UNIVERSITY OF STIRLING

PDMU9L4
DATA SKILLS

WORKBOOK 2 (of 3)
PATH 3

Computing Science & Mathematics
Faculty of Natural Sciences

Academic Year 16/17

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PATH 3:
KEY MATHEMATICAL SKILLS

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Topic 5: Power on!

On a previous sheet we defined the power of a number:

$$3 \times 3 = 3^2$$

$$7 \times 7 \times 7 \times 7 = 7^4$$

$$10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^6 (=1 \text{ million})$$

and:

$$a^n = a \times a \times a .... \times a$$

n denotes the number of factors of 'a' in the product and is called the *exponent* (or *degree* or *index* or *power*).

a^n is said to be the *nth power* of a. Calculating a^n is called taking the nth power of 'a'. (The number 'a' you are taking the power of is sometimes called the *base*.)

For example:

(i) 3^2 is the second power of 3 (with exponent 2). It is also called the square of 3 or 3 squared.

(ii) 8^3 is the third power of 8 (with exponent 3). It is also called the cube of 8 or 8 cubed.

Worked examples:

Evaluate the following expressions:

(i) $4^2 = 4 \times 4 = 16$

(ii) $9^3 = 9 \times 9 \times 9 = 729$

(iii) $4^2 \times 9^3 = 16 \times 729 = 11664$

(iv) $\dfrac{2^3 \times 5^2}{3^4} = \dfrac{(2 \times 2 \times 2) \times (5 \times 5)}{(3 \times 3 \times 3 \times 3)} = 2.47$

Now try question 1 on Worksheet 4a/1.

Negative Powers:

We have defined positive integer powers. (Note: *positive* means greater than zero, *negative* less than zero; an *integer* is a whole number with no fractional or decimal parts, e.g. $-1, 2, 0, 27$ are integers whereas π, $3/7$, 7.23 are not.)

What about negative integer powers? For example what does 3^{-4} mean?
We define a negative power to be just the reciprocal of the positive power.

For example:

$$3^{-4} = \frac{1}{3^4} = \frac{1}{3 \times 3 \times 3 \times 3} = \frac{1}{81} = 0.0123$$

$$7^{-2} = \frac{1}{7^2} = \frac{1}{7 \times 7} = \frac{1}{49} = 0.0204$$

$$10^{-2} = \frac{1}{10^2} = \frac{1}{100} = 0.01$$

and in general:

$$a^{-n} = \frac{1}{n}$$

where n is a positive integer.

Let's mix up positive and negative powers:

$$4^2 \times 5^{-3} = 4^2 \times \frac{1}{5^3} = \frac{4 \times 4}{5 \times 5 \times 5} = \frac{16}{125} = 0.128$$

$$\frac{8^3 \times 5^{-3}}{3^3 \times 5^2} = \frac{8^3}{3^3 \times 5^2 \times 5^3} = \frac{8 \times 8 \times 8}{3 \times 3 \times 3 \times 5 \times 5 \times 5 \times 5 \times 5} = \frac{512}{84375} = 0.00607$$

Now try question 2 on Worksheet 4a/1.

The First Exponent Law

If there are powers of the same number in power expressions then the following exponent law might prove useful:

First Exponent Law: if you *multiply* powers of the same number then this is equivalent to *adding* exponents.

For example:

$3^2 \times 3^4 = (3 \times 3) \times (3 \times 3 \times 3 \times 3) = 3^6$ (i.e. 6 factors of 3 altogether)

$7^3 \times 7^6 = (7 \times 7 \times 7) \times (7 \times 7 \times 7 \times 7 \times 7 \times 7) = 7^9$ (i.e. 9 factors of 7)

$10^2 \times 10^6 = (10 \times 10) \times (10 \times 10 \times 10 \times 10 \times 10 \times 10) = 10^8$

In general:

$$a^m \times a^n = a^{m+n}$$

In the first example: $a = 3$ and $m = 2$, $n = 4$. So $m + n = 6$.

This law still holds with negative powers, remembering that adding the negative of a number is just subtracting that number. For example:

$$5^4 \times 5^{-2} = \frac{5^4}{5^2} = \frac{5 \times 5 \times 5 \times 5}{5 \times 5} = 5 \times 5 = 5^{4-2} = 5^2 = 25$$

$$10^7 \times 10^{-4} = 10^{7-4} = 10^3 = 1000$$

In general:

$$a^n \times a^{-m} = \frac{a \times a \times a \ldots \times a}{a \times a \times \ldots a} = a^{n-m}$$

where m, n are positive integers.

Worked examples:

Evaluate the following expressions:

$$\frac{3^4 \times 5^2}{3^{-2} \times 5^3} = \frac{3^4 \times 3^2}{5^3 \times 5^{-2}} = \frac{3^{4+2}}{5^{3-2}} = \frac{3^6}{5^1} = \frac{729}{5} = 145.8$$

$$\boxed{\frac{6.72 \times 10^6}{5.12 \times 10^3} = \frac{6.72 \times 10^6 \times 10^{-3}}{5.12} = \frac{6.72 \times 10^3}{5.12} = 1.3125 \times 1000 = 1312.5}$$

Now try question 3 on Worksheet 4a/1.

