UNIVERSITY OF
STIRLING

PDMU9L4
DATA SKILLS

WORKBOOK 2 (of 3)
PATH 4

Computing Science & Mathematics
School of Natural Sciences

Academic Year 15/16

4 Fractions and logarithms.

4.1 Dividing with fractions.

For some reason, most people seem to have slept through the lesson on fractions at school. Here are a few reminders.

Examples.

1.

$$\frac{7}{5/3} = 7 \div \frac{5}{3} = 7 \times \frac{3}{5} \quad (\textit{change the sign and flip the fraction})$$
$$= \frac{21}{5}.$$

2.

$$\frac{7/5}{3} \text{ is } \frac{1}{3} \text{ of } \frac{7}{5} = \frac{1}{3} \times \frac{7}{5} \quad (\textit{bring the third out to the front})$$
$$= \frac{7}{15}.$$

3.

$$\frac{7/2}{5/3} = \frac{7}{2} \div \frac{5}{3} = \frac{7}{2} \times \frac{3}{5} = \frac{21}{10} \quad \textit{notice 2 and 5 are on the same level,}$$

the 3 wraps round to the top.

4. $$\frac{1}{2} \times \frac{7}{5/3} = \frac{1}{2} \times \frac{7}{5} \times 3 = \frac{1}{2} \times \frac{21}{5} = \frac{21}{10}.$$

5. $$\frac{1}{2} \times \frac{7/5}{3} = \frac{1}{2} \times \frac{1}{3} \times \frac{7}{5} = \frac{7}{30}.$$

6. $$\frac{1}{2} \times \frac{7/2}{5/3} = \frac{1}{2} \times \frac{7}{2} \times \frac{3}{5} = \frac{21}{20}.$$

7. $$2 \times \frac{7/2}{5/3} = 2 \times \frac{7}{2} \times \frac{3}{5} = 7 \times \frac{3}{5} = \frac{21}{5}.$$

8. $$2 \times \frac{7}{5/3} = 2 \times \frac{7}{5} \times 3 = \frac{42}{5}.$$

9. $$2 \times \frac{7/5}{3} = 2 \times \frac{7}{5} \times \frac{1}{3} = \frac{14}{15}.$$

4.1.1 Exercise.

Evaluate the following and check your answers on your calculator.

1. $3 \times \frac{7/3}{5}$ $\quad 3 \times \frac{7}{3} \div 5 = \frac{21}{3} \div 5 = 7 \div 5 = \frac{7}{5}$

2. $\frac{5/3}{(-2)}$ $\quad \frac{5}{3} \div -2$

3. $(-4) \times \frac{(-7)}{3} \times \frac{1}{2/9}$ $\quad \frac{28}{3} \times \left(1 \div \frac{2}{9}\right)$

4. $\frac{9/2}{3/2}$

5. $\frac{6/5}{(-3)}$

6. $\frac{9}{3/2}$

7. $\frac{8}{1/2}$

8. Half of 8

9. A quarter of 5/3

10. Two thirds of $\frac{7/3}{2/5}$

11. $3 \times \frac{5/2}{9/4}$

12. $\frac{11}{7/4}$

13. $\frac{11/4}{7} \times \frac{1}{2}$

14. $5 \times \frac{9}{3/2} \times \frac{4/5}{(-3)}$

4.2 Manipulating logarithms.

Calculators have only been commonplace for the last thirty or so years. Before that, if you wanted to do calculations with large numbers you used logarithms and consulted log tables. A copy of Cambridge four figure tables is still in 4X2.
There are two main rules for manipulating logs:

Rules.

1. *The log of a product is the sum of the logs.*

   Examples.

   (i) $\ln 6 = \ln(2 \times 3) = \ln 2 + \ln 3$.

   (ii) $\ln 9 = \ln(3 \times 3) = \ln 3 + \ln 3 = 2\ln 3$.

2. *Power law:* $\ln a^b = b \ln a$.

   Examples.

   (i) $\ln 9 = \ln 3^2 = 2\ln 2$.

   (ii) $\ln \frac{1}{3} = \ln(3^{-1}) = -\ln 3$.

Examples.

1.
$$\ln \tfrac{1}{5} + \ln 5 = -\ln 5 + \ln 5 = 0.$$

   Also

$$\ln \tfrac{1}{5} + \ln 5 = \ln\left(\tfrac{1}{5} \times 5\right) = \ln 1 = 0.$$

   This is in accordance with the fact that the log of 1 is zero.

