

Chapter 12: Linear Maps

March 25, 2023

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1 General approach

1.1 Definition

Let E, F two $\mathbb{K} - VS$, and f a mapping from E to F . We say that f is a linear (or f is a linear map) if:

$$\forall(\alpha, X, Y) \in \mathbb{K} \times E \times E, f(\alpha \cdot X + Y) = \alpha \cdot f(X) + f(Y)$$

$$\Longleftrightarrow$$

$$\forall(\alpha, \beta, X, Y) \in \mathbb{K} \times \mathbb{K} \times E \times E, f(\alpha \cdot X + \beta \cdot Y) = \alpha \cdot f(X) + \beta \cdot f(Y)$$

1.2 Notation

We denote $L(E, F)$ the set of all linear maps from E to F .

1.3 Specific Linear Maps

1.3.1 Definition

1. Let $f \in \mathcal{L}(E, F)$: we say f is an endomorphism if $E = F$ we then denote $\mathcal{L}(E)$ the set of all endomorphism of E .
2. Let $f \in \mathcal{L}(E, F)$: we say f is an isomorphism if f is bijective.
3. Let $f \in \mathcal{L}(E, F)$: we say f is an automorphism if f is an endomorphism and an isomorphism. ($E = F$ and bijective)

1.4 Necessary Condition

$$f \in \mathcal{L}(E, F) \implies f(0_E) = 0_F$$

1.4.1 Proof

Let $X \in E$	and	$X \in E.$
$f(0_E) = f(0_R \times X)$	and	$f(0_E) = f(X - X)$
$f(0_E) = 0_R \times f(X)$	and	$f(0_E) = f(X) - f(X)$
$f(0_E) = 0_F$	and	$f(0_E) = 0_F$

2 Kernel and Images

2.1 Definition

Let E and F two $\mathbb{K} - VS$ and $f \in \mathcal{L}(E, F)$. Then:

1. We call kernel of f and denote $Ker(f)$ the subset of E defined as follows:

$$Ker(f) = \{X \in E \mid f(X) = 0_F\} = f^{-1}(\{0_F\})$$

Note: $f^{-1}()$ is NOT the inverse of f because f is not necessarily bijective.

2. We call image of f and denote $Im(f)$ the subset of F defined as follows:

$$Im(f) = \{f(X), X \in E\} = \{Y \in F, \exists X \in E, f(X) = Y\}$$

2.2 Example

$$\begin{aligned} f: R^2 &\longrightarrow R^3 & \textcircled{1} \quad f \in \mathcal{L}(R^2, R^3)? \\ \begin{pmatrix} x \\ y \end{pmatrix} &\longmapsto \begin{pmatrix} x \\ 0 \\ y \end{pmatrix} & \textcircled{2} \quad Kerf = ? \\ & & \textcircled{3} \quad Imf = ? \end{aligned}$$

$$\textcircled{1} \text{ Necessary condition: } f(0_E) = 0_F : f(0_{R^2}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} ?$$

$$\forall (\alpha, X, Y) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2, X = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } Y = \begin{pmatrix} x' \\ y' \end{pmatrix}, x, y, x', y' \in \mathbb{R}$$

$$f(\alpha \cdot X + Y) = \begin{pmatrix} \alpha \cdot x + x' \\ \alpha \cdot y + y' \end{pmatrix} = \begin{pmatrix} \alpha \cdot x + x' \\ 0 \\ \alpha \cdot y + y' \end{pmatrix} = \alpha \cdot \begin{pmatrix} x \\ 0 \\ y \end{pmatrix} + \begin{pmatrix} x' \\ 0 \\ y' \end{pmatrix} = \alpha \cdot f(X) + f(Y)$$

so $\textcircled{1} \quad f \in \mathcal{L}(R^2, R^3) \checkmark$