Chapter 10: Function: Local study

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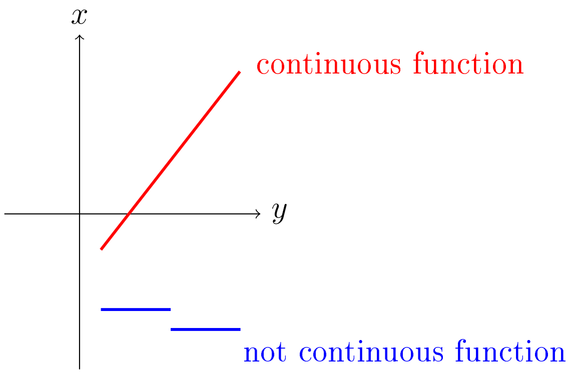
Contents

1 Continuity

1.1 First approach

Let f be a function from $I \subset R$ to R, we say that f is continuous over I if "the graph of f can be drown without taking off the pencil from the paper".

1.1.1 Example



1.2 Definition

1 f continuous at $a \in I$:
we say f is continuous at a - a being a point of I - if and only if:

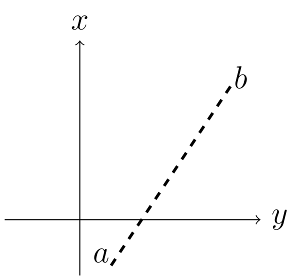
$$f(a) = \lim_{x \to a} f(x)$$
$$f(a) = \lim_{\substack{x \to a \\ x > a}} f(x) = \lim_{\substack{x \to a \\ x < a}} f(x)$$

2 f continuous over I:
we say f is continuous over $I \subset R$ if and only if $\forall a \in I$,f is continuous at a.

1.3 Intermediate value theorem

Let f be a function continuous over $I \subset R$ and $(a, b) \in I^2$. if $f(a)f(b) < 0$ then there exists (at least one) c from $]a, b[$ such that $f(c) = 0$.

1.3.1 Examples



must intersect the x-axis

Second example: $f(x) = x^2 - 2x + 1$
$f(x) = x^2 \cos(x) + x \sin(x) + 1 = 0$ does f have any solutions?
$I = [0, \pi]$ $f(a = 0) = 1$ and $f(b = \pi) = -\pi^2 + 1 < 0$

1. $f : x \mapsto f(x)$ is constant over $I \subset \mathbb{R}$.

2. $f(a) \times f(b) < 0$.

IVT $\Rightarrow \exists c \in]0, \pi[, f(c) = 0$.

2 Image of an interval by a continuous function

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ and $A \subset I$.
We call image of A by f and denote $f(A) : f(A) = \{f(x), x \in A\}$.

2.1 Remark

$\forall y \in f(A), \exists x \in A, y = f(x)$. Example: $f : x \mapsto x^2$ and $A = [-3, 2] \to f(A) = [0, 9]$.

2.2 Proposition

- The image of an interval by a continuous function is an interval.
- The image of a segment ($[a, b]$) by a continuous function is a segment. counter example: $f : x \mapsto \frac{1}{x}$ (continuous over $\mathbb{R}_+^*$) $\quad f(]0, 1] \subset \mathbb{R}_+^*) = [1, +\infty[$

2.3 Corollary

Let f be a continuous function over segment $[a, b]$. Then $f([a, b]) = [m, M]$ where m and M are respectively the minimum and maximum of f over $[a, b]$.

3 Differentiability

All function below are of type $f : I \to \mathbb{R}$ where I is an interval of $\mathbb{R}$ containing at least two points.

3.1 Definition

we say that f is differentiable at a if the following quantity $\frac{f(x)-f(a)}{x-a}$ (the increasing rate of f at a) has a finite limit l at a.
If so, then f is differentiable at a and ...