University of Stirling
Computing Science and Mathematics
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MATU9D2 : PRACTICAL STATISTICS

Chapter 6 Hypothesis Tests and Confidence Intervals for Means and Variances

6.1 : Comparing Means : t tests and t intervals

6.1.1 Z and t distributions

We have seen, that to determine the probability of x exceeding some value x_o we compute the normal deviate (Z score) i.e.

$$z_o = \frac{(x_o - \mu)}{\sigma}$$

and obtain, from Standard Normal Tables (N(0,1) tables)

$$P(z > z_o) = P(x > x_o)$$

<u>Example</u>

Consider a hypothetical patient. Suppose that his diastolic blood pressure is known to be approximately normally distributed with a mean of 60mmHg and a standard deviation of 10mmHg. What is the probability that his diastolic blood pressure on a randomly chosen day will exceed 75mmHg?

Here $x_o = 75$, $\mu = 60$ and $\sigma = 10$, so that

$$z_o = \frac{(x_o - \mu)}{\sigma} = \frac{75 - 60}{10} = 1.5$$

From tables we find that P(z > 1.5) = 1 - P(z < 1.5) = 1 - 0.9332 = 0.0668. There is thus approximately a 7% chance that the diastolic blood pressure on a randomly chosen day will exceed 75.

In the above example, we supposed that the standard deviation was known and equal to σ. In practice it is usually unknown, and we must substitute for it a value of s obtained from a small sample of data.

Suppose then, somewhat more realistically, that although σ is not known exactly, we have a sample standard deviation s = 10 calculated from seven data values.

If the mean diastolic blood pressure is 60 and s = 10 is an estimate of σ, what can we say about the random occurrence of a pressure of 75? Since σ is unknown, we cannot calculate z_o and refer the result to the standard Normal table. Instead, substituting s for σ, we may calculate

$$t_o = \frac{(x_o - \mu)}{s} = \frac{75 - 60}{10} = 1.5$$

On the basis of certain assumption, the quantity $t = {(x-\mu)}/{s}$ has a known distribution.

This important distribution is called the t distribution or Student’s distribution.

The probability points of the t distribution are given in tables.

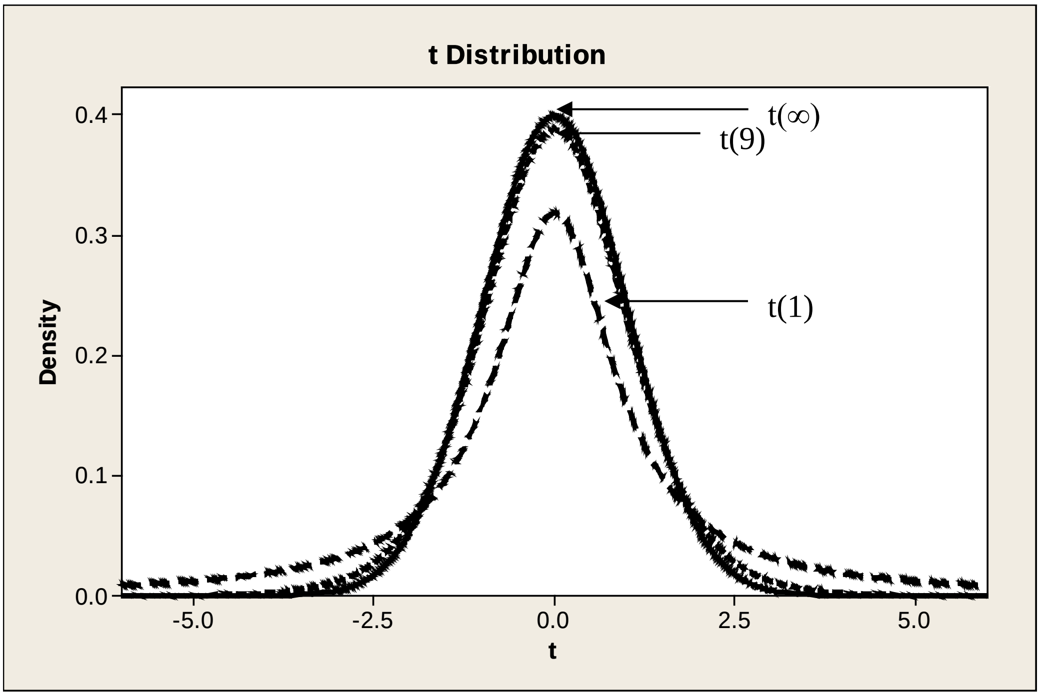
In the present example s has 6 degrees of freedom, and we obtain

$$P(t > 1.5) = 0.09 \quad \text{approximately}$$

As might be expected, the precise form of the t distribution depends on the degree of uncertainty in s^2 which is measured by the degrees of freedom ν on which the statistic s^2 is based. The figure below shows the t distribution for ν =1, 9 and ∞ degrees of freedom.

When ν = ∞, that is, when there is no uncertainty in the estimate s^2, and the t distribution becomes the standard Normal distribution of z. When the number of degrees of freedom is small, however, the possibility of variation in s^2 results in a greater probability of extreme deviations and hence a heavier tailed distribution. The following values for P(t > 2) illustrate the point :

ν	=	∞	P(t > 2)	=	0.023
ν	=	9	P(t > 2)	=	0.038
ν	=	1	P(t > 2)	=	0.148



Except in the extreme tails of the distribution, the normal distribution provides a fair approximation of the t distribution when ν is greater than about 15.

Table : t Distribution

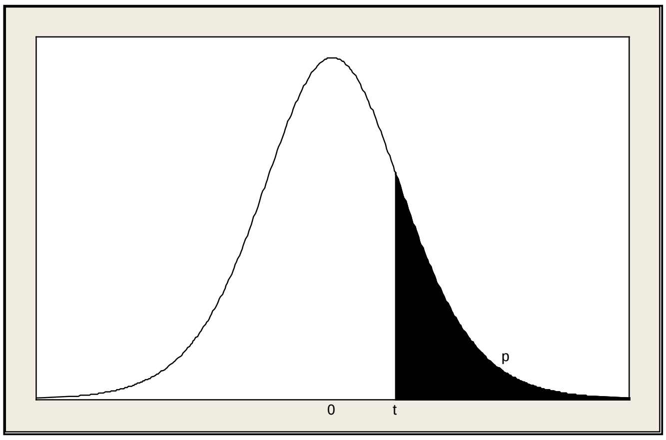


Table of Critical Values [Table entry is the point t with given probability p lying above it

	Tail Probability p											
df	0.25	0.2	0.15	0.1	0.05	0.025	0.02	0.01	0.005	0.0025	0.001	0.0005
1	1.000	1.376	1.963	3.078	6.314	12.71	15.89	31.82	63.66	127.3	318.3	636.6
2	0.816	1.061	1.386	1.886	2.920	4.303	4.849	6.965	9.925	14.09	22.33	31.60
3	0.765	0.978	1.250	1.638	2.353	3.182	3.482	4.541	5.841	7.453	10.21	12.92
4	0.741	0.941	1.190	1.533	2.132	2.776	2.999	3.747	4.604	5.598	7.173	8.610
5	0.727	0.920	1.156	1.476	2.015	2.571	2.757	3.365	4.032	4.773	5.894	6.869
6	0.718	0.906	1.134	1.440	1.943	2.447	2.612	3.143	3.707	4.317	5.208	5.959
7	0.711	0.896	1.119	1.415	1.895	2.365	2.517	2.998	3.499	4.029	4.785	5.408
8	0.706	0.889	1.108	1.397	1.860	2.306	2.449	2.896	3.355	3.833	4.501	5.041
9	0.703	0.883	1.100	1.383	1.833	2.262	2.398	2.821	3.250	3.690	4.297	4.781
10	0.700	0.879	1.093	1.372	1.812	2.228	2.359	2.764	3.169	3.581	4.144	4.587
11	0.697	0.876	1.088	1.363	1.796	2.201	2.328	2.718	3.106	3.497	4.025	4.437
12	0.695	0.873	1.083	1.356	1.782	2.179	2.303	2.681	3.055	3.428	3.930	4.318
13	0.694	0.870	1.079	1.350	1.771	2.160	2.282	2.650	3.012	3.372	3.852	4.221
14	0.692	0.868	1.076	1.345	1.761	2.145	2.264	2.624	2.977	3.326	3.787	4.140
15	0.691	0.866	1.074	1.341	1.753	2.131	2.249	2.602	2.947	3.286	3.733	4.073
16	0.690	0.865	1.071	1.337	1.746	2.120	2.235	2.583	2.921	3.252	3.686	4.015
17	0.689	0.863	1.069	1.333	1.740	2.110	2.224	2.567	2.898	3.222	3.646	3.965
18	0.688	0.862	1.067	1.330	1.734	2.101	2.214	2.552	2.878	3.197	3.610	3.922
19	0.688	0.861	1.066	1.328	1.729	2.093	2.205	2.539	2.861	3.174	3.579	3.883
20	0.687	0.860	1.064	1.325	1.725	2.086	2.197	2.528	2.845	3.153	3.552	3.850
21	0.686	0.859	1.063	1.323	1.721	2.080	2.189	2.518	2.831	3.135	3.527	3.819
22	0.686	0.858	1.061	1.321	1.717	2.074	2.183	2.508	2.819	3.119	3.505	3.792
23	0.685	0.858	1.060	1.319	1.714	2.069	2.177	2.500	2.807	3.104	3.485	3.768
24	0.685	0.857	1.059	1.318	1.711	2.064	2.172	2.492	2.797	3.091	3.467	3.745
25	0.684	0.856	1.058	1.316	1.708	2.060	2.167	2.485	2.787	3.078	3.450	3.725
26	0.684	0.856	1.058	1.315	1.706	2.056	2.162	2.479	2.779	3.067	3.435	3.707
27	0.684	0.855	1.057	1.314	1.703	2.052	2.158	2.473	2.771	3.057	3.421	3.689
28	0.683	0.855	1.056	1.313	1.701	2.048	2.154	2.467	2.763	3.047	3.408	3.674
29	0.683	0.854	1.055	1.311	1.699	2.045	2.150	2.462	2.756	3.038	3.396	3.660
30	0.683	0.854	1.055	1.310	1.697	2.042	2.147	2.457	2.750	3.030	3.385	3.646
40	0.681	0.851	1.050	1.303	1.684	2.021	2.123	2.423	2.704	2.971	3.307	3.551
50	0.679	0.849	1.047	1.299	1.676	2.009	2.109	2.403	2.678	2.937	3.261	3.496
60	0.679	0.848	1.045	1.296	1.671	2.000	2.099	2.390	2.660	2.915	3.232	3.460
80	0.678	0.846	1.043	1.292	1.664	1.990	2.088	2.374	2.639	2.887	3.195	3.416
100	0.677	0.845	1.042	1.290	1.660	1.984	2.081	2.364	2.626	2.871	3.174	3.390
1000	0.675	0.842	1.037	1.282	1.646	1.962	2.056	2.330	2.581	2.813	3.098	3.300
Infinity	0.674	0.842	1.036	1.282	1.645	1.960	2.054	2.326	2.576	2.807	3.090	3.290
	50%	60%	70%	80%	90%	95%	96%	98%	99%	99.50%	99.80%	99.90%
	Confidence Level C											

The t test is a test which is commonly used to test a hypothesis based on sample means.

6.1.2 The Difference with a Claimed Mean : One Sample

Now we have already seen that if samples of size n are taken from a distribution with mean μ and standard deviation σ, then the distribution of the sample means will have a mean μ and standard deviation $\sigma/\sqrt{n}$ (i.e. the standard error). If we only know the sample standard deviation s, then our estimate of the standard error of the mean is $s/\sqrt{n}$ and the quantity

$$t = \frac{\bar{x}-\mu}{s/\sqrt{n}} \sim t(n-1) \quad under\ H_0$$

Examples

1. The level of various substances in the blood of kidney dialysis patients is of concern because kidney failure and dialysis lead to nutritional problems. A researcher performed blood tests on several dialysis patients on six consecutive clinic visits. The variable measured was the phosphate level in blood. Phosphate levels in a single patient tend to vary normally over time. The data on one patient in mg/dl of blood are given below.

