Introduction

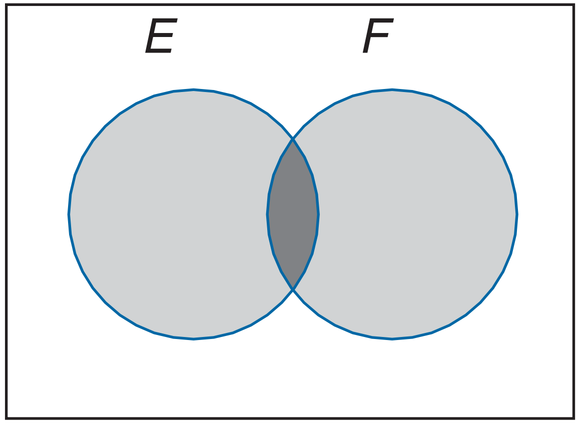
Probability

Sample Space and Event

- Sample Space S: Set of all possible outcomes of an experiment
- Event: Any subset E of the sample space
- Union of Events ($E \cup F$)
  - all outcomes either in E or in F or in both E and F
- Intersection of Events (EF)
  - all outcomes that are in both E and F.
  - event EF will occur only if both E and F occur.
  - If $EF = \emptyset$, implying that E and F cannot both occur, then E and F are said to be mutually exclusive
- Complement of event E (E^c)
  - all outcomes in the sample space S that are not in E
- $E \subset F$ (or equivalently, $F \supset E$):
  - all of the outcomes in *E* are also in *F*

Venn Diagrams

- Graphical representation of events
- Commutative law
  - $E \cup F = F \cup E$
  - $EF = FE$
- Associative law
  - $(E \cup F) \cup G = E \cup (F \cup G)$
  - $(EF)G = E(FG)$
- Distributive law
  - $(E \cup F)G = EG \cup FG$
  - $EF \cup G = (E \cup G)(F \cup G)$



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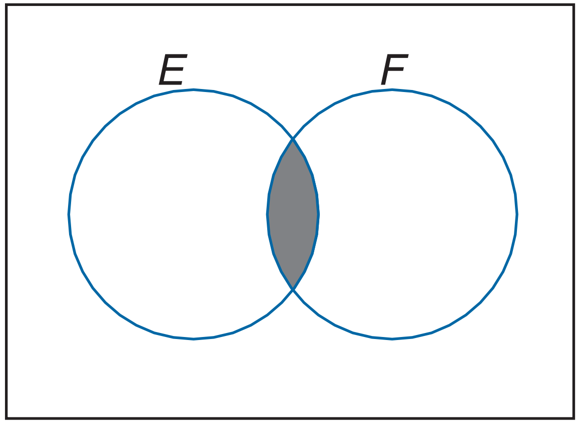

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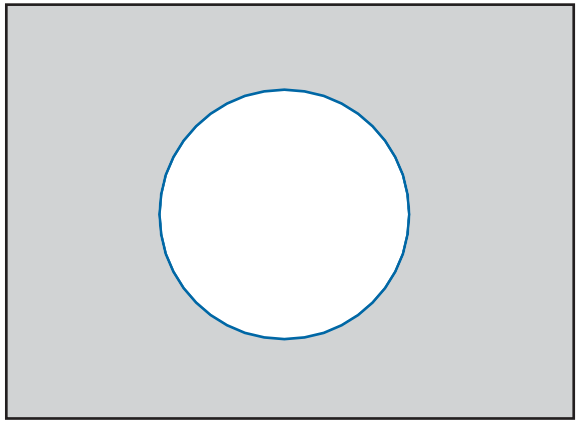

