

# Chapter 1: Numerical Series

September 28, 2023

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# 1 Preamble

## 1.1 Vocabulary

In this chapter, we will use CVG for Convergence and DVG for Divergence. We will also use GT for General Term.

## 1.2 Remark

 Be careful, the series  $\sum U_n$  is not the same as the sequence  $(U_n)_{n \in \mathbb{N}}$ .  $\sum U_n$  is the series of general term  $U_n$  and  $(U_n)_{n \in \mathbb{N}}$  is the sequence  $U_n$ .

# 2 General approach Convergence and Divergence

## 2.1 Definition

Let  $(U_n)_{n \in \mathbb{N}}$  a sequence of real numbers, we call series of general term  $U_k$  and denote  $\sum U_k$  the sequence of partial sums  $(S_n)_{n \in \mathbb{N}}$  where for any integer  $n \in \mathbb{N}$ ,  $S_n = \sum_{k=0}^n U_k$ . We say  $\sum U_k$  is convergent if and only if  $(S_n)_{n \in \mathbb{N}}$  is convergent.


### 2.1.1 Example: the geometric series

Let  $q \in \mathbb{R}^*$  and let us consider the series  $\sum q^k$ . We have:

$$\forall n \in \mathbb{N}, S_n = \sum_{k=0}^n q^k = \begin{cases} \frac{1-q^{n+1}}{1-q} & \text{if } q \neq 1 \\ n+1 & \text{if } q = 1 \end{cases} \Rightarrow \begin{cases} \text{if } -1 < q < 1, \sum_{k=0}^{+\infty} q^k = \frac{1}{1-q} \sum U_k: \text{CVG} \\ \text{if } q > 1 \text{ or } q < -1, \sum U_k: \text{DVG} \\ \sum U_k: \text{DVG} \end{cases}$$

## 2.2 Propositions

Let  $\sum U_k$  and  $\sum V_k$  two series of general terms and  $\lambda \in \mathbb{R}$ . We have:

- If  $[\sum U_k \text{ CVG and } \sum V_k \text{ CVG}]$ , then  $\sum (U_k + V_k) \text{ CVG}$
- If  $[\sum U_k \text{ CVG}]$ , then  $\sum \lambda U_k \text{ CVG}$
- If  $[\sum U_k \text{ CVG and } \sum V_k \text{ DVG}]$ , then  $\sum (U_k + V_k) \text{ DVG}$
-   $\sum U_k \text{ DVG and } \sum V_k \text{ DVG}$  does not imply  $\sum (U_k + V_k) \text{ DVG}$

## 2.3 Sum and Remainder of a convergent series

Let  $\sum U_k$  a convergent series. We call sum of the series  $\sum U_k$  the following real number:  $\sum_{k=0}^{+\infty} U_k = \lim_{n \rightarrow +\infty} S_n$  where  $S_n = \sum_{k=0}^n U_k$ . And we call remainder of the series

$\sum \mathbf{U}_k$  sequence  $(\mathbf{R}_n)$  defined as follows:

$$\forall n \in \mathbb{N}, R_n = \sum_{k=n+1}^{+\infty}$$

### 2.3.1 Example

$$\sum q^k \text{ CVG} \Leftrightarrow -1 < q < 1 : S = \lim_{n \rightarrow +\infty} S_n = \frac{1}{1-q}$$

## 2.4 Convergence necessary condition

### 2.4.1 Proposition

Let  $\sum (\mathbf{U}_k)_{k \in \mathbb{N}}$  a sequence. We have:

$$\sum U_k \text{ CVG} \begin{matrix} \Rightarrow \\ \not\Leftarrow \end{matrix} \left( U_k \xrightarrow[k \rightarrow +\infty]{} 0 \right)$$

### 2.4.2 Example

- Harmonic series:  $\sum \frac{1}{n}, \left(\frac{1}{n}\right) \xrightarrow[n \rightarrow +\infty]{} 0$  but  $\sum \frac{1}{n}$  DVG
- $\sum \frac{e^n}{n^{2023}}, \frac{e^n}{n^{2023}} \xrightarrow[n \rightarrow +\infty]{} +\infty \Rightarrow \sum \frac{e^n}{n^{2023}}$  DVG

