Chapter 13: Matrix

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1 General approach

1.1 Definition

1.1.1 Definition of a matrix

We call matrix of n rows and p columns any mapping in the following form:

$$\begin{aligned} [\![1,n]\!] \times [\![1,p]\!] &\to \mathbb{K} \\ i,j \quad &\quad a_{ij} \end{aligned}$$

We denote such maps as tables of n rows and p columns, and we write:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix}$$

$\forall (i,j) \in [\![1,n]\!] \times [\![1,p]\!]$, we call a_{ij} a coefficient of the matrix. In this case coefficient if i-th row and j-th column.

1.1.2 Notation

We denote $M_{np}(\mathbb{K})$ the set of matrix of n rows and p columns with coefficient from $\mathbb{K}$.

1.1.3 Examples

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \in M_{32}(\mathbb{R})$$

$$B = \begin{pmatrix} i \\ 1+i \\ 3 \end{pmatrix} \in M_{31}(\mathbb{C})$$

1.2 Particular matrices

Let $A \in M_{np}(\mathbb{K})$ then:

1.2.1 Null matrix

1. $[\forall (i,j) \in [\![1,n]\!] \times [\![1,p]\!], a_{ij} = 0] \Rightarrow [A = 0_{np}]$ We say A is the null matrix $M_{np}(\mathbb{K})$.

1.2.1.1 Example

$$A' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \in M_{32}(\mathbb{R})$$

1.2.2 Column matrix

2. $B \in M_{np}(\mathbb{K})$ and $p = 1 \Rightarrow$ B is a column matrix of n rows

1.2.2.1 Example

$$B' = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in M_{31}(\mathbb{R})$$

1.2.3 Row matrix

3. $B \in M_{np}(\mathbb{K})$ and $n = 1 \Rightarrow$ C is a row matrix of p columns

1.2.3.1 Example

$$C' = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \in M_{13}(\mathbb{R})$$

1.2.4 Square matrix

We call square matrix any matrix with same number of rows and columns. We denote $M_n(\mathbb{K})$ the set of square matrix of n rows and columns with coefficient from $\mathbb{K}$.

4. $D \in M_{np}(\mathbb{K})$ and $n = p \Rightarrow$ D is a square matrix denote $M_n(\mathbb{K})$

1.2.4.1 Example

$$D' = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \in M_3(\mathbb{R})$$

1.2.5 Diagonal matrix

5. $\forall E \in M_n(\mathbb{R})$, if $\forall (i,j) \in [\![1,n]\!]^2, i \neq j \Rightarrow a_{ij} = 0$ then we say E is a diagonal matrix

1.2.5.1 Example

$$E' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \in M_2(\mathbb{R})$$

$$E'' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \in M_3(\mathbb{R})$$

1.2.6 Identity matrix

6. $\forall I_n \in M_n(\mathbb{R})$, if $\forall (i,j) \in [\![1,n]\!]^2, i \neq j \Rightarrow a_{ij} = 0$ and $i = j \Rightarrow a_{ij} = 1$ then we say I_n is a identity matrix

1.2.6.1 Example

$$I'_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{R})$$

$$I''_n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in M_3(\mathbb{R})$$

1.2.7 Triangular matrix

6. $\forall F \in M_n(\mathbb{R})$, if $\forall (i,j) \in [\![1,n]\!]^2, i > j \Rightarrow a_{ij} = 0$ then we say F is a lower triangular matrix

7. $\forall G \in M_n(\mathbb{R})$, if $\forall (i,j) \in [\![1,n]\!]^2, i < j \Rightarrow a_{ij} = 0$ then we say G is a upper triangular matrix

1.2.7.1 Example

$$F' = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \in M_2(\mathbb{R})$$

$$G' = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \in M_2(\mathbb{R})$$

1.3 Transposed matrix

1.3.1 Definition

Let $A \in M_{np}(\mathbb{K})$. We call transposed matrix of A (or A transpose) a matrix B from $M_{pn}(\mathbb{K})$ such as:

$$\forall (i,j) \in [\![1,n]\!] \times [\![1,p]\!], a_{ij} = b_{ji}$$

1.3.2 Notation

We denote B as ${}^tA$

1.3.3 Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \in M_{23}(\mathbb{R})$$

$${}^tA = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \in M_{32}(\mathbb{R})$$

