Counting Techniques

(or *how to count without explicitly counting*)

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Warmup

- What do the following problems have in common?
  1. Count the number of ways to select a dozen donuts when there are five types available
  2. Count the number of 16-bit numbers with exactly 4 ones and 12 zeroes

Answer: 1820, why?

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Why do we care?

1. Determining the time and space required to solve a problem always involves solving a counting problem (size of data structures, number of iterations of a loop, number of recursive calls, etc.)
2. Counting is the foundation of discrete probability theory, which plays a central role in the analysis of systems (e.g., networks, cryptography) and the design of randomized algorithms.
3. Two remarkable techniques, the pigeonhole principle and combinatorial proofs depend on counting

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Basic Principles

ADDITION RULE. If an object must be selected either from a pile of p items **or** from another pile of q items, then the selection can be made in $p+q$ different ways.

Example. A lunch special includes soup *or* salad. If there are 4 types of soups and 3 types of salad, then you have $\mathbf{4+3=7}$ options to choose from.

MULTIPLICATION RULE. If two objects must be chosen, the first one from a pile of p items **and** the second one from a pile of q items, then the selection can be made in $p \times q$ different ways.

Example. A dinner special includes soup *and* salad. If there are 4 types of soups and 3 types of salad, then you have $\mathbf{4 \cdot 3 = 12}$ options to choose from.

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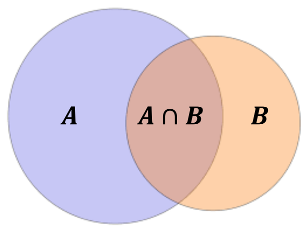
In Terms of Sets…

ADDITION RULE. If A and B are disjoint sets, then $|A \cup B| = |A| + |B|$

MULTIPLICATION RULE. The number of pairs of the form (x, y), where $x \in A$ and $y \in B$ is $|A \times B| = |A| \cdot |B|$

Exercise. What is $|A \cup B|$ if A and B are not necessarily disjoint?



Generalization

- The addition and multiplication rules can, of course, be generalized to any number of sets:

If a set S of objects is *partitioned* into subsets $S_1, S_2, \ldots, S_m$, then
- An object can be chosen from S in $|S_1| + |S_2| + \cdots + |S_m|$ ways
- An m-tuple $(x_1, x_2, \ldots, x_m)$, where $x_i \in S_i$, can be built in $|S_1| \times |S_2| \times \cdots \times |S_m|$ different ways

Examples

Example. You want to recommend an intro programming book to a friend. Since you have four books on Java and three on Python, then you have $4 + 3 = 7$ choices for your recommendation. If, on the other hand, you want to recommend both a book on Java and one on Python, then the number choices is $4 \times 3 = 12$.

Example. The number of entries in the *look-up-table* (LUT) of a frame buffer with eight bits per pixels driving a true color display is $2^8 = 256$. This is the number of different colors that can be displayed at a time in a single frame of animation, each chosen from a palette of 256^3 colors.

Example. The number of vectors that can be constructed to be incident on a set of four points in general position is at most 13 (why?)

Bijection Rule: *how to count one thing by counting another*

BIJECTION RULE. Let X and Y be finite sets. If there is a bijection $f: X \to Y$, then $|X| = |Y|$.

- This principle, as obvious as it appears, when applied to infinite sets plays a fundamental role in logic and computability theory.

  Exercise. How many subsets of $X = \{1, \dots, n\}$ have an even number of elements and how many have and odd number? Use the bijection rule.

Universal Hashing

- Consider a hash table of size m, a prime number
- Each key x is interpreted as consisting of $r+1$ "digits" in base m

$$x = \langle x_r \cdots x_1 x_0 \rangle$$

- The hash function is built using random $(r+1)$-digit base-m number a:

$$a = \langle a_r \cdots a_1 a_0 \rangle$$

$$h_a(x) = \sum_{i=0}^{r} a_i x_i \mod m$$

- The set $\mathcal{H} = \{h_a\}$ of hash functions satisfies: for all keys $x \neq y$, if a is chosen at random, then $\Pr(h_a(x) = h_a(y)) = 1/m$ (proven COMP 3371)

Exercise. How many different hashing functions are there?

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Exercises

1. Let X be a set of size n and Y a set of size m. How many different mappings $f: X \rightarrow Y$ are there?
2. Let X be a set of size n and Y a set of size m. How many different *injective* mappings $f: X \rightarrow Y$ are there?
3. Let X be a set of size n and Y a set of size m. How many different *surjective* mappings $f: X \rightarrow Y$ are there?
4. Describe a bijection between the set of 12-packs of donuts selected from 5 varieties and the set of 16-bit binary sequences with exactly four ones.

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Division Rule

- A generalization of the bijection rule
- A function $f: X \rightarrow Y$ is k-to-1 if it is surjective and for every $y \in Y$, the relation $f^{-1}(y)$ contains exactly k points of X

> DIVISION RULE. Let X and Y be finite sets. If $f: X \rightarrow Y$ is k-to-1, then $|X| = k \cdot |Y|$

Reinterpretation: if each element of a set Y is counted k times, then $|Y| = (\text{total count})/k$

Example. The number of segments of length > 0 that can be constructed to be incident on a set of 4 points is 6.

Example. Let $G(V, E)$ be a simple undirected graph. What is the relation between the number of edges and the sum of degrees of all vertices?

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Common Problem Types

Permutations. The goal is to count *ordered* arrangements or *ordered* selections of objects with or without repetition of objects.

Arrangement a, b, c, b is different from arrangement b, b, a, c

Combinations. The goal is to count *unordered* arrangements or *unordered* selections of objects with or without repetition of objects

Arrangement a, b, c, b is the same as arrangement b, b, a, c

Since repetitions may be allowed, we distinguish between arrangements or selections from a *set* or from a *multiset* (also called a *bag*). For the case of multisets, we specify the number of times an object occurs as illustrated in this example: $M = \{3 \cdot a, 1 \cdot b, 2 \cdot c\}$ denotes the multiset $M = \{a, a, a, b, c, c\}$. Infinity (∞) is allowed as a repetition count

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Example

- How many numbers between 1000 and 9999 have distinct digits?
- Order is important ⇒ problem is of the permutation type
- 9 different values can appear in the thousands, 10 in the hundreds, 10 in the tens, and 10 in the units
  - Is $9 \times 10 \times 10 \times 10 = 9000$ the correct answer?
- There are 9 different choices for the thousands, 9 for the hundreds, 8 for the tens, 7 for the units
  - The correct answer is $9 \times 9 \times 8 \times 7 = 4536$

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Example

- How many *odd* numbers between 1000 and 9999 have distinct digits?
- Order is important ⇒ problem is of the permutation type
- There are 9 possibilities for the thousands, 8 for the hundreds, and 7 for the tens. In how many ways can you choose the units?

$$9 \times 9 \times 8 \times ?$$

- The correct answer is $8 \times 8 \times 7 \times 5 = 2240$. Why?
  - We have 5 different choices for the units, 8 for the thousands, 8 for the hundreds, 7 for the tens

Exercise. How many *even* numbers are there between 1000 and 9999?

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Permutations

- Let $r \in \mathbb{N}$. An *r-permutation* of a set *S* of *n* elements is an ordered arrangement of *r* of the *n* elements.
- A *permutation* of *S* is simply an *n*-permutation of *S*
- $P(n, r)$ = the number of *r*-permutations of *n* elements

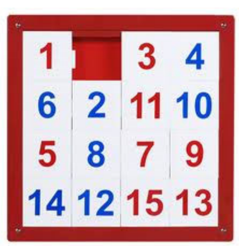
Example. If $S = \{a, b, c\}$, then ab, ac, ba, bc, ca, cb are the six 2-permutations of S and abc, acb, bac, bca, cab, cba are the six permutations of S.

Theorem. For n and r positive integers with $r \leq n$, $P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$.

$$P(n, r) = \frac{n!}{(n-r)!}$$

Examples

1. The number of 4-letter variable names that can be formed using each of the letters *c*, *a*, *r*, *e*, and *s* at most once is $P(5,4) = 120$
   – What is the number of 5-letter variable names using the same letters?
2. The “15-puzzle” consists of 15 sliding squares labeled 1 through 15 and mounted on a 4×4 square frame. The number of different input puzzles is $P(16,16) = 16!$, i.e., $>$20 trillion possibilities



Circular Permutations

- The permutations considered so far are *linear*
- If n objects are arranged in a circle, the number of permutations decreases, as two circular permutations are equal if one can be brought to the other by a rotation

Example. Suppose we arrange 10 children in a circle.

Since each linear permutation gives rise to 9 other equivalent ones, the number of circular permutations is $\frac{10!}{10} = 9!$

Theorem. The number of circular r-permutations of a set of n elements is given by

$$\frac{P(n,r)}{r} = \frac{n!}{r(n-r)!}$$

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Exercises

- What is the number of different 5-letter variables names with at least one upper-case letter?
- What is the number of necklaces that can be made from 17 different spherical beads?

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Combinations

- Let r be a non-negative integer. An r-*combination* of a set S of size n is an r-element subset of S
- The number of r-combinations of a set of size n is denoted by

$$C(n,r) \text{ or } \binom{n}{r}$$

- In particular, $C(n,0) = C(n,n) = 1, C(n,1) = n$ and if $r > n$ then $C(n,r) = 0$
- For convenience we define $C(0,0) \coloneqq 0$

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Basic formula

Theorem. For $r \leq n$, $C(n,r) = \frac{n!}{r!(n-r)!}$

Proof. Follows from $P(n,r) = r!\,C(n,r)$

Corollary. For $r \leq n$, $C(n,r) = C(n,n-r)$

Example. Let S be a set of 25 distinct points on the plane. The maximum number of lines through pairs of points of S is

$$C(25,2) = \frac{25!}{23!2!} = 300.$$

On the other hand, the number of triangles with vertices from S is

$$C(25,3) = \frac{25!}{22!3!} = 2300$$

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Exercise

How many 8-letter variable names can be constructed using the 26 lower-case letters of the alphabet if each word contains 4 or 5 vowels?

Solution. Consider 2 cases as follows:

1) The number of words with 4 vowels is $C(8,4)5^4 21^4 = \frac{8!}{(4!4!)} 5^4 21^4 = 8{,}508{,}543{,}750$
2) The number of words with 5 vowels is $C(8,5)5^5 21^3 = \frac{8!}{(5!3!)} 5^5 21^3 = 1{,}620{,}675{,}000$

Answer: 8508543750+1620675000 = 10,129,218,750

Binomial Coefficients

The numbers $C(n, i)$ are called ***binomial coefficients*** because they appear as the coefficients in the expansion of the n-th power of the binomial $x + y$

Binomial Theorem. Let n be a positive integer. Then, for all x and y

$$(x+y)^n = \binom{n}{0} y^n + \binom{n}{1} x y^{n-1} + \binom{n}{2} x^2 y^{n-2} + \cdots + \binom{n}{n} x^n$$

$$= \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

Proof. Either by induction on *n* or use a *combinatorial argument*

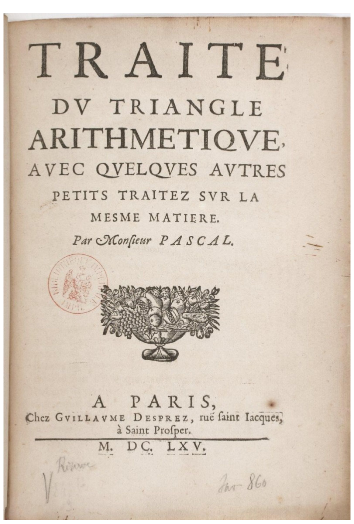
Pascal’s Triangle

Theorem **(*Pascal’s formula*).** For integer n and k with $1 \leq k < n$,
$C(n,k) = C(n-1,k-1) + C(n-1,k)$

Proof. By induction on *n* or by combinatorial argument (partition {1,…,*n*} into two subsets depending on whether they contain *n* or not, and apply the Addition Rule.)

