UNIVERSITY OF STIRLING

PDM9L4
DATA SKILLS

WORKBOOK 1 (of 3)
PATH 4

Computing Science & Mathematics

Academic Year 16/17

1 Powers and Primes.

1.1 How to handle exponents.

Consider 3^4, said *3 to the power (of) 4.*
The superscript 4 is known as the exponent, or power, or index, and is the instruction to multiply 3 by itself 4 times. Thus

$$3^4 = 3 \times 3 \times 3 \times 3.$$

Using the normal rules of multiplication we can derive some rules for the behaviour of exponents.

Examples.

1.

$$3^4 \times 3^7 = 3 \text{ times itself } 11 \text{ times} = 3^{11} = 3^{4+7}.$$

Multiply like things: add the powers.

2.

$$3^5 \div 3^2 = 3^5 \times \frac{1}{3^2} = \frac{3 \times 3 \times 3 \times 3 \times 3}{3 \times 3} = 3 \times 3 \times 3 = 3^3 = 3^{5-2}.$$

Divide like things: take away the powers.

3.

$$3^5 \div 3^5 = \frac{3 \times 3 \times 3 \times 3 \times 3}{3 \times 3 \times 3 \times 3 \times 3} = 1$$

and

$$3^5 \div 3^5 = 3^{5-5} = 3^0.$$

Hence

$$3^0 = 1.$$

Anything to the power zero is 1.

(There is an exception to this rule; what is it?)

4.

$$3^4 \div 3^7 = 3^{4-7} = 3^{-3}.$$

Also

$$3^4 \div 3^7 = 3^4 \times \frac{1}{3^7} = \frac{3 \times 3 \times 3 \times 3}{3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3} = \frac{1}{3^3}.$$

Hence

$$3^{-3} = \frac{1}{3^3}.$$

Change the level; change the sign.

Similarly

$$\frac{1}{3^{-3}} = \frac{1}{1/3^3} = 3^3.$$

5.

$$2^2 \times 3^2 \times 2^4 \times 3^{-5} = 2^2 2^4 3^2 3^{-5} = 2^{(4+2)} \times 3^{(2-5)} = 2^6 \times 3^{-3} = \frac{2^6}{3^3}.$$

6.

$$3^4 \div 2^{-3} \times \frac{1}{3^2} \times \frac{1}{2^5} = 3^4 . \frac{1}{2^{-3}} . \frac{1}{3^2} . \frac{1}{2^5} = \frac{3^4 . 3^{-2}}{2^{-3} . 2^5} = \frac{3^{(4-2)}}{2^{(-3+5)}} = \frac{3^2}{2^2} = \left(\frac{3}{2}\right)^2.$$

If we have two (or more) things to the same power we can take the power outside.

Here

$$\left(\frac{3}{2}\right)^2 = \frac{3}{2} \times \frac{3}{2} = \frac{3 \times 3}{2 \times 2} = \frac{3^2}{2^2}.$$

7.

$$\left(3^2\right)^5 = 3^2 \times 3^2 \times 3^2 \times 3^2 \times 3^2 = 3^{10} = 3^{(2\times 5)}$$

and

$$\left(5^2\right)^{-1} = \frac{1}{5^2} = 5^{-2}.$$

For powers of powers: multiply the powers.

8.

$$3^{5/3} \times 3^2 = 3^{11/3},$$

$$\left(7^{\frac{2}{3}}\right)^3 = 7^{3 \times \frac{2}{3}} = 7^2$$

and

$$8^{1/3} = \left(2^3\right)^{1/3} = 2^{3/3} = 2.$$

The same rules apply for fractional powers.

a) $2^{3/2} = 2^{2/2 + 1/2}$
$= 2^{2/2} . \sqrt{2}$
$= 2\sqrt{2}$

b) $3^{5/2} = 3^{4/2 + 1/2}$
$= 3^{4/2}\sqrt{3}$
$= 9\sqrt{3}$

$\sqrt[3]{x} = x^3$

$\sqrt[3]{216} = 6^3$

1.1.1 Exercise.

Evaluate as far as possible without using a calculator.

1. $1^{-1} = \frac{1}{1^1} = 1$ ✓

2. $5^4 \times 5^3 = 5^7 = 78125$ ✓

3. $5^{-4} \times 5^3 = 5^{-1} = \frac{1}{5}$ ✓

4. $5^{-4} \div 5^3 = 5^{-7} = 0.0000128$ ✓

5. $7^{-1} \times 2^3 \times 7 \times \frac{1}{2^{-2}} = 7^{-1} \times 7 \times 2^3 \times 2^2 = 1 \times 2^5 = 32$ ✓

6. $(2+3)^2 = (2+3)\cdot(2+3) = 4+6+6+9 = 25$ ✓

7. $(4-5+1)^0 = 1$ X

8. $\left(\frac{5}{2}\right)^4 \times 2^{-3} \times \left(\frac{1}{2}\right)^{-6} = \left(\frac{5^4}{2^4}\right) \cdot 2^{-3} \times \frac{1}{\left(\frac{1}{2}\right)^6} = 312.5$ ✓

9. $3^2 \times 7^{-4} \times 3^{-5} \times 7^7 = 3^{-3} \times 7^3 = \frac{1}{3^3} \times 7^3 = 12.70$ ✓

10. $5^2(2^3 + 5^{-2}) = 2^3 \cdot 5^2 + 5^{-2} \cdot 5^2 = 2^3 \cdot 5^2 + 1 = 201$ ✓