2.
$$\ln \frac{5}{7} = \ln\left(5 \times \frac{1}{7}\right) = \ln 5 + \ln \tfrac{1}{7} = \ln 5 - \ln 7$$

   or more simply:

$$\ln(5 \div 7) = \ln 5 - \ln 7.$$

5 Integration.

Integration can be thought of as a way of reversing differentiation and as such we obtain an *antiderivative.* It is often associated with finding the area under a curve and indeed *integration* was developed through considering areas. What is amazing is that the process for finding areas turned out to be the same as the process for finding antiderivatives.

Example.

The derivative of x^2+2 is $2x$ and so an antiderivative of $2x$ is x^2+2. However the derivative of x^2+3 is also $2x$ and so we could also say that an antiderivative of $2x$ is x^2+3. To overcome this ambiguity we say that the antiderivative of $2x$ is x^2+c where c is some constant.

5.1 Notation.

The notation $\int dx$ is the instruction to integrate ($\int$) with respect to x. Thus $\int 2x\,dx$ means integrate $2x$ with respect to x. The thing to be integrated is called the *integrand*; sometimes it is put inside brackets, sometimes not. It is customary to put the dx after the integrand but you may find it useful to put it elsewhere in the integral on occasion.
Also note that a standard abbreviation for "with respect to" is "wrt".

Example.

Using the idea of an antiderivative we can find the integral:

$$\int 2x\ dx = x^2 + c$$

where c is some constant known as *the constant of integration.* Such integrals are called *indefinite integrals.*

5.2 Method.

Examples.

1.

$$\int (x^2+x)\ dx = \frac{x^3}{3} + \frac{x^2}{2} + c.$$

The integral is obtained as follows:

add one onto the power: divide by the new power.

[Compare this with how we described differentiation:

multiply by the power:take one off the power.]

2.

$$\int 3x^5 \; dx = \frac{3x^6}{6} + c = \frac{x^6}{2} + c.$$

3. This method also applies to negative and fractional powers.

$$\begin{aligned} \int 3x^{2/3} + x^{-1/2} \; dx &= \frac{3x^{5/3}}{5/3} + \frac{x^{1/2}}{1/2} \\ &= 3 \times \frac{3x^{5/3}}{5} + 2x^{1/2} \\ &= \frac{9x^{5/3}}{5} + 2x^{1/2} + c. \end{aligned}$$

Note that it is quite common not to add the constant of integration until the end.

With practice you should not need to write down as much working as this.

4.

$$\int \frac{8}{3}x^{1/3} + x^{1/2} \; dx = 2x^{4/3} + \frac{2x^{3/2}}{3} + c$$

If you want to check your integration, differentiate your answer and see if you get what you started with.

5. Trying to differentiate $\frac{1}{x}$ by this method we encounter a problem.

$$\int \frac{1}{x} \; dx = \int x^{-1} \; dx = \frac{x^0}{0},$$

but dividing by zero is not defined. However, we know that the derivative of $\ln x$ is $\frac{1}{x}$ and so, using the idea of the antiderivative, we get

$$\int \frac{1}{x} \; dx = \ln|x| + c.$$

Note $|x|$ means the *absolute value* of x; the value of x ignoring the $\pm$ sign. Thus $|x|$ is always positive. We need this because we cannot have the log of a negative number.

6. We can use the idea of the antiderivative to integrate sine and cosine.

Since

$$\frac{d}{dx}\sin x = \cos x, \quad \int \cos x \; dx = \sin x + c$$

and since

$$\frac{d}{dx}\cos x = -\sin x, \quad \int (-\sin x) \; dx = -\cos x + c.$$

These results are easily remembered:

$$\boxed{\begin{array}{ccc} \textit{diff}\;\Bigg\downarrow & \begin{matrix}\sin\\ \cos\\ -\sin\\ -\cos\\ \sin\end{matrix} & \Bigg\uparrow\;\textit{int}\end{array}}$$

and so on.