   5.5 5.1 4.6 4.8 5.7 6.4

   Is the mean phosphate level greater than 5 mg/dl of blood?

2. In a randomised comparative experiment of dietary calcium on blood pressure, 54 healthy white males were divided at random into two groups. One group received calcium; the other, a placebo. At the beginning of the study, the researchers measured many variables on the subjects. The report on the study gives the mean seated systolic blood pressure of the 27 members of the placebo group as 114.9 mmHg and the standard deviation as 9.3 mmHg.

   Does the mean blood pressure equal 120 mmHg?

Solutions

6.1.3 The Difference between Means : Two Independent Groups

One of the most common questions is to decide whether the means of two samples are significantly different. Now we have already seen that if samples of size n are taken from a distribution with mean μ and standard deviation σ, then the distribution of the sample means will have a mean μ and standard deviation $\sigma/\sqrt{n}$ (i.e. the standard error). If we only know the sample standard deviation s, then our estimate of the standard error of the mean is $s/\sqrt{n}$ and the quantity

$$t = \frac{\bar{x}-\mu}{s/\sqrt{n}} \sim t(n-1)$$

Consider two samples with means $\bar{x}_1$ and $\bar{x}_2$ with standard deviations s_1 and s_2 respectively. Then we have seen that

$$s_1^2 = \frac{\sum (x-\bar{x}_1)^2}{n_1 - 1} \qquad \textit{and} \qquad s_2^2 = \frac{\sum (x-\bar{x}_2)^2}{n_2 - 1}$$

The standard error of the difference between the means is estimated as

$$s_d = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

If we assume that the two samples come from populations having the same variance, then the two samples can be combined to give the best estimate of the population variance, which is called the Pooled Sample Variance, s_p^2

$$s_p^2 = \frac{\sum(x-\bar{x}_1)^2 + \sum(x-\bar{x}_2)^2}{n_1 + n_2 - 2}$$

$$= \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

The estimate is substituted for s_1^2 and s_2^2 to give:

$$s_d = \sqrt{s_p^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}$$

The Null Hypothesis is that the difference between the sample means is not significantly different from d and the test statistic is

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - d}{\sqrt{s_p^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} \qquad \text{with } df = n_1 + n_2 - 2$$

The most common case is for d = 0 and then

$$t = \frac{(\bar{x}_1 - \bar{x}_2)}{\sqrt{s_p^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} \sim t(n_1 + n_2 - 2) \quad under\ H_0$$

Example

3. A bank compares two proposals to increase the amount that its credit card customers charge on their cards. Proposal A offers to eliminate the annual fee for customers who charge £2400 or more during the year. Proposal B offers a small percent of the total charged as a cash rebate at the end of the year. the bank offers each proposal to a sample of 45 of its existing credit card customers.

   At the end of the year, the total amount charged by each customer is recorded. The summary statistics follow :

Group	n	$\bar{x}$	s
A	20	£1987	£392
B	25	£2056	£413

   Do the data show a significant difference between the mean amounts charged by customers offered the two plans ?

Solution

6.1.4 Paired Differences : Two Related Groups

If we make pairs of measurements from a population, we can obtain a sample of n pairs and n differences. These measurements may be measurements made on matched pairs (e.g. subjects of the same age, height, weight etc) or on a subject before and after a particular intervention. This sample of differences will have a sample mean $\bar{x}_d = \frac{\sum x_d}{n}$, where x_d = the difference between the two measurements. This can be used to provide an estimate of the standard deviation of the population of difference values

$$s_d^2 = \frac{\sum (x_d - \bar{x}_d)^2}{n-1} = \frac{\left[\sum x_d{}^2 - \frac{\left(\sum x_d\right)^2}{n} \right]}{n-1}$$

If such sampling is continued, the mean of the distribution of mean differences will equal the mean difference for the population, and the standard error for the distribution of mean differences is estimated as

$$s_\Delta = \frac{s_d}{\sqrt{n}}$$

The Null Hypothesis is that the mean difference is not significantly different from d' and the test statistic is

$$t = \frac{\bar{x}_d - d'}{s_\Delta}$$ where df = n-1 and n = number of pairs of measurements

Again the most common case is for d' = 0 and then $t = \frac{\bar{x}_d}{s_\Delta}$.

Example

4. The design of controls and instruments has a large effect on how easily people can use them. A student project investigated this effect by asking 25 right handed students to turn a knob, with their right hands, that moved an indicator by screw action. There were two identical instruments, one with a right-hand thread (turns clockwise) and the other with a left-hand thread (turn anti-clockwise). The table gives the times required (in seconds to move the indicator a fixed distance.

Subject	Right thread	Left thread
1	113	137
2	105	105
3	130	133
4	101	108
5	138	115
6	118	170
7	87	103
8	116	145
9	75	78
10	96	107
11	122	84
12	103	148
13	116	147
14	107	87
15	118	166
16	103	146
17	111	123
18	104	135
19	111	112
20	89	93
21	78	76
22	100	116
23	89	78
24	85	101
25	88	123

The project hoped to show that right-handed people find right-hand threads easier to use. Does the data support this view?

Solution

6.2 When should we use the t test?

The t test is only valid if

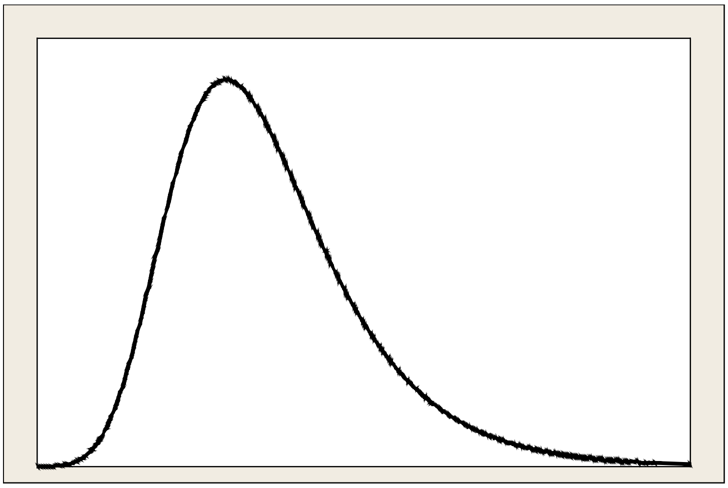
1. The data is a random sample from the population.

2. The samples have been drawn from normally distributed populations.

3. The population variances are equal. (This applies to the two sample case).

If the population is not normally distributed, it is often possible to transform the data into a distribution which is approximately normal.

You should use the following ‘ladder’ of transformation :



$1/x$

$\ln(x)$

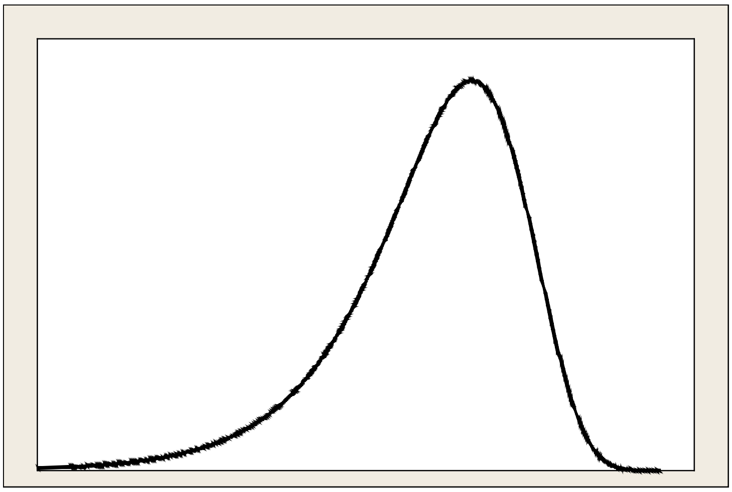
$\sqrt{x}$

x

x^2

x^3

e^x



In order to test assumption 3, an F test should be carried out first. This will be discussed in the next lecture.
With larger samples (n greater than about 30) the assumptions are not critical but should be checked. The test is said to be fairly ‘robust’ to departures from normality.

6.3 Multiple t tests

Up to now we have been talking about t test in which a comparison is being made between two groups (or, in the case of the paired t test, between readings in one set of subjects). Frequently, however, we are comparing, for example, several treatments.

<u>Example</u>

In one investigation of imipramine and lithium in manic depressive illness, three different treatments were given : placebo, lithium and imipramine. Moreover, patients were divided into two separate groups with different types of manic depressive illness. Thus there are 4 possible opportunities to test 'active treatments' against placebo. These are imipramine versus placebo and lithium versus placebo in each type of illness. Finding at least one comparison significant at the 0.05 level is more likely then the p value of 0.05 may lead us to believe.

Under the null hypothesis, each comparison has a 0.95 chance of not being significant at the 0.05 level. If we assume for a moment that the 4 tests are independent, there is 0.95^4 or 0.81 chance of observing none of the tests significant at the 5% level. Here, the 4 tests are unlikely to be independent, since the experience of each placebo group contributes two different tests.

However, an inequality due to Bonferroni implies that the probability that none of the 4 p values falls below 0.05 is at least $1 - (4 \times 0.05) = 0.80$ when the null hypothesis of equal efficacy holds. Thus when we compute four separate p values simultaneously, there is a probability of up to 0.20 that at least one will be smaller than 0.05 even when the treatments do not differ.

In such situations, a significance level of 0.01 is sometimes chosen as the criterion for 'statistical significance'. If an individual p value of 0.01 is used, the significance level of the four tests together is less than 0.04.

<u>Solution - Bonferroni Correction</u>

In general, Bonferroni's inequality implies that when k tests are performed, each with a significance level p, the probability of one or more significant tests by chance alone is at most k p. Thus we sometimes say that the significance level should be multiplied by the number of tests.

e.g. 6 tests, for an overall significance level of 0.05. Use a $0.05 / 6 = 0.008$ significance level for each test.

6.4 Confidence Intervals

In the previous lecture we saw that more information is conveyed by quoting confidence intervals than by simply stating p values. Some commonly used confidence intervals are given below.

Note that the form of the expression is always an appropriate standard error multiplied by an appropriate t value, which for a 95% confidence interval and a large number of degrees of freedom will be slightly less than 2 (in fact, 1.96).

1. <u>Mean of a One Sample</u>

If we have a single sample of mean $\bar{x}$ and standard deviation s then there is a probability (1 - α) that an interval of the form

$$\left[\ \bar{x} \ - \ t(n-1,\ \alpha/2)\sqrt{\frac{s^2}{n}} \quad , \quad \bar{x} \ + \ t(n-1,\ \alpha/2)\sqrt{\frac{s^2}{n}}\ \right]$$

will contain the true value of the population mean, μ.

We refer to the above as a (1-α) confidence interval, i.e. if $\alpha = 0.05$, $1-\alpha = 0.95$ and we have a 95% Confidence Interval.

2. <u>Difference between the Means of Two Samples</u>

Suppose we have two samples, containing n_1 and n_2 values, with means $\bar{x}_1$ and $\bar{x}_2$ and pooled standard deviation s_p (i.e. assume equal variance). Then there is a probability (1 - α) that the true difference between the means $\mu_1 - \mu_2$ lies in the interval

$$\left[\ \bar{x}_1 - \bar{x}_2 \ - \ t(n_1 + n_2 - 2, \alpha/2)\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} \quad , \quad \bar{x}_1 - \bar{x}_2 \ + \ t(n_1 + n_2 - 2, \alpha/2)\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}\ \right]$$

3. <u>Paired Data - Mean Difference</u>

Suppose we have n subjects, each measured on two occasions. If the mean difference between the pairs of readings is $\bar{d}$ and the standard deviation of the difference is s_d, then there is probability (1-α) that the true difference lies in the interval

$$\left[\ \bar{d} \ - \ t(n-1,\ \alpha/2)\sqrt{\frac{s_d^2}{n}} \quad , \quad \bar{d} \ + \ t(n-1,\ \alpha/2)\sqrt{\frac{s_d^2}{n}}\ \right]$$

4. Prediction Intervals

Sometimes we have evaluated the mean $\bar{x}$ and the standard deviation s from a sample of observations and we want to determine a range of values within which you can be reasonably sure that any single future observation will lie. This is called a prediction interval and takes the form

$$\left[\ \bar{x} \ - \ t(n-1,\ \alpha/2)\sqrt{s^2\left(1+\frac{1}{n}\right)} \quad , \quad \bar{x} \ + \ t(n-1,\ \alpha/2)\sqrt{s^2\left(1+\frac{1}{n}\right)}\ \right]$$

5. More than One Comparison

If we are making k comparisons (instead of just 1) then using Bonferroni's inequality we simply replace t (n-1, $\alpha/2$) by t(n-1, $\alpha/2k$).