11. $x^3(x^{-2}+x) = x^{-2} \cdot x^3 + x \cdot x^3 = x + x^4$ ✓

12. $(x+1)(x^2+x^3) = x^3 + x^4 + x^2 + x^3 = x^2 + 2x^3 + x^4$ ✓

13. $(x^3+x^5+x^2) \div x^2 = \frac{x^3}{x^2} + \frac{x^5}{x^2} + \frac{x^2}{x^2} = x^3 \cdot x^{-2} + x^5 \cdot x^{-2} + x^2 \cdot x^{-2} = x + x^3 + 1$ ✓

14. $(x^{-7}+x^6+x^{-4}) \div x^{-3} = \frac{x^{-7}}{x^{-3}} + \frac{x^6}{x^{-3}} + \frac{x^{-4}}{x^{-3}} = \frac{1}{x^7 \cdot x^{-3}} + x^9 + \frac{1}{x^4 \cdot x^{-3}} = \frac{1}{x^4} + x^9 + \frac{1}{x}$ ✓

15. $(xy^{-1} + (xy)^{-1})x^2y^3 = \left(\frac{x}{y} + \frac{1}{xy}\right) x^2y^3 = \left(\frac{x^2}{xy} + \frac{1}{xy}\right) x^2y^3 = \left(\frac{x^2+1}{xy}\right) x^2y^3 = \frac{(x^2+1)x^2y^3}{xy}$
$= (x^2+1)xy^2$ ✓

16. $\left(\frac{1}{5}\right)^{-1} = \frac{5}{1} = 5$ ✓

17. $(2/3+3/2)^{-1} = \left(\frac{2}{3} + \frac{3}{2}\right)^{-1} = \left(\frac{4}{6} + \frac{9}{6}\right)^{-1} = \left(\frac{13}{6}\right)^{-1} = \frac{6}{13}$ ✓

18. $\left(x+x^{1/2}\right)^2 = (x+x^{1/2})\cdot(x+x^{1/2}) = x^2 + x^{3/2} + x^{3/2} + x = 2x^{3/2} + x^2 + x$ ✓

19. $\left(x^{1/2}\right)^{\frac{1}{2}} = \sqrt{(\sqrt{x})} = \sqrt[4]{x}$ $\quad \frac{1}{2}\cdot\frac{1}{2} = \boxed{\frac{1}{4}}$ $\quad$ if $\left(x^{\frac{1}{3}}\right)^{\frac{1}{2}} = \sqrt[6]{x}$ $\quad \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$ ✓

20. $\left(2^{3/2} + 3^{5/2}\right)(2+3)^2 = \left(2^{\frac{3}{2}} + 3^{\frac{5}{2}}\right) 5^2 = (2\sqrt{2} + 9\sqrt{3})\,25$

21. $5^{-3/2} \times 5^{1/2} = 5^{-\frac{3}{2}} \times 5^{\frac{1}{2}} = 5^{-\frac{2}{2}} = 5^{-1} = \frac{1}{5}$ ✓

22. $\left(x^{1/2} - x^{-1/2}\right)^2 = (x^{\frac{1}{2}} - x^{-\frac{1}{2}})\cdot(x^{\frac{1}{2}} - x^{-\frac{1}{2}}) = x - 1 - 1 + x^{-1} = -2 + \frac{1}{x} + x$

23. $(3^3 \times 2^3)^{\frac{1}{3}} = 216^{\frac{1}{3}} = \sqrt[3]{216} = 6$ ✓

24. $(5^3 \times 2^6)^{\frac{1}{3}} = 8000^{\frac{1}{3}} = \sqrt[3]{8000} = 20$ ✓

1.2 Roots and fractional powers.

To find the square root of a number we must find a number which when multiplied by itself twice will give us our original number. Thus the square root of 9 is 3 since $3 \times 3 = 3^2 = 9$. Similarly, since $3^3 = 27$, the cube root of 27 is 3; since $3^4 = 81$ the fourth root of 81 is 3; since $3^5 = 243$ the fifth root of 243 is 3, and so on.
There are two ways we can write the root of a number: the square root of 16 can be written $\sqrt{16}$ or $16^{1/2}$; the cube root of 8 can be written ${}^3\sqrt{8}$ or $8^{1/3}$; the fourth root of 16 can be written ${}^4\sqrt{16}$ or $16^{1/4}$ and so on. Look at your calculator to see which notation is used.
For any root other than the square root it is more common to write fractional power notation.

Examples.

1. The cube root of 8 is 2 since,
$$8^{1/3} = \left(2^3\right)^{1/3} = 2.$$

2. The sixth root of 64 is
$$64^{1/6} = \left(64^{1/2}\right)^{1/3} = \left(\sqrt{64}\right)^{1/3} = 8^{1/3} = 2.$$

3.
$$8^{4/3} = 8^{3/3} \times 8^{1/3} = 8 \times 2 = 16.$$

1.3 Prime numbers.

An integer which is greater than one and which is divisible only by itself is called a *prime number.* The first few primes are 2,3,5,7,11,13,17,19,23. Any number which is not prime is expressible as a product of powers of primes.
(A *product* is what you get when you multiply things together.)

Examples.