When the coefficients are computed bottom-up using Pascal’s formula and displayed for increasing values of *n* (one row per value of *n*), the resulting diagram is called *Pascal’s Triangle*, appearing in Blaise Pascal’s *Traité du triangle arithmétique* (1653)

Traité du triangle arithmétique (1653)


TRAITE
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Par Monsieur PASCAL.

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Chez GVILLAVME DESPREZ, ruë saint Iacques, à Saint Prosper.
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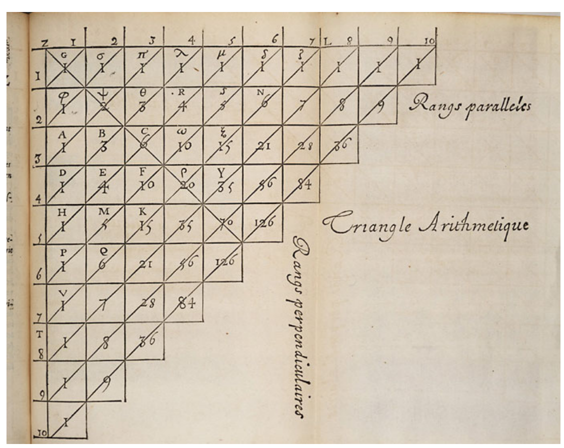

Pascal's Triangle

n	$C(n,0)$	$C(n,1)$	$C(n,2)$	$C(n,3)$	$C(n,4)$	$C(n,5)$	$C(n,6)$	$C(n,7)$
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	20	15	6	1	
7	1	7	21	35	35	21	7	1

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Exercise. Can you provide a combinatorial proof of this identity?

- What other properties can you find?

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A Few Binomial Identities

1) $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$

2) $2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}$

3) $\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots$

4) $n2^{n-1} = 1\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} \cdots + n\binom{n}{n}$

5) $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$

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Permutations of multisets

- If S is a multiset, an r-*permutation* of S is an ordered arrangement of r of the elements of S
- If $|S| = n$, an n-permutation is simply called a *permutation* of S

Example. If $S = \{2 \cdot a, 1 \cdot b, 3 \cdot c\}$ then *acbc* and *cbcc* are 4-permutations of S while *abccca* is a permutation of S.

Theorem. Let S be a multiset with an unlimited supply of each of k distinct objects. Then the number of r-permutations of S is k^r

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Permutations of multisets…

Theorem. Let S be a multiset of size n consisting of k distinct elements with finite repetition counts $n_1, n_2, \dots, n_k$. Let $n = n_1 + n_2 + \cdots + n_k$. Then the number of permutations of S equals

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}$$

Example. The number of permutations of the letters in the word MISSISSIPPI is the same as the number of permutations of the multiset $\{1 \cdot M, 4 \cdot I, 4 \cdot S, 2 \cdot P\}$, which equals $11!/(1!\,4!\,4!\,2!)$.

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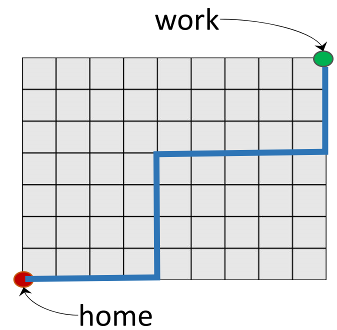
Example

- A secretary works in a building located 9 blocks east and 7 blocks north of his home. Every day he walks the 16 blocks to work. He promised (to himself) that he would retire once he has tried every single shortest route. When would that be?



Solution. A valid path can be viewed as a permutation of the multiset $S = \{9 \cdot E, 7 \cdot N\}$. Therefore, there are $16!/(9!\,7!) = 11440$. At 251 work days per year, exploring all would take 45.58 years.

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Matrix Chain Multiplication

- Consider the product of three matrices A, B, C, of sizes 10×100, 100×5, and 5×50, respectively
- Which one of $\big((AB)C\big)$ or as $\big(A(BC)\big)$ requires fewer scalar multiplications? Does it matter?
  - The number of scalar multiplications is: 5,000 + 2,500 =7,500 for $\big((AB)C\big)$, and 25,000 + 50,000 =75,000 for $\big(A(BC)\big)$
- More generally, when multiplying n matrices, in what order should the partial products be computed?
- Number of ways $P(n)$ to perform the product: $$P(n) = \begin{cases} 1 & \text{if } n = 1 \\ \sum_{k=1}^{n-1} P(k)P(n-k) & \text{if } n \geq 2 \end{cases}$$

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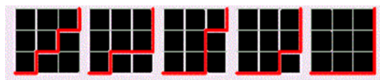
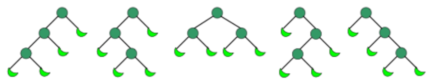
Four Related Problems

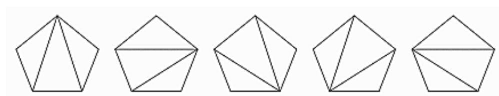
1. Given matrices $A_1, A_2, \ldots, A_n$, in what order should the partial products be computed? How many different choices are there?
   $(A_1(A_2(A_3A_4)))$; $\big(A_1((A_2A_3)A_4)\big)$; $((A_1A_2)\ (A_3A_4))$; $(((A_1A_2)A_3)A_4)$; $\big((A_1(A_2A_3))A_4\big)$
2. Given a regular grid, how many different shortest paths are there from (0,0) to (n, n) that remain below the line $y = x$?



3. How many different binary search trees can be built on a set of keys $x_1 < x_2 < \cdots < x_n$?



4. What is the number of triangulations of a convex $(2 + n)$-polygon?



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n applications of a binary operator

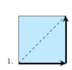
(A_1A_2) $n = 1$

$(A_1(A_2A_3))$ $((A_1A_2)A_3)$ $n = 2$

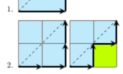
$(A_1(A_2(A_3A_4)))$; $(A_1((A_2A_3)A_4))$; $((A_1A_2)(A_3A_4))$ $n = 3$
$((A_1(A_2A_3))A_4)$; $(((A_1A_2)A_3)A_4)$

$((((A_1A_2)A_3)A_4)A_5)$; $(((A_1(A_2A_3))A_4)A_5)$; $((A_1A_2)((A_3A_4)A_5))$
$((A_1((A_2A_3)A_4))A_5)$; $((A_1(A_2A_3))(A_4A_5))$; $(A_1((A_2(A_3A_4))A_5))$
$(A_1(((A_2A_3)A_4)A_5))$; $(A_1((A_2A_3)(A_4A_5)))$; $(A_1(A_2((A_3A_4)A_5)))$
$((A_1A_2)(A_3(A_4A_5)))$; $(((A_1A_2)A_3)(A_4A_5))$; $(A_1(A_2(A_3(A_4A_5))))$
$(((A_1A_2)(A_3A_4))A_5)$; $((A_1(A_2(A_3A_4)))A_5)$ $n = 4$

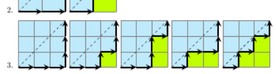
Shortest path in $n \times n$ grid



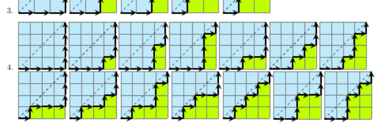
$n = 1$



$n = 2$



$n = 3$



$n = 4$

Triangulations of $(n + 2)$-gon

$n = 1$

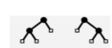
$n = 2$

$n = 3$

$n = 4$

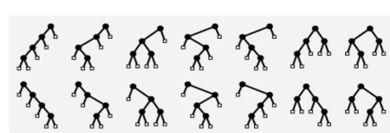
BST for n keys

$n = 1$



$n = 2$

$n = 3$



$n = 4$

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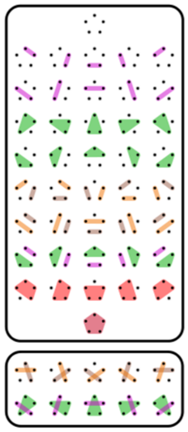
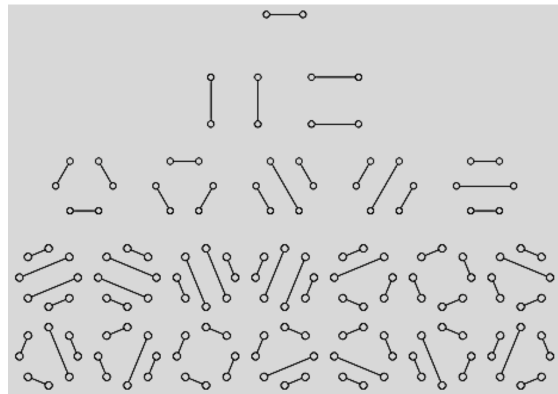
And Two More…

- Non-crossing partitions of *n* items
- Concurrent handshakes of $\boldsymbol{n}$ pairs people





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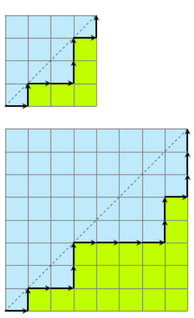
Shortest Constrained Paths on a Grid

- *Goal*: # of shortest paths from $(0,0)$ to (n, n) that avoids locations above the line $y = x$
- A path can be viewed as a *von Dyck word* of length $2n$, i.e., a permutation of $\{n \cdot E, n \cdot N\}$ whose prefixes contain at least as many E's as N's, i.e., at all times $\#E\text{'s} \geq \#N\text{'s}$



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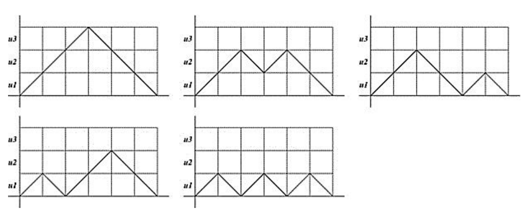

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Shortest Constrained Paths…

- Notation:
  $U_{m,n}$: set of unconstrained paths from $(0,0)$ to (m,n)
  $C_{n,n}$: set of valid constrained paths from $(0,0)$ to (n,n)
  $I_{n,n}$: set of invalid constrained paths from $(0,0)$ to (n,n)
- Answer can be easily computed if we know the number of invalid paths:

  # valid paths = # unconstrained paths − # invalid paths, i.e.,

$$|C_{n,n}| = |U_{n,n}| - |I_{n,n}|$$

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A Useful Transformation

- We establish a bijection $f: I_{n,n} \leftrightarrow U_{n-1,n+1}$ and use the *Bijection Rule* to find $|I_{n,n}|$
- Our bijection f transforms an invalid path by flipping every symbol *after* the first violation

  Example: With $n = 5$, the path ***ENN****NEEEENNE* becomes ***ENN****ENNNEEN*
- In general, an invalid path to (n, n) becomes an unconstrained path to $(n - 1, n + 1)$

Exercise. Verify: (1) if $x \in I_{n,n}$ then $f(x) \in U_{n-1,n}$, (2) f is one-to-one, and (3) f is onto.

