Étale coverings and the fundamental group (SGA 1)

A seminar directed by A. GROTHENDIECK
Including two talks by M. RAYNAUD

Séminaire de Géométrie Algébrique du Bois Marie
1960–1961

Translators' note.

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Introduction

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Étale morphisms

p. 1 To simplify the exposition, we assume that all preschemes in the following are locally Noetherian (at least, starting from Section 2).

1.1 Basics of differential calculus

Let X be a prescheme on Y, and $\Delta_{X/Y}$ the diagonal morphism $X \to X \times_Y X$. This is an immersion, and thus a closed immersion of X into an open subset V of $X \times_Y X$. Let $\mathscr{I}_X$ be the ideal of the closed sub-prescheme corresponding to the diagonal in V (N.B. if one really wishes to do things intrinsically, without assuming that X is separated over Y — a misleading hypothesis — then one should consider the set-theoretic inverse image of $\mathscr{O}_{X\times X}$ in X and denote by $\mathscr{I}_X$ the augmentation ideal in the above . . .). The sheaf $\mathscr{I}_X/\mathscr{I}_X^2$ can be thought of as a quasi-coherent sheaf on X, which we denote by $\Omega^1_{X/Y}$. This sheaf is of finite type if $X \to Y$ is of finite type, and it behaves well with respect to a base change $Y' \to Y$. We also introduce the sheaves $\mathscr{O}_{X\times_Y X}/\mathscr{I}_X^{n+1} = \mathscr{P}^n_{X/Y}$, which are sheaves of *rings* on X, giving us preschemes denoted by $\Delta^n_{X/Y}$ and called the *n-th infinitesimal neighbourhood of X/Y*. The polysyllogism is entirely trivial, even if rather long[1]; it seems wise to not discuss it until we use it for something helpful, with smooth morphisms.

1.2 Quasi-finite morphisms

[1] cf. EGA IV 16.3.

Proposition 2.1. *Let $A \to B$ be a local homomorphism (N.B. all rings are now Noetherian) and $\mathfrak{m}$ the maximal ideal of A. Then the following conditions are equivalent:*

(i) *$B/\mathfrak{m}B$ is of finite dimension over $k = A/\mathfrak{m}$.*

(ii) *$\mathfrak{m}B$ is an ideal of definition, and $B/\mathfrak{r}(B) = \kappa(B)$ is an extension of $k = \kappa(A)$.*

(iii) *The completion $\widehat{B}$ of B is finite over the completion $\widehat{A}$ of A.*

p. 2 We then say that B is *quasi-finite* over A. A morphism $f : X \to Y$ is said to be quasi-finite at x (or the Y-prescheme f is said to be quasi-finite at x) if $\mathcal{O}_x$ is quasi-finite over $\mathcal{O}_{f(x)}$. This is equivalent to saying that x is *isolated in its fibre* $f^{-1}(x)$. A morphism is said to be quasi-finite if it is quasi-finite at each point[2].

Corollary 2.2. *If A is complete, then quasi-finiteness is equivalent to finiteness.*

We could also give the usual polysyllogism (i), (ii), (iii), (iv), (v) for quasi-finite morphisms, but that doesn't seem necessary here.

1.3 Unramified morphisms

Proposition 3.1. *Let $f : X \to Y$ be a morphism of finite type, $x \in X$, and $y = f(x)$. Then the following conditions are equivalent:*

(i) *$\mathcal{O}_x/\mathfrak{m}_y\mathcal{O}_x$ is a finite separable extension of $\kappa(y)$.*

(ii) *$\Omega^1_{X/Y}$ is zero at x.*

(iii) *The diagonal morphism $\Delta_{X/Y}$ is an open immersion on a neighbourhood of x.*

Proof. For the implication (i) $\implies$ (ii), we can use Nakayama to reduce to the case where $Y = \operatorname{Spec}(k)$ and $X = \operatorname{Spec}(k')$, where it is well known, and also trivial by the definition of separable; (ii) $\implies$ (iii) comes from a nice and easy characterisation of open immersions, using Krull; (iii) $\implies$ (i) follows as well from reducing to the case where $Y = \operatorname{Spec}(k)$ and the diagonal morphism is everywhere an open immersion. We must then prove that X is finite with separable ring over k, and this leads us to consider the case where k is algebraically

[2] In EGA II 6.2.3 we further suppose that f is of finite type.

closed. But then every closed point of X is isolated (since it is identical to the inverse image of the diagonal by the morphism $X \to X \times_k X$ defined by x), whence X is finite. We can thus suppose that X consists of a single point, with ring A, and so $A \otimes_k A \to A$ is an isomorphism, hence $A = k$. □

Definition 3.2.

