Randomness

- Cards
- Dice
- Coins
- Radioactivity
- Statistical Mechanics
- Experimental Noise (Photon)

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Random and Monte Carlo

Random number generator

$$x' = (ax + c) \bmod m$$

$x = 1$

$a = 1664525$

$c = 1013904223$

$m = 4294967296$

Example of a linear congruential generator.

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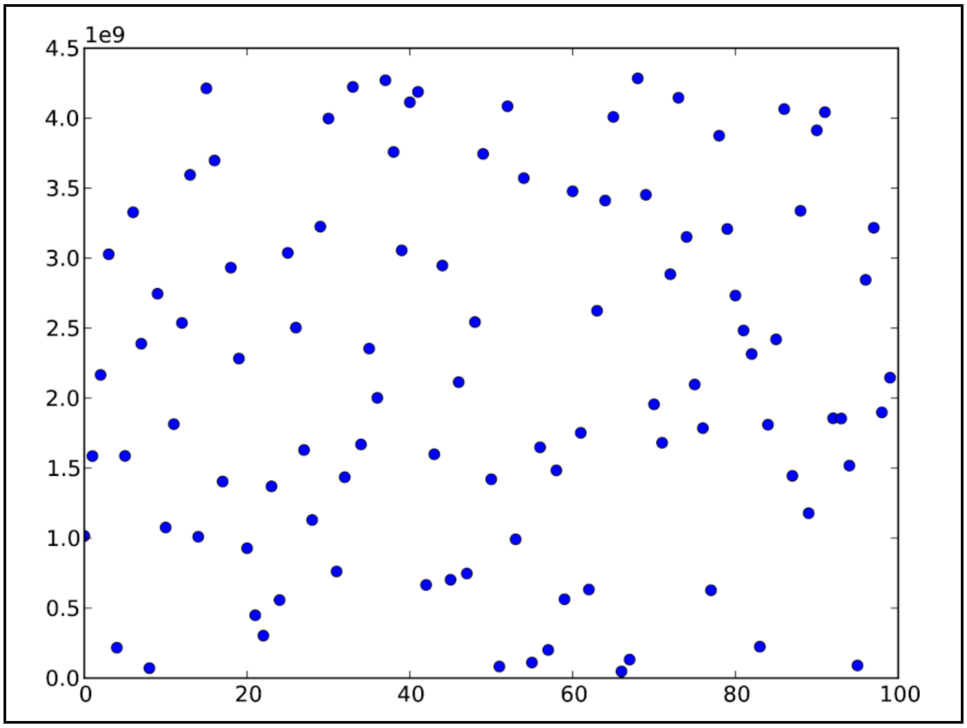

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Psuedorandom numbers

1. Not actually random
2. Number positive and less than m
3. Choices for a, c, and m MATTER!
4. Different x gives different numbers
   - Initial x is called the *seed*

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Psuedorandom numbers

Modern Generators

1. Still different x gives different numbers
   – Initial x is called the *seed*
2. *Mersenne Twister*
   – It has a very long period of $2^{19937} - 1$!!
   – *Somewhat slow*

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Python Random function

- random() - uniform pseudorandom number
  - $0 \leq x < 1$
- random(n) - uniform pseudorandom integer
  - $0 \leq x < n-1$
- random(m, n) - uniform pseudorandom integer
  - $m \leq x < n-1$
- random(m, n, k) - uniform pseudorandom integer
  - $m \leq x < n-1$ in steps of k

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“Repeatable” random numbers

- seed(x) = sets the seed
- Useful for “simple” encryption.
  - One time pad (pad sets a,c,m only pass seed)
  - Keyed encryption

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Random output

- Default: evenly distributed random numbers
  - E.g. $0 \le x < 1$
  - 0.2 probability of getting a random number < 0.2
- What about other distributions?

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Exponential decay

- Radioactive decay

$$x = 1/\mu \ln (1 - z)$$

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Measurement error and S/N

- Experimental data have measurement error.
- Data quality measure as S/N
- N = Noise source can be varied
- Quantum noise = Gaussian Distribution

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Gaussian Random Numbers

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)$$

Substitution gives

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx = z$$

Cannot solve this integral

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Gaussian Random Numbers

Use 2 Gaussians and covert to polar

$$p(x)dx \times p(y)dy = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx \times \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{y^2}{2\sigma^2}\right) dy$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2+y^2}{2\sigma^2}\right) dx\,dy$$