Theorem. $|C_{n,n}| = |U_{n,n}| - |I_{n,n}| = \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n-1)!(n+1)!} = \frac{C(2n,n)}{n+1}$

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Catalan Numbers

- The Catalan numbers are defined by the sequence

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \prod_{i=2}^{n} \frac{n+i}{i}, n = 0,1,2,\ldots$$

- The first few are 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, …
- Named after the Belgian mathematician Eugène Catalan (1814-1894) who used them to express the number of grouping n applications of an associative binary operator

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Matrix Chains Revisited

- We with to show that $P(n) = C_{n-1}$ where

$$P(n) = \begin{cases} 1 & \text{if } n = 1 \\ \sum_{k=1}^{n-1} P(k)P(n-k) & \text{if } n \geq 2 \end{cases}$$

- We establish a bijection between constrained paths and possible ordering of $n-1$ applications of a binary operator (matrix multiplication)

$$\begin{array}{lcll} (((ab)c)d) & & (((abc & EEENNN \\ ((a(bc))d) & & ((a(bc & EENENN \\ ((ab)(cd)) & \Leftrightarrow & ((ab(c \Leftrightarrow & EENNEN \\ (a((bc)d)) & & (a((bc & ENEENN \\ (a(b(cd))) & & (a(b(c & ENENEN \end{array}$$

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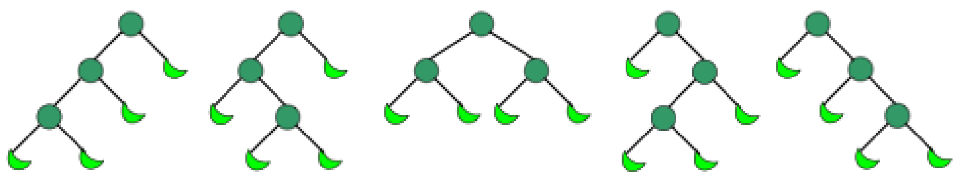
Exercise

- Let b_n denote the number of distinct binary search trees on n keys. Prove that $b_n = C_n$



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Summary (so far)

- r-permutations of n elements: $P(n,r) = \frac{n!}{(n-r)!}$
- Circular r-permutations of n elements: $\frac{P(n,r)}{r} = \frac{n!}{r(n-r)!}$
- r-combinations of n elements: $C(n,r) = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$
- r-permutations of multiset with unlimited supply of k classes: k^r
- If the k classes have repetition counts $n_1, \dots, n_k$ with $n = \sum n_i$: $\frac{n!}{n_1!\,n_2! \cdots n_k!}$
- n-th Catalan number: $C_n = \frac{1}{n+1}\binom{2n}{n}$

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Combinations of multisets

- Order unimportant, repetitions allowed
  Example. $\{1,3,4,3,1,1,7\}$
- If S is a multiset, then an r-combination of S is an unordered selection of r of the elements of S
  - Thus, an r-combination is itself a multiset.
  - If S has n elements, then there is only one n-combination of S, namely S itself
  - If S contains k *distinct* elements, then there are k 1-combinations of S

Example. If $S = \{2 \cdot a, 1 \cdot b, 3 \cdot c\}$ then the following are all the 2-combinations of S: $\{a, a\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, and $\{c, c\}$

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Theorem. Let S be a multiset with k distinct object types, each with unlimited multiplicity. Then the number of r-combinations of S is $C(k - 1 + r, r)$**.**

Proof. Let the distinct objects of *S* be $a_1, a_2 \dots, a_k$. An *r*-combination has the form $\{x_1 \cdot a_1, x_2 \cdot a_2, \dots, x_k a_k\}$ where $x_i \in \mathbb{N}$ and $x_1 + x_2 + \cdots + x_k = r$.
Thus, there is a 1-1 correspondence between the *r*-combinations and non-negative integer solutions to $x_1 + x_2 + \cdots + x_k = r$.
This is the same as the number of permutations of the set $T = \{(k - 1) \cdot 0, r \cdot 1\}$.
The 0's divide the 1's into *k* groups of 1's. The size of the i-th group is the value of x_i.
Example. Assuming $k = 4, r = 5$, the permutation of $\{3 \cdot 0,5 \cdot 1\}$ given by 01110011 corresponds to the solution $x_1 = 0, x_2 = 3, x_3 = 0, x_4 = 2$, a solution to $x_1 + x_2 + x_3 + x_4 = 5$.
Thus, the number of *r*-combinations of S is the same as the number of permutations of T. i.e., $C(k - 1 + r, r)$

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Exercises

- A bakery offers 8 varieties of doughnuts. If a box contains 1 dozen, how many different boxes can you buy?
- A sequence $x_1, x_2, x_3, \ldots$ is monotonically increasing if $x_{i-1} \leq x_i$ for all i. What is the number of monotonically increasing sequences of length 12 whose terms are taken from $\{1,2,3,\ldots,8\}$?

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Theorem. Let S be a multiset with k distinct object types each with unlimited multiplicity. Then the number of r-combinations of S in which each of the k different types occurs at least once equals $C(r-1, k-1)$**.**

Proof.

We seek the number of solutions to $x_1 + x_2 + \cdots + x_n = r$ with positive integers.

Take a sequence of r **1**'s and let the $r-1$ positions between the **1**'s be $T = \{p_1, \ldots, p_{r-1}\}$.

Then, the number of solutions sought is the number of $(k-1)$-combinations of T.

Choose a combination and insert a **0** in each of the chosen p_i's. The number of 1's between consecutive 0's is strictly positive and gives the number of repetitions of the corresponding object type

This number equals $C(r-1, k-1)$, as claimed. ∎

Example. If $r = 6$ and $k = 4$ then **110110101** corresponds to the solution $x_1 = x_2 = 2$, $x_3 = x_4 = 1$.

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Generating Permutations

Problem. Given n, generate all permutations of $\{1, \dots, n\}$

- Component of many algorithms
  - The basis of many "brute force" algorithms
    - Reasonable for small *n* (≤ 20)
    - Processing a permutation takes longer than generating it
  - Structural backbone of backtracking algorithms
  - Get useful insight (average, worst case behavior) into algorithms for various combinatorial problems
    - Sorting, spanning forest, traveling salesman, etc.
- More than 30 different algorithms have been published

Goal. Low average time per permutation, low auxiliary memory requirements, simple state allows you to generate next permutation

Algorithm 1

- Evaluate the following with respect to our goals

$\textsc{Permute}(A, n)$

```
if n = 1
    return {A}
allPerms = { }
for i = 1 to n
    x = A[i]
    rest = A with x removed
    for π ∈ PERMUTE(rest, n − 1)
        t = x followed by π
        allPerms = allPerms ∪ {t}
return allPerms
```

Algorithm 2

- Based on recursive definition $n! = (n-1)! \cdot n$

```
PERMUTE(A, n)
if n = 1
    then PRINTPERMUTATION()
    else for k ← 1 to n
              do SWAP(A, k, n)
                 PERMUTE(A, n − 1)
                 SWAP(A, k, n)
```

Exercise. Why is the algorithm correct?

Claim. Permute requires $\approx 2(e-1)$ swaps/perm.

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Algorithm 3

- Based on the fact that deleting an element from a permutation π of $\{1, \dots, n\}$ results in a permutation σ of $\{1, \dots, n-1\}$
  - n different permutations of $\{1, \dots, n\}$ result in the same permutation σ
- *Basic Idea.* Fix an arbitrary element x. For each $(n-1)$-permutation π of the remaining elements, generate n new permutations by inserting x in all possible positions of π
  - Starting with the single permutation of $\{1\}$, the algorithm generates the permutations of $\{1, \dots, n\}$ recursively in $n-1$ rounds
  - Round $k = 2, \dots, n$ proceeds by listing every permutation π of $\{1, \dots, k-1\}$ and inserting k into π at each possible position
- Only one swap per permutation

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Algorithm 3: Example

Round 2: write each permutation of 1 twice and interlace 2

	1	**2**
2	1	

Round 3: write each permutation of $\{1{,}2\}$ thrice and interlace 3

	1		2	**3**
	1	**3**	2	
3	1		2	
3	2		1	
	2	**3**	1	
	2		1	**3**

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Round 4: write each permutation of {1,2,3} four times and interlace 4

Pros:

- Each permutation requires only 1 swap

Cons:

- finding the right swap is not easy
- A naïve implementation requires large amounts of storage

	1		2		3	**4**
	1		2	**4**	3	
	1	**4**	2		3	
4	1		2		3	
4	1		3		2	
	1	**4**	3		2	
	1		3	**4**	2	
	1		3		2	**4**
	3		1		2	**4**
	3		1	**4**	2	
	3	**4**	1		2	
4	3		1		2	
4	3		2		1	
	3	**4**	2		1	
	3		2	**4**	1	
	3		2		1	**4**
	2		3		1	**4**
	2		3	**4**	1	
	2	**4**	3		1	
4	2		3		1	
4	2		1		3	
	2	**4**	1		3	
	2		1	**4**	3	
	2		1		3	**4**

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An efficient implementation

- In order to determine the next swap, each element is assigned an "intended" direction of movement, e.g., $\vec{2}, \vec{6}, \vec{3}, \overleftarrow{1}, \vec{5}, \vec{4}$
- An element is *movable* if its neighbor in its intended direction has a smaller value
  - In the example above, 6, 3, and 5 are movable
  - Element 1 is never movable
  - Element n is not movable if it is the leftmost (resp. rightmost) element and points to the left (resp. right)

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An efficient implementation…

1. Output $\overleftarrow{1}, \overleftarrow{2}, \overleftarrow{3}, \ldots, \overleftarrow{n}$
2. **while** some element is movable **do**
   a. Swap the largest movable element m with the neighbor it points to
   b. Switch the intended direction of all integers $p > m$

Example. If the current permutation is $\vec{2}, \vec{3}, \overleftarrow{1}, \vec{5}, \vec{6}, \vec{4}$, the next two are $\vec{2}, \vec{3}, \overleftarrow{1}, \vec{5}, \vec{4}, \vec{6}$ and $\vec{2}, \vec{3}, \overleftarrow{1}, \vec{4}, \vec{5}, \overleftarrow{6}$

Exercise. What data structures can be used to keep track of directions, movable elements, and for choosing the pair to swap.

