L E C T U R E 7

Random variables. The law of distribution of a discrete random variable

It is frequently the case when an experiment is performed that we are mainly interested in some function of the outcome as opposed to the actual outcome itself. For instance, in tossing dice we are often interested in the sum of the two dice and are not really concerned about the separate values of each die. That is, we may be interested in knowing that the sum is 7 and not be concerned over whether the actual outcome was (1, 6) or (2, 5) or (3, 4) or (4, 3) or (5, 2) or (6, 1). Also, in coin flipping, we may be interested in the total number of heads that occur and not care at all about the actual head-tail sequence that result. These quantities of interest, or more formally, these real-valued functions defined on the outcome space, are known as *random variables*.

A random variable is understood as a variable which as result of a trial takes one of the possible set of its values (which namely – it is not beforehand known).

We denote random variables by capital letters of Latin alphabet *X, Y, Z*, …, and their values – by the corresponding small letters *x, y, z*, ….

<u>*Example.*</u> The number of the born boys among hundred newborns is a random variable which has the following possible values: 0, 1, 2, …, 100.

<u>*Example.*</u> The distance which will be passed by a shell at shot by a gun is a random variable. Really, the distance depends not only on installation of a sight, but also from many other reasons (force and direction of wind, temperature, etc.) which cannot be completely taken into account. Possible values of this variable belong to some interval (*a*, *b*).

Example. Suppose that our experiment consists of tossing 3 coins. If we let *Y* denote the number of heads appearing, then *Y* is a random variable taking on one of the values 0, 1, 2, 3 with the corresponding probabilities

$$P(Y=0)=P((T,T,T))=\frac{1}{8}$$

$$P(Y=1)=P((T,T,H),\ (T,H,T),\ (H,T,T))=\frac{3}{8}$$

$$P(Y=2)=P((T,H,H),\ (H,T,H),\ (H,H,T))=\frac{3}{8}$$

$$P(Y=3)=P((H,H,H))=\frac{1}{8}$$

A discrete random variable is a random variable which takes on separate, isolated possible values with certain probabilities. The number of possible values of a discrete random variable can be finite or infinite.

For a discrete random variable *X*, we define the *probability mass function p(a)* of *X* by

$$p(a) = P(X = a)$$

A continuous random variable is a random variable which can take all values from some finite or infinite interval. Obviously, the number of possible values of a continuous random variable is infinite.

The most full, exhaustive description of a random variable is its law of distribution.

Any ratio establishing connection between possible values of a random variable and probabilities corresponding to them refers to as the *law of distribution* of the random variable.

About a random variable speak that it «is distributed» under the given law of distribution or «subordinated» to this law of distribution.

For a discrete random variable the law of distribution can be set as a table, analytically (as a formula) and graphically.

The elementary form of assignment of the law of distribution of a discrete random variable *X* is a table (matrix) in which all possible values of a random variable and the probabilities corresponding to them are listed in ascending order, i.e.

x_1	x_2	...	x_i	...	x_n
p_1	p_2	...	p_i		p_n

or

$$X = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix}$$

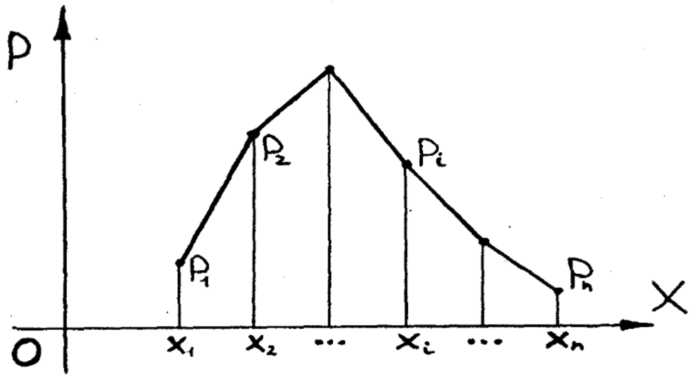
Such a table is called *the series of distribution* of a discrete random variable.

The events $X = x_1$, $X = x_2$, ..., $X = x_n$, consisting in that as a result of trial the random variable X will take on values $x_1, x_2, ..., x_n$ respectively, are incompatible and uniquely possible (because in the table all possible values of a random variable are listed), i.e. form a complete group. Hence, the sum of probabilities is equal to 1. Thus, for every discrete random variable

$$\sum_{i=1}^{n} P(X = x_i) = \sum_{i=1}^{n} p_i = 1$$

(This unit is somehow *distributed* between values of a random variable, therefore from here the term "distribution").