1.4 Symmetric matrix

1.4.1 Symmetric

If ${}^tA = A$ then we say A is symmetric

1.4.1.1 Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} = {}^tA = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix} \in M_3(\mathbb{R})$$

1.4.2 Anti-Symmetric

If ${}^tA = -A$ then we say A is Anti-symmetric

1.4.2.1 Example

$$A = \begin{pmatrix} 0 & -2 & 3 \\ 2 & 0 & -5 \\ -3 & 5 & 0 \end{pmatrix} = -{}^tA = \begin{pmatrix} 0 & 2 & -3 \\ -2 & 0 & 5 \\ 3 & -5 & 0 \end{pmatrix} \in M_3(\mathbb{R})$$

2 Operations on matrices

2.1 Addition and external product

2.1.1 Definition

1. We call internal operation in $M_{np}(\mathbb{K})$ denoted $\oplus$ "internal addition" the one defined as follows:
$$\forall A, B \in M_{np}^2(\mathbb{K}), A + B = (a_{ij} + b_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq p}}$$
$$\text{Where } A = a_{ij\,\substack{1 \leq i \leq n \\ 1 \leq j \leq p}} \text{ and } B = b_{ij\,\substack{1 \leq i \leq n \\ 1 \leq j \leq p}}$$

2. We call "external multiplication" or "multiplication by a scalar" the one defined as follows:
$$\forall A \in M_{np}(\mathbb{K}), \forall \alpha \in \mathbb{K}, \alpha A = (\alpha a_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq p}}$$

2.1.1.1 Example

$$(A, B) \in M_{2,3}(\mathbb{R})^2 \quad A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \text{ and } B = \begin{pmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix}$$
$$A + B = \begin{pmatrix} 1+7 & 2+8 & 3+9 \\ 4+10 & 5+11 & 6+12 \end{pmatrix} = \begin{pmatrix} 8 & 10 & 12 \\ 14 & 16 & 18 \end{pmatrix}$$
$$\alpha = 3, \quad \alpha A = \begin{pmatrix} 3 \times 1 & 3 \times 2 & 3 \times 3 \\ 3 \times 4 & 3 \times 5 & 3 \times 6 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 9 \\ 12 & 15 & 18 \end{pmatrix}$$

2.1.1.2 Proposition

$$(M_{np}, \oplus, \cdot) \text{ is a vector space over } \mathbb{K}$$

2.1.2 Elementary matrix

For $(n,p) \in \mathbb{N}^2, (i,j) \in [\![1,n]\!] \times [\![1,p]\!]$; We denote E_{ij} the matrix from $M_{np}(\mathbb{K})$ such that the ij-th coefficient is 1 and all other coefficient are 0.

$$E_{ij} \text{ are called elementary matrix}$$

2.1.2.1 Example

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
$$E_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$E_{33} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.1.3 Proposition

1. $(E_{ij})_{\substack{1\le i\le n \\ 1\le j\le p}}$ is a basis of $M_{np}(\mathbb{K})$
2. $dim((E_{ij})_{\substack{1\le i\le n \\ 1\le j\le p}}) = np$
   Ex: $M_2(\mathbb{R})$ a $(\mathbb{K})$-VS: B $= \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$
   B is a Standard basis of $M_2(\mathbb{R})$, $dim(M_2(\mathbb{R})) = 2^2 = 4$

2.2 Internal product

2.2.1 Definition

Let $(n,p,q) \in \mathbb{N}^3$ and $A = a_{ij\,\substack{1\le i\le n \\ 1\le j\le p}} \in M_{np}(\mathbb{K}), B = b_{ij\,\substack{1\le i\le p \\ 1\le j\le q}} \in M_{pq}(\mathbb{K})$. We call product of A and B the matrix C form $M_{nq}(\mathbb{K})$ such that:

$$\forall (i,j) \in [\![1,n]\!] \times [\![1,q]\!], c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

