Étale coverings and the fundamental group
(SGA 1)

A seminar directed by A. GROTHENDIECK
Including two talks by M. RAYNAUD

Séminaire de Géométrie Algébrique du Bois Marie
1960–1961

Translators' note.

! translators' note !

Introduction

! introduction !

Contents

1

Étale morphisms

p. 1 To simplify the exposition, we assume that all preschemes in the following are locally Noetherian (at least, starting from Section 2).

1.1 Basics of differential calculus

Let X be a prescheme on Y, and $\Delta_{X/Y}$ the diagonal morphism $X \to X \times_Y X$. This is an immersion, and thus a closed immersion of X into an open subset V of $X \times_Y X$. Let $\mathscr{I}_X$ be the ideal of the closed sub-prescheme corresponding to the diagonal in V (N.B. if one really wishes to do things intrinsically, without assuming that X is separated over Y — a misleading hypothesis — then one should consider the set-theoretic inverse image of $\mathscr{O}_{X\times X}$ in X and denote by $\mathscr{I}_X$ the augmentation ideal in the above ...). The sheaf $\mathscr{I}_X/\mathscr{I}_X^2$ can be thought of as a quasi-coherent sheaf on X, which we denote by $\Omega^1_{X/Y}$. This sheaf is of finite type if $X \to Y$ is of finite type, and it behaves well with respect to a base change $Y' \to Y$. We also introduce the sheaves $\mathscr{O}_{X\times_Y X}/\mathscr{I}_X^{n+1} = \mathscr{P}^n_{X/Y}$, which are sheaves of *rings* on X, giving us preschemes denoted by $\Delta^n_{X/Y}$ and called the *n-th infinitesimal neighbourhood of X/Y*. The polysyllogism is entirely trivial, even if rather long[1]; it seems wise to not discuss it until we use it for something helpful, with smooth morphisms.

1.2 Quasi-finite morphisms

[1] cf. EGA IV 16.3.

Proposition 2.1. *Let $A \to B$ be a local homomorphism (N.B. all rings are now Noetherian) and $\mathfrak{m}$ the maximal ideal of A. Then the following conditions are equivalent:*

(i) $B/\mathfrak{m}B$ *is of finite dimension over* $k = A/\mathfrak{m}$.

(ii) $\mathfrak{m}B$ *is an ideal of definition, and* $B/\mathfrak{r}(B) = \kappa(B)$ *is an extension of* $k = \kappa(A)$.

(iii) The completion $\widehat{B}$ *of* B *is finite over the completion* $\widehat{A}$ *of* A.

p. 2 If any of the above conditions hold, then we say that B is *quasi-finite* over A. A morphism $f : X \to Y$ is said to be quasi-finite *at* x (or the Y-prescheme f is said to be quasi-finite at x) if $\mathcal{O}_x$ is quasi-finite over $\mathcal{O}_{f(x)}$. This is equivalent to saying that x is *isolated in its fibre* $f^{-1}(x)$. A morphism is said to be quasi-finite if it is quasi-finite at each point[2].

Corollary 2.2. *If A is complete, then quasi-finiteness is equivalent to finiteness.*

We could also give the usual polysyllogism (i), (ii), (iii), (iv), (v) for quasi-finite morphisms, but that doesn't seem necessary here.

1.3 Unramified morphisms

Proposition 3.1. *Let* $f : X \to Y$ *be a morphism of finite type,* $x \in X$, *and* $y = f(x)$. *Then the following conditions are equivalent:*

(i) $\mathcal{O}_x/\mathfrak{m}_y\mathcal{O}_x$ *is a finite separable extension of* $\kappa(y)$.

(ii) $\Omega^1_{X/Y}$ *is zero at* x.

(iii) The diagonal morphism $\Delta_{X/Y}$ *is an open immersion on a neighbourhood of* x.

Proof. For the implication (i) $\implies$ (ii), we can use Nakayama to reduce to the case where $Y = \operatorname{Spec}(k)$ and $X = \operatorname{Spec}(k')$, where it is well known, and also trivial by the definition of separable; (ii) $\implies$ (iii) comes from a nice and easy characterisation of open immersions, using Krull; (iii) $\implies$ (i) follows as well from reducing to the case where $Y = \operatorname{Spec}(k)$ and the diagonal morphism is everywhere an open immersion. □

[2] In EGA II 6.2.3 we further suppose that f is of finite type.