L E C T U R E 5
Theorem of addition of probabilities of compatible events

Two events are *compatible* if appearance of one of them doesn't exclude appearance of another event at the same trial.

Example. A – appearance of four aces at tossing a die; B – appearance of an even number of aces. The events A and B are compatible.

Let events A and B be compatible, and the probabilities of these events and the probability of their joint appearance be given. How can we find the probability of the event $A + B$ consisting in that at least one of the events A and B will appear?

Theorem. The probability of appearance of at least one of two compatible events is equal to the sum of the probabilities of these events without the probability of their joint appearance:

$$\boxed{P(A + B) = P(A) + P(B) - P(AB)}$$

Remark 1. Using the obtained formula one should remember that the events A and B can be both independent and dependent.
For independent events: $P(A + B) = P(A) + P(B) - P(A) \cdot P(B)$
For dependent events: $P(A + B) = P(A) + P(B) - P(A) \cdot P_A(B)$

Remark 2. If the events A and B are incompatible then their joint appearance is an impossible event and consequently, $P(AB) = 0$. Then for incompatible events A and B, $P(A + B) = P(A) + P(B)$.

Example. The probabilities of hit in a target at shooting by the first and the second guns are equal to: $p_1 = 0,7$; $p_2 = 0,8$ respectively. Find the probability of hit at one shot (by two guns) by at least one of the guns.
Solution: The probability of hit in the target by each of guns doesn't depend on result of shooting by another gun, therefore the events A (hit by the first gun) and B (hit by the second gun) are independent. The probability of the event AB (both the first and the second guns gave hit) $P(AB) = P(A) \cdot P(B) = 0,7 \cdot 0,8 = 0,56$.
The required probability is:

$$P(A + B) = P(A) + P(B) - P(AB) = 0,7 + 0,8 - 0,56 = 0,94.$$

Remark 3. Since in this example the events A and B are independent, we can use the formula $P(A+B) = 1 - P(\overline{A}) \cdot P(\overline{B})$ (the probability of appearance of at least one of the events). In fact, the probabilities of the events which are opposite to the events A and B, i.e. the probabilities of misses are:

$$P(\overline{A}) = 1 - P(A) = 1 - 0,7 = 0,3;\ \ P(\overline{B}) = 1 - P(B) = 1 - 0,8 = 0,2.$$

The required probability that at least one of guns gives hit at one shot is equal to

$$P(A+B) = 1 - P(\overline{A}) \cdot P(\overline{B}) = 1 - 0,3 \cdot 0,2 = 0,94.$$

Formula of total probability

Let an event A can be happened only in case of appearance of one of incompatible events B_1, B_2, …, B_n which form a complete group. Let both the probabilities of these events and the conditional probabilities $P_{B_1}(A),\ P_{B_2}(A), \ldots,\ P_{B_n}(A)$ of the event A be known. How can we find the probability of the event A? The answer on this question gives the following:

Theorem. The probability of an event A which can be happened only in case of appearance of one of incompatible events $B_1, B_2, \ldots, B_n$ forming a complete group is equal to the sum of products of the probabilities of each of these events on corresponding conditional probability of the event A:

$$\boxed{P(A) = P(B_1)\cdot P_{B_1}(A) + P(B_2)\cdot P_{B_2}(A) + \ldots + P(B_n)\cdot P_{B_n}(A)}$$

This formula is «the formula of total probability».

Example. There are two sets of items. The probability that an item of the first set is standard is equal to 0,8; and of the second set – 0,9. Find the probability that a randomly taken item (from a randomly taken set) is standard.
Solution: Denote by A the event «an extracted item is standard». An item can be extracted from either the first set (the event B_1) or the second set (the event B_2). The probability that an item has been extracted from the first set $P(B_1) = 1/2$. The probability that an item has been extracted from the second set $P(B_2) = 1/2$. The conditional probability that a standard item will be extracted from the first set $P_{B_1}(A) = 0{,}8.$ The conditional probability that a standard item will be extracted from the second set $P_{B_2}(A) = 0{,}9.$ The required probability that a randomly extracted item is standard is equal by the formula of total probability to

$$P(A) = P(B_1)\cdot P_{B_1}(A) + P(B_2)\cdot P_{B_2}(A) = 0{,}5\cdot 0{,}8 + 0{,}5\cdot 0{,}9 = 0{,}85.$$