(a) We then say that f is *unramified* at x, or that X is unramified at x on Y.

(b) Let $A \to B$ be a local homomorphism. We say that it is *unramified*, or that B is a local *unramified* algebra on A, if $B/\mathfrak{r}(A)B$ is a finite separable extension of $A/\mathfrak{r}(A)$, i.e. if $\mathfrak{r}(A)B = \mathfrak{r}(B)$ and $\kappa(B)$ is a separable extension of $\kappa(A)$[3].

p. 3 **Remarks.** The fact that B is unramified over A can be seen at the level of the completions of A and B. Unramified implies quasi-finite.

Corollary 3.3. *The set of points where f is unramified is open.*

Corollary 3.4. *Let X' and X be preschemes of finite type over Y, and $g\colon X' \to X$ a Y-morphism. If X is unramified over Y, then the graph morphism $\Gamma_g\colon X' \to X \times_Y X$ is an open immersion.*

Indeed, this is the inverse image of the diagonal morphism $X \to X \times_Y X$ by

$$g \times_Y \mathrm{id}_{X'}\colon X' \times_Y X \to X \times_Y X.$$

One can also introduce the annihilator ideal $\mathfrak{d}_{X/Y}$ of $\Omega^1_{X/Y}$, called the *different ideal* of X/Y; it defines a closed sub-prescheme of X which, set-theoretically, is the set of point where X/Y is ramified, i.e. not unramified.

Proposition 3.5.

(i) An immersion is ramified.

(ii) The composition of two ramified morphisms is also ramified.

(iii) Base extension of a ramified morphisms is also ramified.

We are rather indifferent about (ii) and (iii) (the second seems more interesting to me). We can, of course, also be more precise, by giving some one-off statements; this is more general only in appearance (except for in the case of definition (b)), and is boring. We obtain, as per usual, the corollaries:

Corollaries 3.6.

[3] cf. regrets in III 1.2.

(iv) *The cartesian product of two unramified morphisms is unramified.*

(v) *If gf is unramified then so too is f.*

(vi) *If f is unramified then so too is f_{red}.*

Proposition 3.7. *Let $A \to B$ be a local homomorphism, and suppose that the residue extension $\kappa(B)/\kappa(A)$ is trivial, with $\kappa(A)$ algebraically closed. In order for B/A to be unramified, it is necessary and sufficient that $\widehat{B}$ be (as an $\widehat{A}$-algebra) a quotient of $\widehat{A}$.*

Remarks.

- In the case where we don't suppose that the residue extension is trivial, we can reduce to the case where it is by taking a suitable finite flat extension of A which destroys the aforementioned extension.

- Consider the example where A is the local ring of an ordinary double point of a curve, and B a point of its normalisation: then $A \subset B$, B is p. 4 unramified over A with trivial residue extension, and $\widehat{A} \to \widehat{B}$ is surjective but *not injective*. We are thus going to strengthen the notion of unramified-ness.

1.4 Étale morphisms. Étale coverings

We are going to suppose that everything concerning flat morphisms that we need to be true is indeed true; these facts will be proved later, if there is time[4].

Definition 4.1.

(a) Let $f : X \to Y$ be a morphism of finite type. We say that f is *étale* at x if f is both flat and unramified at x. We say that f is étale if it is étale at all points. We say that X is étale at x over Y, or that it is a Y-prescheme which is étale at x etc.

(b) Let $f : A \to B$ be a local homomorphism. We say that f is étale, or that B is étale over A, if B is flat and unramified over A[5].

[4] cf. Exp. IV.

[5] cf. regrets in III 1.2.