7. If you are multiplying by a constant you may find it useful to take the constant outside the integration sign:

$$\int 2\pi\sin x \;\, dx = 2\pi \int \sin x \;\, dx = 2\pi(-\cos x) = -2\pi\cos x + c.$$

8. The easiest integral in the world:

$$\int e^x \;\, dx = e^x + c.$$

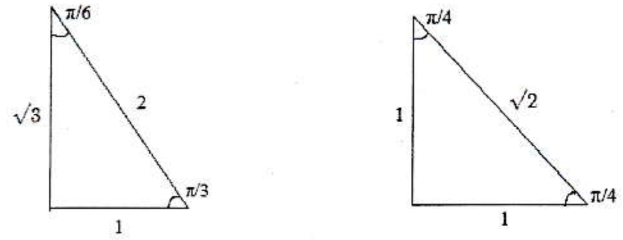
5.2.1 Exercise.

Integrate the following. Check your answers by differentiating.

1. $3x^2 + 2x + 1$
2. $x^{-7/2} + 2x^4$
3. $ax + bx^{4/5}$
4. $x + \dfrac{1}{x}$
5. $(x+1)^2$ [multiply out then integrate]
6. $x + e^x$ remain the Same
7. $x^{-1/2} + \sin x$
8. $\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^2$
9. $2\cos x - \sin x$
10. $16(\sin x - 2\cos x)$
11. $\dfrac{1}{2}x^{-5/2} - \dfrac{1}{3}x^{-3/2}$
12. $\dfrac{x^{2/3}}{3} - \dfrac{x^{1/2}}{4}$
13. $\dfrac{3}{x} + e^x$
14. $x^{-1} + \pi\cos x$

5.3 Sine and Cosine values.

There are certain values of the sine and cosine functions that you should know or should know how to work out quickly. These can be derived from two special triangles.
Draw an equilateral triangle of side two. Thus all sides are of length 2 and all the angles are $60\,^\circ$. Drop a perpendicular down from one corner, i.e. a line that meets the opposite side at $90\,^\circ$. This line will cut the triangle in half giving you the first triangle in the diagram. Notice that the angles are given in radians. You can think of π as $180\,^\circ$ so that $60\,^\circ = 180/3 = \pi/3$ and $30\,^\circ = 180/6 = \pi/6$. Similarly $45\,^\circ = 180/4 = \pi/4$.
The second triangle is a right-angled isoceles triangle of side 1. Thus there are two equal sides of length 1 and two equal angles, both $45\,^\circ = \pi/4$.



Applying SOHCAHTOA to these triangles we get

$$\cos\frac{\pi}{3} = \frac{1}{2} = \sin\frac{\pi}{6}$$

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}$$

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4}$$

5.4 Definite Integrals.

Example.

Evaluate $\displaystyle\int_1^3 (2x+1)\ dx.$

We say this is the integral of $2x+1$ evaluated from 3 to 1. The 3 and the 1 are called the limits of the integral. Such integrals are called *definite integrals* and do not need a constant of integration.
To evaluate these integrals first do the integration. This goes into square brackets with the limits attached. The final answer is *the integral evaluated at the upper limit MINUS the integral evaluated at the lower limit.* To evaluate the integral at a limit substitute that limit in for x.
Here

$$I = \int_1^3 (2x+1)\ dx = \left[x^2+x\right]_1^3 = (3^2+3)-(1^2+1) = 10.$$

Be careful with that minus sign.

$1+e^x$

Examples.

This first example looks difficult but the integration is straightforward and the rest is just messy.

1.

$$\begin{aligned}
\int_1^8 (3x^{1/3} - x^{2/3})\ dx &= \left[\frac{9x^{4/3}}{4} - \frac{3x^{5/3}}{5}\right]_1^8 \\
&= \left(\frac{9}{4}\times 8^{4/3} - \frac{3}{5}\times 8^{5/3}\right) - \left(\frac{9}{4}\times 1^{4/3} - \frac{3}{5}\times 1^{5/3}\right) \\
&= \left(\frac{9}{4}\times 2^4 - \frac{3}{5}\times 2^5\right) - \left(\frac{9}{4} - \frac{3}{5}\right) \\
&= \left(\frac{144}{4} - \frac{96}{5}\right) - \left(\frac{9}{4} - \frac{3}{5}\right)
\end{aligned}$$

at this point it might be a good idea to find out how the fraction button on your calculator works, otherwise

$$\begin{aligned}
&= \left(\frac{144\times 5 - 96\times 4}{20}\right) - \left(\frac{9\times 5 - 3\times 4}{20}\right) \\
&= \left(\frac{(720 - 384 - 45 + 12}{20}\right) = \frac{303}{20}.
\end{aligned}$$

