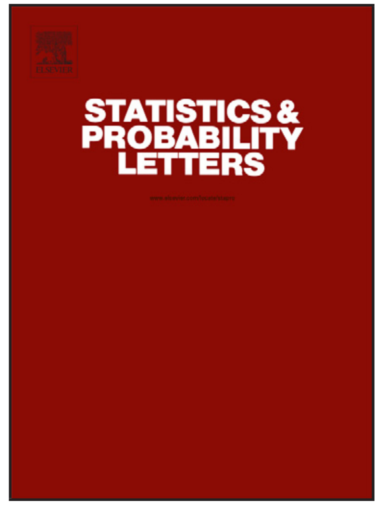

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A result on the first-passage time of an Ornstein-Uhlenbeck process

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Abstract

Consider the first time an Ornstein-Uhlenbeck process starting from zero crosses a constant positive threshold. Assuming that the asymptotic mean is above the threshold, conditions on the asymptotic variance relative to the distance between the threshold and the asymptotic mean are given that ensures the finiteness of the positive Laplace transforms.

Key words: Laplace transform, Martingales, Eigenfunctions, Conditional moments, Hermite polynomials

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1 Introduction

The first-passage time of an Ornstein-Uhlenbeck process through a constant threshold has been used in diverse areas of applied mathematics when it is reasonable to assume an underlying stochastic process in a problem of interest, which eventually will reach a certain level that leads to some observed event. In biology, it has been used as a model for neuronal activity (see e.g. Bulsara et al, 1996; Lansky, Sacerdote and Tomasetti, 1995; Ricciardi and Sacerdote, 1979; Shimokawa et al, 2000; Tuckwell, Wan and Rospars, 2002), in survival analysis the model has been applied by Aalen and Gjessing (2004), and also in mathematical finance it has found applications (see e.g. Jeanblanc and Rutkowski, 2000; Leblanc and Scaillet, 1998; Linetsky, 2004). A large literature has been dedicated to find the distribution of the first-passage time (for an overview see Alili, Patie and Pedersen, 2005).

The theoretical results of the present paper have been used to define moment estimators from observations of first-passage times in the context of neuronal modeling (Ditlevsen and Lansky, 2005). The same techniques were applied in (Ditlevsen and Lansky, 2006) for defining estimators in the Cox-Ingersoll-Ross process (Cox, Ingersoll and Ross, 1985), which in the neuronal literature is called the Feller process (Lansky, Sacerdote and Tomasetti, 1995) because (Feller, 1951) proposed it as a model for population growth.

In this paper a formula for the conditional moments of any order of the Ornstein-Uhlenbeck-process is first derived, and then used to define suitable martingales. The martingales are then applied to give conditions on the parameter space so that the positive Laplace transform of the first-passage time of the process through a constant threshold is finite.

2 The Ornstein-Uhlenbeck process

Consider the Ornstein-Uhlenbeck process:

$$\mathrm{d}X_t = -\beta(X_t - \alpha)\mathrm{d}t + \sigma \mathrm{d}W_t\,; \quad X_0 = x_0 = 0, \tag{1}$$

where W is a Wiener process, and $\theta = (\alpha, \beta, \sigma) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$. The *generator* of (1) is the differential operator

$$L_\theta = \frac{1}{2}\sigma^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} - \beta(x - \alpha)\frac{\mathrm{d}}{\mathrm{d}x}$$

defined for all twice differentiable functions. An *eigenfunction* $\varphi(x;\theta)$ for L_θ with associated *eigenvalue* $\lambda(\theta)$ is a twice continuously differentiable function that fulfills

$$L_\theta \varphi(x;\theta) = -\lambda(\theta)\varphi(x;\theta) \tag{2}$$

for all $x \in \mathbb{R}$. For simplicity consider the centered process $\tilde{X}_t = X_t - \alpha$, which has eigenfunctions

$$\varphi_k = \sum_{m=0}^{[\frac{k}{2}]} \frac{(-1)^m k!}{(k-2m)!m!} \left(\frac{\sigma^2}{4\beta}\right)^m x^{k-2m}$$