## 3 Positive Term Series (P.T.S.)

### 3.1 Definition

Let  $\sum \mathbf{U}_k$  a series. We say  $\sum \mathbf{U}_k$  is a P.T.S., if and only if  $\forall k \in \mathbb{N}, \mathbf{U}_k \geq 0$ .  
We say  $\sum \mathbf{U}_k$  is a P.T.S. from  $\mathbf{p} \in \mathbb{N}$  onwards, if and only if  $\forall k \in \mathbb{N}, k \geq \mathbf{p} \Rightarrow \mathbf{U}_k \geq 0$ .

### 3.2 Propositions

- Let  $\sum \mathbf{U}_k$  a P.T.S. and  $(S_n)_{n \in \mathbb{N}}$  the associated partial sum sequence. Then:

$$\sum U_k \text{ CVG} \Leftrightarrow (S_n)_{n \in \mathbb{N}} \text{ is upper-bounded}$$

- Let  $\sum \mathbf{U}_k$  and  $\sum \mathbf{V}_k$  two series such that:  
 $\forall k \in \mathbb{N}, 0 \leq \mathbf{U}_k \leq \mathbf{V}_k$ . Then:

1. If  $\sum \mathbf{V}_k$  CVG, then  $\sum \mathbf{U}_k$  CVG
2. If  $\sum \mathbf{U}_k$  DVG, then  $\sum \mathbf{V}_k$  DVG

### 3.2.1 Example

What's the nature of  $\sum \frac{1}{|n \cdot \sin(n)|}$  ?

$$\forall n \in \mathbb{N}^*, 0 < |\sin(n)| \leq 1 \Rightarrow 0 < \frac{1}{n} \leq \frac{1}{|n \cdot \sin(n)|}$$

$$\sum \frac{1}{n} \text{ (Harmonic) DVG} \Rightarrow \sum \frac{1}{|n \cdot \sin(n)|} \text{ DVG}$$

### 3.3 Riemann's series

#### 3.3.1 Definition

We call Riemann's series any series of General Terms (GT)  $\sum \frac{1}{n^\alpha}$  where  $\alpha \in \mathbb{R}$ .

#### 3.3.2 Theorem (Riemann)

Let  $\alpha \in \mathbb{R}$ . Then:

$$\sum \frac{1}{n^\alpha} \text{ CVG} \iff \alpha > 1$$

##### 3.3.2.1 Example

- $\sum \frac{1}{\sqrt{2}} = \sum \frac{1}{2^{\frac{1}{2}}} \implies \text{DVG}$
- $\sum \frac{1+\cos(n)}{n^4}$ :  $\forall n \in \mathbb{N}^*, 0 \leq 1 + \cos(n) \leq 2 \implies 0 \leq \frac{1+\cos(n)}{n^4} \leq \frac{2}{n^4}$   
And  $\sum \frac{2}{n^4}$  of same nature as  $\sum \frac{1}{n^4}$  (Riemann's series) CVG  $\implies \sum \frac{1+\cos(n)}{n^4} \text{ CVG}$

### 3.4 Comparison criteria

#### 3.4.1 Proposition

Let  $\sum U_n$  and  $\sum V_n$  two P.T.S.


- ① If  $U_n \underset{+\infty}{\sim} V_n$  then  $\sum U_n$  and  $\sum V_n$  are of same nature
- ② If  $U_n = o(V_n)$  then [If  $\sum V_n \text{ CVG}$  then  $\sum U_n \text{ CVG}$ ]

##### 3.4.1.1 Example

What's the nature of  $\sum U_n$  ?