2.2.1.1 Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \in M_{2,3}(\mathbb{R}) \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in M_{3,4}(\mathbb{R})$$

$$C = A \cdot B = \begin{pmatrix} 1 & 0 & 8 & 11 \\ 4 & 0 & 20 & 32 \end{pmatrix} \in M_{2,4}(\mathbb{R})$$

$$C_{2,3} = 4 \times 1 + 5 \times 2 + 6 \times 1 = 20$$

2.2.2 Remarks

R1 If A, B two matrices: we only can multiply A by B if the number of column of A is equal to the number of row of B.

R2 AB can exists but BA not or the other way around.

2.2.2.1 Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \Rightarrow M_{2,1}(\mathbb{R}), \text{ and exists but } BA \text{ does not exists}$$

R3 In the General case, where AB and BA exists:
$\boxed{AB \neq BA}$ (multiplication of matrix is not commutative)
When $AB = BA$ we say A and B commute.

2.3 Properties of matrix calculus

2.3.1 Properties

1. Let A, B two matrices such that AB exists.
   We can have $AB = 0$ and $(A \neq 0$ or $B \neq 0)$

$$\text{If } A = 0 \text{ or } B = 0 \text{ then } AB = 0$$

$$AB = 0 \nRightarrow A = 0 \text{ or } B = 0$$

2.3.1.1 Example

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

2. $(n,p,q,r) \in \mathbb{N}^4$ and let $(A,B,C) \in M_{np}(\mathbb{K}) \times M_{pq}(\mathbb{K}) \times M_{qr}(\mathbb{K})$

$$A \cdot (B \cdot C) = (A \cdot B) \cdot C \otimes \text{ is commutative}$$

3. $(A,B,C) \in M_{np}(\mathbb{K}) \times M_{pq}^2(\mathbb{K})$

$$A \cdot (B + C) = A \cdot B + A \cdot C$$

$\otimes$ (Matrices multiplication) is distributive over matrix addition ($\oplus$).

4. $A \in M_{np}(\mathbb{K})$ and $B \in M_{pq}(\mathbb{K})$ and $\lambda \in \mathbb{K}$

$$\lambda \cdot (A \cdot B) = A \cdot \lambda \cdot B = A \cdot B \cdot \lambda$$

2.3.2 Case of Square Matrices

1.

$$\forall A \in M_n(\mathbb{K}), A \cdot I_n = I_n \cdot A = A$$

2. Let $(A,B) \in M_n(\mathbb{K})^2$, such that $\boxed{AB = BA}$ Then:

$$(A+B)^n \boxed{=} \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}$$

By convention, $A^0 = B^0 = I_n$

3.

$$\forall (A,B) \in M_n(\mathbb{K})^2, {}^t(A \cdot B) = {}^tA \cdot {}^tB$$

2.4 Inverse of a matrix

2.4.1 Definition

Let $A \in M_n(\mathbb{K})$ we say that A is invertible if:

$$\exists B \in M_n(\mathbb{K}), AB = BA = I_n$$

Then we say that B is the inverse of A and denote $B = A^{-1}$ (B is unique) Hence we have (in case of A invertible): $A \cdot A^{-1} = A^{-1} \cdot A = I_n$

2.4.2 Notation

The set of invertible matrices of $M_n(\mathbb{K})$ is denoted $GL_n(\mathbb{K})$