for $k \in \mathbb{N}$, where $[a]$ denotes the largest integer such that $[a] \leq a$. The associated eigenvalues are $\lambda_k = k\beta$. This can easily be checked by substitution into (2). In Karlin and Taylor (1981, p.333) the eigenfunctions are given for $\sigma = \beta = 1$. Note that $\varphi_k(x) = (\sigma^2/4\beta)^{k/2} H_k(x\sqrt{\beta/\sigma^2})$, where H_k are Hermite polynomials (Lebedev, 1972). For this model we have for $t > s$ that

$$E\left[\varphi_k(\tilde{X}_t;\theta)|\tilde{X}_s\right] = e^{-k\beta(t-s)}\varphi_k(\tilde{X}_s;\theta), \tag{3}$$

(Kessler and Sørensen, 1999), where $E[\cdot]$ denotes expectation under the law of (1). This yields the formula for the conditional moments

$$E[\tilde{X}_t^k \mid \tilde{X}_s] = e^{-k\beta(t-s)} \sum_{m=0}^{[\frac{k}{2}]} \frac{k!}{(k-2m)!m!} \left(\frac{\sigma^2}{4\beta}\right)^m (e^{2\beta(t-s)}-1)^m \tilde{X}_s^{k-2m}. \tag{4}$$

In effect, (4) is obviously true for $k = 1$ and 2. Assume it is true for $k - 2m$ for all $1 \leq m \leq [\frac{k}{2}]$. First note that

$$\begin{aligned} E\left[\varphi_k(\tilde{X}_t;\theta)|\tilde{X}_s\right] &= \sum_{m=0}^{[\frac{k}{2}]} \frac{(-1)^m k!}{(k-2m)!m!} \left(\frac{\sigma^2}{4\beta}\right)^m E\left[\tilde{X}_t^{k-2m}|\tilde{X}_s\right] \\ &= E\left[\tilde{X}_t^k|\tilde{X}_s\right] + \sum_{m=1}^{[\frac{k}{2}]} \frac{(-1)^m k!}{(k-2m)!m!} \left(\frac{\sigma^2}{4\beta}\right)^m E\left[\tilde{X}_t^{k-2m}|\tilde{X}_s\right] \end{aligned}$$

so that (3) yields

$$\begin{aligned} E\left[\tilde{X}_t^k|\tilde{X}_s\right] = e^{-k\beta(t-s)} &\sum_{m=0}^{[\frac{k}{2}]} \frac{(-1)^m k!}{(k-2m)!m!} \left(\frac{\sigma^2}{4\beta}\right)^m \tilde{X}_s^{k-2m} \\ &- \sum_{m=1}^{[\frac{k}{2}]} \frac{(-1)^m k!}{(k-2m)!m!} \left(\frac{\sigma^2}{4\beta}\right)^m E\left[\tilde{X}_t^{k-2m}|\tilde{X}_s\right]. \end{aligned}$$

The last summand on the right hand side can by the induction assumption be written

$$
\begin{aligned}
&-\sum_{m=1}^{[\frac{k}{2}]} \frac{(-1)^m k!}{(k-2m)!m!}\left(\frac{\sigma^2}{4\beta}\right)^m e^{-(k-2m)\beta(t-s)} \times \\
&\sum_{j=0}^{[\frac{k-2m}{2}]} \frac{(k-2m)!}{(k-2(m+j))!j!}\left(\frac{\sigma^2}{4\beta}\right)^j (e^{2\beta(t-s)}-1)^j \tilde{X}_s^{k-2(m+j)} \\
=&-\sum_{n=1}^{[\frac{k}{2}]}\sum_{m=1}^{n} \frac{(-1)^m k!}{(k-2m)!m!}\left(\frac{\sigma^2}{4\beta}\right)^m e^{-(k-2m)\beta(t-s)} \times \\
&\frac{(k-2m)!}{(k-2n)!(n-m)!}\left(\frac{\sigma^2}{4\beta}\right)^{(n-m)} (e^{2\beta(t-s)}-1)^{(n-m)} \tilde{X}_s^{k-2n} \\
=&-\sum_{n=1}^{[\frac{k}{2}]} \frac{k!}{(k-2n)!n!}\left(\frac{\sigma^2}{4\beta}\right)^n e^{-k\beta(t-s)} \tilde{X}_s^{k-2n} \times \\
&\sum_{m=1}^{n} \binom{n}{m} (-e^{2\beta(t-s)})^m (e^{2\beta(t-s)}-1)^{(n-m)} \\
=&-\sum_{n=1}^{[\frac{k}{2}]} \frac{k!}{(k-2n)!n!}\left(\frac{\sigma^2}{4\beta}\right)^n e^{-k\beta(t-s)} \tilde{X}_s^{k-2n}\left((-1)^n - (e^{2\beta(t-s)}-1)^n\right)
\end{aligned}
$$