Application of Powers

Powers are useful when talking about big and small numbers. In normal conversation one uses special words to help us talk about these numbers, but what do these words mean?

1 thousand = 1,000 = $10 \times 10 \times 10 = 10^3$
1 million = 1,000,000 = $10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^6$
1 billion = a thousand million = $10^3 \times 10^6 = 10^9$
1 trillion = a thousand billion = $10^3 \times 10^9 = 10^{12}$

Example 1:
The US Federal Budget Deficit is 1.72 trillion dollars, i.e. \$2,710,000,000,000 = $\$2.71 \times 10^{12}$.
The US population is 250 million, i.e. 250×10^6. So how much per person is the US Government in deficit?

Deficit per citizen =

$$\frac{2.71 \times 10^{12}}{250 \times 10^6} = \frac{2.71 \times 10^{12} \times 10^{-6}}{250} = \frac{2.71 \times 10^6}{250} = \frac{2.71}{250} \times 10^6 = 0.0108 \times 1000 \times 1000 = \$10,800$$

Quite a lot!

Example 2:
The nebula in Andromeda is 15×10^5 light years away.
One light year is how far light travels in one year. In fact:
One light year = 9.46×10^{15} metres.

So how far away from us is this nebula (in metres)?

Answer =
$15 \times 10^5 \times 9.46 \times 10^{15} = 15 \times 9.46 \times 10^5 \times 10^{15} = 142 \times 10^{20}$ metres $= 1.42 \times 10^{22}$ metres

i.e. 14,200,000,000,000,000,000,000 metres
i.e. 14,200,000,000,000,000,000 kilometres.

Quite a way!

How about small numbers?
How much more heavy is a (2 kilo) bag of sugar compared to an electron circling, say, a hydrogen atom?

The electron mass is 9.11×10^{-31} kilos, i.e.

$$0.000000000000000000000000000000911 \text{ kilos}$$

so the ratio is

$$\frac{\text{sugar}}{\text{electron}} = \frac{2}{9.11 \times 10^{-31}} = \frac{2}{9.11} \times 10^{31} = 0.220 \times 10^{31} = 2{,}200{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000.$$

> ***Now try question 4 on Worksheet 5/1***

Scientific Notation

In scientific notation a number is expressed as a power of 10 times a number with one digit to the left of the decimal point. For example:

(i) $342.1 = 3.421 \times 10^2$ (ii) $0.005672 = 5.672 \times 10^{-3}$ (iii) $8.543 \times 10^{-7} = 0.0000008543$

Some calculators and computer languages write 10^n as En to avoid suffixes. So $7.951E6 = 7.951 \times 10^6 = 7951000$ and $3.446E-2 = 3.446 \times 10^{-2} = 0.03446$.

Note: some calculators (like Casio) use a very dangerous notation. If you calculate, for example 326^{10} you will get 1.355738723^{25}. The 25 is NOT an exponent of 1.355738723 but an exponent of 10 which has been left out. In fact, written out properly we have

$$326^{10} = 1.355738723 \times 10^{25} = 1.355738723E25.$$

Appendix

Some useful hints for manipulating expression involving just multiplication and division:

$$\left(\frac{a}{b}\right)\times\left(\frac{c}{d}\right)=\frac{a\times c}{b\times d}$$

(i) For example:

$$\left(\frac{3}{5}\right)\times\left(\frac{12}{7}\right)=\frac{3\times 12}{5\times 7}=\frac{36}{35}=1.029$$

(ii) In a ratio of products we can cancel common factors, for example:

$$\left(\frac{3\times 6}{5\times 6}\right)=\frac{3}{5}$$

but we cannot do this if the x is a +:

$$\left(\frac{3+6}{5+6}\right)\neq\frac{3}{5}$$

(iii) The power of a product is the product of the individual powers:

$(a\times b)^c = a^c \times b^c$

for example:

$(3\times 5)^4 = 3^4 \times 5^4$

(iv) The power of a ratio is the ratio of the individual powers:

$$\left(\frac{a}{b}\right)^c=\frac{a^c}{b^c}$$

for example:

$$\left(\frac{3}{5}\right)^4=\frac{3^4}{5^4}$$

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Topic 5: Power on! – Worksheet 5/1

1. Evaluate the following expressions:

(i) 4^3 $\boxed{64}$

(ii) 7^4 $\boxed{2401}$

(iii) $4^3 \times 7^4$ $\boxed{153664}$

(iv) $\frac{5^2 \times 9^4}{6^3}$ $\boxed{759.4}$

(Do not use the x^y button on your calculator, but you can use the multiplication button!)