2.4.3 How to find the inverse of a matrix

We will use the following system: Where $A \in M_n(\mathbb{K}), (U,V) \in M_{n,1}(\mathbb{K})^2$:

$$A \cdot U = V$$

By solving this system (Gauss elimination algorithm) when A is invertible, we will have:

$$U = A^{-1} \cdot V$$

2.4.3.1 Example

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} A \cdot U = V \Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z - x \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z - x - y \end{pmatrix}$$

$$\iff \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} x - y + z \\ -x + y + z \\ z - x - y \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{x-y+z}{2} \\ \frac{-x+y+z}{2} \\ \frac{x+y-z}{2} \end{pmatrix}$$

$$\iff \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

3 Matrices of Linear Maps

3.1 Definition and examples

3.1.1 Definition

Let $f \in \mathcal{L}(E, F)$, E and F finite dimensional $\mathbb{K}$-vector spaces such that: $dim(E) = p$ and $dim(F) = n$ where $(p, n) \in \mathbb{N}^2$ and $B = (e_1, e_2, \dots, e_n)$ basis of E and $B' = (e'_1, e'_2, \dots, e'_n)$ basis of F

$$\forall U \in E, \exists! (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{K}^p, \quad U = \sum_{i=1}^{p} \lambda_i e'_i$$

We say $\begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{pmatrix}$ is the column matrix of coordinates $A \cdot U$ in B.

3.1.2 Example

Let $E = \mathbb{R}^2$, $U = \begin{pmatrix} 1 \\ 2 \end{pmatrix}_B$ with $B = (\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix})$ and $B' = (\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix})$Then:

$$U = \begin{pmatrix} 2 \\ 1 \end{pmatrix}_B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}_B + \begin{pmatrix} 1 \\ 0 \end{pmatrix}_B \Rightarrow U = \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{B'}$$

3.1.3 Definition

We call matrix of $f \in \mathcal{L}(E, F)$ with respect to basis B and B' denoted $Mat_{BB'}(f)$ the matrix whose $j - th$ column is composed of the coordinates of $f(e_j)$ in B', for all j from $[\![1, p]\!]$. This is a matrix of p columns and n rows:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix}, \forall j \in [\![1, p]\!], f(e_j) = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}_{B'} = \sum_{i=1}^{n} a_{ij} e_i$$

3.1.4 Example

$$f : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^3 \\ \begin{pmatrix} x \\ y \end{pmatrix} & \longmapsto & \begin{pmatrix} x + y \\ 2x + 4y \\ -3y \end{pmatrix} \end{array}$$

1 basis for the domain ($\mathbb{R}^2$): B $= (\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix})$

basis for the codomain ($\mathbb{R}^3$): $B' = (\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix})$

2 $$\begin{pmatrix} x + y \\ 2x + 4y \\ -3y \end{pmatrix} \Rightarrow \begin{cases} f(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}_{B'} \\ f(\begin{pmatrix} 0 \\ 1 \end{pmatrix}) = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}_{B'} \end{cases}$$

3

$$\forall U \in \mathbb{R}^2, U = x \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow f(U) = x \cdot f\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \cdot f\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 0 & -3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 0 & -3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$$

$$\text{so } f\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$$

3.2 Matrix interpretation of Linear Transformation

3.2.1 Proposition

Let E and F two finite dimensional $\mathbb{K}$-VS, B and B' bases of respectively E and F. Let $U \in E$ and $f \in \mathcal{L}(E, F)$. Then:

$$Mat_{B'}(f(u)) = Mat_{BB'}(f) \cdot Mat_B(u)$$

3.2.2 Example

With the same function as before:

$$f : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^3 \\ \begin{pmatrix} x \\ y \end{pmatrix} & \longmapsto & \begin{pmatrix} x+y \\ 2x+4y \\ -3y \end{pmatrix} \end{array}$$

And with the same basis as before: $B = (\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix})$ and $B' = (\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix})$

And with $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ we have:

$$\begin{aligned} Mat_{B'}(f(u)) &= Mat_{B'}(f) \cdot Mat_B(u) \\ &= \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 0 & -3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \\ -6 \end{pmatrix} = f(u) \end{aligned}$$

