1.2.3 Exact solution

- the two stationary sols to the ~~wa~~ wave eq. are real & Im. components of $u(t,x,y) = e^{i(k_x x + k_y y - \omega t)}$ (1.6)

show 1.6 satisfies $u_{tt} = c^2 \Delta u$

$u_t = -\omega i u$, $\underline{u_{tt} = (-\omega i)^2 u = -\omega^2 u}$

$\underline{\Delta u = -(k_x^2 + k_y^2) u}$

$\Rightarrow \quad +\omega^2 u = c^2 (k_x^2 + k_y^2) u$

- $\omega(c, k_x, k_y) = c|k| = c\sqrt{k_x^2 + k_y^2} \Rightarrow \underline{\omega^2 = c^2 (k_x^2 + k_y^2)}$

$\Rightarrow$ 1.6 satisfies $u_{tt} = c^2 \Delta u$. $\square$

1.2.4 Dispersion coefficient.

$m_x = m_y$, $k_x = k_y = k$. $u_{ij}^n = e^{i(kh(i+j) - \tilde{\omega} n \Delta t)}$ (1.7)

Insert (1.7) into $(u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1}) \frac{1}{\Delta t^2} = c^2 \left(\frac{u_{i+1,j}^n - 2u_{ij}^n + u_{i-1,j}^n}{h^2} + \frac{u_{i,j+1}^n - 2u_{ij}^n + u_{i,j-1}^n}{h^2} \right)$

$\hookrightarrow e^{i(kh(i+j) - \tilde{\omega}(n+1)\Delta t)} - 2 \ldots$ $\left(\frac{1}{\frac{c}{a} + \frac{d}{b}} = \frac{ab}{ad+bc} \right)$

$\frac{1}{\sqrt{2}} = \Delta t \frac{c}{h} = \frac{1}{\Delta t}(u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1}) \cdot \frac{(ab)}{ad+bc} \ldots (ab = h^2)$

$\Leftrightarrow c \cdot \mathrm{CFL} = \frac{h^2}{\Delta t}(u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1}) / h(u_{i,j+1}^n - 2u_{ij}^n + u_{i,j-1}^n + u_{i+1,j}^n - 2u_{ij}^n + u_{i-1,j}^n)$

$= \frac{h}{\Delta t} \quad - \, " \, - / (-4u_{ij}^n \ldots)$

$= \frac{h}{\Delta t} \frac{e^{i(kh(i+j) - \tilde{\omega}(n+1)\Delta t)} - 2e^{i(kh(i+j) - \tilde{\omega} n \Delta t)} + e^{i(kh(i+j) - \tilde{\omega}(n-1)\Delta t)}}{-4e^{i(kh(i+j) - \tilde{\omega} n \Delta t)} + e^{i(hk(i+j+1) - \tilde{\omega} n \Delta t)} \cdot 2 + 2e^{i(hk(i+j-1) - \tilde{\omega} n \Delta t)}}$

$\frac{\Delta t c}{h} = \frac{h}{\Delta t 2c} \left(e^{i(kh(i+j))} \left(e^{-i\tilde{\omega}(n+1)\Delta t} - 2e^{-i\tilde{\omega} n \Delta t} + e^{-i\tilde{\omega}(n-1)\Delta t} \right) / e^{-i\tilde{\omega} n \Delta t} \left(-2e^{ikh(i+j)} + e^{ikh(i+j+1)} + e^{ikh(i+j-1)} \right) \right)$

$\frac{\Delta t c}{h} = \frac{h}{\Delta t 2c} \cancel{e^{-i\tilde{\omega} n \Delta t}} \left(e^{-i\tilde{\omega}\Delta t} - 2 + e^{i\tilde{\omega}\Delta t} \right) / \left(-2 + e^{ihk} + e^{-ikh} \right)$

$\Leftrightarrow \frac{\Delta t c}{h} = \frac{h}{\Delta t 2c} \frac{2(\cos(\tilde{\omega}\Delta t) - 1)}{2(\cos(hk) - 1)} \Leftrightarrow 2(\mathrm{CFL})^2 = \frac{\cos(\tilde{\omega}\Delta t) - 1}{\cos(hk) - 1} = 2\left(\frac{1}{\sqrt{2}}\right)^2 = 1$

$\Leftrightarrow \cos(hk) = \cos(\tilde{\omega}\Delta t) \Leftrightarrow \arccos(\cos(hk)) = \arccos(\cos(\tilde{\omega}\Delta t))$

$\Leftrightarrow \frac{hk}{\Delta t} = \tilde{\omega}$. now $C = \frac{c \Delta t}{h} \Rightarrow \frac{h}{\Delta t} = \frac{c}{C} = \sqrt{2} c \rightsquigarrow \tilde{\omega} = k\sqrt{2} c$

and $\omega = c\sqrt{k_x^2 + k_y^2} = c\sqrt{2k^2} = k\sqrt{2} c = \omega$ thus $\underline{\underline{\omega = \tilde{\omega}}}$ for $\mathrm{CFL} = \frac{1}{\sqrt{2}}$. $\square$