A series of distribution can be represented graphically if values of a random variable are postponed on the axis of abscissas, and on the axis of ordinates – their corresponding probabilities. Connecting the received points forms a broken line named a *polygon of distribution of probabilities*.



Example. 100 tickets of a monetary lottery are released. One prize in 50 roubles and ten prizes on 1 rouble are played. Find the law of distribution of a random variable X – cost of a possible prize for an owner of one lottery ticket.

Solution: Write the possible values of X: $x_1 = 50$, $x_2 = 1$, $x_3 = 0$. The probabilities of these possible values are those: $p_1 = 0{,}01$; $p_2 = 0{,}1$; $p_3 = 1 - (p_1 + p_2) = 0{,}89$.

Let's write the required law of distribution:

X	50	1	0
P	0,01	0,1	0,89

Mathematical operations over random variables

Two random variables are *independent* if the law of distribution of one of them does not vary from that which possible values were taken on by another variable. So, if a discrete random variable X can take on values x_i $(i = 1, 2, ..., n)$, and a random variable Y – values y_j $(j = 1, 2, .., m)$ then the independence of the discrete random variables X and Y means the independence of the events $X = x_i$ and $Y = y_j$ for all $i = 1, 2, .., n$ and $j = 1, 2, .., m$. Otherwise, the random variables are *dependent*.

For example, if there are tickets of two different monetary lotteries then the random variables X and Y expressing respectively a prize under each ticket (in monetary units) will be independent as at any prize under a ticket of one lottery the law of distribution of a prize under other ticket will not be changed. If the random variables X and Y express a prize under tickets of one monetary lottery then in this case X and Y are dependent since any prize under one ticket ($X = x_i$) results in change of probabilities of a prize under other ticket ($Y = y_j$), i.e. changes the law of distribution Y.

Let two random variables X and Y be given:

$$X = \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ p_1 & p_2 & \dots & p_n \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 & y_2 & \dots & y_m \\ p'_1 & p'_2 & \dots & p'_m \end{pmatrix}.$$

The product kX of a random variable X on a constant k is the random variable which takes on values kx_i with the same probabilities p_i $(i = 1, 2, .., n)$.

The m-th degree of a random variable X, i.e. X^m is the random variable which takes on values x_i^m with the same probabilities p_i $(i = 1, 2, .., n)$.

Example. Let a random variable X be given: $X = \begin{pmatrix} -2 & 1 & 2 \\ 0{,}5 & 0{,}3 & 0{,}2 \end{pmatrix}$. Find the law of distribution of the random variables: a) $Y = 3X$; b) $Z = X^2$.

Solution: a) The values of the random variable Y will be: $3 \cdot (-2) = -6$; $3 \cdot 1 = 3$; $3 \cdot 2 = 6$ with the same probabilities 0,5; 0,3; 0,2, i.e. $Y = \begin{pmatrix} -6 & 3 & 6 \\ 0{,}5 & 0{,}3 & 0{,}2 \end{pmatrix}$.

b) The values of the random variable Z will be: $(-2)^2 = 4$, $1^2 = 1$, $2^2 = 4$ with the same probabilities 0,5; 0,3; 0,2. Since the value $Z = 4$ can be obtained by squaring the values (-2) with probability 0,5 and $(+2)$ with probability 0,2, under the theorem of addition: $P(Z = 4) = 0{,}5 + 0{,}2 = 0{,}7$. Thus, we have the following law of the random variable Z:

$$Z = \begin{pmatrix} 1 & 4 \\ 0{,}3 & 0{,}7 \end{pmatrix}$$

The sum (*the difference* or *the product*) of random variables X and Y is the random variable which takes on all possible values of kind $x_i + y_j$ ($x_i - y_j$ or $x_i \cdot y_j$) where $i = 1, 2, \dots, n;\ j = 1, 2, \dots, m$ with the probabilities p_{ij} that the random variable X will take on the value x_i, and Y – the value y_j:

$$p_{ij} = P[(X = x_i)(Y = y_j)].$$

If random variables X and Y are independent, i.e. any events $X = x_i$, $Y = y_j$ are independent, then by theorem of multiplication of probabilities for independent events

$$p_{ij} = P(X = x_i) \cdot P(Y = y_j) = p_i \cdot p'_j.$$

(Mathematical) expectation of a discrete random variable

One of the most important concepts in probability theory is the expectation of a random variable. If X is a discrete random variable having a probability mass function $p(x)$, the *(mathematical) expectation* (*the expected value* or *the mean*) of X, denoted by $M(X)$, is defined by

$$\boxed{M(X) = \sum_{x:\ p(x)>0} x \cdot p(x)}$$

In words, the expected value of X is a weighted average of the possible values that X can take on, each value being weighted by the probability that X assumes it. For instance, if the probability mass function of X is given by

$$p(0) = \frac{1}{2} = p(1)$$

then
$$M(X) = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$$

is just the ordinary average of the two possible values 0 and 1 that X can assume. On the other hand, if

$$p(0) = \frac{1}{3} \qquad p(1) = \frac{2}{3}$$

then
$$M(X) = 0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = \frac{2}{3}$$

is a weighted average of the two possible values 0 and 1.

