Quadruple Inequalities:
Between Cauchy–Schwarz and Triangle

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Abstract

We prove a set of inequalities that interpolate the Cauchy–Schwarz inequality and the triangle inequality. Every nondecreasing, convex function with a concave derivative induces such an inequality. They hold in any metric space that satisfies a metric version of the Cauchy–Schwarz inequality, including all CAT(0) spaces and, in particular, all Euclidean spaces. Because these inequalities establish relations between the six distances of four points, we call them quadruple inequalities.

Contents

1 Introduction

1.1 Relating Cauchy–Schwarz and Triangle

The Cauchy–Schwarz inequality states that in any inner product space $(V, \langle\cdot,\cdot\rangle)$, we have

$$\langle u, v\rangle \leq \|u\|\|v\| \tag{1}$$

for all $u, v \in V$, where $\|u\| = \sqrt{\langle u, u\rangle}$. In any metric space $(\mathcal{Q}, d)$ the triangle inequality

$$\overline{y,z} \leq \overline{y,p} + \overline{p,z} \tag{2}$$

is true for all $p, y, z \in \mathcal{Q}$, where we use the short notation $\overline{y,z} := d(y,z)$. Now, consider the inequality

$$\tau(\overline{y,q}) - \tau(\overline{y,p}) - \tau(\overline{z,q}) + \tau(\overline{z,p}) \leq L_\tau\,\overline{q,p}\,\tau'(\overline{y,z}) \tag{3}$$

for $y, z, q, p \in \mathcal{Q}$, a differentiable function $\tau\colon [0,\infty) \to \mathbb{R}$ with derivative τ', and a constant $L_\tau \in [0,\infty)$. We call (3) a *quadruple inequality* [Sch19] as it establishes a relationship between the six distances among four points, see Figure 1. If we plug the identity $\tau = \tau_1 := (x \mapsto x)$ and $L_\tau = 2$ into (3), we obtain

$$\overline{y,q} - \overline{y,p} - \overline{z,q} + \overline{z,p} \leq 2\,\overline{q,p}\,. \tag{4}$$

The triangle inequality (2) implies (4). Furthermore, in a symmetric distance space $(\mathcal{Q}, d)$ [DD16], where d does not necessarily fulfill the triangle inequality, (4) also implies (2) by setting $z = q$. Next, let us evaluate (3) with $\tau = \tau_2 := (x \mapsto x^2)$ and $L_\tau = 1$. We get

$$\overline{y,q}^2 - \overline{y,p}^2 - \overline{z,q}^2 + \overline{z,p}^2 \leq 2\,\overline{q,p}\,\overline{y,z}\,. \tag{5}$$

If we assume that the metric space $(\mathcal{Q}, d)$ is induced by an inner product space $(V, \langle\cdot,\cdot\rangle)$, i.e., $\mathcal{Q} = V$ and $d(q,p) = \|q - p\|$, then (5) becomes

$$2\,\langle q - p\,,\, z - y\rangle \leq 2\|q-p\|\|y-z\|\,. \tag{6}$$

Thus, in this case, (5) is equivalent to (1). Hence, we can consider (5) to be a generalization of the Cauchy–Schwarz inequality to metric spaces. Equation (5) is not true in every metric space. But it is true and characteristic [BN08] in non-positively curved geodesic spaces, which are called $\mathsf{CAT}(0)$ spaces or, if they are complete, Hadamard or NPC spaces. In these spaces, (5) is also known as *four point cosq condition* [BN08] or *Reshetnyak's Quadruple Comparison* [Stu03, Proposition 2.4].

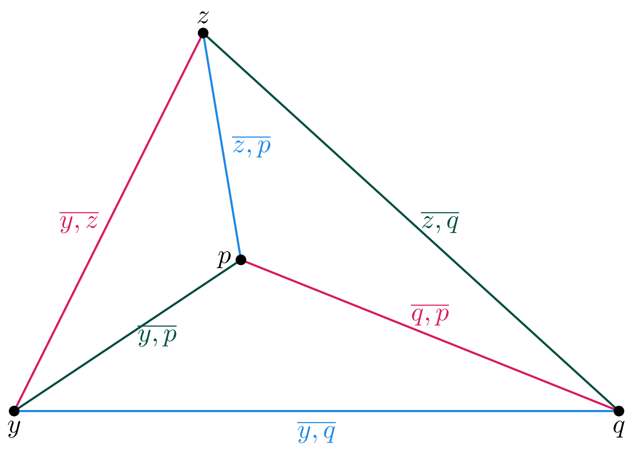

Figure 1: Four points and their six distances.

1.2 Contributions

The functions τ_1, τ_2 are both nondecreasing, convex, and have a concave derivative. They can be considered as edge cases of all functions with these properties: As a linear function, τ_1 can be thought of as "least convex" of all convex functions. Similarly, τ_2, which has a linear and strictly increasing derivative, is a "most convex" function among all functions with a concave derivative. As our main result, we show that in all metric spaces with the property (5), inequality (3) is true for all functions "between" τ_1 and τ_2, i.e., for all nondecreasing, convex functions with a concave derivative. In this sense, we "interpolate" the triangle and the Cauchy–Schwarz inequality.

Theorem 1. Let $(\mathcal{Q}, d)$ be a metric space. Let $y, z, q, p \in \mathcal{Q}$. Assume

$$\overline{y,q}^2 - \overline{y,p}^2 - \overline{z,q}^2 + \overline{z,p}^2 \leq 2\, \overline{q,p}\, \overline{y,z}\,.$$

Let $\tau\colon [0,\infty) \to \mathbb{R}$ be differentiable. Assume τ is nondecreasing, convex and has a concave derivative τ'. Then

$$\tau(\overline{y,q}) - \tau(\overline{y,p}) - \tau(\overline{z,q}) + \tau(\overline{z,p}) \leq 2\, \overline{q,p}\, \tau'(\overline{y,z})\,. \tag{7}$$

We call functions τ that satisfy (3) when (5) is true quadruple transformations:

Definition 2 (Quadruple transformation)**.** Let $\tau\colon (0,\infty) \to \mathbb{R}$ be differentiable. We call τ a *quadruple transformation* if there is a constant $L_\tau \in [0,\infty)$ such that the following condition holds: For every metric space $(\mathcal{Q}, d)$ and pairwise distinct points $y, z, q, p \in \mathcal{Q}$ such that

$$\overline{y,q}^2 - \overline{y,p}^2 - \overline{z,q}^2 + \overline{z,p}^2 \leq 2\, \overline{q,p}\, \overline{y,z}\,,$$

we also have

$$\tau(\overline{y,q}) - \tau(\overline{y,p}) - \tau(\overline{z,q}) + \tau(\overline{z,p}) \leq L_\tau\, \overline{q,p}\, \tau'(\overline{y,z})\,.$$

Define the *quadruple constant* of a quadruple transformation τ as the smallest possible choice of L_τ and denote it as L_τ^*. Let $\mathcal{T}$ denote the *set of all quadruple transformations.*

Let $\mathcal{S}$ be the set of all nondecreasing, convex, and differentiable functions $\tau\colon [0,\infty) \to \mathbb{R}$ with concave derivative. Then, by Theorem 1, $\mathcal{S} \subseteq \mathcal{T}$. We show that, for any $\tau \in \mathcal{S}$, the quadruple constant L_τ^* is in the interval $[1,2]$, see Proposition 10. Further lower bounds on L_τ^* are discussed in Proposition 9. Slightly stronger versions of Theorem 1 are presented in Theorem 23 and Theorem 24.

Let $\mathcal{S}_0$ be set of functions in $\mathcal{S}$ with $\tau(0) = 0$. For $\tau \in \mathcal{S}_0$, the right-hand side of (7) can be bounded by $2\tau(\overline{q,p}) + 2\tau(\overline{y,z})$, see Corollary 25. In inner product spaces, we derive a stronger upper bound:

Theorem 3. Let $(V, \langle \cdot\,, \cdot \rangle)$ be an inner product space with induced metric d. Let $y, z, q, p \in V$. Let $\tau \in \mathcal{S}_0$. Then

$$\tau(\overline{y,q}) - \tau(\overline{y,p}) - \tau(\overline{z,q}) + \tau(\overline{z,p}) \leq \tau(\overline{q,p}) + \tau(\overline{y,z})\,. \tag{8}$$

Note that $\tau(a) + \tau(b)$ can be much larger than $a\tau'(b)$ for $a, b \in [0, \infty)$.

1.3 Related Literature

For a history of the Cauchy–Schwarz inequality and many of its extension, [Ste04] is highly recommended. The book [DD16] is an excellent reference for many metric related concepts.

1.3.1 Convex Analysis

Theorem 3 is related to *Karamata's inequality* [Kar32]: Let $f\colon \mathbb{R} \to \mathbb{R}$ be a convex and nondecreasing function. Let $a_1, \dots, a_n, b_1 \dots, b_n \in \mathbb{R}$ with

$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i \tag{9}$$

for $k = 1, \dots, n$. Then

$$\sum_{i=1}^{n} f(a_i) \leq \sum_{i=1}^{n} f(b_i)\,. \tag{10}$$

If we set $f = \tau$, $n = 4$, $a_1 = \overline{y,q}$, $a_2 = \overline{z,p}$, $a_3 = a_4 = 0$, $b_1 = \overline{q,p}$, $b_2 = \overline{y,z}$, $b_3 = \overline{y,p}$, $b_4 = \overline{z,q}$, then Karamata's inequality proves Theorem 3 for configurations of distances that fulfill (9). But this does not cover all cases.

1.3.2 Quadruple Inequality

Theorem 1 extends [Sch19, Theorem 3], which states that $\tau_\alpha := (x \mapsto x^\alpha)$ with $\alpha \in [1, 2]$ is a quadruple transformation and, together with [Sch19, Appendix E], implies $L^*_{\tau_\alpha} = 2^{2-\alpha}$. Moreover, in [Sch19, Appendix E] it is shown that $\alpha \in [1, 2]$ are the only positive real exponents with $\tau_\alpha \in \mathcal{T}$. Note that $\tau_\alpha \in \mathcal{S}$ for $\alpha \in [1, 2]$. The proof of Theorem 1 requires new ideas compared to the one of [Sch19, Theorem 3], e.g., as we cannot take derivatives with respect to α. Our generalization to all functions $\tau \in \mathcal{S}$ comes at the cost of a larger constant on the right-hand side: Theorem 1 applied to $\tau = \tau_\alpha$ yields a constant factor $L_{\tau_\alpha} = 2$, which is strictly greater than $L^*_{\tau_\alpha}$ for $\alpha > 1$.

1.3.3 Metric Geometry

Aside from $\mathsf{CAT}(0)$ spaces briefly discussed above, some further ideas in metric geometry seem relevant in the context of quadruple inequalities.

A function $\varphi\colon [0, \infty) \to [0, \infty)$ is called *metric preserving*, if $\varphi \circ d$ is a metric for any metric space $(\mathcal{Q}, d)$. See [Cor99] for an overview. As the quadruple inequality (3) with $\tau = \tau_2$ is a condition for Theorem 1, we may think of the main result as stating that a Cauchy–Schwarz-like inequality is persevered under transformation with τ. But note that the right-hand side of (3) is not written in terms of $\tau \circ d$.

A metric space $(\mathcal{Q}, d)$ has the *Euclidean k-point property* [Blu70, Definition 50.1] if any k-tuple of points in $\mathcal{Q}$ has an isometric embedding in the Euclidean space $\mathbb{R}^{k-1}$. If $(\mathcal{Q}, d)$ has the Euclidean

4-point property, then (5) is fulfilled. For $\gamma \in (0,\infty)$, let $\varphi_\gamma(x) := x^\gamma$. This function is metric preserving for $\gamma \le 1$. According to [Blu70, Theorem 52.1], $(\mathcal{Q}, \varphi_\gamma \circ d)$ has the Euclidean 4-point property for all $\gamma \le 1/2$. Furthermore, $\gamma = 1/2$ is the largest exponent with this property. Thus, $(\mathcal{Q}, \varphi_\gamma \circ d)$, fulfills (5) for $\gamma \in (0, 1/2]$. In particular,

$$\overline{y,q}^{2\gamma} - \overline{y,p}^{2\gamma} - \overline{y,p}^{2\gamma} + \overline{y,p}^{2\gamma} \le 2\, \overline{q,p}^{\gamma}\, \overline{y,z}^{\gamma} . \tag{11}$$

As $\tilde{d} = d^{2\gamma}$ is a metric — $x \mapsto x^{2\gamma}$ is metric preserving for $\gamma \in (0, 1/2]$ — we obtain from (4),

$$\overline{y,q}^{2\gamma} - \overline{y,p}^{2\gamma} - \overline{y,p}^{2\gamma} + \overline{y,p}^{2\gamma} \le 2 \min\left(\overline{q,p}^{2\gamma}, \overline{y,z}^{2\gamma}\right) , \tag{12}$$

which also implies (11).

The Euclidean 4-point property can be weakened for $\mathsf{CAT}(0)$ spaces. A metric space $(\mathcal{Q}, d)$ fulfills the $\mathsf{CAT}(0)$ 4-point condition [BH99, Definition II.1.10] if, for all $y, z, q, p \in \mathcal{Q}$, there are $\bar{y}, \bar{z}, \bar{q}, \bar{p} \in \mathbb{R}^2$ such that

$$\overline{y,q} = \|\bar{y} - \bar{q}\| , \qquad \overline{y,p} = \|\bar{y} - \bar{p}\| , \qquad \overline{z,q} = \|\bar{z} - \bar{q}\| , \qquad \overline{z,p} = \|\bar{z} - \bar{p}\| ,$$
$$\overline{q,p} \le \|\bar{q} - \bar{p}\| , \qquad \overline{y,z} \le \|\bar{y} - \bar{z}\| .$$

Every $\mathsf{CAT}(0)$ space fulfills the $\mathsf{CAT}(0)$ four-point condition, see [Reš68] or [BH99, Proposition II.1.11].

Another famous 4-point property is *Ptolemy's inequality*: A metric space $(\mathcal{Q}, d)$ is called *Ptolemaic* if, for all $y, z, q, p \in \mathcal{Q}$, we have

$$\overline{y,q}\,\overline{z,p} + \overline{y,p}\,\overline{z,q} \le \overline{q,p}\,\overline{y,z} . \tag{13}$$

Every inner product space is Ptolemaic. If a normed vector space is Ptolemaic, then it is an inner product space. All $\mathsf{CAT}(0)$ spaces are Ptolemaic. A complete Riemannian manifold is Ptolemaic if and only if it is $\mathsf{CAT}(0)$ [BFW09, Theorem 1.1]. Each geodesically connected metric space satisfying the τ_2-quadruple inequality is Ptolemaic, but a geodesically connected Ptolemaic metric space is not necessarily $\mathsf{CAT}(0)$ [FLS07; BN08].

Strongly related to Theorem 3 is the concept of roundness of a metric space: A value $\alpha \in (0,\infty)$ is called *roundness exponent* of a metric space $(\mathcal{Q}, d)$ if, for all $y, z, q, p \in \mathcal{Q}$,

$$\overline{y,q}^{\alpha} - \overline{y,p}^{\alpha} - \overline{z,q}^{\alpha} + \overline{z,p}^{\alpha} \le \overline{q,p}^{\alpha} + \overline{y,z}^{\alpha} . \tag{14}$$

Let $R = R(\mathcal{Q}, d)$ be the set of all roundness exponents of $(\mathcal{Q}, d)$. The *roundness* $r = r(\mathcal{Q}, d)$ of $(\mathcal{Q}, d)$ is the supremum of the roundness exponents $r := \sup R$. By the triangle inequality and the metric preserving property of $(x \mapsto x^\alpha)$ for $\alpha \in (0, 1]$, we have $(0, 1] \subseteq R$ for all metric spaces. The functions spaces $L_p(0, 1)$ have roundness p for $p \in [1, 2]$ [Enf69]. For a geodesic metric space, roundness $r = 2$ is essentially equivalent to being $\mathsf{CAT}(0)$, see [BN08, Remark 7]. A metric space is called *ultrametric* if the triangle inequality can be strengthened to $\overline{y,z} \le \max(\overline{y,p}, \overline{z,p})$ for all points y, z, p. Every ultrametric space can be isometrically embedded in a Hilbert space, see, e.g., [Fav+14, Corollary 5.4]. A metric space is ultrametric if and only if $r = \infty$, [Fav+14, Theorem 5.1]. Then $R = (0,\infty)$, [Fav+14, Proposition 2.7]. In general, R is not necessarily an interval [Enf70, Remark p. 254]. But if $(\mathcal{Q}, d)$ is a (subset of a) Banach space with the metric d induced by its norm, then $R = (0, r]$ with $r \in [1, 2]$, [Enf70, Proposition 4.1.2]. In particular, (14) holds for $\alpha \in (0, 2]$ in all inner product spaces. A metric space is called *additive* if

$$\overline{y,q} + \overline{z,p} \le \max(\overline{y,p} + \overline{z,q}, \overline{q,p} + \overline{y,z}) \tag{15}$$

for all points y, z, q, p. Every ultrametric space is additive. Every additive metric space is Ptolemaic. Additive metric spaces have roundness $r \ge 2$ [Fav+14, Proposition 4.1].

1.3.4 Martingale Theory

Nondecreasing, convex functions with concave derivative play an important role in the Topchii–Vatutin inequality of martingales, see [TV97, Theorem 2] and [AR03]: For a suitably integrable martingale $(M_n)_{n\in\mathbb{N}_0}$, we have

$$\mathbb{E}[\tau(|M_n|) - \tau(|M_0|)] \le 2 \sum_{k=1}^{n} \mathbb{E}[\tau(|M_k - M_{k-1}|)] \tag{16}$$

for all $\tau \in \mathcal{S}_0$, where $\mathbb{E}[\cdot]$ denotes the expectation. In this context, the functions $\tau \in \mathcal{S}_0$ are named *weakly convex*. Moreover, [TV97, Lemma 6] gives a weaker version of Theorem 3: Let $\tau \in \mathcal{S}_0$. For $a, b \in [0, \infty)$ with $a \geq b$, it was shown that $\tau(a+b) + \tau(a-b) \leq 2\tau(a) + 2\tau(b)$.

