Chapter 11: Vector Spaces

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1 Introduction

1.1 General approach

Studying Vector spaces will allow us to notice general theorems that can be applied to many mathematical structures.

1.2 Definition

A vector space is a set whose elements, often called vectors, may be added together and multiplied ("scaled") by numbers called scalars.

1.3 Notation

A $\mathbb{K} - VectorSpace$ is non-empty set E that has :

- An internal law, which is an map of E x E in E:

$$E \times E \longrightarrow E$$
$$(u, v) \longmapsto u + v$$

- An external law, which is an map of $\mathbb{K}$ x E in E:

$$\mathbb{K} \times E \longrightarrow E$$
$$(\lambda, u) \longmapsto \lambda \cdot u$$

($\mathbb{K}$ is a set, often $\mathbb{R}$)

The elements of E are called vectors
The elements of $\mathbb{K}$ are called scalars
The neutral element 0_E is also called the null vector
The symmetrical $-u$ is also called the opposite
The internal composition law on E, denoted $+$, is the addition
The external composition law on E is the multiplication by a scalar

axioms relative to the internal law :

- 0_E is unique
- $-u$ is unique

1.4 Properties

To know is a space is a vector space, there is properties that need to match up.

- The internal law
- The external law
- Both laws together
- The neutral element
- It's symmetrical

This makes up 8 laws that need to be respected :

1. $u + v = v + u (\forall (u, v) \in E)$
2. $u + (v + w) = (u + v) + w (\forall (u, v, w) \in E)$
3. There exists a neutral element $0_E \in E$ so that $u + 0_E = u (\forall u \in E)$
4. All elements admit a symmetric u' so that $u + u' = 0_E$. This element u' is denoted $-u$
5. $1 \cdot u = u (\forall u \in E)$
6. $\lambda \cdot (\mu \cdot u) = (\lambda\mu) \cdot u (\forall \lambda, \mu \in \mathbb{K}, u \in E)$
7. $\lambda \cdot (v + u) = (\lambda \cdot v) + (\lambda \cdot u) (\forall \lambda \in \mathbb{K}, v, u \in E)$
8. $(\lambda + \mu) \cdot u = (\lambda \cdot u) + (\mu \cdot u) (\forall \lambda, \mu \in \mathbb{K}, u \in E)$

2 Vector Sub-Spaces

A sub space is very useful to prove that a set is a Vector space. We will see that a vector sub-space is vector space.

2.1 Definition

Let E be a vector space, F is a subspace if and only if

- $0_E \in F$
- $u + v \in F \ \forall (u, v) \in F^2$
- $\lambda \cdot u \in F \ \forall \lambda \in \mathbb{K}, v \in F$

2.2 Properties

Showing that a space is a subspace of a bigger (or equal) vector space if enough to prove that it is itself a vector space.

3 Relations between Vs and Vss

3.1 Linear Combinations

Let $v_1, v_2, ..., v_n$, n vectors from a vector space E. Then :

3.2 Vss Intersection

3.3 Vss Sum

3.4 Generated Vss

4 Families

4.1 Free Family

4.2 Spanning Family

5 Basis

6 Dimension of a Vss

7 Proofs

7.1 Intersection of linear subspaces

7.2 Sum of linear subspaces

7.3 Spanned linear subspaces

7.4 Basis

7.5 Existence of a basis in a finite dimention