# L E C T U R E 9
## Basic laws of distribution of discrete random variables

### 1. Binomial law of distribution

A discrete random variable $X$ has *a binomial law of distribution* with parameters $n$ and $p$ if it takes on values 0, 1, 2, …, $m$, …, $n$ with probabilities

$$\boxed{P(X=m)=C_n^m p^m q^{n-m}}$$

where $0 < p < 1$, $q = 1 - p$.

The binomial law of distribution represents the law of distribution of the number $X = m$ of occurrences of an event $A$ in $n$ independent trials in each of which it can occur with the same probability.

The series of distribution of a binomial law has the following form:

| $x_i$ | 0 | 1 | 2 | … | $m$ | … | $n$ |
|---|---|---|---|---|---|---|---|
| $p_i$ | $q^n$ | $C_n^1 p^1 q^{n-1}$ | $C_n^2 p^2 q^{n-2}$ | … | $C_n^m p^m q^{n-m}$ | … | $p^n$ |

Obviously, $\sum_{i=0}^{n} p_i = 1$ because $\sum_{i=0}^{n} p_i$ is the sum of all members of decomposition of Newton binomial:

$$q^n + C_n^1 pq^{n-1} + C_n^2 p^2 q^{n-2} + \ldots + C_n^m p^m q^{n-m} + \ldots + p^n = (q+p)^n = 1^n = 1.$$

Therefore, the law is said to be binomial.

**Theorem.** The mathematical expectation of a random variable $X$ distributed under a binomial law is $M(X) = np$, and its dispersion $D(X) = npq$.

The binomial law of distribution is widely used in the theory and practice of statistical control by a production quality, at the description of functioning of systems of mass service, at modeling the prices of actives, in the theory of shooting and in other areas.

<u>*Example*</u>. Footwear has arrived in a shop from two factories in the ratio 2:3. 4 pairs of footwear have been bought. Find the law of distribution of the number of the bought pairs of footwear made by the first factory. Find the mathematical expectation and the dispersion of this random variable.
*Solution*: The probability that a randomly chosen pair of footwear has been made by the first factory is $p = 2/(2 + 3) = 0{,}4$. The random variable $X$ – the number of bought pairs of footwear made by the first factory among 4 selling pairs – has the binomial law of distribution with parameters $n = 4$, $p = 0{,}4$. The series of distribution of $X$ has the following form:

| $x_i$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $p_i$ | 0,1296 | 0,3456 | 0,3456 | 0,1536 | 0,0256 |

The values $p_i = P(X = m)$, where $m$ = 0, 1, 2, 3, 4, are calculated by the formula:

$$P(X=m)=C_4^m \cdot 0{,}4^m \cdot 0{,}6^{4-m}$$

Find the mathematical expectation and the dispersion of the random variable $X$:

$$M(X) = np = 4 \cdot 0{,}4 = 1{,}6;\ D(X) = npq = 4 \cdot 0{,}4 \cdot 0{,}6 = 0{,}96.$$

**2. The law of Poisson distribution**

A discrete random variable $X$ has *the law of Poisson distribution* with parameter $\lambda > 0$ if it takes on values 0, 1, 2, …, $m$, …(infinite countable set of values) with probabilities

$$\boxed{P(X = m) = \frac{\lambda^m e^{-\lambda}}{m!}}$$

The series of distribution of the Poisson law has the following form:

| $x_i$ | 0 | 1 | 2 | … | $m$ | … |
|---|---|---|---|---|---|---|
| $p_i$ | $e^{-\lambda}$ | $\lambda e^{-\lambda}$ | $\lambda^2 e^{-\lambda}/2!$ | … | $\lambda^m e^{-\lambda}/m!$ | … |

Since the sum of the series

$$\sum_{i=0}^{\infty} p_i = e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2 e^{-\lambda}}{2!} + \ldots + \frac{\lambda^m e^{-\lambda}}{m!} + \ldots =$$

$$= e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} + \ldots + \frac{\lambda^m}{m!} + \ldots\right) = e^{-\lambda} \cdot e^{\lambda} = 1,$$

the basic property of distribution series $\sum_{i=0}^{n} p_i = 1$ holds, and consequently the Poisson law is well-defined.

The Poisson probability distribution was introduced by S.D. Poisson in a book he wrote regarding the application of probability theory to lawsuits, criminal trials, and the like.