1.3.5 Statistics

Theorem 1 can be applied to prove rates of convergence for certain kinds of means [Sch19]: We may want to calculate a mean value of some sample points in a metric spaces. One candidate for this is the *Fréchet mean* [Fré48], also called *barycenter*. It is the (set of) minimizer(s) of the squared distance to the sample points. If Y is a random variable with values in a metric space $(\mathcal{Q}, d)$, the Fréchet mean is $\arg\min_{q\in\mathcal{Q}} \mathbb{E}[\overline{Y,q}^2]$, where we assume $\mathbb{E}[\overline{Y,q}^2] < \infty$ for all $q \in \mathcal{Q}$. Similarly, one can define the Fréchet median [FVJ09] as $\arg\min_{q\in\mathcal{Q}} \mathbb{E}[\overline{Y,q}]$, or a more general τ-Fréchet mean [Sch22] as $\arg\min_{q\in\mathcal{Q}} \mathbb{E}[\tau(\overline{Y,q})]$ for functions $\tau\colon [0,\infty) \to \mathbb{R}$. Given a sequence of independent random variables $Y_1, Y_2, \dots$ with the same distribution as Y, a standard task in statistics is to bound the distance between the sample statistics and its corresponding population version. In our case, assume the τ-Fréchet mean is unique and define

$$m := \arg\min_{q\in\mathcal{Q}} \mathbb{E}[\tau(\overline{Y,q})]\,, \qquad \hat{m}_n := \arg\min_{q\in\mathcal{Q}} \frac{1}{n}\sum_{i=1}^{n} \tau(\overline{Y_i,q})\,.$$

We want to bound $\overline{\hat{m}_n, m}$ depending on n. One can employ quadruple inequalities such as (3) to obtain a suitable upper bound [Sch19, Theorem 1]. This approach is particularly useful, if we do not want to make the assumption that the diameter of the metric space $\sup_{q,p\in\mathcal{Q}} \overline{q,p}$ is finite. With Theorem 1, one can obtain such a bound for τ-Fréchet means with $\tau \in \mathcal{S}$ (under some conditions). We emphasize that this is only possible with (3) and not with (8). Noteworthy examples of $\tau \in \mathcal{S}$ in this context, aside from $\tau = \tau_\alpha$, are the Huber loss $\tau_{\mathsf{H},\delta}$ [Hub64] and the Pseudo-Huber loss $\tau_{\mathsf{pH},\delta}$ [Cha+94] for $\delta \in (0,\infty)$,

$$\tau_{\mathsf{H},\delta}(x) := \begin{cases} \frac{1}{2}x^2 & \text{for } x \leq \delta\,, \\ \delta(x - \frac{1}{2}\delta) & \text{for } x > \delta\,, \end{cases} \qquad \tau_{\mathsf{pH},\delta}(x) := \delta^2\left(\sqrt{1 + \frac{x^2}{\delta^2}} - 1\right)\,,$$

as well as $x \mapsto \ln(\cosh(x))$ [Gre90]. These functions are of great interest in robust statistics and image processing as their respective minimizers combine properties of the classical mean (τ_2-Fréchet mean) and the median (τ_1-Fréchet mean).

1.4 Outline

In the remaining sections, we first discuss the set $\mathcal{T}$, i.e., the set of quadruple transformations, see section 2. We continue with a discussion of the set $\mathcal{S}$, i.e., nondecresing, convex functions with concave derivative, in section 3. Thereafter, we prove our main result, i.e., $\mathcal{S} \subseteq \mathcal{T}$. The basic ideas of the proof and variations of the main result are presented in section 4. The technical details can be found in appendix B and C. The proof of Theorem 3 can be found in appendix A. In section 5 we discuss implications of the main results and open questions.

2 Quadruple Transformations

We explore some properties of quadruple functions $\tau \in \mathcal{T}$ and their quadruple constant L^*_τ.

2.1 Properties

Lemma 4 (Constant functions)**.**

(i) For $c \in \mathbb{R}$, let $\tau_c := (x \mapsto c)$. Then $\tau_c \in \mathcal{T}$ with $L^*_\tau = 0$.

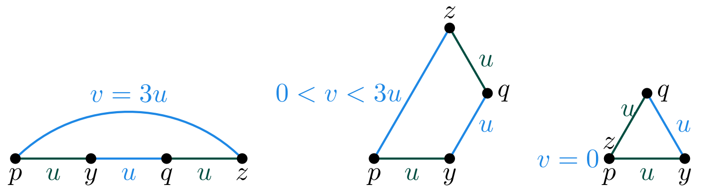

Figure 2: Construction for the proof of Lemma 4 (ii).

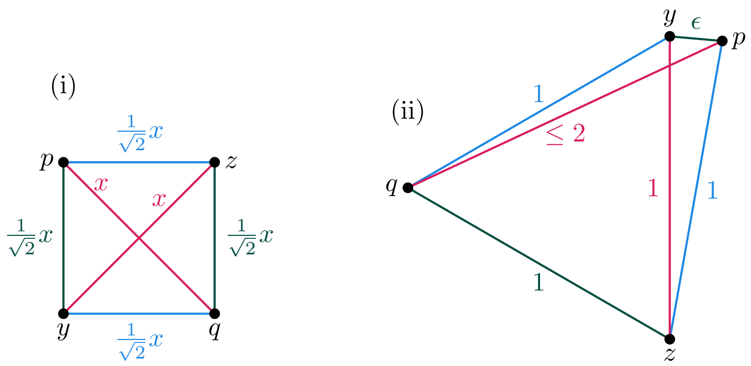

Figure 3: Constructions for the proof of Lemma 5.

(ii) If $\tau \in \mathcal{T}$ with $L_\tau = 0$, then there is $c \in \mathbb{R}$ such that $\tau = \tau_c$.

Proof. The first part is trivial. For the second part, let $u \in (0, \infty)$ and $v \in (0, 3u]$. Take $y, z, q, p \in \mathbb{R}^2$ such that $\overline{y,q} = \overline{y,p} = \overline{z,p} = u$ and $\overline{z,q} = v$, or $\overline{y,q} = \overline{y,p} = \overline{z,q} = u$ and $\overline{z,p} = v$, see Figure 2, to obtain

$$|\tau(u) - \tau(v)| \leq 0 \tag{17}$$

from (3). This can only be fulfilled for constant functions. $\square$

Lemma 5. Let $\tau \in \mathcal{T}$. Then

$\tau'(x) \geq 0$ for all $x \in (0, \infty)$ and

(ii) $\inf_{x \in (0,\infty)} \tau(x) > -\infty$.

Proof. (i) Let $y, z, q, p \in \mathbb{R}^2$ form a square with diagonal $x \in (0, \infty)$, see Figure 3 (i). Then, by (3),

$$0 \leq L_\tau x \tau'(x)\,. \tag{18}$$

As $x > 0$ and $L_\tau \geq 0$ with equality only if τ is constant, we have $0 \leq \tau'(x)$.

(ii) By (i), we only have to check that $\tau(\varepsilon)$ is bounded below for $\varepsilon \in (0, 1]$. Let $\varepsilon > 0$. In the Euclidean plane $\mathbb{R}^2$, set $y = (0,1), z = (0,0), q = (-\sqrt{3}/2, 1/2), p = (\cos(\alpha), \sin(\alpha))$, where $\alpha \in [0, \pi/2]$ is chosen such that $\overline{y,p} = \varepsilon$, see Figure 3 (ii). Then $\overline{y,q} = \overline{z,q} = \overline{y,z} = \overline{z,p} = 1$ and $\overline{q,p} \leq 2$. Then, using (3) and (i), we obtain

$$\tau(1) - \tau(\varepsilon) \leq 2L_\tau \tau'(1)\,. \tag{19}$$

This yields a constant lower bound of $\tau(1) - 2L_\tau \tau'(1) > -\infty$ on $\tau(\varepsilon)$ for all $\varepsilon \in (0, 1]$. □

Next, we extend the domain of $\tau \in \mathcal{T}$ in a consistent way to include 0.

Proposition 6. Let $\tau \in \mathcal{T}$. Define $\tau(0) := \lim_{x \searrow 0} \tau(x)$ and $\tau'(0) := \liminf_{x \searrow 0} \tau'(x)$. Then, given (5), inequality (3) holds for any quadruple of points (not necessarily pairwise distinct).

Proof. By Lemma 5, $\tau(0)$ is well-defined. We have to show that (5) implies (3) in the cases where at least one distance is zero. In the case $y = z$, the left-hand side of (3) vanishes and the right-hand side is nonnegative by Lemma 5 (i). Next, consider the case $y, z, q, p \in \mathcal{Q}$ with $y \neq z$ but at least two points being identical. As any triplet of points in a metric space can be isometrically embedded in the Euclidean plane $\mathbb{R}^2$, this can also be done for (y, z, q, p). Furthermore, we can find a sequence of quadruples of points $(y_n, z_n, q_n, p_n)_{n\in\mathbb{N}} \subseteq (\mathbb{R}^2)^4$ with $\|y_n - z_n\| = \overline{y, z}$ and all other distances strictly positive and convergent to the respective distance of the points y, z, q, p. By continuity of τ, the definition of $\tau(0)$, and the constant value of $\|y_n - z_n\|$, (3) holds in the limit $n \to \infty$. □

Proposition 7 (Power functions)**.** Let $\tau_\alpha = (x \mapsto x^\alpha)$ for $\alpha \in (0, \infty)$. We have $\tau_\alpha \in \mathcal{T}$ if and only if $\alpha \in [1, 2]$. In this case, the quadruple constant is $L^*_{\tau_\alpha} = 2^{2-\alpha}$.

Proof. [Sch19, Theorem 3 and Appendix E]. □

2.2 Lower Bounds on the Quadruple Constant

Lemma 8. The quadruple constant L^*_τ is well-defined in the sense that if $\tau \in \mathcal{T}$ there is a smallest value $L^*_\tau \in [0, \infty)$ such that (3) is true for all metric spaces and quadruples of points therein.

Proof. Let $\tau \in \mathcal{T}$. Let $(L_n)_{n\in\mathbb{N}} \subseteq [0, \infty)$ be a decreasing sequence with $L_\infty := \lim_{n\to\infty} L_n$. Assume (3) is true for all $L_\tau = L_n$. We need to show that (3) is also true for $L_\tau = L_\infty$. If it were false, there would be a metric space $(\mathcal{Q}, d)$ with points $y, z, q, p \in \mathcal{Q}$ such that

$$\tau(\overline{y, q}) - \tau(\overline{y, p}) - \tau(\overline{z, q}) + \tau(\overline{z, p}) > L_\infty \, \overline{q, p} \, \tau'(\overline{y, z}) \,. \tag{20}$$

As the inequality is strict, it would also hold when L_∞ is replaced by a L_n for a sufficiently large n, which is a contradiction. □

Proposition 9. Let $\tau \in \mathcal{T}$. Let $u, v \in (0, \infty)$. Then, assuming the denominator is not 0,

$$L^*_\tau \geq 2\frac{\tau(u) - \tau(0)}{u\tau'(u)} \,, \tag{21}$$

(ii)

$$L^*_\tau \geq \frac{\tau(2u) - \tau(0)}{2u\tau'(u)} \,, \tag{22}$$

(iii)

$$L^*_\tau \geq \frac{\tau'(u) - \tau'(v)}{\tau'(|u - v|)} \,, \tag{23}$$

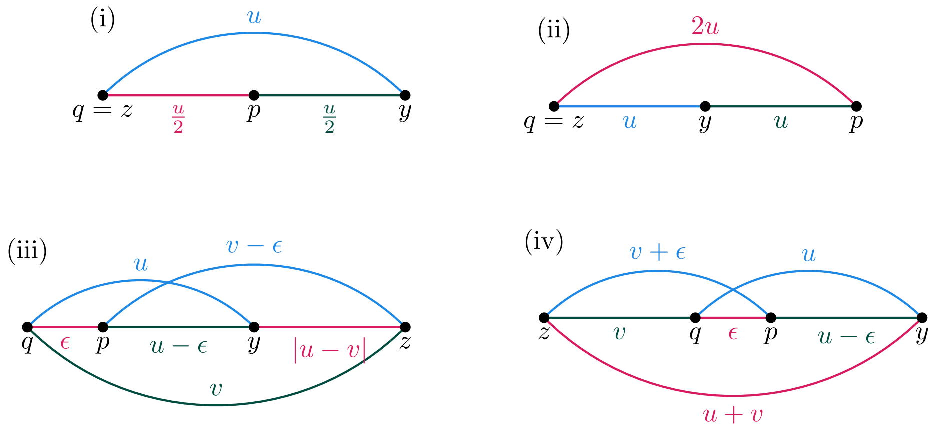

Figure 4: Constructions for the proof of Proposition 9.

(iv)

$$L_\tau^* \geq \frac{\tau'(u) + \tau'(v)}{\tau'(u+v)} \,. \tag{24}$$

Proof. For all parts, we apply (3) in the metric space of the real line with Euclidean metric, see Figure 4.

Set $z = q = 0$, $y = u$, $p = u/2$. Then (3) becomes

$$\tau(u) - \tau(0) \leq L_\tau \frac{u}{2} \tau'(u) \,.$$

(ii) Set $z = q = 0$, $y = u$, $p = 2u$. Then (3) becomes

$$\tau(2u) - \tau(0) \leq 2L_\tau u \tau'(u) \,.$$

(iii) Let $\varepsilon \in (0, \min(u, v))$. Set $q = 0$, $p = \varepsilon$, $y = u$, $z = v$. Then (3) becomes

$$\tau(u) - \tau(u - \varepsilon) - \tau(v) + \tau(v - \varepsilon) \leq L_\tau \, \varepsilon \, \tau'(|u - v|) \,.$$

Thus,

$$L_\tau \tau'(|u - v|) \geq \frac{\tau(u) - \tau(u - \varepsilon)}{\varepsilon} - \frac{\tau(v) - \tau(v - \varepsilon)}{\varepsilon} \xrightarrow{\varepsilon \to 0} \tau'(u) - \tau'(v) \,.$$

(iv) Let $\varepsilon \in (0, \min(u, v))$. Let $q = 0$, $p = \varepsilon$, $y = u$, $z = -v$. Then (3) becomes

$$\tau(u) - \tau(u - \varepsilon) - \tau(v) + \tau(v + \varepsilon) \leq L_\tau \, \varepsilon \, \tau'(u + v) \,.$$

Thus,

$$L_\tau \tau'(u + v) \geq \frac{\tau(u) - \tau(u - \varepsilon)}{\varepsilon} + \frac{\tau(v + \varepsilon) - \tau(v)}{\varepsilon} \xrightarrow{\varepsilon \to 0} \tau'(u) + \tau'(v) \,.$$

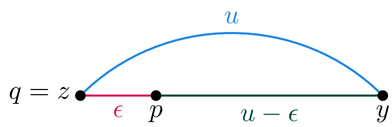

Figure 5: Constructions for the proof of Proposition 10.

□

Proposition 10. Let $\tau \in \mathcal{T}$. Assume τ is not constant. Then $L_\tau^* \geq 1$.

Proof. Let $u \in (0, \infty)$ be such that $\tau'(u) \neq 0$. Then Lemma 5 (i) implies $\tau'(u) > 0$. Let $\varepsilon \in (0, u)$. In the metric space of the real line with Euclidean metric, we choose $z = q = 0$, $p = \varepsilon$, $y = u$, see Figure 5. Then (3) becomes

$$\tau(u) - \tau(u - \varepsilon) - \tau(0) + \tau(\varepsilon) \leq L_\tau \, \varepsilon \, \tau'(u) \,.$$

As τ is nondecreasing by Lemma 5 (i), we have $\tau(0) = \lim_{x \searrow 0} \tau(x) \leq \tau(\varepsilon)$. Thus,

$$L_\tau \tau'(u) \geq \frac{\tau(u) - \tau(u - \varepsilon)}{\varepsilon} \xrightarrow{\varepsilon \to 0} \tau'(u) \,.$$

□

2.3 Stability

We can construct potentially new elements of $\mathcal{T}$ from given ones by taking limits or certain linear combinations, as the next two propositions show.

Proposition 11. Let $(\tau_n)_{n \in \mathbb{N}} \subseteq \mathcal{T}$ with constants $(L_{\tau_n})_{n \in \mathbb{N}} \subseteq [0, \infty)$. Let $\tau \colon (0, \infty) \to \mathbb{R}$ be a differentiable function. Assume $\lim_{n \to \infty} \tau_n(x) = \tau(x)$ and $\limsup_{n \to \infty} \tau_n'(x) \leq \tau'(x)$ for all $x \in (0, \infty)$. Set $L_\tau := \limsup_{n \to \infty} L_{\tau_n}$. Assume $L_\tau < \infty$. Then $\tau \in \mathcal{T}$ with constant L_τ.

Proof. This is a direct consequence of the assumed limit properties of τ, τ', and L_τ, and of the quadruple inequality for τ_n with constant L_{τ_n}:

$$\begin{aligned}
\tau(\overline{y,q}) - \tau(\overline{y,p}) - \tau(\overline{z,q}) + \tau(\overline{z,p}) &= \lim_{n \to \infty} \left(\tau_n(\overline{y,q}) - \tau_n(\overline{y,p}) - \tau_n(\overline{z,q}) + \tau_n(\overline{z,p}) \right) \\
&\leq \lim_{n \to \infty} L_{\tau_n} \, \overline{q,p} \, \tau_n'(\overline{y,z}) \\
&\leq L_\tau \, \overline{q,p} \, \tau'(\overline{y,z}) \,.
\end{aligned}$$

□

Proposition 12.

(i) Let $\tau \in \mathcal{T}$ with constant L_τ. Let $a \in [0, \infty)$. Then $(x \mapsto a\tau(x)) \in \mathcal{T}$ with constant aL_τ.

(ii) Let $\tau, \tilde{\tau} \in \mathcal{T}$ with constant L_τ and $L_{\tilde{\tau}}$, respectively. Then $(x \mapsto \tau(x) + \tilde{\tau}(x)) \in \mathcal{T}$ with constant $L_\tau + L_{\tilde{\tau}}$.

Proof. Both parts follow directly from the definition of $\mathcal{T}$ and linearity of the derivative operator. $\square$

3 Nondecreasing, Convex Functions with Concave Derivative

We define $\mathcal{S}$ to be the set of nondecreasing, convex functions $\tau\colon [0,\infty) \to \mathbb{R}$ with concave derivative $\tau'\colon [0,\infty) \to \mathbb{R}$ where $\tau'(0)$ is the right derivative

$$\tau'(0) = \lim_{h \searrow 0} \frac{\tau(h) - \tau(0)}{h}\,. \tag{25}$$

Let $\mathcal{C}^k([0,\infty))$ denote the space of k-times continuously differentiable functions $[0,\infty) \to \mathbb{R}$, where derivatives at 0 are taken as right derivatives. Denote $\mathcal{S}_0 := \{\tau \in \mathcal{S} \colon \tau(0) = 0\}$. Denote $\mathcal{S}^k := \mathcal{S} \cap \mathcal{C}^k([0,\infty))$ and $\mathcal{S}_0^k := \mathcal{S}_0 \cap \mathcal{C}^k([0,\infty))$ for $k \in \mathbb{N} \cup \{\infty\}$.

3.1 Properties

In this section, we establish some simple properties of functions $\tau \in \mathcal{S}^k$ that will be useful in the proof of the main theorem.