Examples

(Refer back to the 4 questions completed earlier in Chapter 6)

6.6 Comparing Variances : Chisquared Intervals and F Tests

6.6.1 Theoretical Background

In this lecture we shall introduce another very important distribution, the χ^2 (chi squared) distribution. Denote by X_1 the standardised deviate corresponding to the variable x. i.e. $X_1 = \frac{x-\mu}{\sigma}$

Then, X_1^2 is a random variable, whose value must be non-negative.
The distribution of X_1^2 is called the χ^2 distribution on one degree of freedom (1 df) and is often called the $\chi^2(1)$ distribution. This is shown in the figure below. Note that :

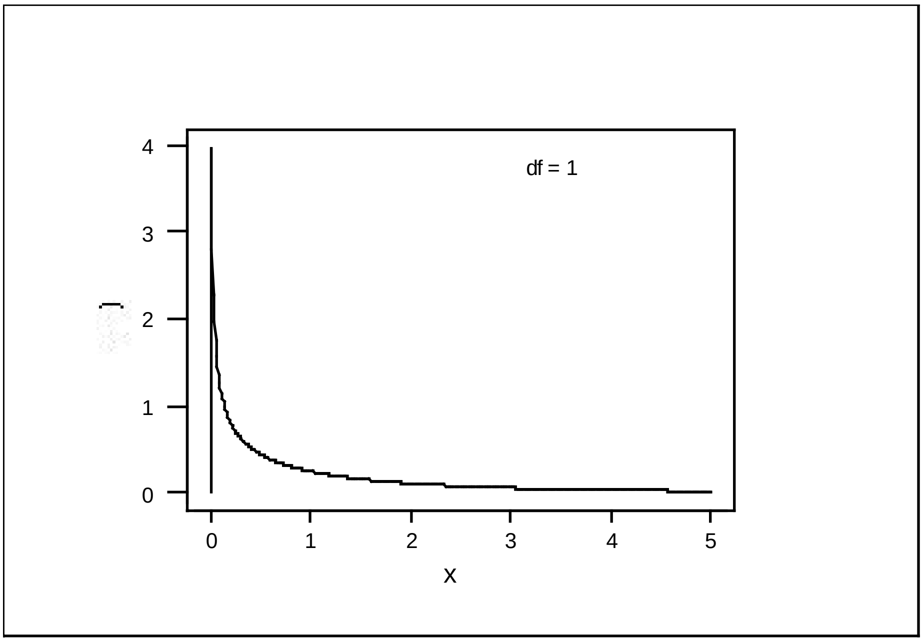
1. The mean value of the distribution is 1.

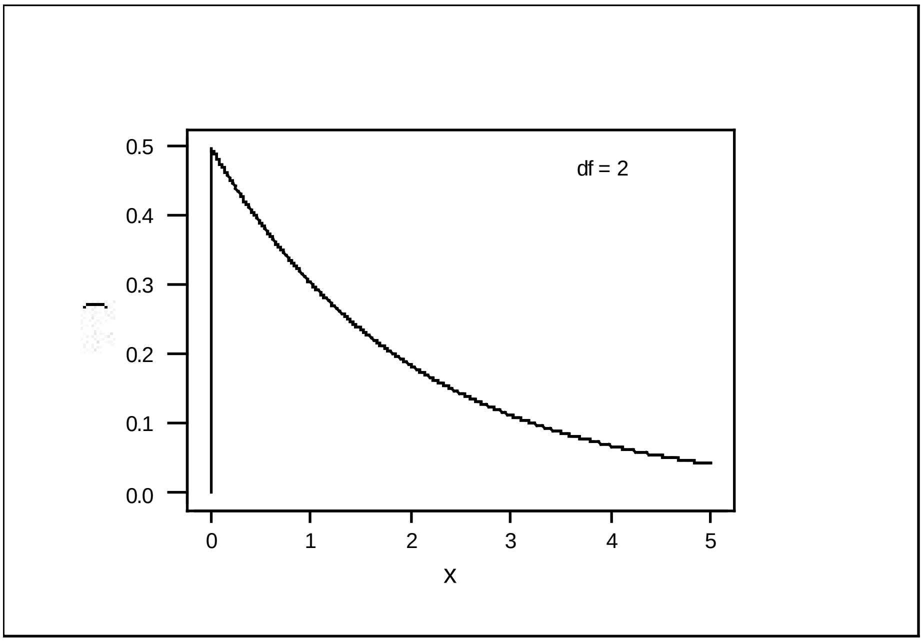
2. From the Table of the Standard Normal distribution we know that there is a probability of 0.05 that $\frac{x-\mu}{\sigma}$ exceeds +1.96 or falls below -1.96. Whenever either of these happens, $\frac{(x-\mu)^2}{\sigma^2}$ =3.84. Thus, the 0.05 level of the $\chi^2(1)$ distribution is 3.84.

Now let x_1 and x_2 be two independent observations on x, define

$$X_2^2 = \frac{(x_1-\mu)^2}{\sigma^2} + \frac{(x_2-\mu)^2}{\sigma^2}$$

X_2^2 follows what is know as the χ^2 distribution on two degrees of freedom. The variable X_2^2, like X_1^2, is necessarily non-negative. Its distribution is also shown in the Figure below. Since X_2^2 is the sum of two independent observations on X_1^2, its mean is 2.





Similarly, in a sample of n independent observations x_i, define

$$X_n^2 \quad = \quad \frac{\sum \left(x_i - \mu\right)^2}{\sigma^2}$$

This follows the χ^2 distribution on n degrees of freedom, with a mean value of n.

As the degrees of freedom increase, the χ^2 distribution becomes more and more symmetric : in fact, $\chi^2(n)$ tends to normality as n increases. The variance of $\chi^2(n)$ distribution is 2n.

6.6.2 Confidence Interval for a Variance / Standard Deviation

One application of the χ^2 distribution is to test whether a variance is significantly different from a given value, or alternatively to work out confidence limits for a measured variance.

Suppose s^2 is the usual estimate of variance in a random sample of size n from a normal distribution with variance σ^2 : the population mean need not be specified. For a test of the null hypothesis that $s^2 = s_0^2$ calculate :

$$X^2 = \frac{(n-1)s^2}{\sigma_o^2}$$

or, equivalently

$$\frac{\sum(x-\bar{x})^2}{\sigma_o^2}$$

and refer this to the χ^2(n-1) distribution. For a two-sided test at significance level α, the critical values for X^2 will be those corresponding to tabulated probabilities of $1 - \alpha/2$ and $\alpha/2$. e.g. for a two-sided 5% level the entries under the headings 0.975 and 0.025 must be used. We denote these by

$$\chi^2(\text{n-1}, 0.975) \qquad \text{and} \qquad \chi^2(\text{n-1}, 0.025)$$

Note that the χ^2 tables that we are using give the probability of observing a value at least as great as that observed i.e. the upper tail of the distribution.

For confidence limits for σ^2 we can argue that the probability is, say, 0.95 that

$$\chi^2(n-1, 0.975) < \frac{(n-1)s^2}{\sigma^2} < \chi^2(n-1, 0.025)$$

and hence

$$\frac{(n-1)s^2}{\chi^2(n-1, 0.025)} < \sigma^2 < \frac{(n-1)s^2}{\chi^2(n-1, 0.975)}$$

Thus a 95% Confidence Interval for σ^2 is

$$\left(\frac{(n-1)s^2}{\chi^2(n-1;0.025)}, \frac{(n-1)s^2}{\chi^2(n-1;0.975)}\right).$$

and a 95% Confidence Interval for σ is

$$\left(\sqrt{\frac{(n-1)s^2}{\chi^2(n-1;0.025)}}\ ,\ \sqrt{\frac{(n-1)s^2}{\chi^2(n-1;0.975)}}\right).$$

EXAMPLE

5. In a randomised comparative experiment of dietary calcium on blood pressure, 54 healthy white males were divided at random into two groups. One group received calcium; the other, a placebo. At the beginning of the study, the researchers measured many variables on the subjects. The report on the study gives the mean seated systolic blood pressure of the 27 members of the placebo group as 114.9 mmHg and the standard deviation as 9.3 mmHg.

   Does the standard deviation of the blood pressure in the placebo group equal 12 mmHg?

6.6.3 Comparison of Two Variances : Two Independent Groups

Suppose that a sample of n_1 observations is randomly drawn from a normal distribution having variance σ_1^2, a second sample of n_2 observations is drawn from a second distribution having variance σ_2^2 and estimates s_1^2 and s_2^2 of the two population variances are calculated.

Then $\dfrac{s_1^2}{\sigma_1^2}$ is distributed as $\dfrac{\chi^2(\nu_1)}{\nu_1}$ and

$\dfrac{s_2^2}{\sigma_2^2}$ is distributed as $\dfrac{\chi^2(\nu_2)}{\nu_2}$

Now the ratio $\dfrac{\dfrac{\chi^2(\nu_1)}{\nu_1}}{\dfrac{\chi^2(\nu_2)}{\nu_2}}$ has an F distribution having ν_1 and ν_2 degrees of freedom, whose probability points are given in tables. Thus

$$\frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2} \sim F(\nu_1 , \nu_2)$$

or, equivalently, $$\frac{s_1^2}{s_2^2} \sim \frac{\sigma_1^2}{\sigma_2^2} F(\nu_1 , \nu_2)$$

The F distribution may thus be used to check the hypotheses concerning the ratio of population variances of two normal populations. In particular, the null hypothesis that the variances are equal may be tested by referring the ratio of the sample variances directly to the F table.

For particular values ν_1 and ν_2, F can clearly assume values on either side of 1, and significant departures from the null hypothesis may be marked either by very small or very large F values. The tabulated critical values are, however, all greater than 1 and refer only to the upper tail of the F distribution. This is not a serious restriction because labelling of the two samples by the numbers 1 and 2 is arbitrary and a mere reversal of the labels will convert a ratio less than 1 into a value greater than 1.

To use the Table we denote by s_1^2 the larger of the two variance estimates : ν_1 is the corresponding number of degrees of freedom (which, of course, is not necessarily the larger of ν_1 and ν_2).

For a two-sided test care should be taken to set the probability to half the two-sided significance level. For a two-sided test at the 5% level, for instance, the entries for $p = 0.025$ are used.

Two connections may be noted between the F distribution and other distributions already met.

1. When $\nu_1 = 1$, the F distribution is that of a quantity following the t distribution on n_2 df.

2. When $\nu_1 = \infty$, the F distribution is the same as that of a χ^2 (ν_1) variable divided by ν_1.

The F test and the associated confidence limits provide an exact treatment of the comparison of two variances estimated from two independent normal samples.

Unfortunately the methods are rather sensitive to the assumption of normality - much more so than the corresponding cases of the t distribution to compare two means. The defect is called a lack of robustness.

EXAMPLE

6. A bank compares two proposals to increase the amount that its credit card customers charge on their cards. Proposal A offers to eliminate the annual fee for customers who charge £2400 or more during the year. Proposal B offers a small percent of the total charged as a cash rebate at the end of the year. The bank offers each proposal to a sample of 45 of its existing credit card customers.

   At the end of the year, the total amount charged by each customer is recorded. The summary statistics follow :

Group	n	$\bar{x}$	s
A	20	£1987	£392
B	25	£2056	£413

   We answered the question : Do the data show a significant difference between the mean amounts charged by customers offered the two plans ?

   However, the test we used assumed that the variances were equal. Was that assumption valid?

6.8 Hypothesis Tests and Confidence Intervals

In the previous lecture we met both Hypothesis Tests and Confidence Intervals based on the t distribution, χ^2 distribution and F distribution.

They are the second in a set of possible tests and intervals that we will meet that will help us answer different types of questions for different types of data.

So far ....