Remark. The concept of expectation is analogous to the physical concept of the *center of gravity* of a distribution of mass. Consider a discrete random variable X having probability mass function $p(x_i)$, $i \geq 1$. If we now imagine a weightless rod in which weights with mass $p(x_i)$, $i \geq 1$, are located at the points x_i, $i \geq 1$, then the point at which the rod would be in balance is known as the center of gravity. For those readers acquainted with elementary statics it is now a simple matter to show that this point is at $M(X)$.

Example. The laws of distribution of random variables X and Y – the numbers of points beaten out by 1-st and 2-nd shooters are known:

$$X = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0{,}25 & 0{,}21 & 0{,}14 & 0{,}15 & 0{,}09 & 0{,}16 \end{pmatrix}$$

$$Y = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0{,}11 & 0{,}13 & 0{,}15 & 0{,}11 & 0{,}21 & 0{,}29 \end{pmatrix}$$

It is necessary to find out who of the shooters shoots better.

Solution: Obviously, a shooter shoots better than another shooter if he beats out more number of points on the average than another one.

$$M(X) = 0 \cdot 0{,}25 + 1 \cdot 0{,}21 + 2 \cdot 0{,}14 + 3 \cdot 0{,}15 + 4 \cdot 0{,}09 + 5 \cdot 0{,}16 = 2{,}10.$$

$$M(Y) = 0 \cdot 0{,}11 + 1 \cdot 0{,}13 + 2 \cdot 0{,}15 + 3 \cdot 0{,}11 + 4 \cdot 0{,}21 + 5 \cdot 0{,}29 = 3{,}05.$$

Thus, the second shooter shoots better than the first one on the average.

If a discrete random variable X takes on an infinite (countable) set of values $x_1, x_2, \dots, x_n, \dots$ then *the mathematical expectation* or *the expected value* of such a discrete random variable is the sum of the following series (if it absolutely converges): $M(X) = \sum_{i=1}^{\infty} x_i p_i$.

Property 1. The mathematical expectation of a constant is equal to the constant:

$$M(C) = C$$

Property 2. A constant multiplier can be taken out for a sign of mathematical expectation, i.e.

$$M(kX) = kM(X)$$

Property 3. The mathematical expectation of the algebraic sum of finitely many random variables is equal to the sum of their mathematical expectations, i.e.

$$M\left(\sum_{i=1}^{s} X_i\right) = \sum_{i=1}^{s} M(X_i).$$

Property 4. The mathematical expectation of the product of finitely many mutually independent random variables is equal to the product of their mathematical expectations:

$$M(\prod_{i=1}^{s} X_i) = \prod_{i=1}^{s} M(X_i)$$

Property 5. The mathematical expectation of deviation of a random variable from its mathematical expectation is equal to zero:

$$M[X - M(X)] = 0$$

Dispersion of a discrete random variable

Although $M(X)$ yields the weighted average of the possible values of X, it does not tell us anything about the variation, or spread, of these values. For instance, although random variables W, Y, and Z, having probability mass functions determined by

$$W = 0 \text{ with probability } 1$$

$$Y = \begin{cases} -1 & \text{with probability } \frac{1}{2} \\ +1 & \text{with probability } \frac{1}{2} \end{cases}$$

$$Z = \begin{cases} -100 & \text{with probability } \frac{1}{2} \\ +100 & \text{with probability } \frac{1}{2} \end{cases}$$

All have the same expectation – namely, 0 – there is much greater spread in the possible value of Y than in those of W (which is a constant) and in the possible values of Z than in those of Y.

As we expect X to take on values around its mean $M(X)$, it would appear that a reasonable way of measuring the possible variation of X would be to look at how far apart X would be from its mean on the average. In practice it is often required to estimate the dispersion (variation) of possible values of a random variable around of its average value. For example, in artillery it is important to know as far as shells will concentrically lie near to the target which should be struck.