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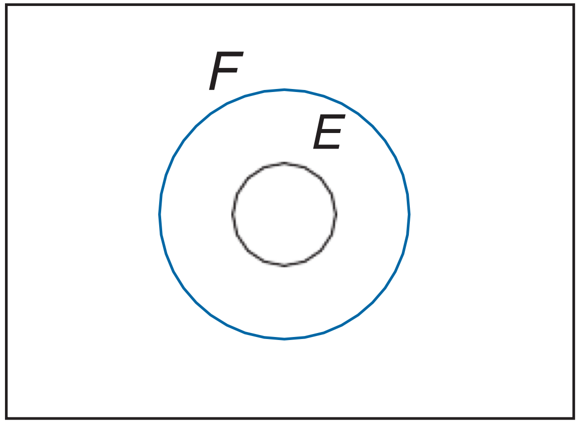

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DeMorgan's Law

$$(E \cup F)^c = E^c F^c$$

$$(EF)^c = E^c \cup F^c$$

Axioms of Probability

- $0 \leq P(E) \leq 1$
- $P(S) = 1$
- For mutually exclusive events $E_1, E_2, \ldots$ ($E_i E_j = \emptyset$, $i \neq j$),

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i), \qquad n = 1, 2, \ldots, \infty$$

- $P(E^c) = 1 - P(E)$ [$P(S) = P(E \cup E^c) = P(E) + P(E^c) = 1$]
- $P(E \cup F) = P(E) + P(F) - P(EF)$

Exercises

- A total of 28 percent of students opt for maths, 7 percent opt for biology, and 5 percent opt for both maths and biology. What percentage of students opt for neither maths nor biology?
- In a race among seven horses {h1, h2, …h7}, the probability of
  - Horse h2 being winner and h5 being the last
  - Horse h3 finishing before h6
  - Horse h2 being neither the first nor the last in the race
- Three balls are taken out, one after another, from an urn containing 3 red, 4 black and 5 white balls, probability of
  - Selecting all balls of different colors
  - Two balls of same color
  - All balls of same color
- Repeat above when all balls are also numbered.

Odds of an Event

- The *odds* of an event *A* is defined by
- Thus the odds of an event *A* tells how much more likely it is that *A* occurs than that it does not occur.
- Example:
  - Odds of picking a red ball from a box containing 6 red and 3 black balls
  - P(R) = 6/9, 1 – P(R) = 3/9
  - Answer: (6/9)÷(3/9) = 2
  - It is 2 times likely that *red ball would be picked.*

Exercise 2

- Sarla wants to arrange 10 books (4 mathematics, 3 chemistry, 2 history and 1 English) on a bookshelf. How many different arrangements are possible if
  - all the books dealing with the same subject are together on the shelf
  - First an last books are history books
  - All mathematics books are placed together
- A class in probability theory consists of 6 boys and 4 girls. An exam is given and the students are ranked according to their performance. Assuming that no two students obtain the same score
  - how many different rankings are possible?
  - What is the probability that all girls outrank boys.
  - What is the probability that two girls are placed in top 3 ranks.
- From a set of n items a random sample of size k is to be selected. What is the probability a given item will be among the k selected?
- A basketball team consists of 6 black and 6 white players. The players are to be paired in groups of two for the purpose of determining roommates. If the pairings are done at random, what is the probability that none of the black players will have a white roommate?

Example 1

- A basketball team consists of 6 students from class XI and 6 from class XII. The players are to be paired in groups of two for the purpose of determining roommates. If the pairings are done at random, what is the probability that none of the class XI players will have a senior partner?

Example 1

- A basketball team consists of 6 students from class XI and 6 from class XII. The players are to be paired in groups of two for the purpose of determining roommates. If the pairings are done at random, what is the probability that none of the class XI players will have a senior partner?