where we have used the identities $\sum_{m=1}^{N}\sum_{j=0}^{N-m} a_{m,j} = \sum_{n=1}^{N}\sum_{m=1}^{n} a_{m,n-m}$ through the change of variable $n = m + j$, and $(a+b)^n = \sum_{i=0}^{n}\binom{n}{i}a^{(n-i)}b^i = a^n + \sum_{i=1}^{n}\binom{n}{i}a^{(n-i)}b^i$, where $a = (e^{2\beta(t-s)} - 1)$ and $b = -e^{2\beta(t-s)}$. This yields

$$
\begin{aligned}
E\left[\tilde{X}_t^k|\tilde{X}_s\right] =& e^{-k\beta(t-s)} \sum_{m=0}^{[\frac{k}{2}]} \frac{(-1)^m k!}{(k-2m)!m!}\left(\frac{\sigma^2}{4\beta}\right)^m \tilde{X}_s^{k-2m} \\
&-\sum_{n=1}^{[\frac{k}{2}]} \frac{k!}{(k-2n)!n!}\left(\frac{\sigma^2}{4\beta}\right)^n e^{-k\beta(t-s)}\left((-1)^n - (e^{2\beta(t-s)}-1)^n\right)\tilde{X}_s^{k-2n}
\end{aligned}
$$

so that the coefficient to $\tilde{X}_s^{k-2i}$ for $1 \le i \le [\frac{k}{2}]$ is

$$
\begin{aligned}
&e^{-k\beta(t-s)}\frac{k!}{(k-2i)!i!}\left(\frac{\sigma^2}{4\beta}\right)^i\left((-1)^i - \left((-1)^i - (e^{2\beta(t-s)}-1)^i\right)\right) \\
=&e^{-k\beta(t-s)}\frac{k!}{(k-2i)!i!}\left(\frac{\sigma^2}{4\beta}\right)^i(e^{2\beta(t-s)}-1)^i
\end{aligned}
$$

which is also the case for $i = 0$. We have finally obtained (4). □

3 The first-passage time through a constant threshold

Define the stopping time

$$T = \inf\{t > 0; X_t \geq S > 0\}, \tag{5}$$

which is the first-passage time of X_t through a constant threshold. The Laplace transform of T is given by the representation

$$E\Big[e^{\lambda\beta T}\Big] = \frac{\exp\{\frac{\alpha^2\beta}{2\sigma^2}\}D_\lambda\left(\alpha\sqrt{2\beta}/\sigma\right)}{\exp\{\frac{(\alpha-S)^2\beta}{2\sigma^2}\}D_\lambda\left((\alpha-S)\sqrt{2\beta}/\sigma\right)} = \frac{H_\lambda\left(\alpha\sqrt{\beta}/\sigma\right)}{H_\lambda\left((\alpha-S)\sqrt{\beta}/\sigma\right)} \tag{6}$$

for $\lambda < 0$, where $D_\lambda(\cdot)$ and $H_\lambda(\cdot)$ are parabolic cylinder and Hermite functions, respectively, see Lebedev (1972, p.98) and Borodin and Salminen (2002, p.542). The result can be extended to $\lambda > 0$ with certain restrictions on the parameter space to ensure that $E\Big[e^{\lambda\beta T}\Big] < \infty$. A decreasing sequence of subsets of the parameter space is found by defining suitable martingales using the conditional moments of (1), which provides conditions on the parameters for which (6) is finite. The expression becomes particularly simple when $k \in \mathbb{N}$, because then (6) reduces to a fraction of Hermite polynomials.

