LECTURE 6
Repetition (recurrence) of trials. The Bernoulli formula

If several trials are made and the probability of an event A for each trial doesn't depend on outcomes of other trials, such trials are called *independent from the event A*.
At various independent trials an event A can have either different probabilities or the same probability. We will further consider only such independent trials in which the event A has the same probability.
We use below the notion of *complex event* meaning by that overlapping of several events, which are called *simple*.
Let n independent trials will be made in each of which an event A can either appear or not to appear. Assume that the probability of the event A for each trial is the same, namely equals p. Consequently, the probability of non-happening the event A in each trial is also constant and equals $q = 1 - p$.
Let's pose the problem of calculating the probability that for n trials the event A will happen exactly k times and consequently will not happen $n - k$ times. It is important to underline that it is not required the event A repeated exactly k times in a certain sequence. For example, if the speech is about appearance of the event A three times in four trials then the following complex events are possible: $AAA\overline{A}$, $AA\overline{A}A$, $A\overline{A}AA$, $\overline{A}AAA$. The entry $AAA\overline{A}$ denotes that the event A happened at the first, second and third trials and it didn't happen at the fourth trial, i.e. the opposite event $\overline{A}$ happened; the remaining entries have the corresponding sense.
Denote the required probability by $P_n(k)$. For example, the symbol $P_5(3)$ denotes the probability that the event will happen exactly 3 times for 5 trials and consequently it will not happen 2 times.
One can solve the posed problem by means of such-called Bernoulli formula.
Deduction of the Bernoulli formula: The probability of one complex event consisting in that for n trials the event A will happen k times and will not happen $n - k$ times is equal by the theorem of multiplication of probabilities of independent events to $p^k q^{n-k}$. There can be such complex events as much as combinations of n elements on k elements can be composed, i.e. C_n^k. Since these complex events are incompatible, by the theorem of addition of probabilities of incompatible events the required probability is equal to the sum of the probabilities of all possible complex events. Since the probabilities of all these complex events are the same, the required probability (of appearance of the event A k times for n trials) is equal to the probability of one complex event multiplied on their number:

$$\boxed{P_n(k) = C_n^k p^k q^{n-k}} \quad \text{or} \quad \boxed{P_n(k) = \frac{n!}{k!(n-k)!} p^k q^{n-k}}$$

Example. The probability that the expense of electric power during one day will not exceed the established norm is equal to p = 0,75. Find the probability that at the closest 6 days the expense of electric power will not exceed the norm for 4 days.
Solution: The probability of normal expense of electric power during each of 6 days is constant and equals 0,75. Consequently, the probability of overexpenditure of electric power for each day is also constant and equals $q = 1 - p = 1 - 0{,}75 = 0{,}25$. The required probability by the Bernoulli formula is equal to

$$P_6(4) = C_6^4 p^4 q^2 = \frac{6 \cdot 5}{1 \cdot 2} \cdot (0{,}75)^4 \cdot (0{,}25)^2 = 0{,}30.$$

The most probable number k_0 of occurrences of an event in independent trials is determined from the double inequality:

$$\boxed{np - q \le k_0 \le np + p}$$

where n – number of trials, p – probability of occurrence of the event in one trial, q – probability of non-occurrence of the event in one trial.

Example. 40 boxes of glassware have been delivered on a warehouse. The probability that all glassware of a randomly taken box appear undamaged is equal to 0,9. Find the most probable number of boxes in which all glassware appears undamaged.
Solution: By the hypothesis n = 40, p = 0,9, q = 0,1. The most probable number of boxes that are not containing damaged glassware is determined by the double inequality: $40 \cdot 0{,}9 - 0{,}1 \le k_0 \le 40 \cdot 0{,}9 + 0{,}9$ or $35{,}9 \le k_0 \le 36{,}9$ Therefore the required most probable number k_0 = 36.