Lemma 13. Let $\tau \in \mathcal{S}$. Then τ is continuously differentiable. Furthermore τ' is nonnegative, continuous, nondecreasing, and differentiable almost everywhere. If $\tau \in \mathcal{S}^2$, then τ'' is nonnegative and nonincreasing. If $\tau \in \mathcal{S}^3$, then τ''' is nonpositive.

Proof. As τ' is concave, τ' is continuous and the differentiable function τ is continuously differentiable. As τ is nondecreasing, τ' is nonnegative. As τ is convex, τ' is nondecreasing. As τ' is concave, τ' is differentiable almost everywhere. Assume τ'' exists. As τ is convex, τ'' is nonnegative. As τ' is concave, τ'' is nonincreasing. Assume τ''' exists. As τ' is concave, τ''' is nonpositive. $\square$

The next lemma shows that all functions $\tau \in \mathcal{S}^3$ are between a nondecreasing linear function and a parabola that opens upward.

Lemma 14 (Polynomial bounds)**.** Let $\tau \in \mathcal{S}^3$. Let $x, y \in [0,\infty)$. Then

(i)
$$\tau(x) + y\tau'(x) \le \tau(x+y) \le \tau(x) + y\tau'(x) + \frac{1}{2}y^2\tau''(x)\,,$$

(ii)
$$\tau'(x) \le \tau'(x+y) \le \tau'(x) + y\tau''(x)\,.$$

Proof. (i) Apply a second and a third order Taylor expansion to $y \mapsto \tau(x+y)$ at 0 and use Lemma 13.

(ii) Apply a first and a second order Taylor expansion to $y \mapsto \tau'(x+y)$ at 0 and use Lemma 13. $\square$

In the following lemma, we provide useful bounds for the proof of the main theorem and gives a hint about the form of the right-hand side of the quadruple inequality (3).

Lemma 15 (Difference bound)**.** Let $\tau \in \mathcal{S}$.

(i) Let $x, y \in [0, \infty)$. Assume $x \geq y$. Then

$$\frac{x-y}{2}\left(\tau'(x)+\tau'(y)\right) \leq \tau(x)-\tau(y) \leq (x-y)\tau'\left(\frac{x+y}{2}\right).$$

(ii) Let $x, y \in [0, \infty)$. Then

$$\tau(x+y)-\tau(|x-y|) \leq 2\min(x,y)\tau'(\max(x,y)).$$

Proof. (i) For the lower bound, as τ' is concave,

$$\begin{aligned}\tau(x)-\tau(y) &= \int_y^x \tau'(u)\mathrm{d}u \\ &\geq (x-y)\int_0^1 (1-t)\tau'(y)+t\tau'(x)\,\mathrm{d}t \\ &= \frac{x-y}{2}\left(\tau'(x)+\tau'(y)\right).\end{aligned}$$

For the upper bound, concavity of τ' implies the existence of an affine linear function h with $h(u) \geq \tau'(u)$ for all $u \in [0, \infty)$ and

$$h\left(\frac{x+y}{2}\right) = \tau'\left(\frac{x+y}{2}\right). \tag{26}$$

Thus,

$$\begin{aligned}\tau(x)-\tau(y) &\leq \int_y^x h(u)\mathrm{d}u \\ &= \frac{x-y}{2}\left(h(x)+h(y)\right) \\ &= (x-y)h\left(\frac{x+y}{2}\right).\end{aligned}$$

(ii) Follows directly from the upper bound in (i).

∎

In further preparation for the main proof, we collect some properties of concave functions such as τ'.

Lemma 16 (Concave derivative)**.** Let $\tau \in \mathcal{S}$.

(i) Let $x, y \in [0, \infty)$. Then

$$\tau'(x+y) \leq \tau'(x)+\tau'(y) \leq 2\tau'\left(\frac{x+y}{2}\right).$$

(ii) Let $a, x \in [0, \infty)$. Then

$$\begin{aligned}\tau'(ax) &\geq a\tau'(x) \text{ for } a \leq 1, \\ \tau'(ax) &\leq a\tau'(x) \text{ for } a \geq 1.\end{aligned}$$

(iii) Let $x, y \in [0, \infty)$. Assume $y \geq x$. Then

$$x\tau'(y) \leq y\tau'(x)\,.$$

Proof. These are all well-known properties of nonnegative, concave functions.

Use Lemma 46 and Jensen's inequality.

(ii) Use $(1-t)\tau'(x_0) + t\tau'(x_1) \leq \tau'((1-t)x_0 + tx_1)$ on points $x_0 = 0$, $x_1 = x$, $t = a$ and on $x_0 = 0$, $x_1 = ax$, $t = 1/a$, respectively, and note that $\tau'(0) \geq 0$.

(iii) Apply (ii) with $a = y/x$.

□

The next lemma provides another tool for the main proof.

Lemma 17 (Square root and derivative)**.** Let $\tau \in \mathcal{S}_0^2$. Let $x, y \in [0, \infty)$. Assume $x \geq y$. Then

$$\tau(\sqrt{xy}) \leq x\tau'\left(\frac{1}{2}y\right)\,.$$

Proof. Define $f(x, y) := x\tau'(\frac{1}{2}y) - \tau(\sqrt{xy})$. Its partial derivative with respect to x is

$$\begin{aligned}\partial_x f(x, y) &= \tau'\left(\frac{1}{2}y\right) - \frac{\sqrt{y}}{2\sqrt{x}}\tau'(\sqrt{xy}) \\ &\geq \tau'\left(\frac{1}{2}y\right) - \tau'\left(\frac{1}{2}y\right) = 0\,,\end{aligned}$$

by Lemma 16 (ii) with $\frac{\sqrt{y}}{2\sqrt{x}} \in [0, 1]$. Thus,

$$\begin{aligned}f(x, y) &\geq f(y, y) \\ &= y\tau'\left(\frac{1}{2}y\right) - \tau(y) \\ &\geq 0\,,\end{aligned}$$

where the last inequality is due to Lemma 15 (i) with $\tau(0) = 0$. □

3.2 Approximation

In the proof of the main theorem, we will first show $\mathcal{S}^3 \subseteq \mathcal{T}$ and then approximate the remaining functions in $\mathcal{S}$ via smooth functions. The following lemma shows that this is possible.

Lemma 18 (Smooth approximation)**.** Let $\tau \in \mathcal{S}$. Then there is a sequence $(\tau_n)_{n\in\mathbb{N}} \subseteq \mathcal{S}^\infty$ such that $\tau(x) = \lim_{n\to\infty} \tau_n(x)$ and $\tau'(x) = \lim_{n\to\infty} \tau_n'(x)$.

Proof. We will smooth τ' by convolution with a mollifier. The convolution is executed in the group of positive real numbers under multiplication endowed with its Haar measure $\mu(A) = \int_A \frac{1}{x}\mathrm{d}x$ for $A \subseteq (0, \infty)$ measurable.

For $n \in \mathbb{N}$, let $\varphi_n \in \mathcal{C}^\infty((0, \infty))$ be a sequence of nonnegative functions with support in

$[\exp(-1/n), \exp(1/n)]$ and

$$\int_0^\infty \frac{\varphi_n(x)}{x} \mathrm{d}x = 1\,. \tag{27}$$

Let $\tau \in \mathcal{S}$ with derivative τ'. For $n \in \mathbb{N}$, $x \in [0, \infty)$, we define

$$\tau_n(x) := \tau(0) + \int_0^x \int_0^\infty \frac{\varphi_n(z)}{z} \tau'\Big(\frac{y}{z}\Big)\, \mathrm{d}z\mathrm{d}y\,. \tag{28}$$

Then, for $y \in [0, \infty)$,

$$\tau_n'(y) = \int_0^\infty \frac{\varphi_n(z)}{z} \tau'\Big(\frac{y}{z}\Big)\, \mathrm{d}z = \int_{\mathbb{R}} \varphi_n(\mathrm{e}^t)\tau'\Big(\mathrm{e}^{\log(y)-t}\Big)\, \mathrm{d}t\,. \tag{29}$$

Thus, $s \mapsto \tau_n'(\mathrm{e}^s)$ is the convolution of $t \mapsto \varphi_n(\mathrm{e}^t)$ with $t \mapsto \tau'(\mathrm{e}^t)$. Using standard results on convolutions, the mollified function has following properties:

(i) τ_n' is infinitely differentiable on $(0, \infty)$, because φ_n is,

(ii) τ_n' is nonnegative, nondecreasing, and concave, because τ' is and φ_n is nonnegative,

(iii) $\lim_{n\to\infty} \tau_n'(x) = \tau'(x)$ because τ' is continuous.

Furthermore, τ_n is convex, as τ_n' is nondecreasing and $\lim_{n\to\infty} \tau_n(x) = \tau(x)$ by dominated convergence. Thus, $(\tau_n)_{n\in\mathbb{N}} \subseteq \mathcal{S}^\infty$, and the sequence has the desired point-wise limits. □

3.3 Stability

The set $\mathcal{S}$ enjoys similar stability properties as $\mathcal{T}$. Thus, after having shown $\mathcal{S} \subseteq \mathcal{T}$, we cannot easily find further functions in $\mathcal{T}$ from the constructions presented in section 2.3.

Proposition 19. Let $(\tau_n)_{n\in\mathbb{N}} \subseteq \mathcal{S}$. Let $\tau\colon [0, \infty) \to \mathbb{R}$ be differentiable. Assume that $\tau(x) = \lim_{n\to\infty} \tau_n(x)$ for all $x \in [0, \infty)$. Then $\tau \in \mathcal{S}$.

Proof. As τ_n is nondecreasing and convex, so is τ. As τ_n' is concave, by [Roc70, Theorem 25.7], for all $x \in (0, \infty)$,

$$\lim_{n\to\infty} \tau_n'(x) = \tau'(x)\,. \tag{30}$$

Thus, as all τ_n' are concave, τ' is concave on $(0, \infty)$. As τ is convex, $\frac{\tau(x+h)-\tau(x)}{h}$ is increasing in both x and h. Thus,

$$\begin{aligned}
\liminf_{x\searrow 0} \tau'(x) &= \inf_{x\in(0,\infty)} \tau'(x) \\
&= \inf_{x,h\in(0,\infty)} \frac{\tau(x+h)-\tau(x)}{h} \\
&= \liminf_{h\searrow 0} \frac{\tau(h)-\tau(0)}{h} \\
&= \tau'(0)\,.
\end{aligned}$$

Therefore, τ' is continuous at 0 and concavity extends to $[0, \infty)$. □

Proposition 20.

(i) Let $\tau \in \mathcal{S}$. Let $a \in [0, \infty)$. Then $(x \mapsto a\tau(x)) \in \mathcal{S}$.

(ii) Let $\tau, \tilde{\tau} \in \mathcal{T}$. Then $(x \mapsto \tau(x) + \tilde{\tau}(x)) \in \mathcal{S}$.

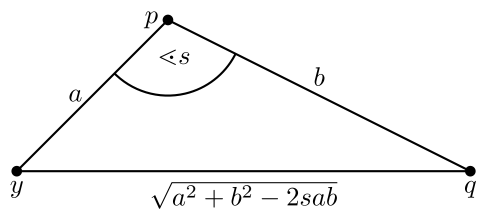

Figure 6: A 3-point parametrization. We denote $\sphericalangle s := \arccos(s)$.

Proof. Both parts follow directly from the definition of $\mathcal{S}$ and linearity of the derivative operator. $\square$

4 Proof Outline

In the first step of the proof of Theorem 1, we represent general 4-point metric spaces with 6 real-valued parameters. We refer to this representation as a *parametrization*. It converts our problem from the domain of metric geometry to the domain of real analysis. The rest of the proof consists of a complex sequence of elementary calculus arguments. This sequence may seem difficult to discover. To aid this process, an extensive application of computer-assisted numerical assessments was employed. The inequality (3) and transformations of it were evaluated on a grid of the parameter space and for a finite set of functions τ. This computational tool played a crucial role in guiding the proof. It helped to identify steps that would not be useful and indicated steps with potential merit.

4.1 Parametrization

There are different ways to parameterize a general k-tuple of points of a metric space. A trivial way is to take the k distances as the k real parameters. Then the parameter space is a subset of $[0,\infty)^k$ with certain constraints that ensure the triangle inequality. In this parametrization, the representation of distances is simple and the parameter space is more complex. We will base our parametrization on the Euclidean cosine-formula. We start with a parametrization of 3 points, see Figure 6. The parameter space for this construction is $\mathbb{R}^2 \times [-1,1]$ without further constraints. In contrast to the purely distance based parametrization described above, this parametrization yields a more complex representation of all distances, but a very simple parameter space.

Proposition 21 (3-point parametrization)**.**

(i) Let $(\mathcal{Q}, d)$ be a metric space and $y, q, p \in \mathcal{Q}$. Set $a := \overline{y,p}$ and $b := \overline{q,p}$. Isometrically embed y, q, p in the Euclidean plane $\mathbb{R}^2$ and set $s := \cos(\measuredangle ypq)$ with the angle $\measuredangle ypq$ measured in the Euclidean plane. Then, $\overline{y,q}^2 = a^2 + b^2 - 2sab$ and $a, b \in [0,\infty), s \in [-1,1]$.

(ii) For all $a, b \in [0,\infty), s \in [-1,1]$ there is a metric space with a triplet of points y, q, p such that $\overline{y,p} = a$, $\overline{q,p} = b$, and $\overline{y,q}^2 = a^2 + b^2 - 2sab$.

Proof. The first part is true by construction. For given parameters $a, b \in [0,\infty), s \in [-1,1]$, we can easily construct a triangle in $\mathbb{R}^2$ with sides $a, b, \sqrt{a^2+b^2-2sab}$ and an angle $\arccos(s)$ between the sides with length a and b. The corners of this triangle are the points y, q, p. $\square$

For the proof of Theorem 1, we use a 4-point parametrization, see Figure 7. It is based on repeated application of the Euclidean cosine formula. Its parameter space is a subset of $[0,\infty)^3 \times [-1,1]^3$ with rather complex constraints. We later relax this parametrization to a construction with a parameter space $[0,\infty)^3 \times [-1,1]^2$ without further constraints. That construction is not a parametrization, but results shown in its parameter space are stronger than those shown in the parametrization.

Proposition 22 (4-point parametrization)**.**

(i) Let $(\mathcal{Q}, d)$ be a metric space and $y, z, q, p \in \mathcal{Q}$. Set $a := \overline{z,p}$, $c := \overline{y,p}$, $b := \overline{q,p}$, $s := \cos(\measuredangle ypq)$, $r := \cos(\measuredangle zpq)$, $t := \cos(\measuredangle ypz)$, with all angles measured as in an isometric 3-point embedding in the Euclidean plane. Then,

$$\overline{y,z}^2 = a^2 + c^2 - 2tac\,, \qquad \overline{y,q}^2 = c^2 + b^2 - 2scb\,, \qquad \overline{z,q}^2 = a^2 + b^2 - 2rab\,.$$

Furthermore, $a, b, c \in [0,\infty)$, $r, s, t \in [-1,1]$, and

$$\begin{aligned} -tac &\leq b^2 - rab - scb + \sqrt{a^2 - 2rab + b^2}\sqrt{c^2 - 2scb + b^2}\,, \\ -rab &\leq c^2 - tac - scb + \sqrt{a^2 - 2tac + c^2}\sqrt{c^2 - 2scb + b^2}\,, \\ -scb &\leq a^2 - rab - tac + \sqrt{a^2 - 2rab + b^2}\sqrt{a^2 - 2tac + c^2}\,. \end{aligned} \tag{31}$$

(ii) For all $a, b, c \in [0,\infty)$, $r, s, t \in [-1,1]$ that fulfill (31), there is a metric space $(\mathcal{Q}, d)$ with a quadruple of points $y, z, q, p \in \mathcal{Q}$ such that

$$\begin{aligned} a &= \overline{z,p}\,, & c &= \overline{y,p}\,, & b &= \overline{q,p}\,, \\ \overline{y,z}^2 &= a^2 + c^2 - 2tac\,, & \overline{y,q}^2 &= c^2 + b^2 - 2scb\,, & \overline{z,q}^2 &= a^2 + b^2 - 2rab\,. \end{aligned}$$

Proof. (i) The inequalities (31) are due to the triangle inequality,

$$\overline{y,z} \leq \overline{y,q} + \overline{z,q}\,, \qquad \overline{z,q} \leq \overline{y,z} + \overline{y,q}\,, \qquad \overline{y,q} \leq \overline{y,z} + \overline{z,q}\,.$$

(ii) Define a four point set $\mathcal{Q}$ with elements named y, z, q, p. Define $d\colon \mathcal{Q} \times \mathcal{Q} \to [0,\infty)$ with the equations given in the lemma, extended by symmetry and $d(x,x) = 0$ for all $x \in \mathcal{Q}$. By construction, d is a semimetric [DD16] (vanishing distance for non-identical points allowed) so that identifying points with distance 0 yields a metric space.

□

With this parametrization and

$$a^2 - c^2 - (a^2 - 2rab + b^2) + (c^2 - 2scb + b^2) = 2b\,(ra - sc)\;, \tag{32}$$

(5) can be expressed as

$$b\,(ra - sc) \leq b\sqrt{a^2 + c^2 - 2tac}\,, \tag{33}$$

and (3) becomes

$$\tau(a) - \tau(c) - \tau_{\surd}\big(a^2 - 2rab + b^2\big) + \tau_{\surd}\big(c^2 - 2scb + b^2\big) \leq L_\tau\, b\, \tau'(a^2 + c^2 - 2tac)\,, \tag{34}$$

where we use the shorthand $\tau_{\surd}(x) := \tau(\sqrt{x})$. Thus, Theorem 1 is equivalent to showing that (33) implies (34) for all $a, b, c \in [0,\infty)$, $r, s, t \in [-1,1]$ that fulfill (31). We will prove a stronger but simpler looking result in section B of the appendix:

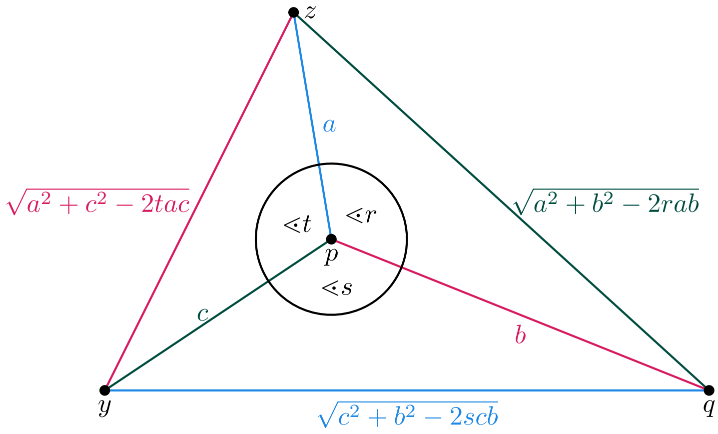

Figure 7: A 4-point parametrization. We denote $\sphericalangle x := \arccos(x)$.