- $U_n = e^{-\sqrt{n}}$ : Step 1:  $n^2 \times U_n = \frac{n^2}{e^{\sqrt{n}}} = \frac{(\sqrt{n})^4}{e^{\sqrt{n}}} \xrightarrow{n \rightarrow +\infty} 0 \implies U_n = o(\frac{1}{n^2})$   
Step 2:  $\sum \frac{1}{n^2} \text{ CVG}$  (Riemann's series  $\alpha = 2 > 1$ )  $\implies \sum U_n \text{ CVG}$

$$\forall n \in \mathbb{N}^*, \frac{n+1}{n} = 1 + \frac{1}{n} \implies \ln(1 + \frac{1}{n}) \underset{+\infty}{=} \frac{1}{n} + o(\frac{1}{n})$$

- $U_n = \ln(\frac{n+1}{n})$ :   $\implies \begin{cases} \textcircled{1} \forall n \in \mathbb{N}, U_n > 0 \text{ since } 1 + \frac{1}{n} > 1 \\ \textcircled{2} U_n \underset{+\infty}{=} \frac{1}{n} \end{cases}$   
 $\implies \sum U_n$  and  $\sum \frac{1}{n}$  of same nature  
and  $\sum \frac{1}{n} \text{ DVG}$  (Harmonic series)

#### 3.4.2 Proposition

Let  $\sum U_n$  a numerical sequence. We have:

$$\sum \overbrace{(U_{n+1} - U_n)}^{w_n} \text{ CVG} \iff (U_n) \text{ CVG}$$

### 3.4.2.1 Example

1.  General Example, limit calculation:

$$\begin{aligned}
 S_n &= \sum_{k=0}^n W_k = \sum_{k=0}^n (U_{k+1} - U_k) = \sum_{k=0}^n U_{k+1} - \sum_{k=0}^n U_k \\
 &= \sum_{k=1}^{n+1} U_k - \sum_{k=0}^n U_k \\
 &= \left( \sum_{k=1}^n U_k + U_{n+1} \right) - \left( U_0 + \sum_{k=1}^n U_k \right) \\
 S_n &= \sum_{k=0}^n (U_{k+1} - U_k) = U_{n+1} - U_0
 \end{aligned}$$

2. 
$$\sum \overbrace{\left( \frac{1}{n+1} - \frac{1}{n} \right)}^{w_n} : \left| \begin{array}{l} \sum W_n \text{ of same nature as } \sum \left( \frac{1}{n} \right)_{n \in \mathbb{N}^*} \left( \begin{array}{l} \frac{1}{n} \xrightarrow{n \rightarrow +\infty} 0 \text{ CVG} \\ \text{So: } \sum W_n \text{ CVG} \end{array} \right) \\ \text{calculation of the limit:} \\ S = \lim_{n \rightarrow +\infty} S_n = \sum_{k=1}^{+\infty} W_k = \lim_{n \rightarrow +\infty} \left( \frac{1}{n+1} - 1 \right) = -1 \end{array} \right.$$

## 3.5 Riemann's Rule

Let  $\sum U_n$  a Positive numerical series. If  $\exists \alpha > 1, n^\alpha \times U_n \xrightarrow{+\infty} 0$  then  $\sum U_n$  CVG

### 3.5.1 Proof

$$\begin{aligned}
 &\exists \alpha > 1, n^\alpha \times U_n \xrightarrow{n \rightarrow +\infty} 0 \implies \frac{U_n}{n^\alpha} \xrightarrow{n \rightarrow +\infty} 0 \\
 \implies &\left\{ \begin{array}{l} U_n = o\left(\frac{1}{n^\alpha}\right) \\ \text{and} \\ \alpha > 1 \\ \text{and} \\ \sum U_n \text{ P.T.S.} \end{array} \right| \left[ \sum \frac{1}{n^\alpha} \text{ CVG (Riemann's series)} \implies \sum U_n \text{ CVG} \right]
 \end{aligned}$$

## 3.6 D'Alembert's Rule

Let  $(U_n)$  be a strictly positive sequence such that:

$$\frac{U_{n+1}}{U_n} \xrightarrow{n \rightarrow +\infty} \ell \in \mathbb{R}_+ \cup \{+\infty\}$$

$$\begin{aligned}
 \ell < 1 &\implies \sum U_n \text{ CVG} \\
 \text{Then: } \ell > 1 &\implies \sum U_n \text{ DVG} \\
 \ell = 1 &\implies \text{no conclusion}
 \end{aligned}$$