Proposition 4.2. *For B/A to be étale, it is necessary and sufficient that $\widehat{B}/\widehat{A}$ be étale.*

p. 5

Proof. Indeed, this is true individually for both "unramified" and "flat". □

Corollary 4.3. *Let $f : X \to Y$ be of finite type, and $x \in X$. The property of f being étale at x depends only on the local homomorphism $\mathscr{O}_{f(x)} \to \mathscr{O}_x$, and in fact only on the corresponding homomorphism for the completions.*

Corollary 4.4. *Suppose that the residue extension $\kappa(A) \to \kappa(B)$ is trivial, or that $\kappa(A)$ is algebraically closed. Then B/A is étale if and only if $\widehat{A} \to \widehat{B}$ is an isomorphism.*

We can combine flatness with 3.7.

Proposition 4.5. *Let $f : X \to Y$ be a morphism of finite type. Then the set of points where f is étale is open.*

Proof. Again, this is in fact true individually for both "unramified" and "flat". □

This proposition shows that we can forget about the "one-off" comments in the study of morphisms of finite type that are somewhere étale.

Proposition 4.6.

(i) An open immersion is étale.

(ii) The composition of two étale morphisms is étale.

(iii) The base extension of an étale morphism is étale.

Proof. Indeed, (i) is trivial, and for (ii) and (iii) it suffices to note that it is true for "unramified" and "flat". □

As a matter of fact, there are also corresponding comments to make about local homomorphisms (without the finiteness condition), which in any case should appear in the multiplodoque‖ (starting with the case of unramified).

‖[Trans.] Grothendieck's *multiplodoque d'algèbre homologique* was the final version of his *Tohoku paper* — see (2.1) in 'Life and work of Alexander Grothendieck' by Ching-Li Chan and Frans Oort for more information.

Corollary 4.7. *The cartesian product of two étale morphisms is étale.*[6]

Corollary. *Let X and X' be of finite type over Y, and $g: X \to X'$ a Y-morphism. If X' is unramified over Y and X is étale over Y, then g is étale.*

Proof. Indeed, g is the composition of the graph morphism $\Gamma_g : X \to X \times_Y X'$ (which is an open immersion by 3.4) and the projection morphism, which is étale since it is just a "change of base" by $X' \to Y$ of the étale morphism $X \to Y$. □

Definition 4.9. We say that a cover of Y is étale (resp. unramified) if it is a Y-scheme X that is finite over Y and étale (resp. unramified) over Y.

The first condition means that X is defined by a coherent sheaf of algebras $\mathscr{B}$ over Y. The second means that $\mathscr{B}$ is locally free over Y (resp. means absolutely nothing) *and*, further, that, for all $y \in Y$, the fibre $\mathscr{B}(y) = \mathscr{B}_y \otimes_{\mathscr{O}_y} \kappa(y)$ is a separable algebra (i.e. a finite composition of finite separable extensions) over $\kappa(y)$.

Proposition 4.10. *Let X be a flat cover of Y of degree n (the definition of this term deserved to figure in 4.9) defined by a locally free coherent sheaf $\mathscr{B}$ of algebras. We define, as usual, the trace homomorphism $\mathscr{B} \to \mathscr{A}$ (that is a homomorphism of $\mathscr{A}$-modules, where $\mathscr{A} = \mathscr{O}_Y$). For X to be étale it is necessary and sufficient that the corresponding bilinear form $\mathrm{tr}_{\mathscr{B}/\mathscr{A}}\, xy$ define an isomorphism of $\mathscr{B}$ over $\mathscr{B}$, or, equivalently, that the* discriminant section

$$d_{X/Y} = d_{\mathscr{B}/\mathscr{A}} \in \Gamma\big(Y, \wedge^n \check{\mathscr{B}} \otimes_{\mathscr{A}} \wedge^n \check{\mathscr{B}}\big)$$

is invertible, or that the discriminant ideal defined by this section is the unit ring.

Proof. We can reduce to the case where $Y = \mathrm{Spec}(k)$, and then it is a well-known criterion of separability, and thus trivial by passing to the algebraic closure of k. □

p. 6 **Remark.** We will have a less trivial statement to make later on, when we do not suppose a priori that X is flat over Y, but instead require some normality hypothesis.

[6] test