Chapter 12: Linear Maps

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Contents

1 General approach

1.1 Definition

Let E, F two $\mathbb{K}-VS$, and f a mapping from E to F. We say that f is a linear (or f is a linear map) if:

$$\forall(\alpha, X, Y) \in \mathbb{K} \times E \times E, f(\alpha \cdot X + Y) = \alpha \cdot f(X) + f(Y)$$

$$\Longleftrightarrow$$

$$\forall(\alpha, \beta, X, Y) \in \mathbb{K} \times \mathbb{K} \times E \times E, f(\alpha \cdot X + \beta \cdot Y) = \alpha \cdot f(X) + \beta \cdot f(Y)$$

1.2 Notation

We denote $L(E, F)$ the set of all linear maps from E to F.

1.3 Specific Linear Maps

1.3.1 Definition

1. Let $f \in \mathcal{L}(E, F)$: we say f is an endomorphism if $E = F$ we then denote $\mathcal{L}(E)$ the set of all endomorphism of E.
2. Let $f \in \mathcal{L}(E, F)$: we say f is an isomorphism if f is bijective.
3. Let $f \in \mathcal{L}(E, F)$: we say f is an automorphism if f is an endomorphism and an isomorphism. ($E = F$ and bijective)

1.4 Necessary Condition

$$f \in \mathcal{L}(E, F) \Longrightarrow f(0_E) = 0_F$$

1.4.1 Proof

Let $X \in E$

$$f(0_E) = f(0_E \times X)$$
$$f(0_E) = 0_F \times f(X)$$
$$f(0_E) = 0_F$$

Let $X \in E$

$$f(0_E) = f(X - X)$$
$$f(0_E) = f(X) - f(X)$$
$$f(0_E) = 0_F$$

2 Kernel and Images

2.1 Definition

Let E and F two $\mathbb{K}-VS$ and $f \in \mathcal{L}(E,F)$. Then:

1. We call kernel of f and denote $Ker(f)$ the subset of E defined as follows:

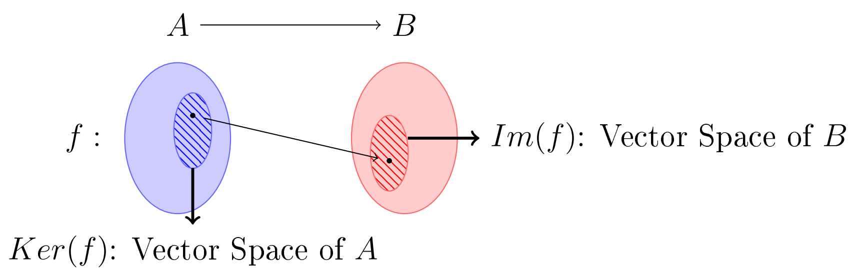
$$Ker(f) = \{X \in E, f(X) = 0_F\} = f^{-1}(\{0_F\})$$

Note: $f^{-1}()$ is NOT the inverse of f beacause f is not necessarily bijective.

2. We call image of f and denote $Im(f)$ the subset of F defined as follows:

$$Im(f) = \{f(X), X \in E\} = \{Y \in F, \exists X \in E, f(X) = Y\}$$

2.2 Graphics representation



2.3 Example

$$f\colon \quad R^2 \longrightarrow R^3$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x \\ 0 \\ y \end{pmatrix}$$

1 $f \in \mathcal{L}(R^2, R^3)$?

2 Kerf $=$?

3 Imf $=$?

1 Necessary condition: $f(0_E) = 0_F$: $f(0_{R^2}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$?

$$\forall (\alpha, X, Y) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2, X = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } Y = \begin{pmatrix} x' \\ y' \end{pmatrix}, x, y, x', y' \in \mathbb{R}$$

$$f(\alpha \cdot X + Y) = \begin{pmatrix} \alpha \cdot x + x' \\ \alpha \cdot y + y' \end{pmatrix} = \begin{pmatrix} \alpha \cdot x + x' \\ 0 \\ \alpha \cdot y + y' \end{pmatrix} = \alpha \cdot \begin{pmatrix} x \\ 0 \\ y \end{pmatrix} + \begin{pmatrix} x' \\ 0 \\ y' \end{pmatrix} = \alpha \cdot f(X) + f(Y)$$

so 1 $f \in \mathcal{L}(R^2, R^3)$ ✓

2 $Ker(f) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in R^2, \begin{pmatrix} x \\ 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

$$③\ Im(f) = \left\{ \begin{pmatrix} x \\ 0 \\ y \end{pmatrix}, (x,y) \in R^2 \right\} = \left\{ x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, (x,y) \in R^2 \right\}$$

