AAQDD – Abstract additive quantum decision diagrams

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February 8, 2024

1 Formal definition

1.1 Abstract states

The purpose of quantum decision diagrams is to provide a more efficient way to store and manipulate quantum states of a finite number of qubits. A n-qubit state is indeed traditionally represented as an element of $\mathbb{C}^{2^n}$ (with norm 1), which takes exponential space as n grows. Abstract states will be in this part defined similarly, but with

The standard definition of real intervals is:

$$\forall a, b \in \mathbb{R}, [a, b] = \{x \in \mathbb{R} / \min(a, b) \leq x \leq \max(a, b)\}$$

Those can be generalised to complex intervals, commonly using the cartesian notation. On that definiton, we can define sums and products:

$$\forall x, y \in \mathbb{C}, [x, y] = \{a + ib; a \in [\Re(x), \Re(y)], b \in [\Im(x), \Im(y)]\}$$

Now let $\mathcal{A}_0 = \{[x, y]; x, y \in \mathbb{C}\}$, we can now define basic operations.

$$\forall \alpha, \beta \in \mathcal{A}_0, \alpha + \beta = \{a + b; a \in \alpha, b \in \beta\}$$

$$\forall \alpha, \beta \in \mathcal{A}_0, \alpha * \beta = \{a * b; a \in \alpha, b \in \beta\}$$

We now have intervals, *abstract elements* of $\mathbb{C}$ represented in $\mathcal{A}_0$. Our abstract elements for a n-qubit quantum state would be in $\mathcal{A}_n = {\mathcal{A}_0}^{2^n}$ for all $n \in \mathbb{N}$. Defining a sum in $\mathcal{A}_n$, and an external product $\alpha * A$ for $\alpha \in \mathcal{A}_0$ and $A \in \mathcal{A}_n$, comes easily.

1.2 Decision diagrams

We inductively define abstract additive quantum decision diagrams, starting from zero-depth decision. The only kind of zero-depth decision diagram is:

$$\begin{array}{c} \Big\downarrow {\scriptstyle [x,x] \in \mathcal{A}_0} \\ \boxed{1} \end{array}$$

Hence, we can define the set of zero-depth decision diagrams $\mathcal{D}_0 = \mathcal{A}_0$. Based on this, we can define inductively higher-depth decision diagrams. Let for every set E be the set of finite subsets of E: $\mathscr{P}_f(E) = \{A \subset E / |A| < \infty\}$. An AAQDD of depth $n+1$ is defined by:

- An incoming abstract amplitude, an element of $\mathcal{A}_0$
- A finite number of left children (diagrams of depth n), an element of $\mathscr{P}_f(\mathcal{D}_n)$
- A finite number of right children (diagrams of depth n), an element of $\mathscr{P}_f(\mathcal{D}_n)$

Defining $\mathcal{D}_{n+1} = \mathcal{A}_0 \times \mathscr{P}_f(\mathcal{D}_n) \times \mathscr{P}_f(\mathcal{D}_n)$ thus comes naturally.

1.3 Diagram evaluation

Now that we defined our decision diagrams, we can evaluate them to get abstract elements. We inductively define our evaluation function for n quibits $\mathcal{E}_n : \mathcal{D}_n \to \mathcal{A}_n$:

$$\forall D \in \mathcal{D}_0, \mathcal{E}_0(D) = D$$

$$\forall \alpha \in \mathcal{A}_0, \forall D, G \in \mathscr{P}_f(\mathcal{D}_n), \mathcal{E}_{n+1}(\alpha, D, G) = \begin{pmatrix} \alpha * \sum(G)_0 \\ ... \\ \alpha * \sum(G)_{2^n-1} \\ \alpha * \sum(D)_0 \\ ... \\ \alpha * \sum(D)_{2^n-1} \end{pmatrix} \quad \text{with}$$

$$\sum : \left| \begin{array}{rcl} \mathscr{P}_f(\mathcal{D}_n) & \longrightarrow & \mathcal{A}_n \\ G & \longmapsto & \displaystyle\sum_{g \in G} \mathcal{E}_n(g) \end{array} \right. \quad \text{and as expected}$$

$$\forall A = \begin{pmatrix} a_0 \\ ... \\ a_{2^n-1} \end{pmatrix} \in \mathcal{A}_n, \forall i \in \{0, ..., 2^n - 1\}, A_i = a_i$$

Since there is no risk of ambiguity, defining $\mathcal{E} : \bigcup \mathcal{D}_n \to \bigcup \mathcal{A}_n$ is not problematic. With this last function, we can now evaluate all our AAQDDs.