2. Evaluate the following expressions:

(i) 4^{-3} $\boxed{\frac{1}{4^3} = \frac{1}{64}}$

(ii) $8^2 \times 4^{-5}$ $\boxed{\frac{8^2}{4^5} = \frac{1}{16}}$

(iii) $\frac{1}{3^{-2}}$ $\boxed{\frac{1}{\frac{1}{3^2}} = 9}$

(iv) $\frac{5^4 \times 4^{-5}}{8^{-1} \times 6^2}$ $\boxed{\frac{\frac{5^4}{4^5 \times 6^2}}{8^1} = 0.14}$

(Do not use the x^y button on your calculator.)

3. Use the first exponent law to simplify and then evaluate:

(i) $4^6 \times 4^{-5}$ $\boxed{4}$

(ii) $4^6 \times 4^{-5} \times 4^7$ $\boxed{4^8 = 65536}$

(iii) $\frac{4^6}{4^3}$ $\boxed{4^6 \times 4^{-3} = 4^3 = 64}$

(iv) $\frac{4^6}{4^{-3}}$ $\boxed{4^9 = 262144}$

(Do not use the x^y button on your calculator.)

4. (i) How many times around the earth's equator is the same distance as the Andromeda Nebula is from the Earth?
(The distance around the Earth on the equator is 25,000 kilometres.)

$\boxed{\frac{142 \times 10^{17}}{25000} = 568 \times 10^{12}}$
5.68×10^{14}

$W = \frac{23}{500}$ cm $H = \frac{23}{500}$ cm $V = 321.000$ cm $L = ?$

$V = H \cdot W \cdot L$ $321000 = \frac{23}{500} \cdot \frac{23}{500} \cdot X$

Power On!

Number of times $\frac{529}{250.000}$ $\frac{529}{250.000}$

$X = 15170.1323.3$ cm

$X = 1517.913233$ Km

(ii) A pipe with square cross-section has a volume of3.21 thousand cubic metres. the width of the pipe is 4.6 centimetres. What is the length of the pipe? (1 centimetre = 10^{-2} metres.)

Answer: 1517 kilometres.

5. *Scientific Notation*
   (a) Express the following numbers in scientific notation:

   (i) 0.0003456 (ii) 37569.21 (iii) 132001987.234

   Answers:

   (i) 3.456×10^{-4} (ii) 3756921×10^{-2} or 3.756921×10^{4} (iii) $132001987234 \times 10^{-3}$ or $1.32001987234 \times 10^{8}$

   (b) Write out the following numbers 'longhand':

   (i) $9.965E-6$ (ii) $5.4373E\,12$ (iii) $6.6667E-9$

   Answers:

   (i) 9.965×10^{-6} 0.000009965 (ii) 5.4373×10^{12} 5437300000000 (iii) 6.6667×10^{-9} 0.0000000066667

6. We defined negative and positive powers of a number. How should we define the zeroth power of a number?

   E.g. what is 3^o, 0.123^o, a^o ? 1

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Topic 6: Boosting Power!

Accompanying the first exponent law is the second exponent law:

Second Exponent Law: if you take the *power* of a power then you *multiply* exponents.

For example:

$(3^2)^4 = (3\times3)\times(3\times3)\times(3\times3)\times(3\times3) = 3^8$ (since there are $8 = 2\times4$ factors of 3 involved)

$(7^4)^3 = (7\times7\times7\times7)\times(7\times7\times7\times7)\times(7\times7\times7\times7) = 7^{12}$ (since there are $12 = 4\times3$ factors of 7 involved)

In general:

$$(a^m)^n = a^{m\times n}$$

This holds even if either m or n or both are negative integers:

$$(3^2)^{-1} = 3^{-2} = 1/9 = 0.111$$
$$(7^{-1})^{-2} = 7^2 = 49$$
$$(3^2)^{-1}(7^{-1})^{-2} = 49/9 = 5.444$$
$$(3^2\times5^3)^{-2} = (9\times125)^{-2} = (1125)^{-2} = 0.0000008 = 8\times10^{-7}$$

Now try Question 1 on Worksheet 4b/1

In calculating powers we often have to calculate powers of a product of powers. In this case it is useful to use one of the properties listed in the Appendix to Topic 4a, namely:

$$(a\times b)^c = a^c\times b^c$$

For example:

$$(3^2\times5^3)^{-2} = (3^2)^{-2}\times(5^3)^{-2} = 3^{-4}\times5^{-6} = 0.01235\times0.000064 = 0.0000008$$

with a identified as $a = 3^2$ and b identified as $b = 5^3$.