$$Im(f) = span \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

2.4 Proposition

1. Let $f \in \mathcal{L}(E,F)$ and $g \in \mathcal{L}(F,G)$. Then:

$$g \circ f \in \mathcal{L}(E,G)$$

2. If f is bijectif then f^{-1} is bijective and $f^{-1} \in \mathcal{L}(F,E)$

3. $\mathcal{L}(E,F)$ is a $\mathbb{K} - VS$:

$$\mathcal{L}(E,F): \begin{array}{l} E \longrightarrow F \\ X \longmapsto B_F \in \mathcal{L}(F,E) \end{array}$$

$$\begin{array}{c} \forall (\alpha, f, g) \in \mathbb{K} \times \mathcal{L}^2(E,F) \\ \alpha \cdot f + g \in \mathcal{L}(E,F) \end{array}$$

4. Let $f \in \mathcal{L}(A,B)$:
   - f is injective $\Longleftrightarrow Ker(f) = \{0_A\}$
   - f is surjective $\Longleftrightarrow Im(f) = B$

3 Projects and Symmetries

3.1 Reminder:

3.1.1 Definition of supllementary subspaces

Let E a $\mathbb{K}$ - VS. Let F and G two supllementary $\mathbb{K}$ - VSS of E:

$$E = F \oplus G \Longleftrightarrow \forall X \in E, \exists!(X_F, X_G) \in F \times G, X = X_F + X_G$$

3.2 Proposition

Let us consider:

$$p: \begin{array}{l} E \longrightarrow \underbrace{F \ \subset E}_{E} \\ X \overset{p}{\longmapsto} X_F \overset{p}{\longmapsto} X_F \end{array}$$

1. $p \in \mathcal{L}(E)$

2. $p \circ p \ (= p^2) = p$

3.
   - $Ker(p) = G$
   - $Im(p) = F$

3.3 Definition

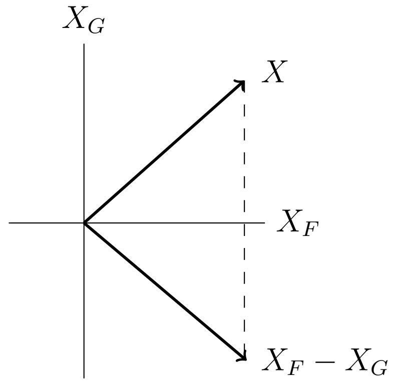
We call projector from $\mathbb{K}$ - VS E over $\mathbb{K}$ - VS F, any endomorphism p of E such that $p \circ p = p$. (We often say that p is a projector over F parallel of/alongside G.)

$$Im(p) \oplus Ker(p) = E$$

Let us consider:

$$S : \begin{array}{ccc} E & \longrightarrow & E \\ X & \longmapsto & (2P - Id_E)(X) = 2p(x) - X \end{array}$$

1. $S \in \mathcal{L}(E)$
2. $\forall X \in E, S(X) = X_E - X_G$
3. $S \circ S = Id_E$



4 Rank Nullity Theorem (RNT)

4.1 Definition of Rank

Let E and F two finite dimensional vector spaces. We call rank of any mapping f from $\mathcal{L}(E, F)$ and denote $rank(f)$ the dimension of $Im(f)$:

$$rank(f) = dim(Im(f))$$

4.2 Proposition

Let E a finite dimensional $\mathbb{K}$ - VS such that $B = (U_1, U_2, \dots, U_n)$ a basis of E. Let F a $\mathbb{K}$ - VS. Then let $f \in \mathcal{L}(E, F)$. We have:

$$Im(f) = sp(\{f(U_1), f(U_2), \dots, f(U_n)\})$$

We then deduce the following theorem:

4.3 Theorem (Rank Nullity Theorem)

Let E a finite dimensional VS and $f \in \mathcal{L}(E, F)$. Then:

$$dim(E) = dim(Ker(f)) + \underbrace{rank(f)}_{dim(Im(f))}$$

4.4 Corollary

Let $f \in \mathcal{L}(E, F)$ where E and F two finite dimensional vector spaces.

$$\text{If } dim(E) = dim(F)(\iff dim(Ker(f)) = 0 \text{ or } dim(Im(f)) = 0)$$

$$\implies$$

$$\text{f injective} \iff \text{f surjective} \iff \text{f bijective}$$

4.5 Involvement

$$\text{f injective} \iff Ker(f) = \{0_E\}$$
$$[\text{f injective} \implies dim(E) \leq dim(F)] \iff [dim(E) > dim(F) \implies \text{f not injective}]$$
$$\text{f surjective} \iff Im(f) = F$$
$$[\text{f surjective} \implies dim(E) \geq dim(F)] \iff [dim(E) < dim(F) \implies \text{f not surjective}]$$