Example. There are 20 radio lamps (including 18 standard ones) in the first box, and 10 radio lamps (including 9 standard ones) in the second box. A lamp has been taken randomly from the second box and placed to the first box. Find the probability that a lamp randomly extracted from the first box is standard.
Solution: Denote by A the event «a standard item has been extracted from the first box». One could be extracted from the second box either a standard item (the event B_1) or a non-standard item (the event B_2). The probability that a standard item has been extracted from the second box $P(B_1) = 9/10$. The probability that a non-standard item has been extracted from the second box $P(B_2) = 1/10$. The conditional probability that a standard item will be extracted from the first box provided that a standard lamp was placed from the second box to the first one $P_{B_1}(A) = 19/21.$ The conditional probability that a standard lamp will be extracted from the first box provided that a non-standard lamp was placed from the second box to the first one $P_{B_2}(A) = 18/21.$ The required probability that a standard lamp will be extracted from the first box is equal by the formula of total probability to

$$P(A) = P(B_1)\cdot P_{B_1}(A) + P(B_2)\cdot P_{B_2}(A) = \frac{9}{10}\cdot\frac{19}{21} + \frac{1}{10}\cdot\frac{18}{21} = 0{,}9.$$

Probability of hypotheses. Bayes's formulas.

Let an event A can happen only in case of appearance of one of incompatible events $B_1, B_2, \ldots, B_n$ forming a complete group. Since it isn't known beforehand which of these events will happen, we call them by *hypotheses*.
The probability of appearance of the event A is defined by the formula of total probability:

$$P(A) = P(B_1)\cdot P_{B_1}(A) + P(B_2)\cdot P_{B_2}(A) + \ldots + P(B_n)\cdot P_{B_n}(A) \quad (*)$$

Assume that a trial has been made in result of which the event A was appeared. Pose the problem to determine how the probabilities of the hypotheses have been changed (in connection with that the event A has already happened). In other words, we will look for the conditional probabilities

$P_A(B_1)$, $P_A(B_2)$, ..., $P_A(B_n)$. Find firstly the conditional probability $P_A(B_1)$. By theorem of multiplication we have $P(AB_1) = P(A)\cdot P_A(B_1) = P(B_1)\cdot P_{B_1}(A)$.
Consequently,

$$P_A(B_1) = \frac{P(B_1)\cdot P_{B_1}(A)}{P(A)}.$$

Replacing $P(A)$ by (*), we obtain

$$P_A(B_1) = \frac{P(B_1)\cdot P_{B_1}(A)}{P(B_1)\cdot P_{B_1}(A) + P(B_2)\cdot P_{B_2}(A) + \ldots + P(B_n)\cdot P_{B_n}(A)}.$$

Formulas determining the conditional probabilities of the remaining hypotheses are deduced analogously, i.e. the conditional probability of any hypothesis B_i ($i = 1, 2, ..., n$) can be calculated by the formula

$$\boxed{P_A(B_i) = \frac{P(B_i)\cdot P_{B_i}(A)}{P(B_1)\cdot P_{B_1}(A) + P(B_2)\cdot P_{B_2}(A) + \ldots + P(B_n)\cdot P_{B_n}(A)}}$$

The obtained formulas are the *Bayes's formulas* (on name of British mathematician who deduced them; published in 1764).
The Bayes's formulas allow overestimating the probabilities of hypotheses after that a trial has been made in result of which the required event has appeared.

Example. Items produced by a factory shop are given for checking them on standard to one of two controllers. The probability that an item will be given to the first controller is equal to 0,6; and to the second – 0,4. The probability that a suitable item will be recognized standard by the first controller is equal to 0,94; and by the second controller – 0,98. A randomly chosen item that is suitable has been recognized standard at checking. Find the probability that this item was checked by the first controller.
Solution: Denote by A the event consisting in that a suitable item has been recognized standard. One can make two assumptions: 1) The item has been checked by the first controller (the hypothesis B_1); 2) The item has been checked by the second controller (the hypothesis B_2).
Find the required probability that the item has been checked by the first controller by the Bayes's formula: $P_A(B_1) = \dfrac{P(B_1)\cdot P_{B_1}(A)}{P(B_1)\cdot P_{B_1}(A) + P(B_2)\cdot P_{B_2}(A)}$

We have: $P(B_1) = 0,6$; $P(B_2) = 0,4$; $P_{B_1}(A) = 0,94$; $P_{B_2}(A) = 0,98$.

The required probability $P_A(B_1) = \dfrac{0,6\cdot 0,94}{0,6\cdot 0,94 + 0,4\cdot 0,98} \approx 0,59.$

We see the probability of the hypothesis B_1 equals *0,6* before the trial, and after that a trial result has become known the probability of this hypothesis (more precise, the conditional probability) has been changed and become 0,59. Thus, using the Bayes's formula allowed overestimating the probability of the considered hypothesis.

Glossary

compatible events – совместные события; **hypothesis** – гипотеза
beforehand – заранее; **to recognize** – признавать, распознавать
a factory shop – цех завода; **suitable item** – годная деталь
to overestimate – переоценить

Exercises for Seminar 5

5.1. There are 20 skiers, 6 bicyclists and 4 runners in a group of sportsmen. The probability to fulfil the corresponding qualifying norm is: for a skier – 0,9, for a bicyclist – 0,8, and for a runner – 0,75. Find the probability that a randomly chosen sportsman will fulfil the norm (to fulfil – выполнить).