Step 1.	Do we have Quantitative Data?	No Yes	- -	See later in Unit Go to Step 2
Step 2.	Is the question about difference in the mean?	No Yes	- -	Go to Step 12 Go to Step 3
Step 3.	Is the data Normally distributed?	No Yes	- -	Go to Step 4 Go to Step 5
Step 4.	Transform the data - Distribution now Normal	Yes No	- -	Go to Step 5. See later in Unit
Step 5.	How many samples do we have?	One Two More	- - -	Go to Step 6 Go to Step 9 See later in Unit
Step 6.	Do we have a large sample or is the population standard deviation known?	No Yes	- -	Go to Step 8 Go to Step 7

Step 7. Either perform a Z test or calculate a Z interval with

Test Statistic
$$Z = \frac{\bar{x} - \mu_o}{\sigma / \sqrt{n}}$$

95% Confidence Interval for μ
$$\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}}$$

Then draw conclusion

Step 8. Either perform a t test or calculate a t interval

Test Statistic $$t = \frac{\bar{x} - \mu_o}{s/\sqrt{n}}$$

95% Confidence Interval for μ $$\bar{x} \pm t(n-1;0.025)\frac{s}{\sqrt{n}}$$

Then draw conclusion

Step 9. Two Samples

Two Samples	Independent	Go to Step 10
	Related	Go to Step 11

Step 10. Unpaired t test

1. Check that the variances are equal.

   Test Statistic $$F = \frac{s_1^2}{s_2^2}$$

2. Perform the test or calculate the t interval

   Test Statistic $$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

   95% Confidence Interval for $\mu_1 - \mu_2$

$$\bar{x}_1 - \bar{x}_2 \pm t(n_1 + n_2 - 2;0.025)\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

Then draw conclusion

Step 11. Paired t test

1. Calculate the differences

2. Perform a One Sample t test on the differences.

Test Statistic $t = \dfrac{\bar{x}_d}{s_d / \sqrt{n}}$

95% Confidence Interval for μ_d

$$\bar{x}_d \quad \pm t(n-1;0.025)\frac{s_d}{\sqrt{n}}$$

Then draw conclusion

Step 12. Is the question one sample about the standard deviation?

If No, see later in Unit
If Yes, Confidence Interval

95% Confidence Interval for σ is

$$\left(\sqrt{\frac{(n-1)s^2}{\chi^2(n-1;0.025)}} \; , \; \sqrt{\frac{(n-1)s^2}{\chi^2(n-1;0.975)}} \right).$$

Table : Chisquare Distribution

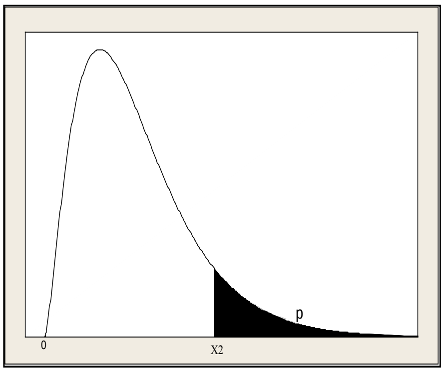


Table c Critical Values [Table entry for the point with probability p lying above it]

	Upper Tail Probability p																
df	0.995	0.99	0.975	0.95	0.9	0.25	0.2	0.15	0.1	0.05	0.025	0.02	0.01	0.005	0.0025	0.001	0.0005
1	0.00	0.00	0.00	0.00	0.02	1.32	1.64	2.07	2.71	3.84	5.02	5.41	6.63	7.88	9.14	10.83	12.12
2	0.01	0.02	0.05	0.10	0.21	2.77	3.22	3.79	4.61	5.99	7.38	7.82	9.21	10.60	11.98	13.82	15.20
3	0.07	0.11	0.22	0.35	0.58	4.11	4.64	5.32	6.25	7.81	9.35	9.84	11.34	12.84	14.32	16.27	17.73
4	0.21	0.30	0.48	0.71	1.06	5.39	5.99	6.74	7.78	9.49	11.14	11.67	13.28	14.86	16.42	18.47	20.00
5	0.41	0.55	0.83	1.15	1.61	6.63	7.29	8.12	9.24	11.07	12.83	13.39	15.09	16.75	18.39	20.51	22.11
6	0.68	0.87	1.24	1.64	2.20	7.84	8.56	9.45	10.64	12.59	14.45	15.03	16.81	18.55	20.25	22.46	24.10
7	0.99	1.24	1.69	2.17	2.83	9.04	9.80	10.75	12.02	14.07	16.01	16.62	18.48	20.28	22.04	24.32	26.02
8	1.34	1.65	2.18	2.73	3.49	10.22	11.03	12.03	13.36	15.51	17.53	18.17	20.09	21.95	23.77	26.12	27.87
9	1.73	2.09	2.70	3.33	4.17	11.39	12.24	13.29	14.68	16.92	19.02	19.68	21.67	23.59	25.46	27.88	29.67
10	2.16	2.56	3.25	3.94	4.87	12.55	13.44	14.53	15.99	18.31	20.48	21.16	23.21	25.19	27.11	29.59	31.42
11	2.60	3.05	3.82	4.57	5.58	13.70	14.63	15.77	17.28	19.68	21.92	22.62	24.73	26.76	28.73	31.26	33.14
12	3.07	3.57	4.40	5.23	6.30	14.85	15.81	16.99	18.55	21.03	23.34	24.05	26.22	28.30	30.32	32.91	34.82
13	3.57	4.11	5.01	5.89	7.04	15.98	16.98	18.20	19.81	22.36	24.74	25.47	27.69	29.82	31.88	34.53	36.48
14	4.07	4.66	5.63	6.57	7.79	17.12	18.15	19.41	21.06	23.68	26.12	26.87	29.14	31.32	33.43	36.12	38.11
15	4.60	5.23	6.26	7.26	8.55	18.25	19.31	20.60	22.31	25.00	27.49	28.26	30.58	32.80	34.95	37.70	39.72
16	5.14	5.81	6.91	7.96	9.31	19.37	20.47	21.79	23.54	26.30	28.85	29.63	32.00	34.27	36.46	39.25	41.31
17	5.70	6.41	7.56	8.67	10.09	20.49	21.61	22.98	24.77	27.59	30.19	31.00	33.41	35.72	37.95	40.79	42.88
18	6.26	7.01	8.23	9.39	10.86	21.60	22.76	24.16	25.99	28.87	31.53	32.35	34.81	37.16	39.42	42.31	44.43
19	6.84	7.63	8.91	10.12	11.65	22.72	23.90	25.33	27.20	30.14	32.85	33.69	36.19	38.58	40.88	43.82	45.97
20	7.43	8.26	9.59	10.85	12.44	23.83	25.04	26.50	28.41	31.41	34.17	35.02	37.57	40.00	42.34	45.31	47.50
21	8.03	8.90	10.28	11.59	13.24	24.93	26.17	27.66	29.62	32.67	35.48	36.34	38.93	41.40	43.77	46.80	49.01
22	8.64	9.54	10.98	12.34	14.04	26.04	27.30	28.82	30.81	33.92	36.78	37.66	40.29	42.80	45.20	48.27	50.51
23	9.26	10.20	11.69	13.09	14.85	27.14	28.43	29.98	32.01	35.17	38.08	38.97	41.64	44.18	46.62	49.73	52.00
24	9.89	10.86	12.40	13.85	15.66	28.24	29.55	31.13	33.20	36.42	39.36	40.27	42.98	45.56	48.03	51.18	53.48
25	10.52	11.52	13.12	14.61	16.47	29.34	30.68	32.28	34.38	37.65	40.65	41.57	44.31	46.93	49.44	52.62	54.95
26	11.16	12.20	13.84	15.38	17.29	30.43	31.79	33.43	35.56	38.89	41.92	42.86	45.64	48.29	50.83	54.05	56.41
27	11.81	12.88	14.57	16.15	18.11	31.53	32.91	34.57	36.74	40.11	43.19	44.14	46.96	49.65	52.22	55.48	57.86
28	12.46	13.56	15.31	16.93	18.94	32.62	34.03	35.71	37.92	41.34	44.46	45.42	48.28	50.99	53.59	56.89	59.30
29	13.12	14.26	16.05	17.71	19.77	33.71	35.14	36.85	39.09	42.56	45.72	46.69	49.59	52.34	54.97	58.30	60.73
30	13.79	14.95	16.79	18.49	20.60	34.80	36.25	37.99	40.26	43.77	46.98	47.96	50.89	53.67	56.33	59.70	62.16
40	20.71	22.16	24.43	26.51	29.05	45.62	47.27	49.24	51.81	55.76	59.34	60.44	63.69	66.77	69.70	73.40	76.10
50	27.99	29.71	32.36	34.76	37.69	56.33	58.16	60.35	63.17	67.50	71.42	72.61	76.15	79.49	82.66	86.66	89.56
60	35.53	37.48	40.48	43.19	46.46	66.98	68.97	71.34	74.40	79.08	83.30	84.58	88.38	91.95	95.34	99.61	102.7
80	51.17	53.54	57.15	60.39	64.28	88.13	90.41	93.11	96.58	101.9	106.6	108.1	112.3	116.3	120.1	124.8	128.3
100	67.33	70.06	74.22	77.93	82.36	109.1	111.7	114.7	118.5	124.3	129.6	131.1	135.8	140.2	144.3	149.4	153.2

Table : F Distribution

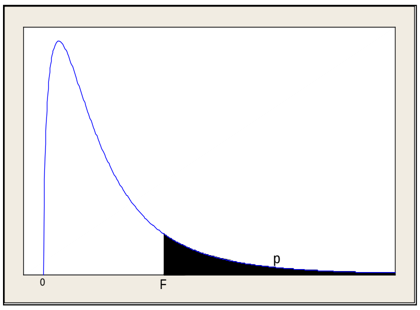


Table of Critical Values Table entry is the point F with probability p lying above it.

