Chapter 13: Matrix

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1 General approach

1.1 Definition

1.1.1 Definition of a matrix

We call matrix of n rows and p columns any mapping in the following form:

$$\begin{array}{cl} [\![1,n]\!] \times [\![1,p]\!] & \to \mathbb{K} \\ i,j & \quad a_{ij} \end{array}$$

We denote such maps as tables of n rows and p columns, and we write:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix}$$

$\forall (i,j) \in [\![1,n]\!] \times [\![1,p]\!]$, we call a_{ij} a coefficient of the matrix. In this case coefficient if i-th row and j-th column.

1.1.2 Notation

We denote $M_{np}(\mathbb{K})$ the set of matrix of n rows and p columns with coefficient from $\mathbb{K}$.

1.1.3 Examples

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \in M_{32}(\mathbb{R})$$

$$B = \begin{pmatrix} i \\ 1+i \\ 3 \end{pmatrix} \in M_{31}(\mathbb{C})$$

1.2 Particular matrices

Let $A \in M_{np}(\mathbb{K})$ then:

1.2.1 Null matrix

1. $[\forall (i,j) \in [\![1,n]\!] \times [\![1,p]\!], a_{ij} = 0] \Rightarrow [A = 0_{np}]$ We say A is the null matrix $M_{np}(\mathbb{K})$.

1.2.1.1 Example

$$A' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \in M_{32}(\mathbb{R})$$

1.2.2 Column matrix

2. $B \in M_{np}(\mathbb{K})$ and $p = 1 \Rightarrow$ B is a column matrix of n rows

1.2.2.1 Example

$$B' = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in M_{31}(\mathbb{R})$$

1.2.3 Row matrix

3. $B \in M_{np}(\mathbb{K})$ and $n = 1 \Rightarrow$ C is a row matrix of p columns

1.2.3.1 Example

$$C' = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \in M_{13}(\mathbb{R})$$

1.2.4 Square matrix

We call square matrix any matrix with same number of rows and columns. We denote $M_n(\mathbb{K})$ the set of square matrix of n rows and columns with coefficient from $\mathbb{K}$.

4. $D \in M_{np}(\mathbb{K})$ and $n = p \Rightarrow$ D is a square matrix denote $M_n(\mathbb{K})$

1.2.4.1 Example

$$D' = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \in M_3(\mathbb{R})$$

1.2.5 Diagonal matrix

5. $\forall E \in M_n(\mathbb{R})$, if $\forall (i,j) \in [\![1,n]\!]^2, i \neq j \Rightarrow a_{ij} = 0$ then we say E is a diagonal matrix

1.2.5.1 Example

$$E' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \in M_2(\mathbb{R})$$

$$E'' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \in M_3(\mathbb{R})$$

1.2.6 Identity matrix

6. $\forall I_n \in M_n(\mathbb{R})$, if $\forall (i,j) \in [\![1,n]\!]^2, i \neq j \Rightarrow a_{ij} = 0$ and $i = j \Rightarrow a_{ij} = 1$ then we say I_n is a identity matrix

1.2.6.1 Example

$$I'_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{R})$$

$$I''_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in M_3(\mathbb{R})$$

1.2.7 Triangular matrix

6. $\forall F \in M_n(\mathbb{R})$, if $\forall (i,j) \in [\![1,n]\!]^2, i > j \Rightarrow a_{ij} = 0$ then we say F is a lower triangular matrix

7. $\forall G \in M_n(\mathbb{R})$, if $\forall (i,j) \in [\![1,n]\!]^2, i < j \Rightarrow a_{ij} = 0$ then we say G is a upper triangular matrix

1.2.7.1 Example

$$F' = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \in M_2(\mathbb{R})$$

$$G' = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \in M_2(\mathbb{R})$$

1.3 Transposed matrix

1.3.1 Definition

Let $A \in M_{np}(\mathbb{K})$. We call transposed matrix of A (or A transpose) a matrix B from $M_{pn}(\mathbb{K})$ such as:

$$\forall (i,j) \in [\![1,n]\!] \times [\![1,p]\!], a_{ij} = b_{ji}$$

1.3.2 Notation

We denote B as ${}^tA$

1.3.3 Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \in M_{23}(\mathbb{R})$$

$${}^tA = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \in M_{32}(\mathbb{R})$$

1.4 Symmetric matrix

1.4.1 Symmetric

If ${}^tA = A$ then we say A is symmetric

1.4.1.1 Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} = {}^tA = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} \in M_3(\mathbb{R})$$

