alggeo24 reading course

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Contents

1 Some basic commutative algebra

This small chapter serves as a short collection on the basis theorem and the Nullstellensatz by Hilbert. This is also a collection of some commutative algebra needed to define affine varieties.

1.1 Hilbert basis theorem

Definition 1.1.1 *A ring* A *is said to be Noetherian if it satisfies the following three equivalent conditions:*

(i) Every non-empty set of ideals in A *has a maximal element.*

(ii) Every ascending chain of ideals in A *is stationary.*

(iii) Every ideal in A *is finitely generated.*

Similarly for a R*-module* M *we call* M *Noetherian if it satisfies the following three equivalent conditions:*

(i) Every non-empty set T *of submodules of* M *has a maximal element.*

(ii) Every ascending chain of submodules of M *is stationary.*

(iii) Every submodule $N \subset M$ *is finitely generated.*

To show the equivalence of the first two properties for rings and modules we can look at the following general result on partially ordered sets.

Proposition 1.1.2 *The following conditions on a partially ordered set* $\mathcal{P}$ *are equivalent:*

(i) Every increasing sequence $x_1 \leq x_2 \leq \cdots$ *in* $\mathcal{P}$ *is stationary, i.e. there is an index* $n \in \mathbb{N}$ *such that for all* $m \geq n$ *we have* $x_m = x_n$*.*

(ii) Every non-empty subset of $\mathcal{P}$ *has a maximal element.*

Proof: Proposition 1.1.2-(i) $\Rightarrow$ Proposition 1.1.2- (ii) Assume that a non-empty subset $T \subset \mathcal{P}$ and that it contains no maximal element. Then we can inductively construct an ascending sequence $(x_i)_{i\in\mathbb{N}}$ such that $x_i \leq x_{i+1}$ which is not stationary, since this would contradict the existence of a maximal element.

Proposition 1.1.2- (ii) $\Rightarrow$ Proposition 1.1.2- (i) For an ascending sequence $(x_i)_{\in\mathbb{N}}$ this set has a maximal element. This means that the sequence has to stabilize. $\square$

Now to show that the third property is also equivalent to the first two properties we just need to show that every submodule is finitely generated, as an ideal is just a special case of a submodule of the ring A considered as a module over itself.

Proposition 1.1.3 *A module* M *over a ring* A *is Noetherian if and only if every submodule* $N \subset M$ *is finitely generated.*

PROOF: Lets first consider the "if" direction. Consider a submodule $N \subset M$ and consider the set $\mathcal{P}$ of all finitely generated submodules of N ordered by inclusion. The set $\mathcal{P}$ is non-empty since $\{0\} \in \mathcal{P}$ and thus $\mathcal{P}$ has a maximal element N_0. If $N \neq N_0$ consider the submodule $N_0 \oplus \mathsf{A}x$ where $x \in N$, $x \notin N_0$. This is still finitely generated and strictly contains N_0, which contradicts the maximality. Thus $N = N_0$ and N is finitely generated.

For the "only if" direction consider an ascending chain of submodules $T := (M_n)_{n\in\mathbb{N}}$ of M. Then the module $\cup_{n\in\mathbb{N}} M_n$ is finitely generated by assumption. Thus $T = \mathsf{A}\langle x_1, \cdots, x_r\rangle$ with $x_i \in M_{n_i}$ and set $n = \max_{i=1}^r n_i$. Then each $x_i \in M_n$ and thus $M_n = M$ which means that the chain is stationary.□

Proposition 1.1.4 *Consider a short exact sequence*

$$0 \longrightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0 \tag{1.1.1}$$

then M is Noetherian if and only if M' and M'' are Noetherian.

PROOF: Consider an ascending chain of submodules $(M_n)_{n\in\mathbb{N}}$ in M. This gives rise to commutative diagrams of the form

$$\begin{tikzcd}
0 \arrow[r] & \alpha^{-1}(M_n) \arrow[r, "\alpha"] \arrow[d, hook] & M_n \arrow[r, "\beta"] \arrow[d, hook] & \beta(M_n) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \alpha^{-1}(M_{n+1}) \arrow[r, "\alpha"] & M_{n+1} \arrow[r, "\beta"] & \beta(M_{n+1}) \arrow[r] & 0
\end{tikzcd} . \tag{1.1.2}$$

In this diagram the rows are exact by construction and since (1.1.1) is exact. Now assume that M is Noetherian. Then we can pick $n \in \mathbb{N}$ big enough such that the middle inclusion is an isomorphism. Then the two other inclusions are isomorphisms as well by the five lemma. Conversely, if M' and M'' are Noetherian, we can pick n big enough such that the chains in M' and M'' have stabilized. Then, again by the five lemma, the middle inclusion is also an isomorphism. □

Now an easy corollary of this is, that direct sums of Noetherian modules are again Noetherian.

