

Perspectives on Probabilistic Graphical Models

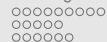
Dong Liu

Information Science and Engineering
KTH - Royal Institute of Technology

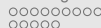


Profile page: <https://firsthandscientist.github.io/>

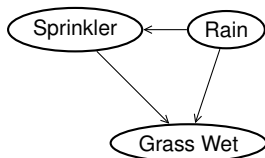
Slide is available at: <https://github.com/FirstHandScientist/phdthesis>



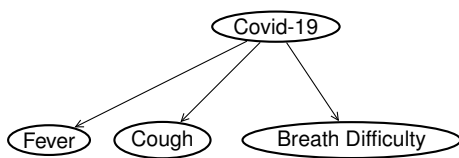
Why are probabilistic graphical models interesting?



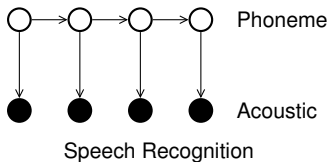
DIRECTED GRAPH REPRESENTATION



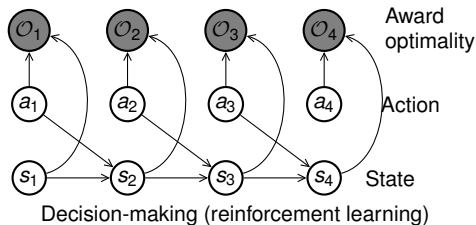
Is the sprinkler working?



Is the person get contagious by COVID?



Speech Recognition

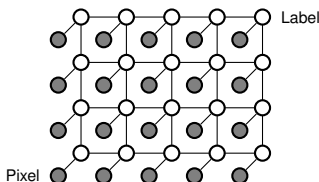


Decision-making (reinforcement learning)

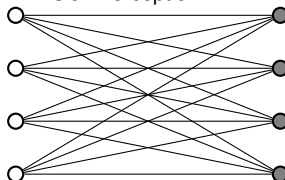


UNDIRECTED GRAPH REPRESENTATIONS

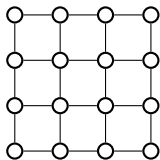
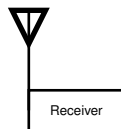
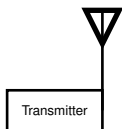
Coco



Vision Perception

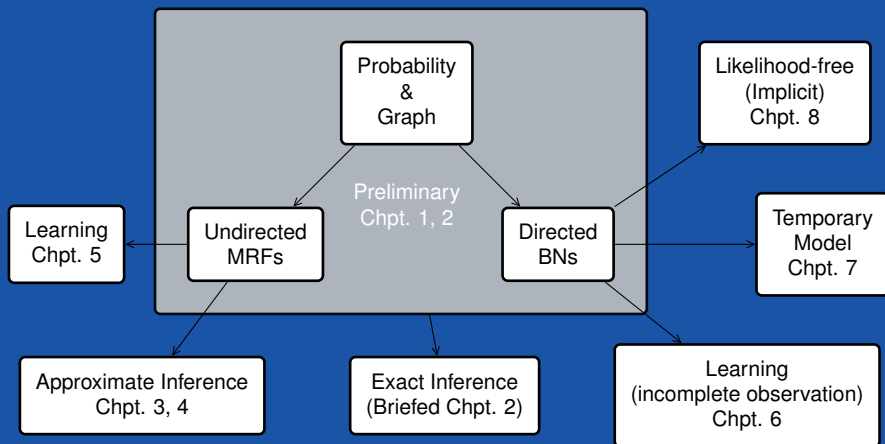


Digital communication

Physics
Ising or Potts

- Error-control codes
- Computational biology
- Natural language processing
- etc.

A GUIDE TO THIS DISSERTATION



WHAT ARE PROBABILISTIC GRAPHICAL MODELS

Informally...

- attributes of our interests in a system \rightarrow variable nodes
- relationship of these factors \rightarrow structures of a graph

Intrinsic property: **reasoning with uncertainty**

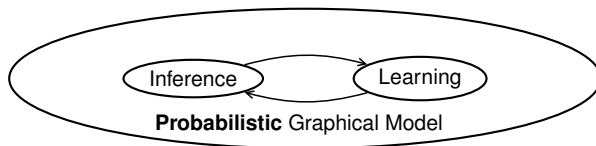
A directed/undirected graph encoding dependencies/independencies of distribution $p(\mathbf{x}; \theta)$:

- A BN/Generative model is a directed graph
 - $p(\mathbf{x}; \theta) = \prod_{n=1}^N p(x_n | \mathcal{P}(x_n))$
 - $\mathcal{P}(\cdot)$ are parent nodes
 - the local functions are proper distributions
- An MRF denoted by an undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$
 - The probability distribution (Gibbs distribution) is $p(\mathbf{x}; \theta) = \frac{1}{Z(\theta)} \prod_{a \in \mathcal{I}} \psi_a(\mathbf{x}_a; \theta_a)$
 - a indexes potential functions $\mathcal{I} = \{\psi_A, \psi_B, \dots, \psi_M\}$
 - $Z(\theta) = \sum_{\mathbf{x}} \prod_a \psi_a(\mathbf{x}_a; \theta_a)$.

USAGE OF GRAPHICAL MODELS

- The common inference problems:
 - Computing the likelihood of observed data.
 - Computing the marginals distribution $p(\mathbf{x}_A)$ over particular subset $A \subset \mathcal{V}$ of nodes
 - Computing the conditional distribution $p(\mathbf{x}_A|\mathbf{x}_B)$,
 - Computing the partition function or the Helmholtz free energy (for MRFs)
- Learning:
 - To model or determine $p(\mathbf{x}; \theta)$.

Two key components interacting with each other:



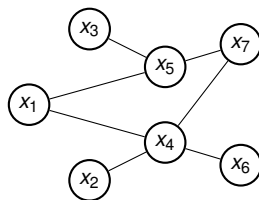
WHAT IS THE STATE OF x ?

A TOY EXAMPLE

Assume that we are interested into the state of node i in an MRF, it can be answered by

- the probability $p(x_i)$, or
- an empirical version, a collection of samples $\{x_i^n\}_{n=1}^N$

It is similar for the case when \mathbf{x} is of interests, instead of x_i .

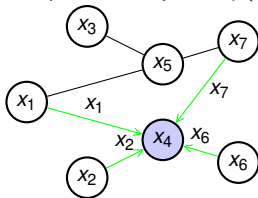


what is the state of x_4

WHAT IS THE STATE OF x ?

Gibbs sampling: let us guess by sampling

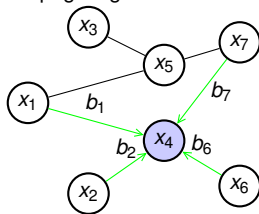
Sample iteratively: $x_i \sim p(x_i | \mathbf{x}_{-i}) \sim p(x_i, \mathbf{x}_{-i})$



Queries by collected samples $\{\mathbf{x}^n\}_1^N$.

Mean Field and BP: *message in form of sample values* \rightarrow *message in form of belief*

Propagating beliefs iteratively

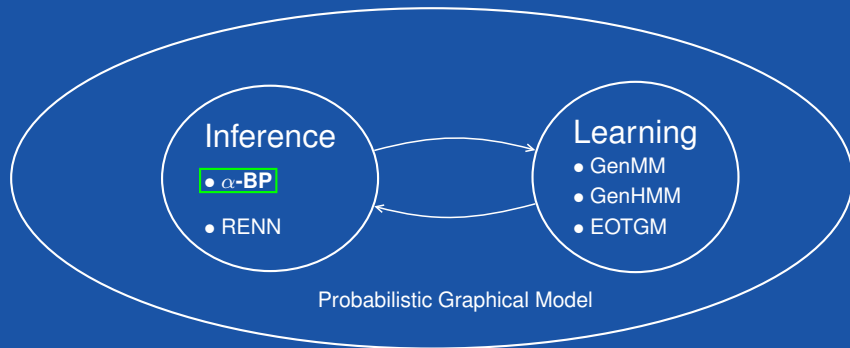


Queries by collected samples $\{b_i\}$.

Intuition from *Gibbs (variational) free energy*

$$F_V(b) = \text{KL}(b(\mathbf{x}) || p(\mathbf{x}; \theta)) - \log Z(\theta)$$

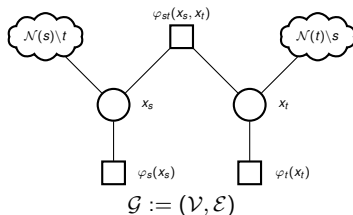
with trial $b(\mathbf{x})$. Instance: Bethe free energy.