3.3 Matrix of g o f

3.3.1 Proposition

Let E, F and G three finite dimensional $\mathbb{K}$-VS, and B, B', B'' bases of respectively E, F and G. Considering $f \in \mathcal{L}(E, F)$ and $g \in \mathcal{L}(F, G)$, we have $g \circ f \in \mathcal{L}(E, G)$ and:

$$Mat_{BB''}(g \circ f) = Mat_{B'B''}(g) \cdot Mat_{BB'}(f)$$

3.3.2 Example

$$f : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} & \longmapsto & \begin{pmatrix} x+y \\ x-y \end{pmatrix} \end{array} \qquad g : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^3 \\ \begin{pmatrix} x \\ y \end{pmatrix} & \longmapsto & \begin{pmatrix} x+2y \\ x \\ -x+y \end{pmatrix} \end{array}$$

And with the following basis: $B = (\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix})$ and $B' = (\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix})$

$$Mat_B(f) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad Mat_{BB'}(g) = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$\forall X = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2, \quad g \circ f(X) = g(f(X)) = g\left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}\right)$$

$$g \circ f(X) = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & 1 \\ 0 & -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

3.4 Matrix of a bijection

3.4.1 Proposition

Let E and F two $\mathbb{K}$-VS of same dimension. B a bases of E and B' a bases of F. Let $f \in \mathcal{L}(E, F)$ Then:

$$f \text{ bijective } \iff Mat_{BB'}(f) \text{ is invertible}$$

3.4.2 Example

$$f : \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} & \longmapsto & \begin{pmatrix} 2x+y \\ x-4y \end{pmatrix} \end{array}$$

To find if f is bijective, we have to show that f is surjective or injective, because f is a endomorphism.
with: $B = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$ and $Mat(f) = \begin{pmatrix} 2 & 1 \\ 1 & -4 \end{pmatrix}$
To prove f is injective, we have to show that $ker(f) = \{0_{\mathbb{R}^2}\}$

$$\begin{aligned} f(X) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} &\iff \begin{pmatrix} 2 & 1 \\ 1 & -4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &\iff \begin{pmatrix} 2 & 1 \\ 0 & -9 \end{pmatrix} : \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &\iff x = y = 0 \Rightarrow S = \{0_{\mathbb{R}^2}\} \\ &\Leftrightarrow f \text{ is injective so } f \text{ is bijective} \end{aligned}$$

Lets find the inverse of f, using two methods:

3.4.2.1 Method 1: Gauss elimination algorithm

$$\begin{aligned}
&\begin{pmatrix} 2 & 1 \\ 1 & -4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \\
\Longleftrightarrow &\begin{pmatrix} 2 & 1 \\ 0 & -9 \end{pmatrix} : \begin{pmatrix} a \\ 2b-a \end{pmatrix} \\
\Longleftrightarrow &\begin{pmatrix} 18 & 0 \\ 0 & -9 \end{pmatrix} : \begin{pmatrix} 8a+2b \\ 2b-a \end{pmatrix} \\
\Longleftrightarrow &\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{4a+b}{9} \\ \frac{a-2b}{9} \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 4a+b \\ a-2b \end{pmatrix} \\
\Longleftrightarrow &\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}
\end{aligned}$$

3.4.2.2 Method 2: System, Gauss-Jordan

Operations on left must be done also on right:

$$\begin{aligned}
\left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \end{array}\right) &\Longleftrightarrow \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & -9 & -1 & 2 \end{array}\right) \\
&\Longleftrightarrow \left(\begin{array}{cc|cc} 18 & 0 & 8 & 2 \\ 0 & -9 & -1 & 2 \end{array}\right) \\
&\Longleftrightarrow \left(\begin{array}{cc|cc} 1 & 0 & \frac{4}{9} & \frac{1}{9} \\ 0 & 1 & -\frac{1}{9} & -\frac{2}{9} \end{array}\right)
\end{aligned}$$

With both methods we have:

$$[Mat_B(f)]^{-1} = \frac{1}{9}\begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix}$$