Theorem 23. Let $a, b, c \geq 0$, $r, s \in [-1, 1]$, and $\tau \in \mathcal{S}_0^3$. Then

$$\tau(a) - \tau(c) - \tau_{\surd}\left(a^2 - 2rab + b^2\right) + \tau_{\surd}\left(c^2 - 2scb + b^2\right) \leq 2b\tau'(\max(ra - sc, |a - c|)) \ . \quad (35)$$

4.2 Remaining Proof Steps

From Theorem 23, we obtain a slightly stronger result than Theorem 1 by relaxing (5):

Theorem 24. Let $\tau \in \mathcal{S}$. Let $(\mathcal{Q}, d)$ be a metric space. Let $y, z, q, p \in \mathcal{Q}$. Assume, there is $L \in [2, \infty)$ such that

$$\overline{y,q}^2 - \overline{y,p}^2 - \overline{z,q}^2 + \overline{z,p}^2 \leq L\,\overline{q,p}\,\overline{y,z}\,. \quad (36)$$

Then

$$\tau(\overline{y,q}) - \tau(\overline{y,p}) - \tau(\overline{z,q}) + \tau(\overline{z,p}) \leq L\,\overline{q,p}\,\tau'(\overline{y,z}) \ . \quad (37)$$

Proof that Theorem 23 implies Theorem 24. Let $\tau \in \mathcal{S}_0^3$. Using (32), the metric version of (35) is

$$\begin{aligned} &\tau(\overline{y,q}) - \tau(\overline{y,p}) - \tau(\overline{z,q}) + \tau(\overline{z,p}) \\ &\leq 2\,\overline{q,p}\,\tau'\left(\max\left(\frac{\overline{y,q}^2 - \overline{y,p}^2 - \overline{z,q}^2 + \overline{z,p}^2}{2\,\overline{q,p}}, |\overline{z,p} - \overline{y,p}|\right)\right) \ . \end{aligned} \quad (38)$$

Using (36) and the triangle inequality, we bound the right-hand side of (38), by

$$2\,\overline{q,p}\,\tau'\left(\max\left(\frac{L\,\overline{y,z}}{2}, \overline{y,z}\right)\right) \ . \quad (39)$$

With Lemma 16 (ii) and $L/2 \geq 1$, we obtain (37).

To extend the result shown for $\tau \in \mathcal{S}_0^3$ to $\tau \in \mathcal{S}$, we use Lemma 18 and Proposition 11 to remove the smoothness requirement, and Lemma 4 (i) and Proposition 12 (ii) to remove the requirement $\tau(0) = 0$. □

Theorem 1 follows from Theorem 24 by fixing $L = 2$. The remaining part of the main proof, i.e., the proof of Theorem 23, is given in the appendix section B. Figure 8 gives an overview of how the different intermediate results presented above and below relate to each other.

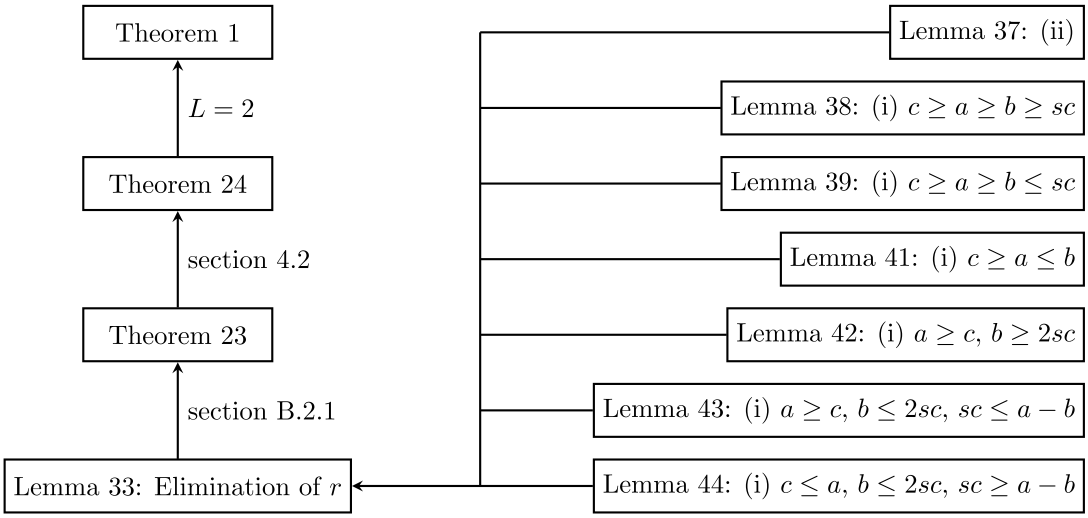

Figure 8: Overview of theorems and lemmas in the main proof.

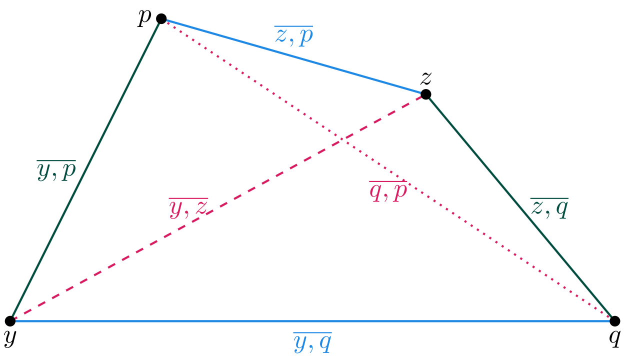

Figure 9: Four points as a quadrilateral. The sides of the quadrilateral show up on the left of the quadruple inequalities (the terms of opposite sides have the same sign); the diagonals form the right-hand side.

5 Implications and Discussion

With Theorem 1, we have shown a new set of rather fundamental inequalities in metric spaces that are related to the Cauchy–Schwarz and the Triangle inequalities. In this section, we discuss some immediate consequences of the main result.

5.1 Symmetries

Figure 9 illustrates the symmetries in quadruple inequalities. Sides of the same color contribute essentially in the same way in the inequality. In (3), the diagonals of Figure 9 come up in non-exchangeable terms. But, they can be swapped in the assumption (5). Thus, if the conditions of Theorem 1 are fulfilled, we have

$$\tau(\overline{y,q}) - \tau(\overline{y,p}) - \tau(\overline{z,q}) + \tau(\overline{z,p}) \leq 2\min(\overline{q,p}\,\tau'(\overline{y,z}), \overline{y,z}\,\tau'(\overline{q,p}))\,. \tag{40}$$

Furthermore, swapping y and z or q and p does not influence the right-hand side but changes the sign on the left-hand side. Thus, assuming

$$\left|\overline{y,q}^2 - \overline{y,p}^2 - \overline{z,q}^2 + \overline{z,p}^2\right| \leq 2\,\overline{q,p}\,\overline{y,z} \tag{41}$$

we get

$$|\tau(\overline{y,q}) - \tau(\overline{y,p}) - \tau(\overline{z,q}) + \tau(\overline{z,p})| \leq 2\min(\overline{q,p}\,\tau'(\overline{y,z}), \overline{y,z}\,\tau'(\overline{q,p}))\,. \tag{42}$$

Further bounds of the right-hand side are shown in the next subsection.

5.2 Bounds for the Right-Hand Side

Corollary 25. Let $\tau \in \mathcal{S}_0$. Let $(\mathcal{Q}, d)$ be a metric space. Let $y, z, q, p \in \mathcal{Q}$. Assume

$$\overline{y,q}^2 - \overline{y,p}^2 - \overline{z,q}^2 + \overline{z,p}^2 \leq 2\,\overline{q,p}\,\overline{y,z}\,. \tag{43}$$

Then the value

$$\tau(\overline{y,q}) - \tau(\overline{y,p}) - \tau(\overline{z,q}) + \tau(\overline{z,p}) \tag{44}$$

is bounded from above by all of the following values:

(i) $2\min(\overline{q,p}\,\overline{y,z})\tau'(\max(\overline{q,p}\,\overline{y,z}))$,

(ii) $2\,\overline{q,p}^{\beta}\overline{y,z}^{1-\beta}\tau'\left(\overline{q,p}^{1-\beta}\overline{y,z}^{\beta}\right)$ for all $\beta \in [0,1]$,

(iii) $2\,(\beta\,\overline{q,p} + (1-\beta)\overline{y,z})\,\tau'((1-\beta)\overline{q,p} + \beta\overline{y,z})$ for all $\beta \in [0,1]$,

(iv) $2\sqrt{\overline{q,p}\,\overline{y,z}}\tau'(\sqrt{\overline{q,p}\,\overline{y,z}})$,

(v) $(\overline{q,p} + \overline{y,z})\,\tau'\left(\frac{\overline{q,p}+\overline{y,z}}{2}\right)$,

(vi) $4\tau(\sqrt{\overline{q,p}\,\overline{y,z}})$,

(vii) $4\tau\left(\frac{\overline{q,p}+\overline{y,z}}{2}\right)$,

(viii) $2\tau(\overline{q,p}) + 2\tau(\overline{y,z})$.

Proof. We first apply Theorem 1 twice, to (y, z, q, p) and to (q, p, y, z), to obtain

$$\tau(\overline{y,q}) - \tau(\overline{y,p}) - \tau(\overline{z,q}) + \tau(\overline{z,p}) \leq 2\min(\overline{q,p}\,\overline{y,z})\tau'(\max(\overline{q,p}\,\overline{y,z}))\,. \tag{45}$$

This shows (i). Let $a, b \in [0,\infty)$ and $\beta \in [0,1]$. Then, by Lemma 16 (ii) and the weighted arithmetic–geometric mean inequality,

$$\begin{aligned}\min(a,b)\tau'(\max(a,b)) &\leq a^{\beta}b^{1-\beta}\tau'(a^{1-\beta}b^{\beta})\\ &\leq (\beta a + (1-\beta)b)\,\tau'((1-\beta)a + \beta b)\,.\end{aligned}$$

Applying these inequalities to (45) shows (ii) and (iii), and their special cases (iv) and (v), where $\beta = 1/2$. By Lemma 15 (i) with $y = 0$, we have

$$x\tau'(x) \leq 2\tau(x) \tag{46}$$

for all $x \in [0,\infty)$. Thus,

$$\begin{aligned}\min(a,b)\tau'(\max(a,b)) &\leq \sqrt{ab}\,\tau'(\sqrt{ab})\\ &\leq 2\tau(\sqrt{ab})\,,\end{aligned}$$

which yields (vi). The remaining parts (vii) and (viii), can be obtained using (46) and Jensen's inequality:

$$\begin{aligned}
\min(a,b)\tau'(\max(a,b)) &\le \frac{a+b}{2}\tau'\left(\frac{a+b}{2}\right) \\
&\le 2\tau\left(\frac{a+b}{2}\right) \\
&\le \tau(a)+\tau(b)\,.
\end{aligned}$$

□

5.3 Corollaries for Special Cases

We apply Theorem 1, Theorem 3, Proposition 7, and Corollary 25 for a triple of points (a quadruple of points with two identical points), on the real line, and for parallelograms in inner product spaces to demonstrate the main results.

Corollary 26 (For three points)**.** Let $(\mathcal{Q}, d)$ be a metric space. Let $y, q, p \in \mathcal{Q}$.

Let $\tau \in \mathcal{S}_0$. Then

$$\tau(\overline{y,q}) - \tau(\overline{y,p}) + \tau(\overline{q,p}) \le 2\,\overline{q,p}\,\tau'(\overline{y,q})\,. \tag{47}$$

(ii) Let $\alpha \in [1,2]$. Then

$$\overline{y,q}^{\alpha} - \overline{y,p}^{\alpha} + \overline{q,p}^{\alpha} \le \alpha 2^{2-\alpha}\,\overline{q,p}\,\overline{y,q}^{\alpha-1}\,. \tag{48}$$

Proof. By the triangle inequality, $|\overline{y,q} - \overline{q,p}| \le \overline{y,p}$. After squaring this inequality, we obtain (5) with $z = q$, which is

$$\overline{y,q}^2 - \overline{y,p}^2 + \overline{q,p}^2 \le 2\,\overline{q,p}\,\overline{y,q}\,. \tag{49}$$

Thus, Theorem 1 implies (47) and Proposition 7 implies (48). □

Corollary 27 (On the real line)**.** Let $a, b, c \in [0, \infty)$.

Let $\tau \in \mathcal{S}_0$. Then

$$\tau(a+b+c) - \tau(a+b) - \tau(b+c) + \tau(b) \le 2c\tau'(a) \tag{50}$$

$$\tau(a+b+c) - \tau(a+b) - \tau(b+c) + \tau(b) \le \tau(c) + \tau(a)\,. \tag{51}$$

(ii) Let $\alpha \in [1,2]$. Then

$$(a+b+c)^{\alpha} - (a+b)^{\alpha} - (b+c)^{\alpha} + b^{\alpha} \le \alpha 2^{2-\alpha} c a^{\alpha-1} \tag{52}$$

$$(a+b+c)^{\alpha} - (a+b)^{\alpha} - (b+c)^{\alpha} + b^{\alpha} \le c^{\alpha} + a^{\alpha}\,. \tag{53}$$

Proof. In $(\mathbb{R}, |\cdot - \cdot|)$, (5) is fulfilled. Choose $y = 0$, $z = a$, $p = a+b$, $q = a+b+c$ and apply Theorem 1 and Proposition 7 to obtain (50). For (51), apply Theorem 3. To get (52), use Proposition 7. The equation (53) is (51) with $\tau = \tau_\alpha = (x \mapsto x^\alpha)$. □

Corollary 28 (In inner product spaces)**.** Let $(V, \langle\cdot\,,\,\cdot\rangle)$ be an inner product space with induced norm $\|\cdot\|$. Let $u, v \in V$.

Let $\tau \in \mathcal{S}_0$. Then

$$\tau(\|u\|) - \tau(\|v\|) \leq \|u - v\| \tau'(\|u + v\|) \,. \tag{54}$$

(ii) Let $\alpha \in [1, 2]$. Then

$$\|u\|^\alpha - \|v\|^\alpha \leq \alpha 2^{1-\alpha} \|u - v\| \|u + v\|^{\alpha - 1} \,. \tag{55}$$

(iii) Let $\tau \in \mathcal{S}_0$. Then

$$\tau(\|u + v\|) + \tau(\|u - v\|) \leq 2\tau(\|u\|) + 2\tau(\|v\|) \,. \tag{56}$$

(iv) Let $\alpha \in [1, 2]$. Then

$$\|u + v\|^\alpha + \|u - v\|^\alpha \leq 2\|u\|^\alpha + 2\|v\|^\alpha \,. \tag{57}$$

Proof. For the first two parts, set $y = 0$, $z = u + v$, $q = u$, $p = v$; for the last two parts, set $y = 0$, $z = v$, $q = u + v$, $p = u$. Then apply Theorem 1, Theorem 3, and Proposition 7. $\square$

Remark 29. Recall the *parallelogram law*: Let $(V, \langle\cdot\,,\,\cdot\rangle)$ be an inner product space with induced norm $\|\cdot\|$. Let $u, v \in V$. Then

$$\|u + v\|^2 + \|u - v\|^2 \leq 2\|u\|^2 + 2\|v\|^2 \,, \tag{58}$$

which is also true with equality. Thus, we can say that Corollary 28 generalizes the parallelogram law.

5.4 Quadruple Constants and Further Research

We summarize our findings on the quadruple inequality and quadruple constant for symmetric and non-symmetric right-hand side, and for power functions as well as nondecreasing, convex functions with concave derivative.

Let $A \subseteq (0, \infty)^6$ be the set of points $x = (x_1, \dots, x_6)$ such that there is a metric space $(\{y_x, z_x, q_x, p_x\}, d)$ with

$$x_1 = \overline{y_x, q_x}\,, \quad x_2 = \overline{y_x, p_x}\,, \quad x_3 = \overline{z_x, q_x}\,, \quad x_4 = \overline{z_x, p_x}\,, \quad x_5 = \overline{q_x, p_x}\,, \quad x_6 = \overline{y_x, z_x} \tag{59}$$

that fulfills (5). For $\alpha \in (0, \infty)$, define

$$\begin{aligned}
L_\alpha &:= \sup_{x \in A} \frac{\overline{y_x, q_x}^\alpha - \overline{y_x, p_x}^\alpha - \overline{z_x, q_x}^\alpha + \overline{z_x, p_x}^\alpha}{\overline{q_x, p_x}\,\overline{y_x, z_x}^{\alpha-1}}\,,\\
K_\alpha &:= \sup_{x \in A} \frac{\overline{y_x, q_x}^\alpha - \overline{y_x, p_x}^\alpha - \overline{z_x, q_x}^\alpha + \overline{z_x, p_x}^\alpha}{\overline{q_x, p_x}^{\frac{\alpha}{2}}\,\overline{y_x, z_x}^{\frac{\alpha}{2}}}\,,\\
J_\alpha &:= \sup_{x \in A} \frac{\overline{y_x, q_x}^\alpha - \overline{y_x, p_x}^\alpha - \overline{z_x, q_x}^\alpha + \overline{z_x, p_x}^\alpha}{\overline{q_x, p_x}^\alpha + \overline{y_x, z_x}^\alpha}\,.
\end{aligned}$$

$\bullet$	$\alpha \in (0,1]$	$\alpha \in [1,2]$	$\alpha \in (2,\infty)$	$\mathcal{S}_0$
$L_\bullet$	∞	$\alpha 2^{2-\alpha}$	∞	2
$K_\bullet$	2	$[2, \alpha 2^{2-\alpha}]$	∞	$[2,4]$
$J_\bullet$	1	$[1, \alpha 2^{1-\alpha}]$	∞	$[1,2]$

Table 1: Range of quadruple constants shown in Proposition 30.

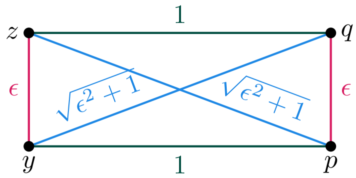

Figure 10: Construction in the proof of Proposition 30.