ALTERNATIVE VIEW OF BP: α -BP

Input:

- A pairwise Markov random field:
 $p(\mathbf{x}) \propto \prod_{s \in \mathcal{V}} \varphi_s(x_s) \prod_{(s,t) \in \mathcal{E}} \varphi_{st}(x_s, x_t)$
- A trial distribution:
 $q(\mathbf{x}) \propto \prod_{s \in \mathcal{V}} \tilde{\varphi}_s(x_s) \prod_{(s,t) \in \mathcal{E}} \tilde{\varphi}_{st}(x_s, x_t)$ with
factorization $\tilde{\varphi}_{s,t}(x_s, x_t) := m_{st}(x_t) m_{ts}(x_s)$
- A metric: α -Divergence

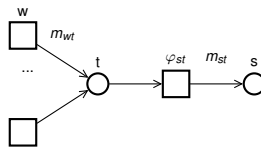


Approximate local minimization:

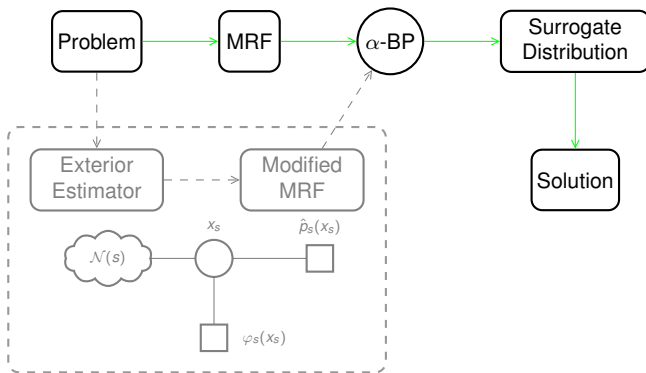
- Direct minimization: $\operatorname{argmin}_{\tilde{\varphi}_{ts}^{\text{new}}(x_t, x_s)} \mathcal{D}_{\alpha_{ts}}(p^{\setminus(t,s)}(\mathbf{x}) \varphi_{ts}(x_t, x_s) \| q^{\setminus(t,s)}(\mathbf{x}) \tilde{\varphi}_{ts}^{\text{new}}(x_t, x_s))$
- Local minimization: $\operatorname{argmin}_{\tilde{\varphi}_{ts}^{\text{new}}(x_t, x_s)} \mathcal{D}_{\alpha_{ts}}(q^{\setminus(t,s)}(\mathbf{x}) \varphi_{ts}(x_t, x_s) \| q^{\setminus(t,s)}(\mathbf{x}) \tilde{\varphi}_{ts}^{\text{new}}(x_t, x_s))$, say you are
updating $\tilde{\varphi}_{ts}^{\text{new}}(x_t, x_s) = m_{ts}^{\text{new}}(x_s) m_{st}(x_t)$

Definition of α -divergence $\mathcal{D}_{\alpha}(p \| q) = \frac{\sum_{\mathbf{x}} \alpha p(\mathbf{x}) + (1-\alpha) q(\mathbf{x}) - p(\mathbf{x})^{\alpha} q(\mathbf{x})^{1-\alpha}}{\alpha(1-\alpha)}$

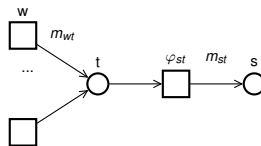
Updating message via α -BP:



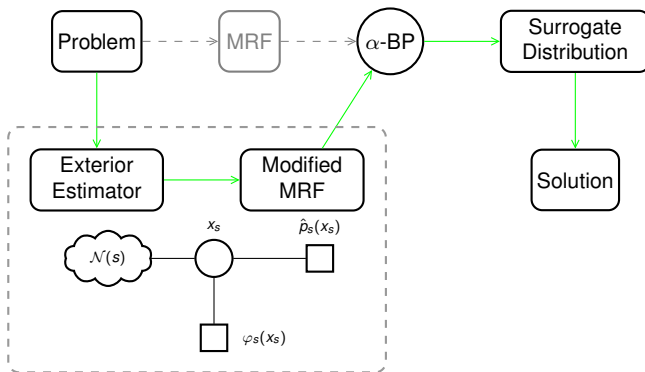
$$\underbrace{m_{ts}^{\text{new}}(x_s)}_{\text{new msg. to s}} \propto \underbrace{m_{ts}(x_s)^{1-\alpha_{ts}}}_{\text{old msg. to s}} \left[\sum_{x_t} \varphi_{ts}(x_t, x_s)^{\alpha_{ts}} \underbrace{m_{st}(x_t)^{1-\alpha_{ts}}}_{\text{old msg. to t}} \underbrace{\varphi_t(x_t) \prod_{w \in \mathcal{N}(t) \setminus s} m_{wt}(x_t)}_{\text{msg. from variable node t to factor } \varphi_{st}} \right].$$



Updating message via α -BP:



$$\underbrace{m_{ts}^{\text{new}}(x_s)}_{\text{new msg. to s}} \propto \underbrace{m_{ts}(x_s)^{1-\alpha_{ts}}}_{\text{old msg. to s}} \left[\sum_{x_t} \varphi_{ts}(x_t, x_s)^{\alpha_{ts}} \underbrace{m_{st}(x_t)^{1-\alpha_{ts}}}_{\text{old msg. to t}} \underbrace{\varphi_t(x_t) \prod_{w \in \mathcal{N}(t) \setminus s} m_{wt}(x_t)}_{\text{msg. from variable node t to factor } \varphi_{st}} \right].$$



INSIGHTS OF α -BP

Connection to standard BP

- $\alpha \rightarrow 0$
- α -divergence reduce KL-divergence
- Update rule of α -BP reduces to $m_{ts}^{\text{new}}(x_s) \propto \sum_{x_t} \varphi_{st}(x_s, x_t) \varphi_t(x_t) \prod_{w \in \mathcal{N}(t) \setminus s} m_{wt}(x_t)$, which is standard BP update rule

Convergence

For an arbitrary pairwise Markov random field over binary variables, if the largest singular value of matrix $\mathbf{M}(\alpha, \theta)$ is less than one, α -BP converges to a fixed point. The associated fixed point is unique.

See Corollary 3.1 for relaxed condition where singular value computation is avoided.

What does that mean

- You can safely use α -BP as an alternative to (loopy) BP
- You can use matrix \mathbf{M} to check if you are guaranteed to get stable solution from α -BP

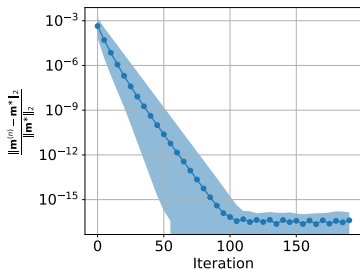
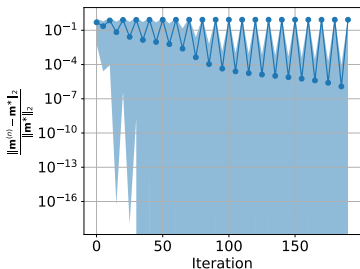
Matrix $\mathbf{M}(\alpha, \theta)$, size $|\vec{\mathcal{E}}| \times |\vec{\mathcal{E}}|$, indexed by directed edges ($t \rightarrow s$), as

For binary, symmetric log-potentials

$$\begin{aligned} \varphi_{st}(x_s, x_t) &= \exp \{ \theta_{st}(x_s, x_t) \}, \\ \varphi_s(x_s) &= \exp \{ \theta_s(x_s) \}. \end{aligned}$$

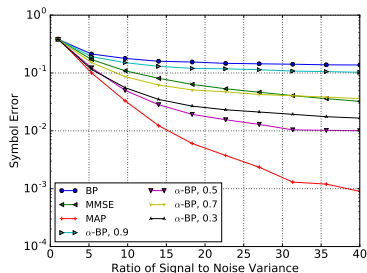
$$M_{(t \rightarrow s), (u \rightarrow v)} = \begin{cases} |1 - \alpha_{ts}|, & u = t, v = s, \\ |1 - \alpha_{ts}| \tanh |\alpha_{ts} \theta_{ts}|, & u = s, v = t, \\ \tanh |\alpha_{ts} \theta_{ts}|, & u \in \mathcal{N}(t) \setminus s, v = t, \\ 0, & \text{otherwise.} \end{cases}$$

SOME NUMERICAL RESULTS

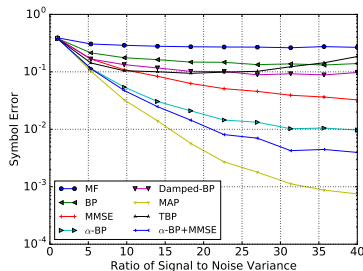


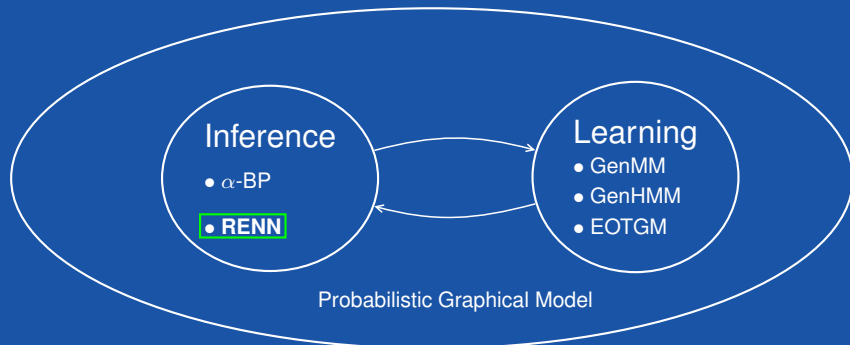
Numerical illustration of convergence, with normalized error $\frac{\|\mathbf{m}^{(n)} - \mathbf{m}^*\|_2}{\|\mathbf{m}^*\|_2}$ versus the number of iterations. Blue region: range of the normalized error of 100 graph realizations. Curve: mean error of the 100 realized graphs.

SOME NUMERICAL RESULTS



Numerical results of α -BP: symbol error of MIMO detection.

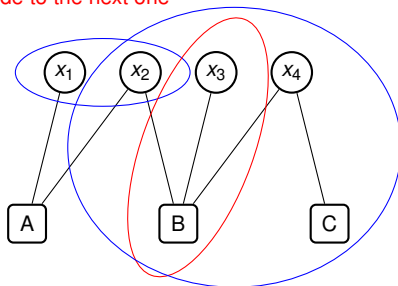




CONTINUING: WHAT IS THE STATE OF x ?