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Example $n = 4$

$\overleftarrow{1}\ \overleftarrow{2}\ \overleftarrow{3}\ \overleftarrow{4}$	$\overleftarrow{3}\ \overleftarrow{1}\ \overleftarrow{2}\ \overleftarrow{4}$	$\overleftarrow{2}\ \overrightarrow{3}\ \overleftarrow{1}\ \overleftarrow{4}$
$\overleftarrow{1}\ \overleftarrow{2}\ \overleftarrow{4}\ \overleftarrow{3}$	$\overleftarrow{3}\ \overleftarrow{1}\ \overleftarrow{4}\ \overleftarrow{2}$	$\overleftarrow{2}\ \overrightarrow{3}\ \overleftarrow{4}\ \overleftarrow{1}$
$\overleftarrow{1}\ \overleftarrow{4}\ \overleftarrow{2}\ \overleftarrow{3}$	$\overleftarrow{3}\ \overleftarrow{4}\ \overleftarrow{1}\ \overleftarrow{2}$	$\overleftarrow{2}\ \overleftarrow{4}\ \overrightarrow{3}\ \overleftarrow{1}$
$\overleftarrow{4}\ \overleftarrow{1}\ \overleftarrow{2}\ \overleftarrow{3}$	$\overleftarrow{4}\ \overleftarrow{3}\ \overleftarrow{1}\ \overleftarrow{2}$	$\overleftarrow{4}\ \overleftarrow{2}\ \overrightarrow{3}\ \overleftarrow{1}$
$\overrightarrow{4}\ \overleftarrow{1}\ \overleftarrow{3}\ \overleftarrow{2}$	$\overrightarrow{4}\ \overrightarrow{3}\ \overleftarrow{2}\ \overleftarrow{1}$	$\overrightarrow{4}\ \overleftarrow{2}\ \overleftarrow{1}\ \overrightarrow{3}$
$\overleftarrow{1}\ \overrightarrow{4}\ \overleftarrow{3}\ \overleftarrow{2}$	$\overrightarrow{3}\ \overrightarrow{4}\ \overleftarrow{2}\ \overleftarrow{1}$	$\overleftarrow{2}\ \overrightarrow{4}\ \overleftarrow{1}\ \overrightarrow{3}$
$\overleftarrow{1}\ \overleftarrow{3}\ \overrightarrow{4}\ \overleftarrow{2}$	$\overrightarrow{3}\ \overleftarrow{2}\ \overrightarrow{4}\ \overleftarrow{1}$	$\overleftarrow{2}\ \overleftarrow{1}\ \overrightarrow{4}\ \overrightarrow{3}$
$\overleftarrow{1}\ \overleftarrow{3}\ \overleftarrow{2}\ \overrightarrow{4}$	$\overrightarrow{3}\ \overleftarrow{2}\ \overleftarrow{1}\ \overrightarrow{4}$	$\overleftarrow{2}\ \overleftarrow{1}\ \overrightarrow{3}\ \overrightarrow{4}$

Generating Combinations

- Generating all combinations (i.e., subsets) of $S = \{0,1,2, \ldots, n-1\}$ is straightforward. Using an n-bit vector $a = \langle a_{n-1}, a_{n-2}, \ldots, a_1, a_0 \rangle$ with $a_i = 1$ iff $i \in S$, each of the 2^n possible bit sequences can be interpreted as a subset of S

- Starting with $a = \mathbf{0}$, generate all sequences by repeatedly incrementing a by 1. The case $n = 3$ is illustrated on the right.

	a_2	a_1	a_0
$\emptyset$	0	0	0
$\{0\}$	0	0	1
$\{1\}$	0	1	0
$\{0,1\}$	0	1	1
$\{2\}$	1	0	0
$\{0,2\}$	1	0	1
$\{1,2\}$	1	1	0
$\{0,1,2\}$	1	1	1

Generating *r*-Combinations

- When $r < n$, it is more efficient to generate *r*-combinations lexicographically
  - We are making use of a total order on the set of permutations

*Example***.** The 4-combinations of $S = \{1, \dots, 6\}$ in lexicographic order are:

1234	1256	2345
1235	1345	2346
1236	1346	2356
1245	1356	2456
1246	1456	3456

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Generating *r*-Combinations

Claim. Let $S = \{1, \dots, n\}$**.** The first r***-combination of*** S in *lexicographic order* is $1, 2, \dots, r,$ and the last is $n - r + 1, n - r + 2, \dots, n.$ Let $A = a_1, a_2, \dots, a_r$ be an r-combination of S, different from the last one. Let j be the largest integer such that $a_j < n$ and $a_j + 1$ is not one of $a_{j+1}, \dots, a_r$ (no repetitions allowed). Then the r-combination that follows A in lexicographic order is

$$a_1, \dots, a_{j-1}, a_j + 1, a_j + 2, \dots, a_j + (r - j + 1)$$

Remark. This result can be easily used to derive an algorithm that systematically generates all r-combinations of $S = \{1, \dots, n\}$

Remark. If we combine the algorithm for generating r-combinations of an n-element set with the algorithm for generating permutations of a set, we obtain an algorithm for generating r-permutations of an n-element set.

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Exercise

- Design an efficient algorithm that generates a permutation of $\langle 1,2,\ldots,n\rangle$ uniformly at random
  - Each of the $n!$ permutations should be equally likely to be selected
  - You may assume access to a function RANDINT(a,b) that generates a random integer between a and b
- How long, in terms of number of calls to RANDINT, does your algorithm require?

Exercise

- Explain how the following algorithm works

```
RANDOMPERMUTE(A, n)
flag = FALSE for i = 1, ..., n
for i = 1 to n
    repeat
        r = RANDINT(1, n)
    until flag(r) = FALSE
    perm(i) = r
    flag(r) = TRUE
return perm
```

- How many times, on average, do you call the random function?

Inclusion-Exclusion Principle

- Technique to count the number of objects in a union of sets, indirectly, by counting how many *are not* in the union
  - Useful when direct counting is difficult or impossible
- Basis for many “fast” exponential algorithms for NP-hard problems, such as determining if a graph contains a Hamiltonian path, or whether n items of sizes $s_1, s_2, \dots, s_n$ can be packed into k bins of capacity C each.
  - Algorithms based on inclusion-exclusion need to consider all subsets of a set, resulting in $\Omega(2^n)$ operations.
  - Expensive but significantly faster than a $\Theta(n!)$ algorithm

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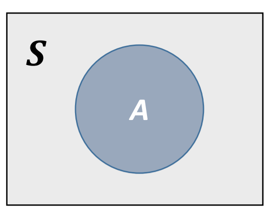
Motivation

- Sometimes, in order to find how many elements belong to a set A, it is easier to find how many *do not* belong to A



$$|A| = |S| - |\bar{A}|$$

Example. Count the number of integers between 1 and 600 that *are not* divisible by 6.

$S = \{1, \dots, 600\}$

$A =$ integers in S *not* divisible by 6

$\bar{A} =$ integers in S divisible by 6

$$|A| = |S| - |\bar{A}| = 600 - 100 = 500$$

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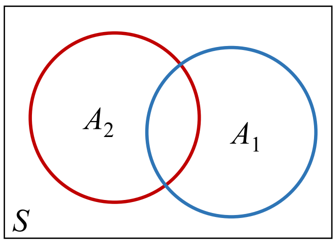
Motivation…

Example. We have a collection S of $n = 100$ polygons. Suppose that 35 polygons are triangles, 40 are rectangles and 25 are pentagons. Furthermore, suppose that 30 are red, out of which 9 are also triangles. *How many polygons are neither triangles nor red?*

Let S = all polygons, A_1 = set of triangles, A_2= set of red polygons.
We want $|\overline{A_1 \cup A_2}| = |\overline{A_1} \cap \overline{A_2}|$.



$$\begin{aligned}|\overline{A_1} \cap \overline{A_2}| &= |\overline{A_1}| - |\overline{A_1} \cap A_2| \\ &= (n - |A_1|) - (|A_2| - |A_1 \cap A_2|) \\ &= |S| - (|A_1| + |A_2|) + |A_1 \cap A_2| \\ &= 100 - (35 + 30) + 9 = 44\end{aligned}$$

$$|A_1 \cup A_2| = (|A_1| + |A_2|) - |A_1 \cap A_2|$$

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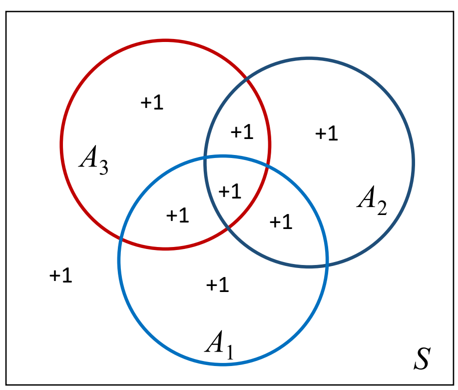

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |S| - (|A_1| + |A_2| + |A_3|)$$

$$+(|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) - (|A_1 \cap A_2 \cap A_3|)$$

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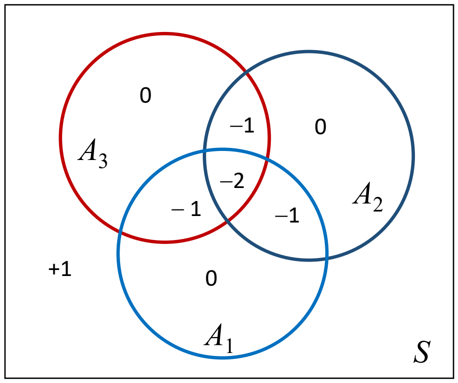

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |S| - (|A_1| + |A_2| + |A_3|)$$

$$+(|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) - (|A_1 \cap A_2 \cap A_3|)$$

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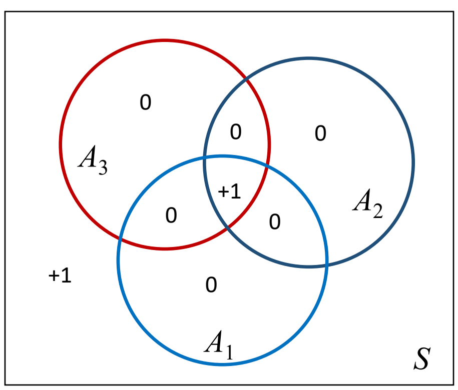

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |S| - (|A_1| + |A_2| + |A_3|)$$

$$+(|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) - (|A_1 \cap A_2 \cap A_3|)$$

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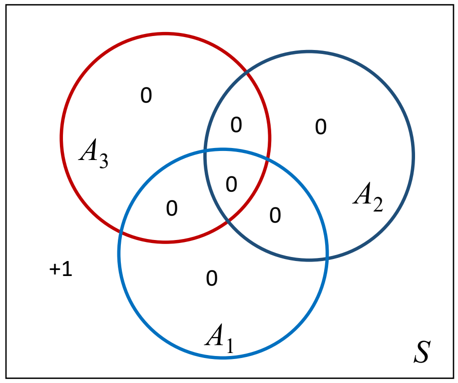

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |S| - (|A_1| + |A_2| + |A_3|)$$
$$+(|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) - (|A_1 \cap A_2 \cap A_3|)$$

$$|A_1 \cup A_2 \cup A_3| = (|A_1| + |A_2| + |A_3|)$$
$$-(|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) + (|A_1 \cap A_2 \cap A_3|)$$

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Generalization

- More generally, let $c_1, \dots, c_t$ be t properties/conditions which each object in S may or may not satisfy. Let A_i denote the subset of objects of S which satisfy condition i (and, possibly, some of the other conditions as well). The following theorem tells us how many objects satisfy *none* of the conditions

Theorem. $$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_t}| = |S| - \sum_{1 \le i \le t} |A_i| + \sum_{1 \le i < j \le t} |A_i \cap A_j| - \sum_{1 \le i < j < k \le t} |A_i \cap A_j \cap A_k| + \cdots + (-1)^t |A_1 \cap \cdots \cap A_t|$$

where the first sum is over all 1-combinations, the second is over all 2-combinations, the third over all 3-combinations of $\{1, \dots, t\}$*, and so on.*

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Generalization…

***Corollary**. The number of objects of S which satisfy at least one of the conditions $c_1, \ldots, c_t$ is given by*

$$|A_1 \cup A_2 \cup \cdots \cup A_t| = \sum_{1 \le i \le t} |A_i| - \sum_{1 \le i < j \le t} |A_i \cap A_j| + \sum_{1 \le i < j < k \le t} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{t-1} |A_1 \cap \cdots \cap A_t|$$

In other words, the size $|A_1 \cup A_2 \cup \cdots \cup A_t|$ of a union of sets is computed as follows: add up the sizes of all the individual sets, the subtract the sizes of all pairwise intersections, now add the sizes of all 3-way intersections, then subtract the sizes of all 4-way intersections, and so on.
In the last step, either add (for t odd) or subtract (for t even) the size of the intersection of all the t sets.