Application:
Find the area of a circle if its radius is 8.41×10^{-3} metres.
Answer:

$$\text{Area} = \pi\times r^2 = 3.142\times(8.41\times10^{-3})^2 = 3.142\times(10^{-3})^2 = 222.2\times10^{-6}$$
$$= 0.0002222 \text{ (square metres)}$$
$$= 222.2 \text{ (square millimetres)}$$

Now try question 2 on Worksheet 4b/1

Non-Integer Powers:
We have defined powers of a number when the exponent was integer. What about powers that are not integer? For example, what do we mean by:

$$3^{\frac{1}{2}}\ ?$$

We define this power to be consistent with the second exponent law:

$$\left(3^{\frac{1}{2}}\right)^2 = 3^{\left(2\times\frac{1}{2}\right)} = 3^1 = 3$$

So $3^{\frac{1}{2}}$ is the number which when squared equals 3, i.e. the square root of 3. In fact there are two such roots. (We will talk only about powers that are positive to avoid this ambiguity.) Similarly:

$$4^{\frac{1}{9}}$$

denotes the 9th root of 4, i.e. the positive number which when raised to the 9th power equals 4. In general:

$$a^{\frac{1}{m}}$$

denotes the positive number which when raised to the mth (integer) power equals a. Your calculator will give non-integer powers. Convert $1/m$ into decimals and key this in as the exponent. For example:

$$3^{\frac{1}{2}} = 3^{0.5} = 1.732$$

$$4^{\frac{1}{9}} = 4^{0.111} = 1.167$$

Now try question 3 on Worksheet 4b/1

Non-Integer Powers Again:
What about a general non-integer power? If we can write the exponent as a 'rational' number, i.e. as a ratio of two integers, then the second exponent law suggests the following definition:

$$a^{\frac{m}{n}} = \left(\frac{1}{a^n}\right)^m$$

i.e. as the mth power of the nth root. The calculator will give us the value of the power, keying in the exponent as a decimal:

$$5^{1.5} = \left(5^{\frac{1}{2}}\right)^3 = (2.236)^3 = 11.18 \textit{ or directly}$$

$$15^{-1.4} = 0.0226 \qquad -\frac{7}{5} \qquad \frac{1}{15^{\frac{7}{5}}} \qquad \left(15^{\frac{1}{5}}\right)^7$$

$$3.1416^{3.1416} = 36.46$$

Now try question 4 on Worksheet 4b/1

Putting it all together –

Power Law

The relationship between an animal's body weight (W) and the amount of heat (H) that body produces is given, to a good approximation, by the formula:

$$H = 60 \times W^{0.79}$$

We can use this formula to obtain a table of H values for given W values and therefore plot the graph of this formula (function).

	cassowary	sheep	human	pig	cow
weight (kgs)	25	50	100	200	500
body heat (cals)	762	1318	2280	3945	8132

Looks familiar?

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Topic 6: Boosting Power! – Worksheet 6/1

1. Evaluate the following expressions using the second exponent law:

(i) $(3^4)^2$ ✓ $\boxed{3^8 = 6561}$

(ii) $(3^{-4})^2$ ✓ $\boxed{3^{-8} = \frac{1}{6561}}$

(iii) $(3^4)^{-2}$ ✓ $\boxed{3^{-8} = \frac{1}{6561}}$

(iv) $(3^{-4})^{-2}$ ✓ $\boxed{3^8 = 6561}$

2. Evaluate the following expressions using the exponent laws:

(i) $(2^3 \times 3^2)^3$ ✓ $\boxed{2^9 \times 3^6 = 373248}$

(ii) $(2^{-3} \times 3^2)^{-3}$ ✓ $\boxed{2^9 \times 3^{-6} = \frac{512}{729}}$

(iii) $(2^3 \times 3^{-2})^{-3}$ ✓ $\boxed{2^{-9} \times 3^6 = 1\frac{217}{512}}$

(iv) $(2^{-3} \times 3^{-2})^3$ ✓ $\boxed{2^{-9} \times 3^{-6} = \frac{1}{373248}}$

3. Evaluate the following expressions: (Check your answers by raising to the requisite power.)

(i) $4^{\frac{1}{2}}$ ✓

$4^{0.5} = 2$

(ii) $27^{\frac{1}{3}}$ ✓ $\boxed{27^{0.333} = 3}$

(iii) $(65536)^{\frac{1}{16}}$ ✓ $\boxed{65536^{0.0625} \quad 2}$

(iv) $4^{-\frac{1}{2}}$ ✓ $\boxed{\frac{1}{4^{0.5}} = \frac{1}{2}}$

(v) $(0.125)^{-\frac{1}{3}}$ ✓ $\boxed{0.125^{-0.333} = 2}$

4. Evaluate the following expressions:

(i) $3.52^{0.96}$ ✓ $\boxed{\left(3.52^{\frac{1}{25}}\right)^{24} = 3.35}$

(ii) $7.123^{-1.03}$ ✓ $\boxed{\left(7.123^{\frac{1}{100}}\right)^{-103}}$

$= 0.13$

(iii) $521.13^{-0.25}$ ✓ $\boxed{\left(521.13^{\frac{1}{4}}\right)^{-1} = 0.21}$

5. Complete the following table using the formula $y = 3.25 * x^{-0.56}$. Plot the graph of this function.

✓

x	0.25	1.25	3.75	5.82	7.25
y	7.06	2.87	1.55	1.21	1.07

$$3.25 \times \left(0.25^{\frac{1}{25}}\right)^{-14} = 7.06$$

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Topic 7: Logs and Powers

So what are logs?

Consider the following problem.