Local theorem of Laplace

It easy to see that using the Bernoulli formula for great values n is sufficiently difficult because for example if $n = 50$, $k = 30$, $p = 0{,}1$ then for finding the probability $P_{50}(30)$ it is necessary to calculate the expression

$$P_{50}(30) = 50!/(30!\cdot 20!) \cdot (0{,}1)^{30} \cdot (0{,}9)^{20}.$$

One arises the question: is it possible to calculate probability interested for us without using the Bernoulli formula? Yes, it can. Local theorem of Laplace gives the asymptotic formula which allows approximately to find the probability of appearance of an event exactly k times for n trials if the number of trials is sufficiently great.
Local theorem of Laplace: If the probability p of appearance of an event A for each trial is constant and differs from 0 and 1 then the probability $P_n(k)$ that the event A will appear for n trials exactly k times is approximately equal to (the more precise, the greater n) the value of the function

$$y = \frac{1}{\sqrt{npq}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = \frac{1}{\sqrt{npq}} \cdot \varphi(x)$$

for $x = (k - np)/\sqrt{npq}.$

There are tables (Appendix 1) with values of the function $\varphi(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}$ corresponding to positive values of the argument x. We use the same tables for negative values of the argument because the function $\varphi(x)$ is even, i.e. $\varphi(-x) = \varphi(x)$.
Thus, the probability that the event A will happen for n independent trials exactly k times is approximately equal to

$$\boxed{P_n(k) \approx \frac{1}{\sqrt{npq}} \cdot \varphi(x)}$$

where $x = (k - np)/\sqrt{npq}.$

Example. Find the probability that an event A will happen exactly 80 times for 400 trials if the probability of appearance of the event for each trial equals 0,2.
Solution: By the hypothesis, $n = 400; k = 80; p = 0,2; q = 0,8.$
Use the asymptotic formula of Laplace:

$$P_{400}(80) \approx \frac{1}{\sqrt{400 \cdot 0,2 \cdot 0,8}} \cdot \varphi(x) = \frac{1}{8} \cdot \varphi(x).$$

Calculate the value x determined by data of the problem:

$$x = \frac{k - np}{\sqrt{npq}} = \frac{80 - 400 \cdot 0,2}{8} = 0.$$

By the table (Appendix 1) we find

$$\varphi(0) = 0,3989 \;\Rightarrow\; P_{400}(80) = \frac{1}{8} \cdot 0,3989 = 0,04986.$$

The Bernoulli formula gives almost the same result: $P_{400}(80) = 0,0498.$

Example. The probability of hit in a target by a shooter at one shot is $p = 0,75$. Find the probability that the shooter will hit in the target 8 times at 10 shots.
Solution: By the hypothesis, $n = 10; k = 8; p = 0,75; q = 0,25.$
Use the asymptotic formula of Laplace:

$$P_{10}(8) \approx \frac{1}{\sqrt{10 \cdot 0,75 \cdot 0,25}} \cdot \varphi(x) = 0,7301 \cdot \varphi(x).$$

Calculate the value x determined by data of the problem:

$$x = \frac{k - np}{\sqrt{npq}} = \frac{8 - 10 \cdot 0,75}{\sqrt{10 \cdot 0,75 \cdot 0,25}} \approx 0,36.$$

By the table (Appendix 1) we find

$$\varphi(0,36) = 0,3739 \;\Rightarrow\; P_{10}(8) = 0,7301 \cdot 0,3739 = 0,273.$$

The Bernoulli formula gives another result: $P_{10}(8) = 0,282.$
Such big difference of results is explained by that n has small value for our example (the Laplace formula gives sufficiently good approximations only for sufficiently great values n).