YEDIDIA, FREEMAN, WEISS: A STEP TO GENERALIZATION

Message among variables & factors → **message among regions** Maybe I should combine this slide to the next one



- Blue region: valid region
- Red region: invalid region

Generalized belief propagation (GBP) generalizes loopy BP

- usual better approximation than LBP
- higher complexity
- sensitive to scheduling of region messages

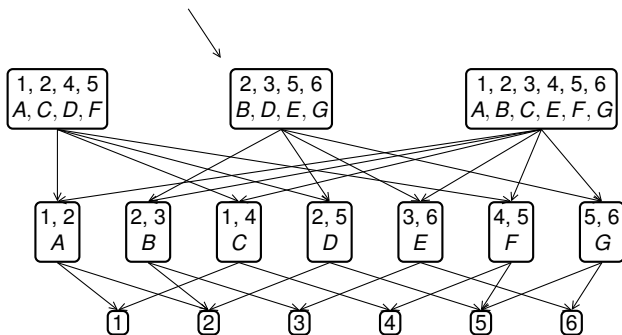
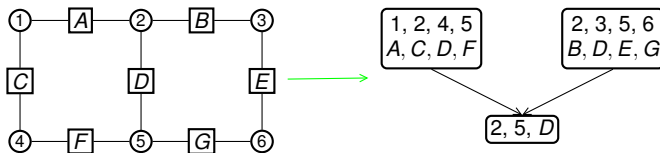
Corresponding to minimization of approximated **variational free energy** $F_V(b)$ with trial b including $\{b_R\}$.

A region R is a set V_R of variables nodes and a set A_R of factor nodes, such that if a factor node ' a ' belongs to A_R , all the variables nodes neighboring a are in V_R .

AN TOY EXAMPLE OF REGION GRAPHS

Factor graph representation of MRF (2-by-3 grid) with factor nodes.

MRF → region graph:



- Clustering nodes
- Valid regions
- Energy Calculated per region
- Msg. Scheduling
- ...
- See Section 4.1

RENN: REGION-BASED ENERGY NEURAL NETWORK

The **region-based free energy** of a region graph is

$$F_R(\mathcal{B}; \theta) = \sum_{R \in \mathcal{R}} \underbrace{c_R}_{\text{Counting number}} \sum_{\mathbf{x}_R} \underbrace{b_R(\mathbf{x}_R)}_{\text{Belief on region R}} \left(\underbrace{E_R(\mathbf{x}_R; \theta_R)}_{\text{Region average energy}} + \ln b_R(\mathbf{x}_R) \right),$$

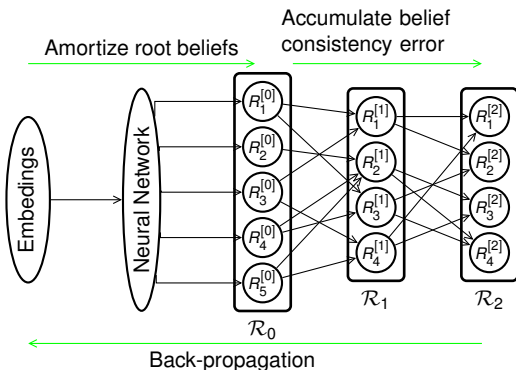
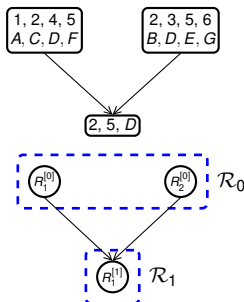
- counting number: balance the contribution of each region
- region average energy: $-\sum_{a \in A_R} \ln \varphi_a(\mathbf{x}_a; \theta_a)$

RENN: REGION-BASED ENERGY NEURAL NETWORK

The **region-based free energy** of a region graph is

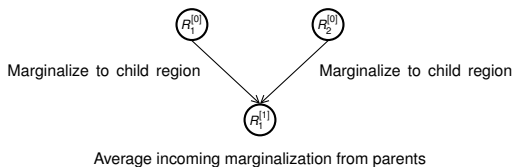
$$F_R(\mathcal{B}; \theta) = \sum_{R \in \mathcal{R}} \underbrace{c_R}_{\text{Counting number}} \sum_{\mathbf{x}_R} \underbrace{b_R(\mathbf{x}_R)}_{\text{Belief on region R}} \left(\underbrace{E_R(\mathbf{x}_R; \theta_R)}_{\text{Region average energy}} + \ln b_R(\mathbf{x}_R) \right),$$

- counting number: balance the contribution of each region
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RENN

Non-root belief:

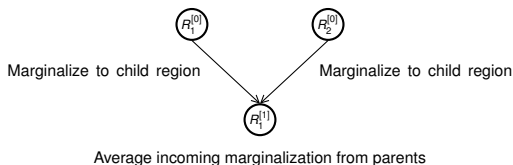


Objective of RENN:

$$\begin{array}{l}
 \text{min} \\
 \text{parameter} \\
 \text{of NN}
 \end{array}
 \underbrace{\text{region-based free energy}(F_R)}_{\text{Assumulated average of all regions}} + \underbrace{\text{panelty on belief consistency}}_{\text{Recursively computed via levels of region graph}}$$

RENN

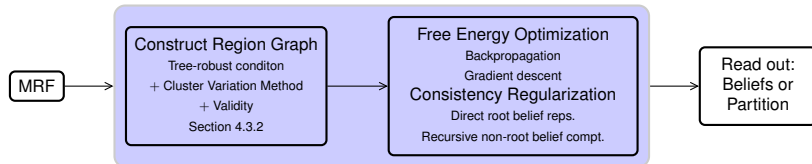
Non-root belief:



Objective of RENN:

$$\min_{\text{parameter of NN}} \underbrace{\text{region-based free energy}(F_R)}_{\text{Assumulated average of all regions}} + \underbrace{\text{panelty on belief consistency}}_{\text{Recursively computed via levels of region graph}}$$

RENN Inference:



RENN

INSIGHTS OF RENN

Generalization

Bethe free energy can be recovered from region-based free energy:

- two-level region graph representation
- constraint that each region can contain at most one factor node

Section 4.2.1

Attributes of RENN

- RENN requires neither sampling technique nor training data (ground-truth marginal probabilities) in performing inference tasks;
- RENN does gradient descent w.r.t. its neural network parameter instead of iterative message-passing, and returns approximation of marginal probabilities and partition estimation in one-shot
- No message propagation, thus no need of message scheduling
- Competitive performance and efficiency

SOME NUMERICAL COMPARISONS

Ising model: $p(\mathbf{x}; \theta) = \frac{1}{Z(\theta)} \exp(\sum_{(i,j) \in \mathcal{E}_F} J_{ij} x_i x_j + \sum_{i \in \mathcal{V}} h_i x_i)$, $\mathbf{x} \in \{-1, 1\}^N$,

- J_{ij} is the pairwise log-potential between node i and j , $J_{ij} \sim \mathcal{N}(0, 1)$
- h_i is the node log-potential for node i , $h_i \sim \mathcal{N}(0, \gamma^2)$

Inference on grid graph ($\gamma = 0.1$).

Metric	n	Mean Field	Loopy BP	Damped BP	GBP	Inference Net	RENN
ℓ_1 error	25	0.271 ± 0.051	0.086 ± 0.078	0.084 ± 0.076	0.057 ± 0.024	0.111 ± 0.072	0.049 ± 0.078
	100	0.283 ± 0.024	0.085 ± 0.041	0.062 ± 0.024	0.064 ± 0.019	0.074 ± 0.034	0.025 ± 0.011
	225	0.284 ± 0.019	0.100 ± 0.025	0.076 ± 0.025	0.073 ± 0.013	0.073 ± 0.012	0.046 ± 0.011
	400	0.279 ± 0.014	0.110 ± 0.016	0.090 ± 0.016	0.079 ± 0.009	0.083 ± 0.009	0.061 ± 0.009
Corre- lation ρ	25	0.633 ± 0.197	0.903 ± 0.114	0.905 ± 0.113	0.923 ± 0.045	0.866 ± 0.117	0.951 ± 0.112
	100	0.582 ± 0.112	0.827 ± 0.134	0.902 ± 0.059	0.899 ± 0.043	0.903 ± 0.049	0.983 ± 0.012
	225	0.580 ± 0.080	0.801 ± 0.078	0.863 ± 0.088	0.869 ± 0.037	0.873 ± 0.037	0.949 ± 0.022
	400	0.596 ± 0.054	0.779 ± 0.059	0.822 ± 0.047	0.852 ± 0.024	0.841 ± 0.028	0.912 ± 0.025
log Z error	25	2.512 ± 1.060	0.549 ± 0.373	0.557 ± 0.369	0.169 ± 0.142	0.762 ± 0.439	0.240 ± 0.140
	100	13.09 ± 2.156	1.650 ± 1.414	1.457 ± 1.365	0.524 ± 0.313	2.836 ± 2.158	1.899 ± 0.495
	225	29.93 ± 4.679	3.348 ± 1.954	3.423 ± 2.157	1.008 ± 0.653	3.249 ± 2.058	4.344 ± 0.813
	400	51.81 ± 4.706	5.738 ± 2.107	5.873 ± 2.211	1.750 ± 0.869	3.953 ± 2.558	7.598 ± 1.146

- ℓ_1 error of beliefs v.s. true
- correlation ρ between true and approximate marginals,
- log Z error, true v.s. free energy approximation.

