Lloyd-Max Scalar Quantizer

The optimal PDF quantizer

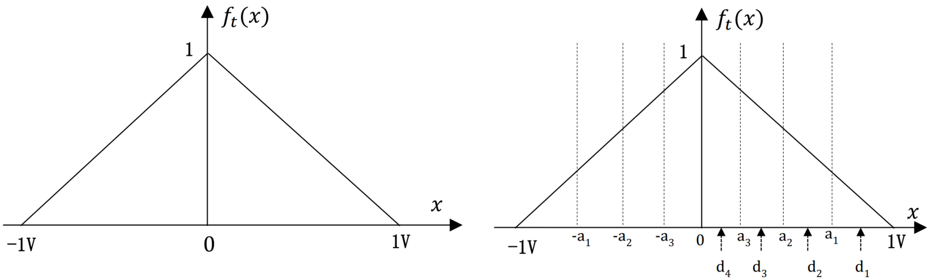
Dexuan He

SJTU

November 13, 2023

Task Introduction

Non-uniform quantification



（1）某模拟信号幅度的概率密度分布　　（2）3 位量化编码方案示意

图 3-4 非均匀匀量化问题探究

Figure. PDF of the signal

- Minimize the quantization noise to signal ratio.
- Closed-form and numerical solution of a_i and d_i.
- Extending the solution to any high-dimensional.

Theorem

Objective Function

To minimize SNR, we consider reduce noise power, that is, the MSE between the quantitative values and the original values. For n quantization steps:

$$\mathcal{L} = MSE = \sum_{i=1}^{M} \int_{a_i}^{a_{i-1}} (x - d_i)^2 f(x) dx \tag{1}$$

where $a_i, d_i, f(x)$ represent quantization interval boundary points, quantization levels, PDF of amplitude x. Now we derive the optimal conditions of this objective function.

Theorem

Interval Condition

$$\frac{\partial \mathcal{L}}{\partial \vec{a}} = 0 \tag{2}$$

$$\frac{\partial}{\partial a_i} \sum_{i=1}^{M} \int_{a_i}^{a_{i-1}} (x - d_i)^2 f(x) dx = 0 \tag{3}$$

$$\frac{\partial}{\partial a_i} \left(\int_{a_i}^{a_{i-1}} (x - d_i)^2 f(x) dx + \int_{a_{i+1}}^{a_i} (x - d_{i+1})^2 f(x) dx \right) = 0 \tag{4}$$

$$-(a_i - d_i)^2 f(a_i) + (a_i - d_{i+1})^2 f(a_i) = 0 \tag{5}$$

$$a_i = \frac{d_i + d_{i+1}}{2} \tag{6}$$

Theorem

Level Condition

$$\frac{\partial \mathcal{L}}{\partial \vec{d}} = 0 \tag{7}$$

$$\frac{\partial}{\partial d_i} \sum_{i=1}^{M} \int_{a_i}^{a_{i-1}} (x - d_i)^2 f(x) dx = 0 \tag{8}$$

$$\sum_{i=1}^{M} \int_{a_i}^{a_{i-1}} \frac{\partial}{\partial d_i} (x - d_i)^2 f(x) dx = 0 \tag{9}$$

$$\sum_{i=1}^{M} \int_{a_i}^{a_{i-1}} -(2x - 2d_i) f(x) dx = 0 \tag{10}$$

$$d_i = \frac{\int_{a_i}^{a_{i-1}} x f(x) dx}{\int_{a_i}^{a_{i-1}} f(x) dx} \tag{11}$$

Theorem 1 (Lloyd-Max)

To minimize SNR under certain amplitude pdf $f(x)$,

$$a_i = \frac{d_i + d_{i+1}}{2} \tag{12}$$

$$d_i = \frac{\int_{a_i}^{a_{i-1}} x f(x) dx}{\int_{a_i}^{a_{i-1}} f(x) dx} \tag{13}$$

i.e. the threshold level is at the midpoint of the adjacent quantization levels, and the quantization level is at the centroid of the quantization interval.

Solution

Closed-form

Let $a_0 = 1V$, $a_4 = 0$, $f(x) = 1 - x$, we get the following recursive formula:

$$a_i = \frac{d_i + d_{i+1}}{2}$$
$$d_i = \frac{1}{3} + \frac{2}{3}(a_{i-1} + a_i) + \frac{1}{3}\frac{a_i a_{i-1} - 1}{1 - \frac{1}{2}(a_{i-1} + a_i)}$$

Due to the cross-directional recurrence and the self-referential term, it is not easy to directly solve the expression. Therefore, we use the numerical iteration optimization method.

SOLUTION

NUMERICAL

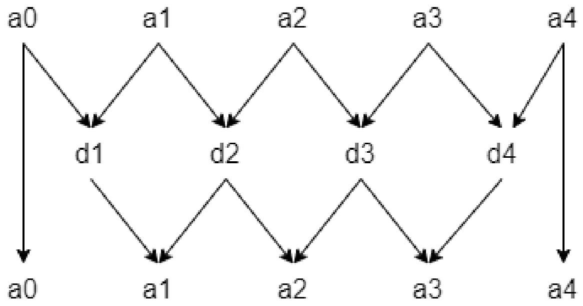


Figure. Propagation

$$\vec{a} = [1, 0.624, 0.392, 0.188, 0]$$
$$\vec{d} = [0.750, 0.499, 0.285, 0.091]$$

$$[d_{128}, d_{127}, d_{50}, d_6, d_1] = [0.0029265, 0.0087910, 0.50897, 0.90390, 0.98093]$$
$$[a_{127}, a_{126}, a_{50}, a_6, a_1] = [0.0058588, 0.0117298, 0.50527, 0.89765, 0.97140]$$