Furthermore, define

$$
\begin{aligned}
L_{\mathcal{S}_0} &:= \sup_{\tau \in \mathcal{S}_0} \sup_{x \in A} \frac{\tau(\overline{y_x, q_x}) - \tau(\overline{y_x, p_x}) - \tau(\overline{z_x, q_x}) + \tau(\overline{z_x, p_x})}{\overline{q_x, p_x}\, \tau'(\overline{y_x, z_x})}\,, \\
K_{\mathcal{S}_0} &:= \sup_{\tau \in \mathcal{S}_0} \sup_{x \in A} \frac{\tau(\overline{y_x, q_x}) - \tau(\overline{y_x, p_x}) - \tau(\overline{z_x, q_x}) + \tau(\overline{z_x, p_x})}{\tau\left(\sqrt{\overline{q_x, p_x}\, \overline{y_x, z_x}}\right)}\,, \\
J_{\mathcal{S}_0} &:= \sup_{\tau \in \mathcal{S}_0} \sup_{x \in A} \frac{\tau(\overline{y_x, q_x}) - \tau(\overline{y_x, p_x}) - \tau(\overline{z_x, q_x}) + \tau(\overline{z_x, p_x})}{\tau(\overline{q_x, p_x}) + \tau(\overline{y_x, z_x})}\,.
\end{aligned}
$$

Proposition 30.

(i) $L_\alpha = \alpha 2^{2-\alpha}$ for $\alpha \in [1,2]$ and $L_\alpha = \infty$ for $\alpha \in (0,\infty) \setminus [1,2]$.

(ii) $K_\alpha = 2$ for $\alpha \in (0,1]$, $K_\alpha \in [2, \alpha 2^{2-\alpha}]$ for $\alpha \in [1,2]$, $K_\alpha = \infty$ for $\alpha \in (2,\infty)$.

(iii) $J_\alpha = 1$ for $\alpha \in (0,1]$, $J_\alpha \in [1, \alpha 2^{1-\alpha}]$ for $\alpha \in [1,2]$, $J_\alpha = \infty$ for $\alpha \in (2,\infty)$.

(iv) $L_{\mathcal{S}_0} = 2$.

(v) $K_{\mathcal{S}_0} \in [2,4]$.

(vi) $J_{\mathcal{S}_0} \in [1,2]$.

Proof. From Proposition 7, we know $L_\alpha = \alpha 2^{2-\alpha}$ for $\alpha \in [1,2]$ and $L_\alpha = \infty$ for $\alpha \in (0,\infty) \setminus [1,2]$. As direct consequences, we obtain $K_\alpha \le \alpha 2^{2-\alpha}$ and $J_\alpha \le \alpha 2^{1-\alpha}$ for $\alpha \in [1,2]$.

If we set $y = p$ and $z = q$ and assume $\overline{y,q} = a \in (0,\infty)$, we have

$$
\begin{aligned}
\tau(\overline{y,q}) - \tau(\overline{y,p}) - \tau(\overline{z,q}) + \tau(\overline{z,p}) &= 2\tau(a)\,, \\
\tau\left(\sqrt{\overline{q,p}\,\overline{y,z}}\right) &= \tau(a)\,, \\
\tau(\overline{q,p}) + \tau(\overline{y,z}) &= 2\tau(a)\,,
\end{aligned}
$$

for any function $\tau \colon (0,\infty) \to \mathbb{R}$ with $\tau(0) = 0$. Thus, $K_\alpha, K_{\mathcal{S}_0} \ge 2$, $J_\alpha, J_{\mathcal{S}_0} \ge 1$.

If τ is metric preserving, like $\tau_\alpha = (x \mapsto x^\alpha)$ with $\alpha \in (0,1]$, then

$$
\tau(\overline{y,q}) - \tau(\overline{y,p}) - \tau(\overline{z,q}) + \tau(\overline{z,p}) \le 2\min(\tau(\overline{q,p}), \tau(\overline{y,z}))\,. \tag{60}
$$

This shows $K_\alpha \le 2$ and $J_\alpha \le 1$ for $\alpha \in (0,1]$.

Let $y=(0,0)$, $z=(0,\epsilon)$, $q=(1,\epsilon)$, $p=(1,0)$ in the Euclidean plane $\mathbb{R}^2$, see Figure 10. Then

$$\begin{gathered}\tau(\overline{y,q})-\tau(\overline{y,p})-\tau(\overline{z,q})+\tau(\overline{z,p})=2\tau(\sqrt{\epsilon^2+1})-2\tau(1)\,,\\ \tau(\sqrt{\overline{q,p}\,\overline{y,z}})=\tau(\epsilon)\,,\\ \tau(\overline{q,p})+\tau(\overline{y,z})=2\tau(\epsilon)\,,\end{gathered}$$

for any function $\tau\colon[0,\infty)\to\mathbb{R}$. Assume $\tau(0)=0$, $\tau'(1)>0$, and $\lim_{\epsilon\searrow 0}\frac{\epsilon}{\tau'(\epsilon)}=\infty$. Then, by l'Hôpital's rule,

$$\lim_{\epsilon\searrow 0}\frac{\tau(\sqrt{\epsilon^2+1})-\tau(1)}{\tau(\epsilon)}=\lim_{\epsilon\searrow 0}\frac{\epsilon\tau'(\sqrt{\epsilon^2+1})}{\sqrt{\epsilon^2+1}\,\tau'(\epsilon)}=\infty\,. \tag{61}$$

In particular, $K_\alpha=\infty$ and $J_\alpha=\infty$ for $\alpha\in(2,\infty)$.

By Theorem 1 and Corollary 25, we have $L_{\mathcal{S}_0}\leq 2$, $K_{\mathcal{S}_0}\leq 4$, and $J_{\mathcal{S}_0}\leq 2$. As here we take the supremum over $\tau\in\mathcal{S}_0$, we also get $L_{\mathcal{S}_0}\geq 2$, e.g., for $\tau=\tau_1=(x\mapsto x)$. $\square$

Remark 31. We have $\alpha 2^{1-\alpha}\in[1,\frac{2}{\mathrm{e}\ln(2)}]$ for $\alpha\in[1,2]$ and $\frac{2}{\mathrm{e}\ln(2)}\approx 1.06$. The maximum is attained at $\alpha=\ln(2)^{-1}\approx 1.44$.

For future work it remains to find the precise values of K_α and J_α for $\alpha\in[1,2]$, and for $K_{\mathcal{S}_0}$ and $J_{\mathcal{S}_0}$. Furthermore, it would be interesting extend the result for an explicit form of the quadruple constant from the functions $\tau_\alpha=(x\mapsto x^\alpha)$, where we have $L^*_{\tau_\alpha}=2^{2-\alpha}$ for $\alpha\in[1,2]$, to functions $\tau\in\mathcal{S}$, where we so far know $L^*_\tau\in[1,2]$.

A Proof of Theorem 3

The proof is inspired by the proofs of [Enf70, Proposition 4.1.1, Proposition 4.1.2].

For any four points $y,z,q,p\in V$, we have

$$\|y-q\|^2-\|y-p\|^2-\|z-q\|^2+\|z-p\|^2=2\,\langle y-z\,,\,p-q\rangle\leq\|q-p\|^2+\|y-z\|^2\,. \tag{62}$$

Let $u,v\in V$. Consider a parallelogram with vertices $(0,(u-v)/2,u,(u+v)/2)$. It has the diagonals u and v and the largest diagonal is not smaller than the largest side. As $\tau\in\mathcal{S}_0$, $\tau_{\surd}$ is nonnegative, nondecreasing, and concave, see Lemma 32. Thus, we can apply Lemma 47 to

$$x_1=x_2=\left\|\frac{u-v}{2}\right\|^2\,,\quad x_3=x_4=\left\|\frac{u+v}{2}\right\|^2\,,\quad x_5=\|u\|^2\,,\quad x_6=\|v\|^2\,, \tag{63}$$

where $x_1+x_2+x_3+x_4\geq x_5+x_6$ is ensured by (62). We obtain

$$\tau(\|u\|)+\tau(\|v\|)\leq 2\tau\left(\left\|\frac{u-v}{2}\right\|\right)+2\tau\left(\left\|\frac{u+v}{2}\right\|\right)\,. \tag{64}$$

To extend the result from parallelograms to any quadrilateral, we note that τ is nondecreasing and convex, and apply Lemma 46: For every $x\in V$,

$$2\tau\left(\left\|\frac{u-v}{2}\right\|\right)\leq\tau(\|x\|)+\tau(\|u-v-x\|)\,, \tag{65}$$

$$2\tau\left(\left\|\frac{u+v}{2}\right\|\right)\leq\tau(\|u-x\|)+\tau(\|v+x\|)\,. \tag{66}$$

By appropriate choice of u,v,x for a given quadrilateral with vertices y,z,q,p, see Figure 11, we have shown

$$\tau(\|y-q\|)+\tau(\|z-p\|)\leq\tau(\|q-p\|)+\tau(\|y-z\|)+\tau(\|y-p\|)+\tau(\|z-q\|) \tag{67}$$

and finished the proof of Theorem 3.

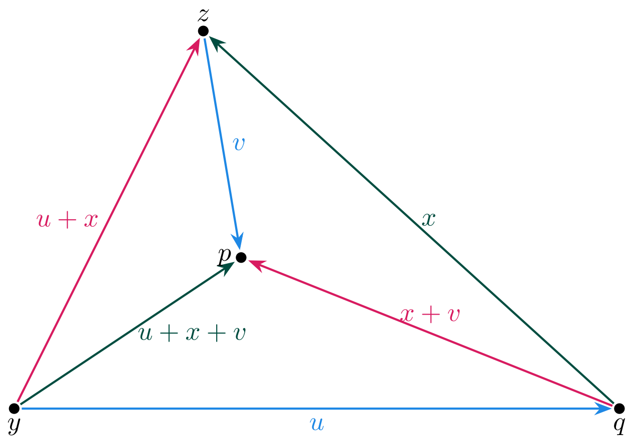

Figure 11: Four points in a vector space V. Their relative position is described by three vectors $u, v, x \in V$.

B Proof of Theorem 23

If we want to show $f(x) \leq 0$ for all $x \in A$, where $A \subseteq \mathbb{R}^n$, for a continuous function $f\colon A \to \mathbb{R}$, it is enough to prove the inequality on a dense subset of A. We use this fact in the following. If we write an expression with a quotient, we silently restrict the domains of the real parameters in all statements about this expression to a domain on which the denominator is not 0. The restricted domain, will always be dense in the unrestricted domain.

B.1 Summary of Properties

Recall $\tau_{\surd}(x) = \tau(\sqrt{x})$. The next lemma shows some simple properties of $\tau_{\surd}$ and its derivatives.

Lemma 32. Let $\tau \in \mathcal{S}_0$.

Then $\tau_{\surd}$ is nonnegative, nondecreasing, and concave.

- If $\tau_{\surd}$ is differentiable, then $\tau'_{\surd}$ is nonnegative and nonincreasing.
- If $\tau_{\surd}$ is twice differentiable, then $\tau''_{\surd}$ is nonpositive.

Proof. As τ is nonnegative, so is $\tau_{\surd}$. We have

$$\tau'_{\surd}(x) = \frac{\tau'(\sqrt{x})}{2\sqrt{x}},$$
$$\tau''_{\surd}(x) = \frac{\tau''(\sqrt{x}) - \frac{\tau'(\sqrt{x})}{\sqrt{x}}}{4x}.$$

As τ' is nonnegative, so is $\tau'_{\surd}$. Thus, τ is increasing. Furthermore, as τ'' is nonincreasing,

$$\tau'(\sqrt{x}) - \tau'(0) = \int_0^{\sqrt{x}} \tau''(u)\mathrm{d}u \geq \sqrt{x}\tau''(x). \tag{68}$$

Thus, with $\tau'(0) \geq 0$, we obtain $\tau'(\sqrt{x}) \geq \sqrt{x}\tau''(x)$. Hence, $\tau''_{\surd}$ is nonpositive, $\tau'_{\surd}$ is nonincreasing, and $\tau_{\surd}$ is concave. □

f	τ	τ'	τ''	τ'''	$\tau_{\surd}$	$\tau'_{\surd}$	$\tau''_{\surd}$	$\tau'''_{\surd}$
$f(0)$	0	≥ 0	≥ 0	≤ 0	0	≥ 0	≤ 0	
$f(x)$	≥ 0	≥ 0	≥ 0	≤ 0	≥ 0	≥ 0	≤ 0	
f monotone	$\nearrow$	$\nearrow$	$\searrow$		$\nearrow$	$\searrow$		
f curvature	$\smile$	$\frown$			$\frown$			

Table 2: Properties of $\tau \in \mathcal{S}_0^3$ and function $\tau_{\surd}(x) = \tau(\sqrt{x})$.

Properties of $\tau \in \mathcal{S}_0^3$ (Lemma 13) and the corresponding $\tau_{\surd}$ (Lemma 32) are summarized in Table 2 for reference. There and in the proof below, we use following shorthand notation for properties of functions $f\colon [0,\infty) \to \mathbb{R}$:

- $f \geq 0$: f is nonnegative.
- $f \leq 0$: f is nonpositive.
- $f \nearrow$: f is nondecreasing.
- $f \searrow$: f is nonincreasing.
- $f \smile$: f is convex.
- $f \frown$: f is concave.

B.2 Lemma 33: Elimination of r

We will show that the following lemma implies Theorem 23.

Lemma 33 (Elimination of r)**.** Let $\tau \in \mathcal{S}_0^3$.

For all $a, b, c \in [0, \infty)$, $s \in [-1, \min(1, 2\frac{a}{c} - 1)]$, we have

$$\tau(a) - \tau(c) - \tau(|a-b|) + \tau_{\surd}(c^2 - 2scb + b^2) \leq 2b\tau'(a - sc) \ . \tag{69}$$

(ii) For all $a, b, c \in [0, \infty)$ with $a \geq c$, we have

$$\tau(a) - \tau(c) - \tau_{\surd}((a-b)^2 - 4bc) + \tau_{\surd}(b + c) \leq 2b\tau'(a - c) \, . \tag{70}$$

B.2.1 Proof that Lemma 33 implies Theorem 23

For this proof, we first show some auxiliary lemmas. We distinguish the cases $ra - sc \leq |a - c|$ and $ra - sc \geq |a - c|$ as well as $a \geq c$ and $c \geq a$. Some trivial implications of these cases are recorded in following lemma.

Lemma 34. Let $a, b, c \geq 0$, $r, s \in [-1, 1]$. Then

$$\begin{aligned} ra - sc \geq a - c \quad &\Leftrightarrow \quad s \leq (r-1)\frac{a}{c} + 1 \quad \Leftrightarrow \quad r \geq (s-1)\frac{c}{a} + 1 \, , \\ ra - sc \geq c - a \quad &\Leftrightarrow \quad s \leq (r+1)\frac{a}{c} - 1 \quad \Leftrightarrow \quad r \geq (s+1)\frac{c}{a} - 1 \, . \end{aligned}$$

Denote

$$\begin{aligned} \ell_\tau(a, b, c, r, s) &:= \tau(a) - \tau(c) - \tau_{\surd}\big(a^2 - 2rab + b^2\big) + \tau_{\surd}\big(c^2 - 2scb + b^2\big) \ , \\ F_\tau(a, b, c, r, s) &:= \ell_\tau(a, b, c, r, s) - 2b\tau'(ra - sc) \ . \end{aligned}$$

For $ra - sc \geq |a - c|$, we want to show $F_\tau(a, b, c, r, s) \leq 0$. Because of the next lemma, we can reduce the number of values of r for which we need to check this inequality.

Lemma 35 (Convexity in r)**.** Let $\tau \in \mathcal{S}_0^3$. Let $a, b, c \geq 0$, $s, r \in [-1,1]$. Assume $ra - sc \geq 0$. Then

$$\partial_r^2 F_\tau(a,b,c,r,s) \geq 0\,.$$

Proof. As $\tau_{\surd}'', \tau''' \leq 0$, we have

$$\partial_r^2 \tau_{\surd}\big(a^2 - 2rab + b^2\big) = 4a^2b^2\tau_{\surd}''(a^2 - 2rab + b^2) \leq 0\,, \partial_r^2 \tau'(ra - sc) = a^2\tau'''(ra - sc) \leq 0\,.$$

Thus,

$$\partial_r^2 F_\tau(a,b,c,r,s) = -\partial_r^2 \tau_{\surd}\big(a^2 - 2rab + b^2\big) - 2b\partial_r^2\tau'(ra - sc) \geq 0\,.$$

□

In the case $|a - c| \geq ra - sc$, the right-hand side of (34) does not depend on r or s. Thus, we will only need to check the inequality with the left-hand side ℓ_τ maximized in r and s.

Lemma 36 (Maximizing the left-hand side for $|a - c| \geq ra - sc$)**.** Let $\tau \in \mathcal{S}_0^3$. Let $a, b, c \in [0, \infty)$, $r, s \in [-1, 1]$. Assume $|a - c| \geq ra - sc$.