SOME NUMERICAL COMPARISONS

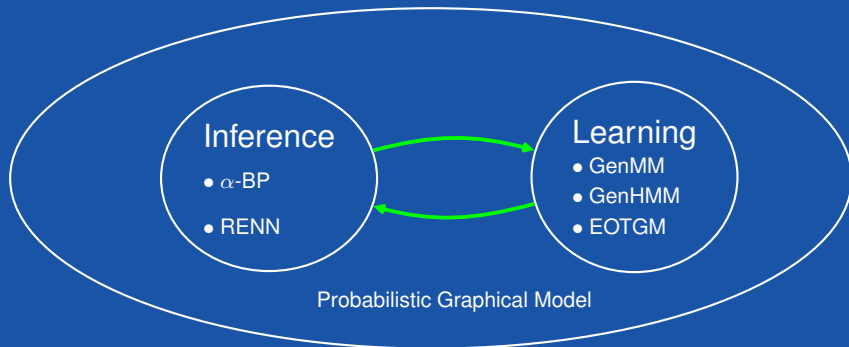
RICHER COMPARISONS

Inference on grid and complete graphs.

		Metric		Mean Field	Loopy BP	Damped BP	GBP	Inference Net	RENN
High Temp -erature	Complete graph N=16	ℓ_1 -	$\gamma = 1$	0.273 ± 0.086	0.239 ± 0.059	0.239 ± 0.059	0.260 ± 0.086	0.249 ± 0.067	0.181 ± 0.092
		error	$\gamma = 4$	0.197 ± 0.049	0.181 ± 0.035	0.180 ± 0.034	0.210 ± 0.070	0.174 ± 0.030	0.125 ± 0.050
	$J_{ij} \sim \mathcal{N}(0, 1)$ $h_i \sim \mathcal{N}(0, \gamma^2)$	log Z	$\gamma = 1$	20.66 ± 5.451	178.7 ± 22.18	178.9 ± 21.88	153.3 ± 25.29	213.6 ± 12.75	14.41 ± 4.135
		error	$\gamma = 4$	10.74 ± 7.385	565.7 ± 73.33	566.1 ± 73.13	106.0 ± 54.43	588.3 ± 62.58	14.72 ± 4.155
Low Temp -erature	Grid graph N=100	ℓ_1	5	0.257 ± 0.065	0.115 ± 0.071	0.120 ± 0.073	0.250 ± 0.024	0.164 ± 0.036	0.100 ± 0.046
		error	15	0.328 ± 0.068	0.228 ± 0.088	0.267 ± 0.147	0.303 ± 0.026	0.279 ± 0.024	0.207 ± 0.054
	$J_{ij} \sim \mathcal{U}(-u, u)$ $h_i \sim \mathcal{U}(-1, 1)$	log Z	5	42.65 ± 17.86	7.346 ± 7.744	5.444 ± 4.811	8.369 ± 7.401	65.60 ± 8.786	11.34 ± 4.724
		error	15	164.9 ± 56.07	58.40 ± 41.36	101.9 ± 54.31	23.10 ± 15.06	224.3 ± 25.52	78.85 ± 15.08

Average consumed time per inference instance (unit: second)

	Mean Field	Loopy BP	Damped BP	GBP	Inference Net	RENN
Grid \mathcal{G} , $N = 400$, $J_{ij} \sim \mathcal{N}(0, 1)$, $h_i \sim \mathcal{N}(0, 0.1^2)$	9.897	425.0	328.3	286.3	74.41	101.0
Complete \mathcal{G} , $N = 16$, $J_{ij} \sim \mathcal{N}(0, 1)$, $h_i \sim \mathcal{N}(0, 1)$	0.457	0.777	1.285	14.29	12.45	16.16
Grid \mathcal{G} , $N = 100$, $J_{ij} \sim \mathcal{U}(-15, 15)$, $h_i \sim \mathcal{U}(-1, 1)$	2.314	253.3	229.3	53.72	103.4	79.38
Complete \mathcal{G} , $N = 9$, $J_{ij} \sim \mathcal{U}(-15, 15)$, $h_i \sim \mathcal{U}(-1, 1)$	0.502	15.86	18.23	3.213	17.21	7.857



TWO DIRECTION IMPACT

Why impact in two direction?

- Learning to Inference:



Stuff we have been taken as default, a given $p(\mathbf{x}; \theta)$

- built by expert knowledge, or
- built by extracting information from evidence (empirical data).
- Inference to Learning:



Model learning: an error trial process that compares inferred 'fact' and actual fact (evidence).

Model learning usually needs inference as a subroutine, which sometimes are replaced by sampling in particle based methods.

INFERENCE ROUTINE IN LEARNING

What is θ in $p(\mathbf{x}; \theta) = \frac{1}{Z(\theta)} \prod_a \psi_a(\mathbf{x}_a; \theta_a)$?

A direct view:

$$\max_{\theta} \log p(\mathbf{x}; \theta) = \max_{\theta} \sum_a \log \psi_a(\mathbf{x}_a; \theta_a) \underbrace{- \log Z(\theta)}_{\text{Helmholtz free energy, can be est. by } F},$$

An alternative view:

$$\frac{\partial \log p(\mathbf{x}; \theta)}{\partial \theta_a} = \frac{\partial \log \varphi_a(\mathbf{x}_a; \theta_a)}{\partial \theta_a} - \underbrace{\mathbb{E}_{p(\mathbf{x}_a; \theta)} \left[\frac{\partial \log \varphi_a(\mathbf{x}_a; \theta_a)}{\partial \theta_a} \right]}_{\text{can be est. by beliefs}}.$$

Remark:

- This essentially requires estimation of Helmholtz free energy or marginal probabilities.
- Stationary points translate into moment matching.

INFERENCE ROUTINE IN LEARNING

What is θ in $p(\mathbf{x}; \theta) = \frac{1}{Z(\theta)} \prod_a \psi_a(\mathbf{x}_a; \theta_a)$?

A direct view:

$$\max_{\theta} \log p(\mathbf{x}; \theta) = \max_{\theta} \sum_a \log \psi_a(\mathbf{x}_a; \theta_a) - \underbrace{\log Z(\theta)}_{\text{Helmholtz free energy, can be est. by } F},$$

An alternative view:

$$\frac{\partial \log p(\mathbf{x}; \theta)}{\partial \theta_a} = \frac{\partial \log \varphi_a(\mathbf{x}_a; \theta_a)}{\partial \theta_a} - \underbrace{\mathbb{E}_{p(\mathbf{x}_a; \theta)} \left[\frac{\partial \log \varphi_a(\mathbf{x}_a; \theta_a)}{\partial \theta_a} \right]}_{\text{can be est. by beliefs}}.$$

Remark:

- This essence
- Stationary



- Two modules are not necessarily coupled
- Each module may be replaced by another algorithm while the other one remains.

al probabilities.

LEARNING MRFs

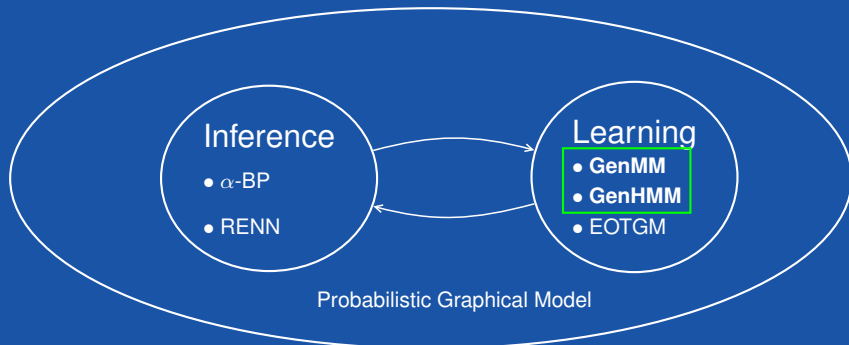
WHAT IS θ IN $p(\mathbf{x}; \theta)$?

Table of negative log-likelihood of learned MRFs

N	True	Exact	Mean Field	Loopy BP	Damped BP	GBP	Inference Net	RENN
Grid Graph								
25	9.000	9.004	9.811	9.139	9.196	10.56	9.252	9.048
100	19.34	19.38	23.48	19.92	20.02	28.61	20. 29	19.76
225	63.90	63.97	69.01	66.44	66.25	92.62	68.15	64.79
Complete Graph								
9	3.276	3.286	9.558	5.201	5.880	10.06	5.262	3.414
16	4.883	4.934	28.74	13.64	18.95	24.45	13.77	5.178

Average consumed time per epoch (unit: second) for two MRF learning cases.