Degrees of Freedom in the Numerator

Degrees of Freedom in the Denominator

DFD	p	1	2	3	4	5	6	7	8	9	10	12	15	20	25	30	40	50	60	120	1000
	0.1	39.86	49.50	53.59	55.83	57.24	58.20	58.91	59.44	59.86	60.19	60.71	61.22	61.74	62.05	62.26	62.53	62.69	62.79	63.06	63.30
	0.05	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54	241.88	243.90	245.95	248.02	249.26	250.10	251.14	251.77	252.20	253.25	254.19
1	0.025	647.79	799.48	864.15	899.60	921.83	937.11	948.20	956.64	963.28	968.63	976.72	984.87	993.08	998.09	1001.40	1005.60	1008.10	1009.79	1014.04	1017.76
	0.01	4052.2	4999.3	5403.5	5624.3	5764.0	5859.0	5928.3	5981.0	6022.4	6055.9	6106.7	6157.0	6208.7	6239.9	6260.4	6286.4	6302.3	6313.0	6339.5	6362.8
	0.001	405312	499725	540257	562668	576496	586033	593185	597954	602245	605583	610352	616074	620842	623703	626087	628471	630379	631332	634193	636101
	0.1	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38	9.39	9.41	9.42	9.44	9.45	9.46	9.47	9.47	9.47	9.48	9.49
	0.05	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.46	19.46	19.47	19.48	19.48	19.49	19.49
2	0.025	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39	39.40	39.41	39.43	39.45	39.46	39.46	39.47	39.48	39.48	39.49	39.50
	0.01	98.50	99.00	99.16	99.25	99.30	99.33	99.36	99.38	99.39	99.40	99.42	99.43	99.45	99.46	99.47	99.48	99.48	99.48	99.49	99.50
	0.001	998.38	998.84	999.31	999.31	999.31	999.31	999.31	999.31	999.31	999.31	999.31	999.31	999.31	999.31	999.31	999.31	999.31	999.31	999.31	999.31
	0.1	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24	5.23	5.22	5.20	5.18	5.17	5.17	5.16	5.15	5.15	5.14	5.13
	0.05	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.63	8.62	8.59	8.58	8.57	8.55	8.53
3	0.025	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47	14.42	14.34	14.25	14.17	14.12	14.08	14.04	14.01	13.99	13.95	13.91
	0.01	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.34	27.23	27.05	26.87	26.69	26.58	26.50	26.41	26.35	26.32	26.22	26.14
	0.001	167.06	148.49	141.10	137.08	134.58	132.83	131.61	130.62	129.86	129.22	128.32	127.36	126.43	125.84	125.44	124.97	124.68	124.45	123.98	123.52
	0.1	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94	3.92	3.90	3.87	3.84	3.83	3.82	3.80	3.80	3.79	3.78	3.76
	0.05	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.70	5.69	5.66	5.63
4	0.025	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.75	8.66	8.56	8.50	8.46	8.41	8.38	8.36	8.31	8.26
	0.01	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.37	14.20	14.02	13.91	13.84	13.75	13.69	13.65	13.56	13.47
	0.001	74.13	61.25	56.17	53.43	51.72	50.52	49.65	49.00	48.47	48.05	47.41	46.76	46.10	45.69	45.43	45.08	44.88	44.75	44.40	44.09
	0.1	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32	3.30	3.27	3.24	3.21	3.19	3.17	3.16	3.15	3.14	3.12	3.11
	0.05	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.52	4.50	4.46	4.44	4.43	4.40	4.37
5	0.025	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.52	6.43	6.33	6.27	6.23	6.18	6.14	6.12	6.07	6.02
	0.01	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.89	9.72	9.55	9.45	9.38	9.29	9.24	9.20	9.11	9.03
	0.001	47.18	37.12	33.20	31.08	29.75	28.83	28.17	27.65	27.24	26.91	26.42	25.91	25.39	25.08	24.87	24.60	24.44	24.33	24.06	23.82
	0.1	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96	2.94	2.90	2.87	2.84	2.81	2.80	2.78	2.77	2.76	2.74	2.72
	0.05	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.83	3.81	3.77	3.75	3.74	3.70	3.67
6	0.025	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.37	5.27	5.17	5.11	5.07	5.01	4.98	4.96	4.90	4.86
	0.01	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56	7.40	7.30	7.23	7.14	7.09	7.06	6.97	6.89
	0.001	35.51	27.00	23.71	21.92	20.80	20.03	19.46	19.03	18.69	18.41	17.99	17.56	17.12	16.85	16.67	16.44	16.31	16.21	15.98	15.77
	0.1	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72	2.70	2.67	2.63	2.59	2.57	2.56	2.54	2.52	2.51	2.49	2.47
	0.05	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.40	3.38	3.34	3.32	3.30	3.27	3.23
7	0.025	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.67	4.57	4.47	4.40	4.36	4.31	4.28	4.25	4.20	4.15
	0.01	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.31	6.16	6.06	5.99	5.91	5.86	5.82	5.74	5.66
	0.001	29.25	21.69	18.77	17.20	16.21	15.52	15.02	14.63	14.33	14.08	13.71	13.32	12.93	12.69	12.53	12.33	12.20	12.12	11.91	11.72
	0.1	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.56	2.54	2.50	2.46	2.42	2.40	2.38	2.36	2.35	2.34	2.32	2.30
	0.05	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.11	3.08	3.04	3.02	3.01	2.97	2.93
8	0.025	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.20	4.10	4.00	3.94	3.89	3.84	3.81	3.78	3.73	3.68
	0.01	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.52	5.36	5.26	5.20	5.12	5.07	5.03	4.95	4.87
	0.001	25.41	18.49	15.83	14.39	13.48	12.86	12.40	12.05	11.77	11.54	11.19	10.84	10.48	10.26	10.11	9.92	9.80	9.73	9.53	9.36
	0.1	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44	2.42	2.38	2.34	2.30	2.27	2.25	2.23	2.22	2.21	2.18	2.16
	0.05	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.89	2.86	2.83	2.80	2.79	2.75	2.71
9	0.025	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.87	3.77	3.67	3.60	3.56	3.51	3.47	3.45	3.39	3.34
	0.01	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	4.96	4.81	4.71	4.65	4.57	4.52	4.48	4.40	4.32
	0.001	22.86	16.39	13.90	12.56	11.71	11.13	10.70	10.37	10.11	9.89	9.57	9.24	8.90	8.69	8.55	8.37	8.26	8.19	8.00	7.84
	0.1	3.29	2.92	2.73	2.61	2.52	2.46	2.41	2.38	2.35	2.32	2.28	2.24	2.20	2.17	2.16	2.13	2.12	2.11	2.08	2.06
	0.05	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.73	2.70	2.66	2.64	2.62	2.58	2.54
10	0.025	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.62	3.52	3.42	3.35	3.31	3.26	3.22	3.20	3.14	3.09
	0.01	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56	4.41	4.31	4.25	4.17	4.12	4.08	4.00	3.92
	0.001	21.04	14.90	12.55	11.28	10.48	9.93	9.52	9.20	8.96	8.75	8.45	8.13	7.80	7.60	7.47	7.30	7.19	7.12	6.94	6.78
	0.1	3.23	2.86	2.66	2.54	2.45	2.39	2.34	2.30	2.27	2.25	2.21	2.17	2.12	2.10	2.08	2.05	2.04	2.03	2.00	1.98
	0.05	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.60	2.57	2.53	2.51	2.49	2.45	2.41
11	0.025	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59	3.53	3.43	3.33	3.23	3.16	3.12	3.06	3.03	3.00	2.94	2.89
	0.01	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.40	4.25	4.10	4.01	3.94	3.86	3.81	3.78	3.69	3.61
	0.001	19.69	13.81	11.56	10.35	9.58	9.05	8.65	8.35	8.12	7.92	7.63	7.32	7.01	6.81	6.68	6.52	6.42	6.35	6.18	6.02
	0.1	3.18	2.81	2.61	2.48	2.39	2.33	2.28	2.24	2.21	2.19	2.15	2.10	2.06	2.03	2.01	1.99	1.97	1.96	1.93	1.91
	0.05	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.50	2.47	2.43	2.40	2.38	2.34	2.30
12	0.025	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.28	3.18	3.07	3.01	2.96	2.91	2.87	2.85	2.79	2.73
	0.01	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01	3.86	3.76	3.70	3.62	3.57	3.54	3.45	3.37
	0.001	18.64	12.97	10.80	9.63	8.89	8.38	8.00	7.71	7.48	7.29	7.00	6.71	6.40	6.22	6.09	5.93	5.83	5.76	5.59	5.44

Table : F Distribution

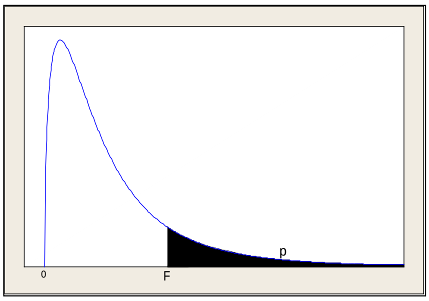


Table of Critical Values Table entry is the point F with probability p lying above it.

Degrees of Freedom in the Denominator

DFD	p	1	2	3	4	5	6	7	8	9	10	12	15	20	25	30	40	50	60	120	1000
		Degrees of Freedom in the Numerator																			
	0.1	3.14	2.76	2.56	2.43	2.35	2.28	2.23	2.20	2.16	2.14	2.10	2.05	2.01	1.98	1.96	1.93	1.92	1.90	1.88	1.85
	0.05	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.41	2.38	2.34	2.31	2.30	2.25	2.21
13	0.025	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31	3.25	3.15	3.05	2.95	2.88	2.84	2.78	2.74	2.72	2.66	2.60
	0.01	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	3.96	3.82	3.66	3.57	3.51	3.43	3.38	3.34	3.25	3.18
	0.001	17.82	12.31	10.21	9.07	8.35	7.86	7.49	7.21	6.98	6.80	6.52	6.23	5.93	5.75	5.63	5.47	5.37	5.30	5.14	4.99
	0.1	3.10	2.73	2.52	2.39	2.31	2.24	2.19	2.15	2.12	2.10	2.05	2.01	1.96	1.93	1.91	1.89	1.87	1.86	1.83	1.80
	0.05	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.34	2.31	2.27	2.24	2.22	2.18	2.14
14	0.025	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21	3.15	3.05	2.95	2.84	2.78	2.73	2.67	2.64	2.61	2.55	2.50
	0.01	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03	3.94	3.80	3.66	3.51	3.41	3.35	3.27	3.22	3.18	3.09	3.02
	0.001	17.14	11.78	9.73	8.62	7.92	7.44	7.08	6.80	6.58	6.40	6.13	5.85	5.56	5.38	5.25	5.10	5.00	4.94	4.77	4.62
	0.1	3.07	2.70	2.49	2.36	2.27	2.21	2.16	2.12	2.09	2.06	2.02	1.97	1.92	1.89	1.87	1.85	1.83	1.82	1.79	1.76
	0.05	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.28	2.25	2.20	2.18	2.16	2.11	2.07
15	0.025	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	2.96	2.86	2.76	2.69	2.64	2.59	2.55	2.52	2.46	2.40
	0.01	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.52	3.37	3.28	3.21	3.13	3.08	3.05	2.96	2.88
	0.001	16.59	11.34	9.34	8.25	7.57	7.09	6.74	6.47	6.26	6.08	5.81	5.54	5.25	5.07	4.95	4.80	4.70	4.64	4.48	4.33
	0.1	3.05	2.67	2.46	2.33	2.24	2.18	2.13	2.09	2.06	2.03	1.99	1.94	1.89	1.86	1.84	1.81	1.79	1.78	1.75	1.72
	0.05	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.23	2.19	2.15	2.12	2.11	2.06	2.02
16	0.025	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05	2.99	2.89	2.79	2.68	2.61	2.57	2.51	2.47	2.45	2.38	2.32
	0.01	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.55	3.41	3.26	3.16	3.10	3.02	2.97	2.93	2.84	2.76
	0.001	16.12	10.97	9.01	7.94	7.27	6.80	6.46	6.20	5.98	5.81	5.55	5.27	4.99	4.82	4.70	4.54	4.45	4.39	4.23	4.08
	0.1	3.03	2.64	2.44	2.31	2.22	2.15	2.10	2.06	2.03	2.00	1.96	1.91	1.86	1.83	1.81	1.78	1.76	1.75	1.72	1.69
	0.05	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.18	2.15	2.10	2.08	2.06	2.01	1.97
17	0.025	6.04	4.62	4.01	3.66	3.44	3.28	3.16	3.06	2.98	2.92	2.82	2.72	2.62	2.55	2.50	2.44	2.41	2.38	2.32	2.26
	0.01	8.40	6.11	5.19	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.46	3.31	3.16	3.07	3.00	2.92	2.87	2.83	2.75	2.66
	0.001	15.72	10.66	8.73	7.68	7.02	6.56	6.22	5.96	5.75	5.58	5.32	5.05	4.78	4.60	4.48	4.33	4.24	4.18	4.02	3.87
	0.1	3.01	2.62	2.42	2.29	2.20	2.13	2.08	2.04	2.00	1.98	1.93	1.89	1.84	1.80	1.78	1.75	1.74	1.72	1.69	1.66
	0.05	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.14	2.11	2.06	2.04	2.02	1.97	1.92
18	0.025	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93	2.87	2.77	2.67	2.56	2.49	2.44	2.38	2.35	2.32	2.26	2.20
	0.01	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.37	3.23	3.08	2.98	2.92	2.84	2.78	2.75	2.66	2.58
	0.001	15.38	10.39	8.49	7.46	6.81	6.35	6.02	5.76	5.56	5.39	5.13	4.87	4.59	4.42	4.30	4.15	4.06	4.00	3.84	3.69
	0.1	2.99	2.61	2.40	2.27	2.18	2.11	2.06	2.02	1.98	1.96	1.91	1.86	1.81	1.78	1.76	1.73	1.71	1.70	1.67	1.64
	0.05	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	2.00	1.98	1.93	1.88
19	0.025	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88	2.82	2.72	2.62	2.51	2.44	2.39	2.33	2.30	2.27	2.20	2.14
	0.01	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.30	3.15	3.00	2.91	2.84	2.76	2.71	2.67	2.58	2.50
	0.001	15.08	10.16	8.28	7.27	6.62	6.18	5.85	5.59	5.39	5.22	4.97	4.70	4.43	4.26	4.14	3.99	3.90	3.84	3.68	3.53
	0.1	2.97	2.59	2.38	2.25	2.16	2.09	2.04	2.00	1.96	1.94	1.89	1.84	1.79	1.76	1.74	1.71	1.69	1.68	1.64	1.61
	0.05	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.07	2.04	1.99	1.97	1.95	1.90	1.85
20	0.025	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84	2.77	2.68	2.57	2.46	2.40	2.35	2.29	2.25	2.22	2.16	2.09
	0.01	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23	3.09	2.94	2.84	2.78	2.69	2.64	2.61	2.52	2.43
	0.001	14.82	9.95	8.10	7.10	6.46	6.02	5.69	5.44	5.24	5.08	4.82	4.56	4.29	4.12	4.00	3.86	3.77	3.70	3.54	3.40
	0.1	2.96	2.57	2.36	2.23	2.14	2.08	2.02	1.98	1.95	1.92	1.87	1.83	1.78	1.74	1.72	1.69	1.67	1.66	1.62	1.59
	0.05	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.94	1.92	1.87	1.82
21	0.025	5.83	4.42	3.82	3.48	3.25	3.09	2.97	2.87	2.80	2.73	2.64	2.53	2.42	2.36	2.31	2.25	2.21	2.18	2.11	2.05
	0.01	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.17	3.03	2.88	2.79	2.72	2.64	2.58	2.55	2.46	2.37
	0.001	14.59	9.77	7.94	6.95	6.32	5.88	5.56	5.31	5.11	4.95	4.70	4.44	4.17	4.00	3.88	3.74	3.64	3.58	3.42	3.28
	0.1	2.95	2.56	2.35	2.22	2.13	2.06	2.01	1.97	1.93	1.90	1.86	1.81	1.76	1.73	1.70	1.67	1.65	1.64	1.60	1.57
	0.05	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.02	1.98	1.94	1.91	1.89	1.84	1.79
22	0.025	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76	2.70	2.60	2.50	2.39	2.32	2.27	2.21	2.17	2.14	2.08	2.01
	0.01	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.12	2.98	2.83	2.73	2.67	2.58	2.53	2.50	2.40	2.32
	0.001	14.38	9.61	7.80	6.81	6.19	5.76	5.44	5.19	4.99	4.83	4.58	4.33	4.06	3.89	3.78	3.63	3.54	3.48	3.32	3.17
	0.1	2.94	2.55	2.34	2.21	2.11	2.05	1.99	1.95	1.92	1.89	1.84	1.80	1.74	1.71	1.69	1.66	1.64	1.62	1.59	1.55
	0.05	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.00	1.96	1.91	1.88	1.86	1.81	1.76
23	0.025	5.75	4.35	3.75	3.41	3.18	3.02	2.90	2.81	2.73	2.67	2.57	2.47	2.36	2.29	2.24	2.18	2.14	2.11	2.04	1.98
	0.01	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.07	2.93	2.78	2.69	2.62	2.54	2.48	2.45	2.35	2.27
	0.001	14.20	9.47	7.67	6.70	6.08	5.65	5.33	5.09	4.89	4.73	4.48	4.23	3.96	3.79	3.68	3.53	3.44	3.38	3.22	3.08
	0.1	2.93	2.54	2.33	2.19	2.10	2.04	1.98	1.94	1.91	1.88	1.83	1.78	1.73	1.70	1.67	1.64	1.62	1.61	1.57	1.54
	0.05	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.97	1.94	1.89	1.86	1.84	1.79	1.74
24	0.025	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70	2.64	2.54	2.44	2.33	2.26	2.21	2.15	2.11	2.08	2.01	1.94
	0.01	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.03	2.89	2.74	2.64	2.58	2.49	2.44	2.40	2.31	2.22
	0.001	14.03	9.34	7.55	6.59	5.98	5.55	5.24	4.99	4.80	4.64	4.39	4.14	3.87	3.71	3.59	3.45	3.36	3.29	3.14	2.99

