L E C T U R E 4
Independent events

An event B is said to be *independent from an event A* if appearance of the event A does not change the probability of the event B, i.e. if the conditional probability of the event B is equal to its unconditional probability:

$$P_A(B) = P(B) \quad (*)$$

Since $P(A) \cdot P_A(B) = P(B) \cdot P_B(A)$, by using (*) we have $P(A) \cdot P(B) = P(B) \cdot P_B(A)$ and consequently $P_B(A) = P(A)$, i.e. the event A does not depend from the event B.
Thus, if an event B does not depend from an event A then the event A does not depend from the event B; this means that *the property of independence is mutual*.

For independent events theorem of multiplication $P(AB) = P(A) \cdot P_A(B)$ has the following form:

$$\boxed{P(AB) = P(A) \cdot P(B)} \quad (**)$$

i.e. the probability of joint appearance of two independent events is equal to the product of the probabilities of these events. The equality (**) is accepted as a definition of independent events.

Two events are *independent* if the probability of their joint appearance is equal to the product of the probabilities of these events; otherwise they are *dependent*.
At practice one concludes on independence of events on the sense of a problem. For example, the probability of hit in a target by each of two guns does not depend on that whether another gun has hit in the target, therefore the events «the first gun has hit in the target» and «the second gun has hit in the target» are independent.

<u>Example.</u> Find the probability of joint hit in a target by two guns if the probability of hit in the target by the first gun (the event A) is equal to 0,8; and by the second gun (the event B) – 0,7.
Solution: The events A and B are independent, therefore by theorem of multiplication, the required probability $P(AB) = P(A) \cdot P(B) = 0{,}8 \cdot 0{,}7 = 0{,}56$.

Remark. If events A and B are independent, the events A and $\overline{B}$, $\overline{A}$ and B, $\overline{A}$ and $\overline{B}$ are also independent.
Proof of Remark:

$$A = A\overline{B} + AB \Rightarrow P(A) = P(A\overline{B}) + P(AB) = P(A\overline{B}) + P(A) \cdot P(B).$$

Then we have $P(A\overline{B}) = P(A) \cdot [1 - P(B)] = P(A) \cdot P(\overline{B}),$ i.e. the events A and $\overline{B}$ are independent. □

Several events are *pairwise independent* if each two of them are independent. For example, events A, B and C are pairwise independent if the events A and B, A and C, B and C are independent.
Several events are *independent in union* (or just *independent*) if each two of them are independent and each event and all possible products of the rest events are independent. For example, if events A_1, A_2 and A_3 are independent in union then the events A_1 and A_2, A_1 and A_3, A_2 and A_3, A_1 and A_2A_3, A_2 and A_1A_3, A_3 and A_1A_2 are independent.
Let's underline that a pairwise independence of several events does not imply their independence in union in general, i.e. the demand of independence of events in union is stronger than the demand of their pairwise independence.

<u>Example.</u> There are 4 coloured balls in an urn: one ball is coloured in red colour (A), one ball is coloured in blue colour (B), one ball is coloured in black colour (C) and one ball – in all these three

colours (*ABC*). Find the probability that a ball extracted at random from the urn has red colour. Since two of four balls have red colour, $P(A) = 2/4 = 1/2$. Reasoning analogously, we have $P(B) = 1/2$, $P(C) = 1/2$. Assume now that the taken ball has blue colour, i.e. the event B has already happened. Is the probability that the extracted ball has red colour changed, i.e. is the probability of the event A changed? One ball of two balls having blue colour has also red colour, therefore the probability of the event A is still equal to 1/2. In other words, the conditional probability of the event A calculated in assumption that the event B has happened is equal to its unconditional probability. Consequently, the events A and B are independent. By analogy we have that the events A and C, B and C are independent. Thus, the events A, B and C are pairwise independent. Are these events independent in union? Let the extracted ball have two colours, for example, blue and black. What is the probability that this ball has also red colour? Since only one ball is coloured in all three colours, therefore the taken ball has also red colour. Thus, assuming that the events B and C have happened, we have the event A will necessarily happen. Consequently, this event is reliable and its probability is equal to 1. In other words, the conditional probability $P_{BC}(A) = 1$ of the event A is not equal to its unconditional probability $P(A) = 1/2$. Thus, the pairwise independent events A, B and C are not independent in union.