	Mean Field	Loopy BP	Damped BP	GBP	Inference Net	RENN
Grid \mathcal{G} , $N=225$	40.09	335.1	525.1	12.37	19.49	16.03
Complete \mathcal{G} , $N=16$	2.499	12.40	5.431	1.387	0.882	2.262



INCOMPLETE OBSERVATION

Partial observation: $\mathbf{x} = [\underbrace{\mathbf{x}_U}_{\text{Unobserved}}, \underbrace{\mathbf{x}_O}_{\text{Observed}}]$

$$l(\mathbf{x}_O; \theta) = \log \sum_{\mathbf{x}_U} p(\mathbf{x}_U, \mathbf{x}_O; \theta) = \log \underbrace{Z(\mathbf{x}_O; \theta)}_{\sum_{\mathbf{x}_U} \tilde{p}(\mathbf{x}; \theta)} - \log Z(\theta),$$

Connect Free Energy to Evidence Lower Bounder:

$$\begin{aligned}
 l(\mathbf{x}_O; \theta) &\geq - \underbrace{F_V(q(\mathbf{x}_U | \mathbf{x}_O))}_{\text{Variational Free Energy}} - \log Z(\theta) \\
 &= \mathbb{E}_{q(\mathbf{x}_U | \mathbf{x}_O)} \left[\log \frac{p(\mathbf{x}_U, \mathbf{x}_O; \theta)}{q(\mathbf{x}_U | \mathbf{x}_O)} \right] \\
 &= \underbrace{\mathbb{E}_{q(\mathbf{x}_U | \mathbf{x}_O)} [\log p(\mathbf{x}_U, \mathbf{x}_O; \theta)] + H(q(\mathbf{x}_U | \mathbf{x}_O))}_{\text{Evidence Lower Bound } F(q, \theta)}
 \end{aligned}$$

Intuition of maximizing $F(q, \theta)$

- Maximizing (incomplete) likelihood
- Minimizing free energy

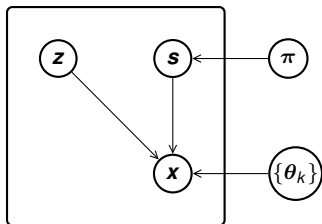
This gives the EM as a coordinate ascent method:

$$\text{E step : } q^{(t+1)} = \underset{q}{\operatorname{argmax}} F(q, \theta^{(t)}),$$

$$\text{M step : } \theta^{(t+1)} = \underset{\theta}{\operatorname{argmax}} F(q^{(t+1)}, \theta).$$

GENERATOR MIXED MODEL

EQUIPPING EM WITH NORMALIZING FLOWS



- Ideal case: The underline true $p^*(\mathbf{x})$ is in hypothesis space \mathcal{H} , i.e. $p^*(\mathbf{x}) \in \mathcal{H}$.
- Out of reach: Test $p^*(\mathbf{x}) \stackrel{?}{\in} \mathcal{H}$
- A general desire:

\mathcal{H} is large \rightarrow candidate $p(\mathbf{x}; \theta)$ is flexible

This brings up the finite **mixture** models.

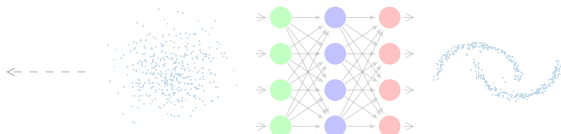
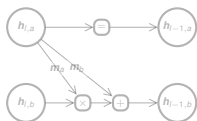
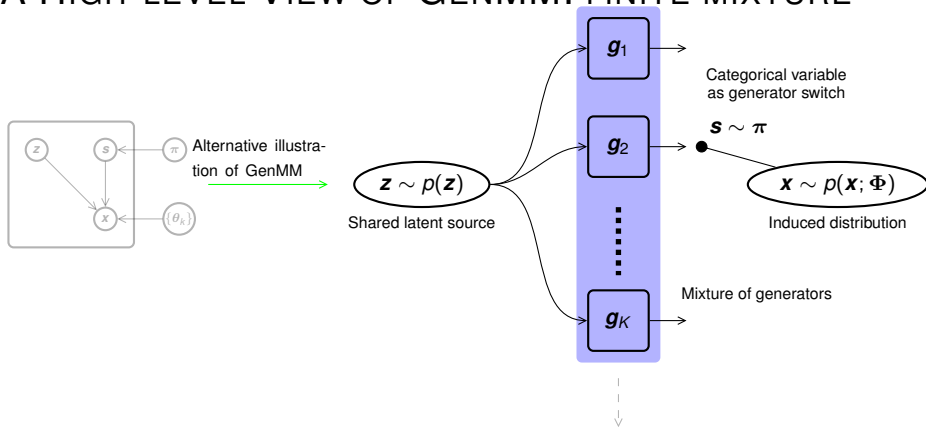
$$p(\mathbf{x}; \Theta) = \sum_{k=1}^K \pi_k p_k(\mathbf{x}) = \sum_{k=1}^K \pi_k p(\underbrace{\mathbf{g}(\mathbf{z}; \theta_k)}_{\text{Variable change via generator } \mathbf{g}})$$

What to expect from GenMM:

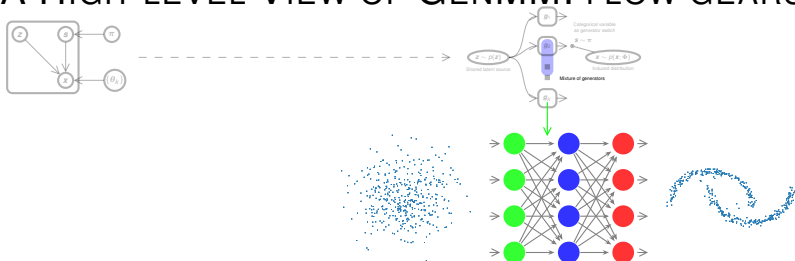
- Flexible and expressive model, enlarging hyperspace \mathcal{H}
- Tractable likelihood
- Compatible with typical statistical models
- Compatible with NN tools/frameworks
- Scale to high-dimensional structured data
- Efficient in sampling (data generation)
- ...

Variable change
via generator \mathbf{g}

A HIGH-LEVEL VIEW OF GENMM: FINITE MIXTURE



A HIGH-LEVEL VIEW OF GENMM: FLOW GEARS



When the k -th generator is selected, i.e., $s_k = 1$ and $s_{k'} = 0$ for $k' \neq k$, say $\tilde{\mathbf{x}} = \mathbf{x}|_{s_k=1}$. By following the [change of variable rule](#)

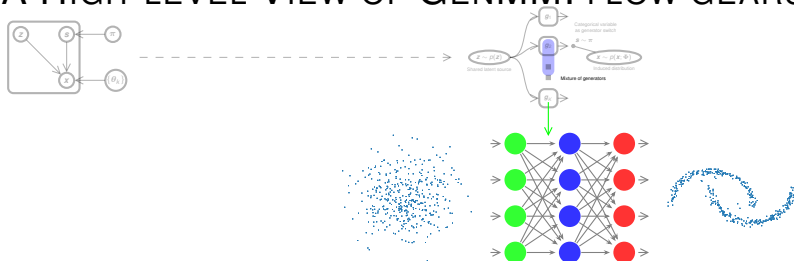


$$\underbrace{p(\tilde{\mathbf{x}})|_{\tilde{\mathbf{x}}=\tilde{\mathbf{g}}(\mathbf{z})}}_{\text{Induced distribution}} = \underbrace{p(\mathbf{z})}_{\substack{\text{Assumed known distribution} \\ \text{easy to sample}}} \cdot \underbrace{\left| \det \left(\frac{\partial \mathbf{z}}{\partial \tilde{\mathbf{x}}} \right) \right|}_{\substack{\text{Computational load} \\ \text{depends on the mapping}}}$$

A toy example:

$$\text{Gaussian linear transform: } Z \sim N(0, 1) \xrightarrow{X=\sigma \cdot Z+\mu} X \sim N(\mu, \sigma)$$

A HIGH-LEVEL VIEW OF GENMM: FLOW GEARS



When the k -th generator is selected, i.e., $s_k = 1$ and $s_{k'} = 0$ for $k' \neq k$, say $\tilde{\mathbf{x}} = \mathbf{x}|_{s_k=1}$. By following the [change of variable rule](#)

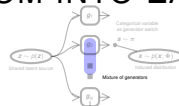
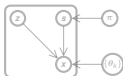


$$\underbrace{p(\tilde{\mathbf{x}})|_{\tilde{\mathbf{x}}=\tilde{\mathbf{g}}(\mathbf{z})}}_{\text{Induced distribution}} = \underbrace{p(\mathbf{z})}_{\substack{\text{Assumed known distribution} \\ \text{easy to sample}}} \cdot \underbrace{\left| \det \left(\frac{\partial \mathbf{z}}{\partial \tilde{\mathbf{x}}} \right) \right|}_{\substack{\text{Computational load} \\ \text{depends on the mapping}}}$$

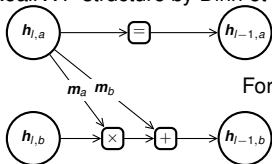
Powering it with a L -layer neural network implementation:

$$\mathbf{z} = \mathbf{h}_0 \xleftrightarrow[\tilde{\mathbf{f}}_1]{\tilde{\mathbf{g}}_1} \mathbf{h}_1 \xleftrightarrow[\tilde{\mathbf{f}}_2]{\tilde{\mathbf{g}}_2} \dots \xleftrightarrow[\tilde{\mathbf{f}}_L]{\tilde{\mathbf{g}}_L} \mathbf{x} = \mathbf{h}_L$$

A HIGH-LEVEL VIEW OF GENMM: ZOOM INTO LAYER

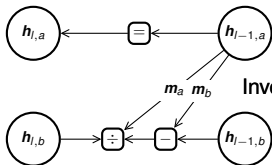


RealNVP structure by Dinh et al



Forward

$$h_{l-1} = \begin{bmatrix} h_{l-1,a} \\ h_{l-1,b} \end{bmatrix} = \begin{bmatrix} h_{l,a} \\ m_a(h_{l,a}) \odot h_{l-1,a} + m_b(h_{l,a}) \end{bmatrix}$$



Inverse

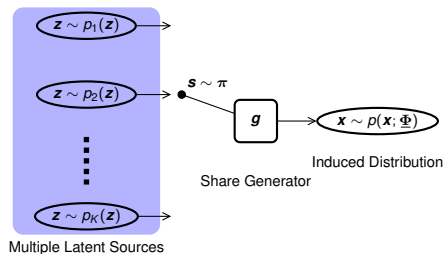
$$h_l = \begin{bmatrix} h_{l,a} \\ h_{l,b} \end{bmatrix} = \begin{bmatrix} h_{l-1,a} \\ (h_{l-1,b} - m_b(h_{l-1,a})) \oslash m_a(h_{l-1,a}) \end{bmatrix}$$

Pick up one layer to zoom in

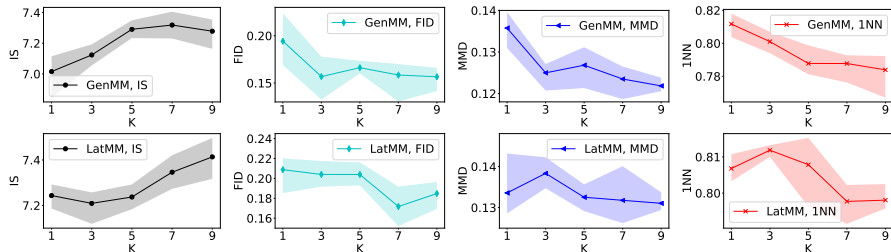
- \odot denotes element-wise product, \oslash denotes element-wise division
- Mapping m_a , m_b can be as complex as possible and not necessary invertible
- Same computation complexity of forward and inverse mapping
- Triangular matrix of Jacobian

Alternative arch. on market: Auto-regressive flow, Glow, ODE, etc.