4 Main result

Theorem 1 *Let X_t and T be given by (1) and (5), respectively, and let $(\alpha, \beta, \sigma) = \theta \in \Theta^{(k)} = \{\theta \,|\, \alpha > S, \sqrt{\sigma^2/\beta} < (\alpha - S)/\lambda^{(k)}\}$ for $k \in \mathbb{N}$, where $\lambda^{(k)}$ is the largest root of the k'th Hermite polynomial. Then*

$$E\left[e^{\lambda\beta T}\right] = \frac{H_\lambda\left(\alpha\sqrt{\beta}/\sigma\right)}{H_\lambda\left((\alpha-S)\sqrt{\beta}/\sigma\right)}$$

for $\lambda \leq k$, where H_λ is the Hermite function.

Note first that this is intuitively true, since for λ a positive integer, (6) is well-defined when θ fulfills the condition given in Theorem 1.

Proof: Define the processes

$$M_t^{(k)} = \sum_{m=0}^{[\frac{k}{2}]} \frac{(-1)^m k!}{(k-2m)!m!}\left(\frac{\sigma^2}{4\beta}\right)^m (1 - e^{-2\beta t})^m e^{k\beta t}\tilde{X}_t^{k-2m} = h_t^k H_k\left(\frac{\tilde{X}_t e^{\beta t}}{2h_t}\right)$$

where $H_k(\cdot)$ are the Hermite polynomials and $h_t = \sqrt{\sigma^2(e^{2\beta t}-1)/4\beta}$. The $M_t^{(k)}$ are martingales with respect to the natural filtration $\mathfrak{F}_t = \sigma(X_s; 0 \le s \le t)$, the sigma-algebra generated by X_s for $0 \le s \le t$. First observe that $M_t^{(k)}$ have finite expectation for all k since they are polynomial functions of a Gaussian variable, which possesses moments of any order. Moreover, for $s < t$ we have

$$
\begin{aligned}
E[M_t^{(k)}|\mathfrak{F}_s] &= E[M_t^{(k)}|\tilde{X}_s] \\
&= \sum_{m=0}^{[\frac{k}{2}]} \frac{(-1)^m k!}{(k-2m)!m!}\left(\frac{\sigma^2}{4\beta}\right)^m (1-e^{-2\beta t})^m e^{k\beta t} E[\tilde{X}_t^{k-2m}|\tilde{X}_s] \\
&= \sum_{m=0}^{[\frac{k}{2}]}\sum_{j=0}^{[\frac{k-2m}{2}]} \frac{(-1)^m k!}{(k-2(m+j))!m!j!}\left(\frac{\sigma^2}{4\beta}\right)^{(m+j)} (1-e^{-2\beta t})^m e^{k\beta s} e^{2m\beta(t-s)}(e^{2\beta(t-s)}-1)^j \tilde{X}_s^{k-2(m+j)} \\
&= \sum_{n=0}^{[\frac{k}{2}]}\sum_{m=0}^{n} \frac{(-1)^m k!}{(k-2n)!m!(n-m)!}\left(\frac{\sigma^2}{4\beta}\right)^n (e^{2\beta(t-s)}-e^{-2\beta s})^m e^{k\beta s}(e^{2\beta(t-s)}-1)^{n-m}\tilde{X}_s^{k-2n} \\
&= \sum_{n=0}^{[\frac{k}{2}]} \frac{k!}{(k-2n)!n!}\left(\frac{\sigma^2}{4\beta}\right)^n e^{k\beta s}\tilde{X}_s^{k-2n} \sum_{m=0}^{n}\binom{n}{m}(e^{-2\beta s}-e^{2\beta(t-s)})^m (e^{2\beta(t-s)}-1)^{n-m} \\
&= \sum_{n=0}^{[\frac{k}{2}]} \frac{(-1)^n k!}{(k-2n)!n!}\left(\frac{\sigma^2}{4\beta}\right)^n e^{k\beta s}\tilde{X}_s^{k-2n}(1-e^{-2\beta s})^n \\
&= M_s^{(k)}.
\end{aligned}
$$