Corollary. The probability of joint appearance of several events that are independent in union is equal to the product of the probabilities of these events:

$$P(A_1A_2...A_n) = P(A_1) \cdot P(A_2) \cdot ... \cdot P(A_n).$$

Remark. If events $A_1, A_2, ..., A_n$ are independent in union, then the opposite to them events $\overline{A_1}, \overline{A_2}, \dots, \overline{A_n}$ are also independent in union.

<u>*Example.*</u> Find the probability of a joint appearance of heads at tossing two coins.
Solution: The probability of appearance of heads on the first coin (the event A) $P(A) = 1/2$. The probability of appearance of heads on the second coin (the event B) $P(B) = 1/2$. The events A and B are independent; therefore the required probability by theorem of multiplication is equal to:

$$P(AB) = P(A) \cdot P(B) = 1/2 \cdot 1/2 = 1/4.$$

<u>*Example.*</u> There are 3 boxes containing 10 items each. There are 8 standard items in the first box, 7 – in the second and 9 – in the third box. One takes at random on one item from each box. Find the probability that all three taken items will be standard.
Solution: The probability that a standard item has been taken from the first box (the event A) $P(A) = 8/10 = 0,8$. The probability that a standard item has been taken from the second box (the event B) $P(B) = 7/10 = 0,7$. The probability that a standard item has been taken from the third box (the event C) $P(C) = 9/10 = 0,9$. Since the events A, B and C are independent in union, the required probability (by theorem of multiplication) is equal to:

$$P(ABC) = P(A) \cdot P(B) \cdot P(C) = 0,8 \cdot 0,7 \cdot 0,9 = 0,504.$$

Probability of appearance of at least one event

Let as a result of a trial n events independent in union or some of them (in particular, only one or none) can appear, so that the probabilities of appearance of each of the events are known. How can we find the probability that at least one of these events will happen? For example, if as a result of a trial three events can appear, then an appearance of at least one of these events means an appearance of either one, or two, or three events. The answer on the posed question is given by the following theorem.
Theorem. The probability of appearance of at least one of the events $A_1, A_2, ..., A_n$ independent in union is equal to the difference between 1 and the product of the probabilities of the opposite events $\overline{A_1}, \overline{A_2}, ..., \overline{A_n}$:

$$\boxed{P(A) = 1 - q_1 q_2 \ldots q_n}$$

where A is the appearance of at least one of the events $A_1, A_2, \ldots, A_n$; $P(\overline{A_i}) = q_i,\ i = \overline{1,n}$.

<u>*Partial case*</u>. If the events $A_1, A_2, \ldots, A_n$ have the same probability which is equal to p then the probability of appearance of at least one of these events:

$$\boxed{P(A) = 1 - q^n}$$

where A is the appearance of at least one of the events $A_1, A_2, \ldots, A_n$; $P(\overline{A_i}) = q = 1 - p,\ i = \overline{1,n}$.

<u>*Example.*</u> The probabilities of hit in a target at shooting by three guns are the following: $p_1 = 0{,}8$; $p_2 = 0{,}7$; $p_3 = 0{,}9$. Find the probability of at least one hit (the event A) at one shot by all three guns.
Solution: The probability of hit in the target by each of the guns doesn't depend on results of shooting by other guns, therefore the considered events A_1 (hit by the first gun), A_2 (hit by the second gun) and A_3 (hit by the third gun) are independent in union. The probabilities of events which are opposite to the events A_1, A_2 and A_3 (i.e. the probabilities of misses) are equal respectively:

$$q_1 = 1 - p_1 = 1 - 0{,}8 = 0{,}2;\quad q_2 = 1 - p_2 = 1 - 0{,}7 = 0{,}3;$$
$$q_3 = 1 - p_3 = 1 - 0{,}9 = 0{,}1.$$

The required probability $P(A) = 1 - q_1q_2q_3 = 1 - 0{,}2 \cdot 0{,}3 \cdot 0{,}1 = 0{,}994$.