Table : F Distribution

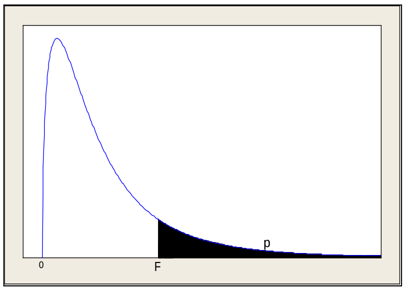


Table of Critical Values Table entry is the point F with probability p lying above it.

Degrees of Freedom in the Denominator

DFD	p	Degrees of Freedom in the Numerator 1	2	3	4	5	6	7	8	9	10	12	15	20	25	30	40	50	60	120	1000
	0.1	2.92	2.53	2.32	2.18	2.09	2.02	1.97	1.93	1.89	1.87	1.82	1.77	1.72	1.68	1.66	1.63	1.61	1.59	1.56	1.52
	0.05	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.84	1.82	1.77	1.72
25	0.025	5.69	4.29	3.69	3.35	3.13	2.97	2.85	2.75	2.68	2.61	2.51	2.41	2.30	2.23	2.18	2.12	2.08	2.05	1.98	1.91
	0.01	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22	3.13	2.99	2.85	2.70	2.60	2.54	2.45	2.40	2.36	2.27	2.18
	0.001	13.88	9.22	7.45	6.49	5.89	5.46	5.15	4.91	4.71	4.56	4.31	4.06	3.79	3.63	3.52	3.37	3.28	3.22	3.06	2.91
	0.1	2.91	2.52	2.31	2.17	2.08	2.01	1.96	1.92	1.88	1.86	1.81	1.76	1.71	1.67	1.65	1.61	1.59	1.58	1.54	1.51
	0.05	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.94	1.90	1.85	1.82	1.80	1.75	1.70
26	0.025	5.66	4.27	3.67	3.33	3.10	2.94	2.82	2.73	2.65	2.59	2.49	2.39	2.28	2.21	2.16	2.09	2.05	2.03	1.95	1.89
	0.01	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09	2.96	2.81	2.66	2.57	2.50	2.42	2.36	2.33	2.23	2.14
	0.001	13.74	9.12	7.36	6.41	5.80	5.38	5.07	4.83	4.64	4.48	4.24	3.99	3.72	3.56	3.44	3.30	3.21	3.15	2.99	2.84
	0.1	2.90	2.51	2.30	2.17	2.07	2.00	1.95	1.91	1.87	1.85	1.80	1.75	1.70	1.66	1.64	1.60	1.58	1.57	1.53	1.50
	0.05	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.92	1.88	1.84	1.81	1.79	1.73	1.68
27	0.025	5.63	4.24	3.65	3.31	3.08	2.92	2.80	2.71	2.63	2.57	2.47	2.36	2.25	2.18	2.13	2.07	2.03	2.00	1.93	1.86
	0.01	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15	3.06	2.93	2.78	2.63	2.54	2.47	2.38	2.33	2.29	2.20	2.11
	0.001	13.61	9.02	7.27	6.33	5.73	5.31	5.00	4.76	4.57	4.41	4.17	3.92	3.66	3.49	3.38	3.23	3.14	3.08	2.92	2.78
	0.1	2.89	2.50	2.29	2.16	2.06	2.00	1.94	1.90	1.87	1.84	1.79	1.74	1.69	1.65	1.63	1.59	1.57	1.56	1.52	1.48
	0.05	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.79	1.77	1.71	1.66
28	0.025	5.61	4.22	3.63	3.29	3.06	2.90	2.78	2.69	2.61	2.55	2.45	2.34	2.23	2.16	2.11	2.05	2.01	1.98	1.91	1.84
	0.01	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.90	2.75	2.60	2.51	2.44	2.35	2.30	2.26	2.17	2.08
	0.001	13.50	8.93	7.19	6.25	5.66	5.24	4.93	4.69	4.50	4.35	4.11	3.86	3.60	3.43	3.32	3.18	3.09	3.02	2.86	2.72
	0.1	2.89	2.50	2.28	2.15	2.06	1.99	1.93	1.89	1.86	1.83	1.78	1.73	1.68	1.64	1.62	1.58	1.56	1.55	1.51	1.47
	0.05	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.89	1.85	1.81	1.77	1.75	1.70	1.65
29	0.025	5.59	4.20	3.61	3.27	3.04	2.88	2.76	2.67	2.59	2.53	2.43	2.32	2.21	2.14	2.09	2.03	1.99	1.96	1.89	1.82
	0.01	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09	3.00	2.87	2.73	2.57	2.48	2.41	2.33	2.27	2.23	2.14	2.05
	0.001	13.39	8.85	7.12	6.19	5.59	5.18	4.87	4.64	4.45	4.29	4.05	3.80	3.54	3.38	3.27	3.12	3.03	2.97	2.81	2.66
	0.1	2.88	2.49	2.28	2.14	2.05	1.98	1.93	1.88	1.85	1.82	1.77	1.72	1.67	1.63	1.61	1.57	1.55	1.54	1.50	1.46
	0.05	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.88	1.84	1.79	1.76	1.74	1.68	1.63
30	0.025	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57	2.51	2.41	2.31	2.20	2.12	2.07	2.01	1.97	1.94	1.87	1.80
	0.01	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.84	2.70	2.55	2.45	2.39	2.30	2.25	2.21	2.11	2.02
	0.001	13.29	8.77	7.05	6.12	5.53	5.12	4.82	4.58	4.39	4.24	4.00	3.75	3.49	3.33	3.22	3.07	2.98	2.92	2.76	2.61
	0.1	2.84	2.44	2.23	2.09	2.00	1.93	1.87	1.83	1.79	1.76	1.71	1.66	1.61	1.57	1.54	1.51	1.48	1.47	1.42	1.38
	0.05	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.78	1.74	1.69	1.66	1.64	1.58	1.52
40	0.025	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45	2.39	2.29	2.18	2.07	1.99	1.94	1.88	1.83	1.80	1.72	1.65
	0.01	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.66	2.52	2.37	2.27	2.20	2.11	2.06	2.02	1.92	1.82
	0.001	12.61	8.25	6.59	5.70	5.13	4.73	4.44	4.21	4.02	3.87	3.64	3.40	3.15	2.98	2.87	2.73	2.64	2.57	2.41	2.25
	0.1	2.81	2.41	2.20	2.06	1.97	1.90	1.84	1.80	1.76	1.73	1.68	1.63	1.57	1.53	1.50	1.46	1.44	1.42	1.38	1.33
	0.05	4.03	3.18	2.79	2.56	2.40	2.29	2.20	2.13	2.07	2.03	1.95	1.87	1.78	1.73	1.69	1.63	1.60	1.58	1.51	1.45
50	0.025	5.34	3.97	3.39	3.05	2.83	2.67	2.55	2.46	2.38	2.32	2.22	2.11	1.99	1.92	1.87	1.80	1.75	1.72	1.64	1.56
	0.01	7.17	5.06	4.20	3.72	3.41	3.19	3.02	2.89	2.78	2.70	2.56	2.42	2.27	2.17	2.10	2.01	1.95	1.91	1.80	1.70
	0.001	12.22	7.96	6.34	5.46	4.90	4.51	4.22	4.00	3.82	3.67	3.44	3.20	2.95	2.79	2.68	2.53	2.44	2.38	2.21	2.05
	0.1	2.79	2.39	2.18	2.04	1.95	1.87	1.82	1.77	1.74	1.71	1.66	1.60	1.54	1.50	1.48	1.44	1.41	1.40	1.35	1.30
	0.05	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.69	1.65	1.59	1.56	1.53	1.47	1.40
60	0.025	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33	2.27	2.17	2.06	1.94	1.87	1.82	1.74	1.70	1.67	1.58	1.49
	0.01	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.50	2.35	2.20	2.10	2.03	1.94	1.88	1.84	1.73	1.62
	0.001	11.97	7.77	6.17	5.31	4.76	4.37	4.09	3.86	3.69	3.54	3.32	3.08	2.83	2.67	2.55	2.41	2.32	2.25	2.08	1.92
	0.1	2.76	2.36	2.14	2.00	1.91	1.83	1.78	1.73	1.69	1.66	1.61	1.56	1.49	1.45	1.42	1.38	1.35	1.34	1.28	1.22
	0.05	3.94	3.09	2.70	2.46	2.31	2.19	2.10	2.03	1.97	1.93	1.85	1.77	1.68	1.62	1.57	1.52	1.48	1.45	1.38	1.30
100	0.025	5.18	3.83	3.25	2.92	2.70	2.54	2.42	2.32	2.24	2.18	2.08	1.97	1.85	1.77	1.71	1.64	1.59	1.56	1.46	1.36
	0.01	6.90	4.82	3.98	3.51	3.21	2.99	2.82	2.69	2.59	2.50	2.37	2.22	2.07	1.97	1.89	1.80	1.74	1.69	1.57	1.45
	0.001	11.50	7.41	5.86	5.02	4.48	4.11	3.83	3.61	3.44	3.30	3.07	2.84	2.59	2.43	2.32	2.17	2.08	2.01	1.83	1.64
	0.1	2.73	2.33	2.11	1.97	1.88	1.80	1.75	1.70	1.66	1.63	1.58	1.52	1.46	1.41	1.38	1.34	1.31	1.29	1.23	1.16
	0.05	3.89	3.04	2.65	2.42	2.26	2.14	2.06	1.98	1.93	1.88	1.80	1.72	1.62	1.56	1.52	1.46	1.41	1.39	1.30	1.21
200	0.025	5.10	3.76	3.18	2.85	2.63	2.47	2.35	2.26	2.18	2.11	2.01	1.90	1.78	1.70	1.64	1.56	1.51	1.47	1.37	1.25
	0.01	6.76	4.71	3.88	3.41	3.11	2.89	2.73	2.60	2.50	2.41	2.27	2.13	1.97	1.87	1.79	1.69	1.63	1.58	1.45	1.30
	0.001	11.15	7.15	5.63	4.81	4.29	3.92	3.65	3.43	3.26	3.12	2.90	2.67	2.42	2.26	2.15	2.00	1.90	1.83	1.64	1.43
	0.1	2.71	2.31	2.09	1.95	1.85	1.78	1.72	1.68	1.64	1.61	1.55	1.49	1.43	1.38	1.35	1.30	1.27	1.25	1.18	1.08
	0.05	3.85	3.00	2.61	2.38	2.22	2.11	2.02	1.95	1.89	1.84	1.76	1.68	1.58	1.52	1.47	1.41	1.36	1.33	1.24	1.11
1000	0.025	5.04	3.70	3.13	2.80	2.58	2.42	2.30	2.20	2.13	2.06	1.96	1.85	1.72	1.64	1.58	1.50	1.45	1.41	1.29	1.13
	0.01	6.66	4.63	3.80	3.34	3.04	2.82	2.66	2.53	2.43	2.34	2.20	2.06	1.90	1.79	1.72	1.61	1.54	1.50	1.35	1.16
	0.001	10.89	6.96	5.46	4.65	4.14	3.78	3.51	3.30	3.13	2.99	2.77	2.54	2.30	2.14	2.02	1.87	1.77	1.69	1.49	1.22

