Chapter 11: Vector Spaces

July 25, 2023

Contents

During this chapter, $\mathbb{K}$ will be either $\mathbb{R}$ or $\mathbb{C}$.

1 General approach

1.1 Structure of a Vector Space

Let E be a set, we define two operations:

- An internal operation:
$$\oplus: \begin{array}{ccc} E \times E & \longrightarrow & E \\ (u, v) & \longmapsto & u + v \end{array}$$
- An external operation:
$$\odot: \begin{array}{ccc} \mathbb{K} \times E & \longrightarrow & E \\ (\lambda, v) & \longmapsto & \lambda u \end{array}$$

1.2 Defintion

We say that $(E, \oplus, \odot)$ is a vector space if $\forall (u, v, w) \in E^3$ we have:

- u + (v + w) = (u + v) + w ($\oplus$ is associative)
- u + v = v + u ($\oplus$ is commutative)
- $\exists 0_E \in E$ such that $u + 0_E = 0_E + u = u$ (Existence of a neutral element for $\oplus$)
- $\exists -u \in E$, $u + (-u) = (-u) + u = 0_E$ (Existence of a symmetrical element for $\oplus$)

And $\forall (u, v) \in E^2$ and $\forall (\alpha, \beta) \in \mathbb{K}^2$ we have:

- $(\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot u$ ($\odot$ is distributive)
- $\alpha(u + v) = \alpha \cdot u + \alpha \cdot v$ ($\odot$ is distributive)
- $(\alpha\beta)u = \alpha(\beta u)$ ($\odot$ is associative)
- $1_{\mathbb{K}} \cdot u = u$ (Where $1_{\mathbb{K}}$ is the neutral element of multiplication of element of $\mathbb{K}$)

Let (E, $\oplus$, $\odot$) be a $\mathbb{K}$-Vector Space,
Any element from E is called a vector and any element from $\mathbb{K}$ is called a scalar.
0_E is called the zero vector.

1.2.1 Property

$\forall u \in E$ and $\forall \alpha \in \mathbb{K}$ we have:

1. $\alpha \cdot 0_E = 0_E$
2. $0_{\mathbb{K}} \cdot u = 0_E$
3. $\alpha \cdot u = 0_E \Leftrightarrow \alpha = 0_{\mathbb{K}}$ or $u = 0_E$

1.2.2 Example

1. $E = \mathbb{R}^2$, $U = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathbb{R}^2$

2. $\mathbb{R}^3, \mathbb{R}^4, \ldots, \mathbb{R}^n$ are $\mathbb{R} - VS$

3. $\mathbb{R}[X]$: Set of polynomials
   $0_{\mathbb{R}[X]} : \mathbb{R}_2[X] = \{aX^2 + bX + c, (a, b, c) \in \mathbb{R}^3\}$
   $\mathbb{R}_2$ means: "at least max2", here "a" can be 0

4. $\mathbb{R}^{\mathbb{N}}$ and $\mathbb{R}^{\mathbb{R}}$ are $\mathbb{R}$-VS

   ex: $\left.\left(\frac{1}{n}\right)_{n \in \mathbb{N}^*}\right\}$ VS: 8 Property are check

1.3 Vector Subspaces (= linear subspaces)

1.3.1 Definition

Let E be a $\mathbb{K}$-Vector Space:

- $F \subset E$ (F is a subset of E)
- $F \neq \emptyset$ (F is non-empty, $0_E \in F$)
- $\forall (u, v) \in F^2$, $\forall \alpha \in \mathbb{K}$, $(\alpha \cdot u + v) \in F$ (F is closed under linear combination)

1.3.2 Propositions

1.3.2.1 Proposition 1

Let E be a $\mathbb{K}$-Vector Space:

$$F \subset E \text{ is a Vector SubSpace of } E \Longrightarrow 0_E \in F$$
$$\not\Longleftarrow$$

1.3.2.2 Example

$E = \mathbb{R}^3$

$F_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3, x + y + z = 1 \right\}$ and $F_1' = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3, x + y + z = 0 \right\}$

