

Étale coverings and the fundamental group (SGA 1)

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1960–1961

Translators' note.

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Introduction

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Étale morphisms

p. 1 | To simplify the exposition, we assume that all preschemes in the following are locally Noetherian (at least, starting from [Section 2](#)).

1.1 Basics of differential calculus

Let X be a prescheme on Y , and $\Delta_{X/Y}$ the diagonal morphism $X \rightarrow X \times_Y X$. This is an immersion, and thus a closed immersion of X into an open subset V of $X \times_Y X$. Let \mathcal{I}_X be the ideal of the closed sub-prescheme corresponding to the diagonal in V (N.B. if one really wishes to do things intrinsically, without assuming that X is separated over Y — a misleading hypothesis — then one should consider the set-theoretic inverse image of $\mathcal{O}_{X \times X}$ in X and denote by \mathcal{I}_X the augmentation ideal in the above ...). The sheaf $\mathcal{I}_X / \mathcal{I}_X^2$ can be thought of as a quasi-coherent sheaf on X , which we denote by $\Omega_{X/Y}^1$. This sheaf is of finite type if $X \rightarrow Y$ is of finite type, and it behaves well with respect to a base change $Y' \rightarrow Y$. We also introduce the sheaves $\mathcal{O}_{X \times_Y X} / \mathcal{I}_X^{n+1} = \mathcal{P}_{X/Y}^n$, which are sheaves of *rings* on X , giving us preschemes denoted by $\Delta_{X/Y}^n$ and called the *n-th infinitesimal neighbourhood of X/Y* . The polysyllogism is entirely trivial, even if rather long¹; it seems wise to not discuss it until we use it for something helpful, with smooth morphisms.

1.2 Quasi-finite morphisms

¹cf. EGA IV 16.3.

Proposition 2.1. *Let $A \rightarrow B$ be a local homomorphism (N.B. all rings are now Noetherian) and \mathfrak{m} the maximal ideal of A . Then the following conditions are equivalent:*

- (i) $B/\mathfrak{m}B$ is of finite dimension over $k = A/\mathfrak{m}$.
- (ii) $\mathfrak{m}B$ is an ideal of definition, and $B/\mathfrak{r}(B) = \kappa(B)$ is an extension of $k = \kappa(A)$.
- (iii) The completion \hat{B} of B is finite over the completion \hat{A} of A .

p. 2 | We then say that B is *quasi-finite* over A . A morphism $f: X \rightarrow Y$ is said to be quasi-finite at x (or the Y -prescheme f is said to be quasi-finite at x) if \mathcal{O}_x is quasi-finite over $\mathcal{O}_{f(x)}$. This is equivalent to saying that x is *isolated in its fibre* $f^{-1}(x)$. A morphism is said to be quasi-finite if it is quasi-finite at each point².

Corollary 2.2. *If A is complete, then quasi-finiteness is equivalent to finiteness.*

We could also give the usual polysyllogism (i), (ii), (iii), (iv), (v) for quasi-finite morphisms, but that doesn't seem necessary here.

1.3 Unramified morphisms

Proposition 3.1. *Let $f: X \rightarrow Y$ be a morphism of finite type, $x \in X$, and $y = f(x)$. Then the following conditions are equivalent:*

- (i) $\mathcal{O}_x/\mathfrak{m}_y\mathcal{O}_x$ is a finite separable extension of $\kappa(y)$.
- (ii) $\Omega_{X/Y}^1$ is zero at x .
- (iii) The diagonal morphism $\Delta_{X/Y}$ is an open immersion on a neighbourhood of x .

Proof. For the implication (i) \implies (ii), we can use Nakayama to reduce to the case where $Y = \text{Spec}(k)$ and $X = \text{Spec}(k')$, where it is well known, and also trivial by the definition of separable; (ii) \implies (iii) comes from a nice and easy characterisation of open immersions, using Krull; (iii) \implies (i) follows as well from reducing to the case where $Y = \text{Spec}(k)$ and the diagonal morphism is everywhere an open immersion. We must then prove that X is finite with separable ring over k , and this leads us to consider the case where k is algebraically

²In EGA II 6.2.3 we further suppose that f is of finite type.

closed. But then every closed point of X is isolated (since it is identical to the inverse image of the diagonal by the morphism $X \rightarrow X \times_k X$ defined by x), whence X is finite. We can thus suppose that X consists of a single point, with ring A , and so $A \otimes_k A \rightarrow A$ is an isomorphism, hence $A = k$. \square

Definition 3.2.

- (a) We then say that f is *unramified* at x , or that X is unramified at x on Y .
- (b) Let $A \rightarrow B$ be a local homomorphism. We say that it is *unramified*, or that B is a local *unramified* algebra on A , if $B/\mathfrak{t}(A)B$ is a finite separable extension of $A/\mathfrak{t}(A)$, i.e. if $\mathfrak{t}(A)B = \mathfrak{t}(B)$ and $\kappa(B)$ is a separable extension of $\kappa(A)$ ³.

p. 3 | **Remarks.** The fact that B is unramified over A can be seen at the level of the completions of A and B . Unramified implies quasi-finite.

Corollary 3.3. *The set of points where f is unramified is open.*

Corollary 3.4. *Let X' and X be preschemes of finite type over Y , and $g: X' \rightarrow X$ a Y -morphism. If X is unramified over Y , then the graph morphism $\Gamma_g: X' \rightarrow X \times_Y X$ is an open immersion.*

Indeed, this is the inverse image of the diagonal morphism $X \rightarrow X \times_Y X$ by

$$g \times_Y \text{id}_{X'}: X' \times_Y X \rightarrow X \times_Y X.$$

One can also introduce the annihilator ideal $\mathfrak{d}_{X/Y}$ of $\Omega_{X/Y}^1$, called the *different ideal* of X/Y ; it defines a closed sub-prescheme of X which, set-theoretically, is the set of point where X/Y is ramified, i.e. not unramified.

Proposition 3.5.

- (i) *An immersion is ramified.*
- (ii) *The composition of two ramified morphisms is also ramified.*
- (iii) *Base extension of a ramified morphisms is also ramified.*

We are rather indifferent about (ii) and (iii) (the second seems more interesting to me). We can, of course, also be more precise, by giving some one-off statements; this is more general only in appearance (except for in the case of definition (b)), and is boring. We obtain, as per usual, the corollaries:

Corollaries 3.6.

³Cf. regrets in III 1.2.

(iv) *The cartesian product of two unramified morphisms is unramified.*

(v) *If gf is unramified then so too is f .*

(vi) *If f is unramified then so too is f_{red} .*

Proposition 3.7. *Let $A \rightarrow B$ be a local homomorphism, and suppose that the residue extension $\kappa(B)/\kappa(A)$ is trivial, with $\kappa(A)$ algebraically closed. In order for B/A to be unramified, it is necessary and sufficient that \hat{B} be (as an \hat{A} -algebra) a quotient of \hat{A} .*

Remarks.

- In the case where we don't suppose that the residue extension is trivial, we can reduce to the case where it is by taking a suitable finite flat extension of A which destroys the aforementioned extension.
- Consider the example where A is the local ring of an ordinary double point of a curve, and B a point of its normalisation: then $A \subset B$, B is unramified over A with trivial residue extension, and $\hat{A} \rightarrow \hat{B}$ is surjective but *not injective*. We are thus going to strengthen the notion of unramified-ness.

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