**Theorem.** The mathematical expectation and the dispersion of a random variable distributed under a Poisson law coincide and are equal to the parameter $\lambda$ of the law, i.e. $M(X) = \lambda$, $D(X) = \lambda$.

The Poisson random variable has a tremendous range of applications in diverse areas because it may be used as an approximation for a binomial random variable with parameters ($n$, $p$) when $n$ is large and $p$ is small enough so that $np$ is a moderate size. In other words, if $n$ independent trials, each of which results in a success with probability $p$, are performed, then, when $n$ is large and $p$ small enough to make $np$ moderate, the number of successes occurring is approximately a Poisson random variable with parameter $\lambda = np$.

Some examples of random variables that usually obey the Poisson probability law follow:

1. The number of misprints on a page (or a group of pages) of a book.
2. The number of people in a community living to 100 years of age.
3. The number of wrong telephone numbers that are dialed in a day.
4. The number of packages of dog biscuits sold in a particular store each day.
5. The number of customers entering a post office on a given day.
6. The number of vacancies occurring during a year in the federal judicial system.
7. The number of $\alpha$-particles discharged in a fixed period of time from some radioactive material.

Each of the preceding, and numerous other random variables, are approximately Poisson for the same reason – namely, because of the Poisson approximation to the binomial. For instance, we can suppose that there is a small probability $p$ that each letter typed on a page will be misprinted. Hence the number of misprints on a page will be approximately Poisson with $\lambda = np$, where $n$ is the number of letters on a page.

*Example.* Suppose that the number of typographical errors on a single page of this book has a Poisson distribution with parameter $\lambda = 1/2$. Calculate the probability that there is at least one error on this page.
*Solution:* Letting $X$ denote the number of errors on this page, we have

$$P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-1/2} = 0{,}393.$$

*Example.* Suppose that the probability that an item produced by a certain machine will be defective is 0,1. Find the probability that a sample of 10 items will contain at most 1 defective item.
*Solution:* The desired probability is

$$P(X = 0) + P(X = 1) = C_{10}^{0}(0{,}1)^{0}(0{,}9)^{10} + C_{10}^{1}(0{,}1)^{1}(0{,}9)^{9} = 0{,}7361,$$

whereas the Poisson approximation yields the value

$$P(X = 0) + P(X = 1) = e^{-1} + e^{-1} \approx 0{,}7358.$$

**3. Geometric distribution**

A discrete random variable $X$ has *a geometric distribution* with the parameter $p$ if it takes on values 1, 2, …, $m$, … (infinite countable set of values) with probabilities

$$\boxed{P(X = m) = pq^{m-1}}$$

where $0 < p < 1$, $q = 1 - p$.

The series of a geometric distribution has the following form:

| | | | | | | |
|---|---|---|---|---|---|---|
| $x_i$ | 1 | 2 | 3 | … | $m$ | … |
| $p_i$ | $p$ | $pq$ | $pq^2$ | … | $pq^{m-1}$ | … |

It is easy to see that the probabilities $p_i$ form the geometric progression with the first member $p$ and denominator $q$ (therefore, the law is said to be geometric).

Since $\sum p_i = p + pq + \ldots + pq^{m-1} + \ldots = p\left(1 + q + \ldots + q^{m-1} + \ldots\right) = p \cdot \dfrac{1}{1-q} = 1,$ the geometric distribution is well-defined.
A random variable $X$ having a geometric distribution represents the number $m$ of trials which have been carried out under Bernoulli circuit with probability $p$ of occurrence of the event in each trial till the first positive outcome.

**Theorem.** The mathematical expectation of a random variable $X$ having the geometrical distribution with parameter $p$ is $M(X) = 1/p$, and its dispersion $D(X) = q/p^2$ where $q = 1 - p$.

*Example*. Testing a big batch of items up to detection of a rejected item (without restriction of the number of tested items) is carried out. Compose the law of distribution of the number of tested items. Find its mathematical expectation and dispersion if it is known that the probability of reject for each item is equal to 0,1.
*Solution*: The random variable $X$ – the number of tested items up to detection of rejected item – has geometrical distribution with the parameter $p$ = 0,1. Therefore, the series of distribution has the following form:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x_i$ | 1 | 2 | 3 | 4 | … | $m$ | … |
| $p_i$ | 0,1 | 0,09 | 0,081 | 0,0729 | … | $0{,}9^{m-1} \cdot 0{,}1$ | … |

$M(X) = 1/p$ = 1/0,1 = 10; $D(X) = q/p^2$ = $0{,}9/(0{,}1)^2$ = 90.