The answer: 0,86.

5.2. The first box contains 20 items and 15 of them are standard; the second – 30 items and 24 of them are standard; the third – 10 items and 6 of them are standard. Find the probability that a randomly extracted item from a randomly taken box is standard.

The answer: 43/60.

5.3. There are radio lamps in two boxes. The first box contains 12 lamps, and 1 of them is non-standard; the second box contains 10 lamps, and 1 of them is non-standard. A lamp is randomly taken from the first box and placed in the second. Find the probability that a randomly extracted lamp from the second box will be non-standard.

The answer: 13/132.

5.4. At a deviation of an automatic device from the normal operating mode the signaling device C-1 acts with the probability 0,8, and the signaling device C-11 acts with the probability 1. The probabilities that the automatic device is supplied with C-1 or C-11 are equal to 0,6 and 0,4 respectively. A signal about cutting the automatic device has been received. What is more probable: the automatic device is supplied with the signaling device C-1 or C-11?

The answer: The probability that the automatic device is supplied with C-1, is equal to 6/11, and C-11 – 5/11.

5.5. The probability for products of a certain factory to satisfy the standard is equal to 0,96. A simplified system of checking on standardness gives positive result with the probability 0,98 for products satisfying the standard, and with the probability 0,05 – for products non-satisfying the standard. A randomly taken product has been recognized as standard at checking. Find the probability that it really satisfies the standard.

The answer: 0,998.

5.6. A digit is firstly randomly chosen from the digits {1, 2, 3, 4, 5}, and then the second digit – from the remaining four digits. Find the probability that an odd digit will be chosen:

a) for the first time; b) for the second time; c) in both times.

The answer: a) 3/5; b) 3/5; c) 3/10.

5.7. Three cards are randomly selected without replacement from an ordinary deck of 52 playing cards. Compute the conditional probability that the first card selected is a spade, given that the second and third cards are spades.

The answer: 0,22.

5.8. Urn A contains 2 white balls and 1 black ball, whereas urn B contains 1white ball and 5 black balls. A ball is drawn at random from urn A and placed in urn B. A ball is then drawn from urn B. It happens to be white. What is the probability that the ball transferred was white?

The answer: 0,8.

Exercises for Homework 5

5.9. A collector has received 3 boxes of items made by the factory № 1, and 2 boxes of items made by the factory № 2. The probability that an item of the factory № 1 is standard is equal to 0,8, and the factory № 2 – 0,9. The collector has randomly extracted an item from a randomly taken box. Find the probability that a standard item has been extracted (a collector – сборщик).

The answer: 0,84.

5.10. There are 4 kinescopes in a television studio. The probabilities that the kinescope will sustain the warranty period of service are equal to 0,8; 0,85; 0,9; 0,95 respectively. Find the probability that a randomly taken kinescope will sustain the warranty period of service (to sustain – выдержать).

The answer: 0,875.

5.11. A die has been randomly extracted from the full set of 28 dice of domino. Find the probability that the second randomly extracted die can be put to the first.

The answer: 7/18.

5.12. For participation in student selective sport competitions 4 students has been directed from the first group, 6 – from the second, 5 – from the third group. The probabilities that a student of the first, second and third group gets in the combined team of institute, are equal to 0,9; 0,7 and 0,8 respectively. A randomly chosen student as a result of competition has got in the combined team. Which of groups is this student most likely belonged to (a combined team – сборная)?

The answer: The probabilities that the student has been chosen from the first, second and third group are equal to 18/59, 21/59, 20/59 respectively.

5.13. English and American spellings are *rigour* and *rigor*, respectively. A man staying at a Parisian hotel writes this word, and a letter taken at random from his spelling is found to be a vowel. If 40 percent of the English-speaking men at the hotel are English and 60 percent are Americans, what is the probability that the writer is an Englishman (rigour (rigor) – суровость; a vowel – гласная)?

The answer: 5/11.

5.14. Urn A has 5 white and 7 black balls. Urn B has 3 white and 12 black balls. We flip a coin. If the outcome is heads, then a ball from urn A is selected, whereas if the outcome is tails, then a ball from urn B is selected. Suppose that a white ball is selected. What is the probability that the coin landed on tails (to flip – подбросить)?

The answer: 12/37.

5.15. There are four urns. The first urn contains 1 white and 1 black ball, the second – 2 white and 3 black balls, the third – 3 white and 5 black balls, and the fourth – 4 white and 7 black balls. The event H_i is the choosing the i-th urn (i = 1, 2, 3, 4). It is known that the probability of choosing the i-th urn is equal to i/10. A ball is randomly extracted from a randomly chosen urn. Find the probability that a randomly extracted ball is white.

The answer: 0,388.