<u>*Example.*</u> There are 4 flat-printing machines at typography. For each machine the probability that it works at the present time is equal to $0{,}9$. Find the probability that at least one machine works at the present time (the event A).
Solution: The events «a machine works» and «a machine doesn't work» (at the present time) are opposite, therefore the sum of their probabilities is equal to 1: $p + q = 1$. Consequently, the probability that a machine doesn't work at the present time is equal to $q = 1 - p = 1 - 0{,}9 = 0{,}1$.
The required probability $P(A) = 1 - q^4 = 1 - (0{,}1)^4 = 0,\ 9999$.

<u>*Example.*</u> A student looks for one formula necessary to him in three directories. The probability that the formula is contained in the first, second and third directories, is equal to 0,6; 0,7 and 0,8 respectively. Find the probability that the formula is contained:

a) only in one directory (the event A);
b) only in two directories (the event B);
c) in all the directories (the event C);
d) at least in one directory (the event D);
e) neither of the directories (the event E).

Solution: Consider elementary events and their probabilities:

A_1 – the formula is in the first directory, $P(A_1) = 0{,}6;\quad P(\overline{A}_1) = 1 - 0{,}6 = 0{,}4;$

A_2 – the formula is in the second directory, $P(A_2) = 0{,}7;\quad P(\overline{A}_2) = 1 - 0{,}7 = 0{,}3;$

A_3 – the formula is in the third directory, $P(A_3) = 0{,}8;\quad P(\overline{A}_3) = 1 - 0{,}8 = 0{,}2.$

Express all the events A–E by the elementary events and their negations, and apply the above-stated theorems:

a) $A = A_1 \cdot \overline{A}_2 \cdot \overline{A}_3 + \overline{A}_1 \cdot A_2 \cdot \overline{A}_3 + \overline{A}_1 \cdot \overline{A}_2 \cdot A_3,$

$$P(A)=P(A_1\cdot\overline{A}_2\cdot\overline{A}_3)+P(\overline{A}_1\cdot A_2\cdot\overline{A}_3)+P(\overline{A}_1\cdot\overline{A}_2\cdot A_3)=$$
$$=P(A_1)\cdot P(\overline{A}_2)\cdot P(\overline{A}_3)+P(\overline{A}_1)\cdot P(A_2)\cdot P(\overline{A}_3)+P(\overline{A}_1)\cdot P(\overline{A}_2)\cdot P(A_3)=$$
$$=0{,}6\cdot 0{,}3\cdot 0{,}2+0{,}4\cdot 0{,}7\cdot 0{,}2+0{,}4\cdot 0{,}3\cdot 0{,}8=0{,}188.$$

b) $B=A_1\cdot A_2\cdot\overline{A}_3+A_1\cdot\overline{A}_2\cdot A_3+\overline{A}_1\cdot A_2\cdot A_3,$

$$P(B)=P(A_1\cdot A_2\cdot\overline{A}_3)+P(A_1\cdot\overline{A}_2\cdot A_3)+P(\overline{A}_1\cdot A_2\cdot A_3)=$$
$$=P(A_1)\cdot P(A_2)\cdot P(\overline{A}_3)+P(A_1)\cdot P(\overline{A}_2)\cdot P(A_3)+P(\overline{A}_1)\cdot P(A_2)\cdot P(A_3)=$$
$$=0{,}6\cdot 0{,}7\cdot 0{,}2+0{,}6\cdot 0{,}3\cdot 0{,}8+0{,}4\cdot 0{,}7\cdot 0{,}8=0{,}452.$$

c) $C=A_1\cdot A_2\cdot A_3,$

$P(C)=P(A_1\cdot A_2\cdot A_3)=P(A_1)\cdot P(A_2)\cdot P(A_3)=0{,}6\cdot 0{,}7\cdot 0{,}8=0{,}336.$

d) $P(D)=1-P(\overline{A}_1)\cdot P(\overline{A}_2)\cdot P(\overline{A}_3)=1-0{,}4\cdot 0{,}3\cdot 0{,}2=0{,}976.$

e) $E=\overline{A}_1\cdot\overline{A}_2\cdot\overline{A}_3,\quad P(E)=0{,}4\cdot 0{,}3\cdot 0{,}2=0{,}024.$

