Exercises

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Contents

1 Exercises Part 1

1.1 Irreducible topological spaces

Exercise 1.1.1 See [2, Exercise 1.3] Determine all irreducible Hausdorff spaces. Determine all noetherian Hausdorff spaces. Show that a topological space is noetherian if and only if every open subspace is quasi-compact.

Exercise 1.1.2 Prove Cayley Hamilton by using the irreducibility of $\mathbb{A}^{n^2}(\Bbbk)$ and the representation of the determinant as a polynomial.

Hint: Use matrices with n distinct Eigenvalues and show that these fulfill the statement of Cayley Hamilton. Then show that this set is open and non-empty.

1.2 Localization of rings and modules

Remark 1.2.1 (Localization of rings and modules) See [3, p. 1.3.3.]. Another important example of a definition by universal property is the notion of localization of a ring. We first review a constructive definition, and then reinterpret the notion in terms of universal property. A multiplicative subset S of a ring A is a subset closed under multiplication containing 1. We define a ring $S^{-1}\mathsf{A}$. The elements of $S^{-1}A$ are of the form $\frac{a}{s}$ where $a \in \mathsf{A}$ and $s \in S$, and where $\frac{a_1}{s_1} = \frac{a_2}{s_2}$ if (and only if) for some $s \in S$, $s(s_2a_1 - s_1a_2) = 0$. We define $\frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{s_2a_1+s_1a_2}{s_1s_2}$, and $\frac{a_1}{s_1} \cdot \frac{a_2}{s_2} = \frac{a_1a_2}{s_1s_2}$. (If you wish, you may check that this equality of fractions really is an equivalence relation and the two binary operations on fractions are well-defined on equivalence classes and make $S^{-1}\mathsf{A}$ into a ring.) We have a canonical ring map $\mathsf{A} \to S^{-1}\mathsf{A}$ given by $a \mapsto \frac{a}{1}$. Note that if $0 \in S$, $S^{-1}\mathsf{A}$ is the 0-ring. There are two particularly important flavors of multiplicative subsets. The first is $\{1, f, f^2, f^3, \cdots\}$ where $f \in \mathsf{A}$. This localization is denoted A_f. (Can you describe an isomorphism $\mathsf{A}_f \to \mathsf{A}[t]/(tf-1)$?) The second is $A \setminus \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal. This localization $S^{-1}\mathsf{A}$ is denoted $\mathsf{A}_\mathfrak{p}$. (Notational warning: If $\mathfrak{p}$ is a prime ideal, then $\mathsf{A}_\mathfrak{p}$ means you're allowed to divide by elements not in $\mathfrak{p}$. However, if $f \in \mathsf{A}$, A_f means you're allowed to divide by f. This can be confusing. For example, if (f) is a prime ideal, then $\mathsf{A}_f \neq \mathsf{A}_{(f)}$.)

Exercise 1.2.1 See [3, Exercise 1.3.C]. Show that $\mathsf{A} \to S^{-1}\mathsf{A}$ is injective if and only if S contains no zerodivisors. (A zerodivisor of a ring A is an element a such that there is a nonzero element b with $ab = 0$. The other elements of A are called non-zerodivisors. For example, an invertible element is never a zerodivisor. Counter-intuitively, 0 is a zerodivisor in every ring but the 0-ring. More generally, if M is an A-module, then $a \in \mathsf{A}$ is a zerodivisor for M if there is a nonzero $m \in M$ with $am = 0$. The other elements of A are called non-zerodivisors for M. Equivalently, and very usefully, $a \in \mathsf{A}$ is a non-zerodivisor for M if and only if $(-) \cdot a \colon M \to M$ is an injection, or equivalently if the sequence

$$0 \longrightarrow M \xrightarrow{(-)\cdot a} M \tag{1.2.1}$$

is exact.) If A is an integral domain and $S = \mathsf{A} \setminus \{0\}$, then $S^{-1}\mathsf{A}$ is called the fraction field of A, which we denote $K(\mathsf{A})$. The exercise shows that A is a subring of its fraction field.

Exercise 1.2.2 See [3, Exercise 1.3.D.] Verify that $\mathsf{A} \to S^{-1}\mathsf{A}$ satisfies the following universal property:

$S^{-1}\mathsf{A}$ is initial among A-algebras $\mathscr{B}$ where every element of S is sent to an invertible element in $\mathscr{B}$.

(Recall: the data of "an A-algebra $\mathscr{B}$" and "a ring map $\mathsf{A} \to \mathscr{B}$" are the same.)

1.3 Algebraic varieties and prevarieties

Exercise 1.3.1 See [2, Exercise 1.4]. Show that the underlying topological space X of a prevariety is a T1-space (i.e., for all $x, y \in X$ there exist open neighborhoods U of x and V of y with $y \notin U$ and $x \notin V$).

Exercise 1.3.2 See [2, Exercise 1.5]. Consider the twisted cubic curve $C = \{(t, t^2, t^3) \mid t \in \Bbbk\} \subset \mathbb{A}^3(\Bbbk)$. Show that C is an irreducible closed subset of $\mathbb{A}^3(\Bbbk)$. Find generators for the ideal $I(C)$. Let $V = V(X^2 - YZ, XZ - X) \subset \mathbb{A}^3(\Bbbk)$. Show that V consists of three irreducible components and determine the corresponding prime ideals.