Integral theorem of Laplace

Suppose that n trials are made in each of which the probability of appearance of the event A is constant and equals p ($0 < p < 1$). How can we calculate probability $P_n(k_1, k_2)$ that the event A will appear for n trials no less than k_1 and no more than k_2 times (for shortness, we will say “from k_1 up to k_2 times”)?
Theorem. If probability p of appearance of an event A for each trial is constant and differs from 0 and 1 then the probability $P_n(k_1, k_2)$ that the event A will appear for n trials from k_1 up to k_2 times is approximately equal to the definite integral

$$P_n(k_1, k_2) \approx \frac{1}{\sqrt{2\pi}} \int_{x'}^{x''} e^{-\frac{z^2}{2}}\, dz \qquad (*)$$

where $x' = (k_1 - np)/\sqrt{npq}$ and $x'' = (k_2 - np)/\sqrt{npq}$.

For solving of problems requiring an application of the integral theorem of Laplace special tables are used because the indefinite integral $\int e^{-\frac{z^2}{2}}dz$ is not expressed by elementary functions.

The table (Appendix 1) for the integral $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_0^x e^{-\frac{z^2}{2}}dz$ is given for positive values *x* and for *x = 0*; it is used the same table for *x < 0* (the function $\Phi(x)$ is odd, i.e. $\Phi(-x) = -\Phi(x)$). There are values of the integral only till *x = 5* in the table because one can assume $\Phi(x)$ *= 0,5* for *x > 5*. The function $\Phi(x)$ is often called the *Laplace function*.

Transform the expression (*) as follows:

$$P_n(k_1,k_2) \approx \frac{1}{\sqrt{2\pi}}\int_{x'}^{0} e^{-\frac{z^2}{2}}dz + \frac{1}{\sqrt{2\pi}}\int_0^{x''} e^{-\frac{z^2}{2}}dz =$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^{x''} e^{-\frac{z^2}{2}}dz - \frac{1}{\sqrt{2\pi}}\int_0^{x'} e^{-\frac{z^2}{2}}dz = \Phi(x'') - \Phi(x')$$

i.e.

$$\boxed{P_n(k_1,k_2) \approx \Phi(x'') - \Phi(x')}$$

where $x' = (k_1 - np)/\sqrt{npq}$ and $x'' = (k_2 - np)/\sqrt{npq}$.

<u>*Example.*</u> The probability that an item has not passed a checking by the quality department is equal to 0,2. Find the probability that there will be unchecked from 70 up to 100 items among 400 randomly selected items.

Solution: By the hypothesis, *p = 0,2; q = 0,8; n = 400;* $k_1 = 70$; $k_2 = 100$.

Use the integral theorem of Laplace: $P_{400}(70,\ 100) \approx \Phi(x'') - \Phi(x')$.

Calculate the lower and the upper limits of integration:

$$x' = \frac{k_1 - np}{\sqrt{npq}} = \frac{70 - 400\cdot 0{,}2}{\sqrt{400\cdot 0{,}2\cdot 0{,}8}} = -1{,}25;\quad x'' = \frac{k_2 - np}{\sqrt{npq}} = \frac{100 - 400\cdot 0{,}2}{\sqrt{400\cdot 0{,}2\cdot 0{,}8}} = 2{,}5.$$

Thus, we have $P_{400}(70,\ 100) = \Phi(2{,}5) - \Phi(-1{,}25) = \Phi(2{,}5) + \Phi(1{,}25)$.

By the table (Appendix 1) we find: $\Phi(2{,}5) = 0{,}4938$; $\Phi(1{,}25) = 0{,}3944$.

The required probability $P_{400}(70,\ 100) = 0{,}4938 + 0{,}3944 = 0{,}8882$.

Glossary

complex event – сложное событие; **overlapping** – совмещение
expense – расход; **electric power** – электроэнергия
overexpenditure – перерасход
quality department – отдел технического контроля
warehouse – товарный склад; **undamaged** – целый (неповрежденный)

Exercises for Seminar 6

6.1. There are 6 motors in a shop. For each motor the probability that it is turned (switched) on at present time is equal to 0,8. Find the probability that at present:

a) 4 motors are turned on;

b) all motors are turned on;
c) all motors are turned off (a shop – цех).
The answer: a) 0,246; b) 0,26; c) 0,000064.