2.

$$\begin{aligned}
\int_{\pi/2}^{\pi} 1 - \sin x\ dx &= \Big[x + \cos x\Big]_{\pi/2}^{\pi} \\
&= (\pi + \cos\pi) - (\tfrac{\pi}{2} + \cos\tfrac{\pi}{2}) \\
&= \pi + (-1) - \tfrac{\pi}{2} - 0 = \tfrac{\pi}{2} - 1.
\end{aligned}$$

3. Sometimes the left square bracket is omitted.

$$\int_e^{e^2} \frac{1}{2x}\ dx = \frac{1}{2}\ln x\Big]_e^{e^2} = \frac{1}{2}\left(\ln e^2 - \ln e\right) = \frac{1}{2}(2-1) = \frac{1}{2}.$$

4. Some integrands must be simplfied before integrating:

$$\begin{aligned}
\int_1^e \frac{x^2-2}{2x}\,dx &= \int_1^e \frac{x^2}{2x}-\frac{2}{2x}\,dx \\
&= \int_1^e \frac{x}{2}-\frac{1}{x}\,dx \\
&= \left[\frac{x^2}{4}-\ln|x|\right]_1^e \\
&= \left(\frac{e^2}{4}-\ln e\right)-\left(\frac{1}{4}-\ln 1\right) \\
&= \frac{e^2}{4}-1-\frac{1}{4}+0=\frac{e^2-5}{4}.
\end{aligned}$$

Be careful when simplifying. For example $\dfrac{2x}{x^2-2}$ does NOT equal $\dfrac{2x}{x^2}-\dfrac{2x}{2}$.

5.4.1 Exercise.

Evaluate the following.

1. $\displaystyle\int_{-1}^{1} x^3-x^{-2}\,dx$
2. $\displaystyle\int_{\pi/6}^{\pi/2} 1+\cos x\,dx$
3. $\displaystyle\int_{1}^{4} x^{1/2}-3x^{-1/2}\,dx$
4. $\displaystyle\int_{0}^{\pi/4} e^x+\sin x\,dx$
5. (*i*) $\displaystyle\int_{\pi/6}^{\pi/3} \sin x-\cos x\,dx$; (*ii*) $\displaystyle\int_{-\pi}^{\pi/4} \cos x-\sin x\,dx$
6. $\displaystyle\int_{-t}^{t} 3x^2-\frac{1}{x^2}\,dx$
7. $\displaystyle\frac{2}{3}\int_{8}^{27} x^{-1/3}-x^{-4/3}\,dx$
8. (*i*) $\displaystyle -\frac{1}{4}\int_{144}^{64}\left(\frac{2+\sqrt{x}}{2\sqrt{x}}\right)dx$; (*ii*) $\displaystyle -\frac{1}{4}\int_{2}^{8}\left(\frac{2+\sqrt{x}}{2\sqrt{x}}\right)dx$
9. (*i*) $\displaystyle\int_{\sqrt{2}}^{2\sqrt{3}} x-5x^2\,dx$; (*ii*) $\displaystyle\int_{-1/3}^{1/2} x-5x^2\,dx$

6 Integration by substitution.

6.1 Indefinte integrals.

Consider the indefinite integral

$$I = \int 4x(x^2+1)^2 \; dx.$$

This can be integrated by two different methods. Firstly we could multiply out then integrate as follows:

$$\begin{aligned} I &= 4\int x(x^2+1)^2 \; dx \\ &= 4\int x^5 + 2x^3 + x \; dx \\ &= 4\left(\frac{x^6}{6} + \frac{2x^4}{4} + \frac{x^2}{2}\right) \\ &= \frac{2x^6}{3} + 2x^4 + 2x^2 + c. \end{aligned}$$

The second method is the *method of substitution.*
Here the aim is to transform an intractable integral into one that we can do.