LATMM: ALTERNATIVE MIXTURE



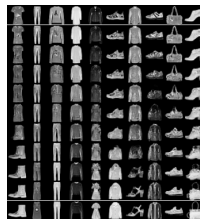
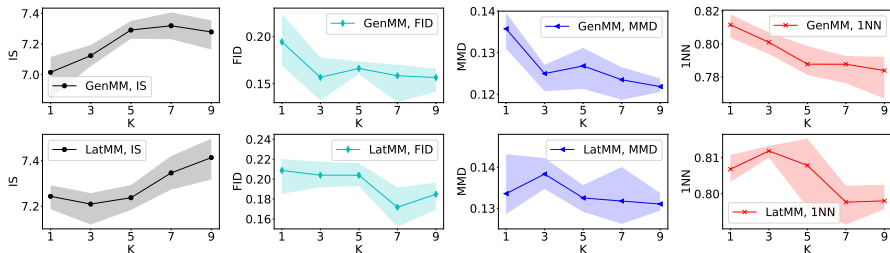
SEMANTIC SCORES AND EXAMPLES



IS, FID, MMD and 1NN of GenMM and LatMM for MNIST dataset.

- IS: Inception Score
- FID: Frechet Inception Distance
- MMD: Maximum Mean Discrepancy
- 1NN: 1-Nearest Neighbor

SEMANTIC SCORES AND EXAMPLES



Generated samples by GenMM and LatMM.

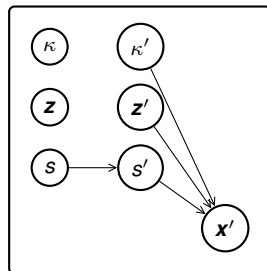
Interpolation in latent space

APPLICATION TO CLASSIFICATION TASKS

Application to classification with maximum likelihood. Test Accuracy Table of GenMM for Classification Task

Dataset	K=1	K=2	K=3	K=4	K=10	K=20	State Of Art
Letter	0.9459	0.9513	0.9578	0.9581	0.9657	0.9674	0.9582
Satimage	0.8900	0.8975	0.9045	0.9085	0.9105	0.9160	0.9090
Norb	0.9184	0.9257	0.9406	0.9459	0.9538	0.9542	0.8920

GENHMM: BRING THE CONCEPT INTO HMM

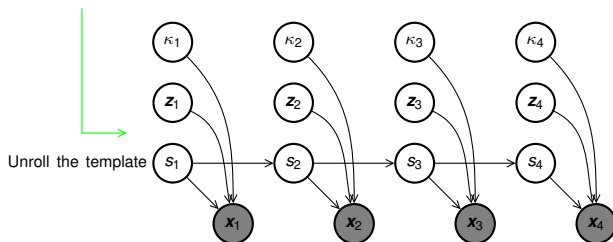


At time t , the probabilistic model of a state $s \in \mathcal{S}$ is then given by

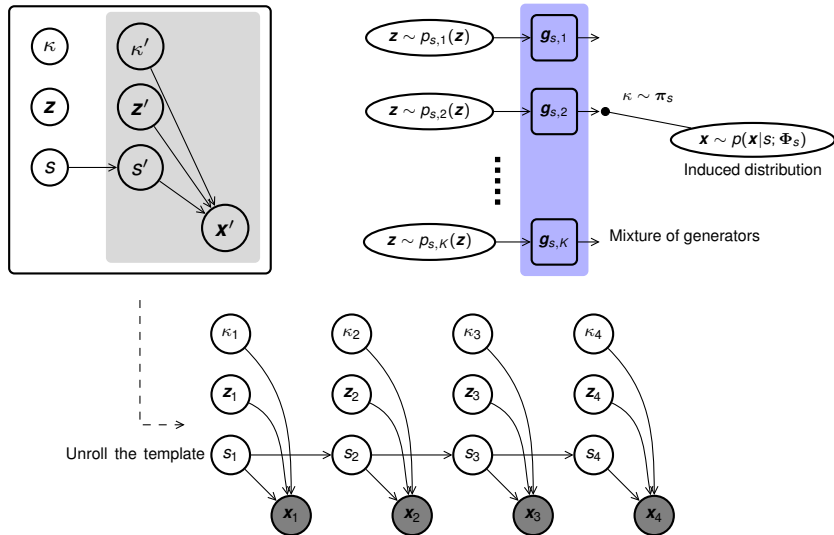
$$p(\mathbf{x}|s; \Phi_s) = \sum_{\kappa=1}^K \pi_{s,\kappa} p(\mathbf{x}|s, \kappa; \theta_{s,\kappa}),$$

where

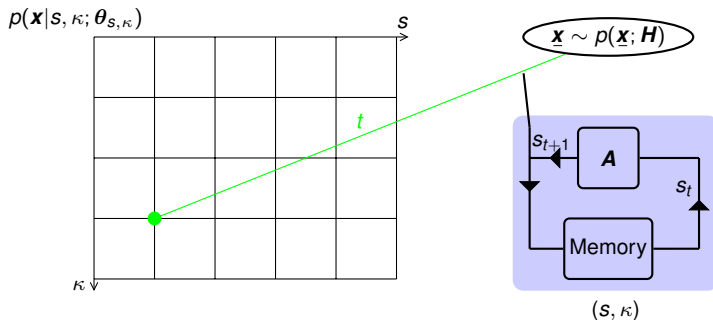
- $\pi_{s,\kappa} = p(\kappa|s; \mathbf{H})$, naturally $\sum_{\kappa=1}^K \pi_{s,\kappa} = 1$
- $p(\mathbf{x}|s, \kappa; \theta_{s,\kappa})$ is induced by the k th generator $\mathbf{g}_k(\mathbf{z}) = \mathbf{g}(\mathbf{z}; \theta_k)$



GENHMM: BRING THE CONCEPT INTO HMM



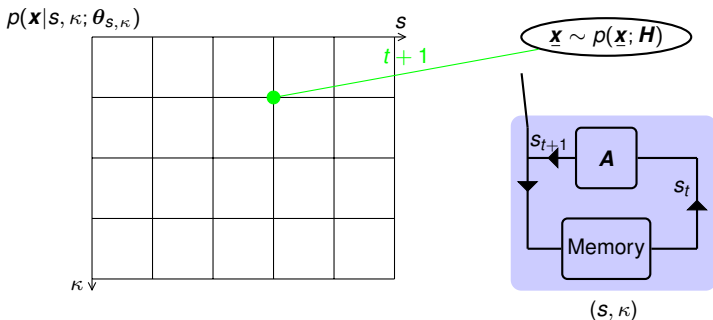
ALTERNATIVE VIEW OF GENHMM



Hypothesis set of GenHMM as $\mathcal{H} := \{\mathbf{H} | \mathbf{H} = \{\mathcal{S}, \mathbf{q}, \mathbf{A}, p(\mathbf{x} | \mathbf{s}; \Phi_{\mathbf{s}})\}\}$

- \mathcal{S} : the set of hidden states of \mathbf{H} .
- $\mathbf{q} = [q_1, \dots, q_{|\mathcal{S}|}]^T$: the initial state distributions of \mathbf{H} . $q_i = p(s_1 = i; \mathbf{H})$.
- \mathbf{A} : the transition matrix of states in \mathbf{H} .
- $\Phi = \{\Phi_s | s \in \mathcal{S}\}$.

