Chapter 11: Vector Spaces

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Contents

During this chapter, $\mathbb{K}$ will be either $\mathbb{R}$ or $\mathbb{C}$.

1 General approach

1.1 Structure of a Vector Space

Let E be a set, we define two operations:

- An internal operation:
$$\oplus : \begin{aligned} E \times E &\longrightarrow E \\ (u, v) &\longmapsto u + v \end{aligned}$$

- An external operation:
$$\odot : \begin{aligned} \mathbb{K} \times E &\longrightarrow E \\ (\lambda, v) &\longmapsto \lambda u \end{aligned}$$

1.2 Defintion

We say that $(E, \oplus, \odot)$ is a vector space if $\forall (u, v, w) \in E^3$ we have:

- u + (v + w) = (u + v) + w ($\oplus$ is associative)
- u + v = v + u ($\oplus$ is commutative)
- $\exists 0_E \in E$ such that $u + 0_E = 0_E + u = u$ (Existence of a neutral element for $\oplus$)
- $\exists -u \in E$, $u + (-u) = (-u) + u = 0_E$ (Existence of a symmetrical element for $\oplus$)

And $\forall (u, v) \in E^2$ and $\forall (\alpha, \beta) \in \mathbb{K}^2$ we have:

- $(\alpha + \beta) \cdot u = \alpha \cdot u + \beta \cdot u$ ($\odot$ is distributive)
- $\alpha(u + v) = \alpha \cdot u + \alpha \cdot v$ ($\odot$ is distributive)
- $(\alpha\beta)u = \alpha(\beta u)$ ($\odot$ is associative)
- $1_{\mathbb{K}} \cdot u = u$ (Where $1_{\mathbb{K}}$ is the neutral element of multiplication of element of $\mathbb{K}$)

Let (E, $\oplus$, $\odot$) be a $\mathbb{K}$-Vector Space,
Any element from E is called a vector and any element from $\mathbb{K}$ is called a scalar.
0_E is called the zero vector.

1.2.1 Property

$\forall u \in E$ and $\forall \alpha \in \mathbb{K}$ we have:

1. $\alpha \cdot 0_E = 0_E$
2. $0_{\mathbb{K}} \cdot u = 0_E$
3. $\alpha \cdot u = 0_E \Leftrightarrow \alpha = 0_{\mathbb{K}}$ or $u = 0_E$

1.2.2 Example

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1.3 Vector Subspaces (= linear subspaces)

1.3.1 Definition

Let E be a $\mathbb{K}$-Vector Space:

- $F \subset E$ (F is a subset of E)
- $F \neq \emptyset$ (F is non-empty, $0_E \in F$)
- $\forall (u, v) \in F^2$, $\forall \alpha \in \mathbb{K}$, $(\alpha \cdot u + v) \in F$ (F is closed under linear combination)

1.3.2 Propositions

1.3.2.1 Proposition 1

Let E be a $\mathbb{K}$-Vector Space:

$$F \subset E \text{ is a Vector SubSpace of } E \Longrightarrow 0_E \in F$$
$$\not\Longleftarrow$$

1.3.2.2 Example

$E = \mathbb{R}^3$
$F_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3, x + y + z = 1 \right\}$ and $F_1' = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3, x + y + z = 0 \right\}$
$0_E \notin F_1 \Longrightarrow F_1$ is not a Vector SubSpace of E
$\forall (u, v) \in {F'}_1^2$, $\forall \alpha \in \mathbb{R}$, $u = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $v = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$
$\alpha u + v = \alpha(x, y, z) + (x', y', z') = 0_E \in F_1' \Longrightarrow F_1'$ is a Vector SubSpace of E TODO A MINIPAGE FOR THE EXAMPLE AND ADD THE OTHER EXAMPLES

1.3.2.3 Proposition 2

Let E be a Vector Space, F and G two Vector SubSpaces of E. Then:

1. $F \cap G \subset E$, $F \cap G$ is a Vector SubSpace of E
2. TO FINISH