Chapter 10: Function: Local study

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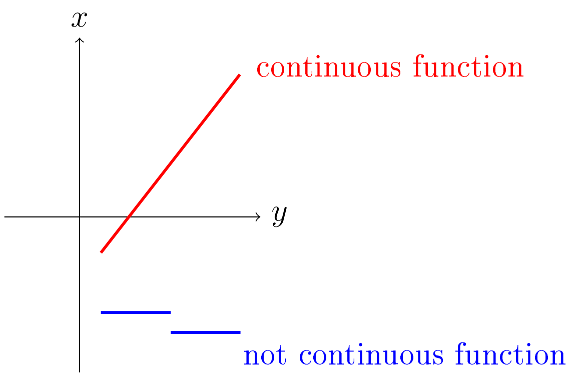
1 Continuity

1.1 First approach

Let f be a function from $I \subset R$ to R, we say that f is continuous over I if "the graph of f can be drown without taking off the pencil from the paper".

1.1.1 Example



1.2 Definition

(1) f continuous at $a \in I$:
we say f is continuous at a - a being a point of I - if and only if:

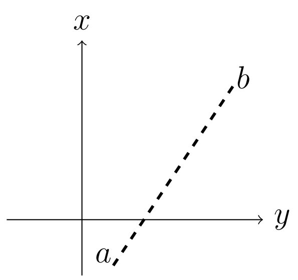
$$f(a) = \lim_{x \to a} f(x)$$
$$f(a) = \lim_{\substack{x \to a \\ x > a}} f(x) = \lim_{\substack{x \to a \\ x < a}} f(x)$$

(2) f continuous over I:
we say f is continuous over $I \subset R$ if and only if $\forall a \in I$,f is continuous at a.

1.3 Intermediate value theorem

Let f be a function continuous over $I \subset R$ and $(a, b) \in I^2$. if $f(a)f(b) < 0$ then there exists (at least one) c from $]a, b[$ such that $f(c) = 0$.

1.3.1 Examples



must intersect the x-axis

Second example: $f(x) = x^2 - 2x + 1$
$f(x) = x^2 \cos(x) + x \sin(x) + 1 = 0$ does f have any solutions?
$I = [0, \pi]$ $f(a = 0) = 1$ and $f(b = \pi) = -\pi^2 + 1 < 0$

1. $f : x \mapsto f(x)$ is constant over $I \subset \mathbb{R}$.
2. $f(a) \times f(b) < 0$.

IVT $\Rightarrow \exists c \in]0, \pi[, f(c) = 0$.

2 Image of an interval by a continuous function

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ and $A \subset I$.
We call image of A by f and denote $f(A)$: $f(A) = \{f(x), x \in A\}$.

2.1 Remark

$\forall y \in f(A), \exists x \in A, y = f(x)$. Example: $f : x \mapsto x^2$ and $A = [-3, 2] \to f(A) = [0, 9]$.

2.2 Proposition

- The image of an interval by a continuous function is an interval.
- The image of a segment ($[a, b]$) by a continuous function is a segment. counter example: $f : x \mapsto \frac{1}{x}$ (continuous over $\mathbb{R}_+^*$) $\quad f(]0, 1] \subset \mathbb{R}_+^*) = [1, +\infty[$

2.3 Corollary

Let f be a continuous function over segment $[a, b]$. Then $f([a, b]) = [m, M]$ where m and M are respectively the minimum and maximum of f over $[a, b]$.

3 Differentiability

All function below are of type $f : I \to \mathbb{R}$ where I is an interval of $\mathbb{R}$ containing at least two points.

3.1 Definition

we say that f is differentiable at a if the following quantity $\frac{f(x)-f(a)}{x-a}$ (the increasing rate of f at a) has a finite limit l at a.
If so, then f is differentiable at a and $f'(a) = l$ is called the derivative number of f at a.

3.1.1 Example

$f : x \mapsto \sqrt{x}$ if $a = 0$ then: $\frac{f(x)-f(0)}{x-0} = \frac{\sqrt{x}-\sqrt{0}}{x-0} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}} \to +\infty$ when $x \to 0$
$\forall a > 0$, $\frac{\sqrt{x}-\sqrt{a}}{x-a} = \frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})} = \frac{1}{\sqrt{x}+\sqrt{a}} \to \frac{1}{2\sqrt{a}}$ when $x \to a$
So $f : x \mapsto \sqrt{x}$ is differentiable at a and $f'(a) = \frac{1}{2\sqrt{a}}$.

3.1.2 Remarks

1. If f is differentiable at a, then the graph of f has a non-vertical tangent at a.
2. If the variation rate tends to $+\infty$ or $-\infty$, then the graph of f admits a vertical tangent.

3.2 Differentiability and continuity

3.2.1 Proposition

If f is differentiable at a then f is continuous at a.

$$\text{Differentiable} \implies \text{Continuity}$$
$$\not\Longleftarrow$$

ex: $\sqrt{x}$ at 0 is continuous but not differentiable.

3.2.2 Proof

$$\begin{aligned}
\text{Differentiable} \implies \exists l \in \mathbb{R}, \lim_{x \to a}\left(\frac{f(x)-f(a)}{x-a}\right) = l &\implies \lim_{x\to a}\left(f(x)-f(a)\right) = 0 \\
&\implies \lim_{x\to a}\left(f(x)\right) = \lim_{x\to a}\left(f(a)\right) \\
&\implies f \text{ is continuous at } a
\end{aligned}$$

4 Taylor Expansion

4.1 Taylor-Young Theorem

Let $n \in \mathbb{N}$ and f be a function of class C^n over I (*id est* f is differentiable at least n times and $f^{(n)}$ is continuous over I).
Let $a \in I$ and $x \in I$. Then:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + o((x-a)^n)$$

4.1.1 Remark

- $n! = n \times (n-1) \times \cdots \times 2 \times 1$, ex: $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
- $f^{(n)} \leftarrow n$-th derivative of f ex: $f^{(3)} = f'''$.
- $o((x-a)^n) \leftarrow$ negligible quantity with respect to

$$(x-a)^n \implies \lim_{x\to a}\left(\frac{o((x-a)^n)}{(x-a)^n}\right) = 0$$

- Theorem mostly used with $a = 0$.

4.2 Definition of Taylor's expansion

Let $n \in \mathbb{N}$, we say that f admit a Taylor expansion of order n at a if there exists real numbers $a_0, a_1, \ldots, a_n$ such that:

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + o((x-a)^n)$$

4.2.1 Remarks

We compute the coefficients $a_0, a_1, \dots, a_n$ using the Taylor-Young theorem for f.

$$\begin{aligned}\text{for ex. } f : x \mapsto e^x &\implies a_0 = f(0) = e^0 = 1\\ &\implies a_1 = f'(0) = e^0 = 1\\ &\implies a_2 = f''(2) = \frac{1}{2!}\end{aligned}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + o(x^n)$$

$$e^x = 1 + x + \underbrace{A(x)}_{o(x)} \qquad e^x = 1 + x + x^2 + \underbrace{B(x)}_{o(x)}$$

4.3 Usual function Taylor expansion at 0 <3

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + o(x^n)$
- $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n})$ even function
- $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+1})$ odd function
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1} \frac{x^n}{n} + o(x^n)$
- $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \cdots + \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} x^n + o(x^n)$
- $\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + o(x^n)$
- $\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^n x^n + o(x^n)$