Therefore also $M_{T\wedge t}^{(k)}$, the processes stopped at T, are martingales (Williams, 1991, p.99), where T is given by (5). This yields

$$\alpha^k = (-1)^k E[M_0^{(k)}] = (-1)^k E[M_{T\wedge t}^{(k)}] = E\left[h_{T\wedge t}^k H_k\left(\frac{(\alpha - X_{T\wedge t})e^{\beta(T\wedge t)}}{2h_{T\wedge t}}\right)\right]. \quad (7)$$

Define $\lambda^{(k)}$ to be the largest positive root of the kth Hermite polynomial. Then $\lambda^{(1)} < \lambda^{(2)} < \cdots < \lambda^{(k)} < \lambda^{(k+1)} < \cdots$ (Szegö, 1975, p.46). The first four are $\lambda^{(1)} = 0, \lambda^{(2)} = 1/\sqrt{2}, \lambda^{(3)} = \sqrt{3/2}$ and $\lambda^{(4)} = \sqrt{(3+\sqrt{6})/2}$. Moreover, $H_k(x)$ is positive and monotonically increasing for $x > \lambda^{(k)}$. Define the decreasing sequence of subsets of the parameter space

$$\Theta^{(k)} = \{\theta \,|\, \alpha > S, \sqrt{\sigma^2/\beta} < (\alpha - S)/\lambda^{(k)}\}$$

where division by 0 is defined to be infinity. The first inequality in the definition of the subsets $\Theta^{(k)}$ defines the *supra-threshold regime*, where the asymptotic mean α of X_t is larger than the threshold S. The second inequality restricts the asymptotic standard deviation $\sqrt{\sigma^2/2\beta}$ of X_t to be smaller than a factor proportional to the distance between the asymptotic mean and the threshold, and inversely proportional to the largest positive root of the kth Hermite polynomial. Thus, when k increases, the variance has to be smaller for the exponential moments to be finite, which is a natural requirement. Assume $\theta \in \Theta^{(k)}$, then

$$\frac{(\alpha - X_{T\wedge t})e^{\beta(T\wedge t)}}{2h_{T\wedge t}} = \frac{(\alpha - X_{T\wedge t})e^{\beta(T\wedge t)}}{\sqrt{\sigma^2/\beta}\sqrt{(e^{2\beta(T\wedge t)} - 1)}} > \frac{(\alpha - S)}{\sqrt{\sigma^2/\beta}} > \lambda^{(k)}$$

for all t, so that if $\theta \in \Theta^{(k)}$, then (7) yields

$$\begin{aligned}
\alpha^k &\geq E\left[h_{T\wedge t}^k H_k\left(\frac{(\alpha - S)e^{\beta(T\wedge t)}}{2h_{T\wedge t}}\right)\right] \\
&= \sum_{m=0}^{[\frac{k}{2}]} g_{km}(\theta)E\left[e^{k\beta(T\wedge t)}(1 - e^{-2\beta(T\wedge t)})^m\right] \\
&= \sum_{m=0}^{[\frac{k}{2}]} g_{km}(\theta)\sum_{i=0}^{m}\binom{m}{i}(-1)^i E\left[e^{(k-2i)\beta(T\wedge t)}\right] \\
&= \sum_{m=0}^{[\frac{k}{2}]} g_{km}(\theta)E\left[e^{k\beta(T\wedge t)}\right] + \sum_{m=1}^{[\frac{k}{2}]} g_{km}(\theta)\sum_{i=1}^{m}\binom{m}{i}(-1)^i E\left[e^{(k-2i)\beta(T\wedge t)}\right]
\end{aligned}$$