Write 100 as a power of 10.	*Answer*: $100 = 10^2$
Write 0.01 as a power of 10.	*Answer*: $0.01 = 10^{-2}$
Write 3.1623 as a power of 10.	*Answer*: $3.1623 = 10^{\frac{1}{2}}$
Write 5 as a power of 10.	*Answer*: $5 = 10^{0.699}$

In general we write:

$$a = 10^p$$

The exponent (power) p is called the log of a to base 10 and written as:

$$p = \log(a) = \log_{10}(a).$$

So:

$$\boxed{a = 10^{\log(a)}} \qquad (1)$$

So to write any (positive) number as a power of 10 all we have to do is find the exponent, i.e. the log of a. This can be found from the calculator.

Since	$\log(100) = 2$	then	$100 = 10^2$
Since	$\log(0.01) = -2$	then	$0.01 = 10^{-2}$
Since	$\log(3.1623) = 0.5$	then	$3.1623 = 10^{0.5}$
Since	$\log(5) = 0.699$	then	$5 = 10^{0.699}$
Since	$\log(27.141 = 1.434$	then	$27.141\ 10^{1.434}$

Before we spell out why logs are useful we need one additional skill. If we know the log of a number how do we find that number? Simple – use the formula (1). For example if the log of a number is:

$$-0.1024$$

then that number is:

$$10^{-0.1024} = 0.79 \text{ (to 2 dps)}$$

If $p = \log(a)$ then a is said to be the antilog of p, i.e.

$$a = \text{antilog}(p) \quad \text{where} \quad p = \log(a).$$

For example $\quad 0.79 = \text{antilog}(0.1024)$ and $-0.1024 = \log(0.79)$

Similarly $\quad 581.23 = \text{antilog}(2.764)$ and $2.764 = \log(581.23)$

Formula (1) tells us how to find the antilog. On your calculator you can obtain the antilog by pressing 'shift' then 'log'.

Now try Question 1 on Worksheet 7/1

So why are logs useful?

Because they simplify arithmetic operations. Why do they simplify arithmetic operations? It is all to do with the exponent laws. First consider multiplication:

Multiplication

If you multiply two powers of the same number then you simply add exponents. For example

$$\boxed{a \times b = 10^p \times 10^q = 10^{p+q} = 10^{(\log(a)+\log(b))}} \qquad (2)$$

Logs therefore reduce multiplication to addition, a much simpler operation. For example:

$$3.721 \times 15.362 = 10^{0.571} \times 10^{1.186} = 10^{1.757} = 57.15$$

$$3.721 \times 0.126 = 10^{0.571} \times 10^{-0.900} = 10^{-0.329} = 0.47$$

So to use logs to do multiplication we have the following procedure:

Procedure 1

To multiply two (positive) numbers:

(i) take the logs of the two numbers,
(ii) add the logs,
(iii) antilog the sum to obtain the product.

[In actual practice we would of course ask the calculator to give us the answer to a multiplication question directly. Our aim here is to understand the properties of logs in prepartion for the final denouement on the next sheet.]

The second exponent law allows us to use logs to reduce the operation of taking powers to the simpler operation of multiplication:

$$\boxed{a^n} = (10^p)^n = 10^{np} = \boxed{10^{n\log(a)}}$$

For example:

$$(6.175)^{-0.54} = (100.791)^{-0.54} = 10^{-0.427} = 0.37.$$

So to take powers with logs we have the following procedure:

Procedure 2

(i) find the log of the number you wish to take the power of,
(ii) multiply the log by the exponent,
(iii) take the antilog of the product to yield the required power.

[In practice of course one would use the x^y button on the calculator.]

Now try Question 2 on Worksheet 7.1

Pushing it a bit further!

(i) Let's look at Procedure 1. It can be summarised as:

$$a \times b = 10^{(\log(a) + \log(b))} \tag{3}$$

But we can also write $a \times b$ as $10^{(\log(a \times b))}$. So, equating exponents on both sides of (3) we get:

$$\boxed{\log(a \times b) = \log(a) + \log(b)} \tag{4}$$

So the log of a product is the sum of the logs of the individual factors in the product.

To check this out consider the product:

8.742×5.328 with $a = 8.742$ and $b = 5.328$.

Now:

$\log(8.742 \times 5.328) = \log(46.577) = 1.67$

while:

$\log(8.742) = 0.9416$, $\log(5.328) = 0.7266$ and their sum is 1.6682.

So:

$\log(8.742) + \log(5.328) = 1.67$.

Checked!

(ii) Let's now look at Procedure 2. It can be summarised as:

$$a^n = 10^{n \times \log(a)} \tag{5}$$

But we can also write a^n as $10^{\log(an)}$. So, equating exponents on both sides of (5) we get:

$$\boxed{\log(a^n) = n \times \log(a)} \tag{6}$$

So the log of a power of a number is the product of the exponent (in that power) and the log of that number.

To check this out consider the power:

$5.284^{1.721}$ with $a = 5.284$ and $n = 1.721$.