Solution: No. of ways of selecting first pair, second pair, … = ${}^{12}C_2\,,{}^{10}C_2\,,{}^{8}C_2$

No. of ways of selecting all pairs

$= {}^{12}C_2{}^{10}C_2.{}^{8}C_2.{}^{6}C_2.{}^{4}C_2.{}^{2}C_2 = (12x11x10x9x8x7x6x5x4x3x2x1) \div 2^6 = 12! \div 2^6$

No. of ways of selecting unordered pairs: $12! \div (6!\ .\ 2^6)$

No. of ways of pairing the XII class players = $6! \div (3!\ .\ 2^3)$

No. of ways of pairing the XI class players = $6! \div (3!\ .\ 2^3)$

No. of ways with no junior-senior pairing = $[6! \div (3!\ .\ 2^3)]^2$

Probability =

$= [\ (6!\ 6!\ )\ (6!\ 2^6\ )] \div [12!\ .\ (3!\ 3!\ 2^3\ 2^3)] = 6!\ 6!\ 6! \div 12!\ 3!\ 3! = 6!\ 5!\ 5! \div 12!$

$= 5!\ 5! \div 12.11.10.9.8.7 = 5! / (11.9.8.7) = 5 / (11.3.7) = 5 / 231 = 0.0216$

Example 2

- If n people are present in a room, what is the probability that no two of them celebrate their birthday on the same day of the year?
- Days in a year: 365
- No. of possibilities: 365^n
- No two people celebrate birthday on same day

  = birthdays are distinct for all persons

  = 365.364.363. … (365-n+1)
- Probability

  = [365.364.363. … (365-n+1)] / 365^n
- Birthday Paradox: Probability of 23 people sharing a birthday is more than ½
  - 50 people sharing a birthday has a 96% chance of happening
  - 100 people sharing a birthday has a 99.99987% chance of happening
- Birthday Attack:
  - Used for brute force cracking of hash

2	0.997260274
3	0.991795834
4	0.983644088
5	0.972864426
6	0.959537516
7	0.943764297
8	0.925664708
9	0.905376166
10	0.883051822
11	0.858858622
12	0.832975211
13	0.805589725
14	0.776897488
15	0.74709868
16	0.716395995
17	0.684992335
18	0.653088582
19	0.620881474
20	0.588561616
21	0.556311665
22	0.524304692
23	**0.492702766**

Birthday Paradox

- Birthday Paradox: Probability of 23 people sharing a birthday is more than
- Why paradox?
  - 23 is much smaller in relation to 365, the number of days of the year.
- For a pair of individuals
  - Possible outcomes = 365x365
  - No. of ways both share a birthday = 365
- Probability for a given pair of individuals $365/(365)^2 = 1/365$
- For 23 people, number of pairings = ${}^{23}C_2 = 253$

Conditional Probability

- Probability of E given that F has occurred
- Notation: P(E|F)
- Conditional probability
  - calculating probabilities when some partial information concerning the result of the experiment is available, or
  - in recalculating them in light of additional information.
- If the event F occurs, then in order for E to occur it is necessary that the actual occurrence be a point in both E and F (i.e., must be in EF). Since we know that F has occurred, it follows that F becomes our new (reduced) sample space and hence
- $P(E|F) = P(EF) / P(F), \; P(F) > 0$
- Example: Suppose that one rolls a pair of dice. We observe that the first die lands on side 3. Given this information, what is the probability that the sum of the two dice equals 8?

  F = sample space = {3,1}, {3,2} {3,3},{3,4},{3,5},{3,6}

  EF = {3,5}

  P(E|F) = 1/6

Example 1

- A bin contains 5 defective (that immediately fail when put in use), 10 partially defective (that fail after a couple of hours of use), and 25 acceptable transistors. A transistor is chosen at random from the bin and put into use. If it does not immediately fail, what is the probability it is acceptable

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- Solution:

  P{acceptable I not defective} = P{acceptable, not defective} / P{not defective}

  = P{acceptable} / P{not defective}

  = (25/40)÷(35/40) = 25/35 = 5/7

Example 2

- The organization that Jones works for is running a father–son dinner for those employees having at least one son. Each of these employees is invited to attend along with his youngest son. If Jones is known to have two children, what is the conditional probability that they are both boys given that he is invited to the dinner? Assume that the sample space S is given by S = {(b, b), (b, g), (g , b), (g , g)} and all outcomes are equally likely [(b, g) means, for instance, that the younger child is a boy and the older child is a girl].