We begin with the integral in terms of x:

$$\boxed{I = 4\int x(x^2+1) \; dx.}$$

We pick part of the integrand for our substitution. The messiest part of the integrand is always a good first choice but there may be more than one possibility, some of which may not work. There is an element of trial and error in integration which is what makes it fun! Here we choose x^2+1 for our substitution.

$$\boxed{\text{Let } u = x^2+1.}$$

This means we can replace $(x^2+1)^2$ with u^2:

$$I = 4\int x \; dx \; \underline{(x^2+1)^2}$$

This leaves us with $x \; dx$ to replace in terms of u. In particular we will need a du in order to complete the instruction to integrate wrt u.
Differentiate u wrt x to get

$$\frac{du}{dx} = 2x.$$

Treat the $\frac{du}{dx}$ as a fraction and take the dx up to the other side.

$$\boxed{\text{Then } du = 2x \; dx.}$$

Thus we can either replace $2x\ \ dx$ with du, or replace $x\ \ dx$ with $\frac{1}{2}\ du$.
Still writing the integral in terms of x

$$I = 2\int \underline{2x\ \ dx}\ \ \underline{(x^2+1)^2}$$

we see that we can now re-write the entire integral in terms of u:

$$I = 2\int\ du\ \ u^2$$

or more commonly

$$\boxed{I = 2\int u^2\ \ du.}$$

This is now easy to integrate wrt u:

$$\boxed{I = 2\times\frac{u^3}{3}+c}$$

and we re-substitute for u to get the integral back in terms of x,

$$\boxed{I = \frac{2}{3}(x^2+1)^3+c.}$$

No matter how tempting, you must NEVER write the integral in terms of u and x together since this would be nonsense. You must find all the bits of the puzzle and then make the transformation in one go.

For completeness we should check that this second answer really is the same as the first.

$$\begin{aligned}\frac{2(x^2+1)^3}{3}+c &= \frac{2}{3}(x^2+1)(x^4+2x^2+1)+c\\ &= \frac{2}{3}(x^6+3x^4+3x^2+1)+c\\ &= \frac{2x^6}{3}+2x^4+2x^2+\frac{2}{3}+c.\end{aligned}$$

The $\frac{2}{3}$ is a constant and as such can be incorporated into the *constant of integration*, and now both answers are the same.

Not all integrals succumb to a substitution attack. Consider the following integral which is very similar to the previous example.

$$I = \int (x^2+1)^2\ \ dx.$$

Trying to do this by substitution we let $u = x^2+1$ so that $du = 2x\ \ dx$. However there is no $2x\ \ dx$ to replace so the method fails. Integrals suitable for this method *must contain the derivative of the substitution.*

6.2 Reversing the chain rule (optional reading).

Recall that we used the chain rule to differentiate composite functions.

If we have a function of a function:

$$f(g(x))$$

then the derivative of this is

derivative of outer function $\times$ *derivative of inner function*

i.e.

$$f'(g(x)) \times g'(x)$$

Thus the form of the derivative is

a function of a function $\times$ *the derivative of the inner function.*

i.e.

$$h(g(x)) \times g'(x) \quad \text{where } h = f'.$$

This means that if we are given a function of this form then it is possible to find an antiderivative by reversing the chain rule. It is this type of function which we can integrate using the substitution method.

Example.

The derivative of $\sin(x^2)$ is $(\cos(x^2) \times 2x)$ and so an antiderivative of $(\cos(x^2) \times 2x)$ is $\sin(x^2)$. In terms of integrals

$$\int 2x\cos(x^2)\ dx = \sin(x^2) + c.$$

Examples.

1.

$$I = \int x^2\sqrt{(x^3+a)}\ dx = \int x^2(x^3+a)^{\frac{1}{2}}\ dx.$$

Let $u = x^3 + a$, (so that $\sqrt{x^3+a} = u^{1/2}$). Then $du = 3x^2\ dx$. Re-writing the integral as

$$I = \frac{1}{3}\int 3x^2\ (x^3+a)^{\frac{1}{2}}\ dx$$

we then make the substitutions and integrate,

$$I = \frac{1}{3}\int u^{1/2}\ du = \frac{1}{3} \times \frac{2u^{3/2}}{3} = \frac{2u^{3/2}}{9} + c.$$

Now substitute back,

$$I = \frac{2}{9}(x^3+a)^{\frac{3}{2}} + c.$$

2.

$$I = \int x^5 (x^3+1)^4 \ dx.$$

Let $u = x^3 + 1$. Then $du = 3x^2 \ dx$. Re-writing the integral as

$$I = \frac{1}{3} \int 3x^2 \ dx \ (x^3+1)^4 \ x^3$$

we see that there is still an x^3 unaccounted for. However we have $u = x^3 + 1$ and so $x^3 = u - 1$, and we have all the pieces. Writing the integral in terms of u we have

$$I = \frac{1}{3} \int du \, u^4 \ (u-1) = \frac{1}{3} \int u^4 (u-1) \ du.$$

Multiply out the integrand and then integrate:

$$I = \frac{1}{3} \int u^5 - u^4 \ du = \frac{1}{3} \left(\frac{u^6}{6} + \frac{u^5}{5} \right) + c.$$