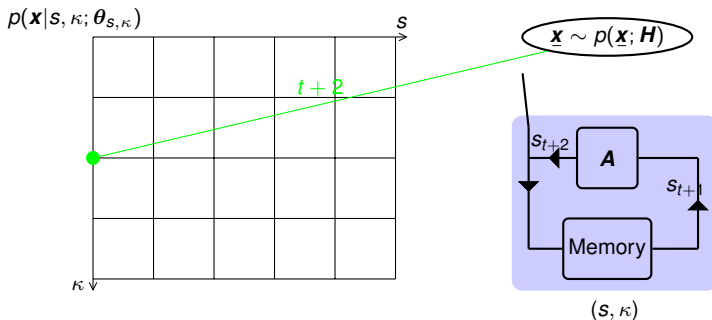
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ALTERNATIVE VIEW OF GENHMM



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LEARNING INTUITION

With empirical distribution $\hat{p}(\underline{\mathbf{x}}) = \frac{1}{N} \sum_n \delta_{\underline{\mathbf{x}}^{(n)}}(\underline{\mathbf{x}})$, learning of GenHMM boils down to

$$\min_{\mathbf{H} \in \mathcal{H}} \text{KL}(\hat{p}(\underline{\mathbf{x}}) \| p(\underline{\mathbf{x}}; \mathbf{H}))$$

Then problem becomes maximum likelihood estimation

$$\hat{\mathbf{H}} = \arg \max_{\mathbf{H} \in \mathcal{H}} \log \prod_i p(\underline{\mathbf{x}}^i; \mathbf{H}),$$

E-step: the \mathcal{Q} function

$$\mathcal{Q}(\mathbf{H}; \mathbf{H}^{\text{old}}) = \underbrace{\mathbb{E}_{\hat{p}(\underline{\mathbf{x}}), p(\underline{\mathbf{s}}, \underline{\kappa} | \underline{\mathbf{x}}; \mathbf{H}^{\text{old}})}}_{\substack{\text{w.r.t. distribution } \hat{p}(\underline{\mathbf{x}}) \\ \text{and } p(\underline{\mathbf{s}}, \underline{\kappa} | \underline{\mathbf{x}}; \mathbf{H}^{\text{old}})}} \underbrace{[\log p(\underline{\mathbf{x}}, \underline{\mathbf{s}}, \underline{\kappa}; \mathbf{H})]}_{\substack{\text{complete lklh.} \\ \text{factorize over time}}},$$

M-step: the maximization step

$$\max_{\mathbf{H}} \mathcal{Q}(\mathbf{H}; \mathbf{H}^{\text{old}}).$$

The M-step can be reformulated as

$$\max_{\mathbf{H}} \mathcal{Q}(\mathbf{H}; \mathbf{H}^{\text{old}}) = \underbrace{\max_{\mathbf{q}} \mathcal{Q}(\mathbf{q}; \mathbf{H}^{\text{old}})}_{\text{Initial State}} + \underbrace{\max_{\mathbf{A}} \mathcal{Q}(\mathbf{A}; \mathbf{H}^{\text{old}})}_{\text{Transition}} + \underbrace{\max_{\Phi} \mathcal{Q}(\Phi; \mathbf{H}^{\text{old}})}_{\text{Generators}}, \quad (1)$$

Maximizing a lower bound of $\log \prod_i p(\underline{\mathbf{x}}^{(i)}; \mathbf{H})$.

LEARNING INTUITION

With empirical distribution $\hat{p}(\underline{\mathbf{x}}) = \frac{1}{N} \sum_n \delta_{\underline{\mathbf{x}}^{(n)}}(\underline{\mathbf{x}})$, learning of GenHMM boils down to

$$\min_{\mathbf{H} \in \mathcal{H}} \text{KL}(\hat{p}(\underline{\mathbf{x}}) \| p(\underline{\mathbf{x}}; \mathbf{H}))$$

Then problem becomes maximum likelihood estimation

$$\hat{\mathbf{H}} = \arg \max_{\mathbf{H}} \log \prod_i p(\underline{\mathbf{x}}^i; \mathbf{H}),$$

E-step: the

- $F = \mathcal{Q} + \text{Entropy}$
- E-step require inference (message-passing)
- No optimality in M-step (NN generators, bash-size gradient descent).
- Still, guaranteed non-decreasing lklh. (c.f. Proposition 7.1)

M-step: the maximization step

$$\max_{\mathbf{H}} \mathcal{Q}(\mathbf{H}; \mathbf{H}^{\text{old}}).$$

The M-step can be reformulated as

$$\max_{\mathbf{H}} \mathcal{Q}(\mathbf{H}; \mathbf{H}^{\text{old}}) = \underbrace{\max_{\mathbf{q}} \mathcal{Q}(\mathbf{q}; \mathbf{H}^{\text{old}})}_{\text{Initial State}} + \underbrace{\max_{\mathbf{A}} \mathcal{Q}(\mathbf{A}; \mathbf{H}^{\text{old}})}_{\text{Transition}} + \underbrace{\max_{\Phi} \mathcal{Q}(\Phi; \mathbf{H}^{\text{old}})}_{\text{Generators}}, \quad (1)$$

Maximizing a lower bound of $\log \prod_i p(\underline{\mathbf{x}}^{(i)}; \mathbf{H})$.

APPLICATION OF GENHMM

Speech Recognition:

Configuration GenHMM in experiments on TIMIT

Latent distribution $p_{s,\kappa}(\mathbf{z})$ $s \in \mathcal{S}, \kappa = 1, 2, \dots, K$	Standard Gaussian
Number of flow blocks	4
Non-linear mapping $\mathbf{m}_a, \mathbf{m}_b$	Multiple layer perception 3 layers and with hidden dimension 24

Phoneme classification / recognition

Model	Criterion	K=1	K=3	K=5
GMM-HMM linear variable change	Accuracy	62.3	68.0	68.7
	Precision	67.9	72.6	73.0
	F1	63.7	69.1	69.7
GenHMM non-linear variable change	Accuracy	76.7	77.7	77.7
	Precision	76.9	78.1	78.0
	F1	76.1	77.1	77.0

Robustness to perturbation of noise.

Model	Criterion	White Noise SNR			
		15dB	20dB	25dB	30dB
GMM-HMM	Accuracy	36.6	44.2	50.8	57.1
	Precision	59.2	64.2	68.4	70.6
	F1	39.9	47.7	53.9	59.9
GenHMM	Accuracy	52.4	62.0	69.7	74.3
	Precision	60.0	65.9	71.7	74.8
	F1	52.5	62.0	69.3	73.5

Application to sepsis detection for infants, c.f. see Section 7.5.

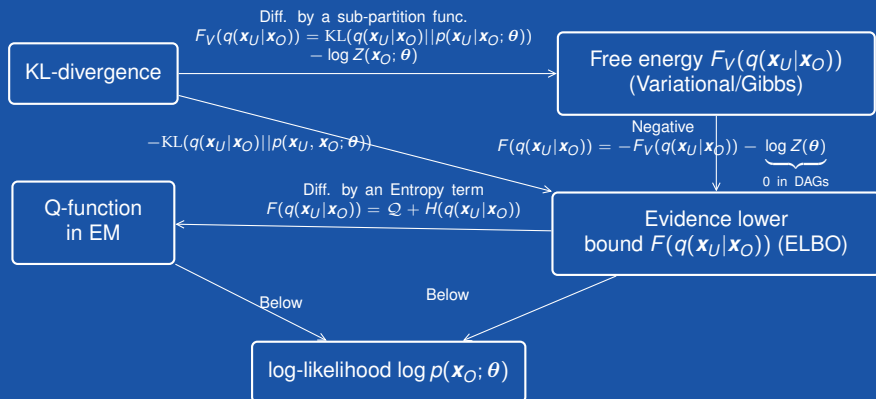
- generative training + discriminative training
- innovative feature engineering: bring derivative features into this problem

REMARK ON GENMM/GENHMM

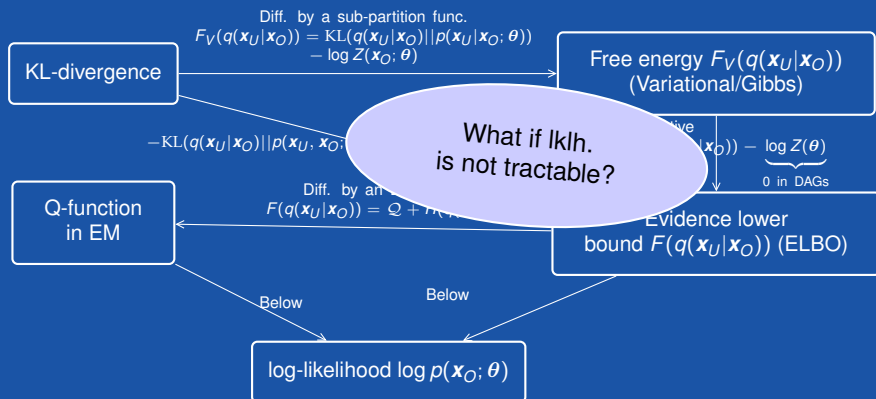
Attributes

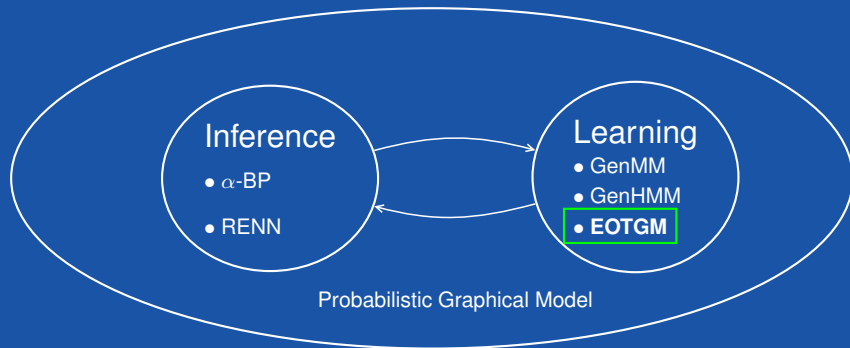
- Free dimension for flexibility: number of mixture + complexity of functional form of neural networks
- Compatible with statistic methods and neural network techniques (error back-propagation, optimizer)
- Embed batch-gradient descent into M-step
- Lack of closed-form update rule and generator changes at each gradient step. We tackle by maintaining old and new generators in EM steps

WHAT HAVE WE BEEN TALKING ABOUT?