Translate

$$x^2 + y^2 = r^2 \quad dx\ dy = r\ dr\ d\theta$$

$$p(r,\theta)\ dr\ d\theta = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{r^2}{2\sigma^2}\right) r\,dr\,d\theta = \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right) dr \times \frac{d\theta}{2\pi}$$

$$= p(r)\,dr \times p(\theta)d\theta$$

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Gaussian Random Numbers

$$p(r)\,dr = \frac{r}{\sigma^2}\left(-\frac{r^2}{2\sigma^2}\right)$$

$$p(r)\,dr = \frac{1}{\sigma^2}\int_0^r \exp\left(-\frac{r^2}{2\sigma^2}\right) r\,dr = 1 - \exp\left(-\frac{r^2}{2\sigma^2}\right) = z$$

$$r = \sqrt{-2\sigma^2 \ln(1-z)} \qquad x = r\cos\theta \qquad y = r\sin\theta$$

X and Y return two Gaussian distributed random numbers

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Data fitting

- Gaussian sampled Monte Carlo
- Fit data (linear, log, exponential regression, χ^2)
- Re-fit data N times
  - Resample data within Gaussian error distribution
- Statistical determination of uncertainties in fitting

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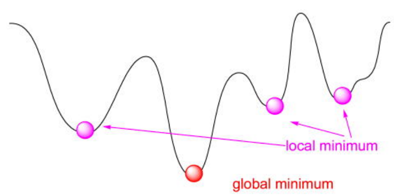
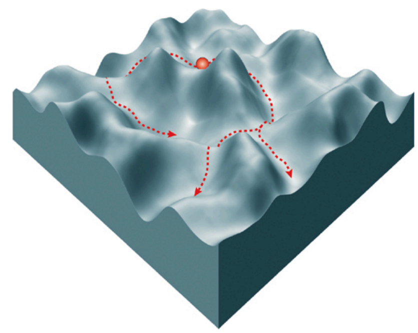
Data fitting

- Key for finding global minimums in fits.





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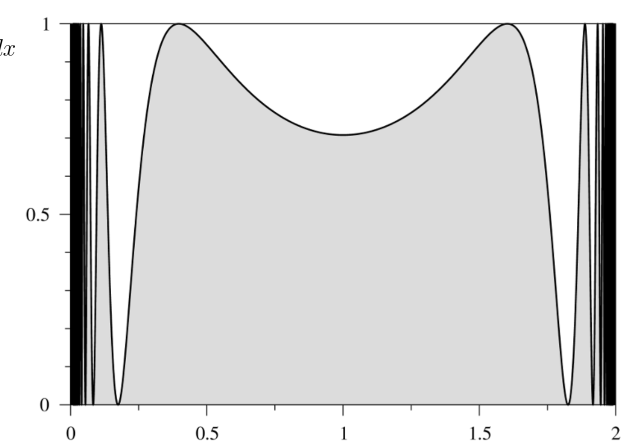
Monte Carlo Integration

$$I = \int_0^2 \sin^2\left[\frac{1}{x(2-x)}\right] dx$$



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Monte Carlo Integration(simple)

- Area A = 2
- p = I /A
- I ≅ kA/N

$$P(k) = \binom{N}{k} p^k (1-p)^{N-k}$$

$$\mathtt{var}\ k = Np(1-p) = N\frac{I}{A}\left(1 - \frac{I}{A}\right)$$

$$\sigma = \sqrt{\mathtt{var}\ k}\frac{A}{N} = \frac{\sqrt{I(A-I)}}{\sqrt{N}}$$

So: N*100
σ * 10

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Monte Carlo Integration (Mean Value)

$$I = \int_a^b f(x)dx$$

$$\langle f \rangle = \frac{1}{b-a}\int_a^b f(x)dx = \frac{I}{b-a}$$

$$I = (b-a)\langle f \rangle$$

$$\langle f \rangle \simeq N^{-1}\sum_{i=1}^{N} f(x_i)$$

$$\boxed{I \simeq \frac{b-a}{N}\sum_{i=1}^{N} f(x_i)}$$

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Monte Carlo Integration (Mean Value)

$$\boxed{I \simeq \frac{b-a}{N}\sum_{i=1}^{N} f(x_i)}$$

$$\texttt{var}\ F = \langle f^2 \rangle - \langle f \rangle^2$$

$$\langle f \rangle \simeq \frac{1}{N}\sum_{i=1}^{N} f(x_i) \qquad \langle f^2 \rangle \simeq \frac{1}{N}\sum_{i=1}^{N} [f(x_i)]^2$$

$$\sigma = \frac{b-a}{N}\sqrt{N, \texttt{var}\ f} = (b-a)\frac{\sqrt{\texttt{var}\ f}}{\sqrt{N}}$$

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