Now:

$\log(5.284^{1.721}) = \log(17.548) = 1.244$

while:

$\log(5.284) = 0.7230$ and $1.721 \times 0.7230 = 1.244$.

Checked!

Now try Question 3 on Worksheet 7/1

Other Logarithms

We have defined our logs with respect to powers of 10. But why 10 besides the act that we have 10 fingers? For example for computers it might be sensible to think in terms of powers of 2 – because computers have two fingers! In fact we can define the log to any positive power. Precisely we define the 'log to base c' of a positive number as the exponent obtained by writing that number as a power of c. Precisely:

$$\boxed{a = c^{\log c(a)}} \tag{7}$$

There is a second logarithm function on your calculator labelled 'ln' for natural logarithm. It is defined with respect to the powers of a very peculiar number:

$$e = 2.7182818\ldots\ldots\ldots\ldots\ldots\ldots$$

This number is as mysterious as the number π arising in trigonometry:

$$\pi = 3.1415927\ldots\ldots\ldots\ldots\ldots\ldots$$

Where e comes from is a story for a later Topic Sheet. Here we will be content with the fact that whatever we do with the log (to base 10) works also for the natural log function 'ln' and indeed for any other logarithm to whatever base. The 'antilog' of the natural logarithm is obtained by '*shift*' 'ln'.

For example, for Procedure 1 we have, with $a = 7.125$, $b = 24.671$:

$$7.125 \times 24.671 = 10^{(\ln(7.125) + \ln(24.671))}$$

but:

$\ln(7.125) = 1.9636$ and $\ln(24.671) = 3.2056$ and the sum is 5.169.

So:

$$7.125 \times 24.671 = 10^{5.169} = 175.8$$

(as can be obtained directly!)

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Topic 7: Logs and Powers – Worksheet 7/1

1.(a) Find the logs (to base 10) of the following numbers:

(i) 62.5 (ii) 1.72 (iii) 0.00284 (iv) 87925.12

Answers:

(i) 1.80 (ii) 0.24 (iii) −2.55 (iv) 4.94

(b) Find the antilogs of the answers to part (a).

Answers: $10^{\boxed{1.80}}$ antilog = 62.5

(i) 63 (ii) 1.74 (iii) 0.00281 (iv) 87096.36

(c) Find the antilogs of the following numbers:

(i) 3.0 (ii) −3.0 (iii) −1.7 (iv) 0.0572

Answers:

(i) 1000 (ii) $\frac{1}{1000}$ (iii) 0.02 (iv) 1.14

2.(a) Use Procedure 1 to calculate the following products:

(i) 17.22×64.08 (ii) 0.173×17.30 (iii) 5.38×8321.67

Answers:

(i) $10^{3.04}$ (ii) $10^{0.48}$ (iii) $10^{4.65}$

$a \times b = 10^{(\log(a) + \log(b))}$

(b) Use Procedure 2 to calculate the following products:

(i) $(3.2281)^{1.32}$ (ii) $(0.163)^{-5.31}$ (iii) $(17.32)^{0.187}$

Answers:

(i) $10^{0.67}$ (ii) $10^{4.18}$ (iii) $10^{0.23}$

$a^n = 10^{n \log(a)}$

3.(a) Check the identity (4) for the following pairs of numbers by evaluating both sides of (4):

(i) $a = 23.56 \quad b = 179.32$ (ii) $a = 0.123 \quad b = 0.778$ (iii) $a = 0.0038 \quad b = 6532.45$

Answers: log(a×b)

(i) $\boxed{3.63}$ (ii) $\boxed{-1,02}$ (iii) $\boxed{1,39}$

(b) Repeat part (a) for:

(i) $a = 3.84 b = 10^3$ (ii) $a = 8.26 \quad b = 10^{-4}$ log(3,84)+ 3 · log(10)

Answers:

(i) $\boxed{3.58}$ (ii) $\boxed{-3.08}$

4. Check the identity (6) for the following pairs of numbers by evaluating both sides of the identity:

(i) $a = 62.34 \quad n = 0.741$ (ii) $a = 0.345 \quad n = 2.75$ (iii) $a = 1.112 \quad n = -1.81$

Answers:

(i) $\boxed{62.34^{0,741} =}$ 1.33 (ii) $\boxed{-1,27}$ (iii) $\boxed{-0,83}$ log(a^n)

5. Repeat questions 1,2,3,4 using natural logarithms rather than logarithms to base 10.

Write answers to question 5 here:

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Topic 8: Logs and Lines

A long time ago we first met logs as a clever way to render a nonlinear graph into a linear one. Remember the body heat vs body weight example? We plotted the data:

weight (kgs)	25	50	100	200	500
body heat (cals)	762	1318	2280	3945	8132

and found the graph nonlinear but plotting the logs of the data gave us a linear graph. Why is this? The answer lies in the formula relating body heat (H) and weight (W) which was revealed in Topic Sheet 6, namely:

$$H = 60 \times W^{0.79} \qquad (1)$$

What we are going to do is to use the two log identities introduced in the previous topic sheet to manipulate this formula. You will remember these two identities stated the following:

Identity 1
The log of a product is the sum of the logs of the individual factors in the product (i.e. multiplication is reduced to addition.)