1.
$$88 = 8 \times 11 = 2^3 \times 11$$

2.
$$88 \times 242 = 8 \times 11 \times 2 \times 121 = 2^3 \times 11 \times 2 \times 11^2 = 2^4 \times 11^3.$$

3. The cube root of 44×242 is
$$(44 \times 242)^{\frac{1}{3}} = (2^2 \times 11 \times 2 \times 11^2)^{\frac{1}{3}} = (2^3 \times 11^3)^{\frac{1}{3}} = 2 \times 11 = 22.$$

4. The square root of 18 is $\sqrt{18} = \sqrt{2 \times 9} = \sqrt{2} \times \sqrt{9} = \sqrt{2} \times 3 = 3\sqrt{2}$.

1.3.1 Exercise.

Express the following as simply as possible. It may be helpful to express some numbers as products of powers of primes. Check your answers with your calculator.

1. $4^{1/2}$ $= \sqrt{4} = 2$
2. $27^{-1/3}$ $= \dfrac{1}{27^{\frac{1}{3}}} = 0.333\ldots$
3. $\dfrac{1}{25^{1/2}}$ $= \dfrac{1}{\sqrt{25}} = \dfrac{1}{5}$
4. $27^{1/3}$ $= \sqrt[3]{27} = 3$
5. $\left(\sqrt{2}\right)^3$ $= (2^{\frac{1}{2}})^3 = 2^{\frac{3}{2}} = 2.83$
6. $16^{3/2}$ $= 16^{2/2} \times 16^{1/2} = 64$
7. $\sqrt{12}$ $= 12^{\frac{1}{2}} = 3.46$
8. $(54)^{2/3} \times 2^{1/3}$ $=$
9. $7^{-1/2} \times 2^{1/2} \times (56)^{1/2}$
10. $81^{3/4}$
11. $125^{-1/3}$
12. $\left(\dfrac{1}{8}\right)^{-1/3}$
13. $216^{1/3}$
14. $225^{1/2}$
15. $96^{-1/2} \times 24^{1/2}$
16. $\dfrac{\sqrt{32760}}{\sqrt{910}}$
17. $\left(\dfrac{169}{49}\right)^{1/2}$

1.4 Notation for functions.

We can think of a function as an input/output model; you put a value in, the function operates on it and you get value out.
If f is a function of x, then x is the input value and $f(x)$ is the value returned by the function. Note that f denotes the function and $f(x)$ denotes the value of that function at x. Popular letters for denoting a function are f, g and h.
We often write $y = f(x)$ and it is convenient to say that y is a function of x. Strictly speaking both x and y are variables: x is the input variable and y is the output variable. We say that x is the *independent variable* and y is the *dependent variable* because its value depends on our choice of x. Popular letters for variables are y (usually for the dependent variable), x, t, θ and r.

2 Differentiation.

Differentiation is most often associated with a rate of change. For example a sled going downhill will accelerate. Its speed will increase over time and we can think of the acceleration as the difference in speed with respect to time. In general differentiation reflects the difference in one variable with respect to another.

2.1 Notation.

The notation $\frac{d}{dx}$ is the instruction to differentiate with respect to x. Thus $\frac{dy}{dx}$ instructs us to differentiate y with respect to x and we say that $\frac{dy}{dx}$ is the first derivative of y. Here y is a function of x and we write $y = f(x)$. An alternative notation for the first derivative is $f'(x)$. Note also that $\frac{d}{dx}(x^2+6)$ is instructing us to differentiate (x^2+6) with respect to x.

$x^3 \times x^2 = x^5$

2.2 Method.

Example.

Let

$$y = x^3 + x^2.$$

x^2 $x^3 + x^2$

Then

$$\frac{dy}{dx} = 3x^2 + 2x.$$

The derivative is obtained by: multiplying each term by its exponent and then reducing the exponent by one.

Bring the power round to the front, then drop one off the power.

This method also applies to negative and fractional powers.

Examples.

1. If

$$y = x^4 + 5x^{-2},$$

then

$$\frac{dy}{dx} = 4x^3 + (-2)(5)x^{-3} = 4x^3 - 10x^{-3}.$$

2. If

$$f(x) = \frac{1}{3}x^{5/2}$$

then

$$f'(x) = \frac{5}{2} \times \frac{1}{3}x^{3/2} = \frac{5}{6}x^{3/2}$$

$y = x^2 + \boxed{a}$

$y' = 2x^1 + 0$

3. If $y = x = x^1$. Then $\dfrac{dy}{dx} = 1 \times x^0 = 1 \times 1 = 1$.

   This could also be written as $\dfrac{d}{dx}(x) = 1$.

   This indicates that we can write $\dfrac{d}{dx}(x)$ as $\dfrac{dx}{dx}$ and treat it like a fraction.

4. *The derivative of a constant is zero,* since any constant can be written as *constant* $\times x^0$. Differentiating this will involve multiplying by zero.

   Thus $\dfrac{d}{dx}(3) = 0$, and $\dfrac{d}{dx}(a) = 0$.

   (Unknown constants are usually denoted by a letter from the beginning of the alphabet.)

   Sometimes it is necessary to re-write the function first.

5. If

$$y = \sqrt{x}, \text{ re}-\text{write as } y = x^{1/2}.$$

   Then

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}.$$

   If possible return your answer in its original form.

