The Matrix Stiffness Method for 2D Trusses

CEE 421L. Matrix Structural Analysis
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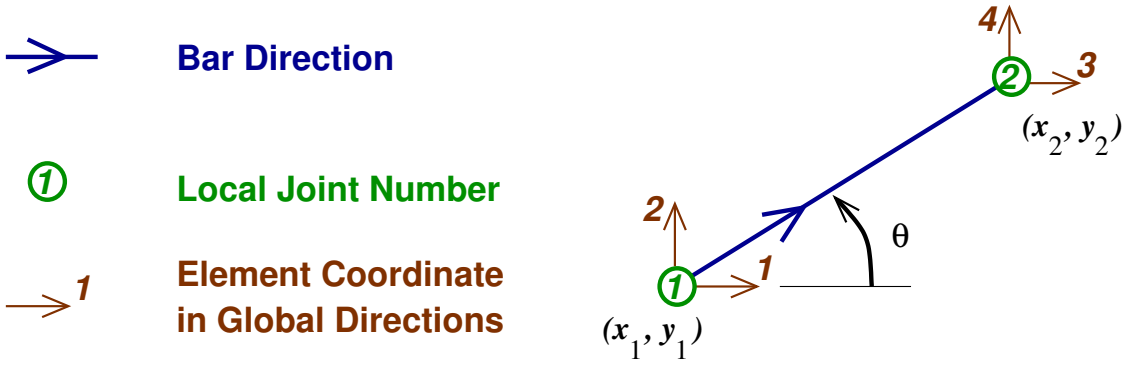
Method

1. Number all of the nodes and all of the elements.

2. Identify the Displacement Degrees of Freedom in Global Directions.
   Number all the structural degrees of freedom in your truss. In a 2D (planar) truss, each node can have a maximum of two degrees of freedom: one in the global X-direction and one in the global Y-direction. If a degree of freedom is restrained by a reaction, then it doesn't get a number.

3. Node Locations.
   Write the (x, y) coordinates of each node using units consistent with E and A. In other words, if E and A are given in kN/cm^2 and cm^2, write the (x, y) coordinates in terms of centimeters.

4. Define each element.
   Draw each element of your truss individually and draw the local coordinates in the global directions. For example if element number N is a diagonal truss element, and the global directions are X: horizontal and Y: vertical, draw element number N like this:



   where 1,2,3,4 are the element coordinates of the truss element in the global directions. The local coordinates are always numbered 1,2,3,4 with 1 and 3 pointing in the global X direction (to the right) and with 2 and 4 pointing in the global Y direction (up). Some or all of these four coordinates will line up with with the structural degrees of freedom that you identified in step 2., above. The angle θ is the counter-clockwise angle from element coordinate 1 to the truss element.

5. Element Stiffness Matrices in Global Coordinates, $\mathbf{K}$.
For each element, find its (4x4) element stiffness matrix, by evaluating the equations below:

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$c = (x_2 - x_1)/L$$

$$s = (y_2 - y_1)/L$$

$$\mathbf{K} = \frac{EA}{L} \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}.$$

or by using the `truss_2d_element.m` script in matlab. You should understand where these equations come from, why this matrix is symmetric, why the diagonal terms are all positive, and what the off-diagonal terms mean.

6. Structural Stiffness Matrix, $\mathbf{K_s}$.
The structural stiffness matrix is a square, symmetric matrix with dimension equal to the number of degrees of freedom. In this step we will fill up the structural stiffness matrix using terms from the element stiffness matrices in global coordinates (from step 5.) This procedure is called *matrix assembly*.

Recall from step 4. how the element degrees of freedom (1,2,3,4) line up with the structural degrees of freedom in your problem. For example, coordinates (1,2,3,4) might line up with degrees of freedom (3,4,7,8) of the truss. In this case, to assemble this element into the structural stiffness matrix,

```
K(1,1) +> Ks(3,3) | K(1,2) +> Ks(3,4) | K(1,3) +> Ks(3,7) | K(1,4) +> Ks(3,8)
------------------|-------------------|-------------------|------------------
K(2,1) +> Ks(4,3) | K(2,2) +> Ks(4,4) | K(2,3) +> Ks(4,7) | K(2,4) +> Ks(4,8)
------------------|-------------------|-------------------|------------------
K(3,1) +> Ks(7,3) | K(3,2) +> Ks(7,4) | K(3,3) +> Ks(7,7) | K(3,4) +> Ks(7,8)
------------------|-------------------|-------------------|------------------
K(4,1) +> Ks(8,3) | K(4,2) +> Ks(8,4) | K(4,3) +> Ks(8,7) | K(4,4) +> Ks(8,8)
```

where the `+>` is short-hand for "is added to" .... `K(1,3)` is added to `Ks(3,7)`.