**4. Hypergeometric distribution**

Hypergeometric distribution is widely used in practice of statistical acceptance control by quality of industrial production, in the problems connected to the organization of sampling inspections, and other areas.

A discrete random variable $X$ has *the hypergeometric distribution* with parameters $n, M, N$ if it takes on values 0, 1, 2, …, $m$, …, $min(n, M)$ with the probabilities

$$\boxed{P(X = m) = \frac{C_M^m C_{N-M}^{n-m}}{C_N^n}}$$

where $M \le N$, $n \le N$; $n, M, N$ are natural numbers.

Let there be $M$ standard items in a batch of $N$ items. $n$ items are randomly selected from the batch (each item can be extracted with the same probability), and the selected item is not replaced in the batch before selecting the next item (therefore the Bernoulli formula here is not applicable). Then the random variable $X$ which is the number $m$ of standard items among $n$ selected items has hypergeometric distribution.

**Theorem.** The mathematical expectation of a random variable $X$ having a hypergeometric distribution with parameters $n$, $M$, $N$ is $M(X) = n\frac{M}{N}$, and its dispersion $D(X) = n\frac{M}{N-1}\left(1-\frac{M}{N}\right)\left(1-\frac{n}{N}\right)$.

<u>*Example*</u>. In a lottery «Sportloto 6 of 45» monetary prizes are received by participants who have guessed 3, 4, 5 and 6 kinds of sports from randomly selected 6 kinds of 45 (the size of a prize increases with an increasing the number of guessed kinds of sports). Find the law of distribution of a random variable $X$ – the number of guessed kinds of sports among randomly selected 6 kinds. What is the probability of receiving a monetary prize? Find the mathematical expectation and the dispersion of the random variable $X$.

*Solution:* Obviously, the number of guessed kinds of sports in the lottery "6 of 45" is a random variable having hypergeometric distribution with the parameters $n$ = 6, $M$ = 6, $N$ = 45. The series of its distribution has the following form:

| $x_i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $p_i$ | 0,40056 | 0,42413 | 0,15147 | 0,02244 | 0,00137 | 0,00003 | 0,0000001 |

The probability of receiving a monetary prize

$$P(3 \le X \le 6) = \sum_{i=3}^{6} P(X = i) = 0{,}02244 + 0{,}00137 + 0{,}00003 + 0{,}0000001 =$$

$$= 0{,}02384 \approx 0{,}024.$$

$M(X) = n \cdot M/N$ = 6 · 6/45 = 0,8; $D(X)$ = 6 · 39/44 (1 – 39/45)(1 – 6/45) = 0,6145.

**Glossary**

**binomial** – биномиальный; **Poisson** – Пуассон
**lawsuit** – судебный процесс; **tremendous** – огромный
**moderate** – небольшой, доступный; **diverse areas** – разнообразные области
**particle** – частица; **to discharge** – разряжать
**circuit** – схема; **hypergeometric** – гипергеометрический

## Exercises for Seminar 9

9.1. A die is tossed three times. Write the law of distribution of the number of appearance of 6.

9.2. Find an average number (mathematical expectation) of typing errors on page of the manuscript if the probability that the page of the manuscript contains at least one typing error is 0,95. It is supposed that the number of typing errors is distributed under the Poisson law (typing error – опечатка; an average number – среднее число).

*The answer*: 3.

9.3. The switchboard of an enterprise serves 100 subscribers. The probability that a subscriber will call on the switchboard within 1 minute is equal 0,02. Which of two events is more probable: 3 subscribers will call or 4 subscribers will call within 1 minute? (Subscriber – абонент, switchboard – коммутатор).

9.4. A die is tossed before the first landing «six» aces. Find the probability that the first appearance of «six» will take place:

(a) at the second tossing the die;
(b) at the third tossing the die;
(c) at the fourth tossing the die.

*The answer*: *(a)* 5/36.

9.5. Suppose that a batch of 100 items contains 6 that are defective and 94 that are non-defective. If $X$ is the number of defective items in a randomly drawn sample of 10 items from the batch, find (a) $P(X = 0)$ and (b) $P(X > 2)$.