If $a \geq c$ and $a^2 \leq c^2 + 2ab - 2cb$, then

$$\ell_\tau(a,b,c,r,s) \leq \tau(a) - \tau(c) - \tau(|a - b|) + \tau(|c - b|)\,.$$

(ii) If $a \geq c$ and $a^2 \geq c^2 + 2ab - 2cb$, then

$$\ell_\tau(a,b,c,r,s) \leq \tau(a) - \tau(c) - \tau_{\surd}((a - b)^2 + 4cb) + \tau(c + b)\,.$$

(iii) If $a \leq c$, then

$$\ell_\tau(a,b,c,r,s) \leq \tau(a) - \tau(c) - \tau_{\surd}(|a - b|) + \tau_{\surd}((c + b)^2 - 4ab)\,.$$

Proof. As $\tau_{\surd} \nearrow$, $s \mapsto \ell_\tau(a,b,c,r,s) \searrow$ and $r \mapsto \ell_\tau(a,b,c,r,s) \nearrow$, i.e., for $s_0, r_0 \in [-1,1]$,

$$\max_{s \geq s_0, r \leq r_0} \ell_\tau(a,b,c,r,s) = \ell_\tau(a,b,c,r_0,s_0)\,.$$

<u>Case 1: $a \geq c$.</u> For $r \in [-1,1]$, set $s_{\mathsf{min}}(r) := (r-1)\frac{a}{c} + 1$, cf. Lemma 34. Define

$$\begin{aligned} f(r) &:= \ell_\tau(a,b,c,r,s_{\mathsf{min}}(r)) \\ &= \tau(a) - \tau(c) - \tau_{\surd}(a^2 - 2rab + b^2) + \tau_{\surd}(c^2 - 2rab + 2ab - 2cb + b^2)\,. \end{aligned}$$

Then

$$\frac{f'(r)}{2ab} = \tau_{\surd}'(a^2 - 2rab + b^2) - \tau_{\surd}'(c^2 - 2rab + 2ab - 2cb + b^2)\,.$$

<u>Case 1.1: $a^2 \leq c^2 + 2ab - 2cb$.</u> As $\tau_{\surd}' \searrow$, we have

$$\begin{gathered} a^2 - 2rab + b^2 \leq c^2 - 2rab + 2ab - 2cb + b^2\,, \\ \tau_{\surd}'(a^2 - 2rab + b^2) \geq \tau_{\surd}'(c^2 - 2rab + 2ab - 2cb + b^2)\,. \end{gathered}$$

Thus, $f'(r) \geq 0$ and f is maximal at $r = r_{\mathsf{max}} = 1$, with $s_{\mathsf{min}}(r) = 1$. Hence,

$$\ell_\tau(a,b,c,r,s) \leq f(1) = \tau(a) - \tau(c) - \tau(|a-b|) + \tau(|c-b|)\,.$$

Case 1.2: $a^2 \geq c^2 + 2ab - 2cb$. As $\tau'_{\surd} \searrow$, we have

$$a^2 - 2rab + b^2 \geq c^2 - 2rab + 2ab - 2cb + b^2\,,$$
$$\tau'_{\surd}(a^2 - 2rab + b^2) \leq \tau'_{\surd}(c^2 - 2rab + 2ab - 2cb + b^2)\,.$$

Thus, $f'(r) \leq 0$ and f is maximal at $r = r_{\mathsf{min}} = 1 - 2\frac{c}{a}$, with $s_{\mathsf{min}}(r) = -1$. Hence,

$$\ell_\tau(a,b,c,r,s) \leq f(r_{\mathsf{min}}) = \tau(a) - \tau(c) - \tau_{\surd}((a-b)^2 + 4cb) + \tau(c+b)\,.$$

Case 2: $a \leq c$. For $r \in [-1,1]$, set $s_{\mathsf{min}}(r) := (r+1)\frac{a}{c} - 1$, cf. Lemma 34. Define

$$\begin{aligned} f(r) &:= \ell_\tau(a,b,c,r,s_{\mathsf{min}}(r)) \\ &= \tau(a) - \tau(c) - \tau_{\surd}(a^2 - 2rab + b^2) + \tau_{\surd}(c^2 - 2rab - 2ab + 2cb + b^2)\,. \end{aligned}$$

Then

$$\frac{f'(r)}{2ab} = \tau'_{\surd}(a^2 - 2rab + b^2) - \tau'_{\surd}(c^2 - 2rab - 2ab + 2cb + b^2)\,.$$

Case 2.1: $a^2 \leq c^2 - 2ab + 2cb$. As $\tau'_{\surd} \searrow$, we have

$$a^2 - 2rab + b^2 \leq c^2 - 2rab - 2ab + 2cb + b^2\,,$$
$$\tau'_{\surd}(a^2 - 2rab + b^2) \geq \tau'_{\surd}(c^2 - 2rab - 2ab + 2cb + b^2)\,.$$

Thus, $f'(r) \geq 0$ and f is maximal at $r = r_{\mathsf{max}} = 1$, with $s_{\mathsf{min}}(r) = 2\frac{a}{c} - 1$. Hence,

$$\ell_\tau(a,b,c,r,s) \leq f(1) = \tau(a) - \tau(c) - \tau_{\surd}(|a-b|) + \tau_{\surd}((c+b)^2 - 4ab)\,.$$

Case 2.2: $a^2 \geq c^2 - 2ab + 2cb$. This cannot happen for $a < c$. Hence, the proof is finished. $\square$

Proof that Lemma 33 implies Theorem 23.
Case 1: $|a-c| \geq ra - sc$.
Case 1.1: $a \geq c$. We can apply Lemma 36 and it suffices to show

$$\tau(a) - \tau(c) - \tau(|a-b|) + \tau(|c-b|) \leq 2b\tau'(a-c) \tag{71}$$

for $a^2 \leq c^2 + 2ab - 2cb$, and for $a^2 \geq c^2 + 2ab - 2cb$,

$$\tau(a) - \tau(c) - \tau_{\surd}((a-b)^2 + 4cb) + \tau_{\surd}(c+b) \leq 2b\tau'(a-c)\,. \tag{72}$$

The latter is exactly Lemma 33 (ii). The former follows from Lemma 33 (i) with $s \in \{-1,1\}$.
Case 1.2: $a \leq c$. We can apply Lemma 36 and it suffices to show

$$\tau(a) - \tau(c) - \tau_{\surd}(|a-b|) + \tau_{\surd}((c+b)^2 - 4ab) \leq 2b\tau'(c-a)\,. \tag{73}$$

This follows from Lemma 33 (i) with $s = 2\frac{a}{c} - 1$.
Case 2: $|a-c| \leq ra - sc$.
Case 2.1: $a \geq c$. In this case, $F_\tau(a,b,c,r,s) \leq 0$ implies (35). As $r \mapsto F_\tau(a,b,c,r,s)$ is convex by Lemma 35, it suffices to show $F_\tau(a,b,c,r,s) \leq 0$ for the extreme values of r in order to establish this inequality for all r. By Lemma 34,

$$r \in \left[(s-1)\frac{c}{a} + 1,\, 1\right]\,. \tag{74}$$

For maximal r, by Lemma 33 (i), we have $F_\tau(a,b,c,1,s) \le 0$. For minimal r, we have $ra-sc = a-c$. Thus, we are in case 1.1.
Case 2.2: $a \le c$. As in case 2.1, it suffices show $F_\tau(a,b,c,r,s) \le 0$ for extreme values of r. By Lemma 34,

$$r \in \left[(s+1)\frac{c}{a} - 1,\, 1\right]. \tag{75}$$

For maximal r, by Lemma 33 (i), we have $F_\tau(a,b,c,1,s) \le 0$. For minimal r, we have $ra-sc = c-a$. Thus, we are in case 1.2. □

B.3 Proof of Lemma 33 (ii)

Lemma 37. Let $\tau \in \mathcal{S}_0^3$. Let $a,b,c \in [0,\infty)$. Assume $a \ge c$. Then

$$\tau(a) - \tau(c) - \tau_{\surd}((a-b)^2 - 4bc) + \tau_{\surd}(b+c) \le 2b\tau'(a-c)\,. \tag{76}$$

Proof. Define

$$f(a,b,c) := \tau(a) - \tau(c) - \tau_{\surd}((a-b)^2 + 4cb) + \tau(c+b) - 2b\tau'(a-c)\,.$$

By Lemma 52 and Lemma 53, $\partial_b f(a,b,c) \le 0$. Thus, as $b \ge 0$, we have

$$f(a,b,c) \le f(a,0,c) = 0\,.$$

□

B.4 Proof of Lemma 33 (i) for $c \ge a \ge b \ge sc$

Lemma 38. Let $\tau \in \mathcal{S}_0^3$. Let $a,b,c \ge 0$, $s \in [-1,1]$. Assume $(s+1)c \le 2a$, $c \ge a \ge b \ge sc$. Then

$$\tau(a) - \tau(c) - \tau(a-b) + \tau_{\surd}(c^2 - 2scb + b^2) \le 2b\tau'(a-sc)\,.$$

Proof. Define

$$\begin{aligned} f(a,b,c,s) &:= \tau(a) - \tau(c) - \tau(a-b) + \tau_{\surd}(c^2 - 2scb + b^2) - 2b\tau'(a-sc)\,,\\ \partial_b f(a,b,c,s) &= \tau'(a-b) + 2(b-sc)\tau'_{\surd}(c^2 - 2scb + b^2) - 2\tau'(a-sc)\,. \end{aligned}$$

By Lemma 54, $\partial_b f(a,b,c,s) \le 0$. Thus, as $b \ge b_{\mathsf{min}} := \max(0,sc)$, we have $f(a,b,c,s) \le f(a,b_{\mathsf{min}},c,s)$. If $s \le 0$, then $b_{\mathsf{min}} = 0$ and

$$f(a,0,c,s) = \tau(a) - \tau(c) - \tau(a) + \tau_{\surd}(c^2) = 0\,.$$

For $s \ge 0$, we have $b_{\mathsf{min}} = sc$. Define

$$g(a,c,s) := f(a,sc,c,s) = \tau(a) - \tau(c) - \tau(a-sc) + \tau_{\surd}((1-s^2)c^2) - 2sc\tau'(a-sc)\,, \tag{77}$$

$$\partial_a g(a,c,s) = \tau'(a) - \tau'(a-sc) - 2sc\tau''(a-sc)\,. \tag{78}$$

Set $d := sc$ and define

$$h(a,d) := \tau'(a) - \tau'(a-d) - 2d\tau''(a-d) = \partial_a g(a,c,s)\,. \tag{79}$$

Then $\partial_a h(a,d) \le 0$ by Lemma 55. As $a \ge b \ge sc = d$, this means

$$h(a,d) \le h(d,d) = \tau'(d) - \tau'(d) - 2(d-d)\tau''(d) = 0\,.$$

Thus, $\partial_a g(a,c,s) = h(a,d) \le 0$. Therefore, as $a \ge a_{\mathsf{min}} := \frac{s+1}{2}c$,

$$\begin{aligned} g(a,c,s) &\le g(a_{\mathsf{min}},c,s) \\ &= \tau(\frac{s+1}{2}c) - \tau(c) - \tau(\frac{1-s}{2}c) + \tau_{\surd}((1-s^2)c^2) - 2sc\tau'(\frac{1-s}{2}c)\,. \end{aligned}$$

Set $u := \frac{1}{2}(c+sc)$, $v := \frac{1}{2}(c-sc)$ with $0 \le v \le u$ due to $s \in [0,1]$, and define

$$\ell(u,v) := \tau(u) - \tau(u+v) - \tau(v) + \tau_{\surd}(4uv) - 2(u-v)\tau'(v) = g(a_{\mathsf{min}},c,s)\,. \tag{80}$$

By Lemma 56, $\partial_u \ell(u,v) \le 0$. Thus, as $u \ge v$, we have

$$\begin{aligned} \ell(u,v) &\le \ell(v,v) \\ &= \tau(v) - \tau(2v) - \tau(v) + \tau_{\surd}(4vv) - 2(v-v)\tau'(v) \\ &= 0\,. \end{aligned}$$

Therefore, we obtain $f(a,sc,c,s) \le g(a,c,s) \le \ell(u,v) \le 0$ and we have finally shown $f(a,b,c,s) \le 0$. □

B.5 Proof of Lemma 33 (i) for $c \ge a \ge b$, $b \le sc$

Lemma 39. Let $\tau \in \mathcal{S}_0^3$. Let $a,b,c \ge 0$, $s \in [-1,1]$. Assume $a \le c$, $(s+1)c \le 2a$, $a \ge b$, and $b \le sc$. Then

$$\tau(a) - \tau(c) - \tau(a-b) + \tau_{\surd}(c^2 - 2scb + b^2) \le 2b\tau'(a-sc)\,.$$

Proof. Define

$$f(a,b,c,s) := \tau(a) - \tau(c) - \tau(a-b) + \tau_{\surd}(c^2 - 2scb + b^2) - 2b\tau'(a-sc)\,.$$

By Lemma 60, $\partial_a f(a,b,c,s) \le 0$. Thus, as $a \ge a_{\mathsf{min}} := (1+s)c/2$, we have

$$\begin{aligned} f(a,b,c,s) &\le f(a_{\mathsf{min}},b,c,s) \\ &= \tau\left(\frac{1}{2}(1+s)c\right) - \tau(c) - \tau\left(\frac{1}{2}(1+s)c - b\right) + \tau_{\surd}(c^2 - 2scb + b^2) - 2b\tau'\left(\frac{1}{2}(1-s)c\right) \\ &=: g(b,c,s)\,. \end{aligned}$$

By Lemma 40, $\partial_b g(b,c,s) + \partial_c g(b,c,s) \le 0$. Set $u := c - b$. Define $h(b,u,s) := g(b,u+b,s)$. Then $\partial_b h(b,u,s) = \partial_b g(b,c,s) + \partial_c g(b,c,s) \le 0$. Thus, as $b \ge 0$, we have

$$h(b,u,s) \le h(0,u,s) = g(0,u,s) = \tau\left(\frac{1}{2}(1+s)u\right) - \tau(u) - \tau\left(\frac{1}{2}(1+s)u\right) + \tau_{\surd}(u^2) = 0\,.$$

Thus,

$$f(a,b,c,s) \le g(b,c,s) = h(b,u,s) \le 0\,.$$

□

Lemma 40. Let $\tau \in \mathcal{S}_0^3$. Let $b, c \geq 0$, $s \in [-1, 1]$. Assume $b \leq sc$. Define

$$g(b,c,s) := \tau\left(\frac{1}{2}(1+s)c\right) - \tau(c) - \tau\left(\frac{1}{2}(1+s)c - b\right) + \tau_{\surd}(c^2 - 2scb + b^2) - 2b\tau'\left(\frac{1}{2}(1-s)c\right).$$

Then

$$\partial_b g(b,c,s) + \partial_c g(b,c,s) \leq 0.$$

Proof. We have

$$\begin{aligned}
\partial_b g(b,c,s) &= \tau'\left(\frac{1}{2}(1+s)c - b\right) - 2(sc-b)\tau'_{\surd}(c^2 - 2scb + b^2) - 2\tau'\left(\frac{1}{2}(1-s)c\right), \\
\partial_c g(b,c,s) &= \frac{1}{2}(1+s)\tau'\left(\frac{1}{2}(1+s)c\right) - \tau'(c) - \frac{1}{2}(1+s)\tau'\left(\frac{1}{2}(1+s)c - b\right) + \\
&\qquad 2(c-sb)\tau'_{\surd}(c^2 - 2scb + b^2) - (1-s)b\tau''\left(\frac{1}{2}(1-s)c\right).
\end{aligned}$$

Define

$$\begin{aligned}
f(b,c,s) := {} & \frac{(1-s)c}{2}\tau'(c-b) + 2(1-s)(b+c)\tau'_{\surd}((c-b)^2) - 2\tau'\left(\frac{(1-s)c}{2}\right) + \\
& - \frac{(1-s)c}{2}\tau'(c) - b(1-s)\tau''\left(\frac{c-b}{2}\right).
\end{aligned}$$

By Lemma 68,

$$\partial_b g(b,c,s) + \partial_c g(b,c,s) \leq f(b,c,s). \tag{81}$$

As $\tau''' \leq 0$, we have

$$\partial_s^2 f(b,c,s) = -2(-c/2)^2\tau'''\left(\frac{(1-s)c}{2}\right) \geq 0,$$

i.e., $f(b,c,s)$ is convex in s. Thus, to show $f(b,c,s) \leq 0$, we only need to check the extremes of s, $s_{\mathsf{min}} := \frac{b}{c}$ and $s_{\mathsf{max}} := 1$. For maximal s, we have

$$f(b,c,s_{\mathsf{max}}) = -2\tau'(0) \leq 0.$$

For minimal s, set $u := c - b \in [0, c]$. We define

$$\begin{aligned}
h(u,c) &:= cf(c-u, c, s_{\mathsf{min}}) \\
&= \frac{4c-u}{2}\tau'(u) - 2c\tau'(u/2) - \frac{u}{2}\tau'(c) - (c-u)u\tau''(u/2).
\end{aligned}$$

By Lemma 66, $\partial_c h(u,c) \leq 0$. Thus, as $c \geq u$, we have $h(u,c) \leq h(u,u)$. By Lemma 67, $h(u,u) \leq 0$. Thus, $cf(b,c,s_{\mathsf{min}}) = h(u,c) \leq h(u,u) \leq 0$. Because $f(b,c,s)$ is convex in s and negative for the extremes of s, we obtain

$$\partial_b g(b,c,s) + \partial_c g(b,c,s) \leq f(b,c,s) \leq 0.$$

□

B.6 Proof of Lemma 33 (i) for $c \geq a$, $a \leq b$

Lemma 41. Let $\tau \in \mathcal{S}_0^3$. Let $a, b, c \geq 0$, $s \in [-1, 1]$. Assume $(s+1)c \leq 2a$, $c \geq a$, $a \leq b$. Then

$$\tau(a) - \tau(c) - \tau(a-b) + \tau_{\surd}(c^2 - 2scb + b^2) \leq 2b\tau'(a - sc).$$

Proof. Define

$$f(a,b,c,s) := \tau(a) - \tau(c) - \tau(b-a) + \tau_{\surd}(c^2 - 2scb + b^2) - 2b\tau'(a - sc)\,.$$

By Lemma 57, $\partial_b f(a,b,c,s) \le 0$. Thus, as $b \ge a$, we have

$$\begin{aligned} f(a,b,c,s) &\le f(a,a,c,s) \\ &= \tau(a) - \tau(c) + \tau_{\surd}(c^2 - 2sca + a^2) - 2a\tau'(a - sc) \\ &=: g(a,c,s)\,. \end{aligned}$$

By Lemma 58, $\partial_a g(a,c,s) \le 0$. Thus, as $a \ge a_{\mathsf{min}} := (s+1)c/2$, we have $g(a,c,s) \le g(a_{\mathsf{min}}, c, s)$. Set $u := (1+s)c/2$, $v := (1-s)c/2$. Then $c = u + v$, $s = (u-v)/(u+v)$, $sc = u - v$, $u - sc = v$, and

$$\begin{aligned} g(a_{\mathsf{min}}, c, s) &= g(u, u+v, (u-v)/(u+v)) \\ &= \tau(u) - \tau(u+v) + \tau_{\surd}(4uv + v^2) - 2u\tau'(v) \\ &:= h(u,v)\,. \end{aligned}$$

By Lemma 59, $\partial_u h(u,v) \le 0$. Thus, as $u \ge 0$, we have

$$h(u,v) \le h(0,v) = \tau(0) - \tau(v) + \tau(v) = 0\,.$$

Thus, $f(a,b,c,s) \le g(a,c,s) \le h(u,v) \le 0$. □

B.7 Proof of Lemma 33 (i) for $a \ge c$, $b \le 2sc$, $sc \ge a - b$

Lemma 42. Let $\tau \in \mathcal{S}_0^3$. Let $a,b,c \in [0,\infty)$, $s \in [-1,1]$. Assume $\frac{1}{2}b \le sc$, $sc \ge a - b$, and $a \ge c$. Then

$$\tau(a) - \tau(c) - \tau(a-b) + \tau_{\surd}(c^2 - 2scb + b^2) \le 2b\tau'\left(\frac{a - sc}{2}\right)\,. \tag{82}$$

Proof. Define

$$x_1^+ := a^2\,, \qquad x_2^+ := c^2 - 2scb + b^2\,, \qquad x_1^- := c^2\,, \qquad x_2^- := (a-b)^2$$

to write the left-hand side of (82) as

$$\tau_{\surd}(x_1^+) + \tau_{\surd}(x_2^+) - \tau_{\surd}(x_1^-) - \tau_{\surd}(x_2^-)\,.$$

As $a \ge c$,

$$x_1^+ + x_2^+ - x_1^- - x_2^- = 2b\,(a - sc) \ge 0\,. \tag{83}$$

Because $a \ge c$ and $b \le 2sc$, we have $a - b \ge a - 2sc \ge a - 2c \ge -c$. Together with $a - b \le a$, we obtain $x_1^+ \ge \max(x_1^-, x_2^-)$.

