Chapter 12: Linear Maps

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1 General approach

1.1 Definition

Let E, F two $\mathbb{K}-VS$, and f a mapping from E to F. We say that f is a linear (or f is a linear map) if:

$$\forall(\alpha, X, Y) \in \mathbb{K} \times E \times E, f(\alpha \cdot X + Y) = \alpha \cdot f(X) + f(Y)$$

$$\Longleftrightarrow$$

$$\forall(\alpha, \beta, X, Y) \in \mathbb{K} \times \mathbb{K} \times E \times E, f(\alpha \cdot X + \beta \cdot Y) = \alpha \cdot f(X) + \beta \cdot f(Y)$$

1.2 Notation

We denote $L(E, F)$ the set of all linear maps from E to F.

1.3 Specific Linear Maps

1.3.1 Definition

1. Let $f \in \mathcal{L}(E, F)$: we say f is an endomorphism if $E = F$ we then denote $\mathcal{L}(E)$ the set of all endomorphism of E.

2. Let $f \in \mathcal{L}(E, F)$: we say f is an isomorphism if f is bijective.

3. Let $f \in \mathcal{L}(E, F)$: we say f is an automorphism if f is an endomorphism and an isomorphism. ($E = F$ and bijective)

1.4 Necessary Condition

$$f \in \mathcal{L}(E, F) \Longrightarrow f(0_E) = 0_F$$

1.4.1 Proof

$$\begin{array}{lcl} \text{Let } X \in E & \text{and} & X \in E. \\ f(0_E) = f(0_R \times X) & \text{and} & f(0_E) = f(X - X) \\ f(0_E) = 0_R \times f(X) & \text{and} & f(0_E) = f(X) - f(X) \\ f(0_E) = 0_F & \text{and} & f(0_E) = 0_F \end{array}$$

2 Kernel and Images

2.1 Definition

Let E and F two $\mathbb{K}-VS$ and $f \in \mathcal{L}(E, F)$. Then:

1. We call kernel of f and denote $Ker(f)$ the subset of E defined as follows:

$$Ker(f) = \{X \in E, f(X) = 0_F\} = f^{-1}(\{0_F\})$$

Note: $f^{-1}()$ is NOT the inverse of f beacause f is not necessarily bijective.

2. We call image of f and denote $Im(f)$ the subset of F defined as follows:

$$Im(f) = \{f(X), X \in E\} = \{Y \in F, \exists X \in E, f(X) = Y\}$$

2.2 Example

$$f: \quad R^2 \longrightarrow R^3$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x \\ 0 \\ y \end{pmatrix}$$

1 $f \in \mathcal{L}(R^2, R^3)$?

2 Kerf = ?

3 Imf = ?

1 Necessary condition: $f(0_E) = 0_F$: $f(0_{R^2}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$?

$$\forall(\alpha, X, Y) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2, X = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } Y = \begin{pmatrix} x' \\ y' \end{pmatrix}, x, y, x', y' \in \mathbb{R}$$

$$f(\alpha \cdot X + Y) = \begin{pmatrix} \alpha \cdot x + x' \\ \alpha \cdot y + y' \end{pmatrix} = \begin{pmatrix} \alpha \cdot x + x' \\ 0 \\ \alpha \cdot y + y' \end{pmatrix} = \alpha \cdot \begin{pmatrix} x \\ 0 \\ y \end{pmatrix} + \begin{pmatrix} x' \\ 0 \\ y' \end{pmatrix} = \alpha \cdot f(X) + f(Y)$$

so 1 $f \in \mathcal{L}(R^2, R^3)$ ✓

2 $Ker(f) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in R^2, \begin{pmatrix} x \\ 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

3 $Im(f) = \left\{ \begin{pmatrix} x \\ 0 \\ y \end{pmatrix}, (x, y) \in R^2 \right\} = \left\{ x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, (x, y) \in R^2 \right\}$

$$Im(f) = span \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

2.3 Proposition

1. Let $f \in \mathcal{L}(E, F)$ and $g \in \mathcal{L}(F, G)$. Then:

$$g \circ f \in \mathcal{L}(E, G)$$

2. If f is objectif then f^{-1} is bijective and $f^{-1} \in \mathcal{L}(F, E)$

3. $\mathcal{L}(E, F)$ is a $\mathbb{K} - VS$:

$$E \to F$$
$$\mathcal{L}(E, F)X \to B_F \in \mathcal{L}(F, E)$$

$$\forall(\alpha, f, g) \in \mathbb{K}\times\mathcal{L}^2(E, F)$$
$$\alpha \cdot f + g \in\mathcal{L}(E, F)$$