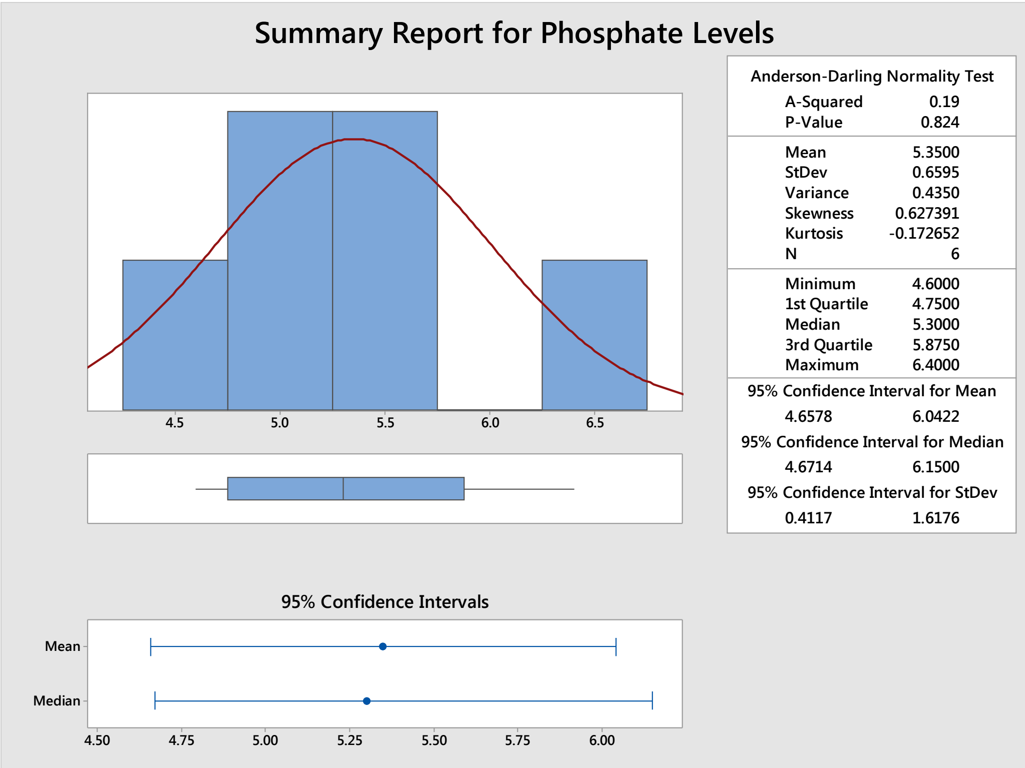
6.9 Minitab Output for Examples

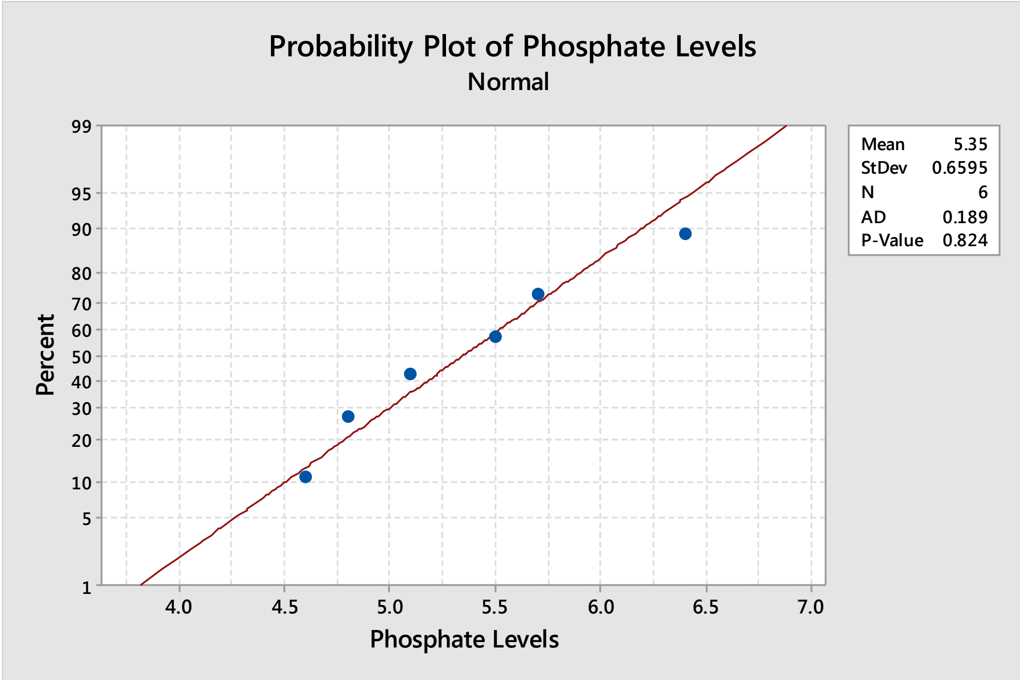
1. The level of various substances in the blood of kidney dialysis patients is of concern because kidney failure and dialysis lead to nutritional problems. A researcher performed blood tests on several dialysis patients on six consecutive clinic visits. The variable measured was the phosphate level in blood. Phosphate levels in a single patient tend to vary normally over time. The data on one patient in mg/dl of blood are given below.

   5.5 5.1 4.6 4.8 5.7 6.4

   Is the mean phosphate level greater than 5 mg/dl of blood?

Solutions





One-Sample T: Phosphate Levels

Test of μ = 5 vs > 5

Variable	N	Mean	StDev	SE Mean	95% Lower Bound	T	P
Phosphate Levels	6	5.350	0.660	0.269	4.807	1.30	0.125

2. In a randomised comparative experiment of dietary calcium on blood pressure, 54 healthy white males were divided at random into two groups. One group received calcium; the other, a placebo. At the beginning of the study, the researchers measured many variables on the subjects. The report on the study gives the mean seated systolic blood pressure of the 27 members of the placebo group as 114.9 mmHg and the standard deviation as 9.3 mmHg.

(i) Does the mean blood pressure equal 120 mmHg?

(ii) Does the standard deviation of the blood pressure in the placebo group equal 12 mmHg?

Solution

(i) **One-Sample T**

```
Test of μ = 120 vs ≠ 120


 N    Mean  StDev  SE Mean       95% CI            T      P
27  114.90   9.30     1.79  (111.22, 118.58)  -2.85  0.008
```

(ii) **Test and CI for One Variance**

```
Method
Null hypothesis         σ = 12
Alternative hypothesis  σ ≠ 12

The chi-square method is only for the normal distribution.
The Bonett method cannot be calculated with summarized data.

Statistics

 N  StDev  Variance
27   9.30      86.5

95% Confidence Intervals
                CI for        CI for
Method          StDev        Variance
Chi-Square  (7.32, 12.75)  (53.6, 162.4)

Tests
             Test
Method      Statistic  DF  P-Value
Chi-Square      15.62  26    0.110
```

Example

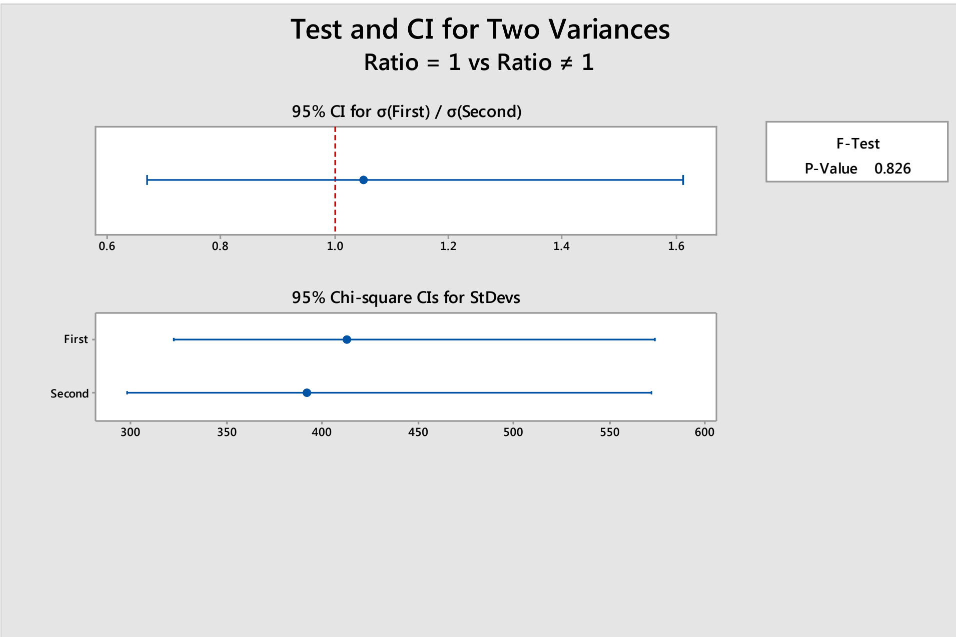
A bank compares two proposals to increase the amount that its credit card customers charge on their cards. Proposal A offers to eliminate the annual fee for customers who charge £2400 or more during the year. Proposal B offers a small percent of the total charged as a cash rebate at the end of the year. the bank offers each proposal to a sample of 45 of its existing credit card customers.

At the end of the year, the total amount charged by each customer is recorded. The summary statistics follow :

Group	n	$\bar{x}$	s
A	20	£1987	£392
B	25	£2056	£413

Do the data show a significant difference between the mean amounts charged by customers offered the two plans?

Solution



Test and CI for Two Variances

Method

Null hypothesis	σ(First) / σ(Second) = 1
Alternative hypothesis	σ(First) / σ(Second) ≠ 1
Significance level	α = 0.05

F method was used. This method is accurate for normal data only.