6. If

$$f(x) = (2\sqrt{x})^3, \quad \text{re}-\text{write as } f(x) = 2^3(\sqrt{x})^3 = 8(x^{1/2})^3 = 8x^{3/2}.$$

   $3 + \frac{1}{2}$

   Then

$$f'(x) = 3/2 \times 8x^{1/2} = 12x^{1/2} = 12\sqrt{x}.$$

$\frac{2}{x^5} = 2x^{-5}$ $2x^{-5}$ 5^2 5×5

2.2.1 Exercise.

Differentiate the following.

1. $f(x) = 6x^2 + 2x^1$
2. $f(x) = x^{-1}$
3. $f(x) = \frac{1}{2}x^4 + 9x^2$
4. $f(x) = x^{-2} + x^2$
5. $f(x) = 6\sqrt{x} + \frac{2}{\sqrt{x}}$
6. $f(x) = ax^2 + b$
7. $y = \frac{1}{x} - \frac{2}{x^2}$ $2x^{-2}$
8. $y = \frac{1}{2}\left(x^{5/3} - x^{-3/2}\right)$
9. $y = \frac{3}{2x^3} + x - 4$
10. $y = ax^{-4} + \frac{2}{3}x^{3/2}$
11. $f(x) = \sqrt{2x}$
12. $f(x) = \sqrt{2}\,x$
13. $y = 12x^{1/4} + 8x^{7/2}$
14. $y = 2ax^2 + b^2x + c$
15. $y = (1 + 2x)^2$
16. $y = \left(\sqrt{x} + \frac{3}{\sqrt{x}}\right)^2 = \left(x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}\right)\left(x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}\right)$
17. $y = (1 + 2x)(2 - 3x^2)$

2.3 Different variables.

Suppose y is a function of t and we write $y = g(t)$. Then the derivative with respect to t is denoted by $g'(t)$ or $\frac{dg}{dt}$ and we differentiate in the usual way. Suppose z is a function of two variables x and y, written $z = f(x,y)$, then we need to specify which variable we are differentiating with respect to. Thus $\frac{dz}{dx}$ is the derivative of z with respect to x, and we differentiate in the same way, treating y as a constant throughout. Similarly $\frac{dz}{dy}$ is the derivative of z with respect to y; here we treat x as a constant.

Examples.

1. Let $g(t) = t^2 + 3t$. Then $\frac{dg}{dt} = 2t + 3$.

2. Let $z = f(x,y) = x^2 + y^2 + x$. Then

Just like
a number = 0

$$\begin{aligned} \frac{dz}{dx} &= 2x+1 \quad \text{and} \\ \frac{dz}{dy} &= 2y \end{aligned}$$

3. Let $z = f(x,y) = x^2y + x$. Then

$$\begin{aligned} \frac{dz}{dx} &= 2xy+1 \quad \text{and} \\ \frac{dz}{dy} &= x^2 \end{aligned}$$

2.3.1 Exercise.

1. Differentiate

   (i) $f(t) = t^3 - 2t^{1/2}$

   (ii) $g(t) = t^{-1/2} + at$

2. Let $z = f(x,y)$. Find $\frac{dz}{dx}$ and $\frac{dz}{dy}$ for the following:

   (i) $z = x^3 + y^{1/2}$

   (ii) $z = ax^2 + xy$

   (iii) $xy^2 + x^3y + ax$

2.4 Product Rule.

Suppose the function $f(x)$ can be written as the product of two other functions of x so that $f(x) = u(x)v(x)$.
Then

$$f'(x) = u(x)v'(x) + v(x)u'(x),$$

sometimes written as

$$\boxed{f'(x) = uv' + u'v.}$$

Thus the derivative of $f(x)$ is

1st by diff of 2nd +2nd by diff of 1st.

Examples.

1. Let $y = (1+2x)(2-3x^2)$.
   Then
   $$\begin{aligned} \frac{dy}{dx} &= (1+2x) \times \frac{d}{dx}(2-3x^2) + (2-3x^2) \times \frac{d}{dx}(1+2x) \\ &= (1+2x)(-6x) + (2-3x^2)(2) \\ &= -6x - 12x^2 + 4 - 6x^2 = 4 - 6x - 18x^2. \end{aligned}$$

2. Let $f(x) = (x^2+3)(2x^{-1}+6x)$.
   $(x^2+3)(2x^{-1}+6x)' + (2x^{-1}+6x)\cdot(x^2+3)'$
   $(x^2+3)(-2x^{-2}+6) + (2x^{-1}+6x)\cdot(2x)$
   Then
   $$\begin{aligned} f'(x) &= (x^2+3)(-2x^{-2}+6) + (2x^{-1}+6x)(2x) \\ &= -2 + 6x^2 - 6x^{-2} + 18 + 4 + 12x^2 \\ &= 18x^2 + 20 - 6x^{-2}. \end{aligned}$$

2.5 Chain Rule.

Example.

Suppose $f(x) = (x^3+2x+5)^3$.

For functions of this form we *differentiate the bracket as a single item and then multiply by the derivative of what is inside the bracket.*

Thus

$$f'(x) = 3(x^3+2x+5)^2 \times \frac{d}{dx}(x^3+2x+5) = 3(x^3+2x+5)^2(3x^2+2).$$

Example

Let $y = (1+x^2)^{3/2}$.
Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{3}{2}(1+x^2)^{1/2}(2x) \\ &= 3x(1+x^2)^{1/2} \end{aligned}$$

2.6 Quotient Rule.

Suppose the function $f(x)$ can be written as the quotient of two other functions of x so that $f(x) = \dfrac{u(x)}{v(x)}$.
Then

$$f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{(v(x))^2},$$

sometimes written as

$$\boxed{f'(x) = \frac{vu' - uv'}{v^2}.}$$

Thus the derivative of $f(x)$ is

$$\frac{\textit{bottom by diff of top MINUS top by diff of bottom}}{\textit{bottom squared}}$$

Example

Let $f(x) = \dfrac{(x^2+1)}{(1-x)}$.