4.6 Proof of corollary

Hypothesis: $dim(E) = dim(F)$

$$\begin{aligned}
\text{f injective} &\iff Ker(f) = \{0_E\} \\
&\iff dim(Ker(f)) = 0 \\
&\iff dim(E) = dim(Ker(f)) + dim(Im(f)) \\
&\iff dim(E) = dim(Im(f)) \\
&\iff dim(F) = dim(Im(f)) \\
\text{f surjective} &\iff Im(f) = F
\end{aligned}$$

5 Important Proof

5.1 Proposition (Kernel and image).

Let E and F be two vector spaces over $\mathbb{R}$ and $f \in \mathcal{L}(E, F)$.

1. $Ker(f)$ is a linear subspace of E.
2. $Im(f)$ is a linear subspace of F.

5.1.1 Proof:

1. $Ker(f)$ is a linear subspace of E.
   - By definition, $Ker(f) \subset E$. Since f is a linear map from E to F, we know that $f(0_E) = 0_F$, that is, $0_E \in Ker(f)$.

- Let $(u,v) \in (Ker(f))^2$ and $\alpha \in \mathbb{R}$.

$$\begin{aligned} f(\alpha u + v) &= \alpha f(u) + f(v) \text{ because } f \text{ is a linear map} \\ &= \alpha(0_F) + (0_F) \\ &= 0_F \end{aligned}$$

  Thus, $\alpha u + v \in Ker(f)$. $Ker(f)$ is hence a linear subspace of E.

2. $Im(f)$ is a linear subspace of F.
   - By definition, $Im(f) \subset F$. Since f is a linear map from E to F, we know that $0_F = f(0_E)$. Thus, $0_F \in Im(f)$.
   - Let $(v,v') \in (Im(f))^2$ and $\alpha \in \mathbb{R}$. Then:

$$v \in Im(f) \Leftrightarrow \exists w \in E, v = f(w) \quad \text{and} \quad v' \in Im(f) \Leftrightarrow \exists w' \in E, v' = f(w')$$

     We hence get:

$$\begin{aligned} \alpha v + v' &= \alpha f(w) + f(w') \\ &= f(\alpha w + w) \text{ since } f \text{ is a linear map} \end{aligned}$$

     This proves that $\alpha v + v' \in Im(f)$. $Im(f)$ is hence a linear subspace of F.

5.2 Proposition (Characterizing injective and surjective linear maps).

Let E and F be two vector spaces over $\mathbb{R}$ and $f \in \mathcal{L}(E,F)$.

1. f is injective if and only if $Ker(f) = \{0_E\}$.
2. f is surjective if and only if $Im(f) = F$.

5.2.1 Proof:

1. $\boxed{\Longrightarrow}$ Assume that f is injective. We hence know that

$$\forall (u,u') \in E^2, f(u) = f(u') \Rightarrow u = u'$$

   - Let $u \in Ker(f)$.
     $f(u) = 0_F$ and $0_F = f(0_E)$. (because f is a linear map) $\Rightarrow f(u) = f(0_E) \Rightarrow u = 0_E$ using the injectivity definition. This proves that $Ker(f) \subset \{0_E\}$.
   - Since $\{0_E\} \subset Ker(f)$, we get $Ker(f) = \{0_E\}$.

   $\boxed{\Longleftarrow}$ Assume that $Ker(f) = \{0_E\}$.
   Let $(u,u') \in E^2$ such that $f(u) = f(u')$. Then

$$\begin{aligned} f(u) = f(u') &\Rightarrow f(u) - f(u') = 0_F \\ &\Rightarrow f(u-u') = 0_F \text{ because } f \text{ is a linear map} \\ &\Rightarrow u - u' \in Ker(f) \\ &\Rightarrow u - u' = \{0_E\} \text{ because } Ker(f) = \{0_E\} \\ &\Rightarrow u - u' = 0_E \\ &\Rightarrow u = u' \end{aligned}$$

f is hence injective.

2. For the surjectivity, we have:

$$\begin{aligned} f \text{ surjective} &\iff \forall v \in F, \exists u \in E, v = f(u) \\ &\iff \forall v \in F, v \in Im(f) \\ &\iff F \subset Im(f) \\ &\iff Im(f) = F \text{ since the inclusion } Im(f) \subset F \text{ is always true} \end{aligned}$$