One possible way to measure this would be to consider the quantity $M(|X - a|)$, where $a = M(X)$. However, it turns out to be mathematically inconvenient to deal with this quantity, and so a more tractable quantity is usually considered – namely, the expectation of the square of the difference between X and its mean. We thus have the following definition:

If X is a random variable with expectation $M(X)$, then *the dispersion (variance)* of X, denoted by $D(X)$, is defined by

$$\boxed{D(X) = M[(X - M(X))^2]}$$

An alternative formula for $D(X)$ is derived as follows:

$$D(X) = M[(X - M(X))^2] = \sum_x (x - M(X))^2 p(x) = \sum_x (x^2 - 2M(X)\cdot x +$$
$$+ (M(X))^2) p(x) = \sum_x x^2 p(x) - 2M(X)\sum_x xp(x) + (M(X))^2 \sum_x p(x) =$$
$$= M[X^2] - 2(M(X))^2 + (M(X))^2 = M[X^2] - (M(X))^2.$$

That is,

$$\boxed{D(X) = M[X^2] - (M(X))^2}$$

In words, the dispersion of X is equal to the expected value of X^2 minus the square of its expected value. This is, in practice, often the easiest way to compute $D(X)$.

Thus, for our example we have $D(W) = 0$, $D(Y) = 1$ and $D(Z) = 10000$.

<u>*Example.*</u> Calculate $D(X)$ if X represents the outcome when a die is rolled.
Solution: We have the following law of distribution:

X	1	2	3	4	5	6
p	1/6	1/6	1/6	1/6	1/6	1/6

Consequently,

$$M(X) = 1\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + 3\cdot\frac{1}{6} + 4\cdot\frac{1}{6} + 5\cdot\frac{1}{6} + 6\cdot\frac{1}{6} = \frac{7}{2}.$$

Also,

$$M(X^2) = 1^2\cdot\frac{1}{6} + 2^2\cdot\frac{1}{6} + 3^2\cdot\frac{1}{6} + 4^2\cdot\frac{1}{6} + 5^2\cdot\frac{1}{6} + 6^2\cdot\frac{1}{6} = \frac{91}{6}.$$

Hence,

$$D(X) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}.$$

Remark. Analogous to the mean being the center of gravity of a distribution of mass, the dispersion (variance) represents, in the terminology of mechanics, *the moment of inertia*.

The mean square deviation (*the standard deviation*) $\sigma(X)$ of a random variable X is the arithmetic value of the square root of its dispersion:

$$\boxed{\sigma(X) = \sqrt{D(X)}}$$

Property 1. The dispersion of a constant is equal to zero: $D(C) = 0$.
Property 2. A constant multiplier can be taken out from the argument of the dispersion involving it in square:

$$D(kX) = k^2 D(X)$$

Property 3. The dispersion of a random variable is equal to the difference between the mathematical expectation of the square of the random variable and the square of its mathematical expectation:

$$D(X) = M(X^2) - [M(X)]^2$$

Property 4. The dispersion of the algebraic sum of finitely many mutually independent random variables is equal to the sum of their dispersions:

$$D\left(\sum_{i=1}^{s} X_i\right) = \sum_{i=1}^{s} D(X_i).$$

Observe that *the dispersion of both the sum and the difference of independent random variables X and Y is equal to the sum of their dispersions*, i.e.

$$D(X + Y) = D(X - Y) = D(X) + D(Y).$$

The mathematical expectation, the dispersion and the mean square deviation are *numerical characteristics* of a random variable.

Glossary

shell – артиллерийский снаряд; **aim, sight** – прицел
exhaustive – исчерпывающий; **assignment** – задание

ascending order – возрастающий порядок
mathematical expectation, mean value – математическое ожидание
rod – стержень; **on the average** – в среднем; **deviation** – отклонение
concentration – кучность; **to strike** – поражать
dispersion, variance – дисперсия, рассеяние
mean square deviation – среднее квадратическое отклонение
numerical characteristic – числовая характеристика

Exercises for Seminar 7

7.1. Two balls are chosen randomly from an urn containing 8 white, 4 black and 2 orange balls. Suppose that we win $2 for each black ball selected and we lose $1 for each white ball selected. Let X denote our winnings. What are the possible values of X, and what are the probabilities associated with each value?

7.2. The probability of working each of four combines without breakages during a certain time is equal to 0,9. Compose the law of distribution of a random variable X – the number of combines working trouble-free. Find the mathematical expectation, the dispersion and the mean square deviation of the random variable X.

The answer: $M(X) = 3,6;\ D(X) = 0,36;\ \sigma(X) = 0,6.$

7.3. The probability of birth of a boy in a family is equal to 0,515. Compose the law of distribution of a random variable X – the number of boys in families having four children. Find the mathematical expectation, the dispersion and the mean square deviation.