Identity 2
The log of a power of a number is the product of the exponent (in that power) and the log of that number (i.e. taking powers is reduced to multiplication).

This is what we are going to do with formula (1):

1. Take logs of both sides:
$$\log(H) = \log(60 \times W^{0.79})$$

2. Expand the right hand side using identity 1:
$$\log(H) = \log(60) + \log(W^{0.79})$$

3. Expand the second term on the right hand side using identity 2:
$$\log(H) = \log(60) + 0.79 * \log(W) = 1.778 + 0.79 * \log(W)$$

4. If we relabel the logs:
$$y = \log(H) \text{ and } x = \log(W)$$
we get the line:
$$y = 0.79 * x + 1.778$$

The gradient of the log-log graph line for the heat vs weight problem should, therefore, be the exponent 0.79 in the original power formula and the intercept should be log(60), i.e. the log of the multiplicative (constant) factor in that formula.

If we apply the same argument to the general power formula:

$$\boxed{Y = c * X^b} \qquad (2)$$

relating independent variable big X to dependent variable big Y then we can show that the log-log data lies on the line with equation:

$$\boxed{y = b * x + \log(c)}$$

where $y = \log(Y), x = \log(X)$. So the gradient is the exponent and the intercept is the log of the constant multiplicative factor c in (2).

We can turn the argument around. If we have some data which generates a linear graph when the log values are plotted then we can deduce that the original data satisfies a formula of the form (2) where the exponent (power) is equal to the gradient and the multiplicative factor (c) is the antilog of the intercept. For example consider the data set of question 1 of Worksheet 2/1:

stimulus	0	1	2	3	4	5
response	0	1.5	2.12	2.6	3	3.35

You found the gradient to be roughly 0.5 and the intercept to be 0.175. Since the antilog of the intercept is given by

antilog(0.175) = 1.496 (say 1.5).

We conclude that the original data satisfies the (nonlinear) power formula:

$$\boxed{Y = 1.5 * X^{0.5}}$$

while the logs of the data satisfy the (linear) formula:

$$\boxed{0.175 + 0.5 * x}$$

where $y = \log(Y)$ and $x = \log(X)$.

Geometric Growth

We found that in some cases we could obtain a linear graph by just taking logs of the dependent variable. One such example is world population growth with population as the dependent variable and time as the independent variable. Let's consider this example in more detail.

The basic facts given to us are that the world population in 1990 is 5.6 billion and that each decade it is increasing by 20%. So in the year 2000 it is 5.60 billion multiplied by the factor 1.20 i.e. 6.72 billion. In the year 2010 the population will be the population in 2000multiplied by the 'growth factor' 1.20 if (as we are to assume) population continues to increase at 20% per decade. So in the year 2010 the population will be $6.72 \times 1.20 = 8.064$ billion. So:

Population 1990 = 5.6
Population 2000 = 5.6×1.2
Population 2010 = $5.6 \times 1.2 \times 1.2 = 5.6 \times (1.2)^2$
Population 2020 = $5.6 \times (1.2)^3$

Population after t decades is therefore $P = 5.6 \times (1.2)^t$.

This is an example of 'geometric growth' with 'growth fctor' 1.2 and 'initial value' 5.6.

In general the formula for geometric growth is given by

$$\boxed{Y = c * a^x} \qquad (3)$$

reverting to the big X and big Y labelling. Here a is the growth factor and c the initial value. Let's see how we get a straight line out of this formula:

1. Take logs of both sides of this formula:
$$\log(Y) = \log(c * a^x)$$

2. Use identity 1:
$$\log(Y) = \log(c) + \log(a^x)$$

3. Use identity 2:
$$\log(Y) = \log(c) + X * \log(a)$$

4. Relabelling, $y = \log(Y), x = X$, then
$$y = \log(a * x + \log(c)$$

So the gradient is the log of the growth factor and the intercept is the log of the initial value.

We can turn the argument around. If we have some data which generate a linear graph when the logs of the dependent variable are plotted against the independent variable values, then we can say that the original data satisfy a formula of the form (3) where the growth factor (a) is equal to the antilog of the gradient and the multiplicative factor (c) is the antilog of the intercept. For example consider the boiling water set of data plotted in Topic sheet 2:

Altitude (metres)	0	300	600	850	1000
Temperature (centigrade)	100	50.2	25.2	14.2	10.1

We found the gradient to be −0.001 and the intercept to be 2.0. Since the antilog of the gradient is 0.9977 and the antilog of the intercept is 100 we conclude that the original data satisfy geometric growth:

$$Y = 100 * (0.9977)^x$$

while the logs (y) of the dependent variable satisfy the (linear) formula:

$$y = -0.001 * x + 2.0 .$$

$Y = cx^b$

$\log(c) = \text{Intercept}$

$\log(c) = 0$

$\cancel{\log}(c) = \boxed{10}^0$

$\boxed{c = 1}$

$Y = 1 \times X^{2.05}$

$Y = c\,a^x$

$\log(a) = -0{,}13$

$a = 10^{-0{,}13}$

$= 0{,}74$

$\log(c) = 1.49$

$c = 10^{1.49}$

$= 30.90$

$Y = 30.9 \cdot 0.74^x$

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Topic 8: Logs and Lines – Worksheet 8/1