Then

$(1-x)(x^2+1)' - (x^2+1)(1-x)'$

$(1-x)(2x) - (x^2+1)(-1)$

$2x - 2x^2 + x^2 + 1$

$$\begin{aligned} f'(x) &= \frac{(1-x)(2x) - (x^2+1)(-1)}{(1-x)^2} \\ &= \frac{2x - 2x^2 + x^2 + 1}{(1-x)^2} \\ &= \frac{2x - x^2 + 1}{(1-x)^2} \end{aligned}$$

Notice that the top (the numerator) gets tidied up but the bottom (the denominator) is left.

Sometimes we need to use more than one technique at a time.

Examples.

1. Let $y = 2x^2(x^3-2x)^3$. We need to use the product rule and the chain rule.

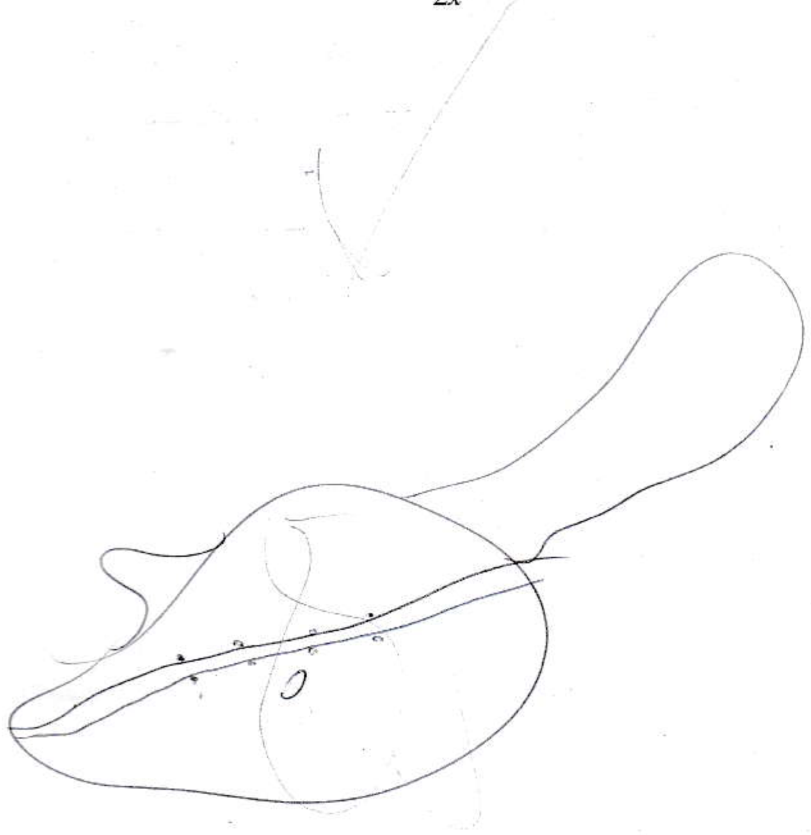
   Then

$$\begin{aligned} \frac{dy}{dx} &= (2x^2)\frac{d}{dx}(x^3-2x)^3 + (x^3-2x)^3\frac{d}{dx}(2x^2) \\ &= (2x^2)(3)(x^3-2x)^2(3x^2-2) + (x^3-2x)^3(4x) \\ &= 6x^2(x^3-2x)^2(3x^2-2) + 4x(x^3-2x)^3 \\ &= (x^3-2x)^2\{6x^2(3x^2-2) + 4x(x^3-2x)\} \\ &= (x^3-2x)^2\{18x^4 - 12x^2 + 4x^4 - 8x^2\} \\ &= (x^3-2x)^2(22x^4 - 20x^2) \\ &= 2x^2(x^3-2x)^2(11x^2-10) \end{aligned}$$

2. Let $f(x) = \dfrac{(x^3-1)^2}{2x}$. Here we need the quotient rule and the chain rule.

Then

$$
\begin{aligned}
f'(x) &= \frac{(2x)(2)(x^3-1)(3x^2)-(x^3-1)^2(2)}{4x^2} \\
&= \frac{2(x^3-1)(6x^3-x^3+1)}{4x^2} \\
&= \frac{(x^3-1)(5x^3+1)}{2x^2}
\end{aligned}
$$



2.6.1 Exercise.

Differentiate the following functions, using product, chain or quotient rule as appropriate.