Glossary

independent events – независимые события
mutual – взаимный; **independent in union** – независимые в совокупности
flat-printing machine – плоскопечатная машина
directory – справочник

Exercises for Seminar 4

4.1. The probability that a shooter hit in a target at one shot is equal to 0,9. The shooter has made 3 shots. Find the probability that all 3 shots will strike the target.

4.2. A coin and a die are tossed. Find the probability of joint appearance of the following events: «the coin lands on heads» and «the die lands on 6».

4.3. What is the probability that at tossing three dice 6 aces will appear at least on one of the dice (the event A)?

The answer: 0,421.

4.4. There are 8 standard items in a batch of 10 items. Find the probability that there is at least one standard item among two randomly taken items.

4.5. Two dice are rolled. What is the conditional probability that at least one lands on 6 given that the dice land on different numbers?

4.6. The probability of hit in a target by the first shooter at one shot is equal to 0,8, and by the second shooter – 0,6. Find the probability that the target will be struck only with one shooter.

The answer: 0,44.

4.7. The probability to receive high dividends under shares at the first enterprise – 0,2; on the second – 0,35; on the third – 0,15. Determine the probability that a shareholder having shares of all the enterprises will receive high dividends:

a) only at one enterprise;

b) at least on one enterprise (a share – акция).

The answer: a) 0,4265; b) 0,558.

4.8. The first brigade has 6 tractors, and the second – 9. One tractor demands repair in each brigade. A tractor is chosen at random from each brigade. What is the probability that:

a) both chosen tractors are serviceable;

b) one of the chosen tractors demands repair (serviceable – исправный).

The answer: a) 20/27; b) 13/54.

Exercises for Homework 4

4.9. There are items in two boxes: in the first – 10 (3 of them are standard), in the second – 15 (6 of them are standard). One takes out at random on one item from each box. Find the probability that both items will be standard.

The answer: 0,12.

4.10. There are 3 television cameras in a TV studio. For each camera the probability that it is turned on at present, is equal to p = 0,6. Find the probability that at least one camera is turned on at present (the event A).

The answer: 0,936.

4.11. What is the probability that at least one of a pair of dice lands on 6, given that the sum of the dice is 8?

The answer: 0,4.

4.12. 10 of 20 savings banks are located behind a city boundary. 5 savings banks are randomly selected for an inspection. What is the probability that among the selected banks appears inside the city:

a) 3 savings banks; b) at least one?

The answer: a) 0,348; b) 0,984.

4.13. There are 16 items made by the factory № 1 and 4 items of the factory № 2 at a collector. Two items are randomly taken. Find the probability that at least one of them has been made by the factory № 1.

The answer: 92/95.

4.14. Three buyers went in a shop. The probability that each buyer makes purchases is equal to 0,3. Find the probability that:

a) two of them will make purchases;

b) all three will make purchases;

c) only one of them will make purchases.

The answer: a) 0,189; b) 0,027; c) 0,441.

4.15. Three students pass an exam. The probability that the exam will be passed on "excellent" by the first student is equal to 0,7; by the second – 0,6; and by the third – 0,2. What is the probability that the exam will be passed on "excellent" by:

a) only one student; b) two students;

c) at least one; d) neither of the students?

The answer: a) 0,392; b) 0,428; c) 0,904; d) 0,096.

4.16. Three shots are made in a target. The probability of hit at each shot is equal to 0,6. Find the probability that only one hit will be in result of these shots.

The answer: 0,288.