1. (i) Plot the logs of the following data:

X	0.1	0.75	1.37	2.14	3.44
Y	0.01	0.54	1.79	4.36	11.28

(ii) Find the gradient and intercept of the line obtained in part (i).

b gradient $\boxed{2.05}$ intercept $\boxed{0}$ log(c)

(iii) Hence find a power formula for the original data:

formula $\boxed{Y = 1 \cdot X^{2.05}}$

2. Repeat question 1 for the following data set:

X	0.5	1.5	2.5	3.5	4.5
Y	4	1.333	0.8	0.571428571	0.444444444

Answers:

gradient $\boxed{-1}$ ✓ intercept $\boxed{0.3}$ ✓

formula $\boxed{Y = 2 \cdot X^{-1}}$ ✓

3. (i) Plot the log of the dependent variable values against the independent variable values:

X	0	0.5	1	2	4
Log only → Y	31	27	23	17	10

(ii) Find the gradient and intercept of the line obtained in part (i):

log(a) gradient $\boxed{-0.13}$ ✓ intercept $\boxed{1.49}$ ✓ log(c)

(iii) Hence find the geometric law satisfied by the original data:

formula $\boxed{Y = 30.9 \cdot 0.74^X}$ ✓

4. (i) Repeat question 3 for the data set:

X	0	1	1.5	2	2.5
Y	2	16	45	128	362

(ii) Find the gradient and intercept of the line obtained in part (i):

gradient $\boxed{0.9}$ ✓ intercept $\boxed{0.3}$ ✓

(iii) Hence find the geometric law satisfied by the original data:

formula $\boxed{Y = 2 \cdot 8^{X}}$ ✓

---

iii) $2.37 \cdot (1.52)^n = (5.32)^{n-3}$

$\log 2.37 + n \log 1.52 = (n-3) \cdot \log(5.32)$

$\log 2.37 + n \log 1.52 = n \log 5.32 - 3 \log 5.32$

$n \log 1.52 - n \log 5.32 = -\log(2.37) - 3 \log(5.32)$

$$\frac{n\,\cancel{(\log 1.52 - \log 5.32)}}{\cancel{\log 1.52 - \log 5.32}} = \frac{-\log(2.37) - 3\log(5.32)}{\log 1.52 - \log 5.32}$$

$$n = \frac{-0.3747 - 2.1777}{-0.5440}$$

$$\boxed{n = 4.6915}$$

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Topic 8: Logs and Lines – Worksheet 8/2

1. Solve the equation $2 * 3^n = 5$ for n
   (i) by trial and error (ii) by first taking logs of both sides of the equation.

   $n =$ $\boxed{0.834}$

$2 \cdot 3^n = 5$
$\log(2 \cdot 3^n) = \log(5)$
$\log(2) + \log(3^n) = \log(5)$
$\log(2) + n \cdot \log(3) = \log(5)$
$\frac{n \cdot \log(3)}{\log(3)} = \frac{\log(5) - \log(2)}{\log(3)}$
$n = 0.834$

2. Repeat question 1 for the equations:

   (i) $(2.71)^n = 6.24$

   (ii) $18.53 * (5.21)^{-n} = 1.73$

   (iii) $2.37 * (1.52)^n = (5.32)^{n-3}$

   Answers:

   (i) $\boxed{1.836}$ ✓ (ii) $\boxed{1.436}$ ✓ (iii) $\boxed{4.691}$

3. (i) If $375.2 = 25.8 + (1.2)^n * 4.73$ find n. $n =$ $\boxed{23.5972}$

   First

   (ii) If $52.6 = 10.1 + (1.2)^5(x - 10.1)$ find x. $x =$ $\boxed{27.1797}$

4. (i) If $\log(y) = 0.56 + 0.24x$ express y in terms of x. $y =$ $\boxed{3.63 \cdot (1.738)^x}$

   (ii) If $\log(y) = -0.26 - 0.83 * \log(x)$ express y in terms of x.

   $x =$ $\boxed{0.549 \cdot x^{-0.83}}$

$y = 10^{-0.26 - 0.83 \log(x)}$
$= 10^{-0.26} \cdot 10^{-0.83 \log x}$
$= 0.5495 \cdot \cancel{10^{\log(x)}}^{-0.83}$
$= 0.5495 \cdot x^{-0.83}$

$\log(y) = 0.56 + 0.24x$
$y = 10^{0.56 + 24x}$
$= 10^{0.56} \cdot 10^{0.24x}$
$= 3.63 \cdot 10^{0.24x} = 3.63 \cdot (1.738)^x$