where $g_{km}(\theta) = \dfrac{(-1)^m k!}{(k-2m)!\,m!}\left(\dfrac{\sigma^2}{4\beta}\right)^m (\alpha - S)^{k-2m}$. The coefficient to $E\left[e^{k\beta(T\wedge t)}\right]$ is $\sum_{m=0}^{[\frac{k}{2}]} g_{km}(\theta) = \left(\dfrac{\sigma^2}{4\beta}\right)^{\frac{k}{2}} H_k\left(\dfrac{(\alpha - S)\sqrt{\beta}}{\sigma}\right)$, which is positive when $\theta \in \Theta^{(k)}$. We can therefore rearrange

$$\frac{\alpha^k - \sum_{m=1}^{[\frac{k}{2}]} g_{km}(\theta)\sum_{i=1}^{m}\binom{m}{i}(-1)^i E\left[e^{(k-2i)\beta(T\wedge t)}\right]}{\sum_{m=0}^{[\frac{k}{2}]} g_{km}(\theta)} \geq E\left[e^{k\beta(T\wedge t)}\right] \tag{8}$$

This is valid for all $k \in \mathbb{N}$. For $k = 1$ we obtain $\alpha/(\alpha - S) \geq E\left[e^{\beta(T\wedge t)}\right]$ for $\alpha > S$, and when $k = 2$ then (8) yields $(\alpha^2 - \sigma^2/2\beta)\,/\,((\alpha - S)^2 - \sigma^2/2\beta) \geq E\left[e^{2\beta(T\wedge t)}\right]$ for $\alpha > S$ and $\sigma^2/2\beta < (\alpha - S)^2$. Therefore, by induction and because $\Theta^{(k)} \subset \Theta^{(k-1)}$, the left hand side will be finite, say less than some constant K_k depending on k. Taking limits on both sides we obtain

$$K_k \geq \lim_{t\to\infty} E\left[e^{k\beta(T\wedge t)}\right] = E\left[e^{k\beta T}\right]$$

by monotone convergence. We have thus obtained conditions on the parameter space for which $E[e^{k\beta T}] < \infty$, so that (6) can be applied. $\square$

Note that substituting $E\left[e^{(k-2i)\beta(T\wedge t)}\right]$ in (8) with expression (6) for $\lambda = k-2i$ we obtain that in the limit when $t \to \infty$, (8) is an equality, using the same techniques as in the proof of (4). This implies that when $\theta \in \Theta^{(k)}$, then $E[M_0^{(k)}] = (-\alpha)^k = E[M_T^{(k)}]$.

In Table 1 the first four moments are given with indication of the subset of the parameter space in which the expressions are valid.

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k	$\lambda^{(k)}$	$E\left[e^{k\beta T}\right]$
1	0	$\dfrac{\alpha}{(\alpha - S)}$
2	$\dfrac{1}{\sqrt{2}}$	$\dfrac{\alpha^2 - \dfrac{\sigma^2}{2\beta}}{(\alpha - S)^2 - \dfrac{\sigma^2}{2\beta}}$
3	$\sqrt{\dfrac{3}{2}}$	$\dfrac{\alpha}{(\alpha - S)} \left(\dfrac{\alpha^2 - \dfrac{3\sigma^2}{2\beta}}{(\alpha - S)^2 - \dfrac{3\sigma^2}{2\beta}} \right)$
4	$\sqrt{\dfrac{3+\sqrt{6}}{2}}$	$\dfrac{\left(\alpha^2 - \dfrac{3\sigma^2}{2\beta}\right)^2 - \dfrac{3\sigma^4}{2\beta^2}}{\left((\alpha - S)^2 - \dfrac{3\sigma^2}{2\beta}\right)^2 - \dfrac{3\sigma^4}{2\beta^2}}$

Table 1
The first 4 moments of $e^{\beta T}$, where T is the first-passage time of an Ornstein-Uhlenbeck process with parameters $\theta = (\alpha, \beta, \sigma)$ through a constant threshold S, which is valid for $\theta \in \Theta^{(k)} = \{\theta \,|\, \alpha > S, \sqrt{\sigma^2/\beta} < (\alpha - S)/\lambda^{(k)}\}$.