Thus

$$I = \frac{1}{3} \left(\frac{(x^3+1)^6}{6} + \frac{(x^3+1)^5}{5} \right) + c.$$

3.

$$I = \int \cos ax \ dx.$$

Let $u = ax$. Then $du = a \ dx$, so that $dx = \frac{1}{a} \ du$. Then

$$I = \frac{1}{a} \int \cos u \ du = \frac{1}{a} \sin u = \frac{1}{a} \sin ax + c.$$

This gives us a *standard integral*:

$$\int \cos ax \ dx = \frac{1}{a} \sin ax + c.$$

Similarly

$$\int \sin ax \ dx = -\frac{1}{a} \cos ax + c.$$

Any Calculus textbook will contain a list of *standard integrals*. These are either very common integrals or very complicated ones which you will not be expected to work out for yourselves but may need to use at some point.

4. This next example uses a trigonometric identity and also splits up the integral.

$$\begin{aligned} I &= \int \cos^2 x \ dx \\ &= \frac{1}{2} \int 1 + \cos 2x \ dx \\ &= \frac{1}{2} \int 1 \ dx + \frac{1}{2} \int \cos 2x \ dx \\ &= \frac{x}{2} + \frac{1}{4} \sin 2x + c, \end{aligned}$$

using a standard integral. Notice that we only need add one constant of integration.

5.
$$I = \int (t^2+1)\exp(t^3+3t)\ dt$$

Recall that $\exp(t^3+3t)$ is another way of writing $e^{(t^3+3t)}$ and so we could also write

$$I = \int (t^2+1)e^{t^3+3t}\ dt.$$

Let $u = t^3+3t$. Then
$$du = 3t^2+3\ dt = 3(t^2+1)\ dt$$

and so
$$(t^2+1)\ dt = \tfrac{1}{3}\ du.$$

Hence
$$I = \tfrac{1}{3}\int e^u\ du = \tfrac{1}{3}e^u = \tfrac{1}{3}e^{t^3+3t}+c.$$

6.
$$I = \int \frac{\ln x^2}{x}\ dx.$$

Note that we CANNOT cancel the x on the bottom with the x^2. The x^2 belongs to the log function, the x does not.

We know that the derivative of $\ln x$ is $\frac{1}{x}$ but the x^2 seems to pose a problem. However we can use the power law for logarithms and re-write the integral:

$$I = \int \frac{2\ln x}{x}\ dx = 2\int \frac{\ln x}{x}\ dx.$$

Let $u = \ln x$. Then $du = \dfrac{1}{x}\ dx$ and

$$I = 2\int u\ du = 2\left(\frac{u^2}{2}\right) = u^2 = (\ln x)^2+c.$$

(Strictly speaking we need to stipulate that $x > 0$. Why?)

7. Sometimes we need to make more than one substitution.

$$I = \int 2x\sin^2(x^2+1)\cos(x^2+1)\ dx$$

Let $u = x^2+1$. Then $du = 2x\ dx$, and

$$I = \int \sin^2 u\cos u\ du.$$

(Choosing the next substitution is a little tricky. Letting $t = \cos u$ does not work (you should verify this) and so we will use the substitution $t = \sin u$. The subsitution $t = \sin^2 u$ will also work.)

Let $t = \sin u$. Then $dt = \cos u \ du$ and

$$I = \int t^2 \ dt = \frac{t^3}{3} + c.$$

Now we must substitute for t and then u:

$$I = \frac{t^3}{3} = \frac{\sin^3 u}{3} = \frac{\sin^3(x^2+1)}{3} + c.$$

6.3 Two more standard integrals.

During the next exercise you should find yourself using these two standard integrals.

1.
$$\int e^{ax} \ dx = \frac{1}{a}e^{ax} + c,$$

and

2.
$$\int \frac{g'(x)}{g(x)} \ dx = \ln|g(x)| + c$$

a function with its derivative on top integrates to the log of that function.

After the next exercise you will be ready for quiz 2.