9.6. There are 7 standard items in a set of 10 items. 4 items are randomly taken from the set. Find the law of distribution of the random variable $X$ equal to the number of standard items among the taken items.

9.7. An urn contains 5 white and 20 black balls. 3 balls are randomly taken from the urn. Compose the law of distribution of the random variable $X$ equal to the number of taken out white balls.

9.8. At horse-racing competitions it is necessary to overcome four obstacles with the probabilities equal 0,9; 0,8; 0,7; 0,6 respectively. At the first failure the sportsman in the further competitions does not participate. Compose the law of distribution of a random variable $X$ – the number of taken obstacles. Find the mathematical expectation of the random variable $X$ (obstacle – препятствие).

*The answer:* $M(X) = 2{,}4264$.

9.9. Two shooters make on one shot in a target. The probability of hit by the first shooter at one shot is 0,5, and by the second shooter – 0,4.

(a) Find the law of distribution of the random variable $X$ – the number of hits in the target;
(b) Find the probability of the event $X \geq 1$.

*The answer:* b) 0,7.

9.10. A set of families has the following distribution on number of children:

| $x_i$ | $x_1$ | $x_2$ | 2 | 3 |
|---|---|---|---|---|
| $p_i$ | 0,1 | $p_2$ | 0,4 | 0,35 |

Determine $x_1$, $x_2$, $p_2$, if it is known that $M(X) = 2$, $D(X) = 0{,}9$.

## Exercises for Homework 9

9.11. Compose the law of distribution of probabilities of the number of appearances of the event A in three independent trials if the probability of appearance of the event is 0,6 for each trial.

9.12. Let $X$ be a random variable equal to the number of boys in families with five children. Assume that probabilities of births of both boy and girl are the same. Find the law of distribution of the random variable $X$. Find the probabilities of the following events:

(a) there are 2-3 boys in a family;
(b) no more than three boys;
(c) more than 1 boy.
*The answer:* a) 5/8; b) 13/16; c) 13/16.

9.13. A factory has sent 5000 suitable items to its warehouse. The probability that an item is broken during a transportation is 0,0002. Find the probability that 3 non-suitable items will be arrived at the warehouse.

*The answer*: 0,06.

9.14. Suppose that the number of accidents occurring on a highway each day is a Poisson random variable with parameter $\lambda = 3$.

(a) Find the probability that 3 or more accidents occur today.
(b) Repeat part (a) under the assumption that at least 1 accident occurs today.
*The answer:* a) 0,577; b) 0,627.

9.15. A hunter shoots on a game before the first hit, but he managed to make no more than four shots. The probability of hit by him at one shot is 0,9.

(a) Find the law of distribution of a random variable $X$ – the number of misses;
(b) Find the probability of the following events: $X < 2$, $X \le 3$, $1 < X \le 3$ (hunter – охотник; game – дичь).
*The answer:* b) 0,99; 0,9999; 0,0099.

9.16. There are 11 students in a group, and 5 of them are girls. Compose the law of distribution of the random variable $X$ – the number of girls from randomly selected three students.

9.17. There are 8 pencils in a box, and 5 of them are green. 3 pencils are randomly taken from the box.

(a) Find the law of distribution of a random variable $X$ equal to the number of green pencils among taken.
(b) Find the probability of the event: $0 < X \le 2$.
*The answer:* b) 45/56.

9.18. There are 20 products in a set, and 4 of them are defective. 3 products are randomly chosen for checking their quality. Compose the law of distribution of a random variable $X$ – the number of defective products contained in the sample.

9.19. The probability of successful passing an exam by the first student is 0,7, and by the second – 0,8. Compose the law of distribution of a random variable $X$ – the number of the students successfully passed the exam if each of them can retake only once the exam if he didn't pass it at the first time. Find the mathematical expectation of the random variable $X$.

*The answer*: $M(X) = 1,87$.

9.20. A discrete random variable $X$ is given by the following law of distribution:

| | | | | |
|---|---|---|---|---|
| $x_i$ | 1 | $x_2$ | $x_3$ | 3 |
| $p_i$ | 0,1 | $p_2$ | 0,5 | 0,1 |

Determine $x_2$, $x_3$, $p_2$, if it is known that $M(X) = 4$, $M(X^2) = 20,2$.
*The answer:* $x_2 = 2$; $x_3 = 6$ or $x_2 = 7$; $x_3 = 3$.