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Example

- A certain town has 3 clubs. The tennis club has 20 members, the stamp collectors club 15 members, and the Egyptology club numbers 8. There are 2 tennis players and 3 stamp collectors among the Egyptologists, 6 people both play tennis and collect stamps, and there is even one busy person that belongs to all three clubs. *How many people are engaged in the club life?*

$$|C \cup T \cup E| = |C| + |T| + |E|$$

$$-|C \cap T| - |C \cap E| - |E \cap T| + |C \cap T \cap E|$$

$$= (15 + 20 + 8) - (2 + 3 + 6) + 1 = 33$$

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Notation

$$N(c_{i_1} c_{i_2} \ldots c_{i_k}) = |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}|$$
$$\overline{N} = |\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_t}|$$
$$S_0 = |S| = N$$
$$S_1 = |A_1| + |A_2| + \cdots + |A_t| = \sum N(c_i)$$
$$S_2 = |A_1 \cap A_2| + \cdots + |A_1 \cap A_t| + \cdots + |A_{t-1} \cap A_t| = \sum N(c_i c_j)$$

and in general

$$S_k = \sum |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|$$

where the selection is made over all $\binom{t}{k}$ k-combinations of the set of t conditions. Then,

$$\overline{N} = |\overline{A_1} \cap \cdots \cap \overline{A_t}| = S_0 - S_1 + S_2 - S_3 + \cdots + (-1)^t S_t$$
$$N - \overline{N} = |A_1 \cup \cdots \cup A_t| = S_1 - S_2 + S_3 + \cdots + (-1)^{t-1} S_t$$

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Example

Find the number of integers between 1 and 1000, inclusive, which are divisible by none of 5, 6, and 8.

- c_1: property of being divisible by 5
- c_2: property of being divisible by 6
- c_3: property of being divisible by 8
- $S = \{1,2, \ldots, 1000\}$
- A_i: subset of S satisfying c_i

We wish to find $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}|$

$$|A_1| = \left\lfloor \frac{1000}{5} \right\rfloor = 200 \qquad |A_1 \cap A_2| = \left\lfloor \frac{1000}{30} \right\rfloor = 33$$
$$|A_2| = \left\lfloor \frac{1000}{6} \right\rfloor = 166 \qquad |A_1 \cap A_3| = \left\lfloor \frac{1000}{40} \right\rfloor = 25 \qquad |A_1 \cap A_2 \cap A_3| = \left\lfloor \frac{1000}{120} \right\rfloor = 8$$
$$|A_3| = \left\lfloor \frac{1000}{8} \right\rfloor = 125 \qquad |A_2 \cap A_3| = \left\lfloor \frac{1000}{24} \right\rfloor = 41$$

$$\overline{N} = S_0 - S_1 + S_2 - S_3 = 1000 - (200 + 166 + 125) + (33 + 25 + 41) - 8 = 600$$

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Example

Find the number of non-negative integer solutions to $x_1 + x_2 + x_3 + x_4 = 20$ subject to $x_i \leq 8, 1 \leq i \leq 4$.

- $S =$ non-negative integers solutions to $x_1 + x_2 + x_3 + x_4 = 20$ $\Rightarrow |S| =$ #permutations of $\{20 \cdot 1, 3 \cdot 0\} = C(23,20) = C(23,3)$
- $c_i =$ property that $x_i > 8 \Rightarrow A_i =$ solutions satisfying c_i
- What is $|A_i|$?
  - Bijection from solutions of $x_1 + x_2 + x_3 + x_4 = 11$ **to** solutions in A_i
  - Then $|A_i| = C(11 + 3, 11)$
- Similarly, $|A_i \cap A_j| =$ #solutions to $x_1 + x_2 + x_3 + x_4 = 2$, i.e., $C(2 + 3, 2)$

$$N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4) = S_0 - S_1 + S_2 - S_3 + S_4 =$$
$$C(23,20) - 4C(14,11) + 6C(5,2) - 0 + 0 = 375$$

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Example

Consider a region with five villages. You wish to build a system of roads so that no village is completely isolated, although it may remain disconnected from specific villages (i.e., the resulting graph may not be connected). *In how many ways can this be done?*

- S: set of all graphs on $V = \{1,2,3,4,5\}$. How many?
- $|S| = 2^{\binom{5}{2}} = 2^{10} = 1024$
- c_i: village i is isolated; A_i: graphs in S satisfying c_i
- We want $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4} \cap \overline{A_5}|$
- Can you find S_1, S_2, S_3, S_4, S_5 ?

$$\overline{N} = S_0 - S_1 + S_2 - S_3 + S_4 - S_5 =$$
$$2^{10} - \binom{5}{1}2^6 + \binom{5}{2}2^3 - \binom{5}{3}2^1 + \binom{5}{4}2^0 - \binom{5}{5}1 = 768$$

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Example

In how many ways can six married couples be seated at a circular table so that no one sits next to their spouse?

- S: set of all sitting arrangements. How many?

$$|S| = 11!$$

- c_i: members of couple i sit together. $N(c_1)$?

$$S_1 = 6(2)(10!) \quad S_2 = 2^2C(6,2)(9!) \quad S_3 = C(6,3)2^3(8!)$$

$$S_4 = C(6,4)2^4(7!) \quad S_5 = C(6,5)2^5(6!) \quad S_6 = C(6,6)2^6(5!)$$

$$\overline{N} = \sum_{i=0}^{6}(-1)^i C(6,i)2^i(11-i)! = 12,771,840$$

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r-Combinations of multisets with finite repetition counts

- Previously we computed the number $C(n,r)$ of r-combinations of a set S and the number $C(r+k-1,r)$ of r-combinations of a multiset with k classes of unlimited multiplicity.
- We now look at multisets with finite repetition counts

*Exercise***.** Compute the number of 10-combinations of the multiset $\{3 \cdot r, 4 \cdot g, 5 \cdot b\}$.

We can do this by imposing additional conditions on the 10-combinations of $\{\infty \cdot r, \infty \cdot g, \infty \cdot b\}$

c_1: 10-combination has more than 3 r's.
c_2: 10-combination has more than 4 g's.
c_3: 10-combination has more than 5 b's.

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Generalized Inclusion-Exclusion

- Given t properties $c_1, \dots, c_t$, and a set S, let E_m (resp. L_m) denote the number of elements of S that satisfy exactly (resp. at least) m of the properties.

  Note: so far, we have only considered the cases E_0 and L_1

Theorem. *Under the same hypothesis as the inclusion-exclusion principle, for each $1 \le m \le t$, the number of elements of S that satisfy exactly m of the properties $c_1, \dots, c_t$ is*

$$E_m = S_m - C(m+1,1)S_{m+1} + C(m+2,2)S_{m+2} - \dots + (-1)^{t-m}C(t, t-m)S_t$$

Corollary.

$$L_m = S_m - C(m, m-1)S_{m+1} + C(m+1, m-1)S_{m+2} - \cdots + (-1)^{t-m}C(t-1, m-1)S_t$$

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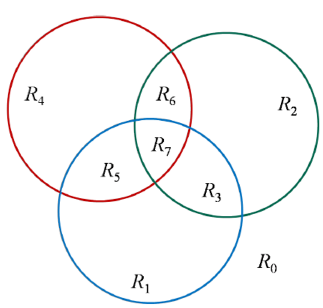
Example

- Verify the special case E_m, for $t = 3, m = 1$, using Venn diagrams (each region has been numbered for easy reference)



- S_1 overestimates the actual number as regions 3, 5, 6 are counted twice and region 7 is counted 3 times
- If we subtract $2S_2$ we end with a net loss of 3 for region 7 which needs to be added back
- Correct formula is $S_1 - 2S_2 + 3S_3$

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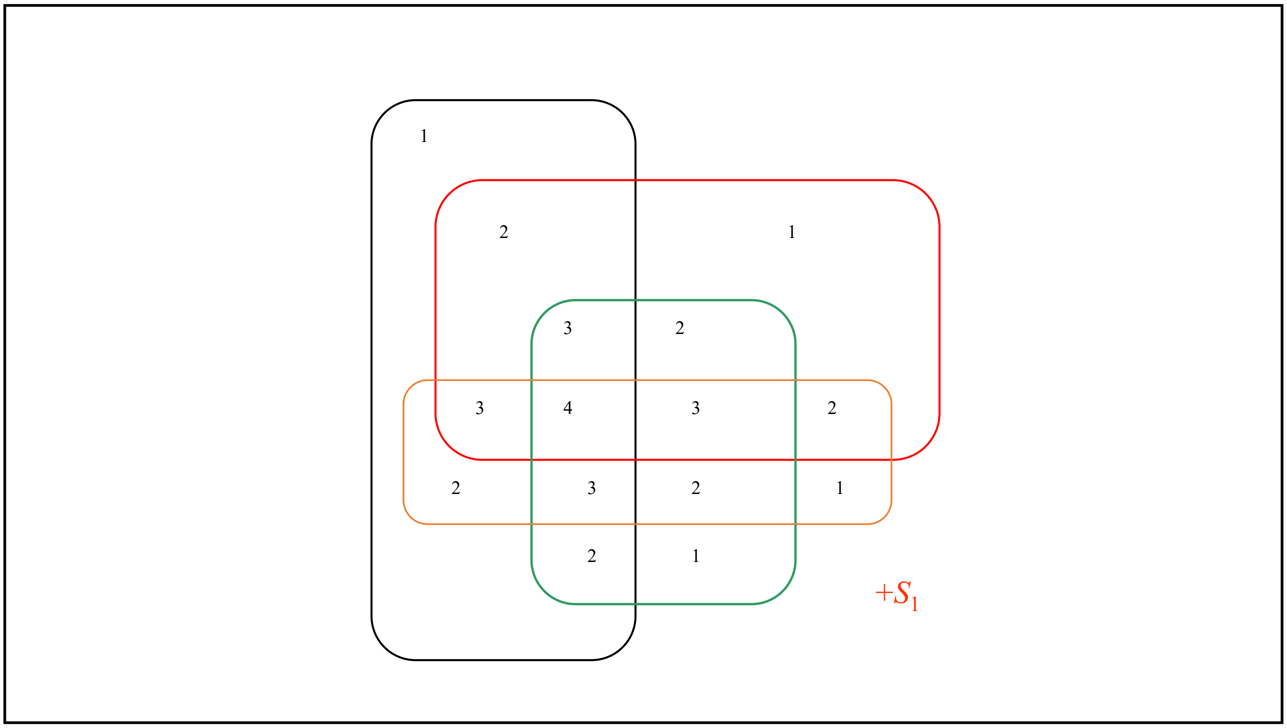