Case 1: $x_2^+ \ge \min(x_1^-, x_2^-)$: By first applying Lemma 49 and then using (83), we obtain

$$\tau_{\surd}(x_1^+) + \tau_{\surd}(x_2^+) - \tau_{\surd}(x_1^-) - \tau_{\surd}(x_2^-) \le 2\tau_{\surd}(b\,(a - sc))\,.$$

Case 2: $x_2^+ \leq \min(x_1^-, x_2^-)$: By (83), we have

$$x_1^+ + x_2^+ \geq x_1^- + x_2^- \,. \tag{84}$$

By first applying Lemma 50, then using (83), and finally $\tau_{\surd} \frown$, we obtain

$$\begin{aligned}
&\tau_{\surd}(x_1^+) + \tau_{\surd}(x_2^+) - \tau_{\surd}(x_1^-) - \tau_{\surd}(x_2^-) \\
&\leq \tau_{\surd}(2b\,(a - sc)) \\
&\leq 2\tau_{\surd}(b\,(a - sc)) \,.
\end{aligned}$$

Finally: The condition $0 \leq a - sc \leq b$ together with Lemma 17 implies

$$\tau_{\surd}(b\,(a - sc)) \leq b\tau'\left(\frac{a - sc}{2}\right) .$$

□

B.8 Proof of Lemma 33 (i) for $a \geq c$, $b \geq 2sc$

Lemma 43. Let $\tau \in \mathcal{S}_0^3$. Let $a, b, c[0, \infty)$, $s \in [-1, 1]$. Assume $b \geq 2sc$, $a \geq sc$. Then

$$\tau(a) - \tau(c) - \tau(|a - b|) + \tau(c^2 - 2scb + b^2) \leq 2\tau'\left(\frac{a - sc}{2}\right) .$$

Proof. The function $x \mapsto \tau_{\surd}(x + u) - \tau_{\surd}(x)$, $u \geq 0$ is $\searrow$ as $\tau'_{\surd} \searrow$. Thus,

$$\begin{aligned}
-\tau_{\surd}(c^2) + \tau_{\surd}(c^2 - 2scb + b^2) &\leq -\tau_{\surd}((sc)^2) + \tau_{\surd}((sc)^2 - 2scb + b^2) \\
&= -\tau|sc| + \tau|sc - b| \,.
\end{aligned}$$

We use this and then apply Lemma 51 to obtain,

$$\begin{aligned}
\tau(a) - \tau(c) - \tau|a - b| + \tau(c^2 - 2scb + b^2) &\leq \tau(a) - \tau|sc| - \tau|a - b| + \tau|sc - b| \\
&\leq 2\left(\tau\left(\frac{a - sc + b}{2}\right) - \tau\left|\frac{a - sc - b}{2}\right|\right) .
\end{aligned}$$

As $a - sc + b \geq |a - sc - b|$, Lemma 15 (ii) yields

$$\begin{aligned}
&\tau\left(\frac{a - sc + b}{2}\right) - \tau\left|\frac{a - sc - b}{2}\right| \\
&\leq \left(\frac{a - sc + b}{2} - \frac{|a - sc - b|}{2}\right)\tau'\left(\frac{a - sc + b}{4} + \frac{|a - sc - b|}{4}\right) \\
&\leq \min(a - sc, b)\tau'\left(\frac{\max(a - sc, b)}{2}\right) .
\end{aligned}$$

By Lemma 16 (iii),

$$\min(a - sc, b)\tau'\left(\frac{\max(a - sc, b)}{2}\right) \leq b\tau'\left(\frac{a - sc}{2}\right) . \tag{85}$$

□

B.9 Proof of Lemma 33 (i) for $a \geq c$, $b \leq 2sc$, $sc \leq a - b$

Lemma 44. Let $\tau \in \mathcal{S}_0^3$. Let $a, b, c \in [0, \infty)$, $s \in [-1, 1]$. Assume $\frac{1}{2}b \le sc \le a - b$. Then

$$\tau(a) - \tau(c) - \tau(a-b) + \tau_{\surd}(c^2 - 2scb + b^2) \le 2b\tau'\left(\frac{a - sc}{2}\right) .$$

Proof. Set $x := a - b$. Define

$$f(x, b, c, s) := \tau(x+b) - \tau(c) - \tau(x) + \tau_{\surd}(c^2 - 2scb + b^2) - 2\left(\tau\left(\frac{x - sc}{2} + b\right) - \tau\left(\frac{x - sc}{2}\right)\right) .$$

Then

$$\partial_x f(x, b, c, s) = \tau'(x+b) - \tau'(x) - \tau'\left(\frac{x - sc}{2} + b\right) + \tau'\left(\frac{x - sc}{2}\right) .$$

We have

$$x + b \ge x , \qquad \frac{x - sc}{2} \le \frac{x - sc}{2} + b , \qquad (x+b) + \frac{x - sc}{2} = x + \left(\frac{x - sc}{2} + b\right) .$$

Thus, Lemma 48 implies

$$\tau'(x+b) + \tau'\left(\frac{x - sc}{2}\right) \le \tau'(x) + \tau'\left(\frac{x - sc}{2} + b\right) .$$

Thus, $\partial_x f(x, b, c, s) \le 0$. Thus, as $x \ge sc$, we have $f(x, b, c, s) \le f(sc, b, c, s)$. By Lemma 45, $f(sc, b, c, s) \le 0$. Thus, $f(x, b, c, s) \le 0$. Then we obtain, using Lemma 15 (i) and $\tau' \nearrow$,

$$\tau(a) - \tau(c) - \tau(a-b) + \tau_{\surd}(c^2 - 2scb + b^2) \le 2b\tau'\left(\frac{a - sc}{2}\right) .$$

□

Lemma 45. Let $\tau \in \mathcal{S}_0^3$. Let $x, b, c \in [0, \infty)$. Assume $b \le 2x$, $x + b \ge c$, $x \le c$. Then

$$\tau(x+b) + \tau_{\surd}(c^2 - 2xb + b^2) \le \tau(c) + \tau_{\surd}(x) + 2\tau(b) .$$

Proof. Define

$$f(x, b, c) := \tau(x+b) + \tau_{\surd}(c^2 - 2xb + b^2) - \tau(c) - \tau(x) - 2\tau(b) ,$$
$$g(x, b, c) := \partial_x f(x, b, c) = \tau'(x+b) - \tau'(x) - 2b\tau'_{\surd}(c^2 - 2xb + b^2) .$$

By Lemma 65, $\partial_x g(x, b, c) \le 0$. Thus, as $x \ge x_{\mathsf{min}} := \max(\frac{b}{2}, c-b)$, we have $g(x, b, c) \le g(x_{\mathsf{min}}, b, c)$.
Case 1, $x_{\mathsf{min}} = c - b$:
In this case $c - b \ge b/2$, i.e., $2c \ge 3b$. By Lemma 61, $g(c - b, b, c) \le 0$. Thus, as $x \ge x_{\mathsf{min}} = c - b$, we have $f(x, b, c) \le f(c - b, b, c)$. By Lemma 62, $f(c - b, b, c) \le 0$. Thus, $f(x, b, c) \le 0$.
Case 2, $x_{\mathsf{min}} = \frac{b}{2}$:
In this case $c - b \le b/2$, i.e., $2c \le 3b$. By Lemma 63, $g(b/2, b, c) \le 0$. Thus, as $x \ge x_{\mathsf{min}} = b/2$, we have $f(x, b, c) \le f(b/2, b, c)$. By Lemma 64, $f(b/2, b, c) \le 0$. Thus, $f(x, b, c) \le 0$. □

C Auxiliary Results

To make the proofs presented in appendix A and B more readable, we have extracted some calculations to this section.

C.1 Merging Terms

Lemma 46.

(i) Let $f\colon [0,\infty) \to \mathbb{R}$. Assume f is concave. Let $a, b \in [0,\infty)$ with $a \geq b$. Then $x \mapsto f(a+x) + f(b-x)$ is nonincreasing. If additionally $f(0) \geq 0$, then f is subadditive.

(ii) Let $f\colon [0,\infty) \to \mathbb{R}$. Assume f is convex. Let $a, b \in [0,\infty)$ with $a \geq b$. Then $x \mapsto f(a+x) + f(b-x)$ is nondecreasing.

Proof. We prove the first part; the second part is analogous. As f is concave, we have

$$\begin{aligned} f(a) &\geq \frac{a-b+x}{a-b+2x} f(a+x) + \frac{x}{a-b+2x} f(b-x)\,, \\ f(b) &\geq \frac{x}{a-b+2x} f(a+x) + \frac{a-b+x}{a-b+2x} f(b-x) \end{aligned}$$

for $x \in [0,b]$. Adding the two inequalities yields

$$f(a) + f(b) \geq f(a+x) + f(b-x)\,. \tag{86}$$

As this inequality also applies to $\tilde{a} = a+x$, $\tilde{b} = b-x$, we have that $x \mapsto f(a+x) + f(b-x)$ is nonincreasing. Subadditivity follows by setting $x = b$. $\square$

Lemma 47. Let $f\colon [0,\infty) \to \mathbb{R}$. Assume $f(0) \geq 0$, f is nondecreasing, and f is concave. Let $x_1, \ldots, x_6 \in [0,\infty)$. Assume $\max(x_1,x_2,x_3,x_4) \leq \max(x_5,x_6)$ and $x_1+x_2+x_3+x_4 \geq x_5+x_6$. Then

$$f(x_1) + f(x_2) + f(x_3) + f(x_4) \geq f(x_5) + f(x_6)\,. \tag{87}$$

Proof. Without loss of generality assume $x_1 \geq x_2 \geq x_3 \geq x_4$ and $x_5 \geq x_6$.

First consider the case $x_1 \geq x_6$. We decrease x_5 and increase x_6 while holding $x_5 + x + 6$ constant until one $x_\bullet$ on the right-hand side coincides with one $x_\bullet$ one the left-hand side. By Lemma 46, this can only increase the right-hand side of (87). If $\{x_1,x_2,x_3,x_4\} \cap \{x_5,x_6\} \neq \emptyset$, we can subtract the term with the value in the intersection from (87). The inequality of the form $f(x_1)+f(x_2)+f(x_3) \geq f(x_1+x_2+x_3) \geq f(x_5)$ for $x_5 \geq x_1+x_2+x_3$ is obtained using subadditivity of f, see Lemma 46, and the assumption that f is nondecreasing.

Now consider the case $x_1 < x_6$. Set $s := (x_5+x_6)/2$. Using Lemma 46, we obtain $f(x_5) + f(x_6) \leq 2f(s)$. Furthermore $x_1 \leq s$ and $x_1+x_2+x_3+x_4 \leq 2s$. Thus, again using Lemma 46 and the assumption that f is nondecreasing, we can increase x_1 and x_2 while decreasing x_3 and x_4 to 0 to get

$$f(x_1) + f(x_2) + f(x_3) + f(x_4) \geq 2f(s) + 2f(0)\,. \tag{88}$$

As $f(0) \geq 0$, we arrive at the desired result. $\square$

Lemma 48. Let $\tau \in \mathcal{S}$. Let $a,b,c,d \in [0,\infty)$. Assume $a \geq b \geq c \geq d$ and $a+d \leq b+c$. Then $\tau'(a) + \tau'(d) \leq \tau'(b) + \tau'(c)$.

Proof. As τ' is concave, Lemma 46 applies. $\square$

Lemma 49. Let $\tau \in \mathcal{S}_0^3$. Let $a \geq b \geq 0$, $d \geq c \geq 0$. Then

$$\tau_{\surd}(a) - \tau_{\surd}(b) - \tau_{\surd}(c) + \tau_{\surd}(d) \leq 2\tau_{\surd}\left(\frac{1}{2}(a-b+d-c)\right).$$

Proof. As $a \geq b$, $d \geq c$, subadditivitiy of $\tau_{\surd}$

$$\tau_{\surd}(a) - \tau_{\surd}(b) - \tau_{\surd}(c) + \tau_{\surd}(d) \leq \tau_{\surd}(a-b) + \tau_{\surd}(d-c).$$

Furthermore, by concavity of $\tau_{\surd}$,

$$\frac{1}{2}\tau_{\surd}(a-b) + \frac{1}{2}\tau_{\surd}(d-c) \leq \tau_{\surd}\left(\frac{1}{2}(a-b+d-c)\right).$$

□

Lemma 50. Let $\tau \in \mathcal{S}_0^3$. Let $a \geq b \geq c \geq d \geq 0$, $a+d \geq b+c$. Then

$$\tau_{\surd}(a) - \tau_{\surd}(b) - \tau_{\surd}(c) + \tau_{\surd}(d) \leq \tau_{\surd}(a-b-c+d).$$

Proof. Define $f(x,y) = \tau_{\surd}(x)+\tau_{\surd}(y)-\tau_{\surd}(x+y)$ for $x, y \geq 0$. Then $\partial_x f(x,y) = \tau_{\surd}'(x)-\tau_{\surd}'(x+y) \geq 0$ and similarly $\partial_y f(x,y) \geq 0$. Set $\delta := a-b$ and $\epsilon := c-d$. The assumptions ensure $\delta \geq \epsilon \geq 0$. Then,

$$f(b,\delta) \geq f(b,\epsilon) \geq f(d,\epsilon).$$

Thus,

$$\begin{aligned}
0 &\geq f(d,\epsilon) - f(b,\delta) \\
&= \tau_{\surd}(d) + \tau_{\surd}(\epsilon) - \tau_{\surd}(d+\epsilon) - \tau_{\surd}(b) - \tau_{\surd}(\delta) + \tau_{\surd}(b+\delta) \\
&= \tau_{\surd}(d) + \tau_{\surd}(\epsilon) - \tau_{\surd}(c) - \tau_{\surd}(b) - \tau_{\surd}(\delta) + \tau_{\surd}(a).
\end{aligned}$$

With this we get

$$\begin{aligned}
\tau_{\surd}(d) - \tau_{\surd}(c) - \tau_{\surd}(b) + \tau_{\surd}(a) &\leq \tau_{\surd}(\delta) - \tau_{\surd}(\epsilon) \\
&\leq \tau_{\surd}(\delta - \epsilon) \\
&= \tau_{\surd}(a-b-c+d).
\end{aligned}$$

□

Lemma 51 (Simple Merging Lemma)**.** Let $\tau \in \mathcal{S}_0^3$. Let $b \geq 0$, $a, c \in \mathbb{R}$. Then

$$\tau(|a|) - \tau(|c|) - \tau(|a-b|) + \tau(|c-b|) \leq 2\left(\tau\left(\frac{a-c+b}{2}\right) - \tau\left(\frac{|a-c-b|}{2}\right)\right)\mathbb{1}_{a>c}.$$

Proof. Define $f(x) := \tau(|x|) - \tau(|x-y|)$. Then $f'(x) = \mathsf{sgn}(x)\tau'(|x|) - \mathsf{sgn}(x)\tau'(|x-y|)$. If $y \geq 0$ then: if $x \geq 0$ then $|x| \geq |x-y|$, if $x \leq 0$ then $|x| \leq |x-y|$. Thus, $f'(x) \geq 0$, as τ' is increasing.

Hence, $f(x)$ is increasing. Thus, if $a \leq c$, then

$$\tau(|a|) - \tau(|a-b|) \leq \tau(|c|) - \tau(|c-b|)\,.$$

This shows the inequality for the case $a \leq c$.

Now assume $a > c$. Set $q := a - b$ and define

$$g(b) := \tau(|q+b|) - \tau(|c|) - \tau(|q|) + \tau(|c-b|) - 2\left(\tau\left(\frac{q-c}{2} + b\right) - \tau\left(\frac{q-c}{2}\right)\right)\,.$$

We have

$$g'(b) = \mathsf{sgn}(q+b)\tau'(|q+b|) - \mathsf{sgn}(c-b)\tau'(|c-b|) - 2\tau'\left(\frac{q-c}{2} + b\right)\,.$$

Case 1: $\mathsf{sgn}(q+b) = +1$, $\mathsf{sgn}(c-b) = +1$:

$$g'(b) = \tau'(q+b) - \tau'(c-b) - 2\tau'\left(\frac{q-c}{2} + b\right)\,,$$

As τ' is concave, it is subadditive and $\tau'(2x) \leq 2\tau'(x)$. Furthermore, $q + b = a > c \geq c - b$. Thus,

$$\tau'(q+b) - \tau'(c-b) \leq \tau'(q+b-c+b) \leq 2\tau'\left(\frac{q-c}{2} + b\right)\,.$$

Case 2: $\mathsf{sgn}(q+b) = -1$, $\mathsf{sgn}(c-b) = -1$:

$$g'(b) = -\tau'(-q-b) + \tau'(b-c) - 2\tau'\left(\frac{q-c}{2} + b\right)\,,$$

Similarly to the first case, we have $b - c \geq -c > -a = -q - b$ and

$$\tau'(b-c) - \tau'(-q-b) \leq \tau'(b-c+q+b) \leq 2\tau'\left(\frac{q-c}{2} + b\right)\,.$$

Case 3: $\mathsf{sgn}(q+b) = +1$, $\mathsf{sgn}(c-b) = -1$:

$$\tau'(q+b) + \tau'(b-c) - 2\tau'\left(\frac{q-c}{2} + b\right)\,,$$

τ' is concave, thus

$$\frac{1}{2}\tau'(q+b) + \frac{1}{2}\tau'(b-c) \leq \tau'\left(\frac{q-c}{2} + b\right)\,.$$

Case 4: $\mathsf{sgn}(q+b) = -1$, $\mathsf{sgn}(c-b) = +1$:

$$-\tau'(-q-b) - \tau'(c-b) - 2\tau'\left(\frac{q-c}{2} + b\right)\,,$$

$$-\tau'(-q-b) - \tau'(c-b) \leq 0\,.$$

Together: In every case, we have $g'(b) \leq 0$ and $g(0) = 0$. Thus,

$$g(b) \leq 0\,.$$

□

C.2 Mechanical Proofs

The following auxiliary results consist of simple term transformations. Their proofs are not commented further.