### 3.6.1 Example

$$\sum \frac{1}{n!} : \forall n \in \mathbb{N}, \frac{1}{n!} > 0 \text{ (P.T.S.) and } \frac{U_{n+1}}{U_n} = \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{1}{n+1} \xrightarrow{n \rightarrow +\infty} 0 < 1 \implies \sum \frac{1}{n!} \text{ CVG}$$

## 3.7 Cauchy's Rule

Let  $(U_n)$  be a strictly positive sequence such that:

$$\sqrt[n]{U_n} \xrightarrow{n \rightarrow +\infty} \ell \in \mathbb{R}_+ \cup \{+\infty\}$$

$$\begin{aligned} \ell < 1 &\implies \sum U_n \text{ CVG} \\ \text{Then: } \ell > 1 &\implies \sum U_n \text{ DVG} \\ \ell = 1 &\implies \text{no conclusion} \end{aligned}$$

### 3.7.1 Example

$$\begin{aligned} \sum \left(\frac{n}{n+1}\right)^{n^2} : \forall n \in \mathbb{N}, \left(\frac{n}{n+1}\right)^{n^2} > 0 \text{ (P.T.S.)}, \sqrt[n]{U_n} &= \left(\left(\frac{n}{n+1}\right)^{n^2}\right)^{\frac{1}{n}} = \left(\frac{n}{n+1}\right)^n = e^{n \ln(1 - \frac{n}{n+1})} \\ \ln(1 - \frac{n}{n+1}) \sim n \times \left(-\frac{n}{n+1}\right) \xrightarrow{n \rightarrow +\infty} -1 < 0 &\implies \sqrt[n]{U_n} \xrightarrow{n \rightarrow +\infty} e^{-1} = \frac{1}{e} < 1 \xRightarrow{\text{Cauchy}} \sum U_n \text{ CVG} \end{aligned}$$

## 3.8 Examples

# 4 Alternating Series

## 4.1 Definition

Let  $(U_n) \in \mathbb{R}^{\mathbb{N}}$ , we say  $(U_n)$  is an alternating sequence thus  $\sum U_n$  an alternating series, if there exists  $\left| \begin{array}{l} \text{a positive} \\ \text{a negative} \end{array} \right.$  sequence  $(a_n)$  such that:

$$\forall n \in \mathbb{N}, \left| \begin{array}{l} U_n = (-1)^n \times a_n \\ U_n = (-1)^{n+1} \times a_n \end{array} \right.$$

### 4.1.1 Example

$$\sum \frac{(-1)^n}{n} \text{ is an alternating series because } \forall n \in \mathbb{N}, \frac{(-1)^n}{n} = (-1)^n \times \frac{1}{n}$$

## 4.2 Alternating Series Special Criteria (A.S.S.C.)

### 4.2.1 Theorem

Let  $(U_n)$  an alternating sequence, such that:

$$\left[ \begin{array}{l} U_n \xrightarrow{n \rightarrow +\infty} 0 \\ (|U_n|)_{n \in \mathbb{N}} \text{ is decreasing} \end{array} \right] \implies \sum U_n \text{ CVG}$$

### 4.2.2 Explanation

An alternating sequence is of the form

$$\begin{array}{lll} \mathbf{U}_n = (-1)^n \times \mathbf{a}_n & \text{or} & \mathbf{U}_n = (-1)^{n+1} \times \mathbf{a}_n \\ |\mathbf{U}_n| = |(-1)^n \times \mathbf{a}_n| = |\mathbf{a}_n| & \text{or} & |\mathbf{U}_n| = |(-1)^{n+1} \times \mathbf{a}_n| = |\mathbf{a}_n| \\ & \text{So } (|\mathbf{U}_n|)_{n \in \mathbb{N}} = (\mathbf{a}_n)_{n \in \mathbb{N}} \end{array}$$