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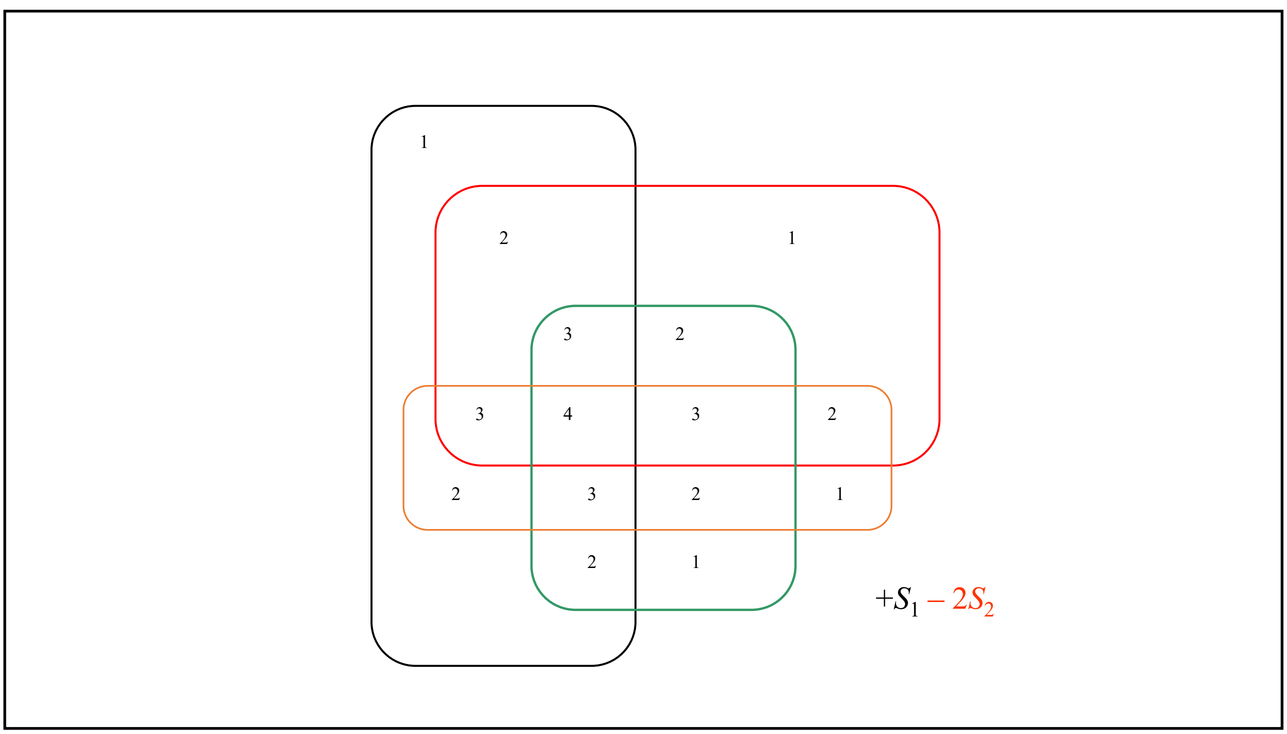

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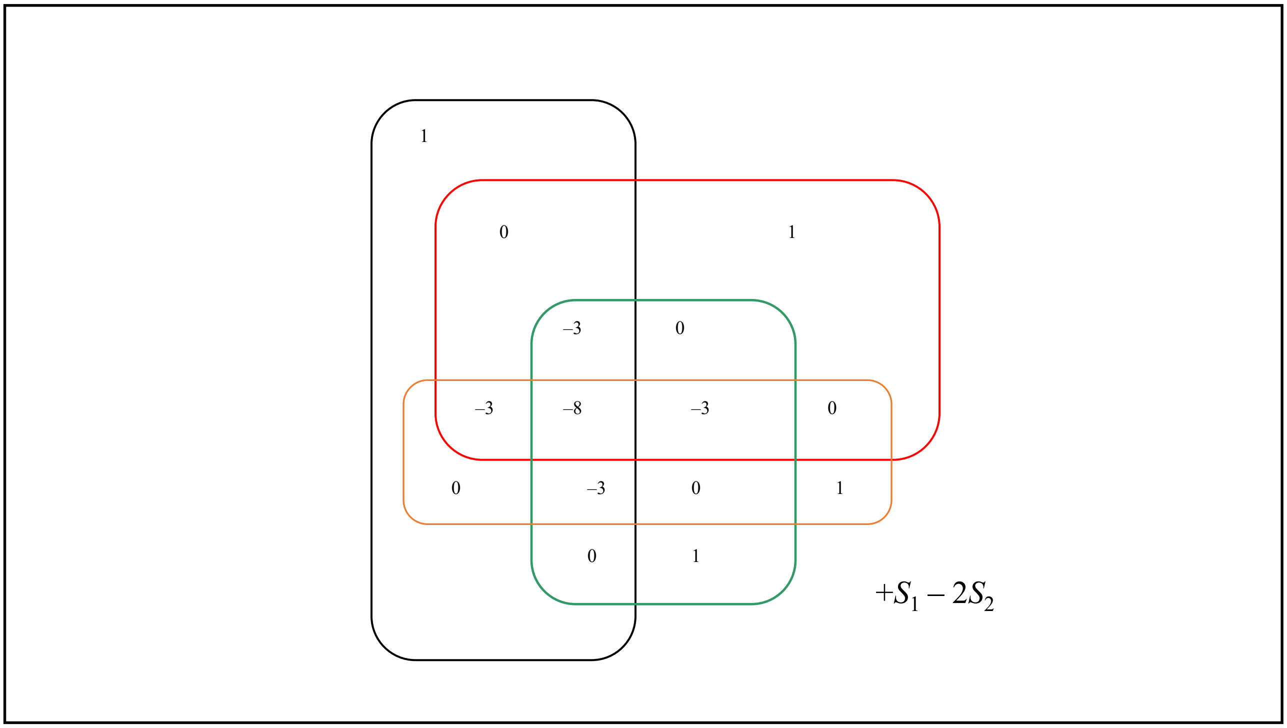

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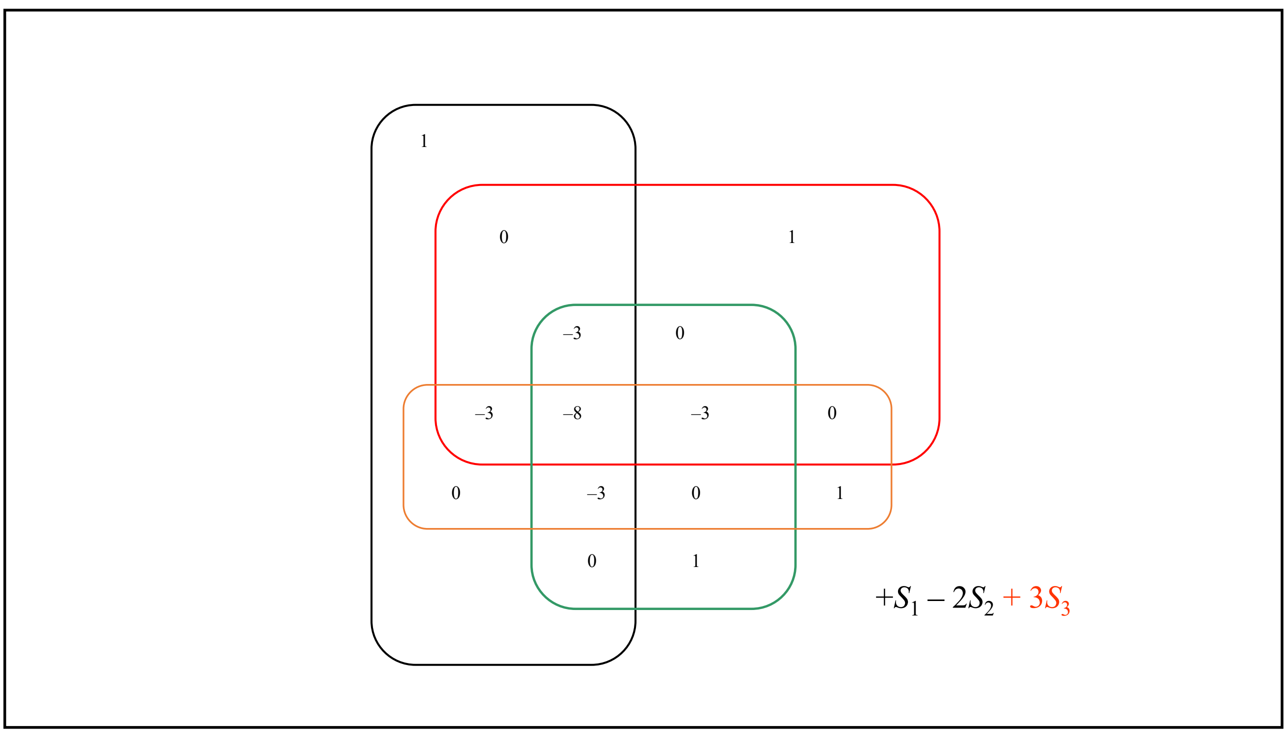

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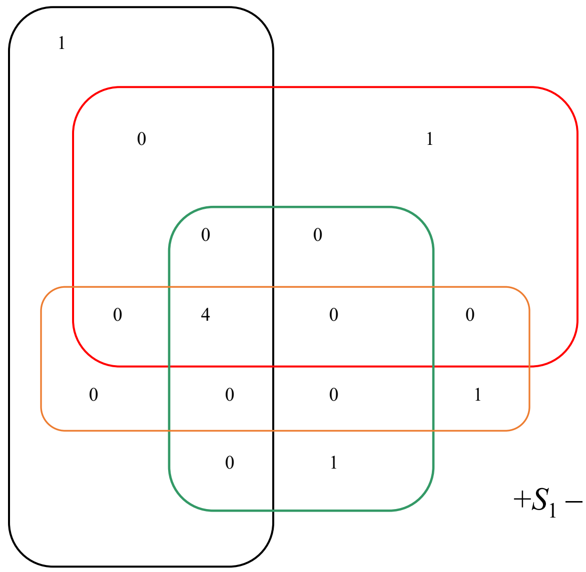

$+S_1 - 2S_2 + 3S_3$

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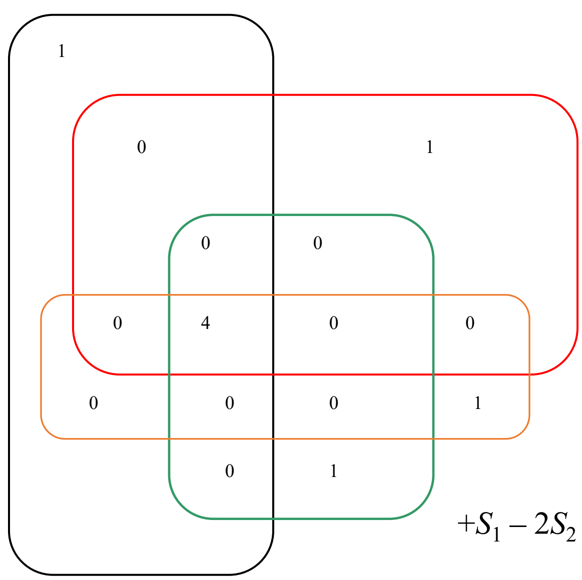