Lemma 52. Let $\tau \in \mathcal{S}_0^3$. Let $a, b, c \in [0, \infty)$. Assume $a \geq c$, $a - b - 2c \geq 0$. Then

$$2(a-b-2c)\tau'_{\surd}((a-b)^2 + 4cb) + \tau'(c+b) - 2\tau'(a-c) \leq 0\,. \tag{89}$$

Proof.

$$\begin{aligned}
& 2(a-b-2c)\tau'_{\surd}((a-b)^2+4cb) + \tau'(c+b) - 2\tau'(a-c) \\
\tau' \geq 0 \qquad & \leq 2(a-b-2c)\tau'_{\surd}((a-b)^2+4cb) - \tau'(a-c) \\
& = 2(a-b)\tau'_{\surd}((a-b)^2+4cb) - 4c\tau'_{\surd}((a-b)^2+4cb) - \tau'(a-c) \\
a \geq b, \tau'_{\surd} \searrow \qquad & \leq 2(a-b)\tau'_{\surd}((a-b)^2) - 4c\tau'_{\surd}((a-b)^2+4cb) - \tau'(a-c) \\
& = \tau'(a-b) - 4c\tau'_{\surd}(a^2 - b((a-2c) + (a-b-2c))) - \tau'(a-c) \\
a - 2c \geq b \geq 0, \tau'_{\surd} \searrow \qquad & \leq \tau'(a-b) - 4c\tau'_{\surd}(a^2) - \tau'(a-c) \\
& = \tau'(a-b) - \frac{2c}{a}\tau'(a) - \tau'(a-c) \\
\tau' \nearrow \qquad & \leq \tau'(a) - \frac{2c}{a}\tau'(a) - \tau'(a-c) \\
& = \left(1 - \frac{2c}{a}\right)\tau'(a) - \tau'(a-c) \\
a \geq 2c, \tau' \frown \qquad & \leq \tau'\left(\left(1 - \frac{2c}{a}\right)a\right) - \tau'(a-c) \\
& = \tau'(a-2c) - \tau'(a-c) \\
\tau' \nearrow \qquad & \leq 0\,.
\end{aligned}$$

$\square$

Lemma 53. Let $\tau \in \mathcal{S}_0^3$. Let $a, b, c \in [0, \infty)$. Assume $a \geq c$, $a - b - 2c \leq 0$. Then

$$-2(b+2c-a)\tau'_{\surd}((a-b)^2 + 4cb) + \tau'(c+b) - 2\tau'(a-c) \leq 0\,. \tag{90}$$

Proof.

$$\begin{aligned}
& -2(b+2c-a)\tau'_{\surd}((a-b)^2+4cb) + \tau'(c+b) - 2\tau'(a-c) \\
& = -2(b+2c-a)\tau'_{\surd}(a^2 + b(-2a+b+4c)) + \tau'(c+b) - 2\tau'(a-c) \\
a \geq c, b+2c \geq a, \tau'_{\surd} \searrow \qquad & \leq -2(b+2c-a)\tau'_{\surd}(a^2 + b(-2a+b+4a)) + \tau'(c+b) - 2\tau'(a-c) \\
& = -2(b+2c-a)\tau'_{\surd}((a+b)^2) + \tau'(c+b) - 2\tau'(a-c) \\
& = -\frac{b+2c-a}{a+b}\tau'(a+b) + \tau'(c+b) - 2\tau'(a-c) \\
a \geq c, \tau' \nearrow \qquad & \leq -\frac{b+2c-a}{a+b}\tau'(a+b) + \tau'(a+b) - 2\tau'(a-c) \\
& = \frac{2(a-c)}{a+b}\tau'(a+b) - 2\tau'(a-c) \\
0 \leq a - c \leq a + b, \tau' \frown \qquad & \leq 2\tau'(a-c) - 2\tau'(a-c) \\
& = 0\,.
\end{aligned}$$

$\square$

Lemma 54. Let $\tau \in \mathcal{S}_0^3$. Let $a, b, c \geq 0$, $s \in [-1, 1]$. Assume $a \geq b \geq sc$. Then

$$\tau'(a-b) + 2(b-sc)\tau'_{\surd}(c^2 - 2scb + b^2) - 2\tau'(a-sc) \leq 0\,.$$

Proof.

$$\begin{aligned}
& && \tau'(a-b) + 2(b-sc)\tau'_{\surd}(c^2 - 2scb + b^2) - 2\tau'(a-sc) \\
c \geq sc, \tau'_{\surd} \searrow & && \leq \tau'(a-b) + 2(b-sc)\tau'_{\surd}(s^2c^2 - 2scb + b^2) - 2\tau'(a-sc) \\
& && = \tau'(a-b) + \tau'(b-sc) - 2\tau'(a-sc) \\
a \geq b \geq sc, \tau' \nearrow & && \leq \tau'(a-sc) + \tau'(a-sc) - 2\tau'(a-sc) \\
& && = 0\,.
\end{aligned}$$

$\square$

Lemma 55. Let $\tau \in \mathcal{S}_0^3$. Let $a, d \in [0, \infty)$. Assume $a \geq d$. Then

$$\tau''(a) - 2\tau''(d) \leq 0\,.$$

Proof.

$$\begin{aligned}
& && \tau''(a) - 2\tau''(d) \\
\tau'' \searrow, d \leq a & && \leq -\tau''(d) \\
\tau'' \geq 0 & && \leq 0\,.
\end{aligned}$$

$\square$

Lemma 56. Let $\tau \in \mathcal{S}_0^3$. Let $u, v \in [0, \infty)$. Assume $u \geq v$. Then

$$\tau'(u) - \tau'(u+v) + 4v\tau'_{\surd}(4uv) - 2\tau'(v) \leq 0\,.$$

Proof.

$$\begin{aligned}
& && \tau'(u) - \tau'(u+v) + 4v\tau'_{\surd}(4uv) - 2\tau'(v) \\
\tau' \nearrow & && \leq 4v\tau'_{\surd}(4uv) - 2\tau'(v) \\
u \geq v, \tau'_{\surd} \searrow & && \leq 4v\tau'_{\surd}(4vv) - 2\tau'(v) \\
& && = \tau'(2v) - 2\tau'(v) \\
\tau' \frown & && \leq 0\,.
\end{aligned}$$

$\square$

Lemma 57. Let $\tau \in \mathcal{S}_0^3$. Let $a, b, c \geq 0$, $s \in [-1, 1]$. Assume $(s+1)c \leq 2a$, $a \leq b$. Then

$$-\tau'(b-a) + 2(b-sc)\tau'_{\surd}(c^2 - 2scb + b^2) - 2\tau'(a-sc) \leq 0\,.$$

Proof.

$$\begin{aligned}
& -\tau'(b-a) + 2(b-sc)\tau'_{\surd}(c^2 - 2scb + b^2) - 2\tau'(a-sc) \\
\tau' \frown \text{(subadditiv)} \qquad & \leq -\tau'(b-a+2(a-sc)) + 2(b-sc)\tau'_{\surd}(c^2 - 2scb + b^2) \\
& = -\tau'(b+a-2sc) + 2(b-sc)\tau'_{\surd}(c^2 - 2scb + b^2) \\
a \geq \frac{1+s}{2}c, \tau' \nearrow \qquad & \leq -\tau'(b + \frac{1+s}{2}c - 2sc) + 2(b-sc)\tau'_{\surd}(c^2 - 2scb + b^2) \\
& = -\tau'(b + \frac{1-s}{2}c - sc) + 2(b-sc)\tau'_{\surd}(c^2 - 2scb + b^2) \\
1-s \geq 0, \tau' \nearrow \qquad & \leq -\tau'(b-sc) + 2(b-sc)\tau'_{\surd}(c^2 - 2scb + b^2) \\
sc \leq c, \tau'_{\surd} \searrow \qquad & \leq -\tau'(b-sc) + 2(b-sc)\tau'_{\surd}((sc)^2 - 2scb + b^2) \\
& = -\tau'(b-sc) + \tau'(b-sc) \\
& = 0\,.
\end{aligned}$$

□

Lemma 58. Let $\tau \in \mathcal{S}_0^3$. Let $a, c \in [0, \infty)$, $s \in [-1, 1]$. Assume $a \geq sc$. Then

$$\tau'(a) + 2(a-sc)\tau'_{\surd}(c^2 - 2sca + a^2) - 2\tau'(a-sc) - 2a\tau''(a-sc) \leq 0\,.$$

Proof.

$$\begin{aligned}
& \tau'(a) + 2(a-sc)\tau'_{\surd}(c^2 - 2sca + a^2) - 2\tau'(a-sc) - 2a\tau''(a-sc) \\
a \geq sc, sc \leq c, \tau'_{\surd} \searrow \qquad & \leq \tau'(a) + 2(a-sc)\tau'_{\surd}((sc)^2 - 2sca + a^2) - 2\tau'(a-sc) - 2a\tau''(a-sc) \\
& = \tau'(a) + \tau'(a-sc) - 2\tau'(a-sc) - 2a\tau''(a-sc) \\
& = \tau'(a) - \tau'(a-sc) - 2a\tau''(a-sc) \\
sc \leq 2a, \tau'' \geq 0 \qquad & \leq \tau'(a) - \tau'(a-sc) - sc\tau''(a-sc) \\
\text{Lemma 14 (ii)} \qquad & \leq 0\,.
\end{aligned}$$

□

Lemma 59. Let $\tau \in \mathcal{S}_0^3$. Let $u, v \in [0, \infty)$. Then

$$\tau'(u) - \tau'(u+v) + 4v\tau'_{\surd}(4uv + v^2) - 2\tau'(v) \leq 0\,.$$

Proof.

$$
\begin{aligned}
& && \tau'(u) - \tau'(u+v) + 4v\tau'_{\surd}(4uv + v^2) - 2\tau'(v) \\
& && = \tau'(u) - \tau'(u+v) + \frac{1}{\sqrt{u/v + 1/4}}\tau'(2v\sqrt{u/v+1/4}) - 2\tau'(v) \\
& 2\sqrt{u/v+1/4} \geq 1, \tau' \frown && \leq \tau'(u) - \tau'(u+v) + \frac{2\sqrt{u/v+1/4}}{\sqrt{u/v+1/4}}\tau'(v) - 2\tau'(v) \\
& && = \tau'(u) - \tau'(u+v) \\
& v \geq 0, \tau' \nearrow && \leq 0\,.
\end{aligned}
$$

□

Lemma 60. Let $\tau \in \mathcal{S}_0^3$. Let $a, b, c \in [0, \infty)$ and $s \in [-1, 1]$. Assume $a \geq sc \geq b$. Then

$$\tau'(a) - \tau'(a-b) - 2b\tau''(a - sc) \leq 0\,.$$

Proof.

$$
\begin{aligned}
& && \tau'(a) - \tau'(a-b) - 2b\tau''(a-sc) \\
& b \leq sc, \tau'' \searrow && \leq \tau'(a) - \tau'(a-b) - 2b\tau''(a-b) \\
& \text{Lemma 14 (ii)} && \leq b\tau''(a-b) - 2b\tau''(a-b) \\
& && = -b\tau''(a-b) \\
& \tau'' \geq 0 && \leq 0\,.
\end{aligned}
$$

□

Lemma 61. Let $\tau \in \mathcal{S}_0^3$. Let $b, c \in [0, \infty)$. Assume $2c \geq 3b$. Then

$$\tau'(c) - \tau'(c-b) - 2b\tau'_{\surd}(c^2 - 2(c-b)b + b^2) \leq 0\,.$$

Proof.

$$
\begin{aligned}
& && \tau'(c) - \tau'(c-b) - 2b\tau'_{\surd}(c^2 - 2(c-b)b + b^2) \\
& && = \tau'(c) - \tau'(c-b) - 2b\tau'_{\surd}(c^2 - b(2c - 3b)) \\
& 2c \geq 3b, \tau'_{\surd} \searrow && \leq \tau'(c) - \tau'(c-b) - 2b\tau'_{\surd}(c^2) \\
& && = \tau'(c) - \tau'(c-b) - \frac{b}{c}\tau'(c) \\
& && = \frac{c-b}{c}\tau'(c) - \tau'(c-b) \\
& \frac{c-b}{c} \in [0,1], \tau' \frown && \leq \tau'(c-b) - \tau'(c-b) \\
& && = 0\,.
\end{aligned}
$$

□

Lemma 62. Let $\tau \in \mathcal{S}_0^3$. Let $b, c \in [0,\infty)$. Assume $c \geq b$. Then

$$\tau_{\sqrt{}}((c-b)^2 + 2b^2) - \tau(c-b) - 2\tau(b) \leq 0\,.$$

Proof.

$$\begin{aligned}
& && \tau_{\sqrt{}}((c-b)^2 + 2b^2) - \tau(c-b) - 2\tau(b) \\
& \tau_{\sqrt{}} \frown \ \text{(subadditive)} && \leq \tau_{\sqrt{}}((c-b)^2) + \tau_{\sqrt{}}(2b^2) - \tau(c-b) - 2\tau(b) \\
& && = \tau_{\sqrt{}}(2b^2) - 2\tau(b) \\
& \tau_{\sqrt{}} \frown \ \text{(subadditive)} && \leq \tau_{\sqrt{}}(b^2) + \tau_{\sqrt{}}(b^2) - 2\tau(b) \\
& && = 0\,.
\end{aligned}$$

□

Lemma 63. Let $\tau \in \mathcal{S}_0^3$. Let $b, c \in [0,\infty)$. Assume $2c \leq 3b$. Then

$$\tau'\left(\frac{3}{2}b\right) - \tau'\left(\frac{1}{2}b\right) - 2b\tau'_{\sqrt{}}(c^2) \leq 0\,.$$

Proof.

$$\begin{aligned}
& && \tau'\left(\frac{3}{2}b\right) - \tau'\left(\frac{1}{2}b\right) - 2b\tau'_{\sqrt{}}(c^2) \\
& c \leq \frac{3}{2}b, \tau'_{\sqrt{}} \searrow && \leq \tau'\left(\frac{3}{2}b\right) - \tau'\left(\frac{1}{2}b\right) - 2b\tau'_{\sqrt{}}\left(\left(\frac{3}{2}b\right)^2\right) \\
& && = \tau'\left(\frac{3}{2}b\right) - \tau'\left(\frac{1}{2}b\right) - \frac{2}{3}\tau'\left(\frac{3}{2}b\right) \\
& && = \frac{1}{3}\tau'\left(\frac{3}{2}b\right) - \tau'\left(\frac{1}{2}b\right) \\
& \frac{1}{3} \leq 1, \tau' \frown && \leq \tau'\left(\frac{1}{2}b\right) - \tau'\left(\frac{1}{2}b\right) \\
& && = 0\,.
\end{aligned}$$

□

Lemma 64. Let $\tau \in \mathcal{S}_0^3$. Let $b \in [0,\infty)$. Then

$$\tau(\frac{3}{2}b) - \tau(\frac{1}{2}b) - 2\tau(b) \leq 0\,.$$

Proof.

$$
\begin{aligned}
& && \tau(\tfrac{3}{2}b) - \tau(\tfrac{1}{2}b) - 2\tau(b) \\
& && = \tau_{\surd}(\tfrac{9}{4}b^2) - \tau_{\surd}(\tfrac{1}{4}b^2) - 2\tau(b) \\
&\tau_{\surd} \frown \ \text{(subadditive)} && \leq \tau_{\surd}(\tfrac{8}{4}b^2) - 2\tau(b) \\
& && = \tau_{\surd}(2b^2) - 2\tau(b) \\
&\tau_{\surd} \frown \ \text{(subadditive)} && \leq \tau_{\surd}(b^2) + \tau_{\surd}(b^2) - 2\tau(b) \\
& && = 0\,.
\end{aligned}
$$

□

Lemma 65. Let $\tau \in \mathcal{S}_0^3$. Let $x, b, c \in [0, \infty)$. Assume $x \leq c$. Then

$$\tau''(x+b) - \tau''(x) + 4b^2\tau''_{\surd}(c^2 - 2xb + b^2) \leq 0\,.$$

Proof.

$$
\begin{aligned}
& && \tau''(x+b) - \tau''(x) + 4b^2\tau''_{\surd}(c^2 - 2xb + b^2) \\
&\tau''_{\surd} \leq 0 && \leq \tau''(x+b) - \tau''(x) \\
&\tau'' \searrow && \leq 0\,.
\end{aligned}
$$

□

Lemma 66. Let $\tau \in \mathcal{S}_0^3$. Let $u, c \in [0, \infty)$. Then

$$2\tau'(u) - 2\tau'(u/2) - \frac{u}{2}\tau''(c) - u\tau''(u/2) \leq 0\,.$$

Proof.

$$
\begin{aligned}
& && 2\tau'(u) - 2\tau'(u/2) - \frac{u}{2}\tau''(c) - u\tau''(u/2) \\
&\tau'' \geq 0 && \leq 2\tau'(u) - 2\tau'(u/2) - u\tau''(u/2) \\
&\text{Lemma 14 (ii)} && \leq u\tau''(u/2) - u\tau''(u/2) \\
& && = 0\,.
\end{aligned}
$$

□

Lemma 67. Let $\tau \in \mathcal{S}_0^3$. Let $u \in [0, \infty)$. Then

$$u\tau'(u) - 2u\tau'(u/2) \leq 0\,.$$

Proof.

$$\begin{aligned} & u\tau'(u) - 2u\tau'(u/2) \\ \tau' \frown \quad & \leq 2u\tau'(u/2) - 2u\tau'(u/2) \\ & = 0\,. \end{aligned}$$

□

Lemma 68. Let $\tau \in \mathcal{S}_0^3$. Let $b, c \in [0, \infty)$, $s \in [-1, 1]$. Assume $b \leq sc$. Then

$$\begin{aligned} & \frac{(1-s)c}{2}\tau'\left(\frac{(1+s)c}{2} - b\right) + 2(1-s)(b+c)\tau'_{\surd}(c^2 - 2scb + b^2) - 2\tau'\left(\frac{(1-s)c}{2}\right) + \\ & \quad \frac{(1+s)c}{2}\tau'\left(\frac{(1+s)c}{2}\right) - \tau'(c) - b(1-s)\tau''\left(\frac{(1-s)c}{2}\right) \\ & \leq \frac{(1-s)c}{2}\tau'(c-b) + 2(1-s)(b+c)\tau'_{\surd}((c-b)^2) - 2\tau'\left(\frac{(1-s)c}{2}\right) + \\ & \quad - \frac{(1-s)c}{2}\tau'(c) - b(1-s)\tau''\left(\frac{c-b}{2}\right)\,. \end{aligned}$$

Proof. Apply

$$\begin{aligned} \tau'((s+1)c/2 - b) &\leq \tau'(c-b)\,, && s \leq 1, \tau' \nearrow\,, \\ \tau'_{\surd}(c^2 - 2scb + b^2) &\leq \tau'_{\surd}((c-b)^2)\,, && s \leq 1, \tau'_{\surd} \searrow\,, \\ \tau'((s+1)c/2) &\leq \tau'(c)\,, && s \leq 1, \tau' \nearrow\,, \\ -\tau''((1-s)c/2) &\leq -\tau''((c-b)/2)\,, && b \leq sc, \tau'' \searrow\,. \end{aligned}$$

□

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I want to thank Christophe Leuridan for his very helpful answer on math overflow[1]. His quick response to my post allowed me to obtain the proof of Lemma 18 very efficiently.

[1] `https://mathoverflow.net/questions/447718/smooth-approximation-of-nonnegative-nondecreasing-concave-functions/447722`