1. $(x-2)(x^2+3)$
2. $(\sqrt{x}-1)(x^2-2)$
3. $(x^{5/3}+x^2)^4$
4. $(x^2+2)^{-1}$
5. $\dfrac{1}{x^2+2}$
6. $\dfrac{a-\sqrt{x}}{x}$
7. $\left(\dfrac{3}{2}x^2+4x\right)(2x+1)$
8. $(x+1)^3(x-1)$
9. $\dfrac{(x+1)^3}{x^2-1}$
10. $(\sqrt{x}+1)^3$
11. $\dfrac{x^2-4}{x+2}$
12. $x^{3/2}(\sqrt{x}+x^2)$
13. $x^{3/2}(\sqrt{x}+x^2)^2$
14. $x(x+1)(x+4)$
15. $(x^{1/2}+1)(x^2+2x)$
16. $\dfrac{x^{1/2}-1}{2x+1}$
17. $\dfrac{-1}{\sqrt{x^2+a^2}}$
18. $(x^{3/2}+a)^6$
19. $(x^{-2}-3x^{-1})^{-2}$
20. $(x^{-1}+2)(3x-1)$
21. $\dfrac{x^{-1}+2}{3x-1}$
22. $(x^3+4x)^7$
23. $\left(x-\dfrac{1}{\sqrt{x}}\right)^3$
24. $(3-9x-2x^2)^3$
25. $(2-9x)^2(1+4x^2)$
26. $\dfrac{x^2+a}{x^{-2}-a}$
27. $(x^3+ax)^{1/3}$

3 Differentiating other types of functions.

3.1 Trigonometric functions: sine and cosine.

The derivative of the sine function is the cosine function and the derivative of the cosine function is the negative sine function.

sin goes to cos, cos goes to minus sin

or

$$sin \rightarrow cos \rightarrow (-sin) \rightarrow (-cos) \rightarrow sin \rightarrow \ldots$$

Formally

$$\boxed{\frac{d}{dx}\sin x = \cos x \qquad \text{and} \qquad \frac{d}{dx}\cos x = -\sin x.}$$

Other trig functions you should know about are tangent and secant.

Tangent is sine over cosine: $\boxed{\tan = \frac{\sin}{\cos}}$

secant is one over cosine: $\boxed{\sec = \frac{1}{\cos}}$

3.2 Exponential.

Just as π is a real number, approximately 3.142, e is also a real number, approximately 2.718. Thus e^2 is approximately $2.718^2 = 7.389$. The exponential function $y = e^x$ gives the xth power of $2.718\ldots$ for any x. Note that e or any particular power of e is just a constant.

The derivative of the exponential function, $y = e^x$, *is* $\frac{dy}{dx} = e^x$.

The exponential function can also be denoted by "exp". Thus e^x can also be written $\exp(x)$.

3.3 Natural logarithms.

If we can write a number as a power of e then the natural logarithm of that number, written "ln", is the power. Recall 7.389 was e^2 and so $\ln 7.389 = 2$. Similarly the natural log of (e^7), written $\ln e^7$, is 7: $\ln(e^7) = 7$. We can think of the natural logarithm as undoing the exponential. Similarly the exponential undoes the natural logarithm. Thus $e^{\ln 5}$ (also written as "$\exp(\ln 5)$") is 5.

The derivative of the natural logarithm, $y = \ln x$, *is* $\frac{dy}{dx} = \frac{1}{x}$.

All the usual rules apply to these functions.

$2 \quad -\sin x$

Examples.

1. Let $y = 2x\cos x$. This is a product; $2x \times \cos x$ and so we appy the product rule. Thus
$$\frac{dy}{dx} = 2x(-\sin x) + \cos x(2).$$
Note that we CANNOT leave the answer like this. The 2 after the cosine may be interpreted as belonging to the cosine function when in fact it does not. This second term should read "cos x times 2". To make this clear the 2 MUST be brought round to the front so that it reads "2 times cos x". It is also good practice to bring the minus sign out in front of the 2x. However this means that the answer will now start with a minus sign and it is generally better to start an expression with a positive term if possible. The best layout for the final answer is:
$$\frac{dy}{dx} = 2\cos x - 2x\sin x.$$

2. Let $f(x) = \dfrac{e^x}{x}$. This is a quotient so we use the quotient rule, and then tidy up a bit to make the answer look nice. Thus
$$f'(x) = \frac{xe^x - e^x}{x^2} = e^x(x-1)x^{-2}.$$

3. Let $f(x) = x^2 \ln x$. Then $x^2 \ln(x)' + \ln(x)(x^2)' = x^2 \cdot \frac{1}{x} + \ln x \cdot 2x =$
$$f'(x) = x^2 \times \frac{1}{x} + \ln x \times 2x = x + 2x\ln x.$$
Again notice that we brought the 2x round to the front of the ln x.

4. Let $f(x) = (x^2+3)^2 \sin x$. For this we need the product rule and chain rule. Thus
$$\begin{aligned} f'(x) &= (x^2+3)^2 \times \frac{d}{dx}(\sin x) + \sin x \times \frac{d}{dx}(x^2+3)^2 \\ &= (x^2+3)^2 \times (\cos x) + \sin x \times 2(x^2+3) \times \frac{d}{dx}(x^2+3) \\ &= (x^2+3)^2 \times (\cos x) + \sin x \times 2(x^2+3) \times 2x \\ &= (x^2+3)^2 \cos x + 4x(x^2+3)\sin x. \end{aligned}$$
You should not find it necessary to write down the first two lines of working. Notice how the trig functions are written at the end of each term.

5. Let $y = \tan x = \dfrac{\sin x}{\cos x}$. We use the quotient rule. Thus $\qquad \cos^2 x + \sin^2 x = 1$
$$\begin{aligned} \frac{dy}{dx} &= \frac{\cos x \times \cos x - \sin x \times (-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x. \end{aligned}$$

Notice that $(\cos x)^2$ is written $\cos^2 x$. Similarly $\sin^5 x$ means $\sin x$ to the fifth power.