$+S_1 - 2S_2 + 3S_3 - 4S_4$

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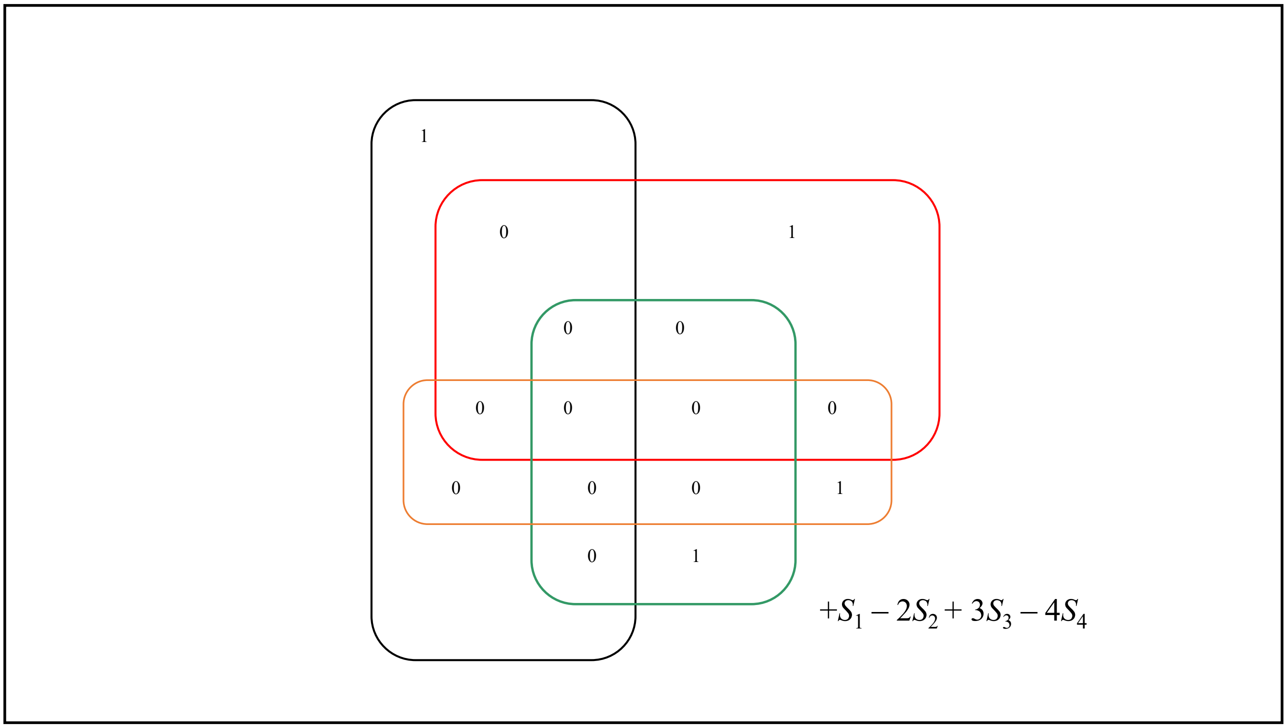

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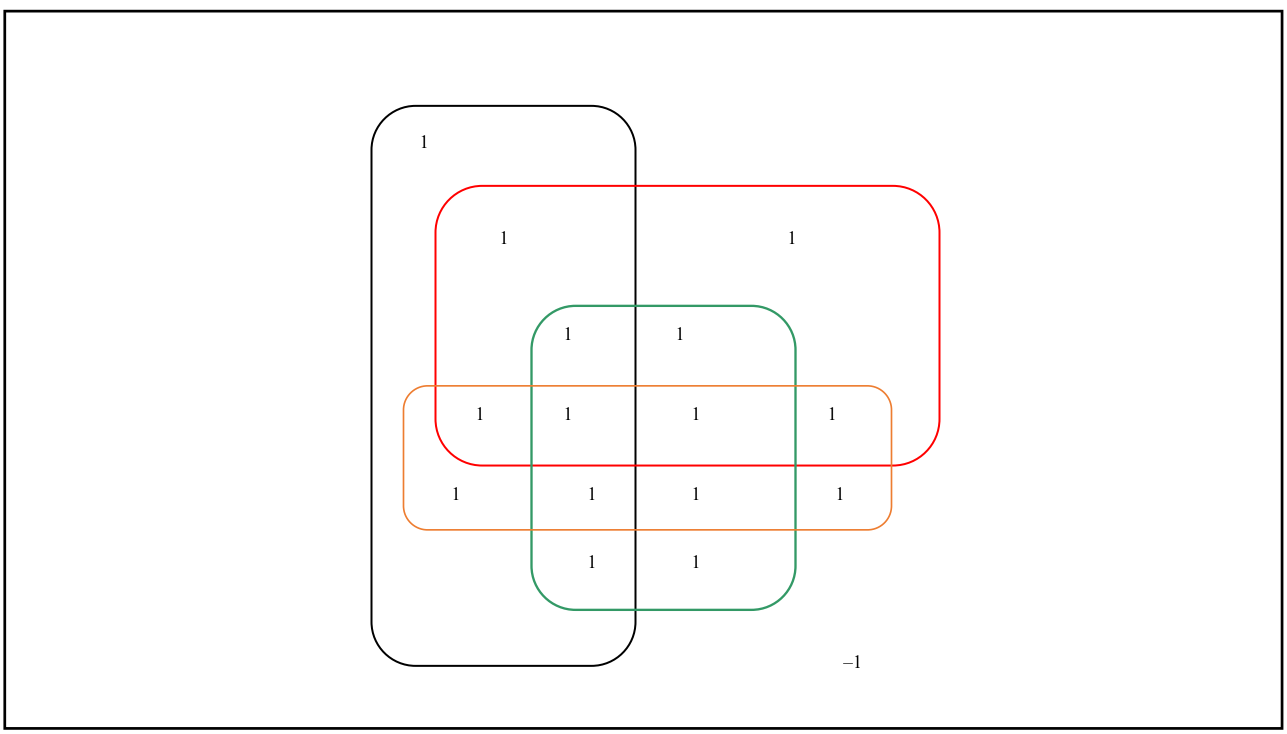

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Warmup: Nothing in its Place

- Eight CDs lie on a table outside their case next to the eight empty cases. Each CD is placed on a case at random, one CD per case. *What is the probability that no CD is returned to its case?*
- Is it more or less likely that no CD is returned to its case when instead of 8 CDs we have 80 CDs?

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Derangements

- If we view a permutation as a mapping, a ***derangement*** is a permutation in which no element maps to itself, such as 31254, 23451, 54321, etc.
- *Examples*
  - In how many ways can you place bets on the arriving positions of 8 race cars so that you lose *all* your bets?
  - In how many ways can you return 8 CDs to their cases so that no CD is in the right case?
  - Eight couples arrive at a party. In how many ways can they choose dancing partners so that no one dances with their spouse?

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Derangements…

- We denote by d_n the number of derangements of $\{1,2,\ldots,n\}$

  $n = 1$: $d_1 = 0$, as **1** is the only permutation

  $n = 2$: $d_2 = 1$, as **21** is the only derangement

  $n = 3$: $d_3 = 2$, from **231** and **312**

  $n = 4$: $d_4 = 9$, from the following permutations

2143	**2341**	**2413**
3142	**3412**	**3421**
4123	**4312**	**4321**

We get the sequence of d_i's: $0, 1, 2, 9, 44, 265, 1854, 14833, 133496, \ldots$

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The number of derangements

Theorem. For $n \geq 1$, $\quad d_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$

Proof. Here, S is the set of permutations of $1, \ldots, n$, so $S_0 = n!$.
For $j = 1, \ldots, n$, c_j is the condition that in a permutation, j is in its natural position.
For example, a permutation $i_1 i_2 \ldots i_n$ satisfies c_1 if it has the form $1 i_2 \ldots i_n$, i.e., if $i_1 = 1$.
Thus, $N(c_1) = (n-1)!$ and $S_1 = C(n,1)(n-1)! = n!$.
Similarly, permutations in $A_1 \cap A_2$ are of the form $12 i_3 \ldots i_n$ and $N(c_1 c_2) = (n-2)!$.
Therefore, $S_2 = C(n,2)(n-2)! = n!/2!$.
More generally, for any $1 \leq k \leq n$, the permutations in $A_1 \cap A_2 \cap \cdots \cap A_k$ have the form $12 \ldots k i_{k+1} \ldots i_n$, where $i_{k+1} \ldots i_n$ is an arbitrary permutation of $k+1, \ldots, n$.
Thus, $N(c_1 c_2 \cdots c_k) = (n-k)!$ and $S_k = C(n,k)(n-k)!$.
A direct application of the inclusion-exclusion principle yields:

$$d_n = S_0 - S_1 + \cdots + (-1)^n S_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right) \quad \blacksquare$$

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How does d_n change as n grows?

- Recall from basic calculus:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^k}{k!} + \cdots$$

- In particular, for $x = -1$

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots$$

- Therefore

$$\lim_{n \to \infty} \frac{d_n}{n!} = e^{-1} \quad \text{and} \quad \left| \frac{d_n}{n!} - \frac{1}{e} \right| \le \frac{1}{(n+1)!}$$

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Derangements in Random Permutations

- Since d_n is the # of permutations of $\{1, \ldots, n\}$ that are derangements then the probability that a *random* permutation is a derangement is

$$\frac{d_n}{n!} \approx e^{-1} \approx 0.3679$$

- This probability is largely independent of n

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Recursive Structure

Theorem. $d_n = (n-1)(d_{n-1} + d_{n-2})$ for $n = 3,4,5, \ldots$

$$\begin{aligned} d_3 = 2(d_1 + d_2) &= 2(0+1) = 2 \\ d_4 = 3(d_2 + d_3) &= 3(1+2) = 9 \\ d_5 = 4(d_3 + d_4) &= 4(2+9) = 44 \\ d_6 = 5(d_4 + d_5) &= 5(9+44) = 265 \end{aligned}$$

Proof (Euler)**.** The d_n derangements of $\{1, \ldots, n\}$ can be partitioned into $n-1$ classes depending on which of $2,3, \ldots, n$ appears in the first position. Since each class "looks the same", $d_n = (n-1)t_n$ where t_n is the number of derangements of the form $2i_2i_3 \ldots i_n$. Further partition the t_n derangements into two types: t_n' derangements of the form $21i_3i_4 \ldots i_n$; and t_n'' derangements of the form $2i_2i_3i_4 \ldots i_n$ with $i_2 \neq 1, i_3 \neq 3, i_4 \neq 4, \ldots i_n \neq n$. But $t_n' = d_{n-2}$ and, by relabeling, we can see that $t_n'' = d_{n-1}$, which concludes the proof.

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$d_n = (n-1)(d_{n-1} + d_{n-2})$

$$\begin{aligned} d_n - nd_{n-1} &= -[d_{n-1} - (n-1)d_{n-2}] \\ &= (-1)^2[d_{n-2} - (n-2)d_{n-3}] \\ &= (-1)^3[d_{n-3} - (n-3)d_{n-4}] \\ \ldots \\ &= (-1)^{n-2}[d_2 - 2d_1] \end{aligned}$$

Since $d_2 = 1$ and $d_1 = 0$ we obtain the equivalent formula

$$d_n = nd_{n-1} + (-1)^n, \text{ for } n = 2,3,4\ldots$$

By iterating this recurrence, you get an alternative proof of

$$d_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$$

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Exercise

A class of n children takes a walk every day in a single file. every child, except the first is preceded by another child. Since it is not very interesting to always see the same person in front of you, on the second day they decide to switch positions so that no child is preceded by the same child from the previous day. *In how many ways can the children switch positions?*

Hint: we want the permutations that *do not* contain any of the patterns $12, 23, 34, \dots, (n-1)n$

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Relative Forbidden Positions

- The number of permutations of $\{1, \dots, n\}$ that *do not* contain any of the patterns $12, 23, 34, \dots, (n-1)n$ is denoted by q_n

$n = 2$: only 21 is allowed.

$n = 3$: 132, 213, 321 are allowed.

$n = 4$: 4132, 3214, 2431, 1324, 4321, 3241, 2413, 1432, 4213, 3142, 2143.