1.4.2 Anti-Symmetric

If ${}^tA = -A$ then we say A is Anti-symmetric

1.4.2.1 Example

$$A = \begin{pmatrix} 0 & -2 & 3 \\ 2 & 0 & -5 \\ -3 & 5 & 0 \end{pmatrix} = {}^tA = \begin{pmatrix} 0 & 2 & -3 \\ -2 & 0 & 5 \\ 3 & -5 & 0 \end{pmatrix} \in M_3(\mathbb{R})$$

2 Operations on matrices

2.1 Addition and external product

2.1.1 Definition

1. We call internal operation in $M_{np}(\mathbb{K})$ denoted $\oplus$ "internal addition" the one definde as follows:

$$\forall A, B \in M_{np}^2(\mathbb{K}), A + B = (a_{ij} + b_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le p}}$$

$$\text{Where } A = a_{ij}{}_{\substack{1 \le i \le n \\ 1 \le j \le p}} \text{ and } B = b_{ij}{}_{\substack{1 \le i \le n \\ 1 \le j \le p}}$$

2. We call "external multiplication" or "multiplication by a scalar" the one defined as follows:

$$\forall A \in M_{np}(\mathbb{K}), \forall \alpha \in \mathbb{K}, \alpha A = (\alpha a_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le p}}$$

2.1.1.1 Example

$$(A, B) \in M_{2,3}(\mathbb{R})^2 \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \text{ and } B = \begin{pmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 1+7 & 2+8 & 3+9 \\ 4+10 & 5+11 & 6+12 \end{pmatrix} = \begin{pmatrix} 8 & 10 & 12 \\ 14 & 16 & 18 \end{pmatrix}$$

$$\alpha = 3, \quad \alpha A = \begin{pmatrix} 3 \times 1 & 3 \times 2 & 3 \times 3 \\ 3 \times 4 & 3 \times 5 & 3 \times 6 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{pmatrix}$$

2.1.1.2 Proposition

$$(M_{np}, \oplus, \cdot) \text{ is a vector space over } \mathbb{K}$$

2.1.2 Elementary matrix

For $(n,p) \in \mathbb{N}^2, (i,j) \in [\![1,n]\!] \times [\![1,p]\!]$; We denote E_{ij} the matrix from $M_{np}(\mathbb{K})$ such that the ij-th coefficient is 1 and all other coefficient are 0.

$$E_{ij} \text{ are called elementary matrix}$$

2.1.2.1 Example

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
$$E_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$E_{33} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.1.3 Proposition

1. $(E_{ij})_{\substack{1\leq i\leq n \\ 1\leq j\leq p}}$ is a basis of $M_{np}(\mathbb{K})$
2. $dim((E_{ij})_{\substack{1\leq i\leq n \\ 1\leq j\leq p}}) = np$
   Ex: $M_2(\mathbb{R})$ a ($\mathbb{K}$)-VS: B $= \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$
   B is a Standard basis of $M_2(\mathbb{R})$, $dim(M_2(\mathbb{R})) = 2^2 = 4$

2.2 Internal product

2.2.1 Definition

Let $(n,p,q) \in \mathbb{N}^3$ and $A = a_{ij}{}_{\substack{1\leq i\leq n \\ 1\leq j\leq p}} \in M_{np}(\mathbb{K}), B = b_{ij}{}_{\substack{1\leq i\leq p \\ 1\leq j\leq q}} \in M_{pq}(\mathbb{K})$. We call product of A and B the matrix C form $M_{nq}(\mathbb{K})$ such that:

$$\forall (i,j) \in [\![1,n]\!] \times [\![1,q]\!], c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

2.2.1.1 Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \in M_{2,3}(\mathbb{R}) \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in M_{3,4}(\mathbb{R})$$

$$C = A \cdot B = \begin{pmatrix} 1 & 0 & 8 & 11 \\ 4 & 0 & 20 & 32 \end{pmatrix} \in M_{2,4}(\mathbb{R})$$

$$C_{2,3} = 4 \times 1 + 5 \times 2 + 6 \times 1 = 20$$

2.2.2 Remarks

R1 If A, B two matrices: we only can multiply A by B if the number of column of A is equal to the number of row of B.

R2 AB can exists but BA not or the other way around.

2.2.2.1 Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \Rightarrow M_{2,1}(\mathbb{R}), \text{ and exists but } BA \text{ does not exists}$$

R3 In the $\boxed{\text{General case}}$, where AB and BA exists:
$\boxed{\text{AB} \neq \text{BA}}$ (multiplication of matrix is not commutative)
When $AB = BA$ we say A and B commute.

2.3 Properties of matrix calculus

2.3.1 Properties