Exercise 1.3.3 See [2, Exercise 1.14]. Let X be a prevariety and let Y be an affine variety. Show that the map

$$\mathrm{Hom}(X, Y) \to \mathrm{Hom}_{\Bbbk-\mathsf{alg}}(\Gamma(Y), \Gamma(X)),\ f \mapsto (f^* \colon \phi \mapsto \phi \circ f),$$

is bijective. Deduce that $\mathrm{Hom}(X, \mathbb{A}^n(\Bbbk)) = \Gamma(X)^n$.

Proof: Let us first consider the case that X and Y are affine varieties. Then a map $f \colon X \to Y$ is a component-wise polynomial map $f \colon \Bbbk^n \supset X \to Y \subset \Bbbk^m$ with

$$f(x_1, \cdots, x_n) = (f_1(x_1, \cdots, x_n), \cdots, f_m(x_1, \cdots, x_n)).$$

Then the result to show is that the contravariant functor

$$\Gamma(\cdot) \colon \mathrm{AffineVarieties} \to \mathrm{int} - \mathrm{Alg}, \quad (f \colon X \to Y) \mapsto (f^* \colon \Gamma(Y) \to \Gamma(X))$$

from affine varieties into integral $\Bbbk$-algebras is full and faithful. We show this by constructing explicit isomorphisms

$$\Phi \colon \mathrm{Hom}(X, Y) \leftrightarrow \mathrm{Hom}_{\Bbbk-\mathsf{alg}}(\Gamma(Y), \Gamma(X)) \colon \Psi.$$

We define $\Phi(f) = f^*$. To define Ψ we want to find for any $\alpha \colon \Gamma(Y) \to \Gamma(X)$ an element $f \colon X \to Y$ such that $f^* = \alpha$, since then $\Psi(\Phi(f)) = \Psi(\alpha) = f$ would be a left inverse map.

For this we define

$$\Psi(\alpha) = (\alpha(Y_1), \cdots, \alpha(Y_m))$$

where we denote with Y_n the map $\mathrm{pr}_n\big|_Y \colon \Bbbk^m \supset Y \to \Bbbk$ which is a morphism of affine varieties between Y and $\mathbb{A}^1$. Note, that we need to show that this map is well defined. I.e. that for the prime-ideal $\mathfrak{p}$ such that $Y = V(\mathfrak{p})$ we have $p(\Psi(\alpha)(x)) = 0$ for any $x \in X$ and $p \in \mathfrak{p}$. This however is clear since

$$(Y_1, \cdots, Y_m) \colon Y \to Y$$

is just the identity map and by the $\Bbbk$-algebra morphism property of α we get

$$p(\Psi(\alpha)(x)) = p(\alpha(Y_1)(x), \cdots, \alpha(Y_m)(x))$$

$$
\begin{aligned}
&= \alpha(p \circ (Y_1, \cdots, Y_m))(x) \\
&= \alpha(p)(x) \\
&= \alpha(0)(x)
\end{aligned}
$$

since any $p \in \mathfrak{p}$ is the 0-map restricted to Y.

Now to show that Φ and Ψ are bijections we show that they are mutually inverse.

For $f = (f_1, \cdots, f_m)\colon X \to Y$ with $f_i \in \Bbbk[T_1, \cdots, T_n]$ we calculate

$$
\begin{aligned}
\Psi(\Phi(f)) =& \Psi(f^*) \\
=& (f^* Y_1, \cdots, f^* Y_m) \\
=& (f_1, \cdots, f_m) \\
=& f.
\end{aligned}
$$

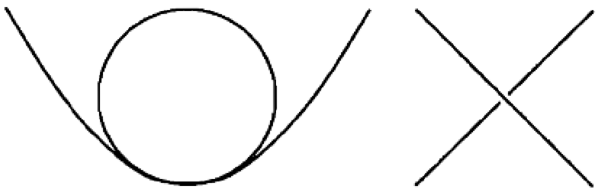
Conversely for $\alpha\colon \Gamma(Y) \to \Gamma(X)$ it is enough to show $\Phi \circ \Psi = \mathrm{id}$ on generators of $\Gamma(Y)$. These are given by the component functions Y_n. We calculate

$$
\begin{aligned}
\Phi(\Psi(\alpha))(Y_n) &= \Phi(\alpha(Y_1), \cdots, \alpha(Y_m))(Y_n) \\
&= (\alpha(Y_1), \cdots, \alpha(Y_m))^*(Y_n) \\
&= \mathrm{pr}_n(\alpha(Y_1), \cdots, \alpha(Y_m)) \\
&= \alpha(Y_n).
\end{aligned}
$$

Since we never needed that X is actually an affine variety, we can easily replace it with $\cup X_i$, where every X_i is an affine variety and the construction follows analogously.

Now for $\mathbb{A}^n$ this means that $\mathrm{Hom}(X, \mathbb{A}^n) \cong \mathrm{Hom}(\Gamma(\mathbb{A}^n), \Gamma(X)) \cong \mathrm{Hom}(\Bbbk[T_1, \cdots, T_n], \Gamma(X)) \cong \mathrm{Hom}(\Bbbk[x], \Gamma(X))^n \cong \mathrm{Hom}(\Gamma(\mathbb{A}^1), \Gamma(X))^n$. $\square$

Exercise 1.3.4 See [1, Exercise 1.10]. Find rings to represent the following figures.



The first represents the union of a circle and a parabola in the plane and the second shows the union if two skew lines in 3-space. (You may use the Nullstellensatz to prove your answer is right.)

Exercise 1.3.5 When is $\mathbb{S}^1 \subset \mathbb{R}^2$ irreducible?

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