Chapter 11: Vector Spaces

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1 Introduction

1.1 General approach

Studying Vector spaces will allow us to notice general theorems that can be applied to many mathematical structures.

1.2 Definition

A vector space is a set whose elements, often called vectors, may be added together and multiplied ("scaled") by numbers called scalars.

1.3 Notation

A $\mathbb{K}-VectorSpace$ is non-empty set E that has :

- An internal law, which is an map of E x E in E:

$$\begin{aligned} E \times E &\longrightarrow E \\ (u,v) &\longmapsto u+v \end{aligned}$$

- An external law, which is an map of $\mathbb{K}$ x E in E:

$$\begin{aligned} \mathbb{K} \times E &\longrightarrow E \\ (\lambda,u) &\longmapsto \lambda \cdot u \end{aligned}$$

($\mathbb{K}$ is a set, often $\mathbb{R}$)

The elements of E are called vectors
The elements of $\mathbb{K}$ are called scalars
The neutral element 0_E is also called the null vector
The symmetrical $-u$ is also called the opposite
The internal composition law on E, denoted $+$, is the addition
The external composition law on E is the multiplication by a scalar

axioms relative to the internal law :

- 0_E is unique
- $-u$ is unique

1.4 Properties

To know is a space is a vector space, there is properties that need to match up.

- The internal law
- The external law
- Both laws together
- The neutral element
- It's symmetrical

This makes up 8 laws that need to be respected :

1. $u + v = v + u(\forall(u, v) \in E)$
2. $u + (v + w) = (u + v) + w(\forall(u, v, w) \in E)$
3. There exists a neutral element $0_E \in E$ so that $u + 0_E = u(\forall u \in E)$
4. All elements admit a symmetric u' so that $u + u' = 0_E$. This element u' is denoted $-u$
5. $1 \cdot u = u(\forall u \in E)$
6. $\lambda \cdot (\mu \cdot u) = (\lambda\mu) \cdot u(\forall \lambda, \mu \in \mathbb{K}, u \in E)$
7. $\lambda \cdot (v + u) = (\lambda \cdot v) + (\lambda \cdot u)(\forall \lambda \in \mathbb{K}, v, u \in E)$
8. $(\lambda + \mu) \cdot u = (\lambda \cdot u) + (\mu \cdot u)(\forall \lambda, \mu \in \mathbb{K}, u \in E)$

2 Vector Sub-Spaces

A sub space is very useful to prove that a set is a Vector space. We will see that a vector sub-space is vector space.

2.1 Definition

Let E be a vector space, F is a subspace if and only if

- $0_E \in F$
- $u + v \in F \ \forall(u, v) \in F^2$
- $\lambda \cdot u \in F \ \forall \lambda \in \mathbb{K}, v \in F$

2.2 Properties

Showing that a space is a subspace of a bigger (or equal) vector space if enough to prove that it is itself a vector space.

3 Relations between V. Space and V. SubSpace

3.1 Linear Combinations

Let $v_1, v_2, ..., v_n$, n vectors from a vector space E. Then :
Any vector of the form :

$$u = \lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_n v_n$$

is called a Linear Combination of the vectors $v_1, v_2, ..., v_n$

The scalars $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K}$ are the coefficients of the linear combination.

Let E be a vector space. Then F is a vector sub space if and only if all linear combination of two element of F also belongs to E.

3.2 Vector SubSpace Intersection

- the intersection $\cap$ of two vector sub spaces **is** also a sub space
- the union $\cup$ of two vector sub spaces **is not** a sub space

3.3 Vector SubSpace Sum

Let F and G be two vectorial sub spaces of E
The sum of two vectorial sub spaces is also a vss, in fact, it is the smallest vss including both F and G
F and G are in direct sum in E if

- $F \cap G = \{0_E\}$
- $F + G = E$

We then denote $F \oplus G = E$
F and G are called additional sub-spaces

F and G are additional sub-spaces in E if and only if any element of E is uniquely written as the sum of an element of F and an element of G
SYSTEM TO DO HERE

4 Families

4.1 Free Family

A family $\{v_1, v_2, ..., v_p\}$ of E is called a *free family* (or *linearly independent family*) if all null linear combination

$$\lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_p v_p = 0$$

is such that all λ coefficients are null.

The opposite (there exists a null linear combination with at least 1 coefficient that is not null) is called a linked family or a linearly dependent family.

Let E a $\mathbb{K}$-vector space,
A family $F = \{v_1, v_2, ..., v_p\}$ with $p \geq 2$ vectors of E is a linked family iff at least one vector is a linear combination of the other vectors

4.2 Spanning Family

A family $\{v_1, v_2, ..., v_p\}$ is a generative family of E if all vector of E is a linear combination of the vectors $v_1, v_2, ..., v_p$

$$\forall x \in E, \exists(\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{K}^n : x = \lambda_1 x_1 + \lambda_2 x_2 + ... + +\lambda_n x_n$$

To prove that a family spans a vector space, we need to find the solutions $\lambda_1, ..., \lambda_n$ of the equation given just above.

5 Basis

6 Dimension of a Vector SubSpace

7 Proofs

7.1 Intersection of linear subspaces

7.2 Sum of linear subspaces

7.3 Spanned linear subspaces

7.4 Basis

7.5 Existence of a basis in a finite dimention