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- Solution:

  $P(\{(b, b)\}) \div P(\{(b, b), (b, g), (g, b)\})$

  $= 1/3$

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Example 3

- There is a 30 percent chance that a bank will set up a branch office in Jaipur. If it does, Rama is 60 percent certain that she will be made manager of this new operation. What is the probability that Rama will be a Jaipur branch manager?

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- Solution:

$$P(M|B) = 0.6$$

$$P(B) = 0.3$$

$$P(M|B) = P(MB)/P(B)$$

$$P(MB) = P(M|B)P(B) = 0.6 * 0.3 = 0.18$$

Bayes' Formula

- Let E and F be events.
- $E = EF \cup EF^c$
- For a point to be in E, it must either be in both E and F or be in E but not in F.
- $P(E) = P(EF) + P(EF^c)$
- $= P(E|F)P(F) + P(E|F^c)P(F^c)$
- $= P(E|F)P(F) + P(E|F^c)[1 - P(F)]$

Example 4

- Statistics of an insurance company shows that an accident-prone person will have an accident with 0.4 probability at some time within a fixed 1-year period and for a non-accident-prone person, this probability is 0.2. If we assume that 30 percent of the population is accident prone, what is the probability that a new policy holder will have an accident within a year of purchasing a policy?

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- Solution:

$P(F) = 0.3$, $1-P(F) = 0.7$,   F:Person being accident prone

$P(A) = P(A|F)P(F) + P(A|F') P(F')$

$= 0.4*0.3 + 0.2*0.7 = 0.12 + 0.14 = 0.26$

- A new policy holder has an accident within a year of purchasing his policy. What is the probability that he is accident prone?
- Solution:

$P(F|A) = P(A|F)P(F)/P(A) = 0.4 * 0.3 / 0.26 = 0.12/0.26 = 0.4615$

Example 5

- A laboratory blood test is 99% effective in detecting a certain disease when it is, present. However, the test also yields a “false positive” result for 1 percent of the healthy persons tested. (i.e. there is 0.01 probability that the test shall indicate that a healthy person has the disease.) If 0.5 % of the population actually has the disease, what is the probability a person has the disease given that his test result is positive?

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- Solution:

$P(T|H) = 0.01,\ P(T|D) = 0.99$

$P(T) = P(T|H)P(H) + P(T|D)P(D)$

$= 0.01*0.995 + 0.99*0.005 = 0.00995 + 0.00495 = 0.0149$

$P(D|T) = P(T|D)P(D)/P(T)$

$= 0.99*0.005/0.0149 = 0.00495/0.0149 = 0.3322$

Example 6

- At a certain stage of a criminal investigation, the inspector in charge is 60 percent convinced of the guilt of a certain suspect. Suppose now that a new piece of evidence that shows that the criminal has a certain characteristic (such as left-handedness, baldness, brown hair, etc.) is uncovered. If 20 percent of the population possesses this characteristic, how certain of the guilt of the suspect should the inspector now be if it turns out that the suspect is among this group?

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- Solution:

$P(GC) = P(G)P(C|G) = (.6)(1) = .6$

$P(C) = P(C|G)P(G)+P(C|G^c)P(G^c) = (1)(.6)+(.2)(.4) = .68$

where we have supposed that the probability of the suspect having the characteristic if he is, in fact, innocent is equal to .2, the proportion of the population possessing the characteristic. Hence

$P(G|C) = 60/68 = .882$

and so the inspector should now be 88 percent certain of the guilt of the suspect.

Independent Events

- Two events are said to be independent if
- Two events E and F are said to be independent if
- $P(EF) = P(E)P(F)$
- If E and F are independent, then so are E and F^c.
  Prove this statement.

Thank you.