6.2. Find the probability that an event *A* will appear in five independent trials no less than two times if the probability of occurrence of the event *A* for each trial is equal to 0,3.
The answer: 0,472.

6.3. A coin is tossed 6 times. Find the probability that the coin lands on heads:
a) less than two times;
b) no less than two times.
The answer: a) 7/64; b) 57/64.

6.4. Find approximately the probability that an event will happen exactly 104 times at 400 trials if in each trial the probability of its occurrence is equal to 0,2.
The answer: 0,0006.

6.5. The probability of striking a target by a shooter at one shot is equal to 0,75. Find the probability that at 100 shots the target will be struck:
a) no less than 70 and no more 80 times;
b) no more than 70 times.
The answer: a) 0,7498; b) 0,1251.

6.6. The probability that an event *A* will appear at least once at two independent trials is equal to 0,75. Find the probability of appearance of the event in one trial (it is supposed that the probability of appearance of the event in both trials is the same).
The answer: 0,5.

6.7. A coming up a potato is equal to 80 %. How many is it necessary to plant potatoes that the most probable number of came up potatoes of them was equal to 100 (to come up – всходить (*о растении*))?
The answer: 124 or 125.

6.8. There are 4000 bees in a bee family. The probability of illness within a day is equal to 0,002 for each bee. Find the probability that more than one bee will be ill within a day (a bee – пчела).
The answer: 0,99.

6.9. A coming up a grain stored in a warehouse is equal to 80%. What is the probability that the number of came up grains among 100 ones will make from 68 up to 90 pieces (a grain – зерно)?
The answer: 0,992.

6.10. The probability of receiving an excellent mark at an exam is equal to 0,2. Find the most probable number of excellent marks and the probability of this number if 50 students pass the exam.
The answer: k_0 = 10 and $P_n(k_0)$ = 0,141.

Exercises for Homework 6

6.11. An event B will appear in case when an event A will appear no less than two times. Find the probability that the event B will happen if 6 independent trials will be made in each of which the probability of occurrence of the event A is equal to 0,4.
The answer: 0,767.

6.12. 8 independent trials have been made in each of which the probability of occurrence of an event A is equal to 0,1. Find the probability that the event A will appear at least 2 times.

The answer: 0,19.

6.13. A factory has sent 5000 good-quality products. The probability that one product has been damaged at a transportation is 0,002. Find the probability that at the transportation it will be damaged:

a) 3 products; b) 1 product; c) no more than 3 products.

The answer: a) 0,0107; b) 0,00218.

6.14. A shooter has made 400 shots, and the probability of hit in a target is 0,8. Find the probability that he hits from 310 up to 325 times.

The answer: 0,6284.

6.15. The number of workers of an enterprise is 500 persons. The probability of absence on the work because of illness is equal to 0,01 for each worker of the enterprise. Determine the probability that at least one of workers will not come to work at the nearest day.

The answer: 0,985.

6.16. How many times is it necessary to toss a die in order that the most probable number of landing 6 aces was equal to 50?

The answer: $299 \le n \le 305$.

6.17. A shooter hits in a target with the probability 0,6. He is going to make 10 shots. Find the probability that he hits in the target:

a) three times; b) at least once.

The answer: a) 0,0425; b) 0,9999.

6.18. A coming up seeds makes 80%. What is the probability that from 780 up to 820 seeds will come up of 1000 sown seeds?

The answer: 0,8858.

6.19. There are 70 automobiles in a park. The probability of breakage of an automobile is equal to 0,2. Find the most probable number of serviceable automobiles and the probability of this number.

The answer: $k_0 = 14$ and $P_n(k_0) = 0,119$.

6.20. 900 students are studying at a faculty. The probability of birthday in a given day is equal to 1/365 for each student. Find the probability that there will be three students with the same birthday.

The answer: 0,24.