Add each element into the structural stiffness matrix in this way to get $\mathbf{K_s}$. Note that if one end of the truss element is fully restrained in both the the X- and Y- directions, you will need to place only four of the sixteen terms of the element's 4x4 stiffness matrix.

7. Forces, $\mathbf{p}$.
Create the force vector $\mathbf{p}$, by finding the components of each applied force in the directions of the global degrees of freedom. Create the force vector by placing these force components into the force vector at the proper coordinates.

8. Deflections, **d**.
   Find the deflections by inverting the stiffness matrix and multiplying it by the load vector. You can do this easily in matlab: `d = Ks \ p`

9. Internal bar forces, T.
   Again, recall how the global degrees of freedom line up with each element's coordinates (1,2,3,4). For example, in element number N from step 6., the local element deflections in the global directions, v_1, v_2, v_3, v_4 line up with the structural deflections deflections d_3, d_4, d_7, d_8. The internal bar forces can be computed from:

$$T = \frac{EA}{L}[(v_3 - v_1)c + (v_4 - v_2)s] = \frac{EA}{L}[(d_7 - d_3)c + (d_8 - d_4)s]$$

   where c and s are the direction cosine and sine for the element from step 5.

   You should be able to derive this equation.

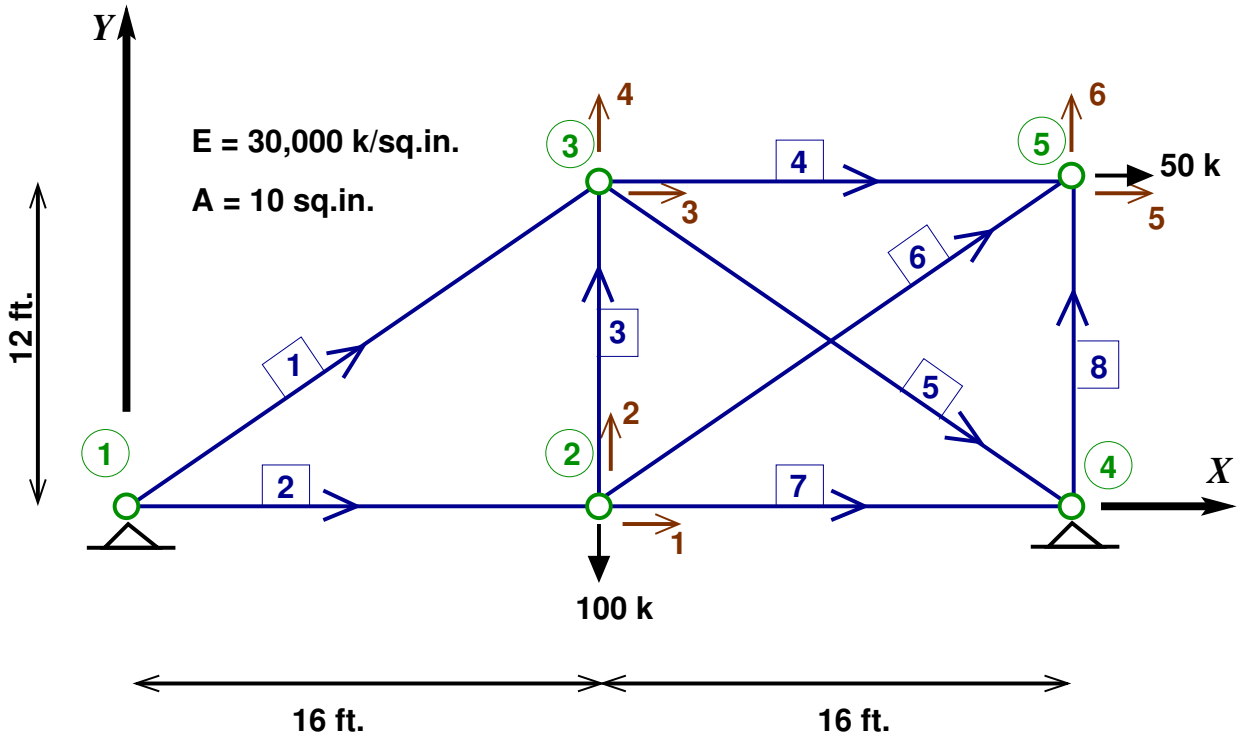
   Knowing what each bar force is, the reactions can be easily computed with equilibrium equations.