Corollary 1.1.5 *Let M_i, $i \in \{1, \cdots, n\}$ be Noetherian modules. Then $\bigoplus_{i=1}^n M_i$ is also Noetherian.*

PROOF: This is an immediate consequence of Proposition 1.1.4 since the sequence

$$0 \longrightarrow M_k \longrightarrow \bigoplus_{i=1}^k M_i \longrightarrow \bigoplus_{i=1}^{k-1} M_i \longrightarrow 0 \tag{1.1.3}$$

is exact, the statement follows by induction. □

Another nice feature of finitely generated modules is, that they are quotients of A^n for some $n \in \mathbb{N}$.

Proposition 1.1.6 *Let M be a A-module. Then M is finitely generated if and only if it is a quotient of A^n for some $n \in \mathbb{N}$.*

PROOF: Let $M = \mathsf{A}\langle x_1, \cdots, x_n\rangle$ be finitely generated. Then we define a A-module morphism by defining

$$\phi\colon \mathsf{A}^n \to M, \quad \phi(a_1, \cdots, a_n) := a_1x_1 + \cdots + a_nx_n.$$

This map is obviously surjective. Conversely, if we have quotient map $\phi\colon \mathsf{A}^n \to M$, then M is generated by $\phi(e_i)$, where $(e_i)_j = \delta_{i,j}1$ for $i = \{1, \cdots, n\}$. □

And last but not least we can get a nice result, that all finitely generated A-modules over a Noetherian ring are also Noetherian.

Proposition 1.1.7 *Let M be a finitely generated A-module and A Noetherian. Then M is Noetherian.*

Proof: This proof follows immediately from Proposition 1.1.6 since the corresponding quotient map fits into the short exact sequence

$$0 \longrightarrow \ker(\phi) \hookrightarrow \mathsf{A}^n \xrightarrow{\phi} M \longrightarrow 0 \tag{1.1.4}$$

from which the statement follows by Proposition 1.1.4. □

Now we have all tools we need to prove Hilbert basis theorem.

Theorem 1.1.8 (Hilbert basis theorem) *See [1]. Let A be a Noetherian ring, then the polynomial ring $\mathsf{A}[x]$ is Noetherian.*

Proof: Consider an ideal $\mathfrak{a} \subset \mathsf{A}[x]$. We want to show that $\mathfrak{a}$ is finitely generated. Then, by Definition 1.1.1, the ring $\mathsf{A}[x]$ would be Noetherian.

We first consider the leading coefficients of polynomials p in $\mathfrak{a}$ yield an ideal $I \subset \mathsf{A}$. Since A is assumed to be Noetherian, this ideal is finitely generated, i.e. $I = \mathsf{A}\langle a_1, \cdots, a_n\rangle$. Now we can pick the corresponding polynomials $f_i = a_i x^{r_i} +$ (lower terms). For the ideal generated by these polynomials we get $\mathfrak{a}' := \mathsf{A}\langle f_1, \cdots, f_n\rangle \subset \mathfrak{a}$.

We want to show that $\mathfrak{a} = \mathfrak{a}'$. For this we pick some $f = ax^m +$ (lower terms) for $a \in I$, i.e. $f \in \mathfrak{a}$. If $m \geq r$ for $r = \max_{i=1}^n r_i = \max_{i=1}^n \deg(f_i)$ we can write $a = \sum_{i=1}^n u_i a_i$ with $u_i \in \mathsf{A}$, $i = 1, \cdots, n$ since $a \in I$ and I is finitely generated. We can thus consider

$$f - \sum_{i=1}^{n} u_i f_i x^{m-r_i}.$$

This is an element in $\mathfrak{a}$ and has degree $< m$.

We can thus write $f = g + h$ for a polynomial g with $\deg(g) < r$ and $h \in \mathfrak{a}'$.

Now consider $M = \mathsf{A}\langle 1, x, x^2, \cdots, x^{r-1}\rangle$, then we have shown that $\mathfrak{a} = (\mathfrak{a} \cap M) + \mathfrak{a}'$. Since M is finitely generated as a A-module, it is quotient of A^n which means that it is Noetherian as well by Proposition 1.1.7. Since $\mathfrak{a} \cup M$ is a submodule of M it is thus also finitely generated. Finally if we have generators $\{f_i\}_{i=1,\cdots,n}$ of $\mathfrak{a}'$ and generators $\{g_i\}_{i=1,\cdots,m}$ of $\mathfrak{a} \cup M$ they generate $\mathfrak{a}$ together.

This shows that any ideal $\mathfrak{a}$ is finitely generated, even as a A-module and hence $\mathsf{A}[x]$ is Noetherian. □

Corollary 1.1.9 *For a Noetherian ring A the polynomials $\mathsf{A}[x_1, \cdots, x_n]$ is Noetherian.*

Proof: This is just an inductive application of Hilbert basis theorem on $\mathsf{A}[x, y] = (\mathsf{A}[x])[y]$. □

Bibliography

[1] M.F. Atiyah and I.G. MacDonald. *Introduction to Commutative Algebra*. Addison-Wesley Series in Mathematics. Avalon Publishing, 1994. ISBN: 978-0-8133-4544-4. URL: https://books.google.de/books?id=HOASFid4x18C.