This solution used a trig identity. It would be nice to know all the trig identities, however you *should* know the following:

3.4 Trigonometric identities.

1. $\sin^2 x + \cos^2 x = 1$.
2. $\sin 2x = 2\sin x \cos x$.
3. $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$.

$y = \sin x$ $y' = \cos x$	$y = \cos x$ $y' = -\sin x$
$\tan = \frac{\sin}{\cos}$	$\sec = \frac{1}{\cos}$
$y = \ln x$ $y' = \frac{1}{x}$	$y = e^x$ $y' = e^x$

3.4.1 Exercise.

Differentiate the following.

1. $y = 2\sin x + \cos x$
2. $y = \cos x + x^2$
3. $y = (1+x)e^x$
4. $y = \dfrac{1-\sin x}{1+x}$
5. $y = \dfrac{1-\cos x}{1-x}$
6. $y = (1-x)\cos x$
7. $y = x\ln x + \ln x$
8. $f(x) = \dfrac{\ln x}{\sin x}$
9. $f(x) = \sin x \ln x + x\cos x$
10. $f(x) = \sin x \cos x$
11. $f(x) = \dfrac{(x^2+2)^{1/2}}{\cos x}$
12. $y = \sqrt{(x+1)}\,\ln x$
13. $y = e^x \ln x$
14. $y = \dfrac{2x}{e^x}$
15. $f(x) = (\sin x)(\sin x)$
16. $f(x) = (\cos x)(\cos x)$

3.5 Composite functions and the chain rule.

Go back and read the section on the chain rule.

The function $f(x) = (x^3+2x+5)^3$ is a composite function; a function of a function. We can think of it as one function nested inside another. The outer function is "to the power of 3" and the inner function is "x^3+2x+5". Using the chain rule to differentiate we

differentiate the outer function and then multiply by the derivative of the inner function.

This results in

$$f'(x) = 3(x^3+2x+5)^2 \times \frac{d}{dx}(x^3+2x+5).$$

Formally if $y = f(g(x))$ then

$$\frac{dy}{dx} = f'(g(x)) \times g'(x).$$

This principle can be applied to any composite function.

Examples.

1. Let $y = \sin(2x+1)$. This is sine of $(2x+1)$ and so the outer function is sine [with derivative cosine] and the inner function is $(2x+1)$.

   Thus

$$\begin{aligned}\frac{dy}{dx} &= \cos(2x+1) \times \frac{d}{dx}(2x+1) \\ &= \cos(2x+1) \times 2 \\ &= 2\cos(2x+1).\end{aligned}$$

   The part belonging to a function is called the *argument* of that function. You will notice that sometimes it is put inside brackets.

   It is important that

   the derivative of the outer function keeps the same argument.

2. Let $y = e^{2x}$. This is the exponential of $(2x)$ and so the outer function is the exponential and the inner function is $2x$.

   Thus

$$\frac{dy}{dx} = e^{2x} \times \frac{d}{dx}(2x) = 2e^{2x}.$$

   With practice you should not need to include the first line of working.

3. Let $f(x) = \ln x^2$. Then $f'(x) = \frac{1}{x^2} \times 2x = \frac{2}{x}$.

4. Let $y = \sin^3 x$. This is the third power of $(\sin x)$.
   If we write it as $y = (\sin x)^3$ it is easy to see that

$$\begin{aligned}
\frac{dy}{dx} &= 3(\sin x)^2 \times \frac{d}{dx}(\sin x) \\
&= 3(\sin x)^2(\cos x).
\end{aligned}$$

   However this should be written as $3\sin^2 x \cos x$.

5. Let $f(x) = \cos^4 x$. Then $f'(x) = 4\cos^3 x \times (-\sin x) = -4\sin x \cos^3 x$.

   This process can also be applied to a function of a function of a function.

6. Let $f(x) = \sin^5\left(3x^2\right)$. This is the fifth power of (the sine of $\left(3x^2\right)$). Thus the derivative is

$$\begin{aligned}
f'(x) &= 5\sin^4\left(3x^2\right) \times \frac{d}{dx}\left(\sin\left(3x^2\right)\right) \\
&= 5\sin^4\left(3x^2\right) \times \cos\left(3x^2\right) \times \frac{d}{dx}\left(3x^2\right) \\
&= 5\sin^4\left(3x^2\right) \times \cos\left(3x^2\right) \times 6x \\
&= 30x\,\sin^4\left(3x^2\right)\cos\left(3x^2\right)
\end{aligned}$$

   We can now differentiate quite bizarre functions.

7. Let $y = \ln\sin x^2$. This is the natural log of (sine of (x^2)), <u>not</u> the log function multiplied by the sine function.
   You may find it clearer to write it as $y = \ln(\sin x^2)$. The derivative is

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{\sin x^2} \times \frac{d}{dx}\left(\sin x^2\right) \\
&= \frac{1}{\sin x^2} \times \cos x^2 \times \frac{d}{dx}\left(x^2\right) \\
&= \frac{1}{\sin x^2} \times \cos x^2 \times 2x \\
&= \frac{2x\,\cos x^2}{\sin x^2} \\
&= \frac{2x}{\tan x^2}.
\end{aligned}$$