6.3.1 Exercise.

Evaluate the following integrals using the substitution method or an appropriate standard integral.

1. $\int \sin^2 u \cos u \; du$, using the substitution $t = \sin^2 u$
2. $\int \frac{1}{\sqrt{x-1}} \; dx$
3. $\int \frac{2}{(x+1)^3} \; dx$
4. $\int (2x^3+5x)(6x^2+5) \; dx$
5. $\int 6(2x^3+5x)(x^2+1) \; dx$
   [split the integral and use the previous answer]
6. $\int \sin \pi x - \cos \pi x \; dx$
7. $\int \cos \frac{\pi}{2}t - \sin \frac{\pi}{2}t \; dt$
8. $\int t(t^2+1)^{\frac{1}{2}} \; dt$
9. $\int 3x^2(x^3+1) \; dx$
10. $\int \sin 5x \; dx$
11. $\int \cos(5x+1) \; dx$
12. $\int \sin(3t+1) \; dt$
13. $\int (2x+5)^{\frac{7}{2}} \; dx$
14. $\int \frac{dt}{\sqrt{5-3t}}$
15. $\int e^{5t} \; dt$
16. $\int e^{3x+1} \; dx$
17. $\int \frac{e^{\sqrt{x}}}{2\sqrt{x}} \; dx$
18. $\int 4te^{\pi t^2} \; dt$
19. $\int \frac{\ln(x^{-1})}{x} \; dx$
20. $\int \frac{\ln \sqrt{x}}{x} \; dx$
21. $\int \frac{\ln(4x^2)}{x} \; dx$
22. $\int \frac{dx}{x-2}$
23. $\int \frac{6t+2}{3t^2+2t} \; dt$
24. $\int \frac{5x^{2/3}-6x^2}{3x^{5/3}-2x^3} \; dx$
25. $\int \tan x \; dx$
26. $\int \frac{\sec^2 x}{\tan x} \; dx$
27. $\int 3x \sin(3x^2+1) \; dx$
28. $\int \frac{\cos(3\sqrt{x})}{\sqrt{x}} \; dx$
29. $\int 3\pi \sin^2(\pi x) \cos(\pi x) \; dx$
30. $\int 5x^2 \cos(x^3) \; dx$
31. $\int t \cos(t^2+1) \; dt$
32. $\int t^2 \exp(t^3+3) \; dt$
    [remember $\exp(x) = e^x$]

33. $\int \sin x \cos x \; dx$,
using the substitutions

(i) $u = \sin x$

(ii) $u = \cos x$

34. $\int \sin x \cos x \; dx$, using the trigonometric identity $\sin 2x = 2 \sin x \cos x$

35. $\int \sin^2 x \; dx$,
using a trigonometric identity

36. $\int \cos^2 3x \; dx$,
using a trigonometric identity

37. $\int x^{-2} \sin^2(x^{-1}) \; dx$

38. $\int x^3 \sqrt{x^2+3} \; dx$

39. $\int x^{1/3} \sqrt{x^{2/3}-1} \; dx$

40. $\int 6x^5 (x^3+1)^{-2} \; dx$

41. $\int \frac{x \exp \sqrt{x^2+1}}{\sqrt{x^2+1}} \; dx$

42. $\int \frac{dx}{\sqrt{x}(1+\sqrt{x})}$

43. $\int \frac{1}{x+\sqrt{x}} \; dx$

44. $\int \frac{1}{\sqrt{x}(1+\sqrt{x})^{1/2}} \; dx$

45. $\int \frac{3}{4\sqrt{x}(1-\sqrt{x})^{5/2}} \; dx$

46.
$$\int 6(2x^3+1)\cos^2(x^4+2x)\sin(x^4+2x)dx$$

47. and finally, something a little different.

$$\int \frac{dx}{\sqrt{1-x^2}}$$

using the substitution $x = \sin u$.

For this you need to know the notation for the inverse sine function, i.e. how to write the instruction to undo the sine function. The most common notation nowadays is $\sin^{-1}$ and so if $x = \sin u$, $\sin^{-1} x = \sin^{-1}(\sin u) = u$.

Unfortunately $\sin^{-1}$ can be easily misinterpreted as $\frac{1}{\sin}$ and it is much safer to use the old-fashioned notation *arc sin*. Thus if $x = \sin u$ then $u = \arcsin x$. Sadly the arcsin notation is not used on calculators; it doesn't fit....