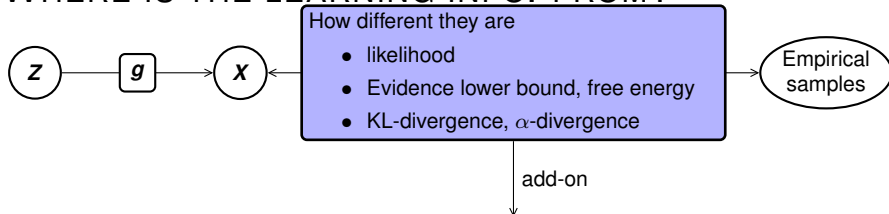


WHAT HAVE WE BEEN TALKING ABOUT?





WHERE IS THE LEARNING INFO. FROM?



Optimal transport (OT): moving mass from a dist. to another

$$T(p^*, p) = \min_{\pi \in \Pi(p^*, p)} \left\langle \underbrace{\pi}_{\text{marginalize to } p^*, p}, \underbrace{M}_{\substack{\text{cost matrix} \\ \text{sample difference}}} \right\rangle,$$

Key attributes:

- Does not require tractable lklh.
- Learning gradient info. from sample comparison
- High complexity, each evaluation is solving an optimization problem

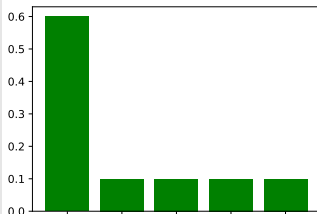
WHERE IS THE LEARNING INFO. FROM?

How different they are

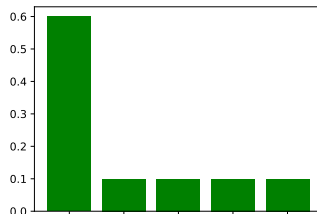
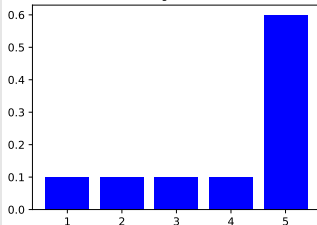
- likelihood

Empirical

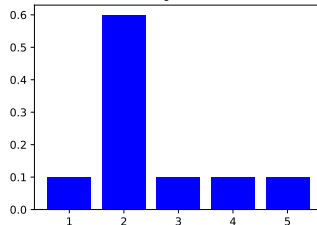
z



OT distance 2.0
KL divergence 0.8959

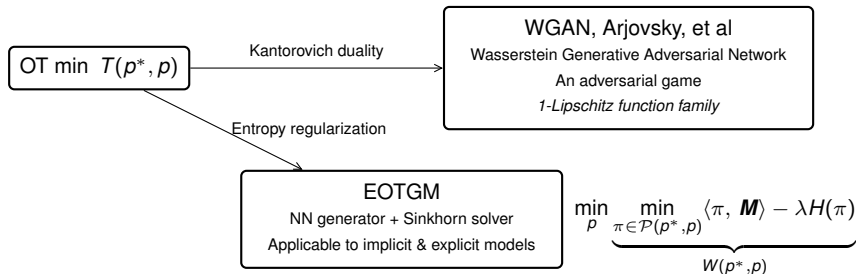


OT distance 0.5
KL divergence 0.8959



A toy example

EOTGM: EOT BASED GENERATIVE MODEL

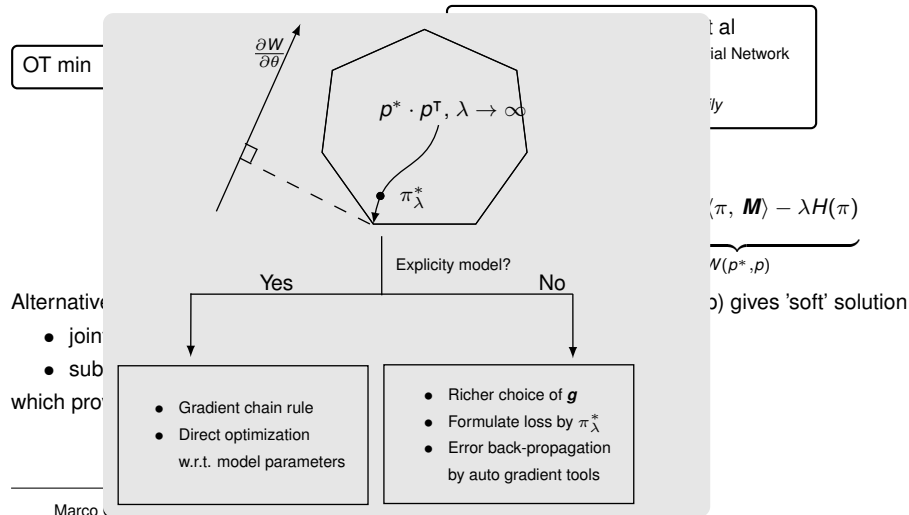


Alternatively scale the rows & columns of matrix $e^{-\mathbf{M}/\lambda}$ (Sinkhorn & Knopp) gives 'soft' solution

- joint distribution π^*
- subgradient β^*

which provides the gradient information for adjusting \mathbf{g} .

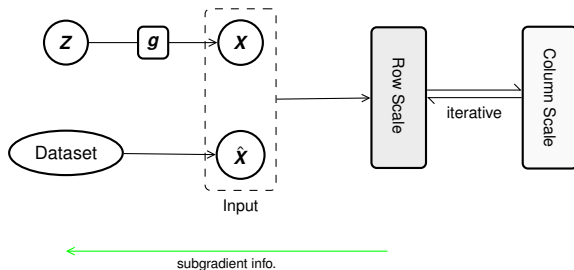
EOTGM: EOT BASED GENERATIVE MODEL



EOTGM AND EOTGAN

EOTGM

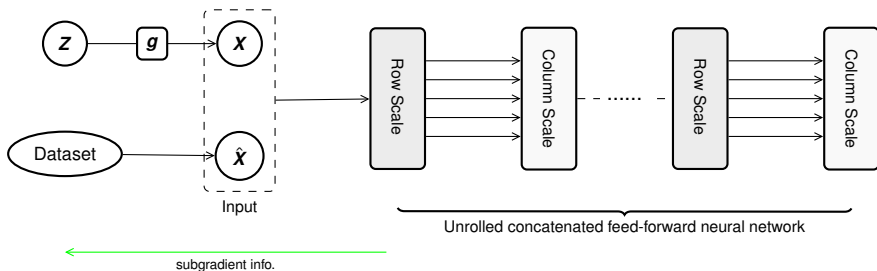
generated v.s. empirical sample comparison



EOTGM AND EOTGAN

EOTGM

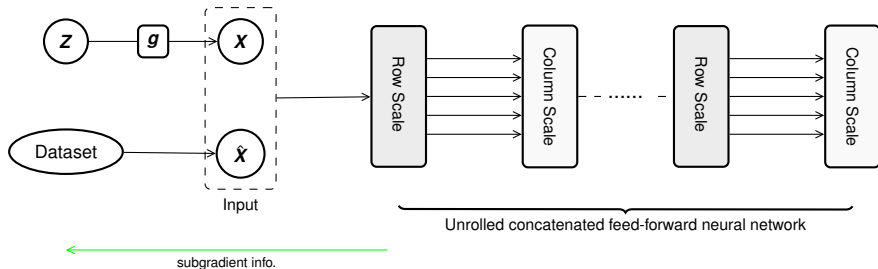
generated v.s. empirical sample comparison



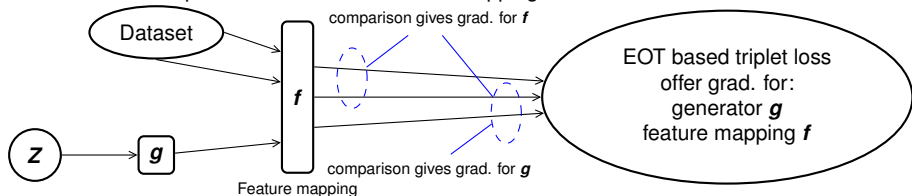
EOTGM AND EOTGAN

EOTGM

generated v.s. empirical sample comparison



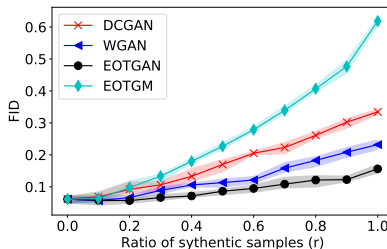
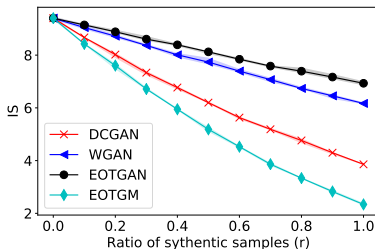
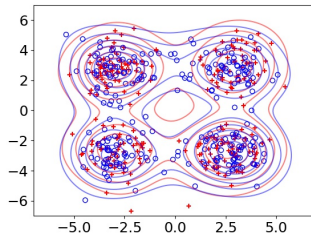
EOTGAN: learn implicit distribution with feature mapping



NUMERICAL

→ Toy distribution learning (target at 4-mixture Gaussians) using EOTGM. Real samples (red '+') and contour (red curve), versus generated samples (blue 'o') and contour (blue curve) by g .

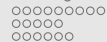
↓ Comparison of IS and FID (on MNIST) versus mixing ratio r .



SUMMARY

Wrap-up

- Inference with message-passing and analysis
- Inference with free energy minimization by neural networks
- Inference .. Learning: their interactions
- Neural network generators in EM for more flexible modeling; A Further step into temporal models
- A bonus modeling method for likelihood-free learning



Thank you for your attention.
Q&A.