3.4.3 Proposition

Let E and F two $\mathbb{K}$-VS of same dimension. B a bases of E and B' a bases of F. Let $f \in \mathcal{L}(E, F)$ Then:

$$f \text{ bijective } \Longleftrightarrow Mat_{BB'}(f) \text{ is invertible}$$

And in this case we have:

$$[Mat_{BB'}(f)]^{-1} = Mat_{BB'}(f^{-1})$$

3.4.4 Examples

3.4.4.1 Example 1

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

1 Is A invertible, compute the inverse of A

$$\begin{aligned}
&\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \\
\Longleftrightarrow &\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \text{There are pivot (circled) so A is invertible} \\
\Longleftrightarrow &\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \\
\Longleftrightarrow &\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \\
\Longleftrightarrow & A^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}
\end{aligned}$$

We can also compute A^{-1} this way: (if A is invertible we have:)

$$\begin{aligned}
AX = U &\Longleftrightarrow X = A^{-1}U \\
&\Longleftrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{array}{c} R_1 \\ R_2 - R_1 \\ R_3 - R_1 \end{array} \\
&\Longleftrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} a \\ b - a \\ c - a \end{pmatrix} \begin{array}{c} R_1 \\ R_2 - R_3 \\ R_3 \end{array} \\
&\Longleftrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{array}{c} R_1 + R_2 \\ -R_2 \\ R_3 \end{array} \\
&\Longleftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} a + b - c \\ -b + c \\ c - a \end{pmatrix} \\
&\Longleftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{array}{c} a \\ b \\ c \end{array}
\end{aligned}$$

2 Let $f \in \mathcal{L}(\mathbb{R}^3)$, Show that f is an automorphism and determine f^-1

$$f : \begin{array}{ccc} \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \mapsto & \begin{pmatrix} x+y \\ x+z \\ x+y+z \end{pmatrix} \end{array} \qquad B = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

$A = MatB(f)$ so A invertible $\iff f$ is an automorphism
so $A^{-1} = MatB(f^{-1})$

$$f^{-1} \in \mathcal{L}(R^3): \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} x+y-z \\ -y+z \\ -x+z \end{pmatrix} \quad \text{(using } A^{-1}\text{)}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto A^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

3.4.4.2 Example 2

$$\phi : \begin{array}{ccc} \mathbb{R}_2[X] & \longrightarrow & \mathbb{R}_2[X] \\ P & \longmapsto & XP' + P(X+1) \end{array} \qquad B = (1, X, X^2)$$

So we have: $\phi(1) = 1$ and $\phi(X) = 2X + 1$ and $\phi(X^2) = 3X^2 + 2X + 1$

$$\text{So: } A = Mat_B(\phi) = \begin{array}{c} \begin{array}{ccc} (\phi(1) & \phi(X) & \phi(X^2)) \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix} \end{array} \begin{array}{c} \\ 1 \\ X \\ X^2 \end{array}$$

an invertible matrix (upper right triangle with non zero diagonal coefficient
We have to find A^{-1} to find ϕ^{-1}

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array} \right) \begin{array}{c} 3R_1 - R_3 \\ R_2 - 2R_3 \\ R_3 \end{array} \iff \left(\begin{array}{ccc|ccc} 3 & 3 & 0 & 3 & 0 & -1 \\ 0 & 6 & 0 & 0 & 3 & -1 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array} \right) \begin{array}{c} 2R_1 - R_2 \\ R_2 \\ R_3 \end{array}$$

$$\iff \left(\begin{array}{ccc|ccc} 6 & 0 & 0 & 6 & -3 & 0 \\ 0 & 6 & 0 & 0 & 3 & -2 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array} \right) \begin{array}{c} \frac{1}{6}R_1 \\ \frac{1}{6}R_2 \\ \frac{1}{3}R_3 \end{array}$$

$$\implies [Mat_B(\phi)]^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$$

$$\implies [Mat_B(\phi)]^{-1} = \frac{1}{6} \begin{pmatrix} 6 & -3 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{pmatrix}$$