Notation

$\mathbf{u}$	=	Element deflection vector in the Local coordinate system
$\mathbf{q}$	=	Element force vector in the Local coordinate system
$\mathbf{k}$	=	Element stiffness matrix in the Local coordinate system
		... $\mathbf{q} = \mathbf{k}\,\mathbf{u}$
$\mathbf{T}$	=	Coordinate Transformation Matrix (orthonormal)
		... $\mathbf{T}^{-1} = \mathbf{T}^{\mathsf{T}}$
$\mathbf{v}$	=	Element deflection vector in the Global coordinate system
		... $\mathbf{u} = \mathbf{T}\,\mathbf{v}$
$\mathbf{f}$	=	Element force vector in the Global coordinate system
		... $\mathbf{q} = \mathbf{T}\,\mathbf{f}$
$\mathbf{K}$	=	Element stiffness matrix in the Global coordinate system
		... $\mathbf{K} = \mathbf{T}^{\mathsf{T}}\,\mathbf{k}\,\mathbf{T}$
$\mathbf{d}$	=	Structural deflection vector in the Global coordinate system
$\mathbf{p}$	=	Structural load vector in the Global coordinate system
$\mathbf{K_s}$	=	Structural stiffness matrix in the Global coordinate system
		... $\mathbf{p} = \mathbf{K_s}\,\mathbf{d}$

Coordinate System	**Local**	**Global**
Element Deflection	$\mathbf{u}$	$\mathbf{v}$
Element Force	$\mathbf{q}$	$\mathbf{f}$
Element Stiffness	$\mathbf{k}$	$\mathbf{K}$
Structural Deflection	-	$\mathbf{d}$
Structural Loads	-	$\mathbf{p}$
Structural Stiffness	-	$\mathbf{K_s}$

Example



1 Global Joint Number

2 Bar Number

Bar Direction

1 Structural Degree of Freedom

Applied Force

1 Local Joint Number

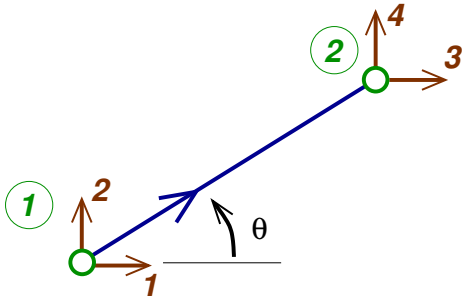
1 Local Degree of Freedom

Global Coordinate System

System Equation: {f} = [K]{d}

{d} : displacement vector
= { d1 d2 d3 d4 d5 d6 }'

{f} : load vector = { 0 –100 0 0 50 0 }'