Statistics

Sample	N	StDev	Variance	95% CI for StDevs
First	25	413.000	170569.000	(322.482, 574.546)
Second	20	392.000	153664.000	(298.112, 572.544)

Ratio of standard deviations = 1.054
Ratio of variances = 1.110

95% Confidence Intervals

Method	CI for StDev Ratio	CI for Variance Ratio
F	(0.673, 1.613)	(0.453, 2.603)

Tests

Method	DF1	DF2	Test Statistic	P-Value
F	24	19	1.11	0.826

Two-Sample T-Test and CI

Sample	N	Mean	StDev	SE Mean
1	20	1987	392	88
2	25	2056	413	83

Difference = μ (1) - μ (2)
Estimate for difference: -69
95% CI for difference: (-313, 175)
T-Test of difference = 0 (vs ≠): T-Value = -0.57 P-Value = 0.572 DF = 43
Both use Pooled StDev = 403.8556

Example

The design of controls and instruments has a large effect on how easily people can use them. A student project investigated this effect by asking 25 right handed students to turn a knob, with their right hands, that moved an indicator by screw action. There were two identical instruments, one with a right-hand thread (turns clockwise) and the other with a left-hand thread (turn anti-clockwise). The table gives the times required (in seconds to move the indicator a fixed distance.

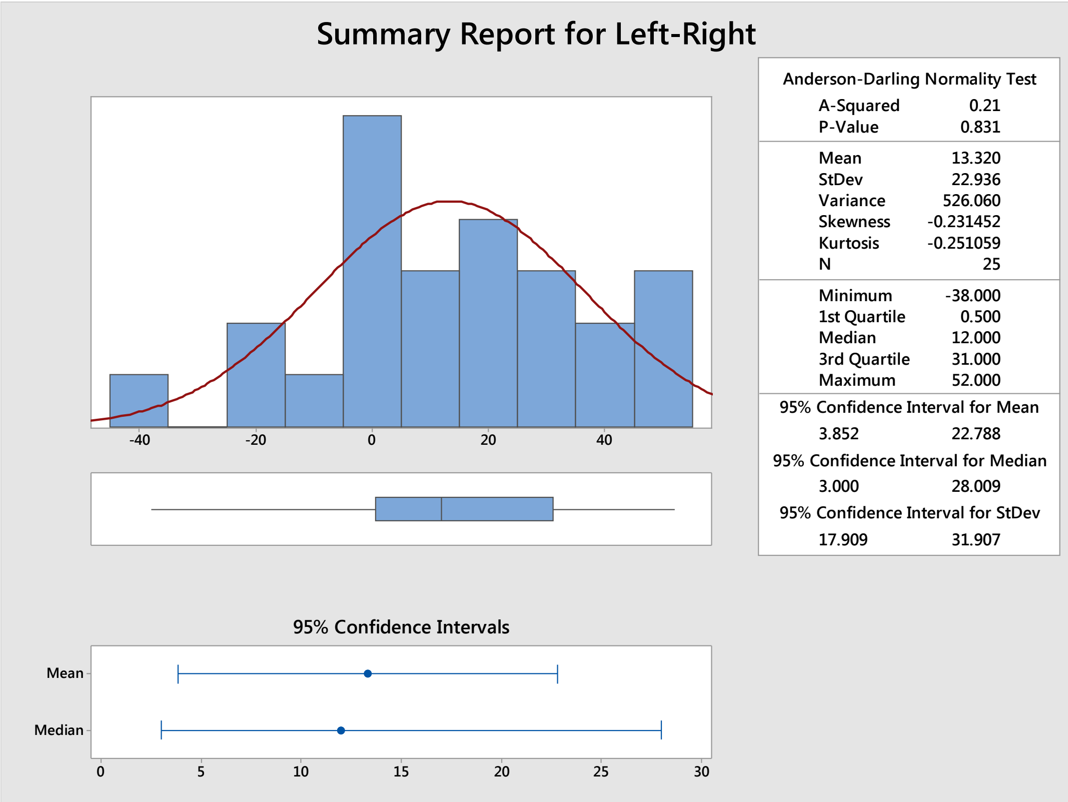
Subject	Right thread	Left thread
1	113	137
2	105	105
3	130	133
4	101	108
5	138	115
6	118	170
7	87	103
8	116	145
9	75	78
10	96	107
11	122	84
12	103	148
13	116	147
14	107	87
15	118	166
16	103	146
17	111	123
18	104	135
19	111	112
20	89	93
21	78	76
22	100	116
23	89	78
24	85	101
25	88	123

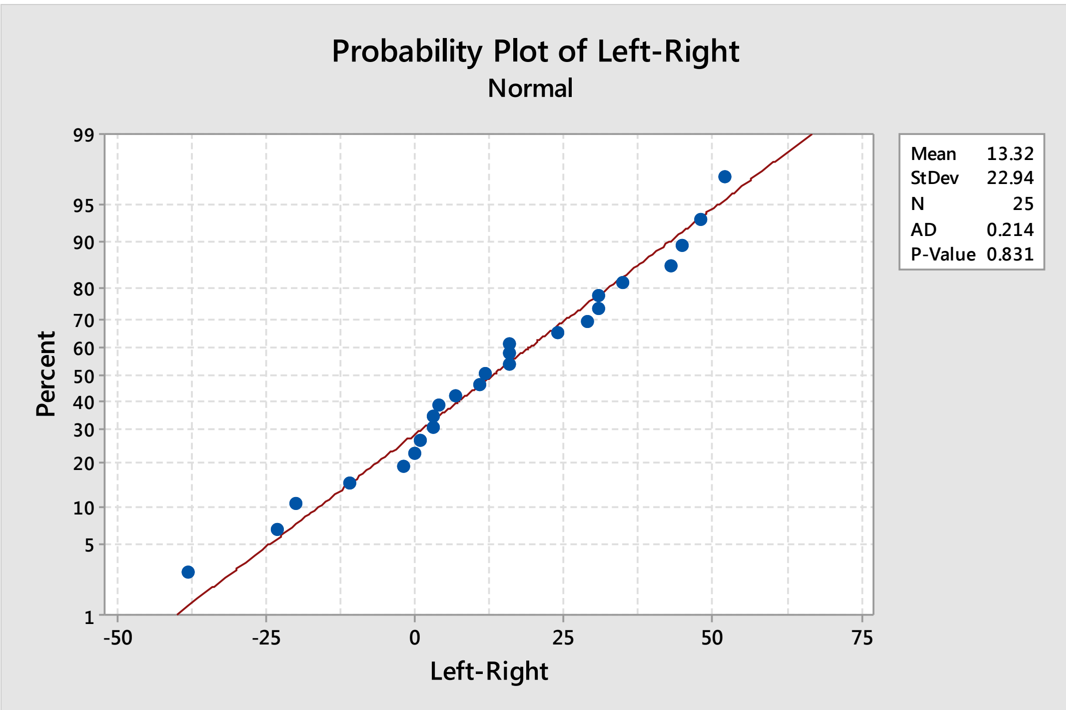
The project hoped to show that right-handed people find right-hand threads easier to use. Does the data support this view?

Solution

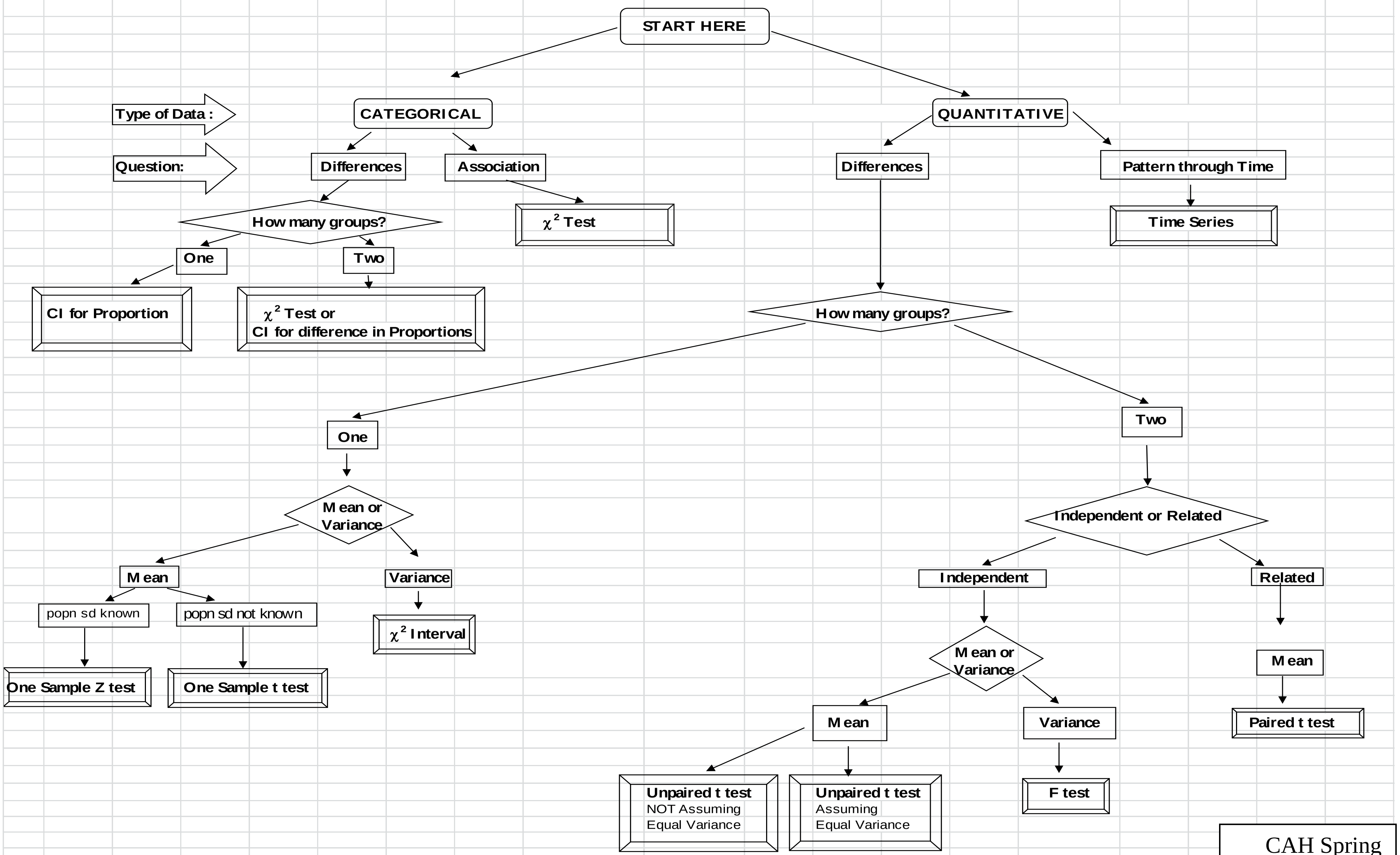
Paired T-Test and CI: Right, Left

```
Paired T for Right - Left

             N    Mean  StDev  SE Mean
Right       25  104.12  15.80     3.16
Left        25  117.44  27.26     5.45
Difference  25  -13.32  22.94     4.59


95% upper bound for mean difference: -5.47
T-Test of mean difference = 0 (vs < 0): T-Value = -2.90  P-Value = 0.004
```


Summary Report for Left-Right
Anderson-Darling Normality Test
A-Squared 0.21
P-Value 0.831
Mean 13.320
StDev 22.936
Variance 526.060
Skewness -0.231452
Kurtosis -0.251059
N 25
Minimum -38.000
1st Quartile 0.500
Median 12.000
3rd Quartile 31.000
Maximum 52.000
95% Confidence Interval for Mean
3.852 22.788
95% Confidence Interval for Median
3.000 28.009
95% Confidence Interval for StDev
17.909 31.907
-40 -20 0 20 40
95% Confidence Intervals
Mean
Median
0 5 10 15 20 25 30


Probability Plot of Left-Right
Normal
Percent
99 95 90 80 70 60 50 40 30 20 10 5 1
-50 -25 0 25 50 75
Left-Right
Mean 13.32
StDev 22.94
N 25
AD 0.214
P-Value 0.831

Choosing the Appropriate Statistical Test (Part 1)



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