$0_E \notin F_1 \Longrightarrow F_1$ is not a Vector SubSpace of E

$\forall (u, v) \in F'^2_1$, $\forall \alpha \in \mathbb{R}$, $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $v = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$

$\alpha u + v = \alpha(x, y, z) + (x', y', z') = 0_E \in F_1' \Longrightarrow F_1'$ is a Vector SubSpace of E

$F_2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \begin{cases} \in \mathbb{R}^3 \\ \ni 0_E \end{cases}, 0_E \in xy \geq 0 \right\}$

$\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}\right) \in {F_2}^2$

$\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\-1\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix} \notin {F_2}^2$ Not closed by linear combination

Exercise on your own: $F_3 = \{P \in \mathbb{R}[X], P(2) = 0\}$

1.3.2.3 Proposition 2

Let E be a Vector Space, F and G two Vector SubSpaces of E. Then:

1. $F \cap G \subset E$, $F \cap G$ is a Vector SubSpace of E
2. $\left.\begin{array}{l} F \text{ VSS of } E \implies 0_E \in F \\ G \text{ VSS of } E \implies 0_E \in G \end{array}\right| \implies 0_E \in F \cap G \implies F \cap G \neq \emptyset$
3. $\forall (u, v) \in F \cap G, \forall \alpha \in \mathbb{K}$ Then: F close under linear combination
   $\left.\begin{array}{l} \implies \alpha u + v \in F. \\ \text{So far for G} \implies \alpha u + v \in G \end{array}\right\} \alpha u + v \in F \cap G$
   and $F \cap G$ is closed under linear combination.

1.3.2.4 Proposition 3

Let E be a VS: $\{0_E\}$ (sigleton) is a VSS of E

1.3.2.5 Proof:

1. $\{0_E\} \subset E$
2. $0_E \in \{0_E\} \implies \{0_E\} \neq \emptyset$
3. $\forall (u, v) \in \{0_E\}^2, u = v = 0_E$ and $\forall \alpha \in \mathbb{K}, \alpha u = 0_E \implies \alpha u + v = 0_E \implies (\alpha u + v) \in \{0_E\}$

1.3.2.6 Proposition 4

Let $(E, \oplus, \odot)$ be a VS and F a VSS of E, then F is a VS

1.4 Sum of VSS

Let E a $\mathbb{K}$-VS, F and G VSS of E, we say H is the sum of F and G if:

$$H = \{u \in E, \exists (v, w) \in F \times G, u = v + w\}$$

1.5 Examples

1. $E = \mathbb{R}^2, G = \left\{\begin{pmatrix}x\\0\end{pmatrix}, x \in \mathbb{R}\right\}, H = \left\{\begin{pmatrix}0\\y\end{pmatrix}, y \in \mathbb{R}\right\}$
   $G + H = \left\{\begin{pmatrix}x\\y\end{pmatrix}, (x, y) \in \mathbb{R}^2 = \mathbb{R}^2\right\}$

2. $$\left.\begin{aligned} F &= \left\{ \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}, x \in \mathbb{R} \right\} \\ G &= \left\{ \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}, z \in \mathbb{R} \right\} \end{aligned}\right\} \text{VSS of } \mathbb{R}^3$$
$$F + G = \left\{ \begin{pmatrix} x \\ 0 \\ z \end{pmatrix}, (x, z) \in \mathbb{R}^2 \right\} \in \mathbb{R}^3 \neq \mathbb{R}^3$$

1.6 Direct Sum

1.6.1 Definition

Let E a $\mathbb{K}$-VS, F and G two VSS of E,
we say that F and G are in direct sum if:

$$F \cap G = \{0_E\}$$

1.6.2 Notation

In that case we denote $F \oplus G$ instead of $F + G$

1.6.3 Examples

1. Both of previous examples
2. $$\left.\begin{aligned} F &= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix}, z = 0, x, y \in \mathbb{R}^2 \right\} \\ G &= \left\{ \begin{pmatrix} x \\ 0 \\ z \end{pmatrix}, (x, y) \in \mathbb{R}^2 \right\} \end{aligned}\right\} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in F \cap G$$
   $F + G$ Check
   $F \oplus G$ Not bc $F \cap G \neq \{0_E\}$