year:

1. Two sequences (u_n) and (v_n) are adjacent if they satisfy the conditions:
   - One of them is increasing and the other one is decreasing.
   - The sequence $(u_n - v_n)$ converges to 0.
2. Properties of adjacent sequences: if two sequences (u_n) and (v_n) are adjacent, then:
   - Both converge. Furthermore, they admit an **identical** limit ℓ.
   - If (u_n) is the increasing sequence and (v_n) the decreasing one, then:

$$\forall n \in \mathbb{N}, \quad u_n \leq u_{n+1} \leq \ell \leq v_{n+1} \leq v_n$$

3. Property about subsequences: consider a sequence (u_n) such that the subsequences (u_{2n}) and (u_{2n+1}) both converge to an **identical** limit ℓ. Then, (u_n) converges to ℓ.

6.2.2.2 Proof of the theorem

Let (u_n) be an alternating sequence. Then there exists a positive sequence (a_n) such that:

$$(u_n) = ((-1)^n \times a_n) \quad \text{or} \quad (u_n) = (-(-1)^n \times a_n)$$

For the proof, we can assume that we are in the first case $(u_n) = ((-1)^n \times a_n)$. If not, just replace (u_n) by $(-u_n)$. The positive sequence (a_n) is in fact the sequence $(|u_n|)$: the theorem hypothesis state that it decreases and converges to 0.

Let (S_n) be the partial sums of $\sum u_n$: for all $n \in \mathbb{N}$,

$$S_n = a_0 - a_1 + a_2 - a_3 + \cdots + (-1)^n a_n$$

To start with, let us prove that the sequences (S_{2n}) and (S_{2n+1}) are adjacent.

1. Monotony of (S_{2n}): this subsequence contains the terms of even ranks. The term following S_{2n} is hence $S_{2(n+1)} = S_{2n+2}$. Thus, for all $n \in \mathbb{N}$:

$$\left\{\begin{array}{rl} S_{2n} &= a_0 - a_1 + a_2 - a_3 + \cdots + a_{2n} \\ S_{2(n+1)} &= a_0 - a_1 + a_2 - a_3 + \cdots + a_{2n} - a_{2n+1} + a_{2n+2} \\ \hline S_{2(n+1)} - S_{2n} &= -a_{2n+1} + a_{2n+2} \end{array}\right.$$

   Since (a_n) is decreasing, $-a_{2n+1} + a_{2n+2}$ is negative. The sequence (S_{2n}) is hence decreasing.

2. Monotony of (S_{2n+1}): this subsequence contains the terms of odd ranks. The term following S_{2n+1} is hence $S_{2(n+1)+1} = S_{2n+3}$. Thus, for all $n \in \mathbb{N}$:

$$\begin{cases} \quad S_{2n+1} = a_0 - a_1 + a_2 - a_3 + \cdots + a_{2n} - a_{2n+1} \\ \underline{S_{2(n+1)+1} = a_0 - a_1 + a_2 - a_3 + \cdots + a_{2n} - a_{2n+1} + a_{2n+2} - a_{2n+3}} \\ S_{2(n+1)+1} - S_{2n+1} = a_{2n+2} - a_{2n+3} \end{cases}$$

Since (a_n) is decreasing, $a_{2n+2} - a_{2n+3}$ is positive. The sequence (S_{2n+1}) is hence increasing.

3. Study of $S_{2n+1} - S_{2n}$: for all $n \in \mathbb{N}$,

$$\begin{cases} \quad S_{2n} = a_0 - a_1 + a_2 - a_3 + \cdots + a_{2n} \\ \underline{S_{2n+1} = a_0 - a_1 + a_2 - a_3 + \cdots + a_{2n} - a_{2n+1}} \\ S_{2n+1} - S_{2n} = -a_{2n+1} \end{cases}$$

Since (a_n) converges to 0, $(S_{2n+1} - S_{2n})$ converges to 0 too.

We hence proved that (S_{2n}) and (S_{2n+1}) are adjacent. From this, we know that they both converge and admit an identical limit ℓ. The we get:

$$\left.\begin{array}{l} S_{2n} \xrightarrow[n\to+\infty]{} \ell \\ S_{2n+1} \xrightarrow[n\to+\infty]{} \ell \end{array}\right\} \implies S_n \xrightarrow[n\to+\infty]{} \ell$$

This prove that (S_n) converges, that is, $\sum u_n$ converges.

Now let us prove that for all n $\in \mathbb{N}$, $|R_n| \leq |u_{n+1}|$: the sequences (S_{2n}) and (S_{2n+1}) being adjacent, we know that for all $n \in \mathbb{N}$:

$$S_{2n+1} \leq S_{2n+3} \leq \ell \leq S_{2n+2} \leq S_{2n}$$

Thus, $|R_{2n}| = S_{2n} - \ell \leq S_{2n} - S_{2n+1} = u_{2n+1}$

and $|R_{2n+1}| = \ell - S_{2n+1} \leq S_{2n+2} - S_{2n+1} = u_{2n+2}$.

Thus, for all $n \in \mathbb{N}$, $|R_n| \leq |u_{n+1}|$.