Now find $Q \in \mathbb{R}_2[X], \phi(Q)(X) = 1 + (X-1)^2$
$\phi^{-1}(1+(X-1)^2)$? we use the matrix to find the coordinates: $1+(X-1)^2 = 2-2X+X^2$

$$\frac{1}{6}\begin{pmatrix} 6 & -3 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}_B = \frac{1}{6}\begin{pmatrix} 18 \\ -8 \\ 2 \end{pmatrix}_B = \begin{pmatrix} 3 \\ -\frac{4}{3} \\ \frac{1}{3} \end{pmatrix}_B$$

$$\exists! Q \in \mathbb{R}[X], \phi(Q) = \mathbb{R} \quad Q(X) = 3 - \frac{4}{3}X - \frac{1}{3}X^2$$

4 Important Proof

4.1 Proposition

Let n and p be two non-zero natural numbers, E and F two finite-dimensional vector spaces over $\mathbb{R}$ such that $dim(E) = p$ and $dim(F) = n$.
Let $B = (e_1, \ldots, e_p)$ be a basis of E and $B' = (\epsilon_1, \ldots, \epsilon_n)$ a basis of F.
Consider $f \in \mathcal{L}(E, F)$ and $A = Mat_{B,B'}(f) = (a_{ij})$.
For all $u \in E$, let X denote the column matrix containing the coordinates of u in basis B and Y the coordinates of $f(u)$ in basis B'.
Then,

$$Y = AX$$

4.1.1 Remarks:

1. Note that $Y = AX \iff Mat_{B'}(f(u)) = Mat_{B,B'}(f) \cdot Mat_B(u)$
2. $A = Mat_{B,B'}(f) = (a_{ij})$. The columns of A contain the coordinates of the vectors $f(e_j)$ in basis B':

$$\begin{array}{c}
\begin{array}{ccc} f(e_1) & & f(e_p) \\ \downarrow & & \downarrow \end{array} \\
\begin{pmatrix} a_{1,1} & \ldots & a_{1,p} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \ldots & a_{n,p} \end{pmatrix}
\end{array}
\begin{array}{l} \\ \\ \leftarrow \text{ coefficient of } \epsilon_1 \\ \\ \leftarrow \text{ coefficient of } \epsilon_n \end{array}$$

   that is, $\forall j \in [\![1, p]\!]$, $f(e_j) = a_{1,j}\epsilon_1 + \cdots + a_{n,j}\epsilon_n$.
3. $A \in \mathcal{M}_{n,p}(\mathbb{R})$, $X \in \mathcal{M}_{p,1}(\mathbb{R})$ and $Y \in \mathcal{M}_{n,1}(\mathbb{R})$.

4.1.2 Proof:

Let $u \in E$.

- Since B is a basis of E:

$$\exists!(\lambda_1, \ldots, \lambda_p) \in \mathbb{R}^p, u = \lambda_1 e_1 + \cdots + \lambda_p e_p$$

  Then $X = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_p \end{pmatrix}$.

- Using the linearity of f, we get:

$$\begin{aligned} f(u) &= \lambda_1 f(e_1) + \cdots + \lambda_p f(e_p) \\ &= \lambda_1(a_{1,1}\epsilon_1 + \cdots + a_{n,1}\epsilon_n) + \cdots + \lambda_p(a_{1,p}\epsilon_1 + \cdots + a_{n,p}\epsilon_n) \\ &= (\lambda_1 a_{1,1} + \cdots + \lambda_p a_{1,p})\epsilon_1 + \cdots + (\lambda_1 a_{n,1} + \cdots + \lambda_p a_{n,p})\epsilon_n \end{aligned}$$

It follows that $Y = \begin{pmatrix} \lambda_1 a_{1,1} + \dots + \lambda_p a_{1,p} \\ \vdots \\ \lambda_1 a_{n,1} + \dots + \lambda_p a_{n,p} \end{pmatrix} = \begin{pmatrix} a_{1,1} & \dots & a_{1,p} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,p} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_p \end{pmatrix} = AX.$