The answer: $M(X) = 2,06;\ D(X) = 0,999;\ \sigma(X) = 1,0.$

7.4. There are 6 masters of sports in a group of 10 sportsmen. One selects (under the circuit without replacement) 3 sportsmen. Compose the law of distribution of a random variable X – the number of masters of sports of the selected sportsmen. Find the mathematical expectation of the random variable X.

The answer: $M(X) = 1,8$.

7.5. A shooter makes shots in a target before the first hit. The probability of hit in the target at each shot is equal to 0,7. Compose the law of distribution of a random variable X – the number of shots made by the shooter. Find the most probable number of cartridges (patrons) given to the shooter.

The answer: $k_0 = 1$.

7.6. The mathematical expectation of a random variable X is equal to 8. Find the mathematical expectation of the following random variables: a) $X – 4$; b) $3X + 4$.

7.7. The dispersion of a random variable X is equal to 8. Find the dispersion of the following random variables: a) $X – 2$; b) $3X + 2$.

7.8. Independent random variables X and Y have the following distributions:

X	2	4	6
p	0,3	0,5	0,2

Y	3	4
p	0,4	0,6

Compose the law of distribution of the random variable $Z = X + Y$. Find the mathematical expectation, the dispersion and the mean square deviation of the random variable Z.

The answer: $M(Z) = 7,4$; $D(Z) = 2,2$.

7.9. Find the mathematical expectation and the dispersion of random variable $Z = 4X - 2Y$ if $M(X) = 5$, $M(Y) = 3$, $D(X) = 4$, $D(Y) = 6$. The random variables X and Y are independent.

The answer: $M(Z) = 14$; $D(Z) = 88$.

7.10. A total of 4 buses carrying 148 students from the same school arrives at a football stadium. The buses carry, respectively, 40, 33, 25, and 50 students. One of the students is randomly selected. Let X denote the number of students that were on the bus carrying this randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on his bus. Which of $M(X)$ or $M(Y)$ do you think is larger? Why? Compute $M(X)$ and $M(Y)$.

Exercises for Homework 7

7.11. Two dice are rolled. Let X equal the sum of the 2 dice. What are the possible values of X, and what are the probabilities associated with each value?

7.12. The probability that a buyer will make a purchase in a shop is equal to 0,4. Compose the law of distribution of a random variable X – the number of buyers who have made a purchase if the shop was visited by 3 buyers. Find the mathematical expectation, the dispersion and the mean square deviation of the random variable X.

The answer: $M(X) = 1{,}2$; $D(X) = 0{,}72$; $\sigma(X) = 0{,}85$.

7.13. A buyer attends shops for purchasing the necessary goods. The probability that the goods are in a certain shop is equal to 0,4. Compose the law of distribution of a random variable X – the number of shops which will be attended by the buyer from four possible. Find the most probable number of shops which will be visited by the buyer.

The answer: $1 \le k_0 \le 2$.

7.14. A sample of 3 items is selected at random from a box containing 20 items of which 4 are defective. Find the expected number (mathematical expectation) of defective items in the sample.

The answer: 0,6.

7.15. A box contains 5 red and 5 blue marbles. Two marbles are withdrawn randomly. If they are the same color, then you win \$1.10; if they are different colors, then you win – \$1.00 (that is, you lose \$1.00). Calculate the mathematical expectation and the dispersion of the amount you win (marble – мрамор; to withdraw – извлекать).

The answer: $M(X) = -1/15$; $D(X) = 49/45$.

7.16. The mathematical expectation of a random variable X is equal to 7. Find the mathematical expectation of the following random variables:

a) $X + 6$; b) $4X - 3$.

7.17. The dispersion of a random variable X is equal to 9. Find the dispersion of the following random variables: a) $X + 6$; b) $2X - 7$.

7.18. Independent random variables X and Y have the following distributions:

X	2	4	6
p	0,3	0,5	0,2

Y	3	4
p	0,4	0,6

Compose the law of distribution of the random variable $V = XY$. Find the mathematical expectation, the dispersion and the mean square deviation of the random variable V.

The answer: $M(V) = 13{,}68$; $D(V) = 29{,}3376$.

7.19. Find the mathematical expectation and the dispersion of random variables:

a) $Z = 2X - 4Y$; b) $Z = 3X + 5Y$

if $M(X) = 5$, $M(Y) = 3$, $D(X) = 4$, $D(Y) = 6$. The random variables X and Y are independent.

The answer: a) $M(Z) = -2$; $D(Z) = 112$; b) $M(Z) = 30$; $D(Z) = 186$.