Theorem. For $n \geq 1$

$$q_n = n! - \binom{n-1}{1}(n-1)! + \binom{n-1}{2}(n-2)! - \binom{n-1}{3}(n-3)! + \cdots + (-1)^n \binom{n-1}{n-1} 1!$$

Theorem. $q_n = d_n + d_{n-1}$

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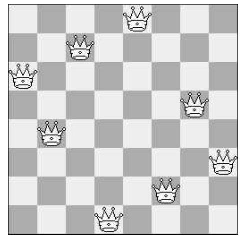
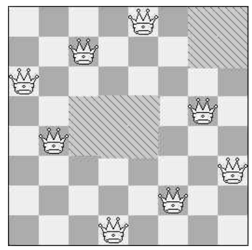
Rook polynomials: Warmup

- In how many ways can you place 8 rooks in a 8×8 chessboard so that no two of them can take each other?



- How about if some positions (shaded blocks) are forbidden?



Rook polynomials

- Many problems that have nothing to do with chess can be modeled using arrangements of rooks on a board

Example. Professor Rook has to grade five finals and has five students to help with grading. Each grader should get exactly one exam in a topic they know. *In how many ways can this be done?*

Model as 5×5 chessboard with forbidden rook positions

	C	Fortran	Java	Lisp	Python
Chuck		$\times$			
Jane		$\times$			
Pete	$\times$		$\times$		
Sam	$\times$			$\times$	
Tom		$\times$			$\times$

a forbidden position

Notation

- Let C denote a specific $m \times n$ board. Then $r_k(C)$ denotes the number of ways in which k rooks can be placed in C so that no two attack each other, i.e., no two occupy either the same row or the same column
- For convenience, we define $r_0(C) = 1$
- The ***rook polynomial*** of C, denoted $r(C, x)$, is defined as

$$r(C,x) = \sum_{k=0}^{\infty} r_k(C) x^k$$

Note: the coefficient of x^k is the number of ways in which k non-attacking rooks can be placed in C

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Example

- Consider a chessboard C of size $n \times n$ with rook polynomial $\mathcal{R}_n(x) = r(C, x)$. Then

$$\mathcal{R}_1(x) = 1 + x$$

$$\mathcal{R}_2(x) = 1 + 4x + 2x^2$$

$$\mathcal{R}_3(x) = 1 + 9x + 18x^2 + 6x^3$$

Exercise. What is $\mathcal{R}_4(x)$?

Exercise. What is $\mathcal{R}_n(x)$?

Exercise. What is $\mathcal{R}_{m,n}(x)$?

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Example

- In this example, the board contains forbidden positions, marked with ×

3	2	1
4	×	×
×	5	6

$r_1 = 6$: the number of unshaded squares

$r_2 = 8$: $(1,4), (1,5), (2,4), (2,6), (3,5), (3,6), (4,5), (4,6)$.

$r_3 = 2$: $(1,4,5), (2,4,6)$

$r_k = 0$ for $k > 3$

$$r(C, x) = 1 + 6x + 8x^2 + 2x^3$$

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Analysis Techniques

- The case-by-case analysis used so far quickly becomes unmanageable for large boards
- Two techniques, ***decomposition*** and ***reduction*** make the analysis simpler
  - If a large board C can be partitioned into k independent sub-boards $C_1, C_2, \ldots, C_k$, decomposition allows you to compute the polynomial of C by multiplying the polynomials of $C_1, C_2, \ldots, C_k$
  - Reduction uses the Addition Rule to compute the polynomial of C from the polynomials of two smaller boards that result from either placing a rook or not at a arbitrarily chosen cell of C

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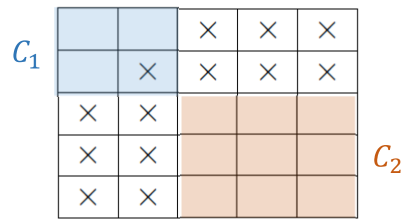
Decomposition

- Given a board C, two sub-boards are *independent* if they share no open cells in the same row or same column



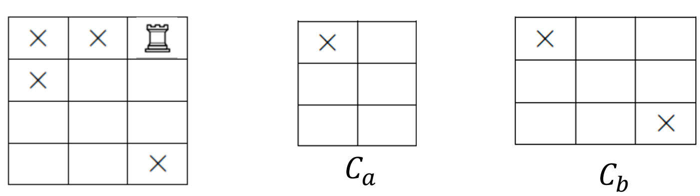
$r(C_1, x) = 1 + 3x + x^2$ $\quad$ $r(C_2, x) = 1 + 9x + 18x^2 + 6x^3$

$r(C, x) = r(C_1, x) \cdot r(C_2, x) = 1 + 12x + 46x^2 + 69x^3 + 36x^4 + 6x^5$

Theorem. If C is a board made up of pairwise independent sub-boards $C_1, \ldots, C_k$ then $r(C, x) = r(C_1, x) \cdot r(C_2, x) \cdots r(C_k, x)$

Reduction

- Uses the addition rule by deciding whether to place a rook or not on a given cell of C



- Since the two cases cover all possibilities and are mutually disjoint, by the addition rule, $r_k(C) = r_{k-1}(C_a) + r_k(C_b)$

$$1 + \sum_{k=1}^{n} r_k(C)x^k = x \sum_{k=1}^{n} r_{k-1}(C_a)x^{k-1} + 1 + \sum_{k=1}^{n} r_k(C_b)x^k$$

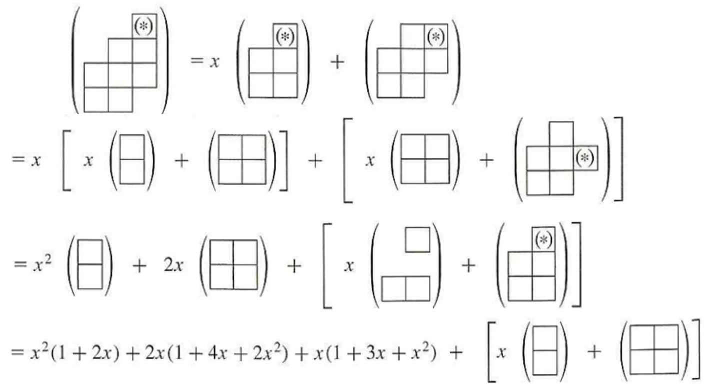
$$r(C, x) = x \cdot r(C_a, x) + r(C_b, x)$$

Example

- Compute the rook polynomial of the board:

×	×	♖
×		
		×

- If B is a board, (B) denotes $r(B, x)$



$$= 3x + 12x^2 + 7x^3 + x(1+2x) + (1+4x+2x^2) = 1 + 8x + 16x^2 + 7x^3$$

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Example

- Back to Prof. Rook example. He needs to assign exactly one grader to each exam

	C	F	J	L	P
C	×				
J		×		×	
P			×		
S		×		×	
T	×				×

- After rearranging columns first, and then rows, we get:

	L	F	J	C	P
C				×	
J	×	×			
P			×		
S	×	×			
T				×	×

	L	F	J	C	P
S	×	×			
J	×	×			
P			×		
C				×	
T				×	×

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Counting # of Valid Assignments

- Once again, we use inclusion-exclusion.
  - Let $S =$ unconstrained assignments, one grader per exam
  - Let $c_i =$ i-th grader is assigned to a forbidden exam
  - We want $N(\overline{c_1}\overline{c_2}\overline{c_3}\overline{c_4}\overline{c_5}) = |\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4} \cap \overline{A_5}| = S_0 - S_1 + S_2 - S_3 + S_4 - S_5$
- Clearly, $S_0 = 5! = 120$
- S_1 is given by

$$S_1 = N(c_1) + N(c_2) + N(c_3) + N(c_4) + N(c_5)$$

- S_2 is given by

$$N(c_1c_2) + N(c_1c_3) + N(c_1c_4) + N(c_1c_5) + N(c_2c_3) + N(c_2c_4) + N(c_2c_5) + N(c_3c_4) + N(c_3c_5) + N(c_4c_5)$$

- Proceed similarly for S_3, S_4, S_5

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Computing S_1

- $N(c_1) = 4! + 4!$ (assigning Sam to each of List or Fortran)
- $N(c_2) = 4! + 4!$ (assigning Jane to each of List or Fortran)
- $N(c_3) = 4!$ (assigning Pete to Java)
- $N(c_4) = 4!$ (assigning Chuck to C)
- $N(c_5) = 4! + 4!$ (assigning Tom to each of C or Python)

Exams

Graders	L	F	J	C	P
S	×	×			
J	×	×			
P			×		
C				×	
T				×	×

- Therefore, $S_1 = 8(4!) = 192$

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Computing S_2

- $N(c_1 c_2) = 2 \cdot 3!$, as there are 2 ways to assign Sam and Jane to two different exams with both occupying a forbidden position, and 3! ways to assign the rest to different exams without regard to constraints
- $N(c_1 c_5) = (2 \cdot 2) \cdot 3!$, as Sandra and Tom can be assigned freely to any of their forbidden positions

	L	F	J	C	P
S	×	×			
J	×	×			
P			×		
C				×	
T				×	×

- For the remaining 2-combinations we get:

c_1c_2	c_1c_3	c_1c_4	c_1c_5	c_2c_3	c_2c_4	c_2c_5	c_3c_4	c_3c_5	c_4c_5
$2 \cdot 3!$	$2 \cdot 3!$	$2 \cdot 3!$	$4 \cdot 3!$	$2 \cdot 3!$	$2 \cdot 3!$	$4 \cdot 3!$	$1 \cdot 3!$	$2 \cdot 3!$	$1 \cdot 3!$

- Thus, $S_2 = 22(3!) = 132$

Complement Board

- There is a relationship between the S_i's and the rook polynomial for the *complement board* C_f of forbidden positions obtained by swapping the state of each cell
- Each permutation in S_i consists of i forbidden choices followed by $n - i$ arbitrary choices
- Thus $S_1 = r_1(C_f)(5-1)!$, $S_2 = r_2(C_f)(5-2)!$ and, in general, $S_i = r_i(C_f)(5-i)!$

×	×			
×	×			
		×		
			×	
			×	×

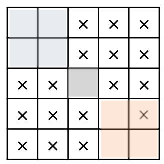
C

		×	×	×
		×	×	×
×	×		×	×
×	×	×		×
×	×	×		

$\boldsymbol{C_f}$

Complement Board…

- We can greatly simplify our work and expedite the solution by first computing the polynomial $r(C_f, x)$
- Using the decomposition principle, we can partition C_f into three independent sub-boards $\boldsymbol{C_1}, \boldsymbol{C_2}, \boldsymbol{C_3}$



$r(C_1, x) = 1 + 4x + 2x^2$

$r(C_2, x) = 1 + x$

$r(C_3, x) = 1 + 3x + x^2$

$$r(C_f, x) = r(C_1, x) \cdot r(C_2, x) \cdot r(C_3, x)$$
$$= 1 + 8x + 22x^2 + 25x^3 + 12x^4 + 2x^5$$
$$N(\overline{c}_1\overline{c}_2\overline{c}_3\overline{c}_4\overline{c}_5) = S_0 - S_1 + S_2 - S_3 + S_4 - S_5$$
$$= 5! - 8(4!) + 22(3!) - 25(2!) + 12(1!) - 2(0!) = 20$$

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Exercise

Let $A = \{1,2,3,4\}$ and $B = \{u, v, w, x, y, z\}$. How many 1-1 functions $f: A \rightarrow B$ satisfy:

$f(1) \in \{w, x, y, z\}$,
$f(2) \in \{u, v, x, y, z\}$,
$f(3) \in \{u, v, x\}$,
$f(4) \in \{u, v, x\}$

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