Element Coordinates in Global Directions

```
function [ K, L ] = truss_2d_element ( x1, y1, x2, y2, EA )
% [ K, L ] = TRUSS_ELEMENT_2D ( X1, Y1, X2, Y2, EA, T )
% Compute the element stiffness matrix for a 2D truss bar in global coordinates
%
% INPUT DATA:
%      X1,Y1   is the location of joint 1 of the truss bar
%      X2,Y2   is the location of joint 2 of the truss bar
%      EA      is the product of the elastic modulus and the section area
%
% OUTPUT DATA:
%       K      is the 4x4 truss bar element stiffness matrix in global element coord's
%       L      is the length of the truss bar

 L = sqrt( (x2-x1)^2 + (y2-y1)^2 );            % length of the bar

 c = ( x2 - x1 ) / L;                         % cosine of bar angle
 s = ( y2 - y1 ) / L;                         %   sine of bar angle

 K  = (EA/L) *  [ c^2   c*s  -c^2  -c*s ;
                  c*s   s^2  -c*s  -s^2 ;
                 -c^2  -c*s   c^2   c*s ;
                 -c*s  -s^2   c*s   s^2 ];
% ------------------------------------------------------------ TRUSS_ELEMENT_2D

>> format bank                          % two decimal places after the .
>> E = 3e4;                             % modulus of elasticity
>> A = 10;                              % area of cross section
>> EA = E*A;

>> [ K1, L(1) ] = truss_2d_element( 0, 0, 12*16, 12*12, EA )           % truss element 1
K1 =

    800.00    600.00   -800.00   -600.00
    600.00    450.00   -600.00   -450.00
   -800.00   -600.00    800.00    600.00
   -600.00   -450.00    600.00    450.00




>> [ K2, L(2) ] = truss_2d_element( 0, 0, 12*16, 0, EA )               % truss element 2
K2 =

   1562.50      0.00  -1562.50      0.00
      0.00      0.00      0.00      0.00
```

```
  -1562.50      0.00   1562.50      0.00
      0.00      0.00      0.00      0.00


>> [ K3, L(3) ] = truss_2d_element( 12*16, 0, 12*16, 12*12, EA )        % truss element 3
K3 =

      0.00      0.00      0.00      0.00
      0.00   2083.33      0.00  -2083.33
      0.00      0.00      0.00      0.00
      0.00  -2083.33      0.00   2083.33


>> [ K4, L(4) ] = truss_2d_element( 12*16, 12*12, 12*32, 12*12, EA )    % truss element 4
K4 =

   1562.50      0.00  -1562.50      0.00
      0.00      0.00      0.00      0.00
  -1562.50      0.00   1562.50      0.00
      0.00      0.00      0.00      0.00


>> [ K5, L(5) ] = truss_2d_element( 12*16, 12*12, 12*32, 0, EA )        % truss element 5
K5 =

    800.00   -600.00   -800.00    600.00
   -600.00    450.00    600.00   -450.00
   -800.00    600.00    800.00   -600.00
    600.00   -450.00   -600.00    450.00


>> [ K6, L(6) ] = truss_2d_element( 12*16, 0, 12*32, 12*12, EA )        % truss element 6
K6 =

    800.00    600.00   -800.00   -600.00
    600.00    450.00   -600.00   -450.00
   -800.00   -600.00    800.00    600.00
   -600.00   -450.00    600.00    450.00

>> [ K7, L(7) ] = truss_2d_element( 12*16, 0, 12*32, 0, EA )            % truss element 7
K7 =

   1562.50      0.00  -1562.50      0.00
      0.00      0.00      0.00      0.00
  -1562.50      0.00   1562.50      0.00
      0.00      0.00      0.00      0.00
```

```
>> [ K8, L(8) ] = truss_2d_element( 12*32, 0, 12*32, 12*12, EA )        % truss element 8
K8 =

      0.00      0.00      0.00      0.00
      0.00   2083.33      0.00  -2083.33
      0.00      0.00      0.00      0.00
      0.00  -2083.33      0.00   2083.33



% ---------------------------    assemble the global structural stiffness matrix ...

>> Ks = [ 3925   600       0       0    -800   -600    ;
>          600  2533.33    0   -2083.33 -600   -450    ;
>            0     0    3162.5     0   -1562.5    0    ;
>            0 -2083.33    0    2983.33    0      0    ;
>         -800  -600   -1562.5     0    2362.5  600    ;
>         -600  -450       0       0     600   2533.33 ]



Ks =
   3925.00    600.00      0.00      0.00   -800.00   -600.00

    600.00   2533.33      0.00  -2083.33   -600.00   -450.00

      0.00      0.00   3162.50      0.00  -1562.50      0.00

      0.00  -2083.33      0.00   2983.33      0.00      0.00

   -800.00   -600.00  -1562.50      0.00   2362.50    600.00

   -600.00   -450.00      0.00      0.00    600.00   2533.33



>>  find(Ks-Ks')          % check to see if Ks is symmetric ...
ans = [](0x0)             % It is!



>> p = [ 0 -100 0 0 50 0 ]'        % input the load vector

p =

     0.00
  -100.00
     0.00
```

```
     0.00
    50.00
     0.00

>> format                % change formats for more sig. fig's


>> d = Ks \ p            % compute the joint displacements

d =

   0.0146067
  -0.1046405
   0.0027214
  -0.0730729
   0.0055080
  -0.0164325
```