Chapter 2: Generating Function

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Contents

1 Preamble

This chapter is about generating function in probability. To understand this chapter, you need to have a good understanding of the chapter 4 of the first year about probability. This chapter is also related to the next chapter about Power Series. You can here a little summary of the chapter 4 of the first year:

1.1 Reminder

1.1.1 Definition

A random variable is a function from a probability space to the real numbers.
X a random variable is a function:

$$X: \begin{array}{l} \Omega \longrightarrow \mathbb{R} \\ w \longmapsto X(w) \end{array}$$

We denote the range of X by $X(\Omega)$.

1.1.2 Expectation

Let X be a random variable, we define the expectation of X by:

$$E(X) = \sum_{k \in X(\Omega)} P(X = k) \cdot k$$

$$E(g(x)) = \sum_{k \in X(\Omega)} P(X = k) \cdot g(k)$$

1.1.3 Variance

Let X be a random variable, we define the variance of X by:

$$Var(X) = E((X - E(X))^2)$$

$$Var(X) = E(X^2) - E(X)^2$$

2 Generating Function

The Generating Function of a discrete random variable contains all the distribution data. Thus we can in particular compute its Expected Value and Variance.

2.1 Definition

Let X be a finite integer random variable, with $X(\Omega) = [\![0, n]\!]$, we call the Generating Function of the following polynomial:

$$G_X: \begin{array}{l} \mathbb{R} \longrightarrow \mathbb{R} \\ t \longmapsto \sum_{k=0}^{n} P(X = k) \cdot t^k = E(t^X) \end{array}$$

2.2 Remark

Let X and Y be two Finite Integer Random Variables (F.I.R.V.) such that $G_X = G_Y$:

$$\begin{aligned} & X(\Omega) = Y(\Omega) = [\![0, n]\!] \\ \implies & \forall k \in [\![0, n]\!], P(X = k) = P(Y = k) \\ \implies & X \text{ and } Y \text{ have the same distribution} \end{aligned}$$

2.3 Example

Roll a dice with X is "pick a 6" and Y is "pick a 1".
X and Y have the same distribution $= \frac{1}{6}$ but $X \neq Y$.

Example 2: Bernoulli random variable (A random variable with a Bernoulli distribution)

$$X \sim B(p) \implies \left| \begin{array}{l} X(\Omega) = \{0, 1\} \\ P(X = 0) = 1 - p \quad , p \in]0, 1[\\ P(X = 1) = p \end{array} \right.$$

Then we have:

$$G_X \colon \begin{array}{rcl} \mathbb{R} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \displaystyle\sum_{k=0}^{1} P(X = k) \cdot t^k = P(X = 0) \cdot t^0 + P(X = 1) \cdot t^1 \\ & & \qquad\qquad\qquad\qquad = (1 - p) \cdot t^0 + p \cdot t^1 \end{array}$$