3.5.1 Exercise.

Differentiate the following

1. $y = \sin x^3$.
2. $y = \cos(2x^2 + 1)$
3. $y = \cos 3x$
4. $y = \sin\left(\frac{1}{2}x\right)$
5. $y = \cos(3x + 1)$
6. $y = \sin^2(ax)$
7. $y = e^{3x}$
8. $y = e^{(3x^2+1)} = \exp(3x^2 + 1)$
9. $y = \sin 2x - \cos 3x$
10. $y = xe^{2x}$
11. $y = x(e^x)^2$ [think about it!]
12. $y = \dfrac{1}{2}\ln x$
13. $y = \ln\sqrt{x}$
14. $y = \dfrac{1}{\sin^2 x}$
15. $y = (\cos x)^{-1/2}$
16. $y = \ln(x/2)$
17. $y = x^2 e^{x/2}$
18. $y = e^{-1}$
19. $f(x) = x\cos 2x$
20. $f(x) = \cos ax$
21. $f(x) = \sin \pi x$
    [remember π is a constant.]
22. $f(x) = e^{a^2 x}$
23. $f(x) = \ln(e^{2x})$
24. $f(x) = \sin\cos 2x$
25. $f(x) = \cos^4 \pi x$
26. $f(x) = \pi \sin \pi x$
27. $f(x) = \cos^2 x$
28. $f(x) = \cos^2(2x^2)$
29. $f(x) = x^{7/2} - x\sin 3x$
30. $f(x) = (x - \sin x)^{1/2}$
31. $f(x) = \ln(\tan x)$
32. $f(t) = \dfrac{\sin 2t}{\cos t}$
33. $f(t) = \dfrac{e^{2t}}{t^2}$
34. $f(t) = \ln(t^2 - 1)$
35. $f(t) = t^3 - 3e^t$
36. $g(t) = \sin^3(4t^2)$
37. $g(t) = \cos 3t - \sin 3t$
38. $y = \dfrac{\ln t}{t^2}$
39. $y = \dfrac{\sin 2t}{\cos t^2}$
40. $y = \dfrac{e^x}{2x^2}$
41. $y = \dfrac{\cos 3x}{e^{2x}}$
42. $y = \dfrac{\cos 2x}{\sin 2x}$
43. $y = \dfrac{\cos 3x}{(x+1)^2}$

3.6 Implicit differentiation.

Consider the function $2xy = y^3$ where y is a function of x. This should not be confused with functions of the type $z = 2xy - y^3$ where z is a function of two variables x and y.
To differentiate $2xy = y^3$ we differentiate both sides with respect to x, remembering that y is a function of x, not a constant.
To differentiate $2xy = (2x) \times (y)$ we use the product rule and the derivative is

$$2x \times \frac{d}{dx}(y) + y \times 2 = 2x\frac{dy}{dx} + 2y.$$

Notice that we can write $\frac{d}{dx}(y)$ as $\frac{dy}{dx}$.
To differentiate y^3 we use the chain rule; the third power is the outer function, y is the inner function and the derivative is

$$3y^2 \times \frac{d}{dx}(y) = 3y^2\frac{dy}{dx}.$$

Equating the two sides we obtain

$$2x\frac{dy}{dx} + 2y = 3y^2\frac{dy}{dx}.$$

Re-arranging this so that all the terms with $\frac{dy}{dx}$ are on one side and everything else on the other we obtain

$$3y^2\frac{dy}{dx} - 2x\frac{dy}{dx} = 2y.$$

Taking $\frac{dy}{dx}$ out as a common factor we get

$$\frac{dy}{dx}(3y^2 - 2x) = 2y.$$

We want $\frac{dy}{dx}$ on its own and so we take the $(3y^2 - 2x)$ over the other side and divide by it:

$$\frac{dy}{dx} = \frac{2y}{3y^2 - 2x}.$$

Note that for some of these expressions, certain restriction on the values of x and y may be necessary for them to be functions. We shall assume that these restrictions are given.

Examples.

1. Let y be a function of x and let $2x^2 + y = y^2$. Find $\frac{dy}{dx}$.

   Solution. Differentiating with respect to x:

$$\begin{aligned} 4x + \frac{dy}{dx} &= 2y\frac{dy}{dx}, \\ 4x &= \frac{dy}{dx}(2y-1), \\ \frac{dy}{dx} &= \frac{4x}{2y-1}. \end{aligned}$$

2. Let y be a function of x and let $3x^2y = xy^2$. Find $\frac{dy}{dx}$.

Solution. We need the product rule for the LHS, and product and chain rule for the RHS. Differentiating with respect to x:

$$\begin{aligned}
3x^2 \times \frac{dy}{dx} + y \times 6x &= x \times 2y\frac{dy}{dx} + y^2, \\
3x^2\frac{dy}{dx} - 2xy\frac{dy}{dx} &= y^2 - 6xy, \\
(3x^2 - 2xy)\frac{dy}{dx} &= y^2 - 6xy, \\
\frac{dy}{dx} &= \frac{y^2 - 6xy}{3x^2 - 2xy}.
\end{aligned}$$

An inaccurate but easy way to remember how to do this is to differentiate the x and the y with respect to themselves using the usual rules. Then in every term you differentiate y, multiply by $\frac{dy}{dx}$.

3.6.1 Exercise.

Use implicit differentiation to find an expression for $\frac{dy}{dx}$ for the following.

1. $4y^3 + xy = x^2$
2. $y + xy = x^3$
3. $x - y^2 = x^2 + xy$
4. $(x-1)y^2 = x + y$
5. $x^2 - y^2 = 4$
6. $x^2 + y^3 - 2xy = 0$

You are now ready for quiz 1.