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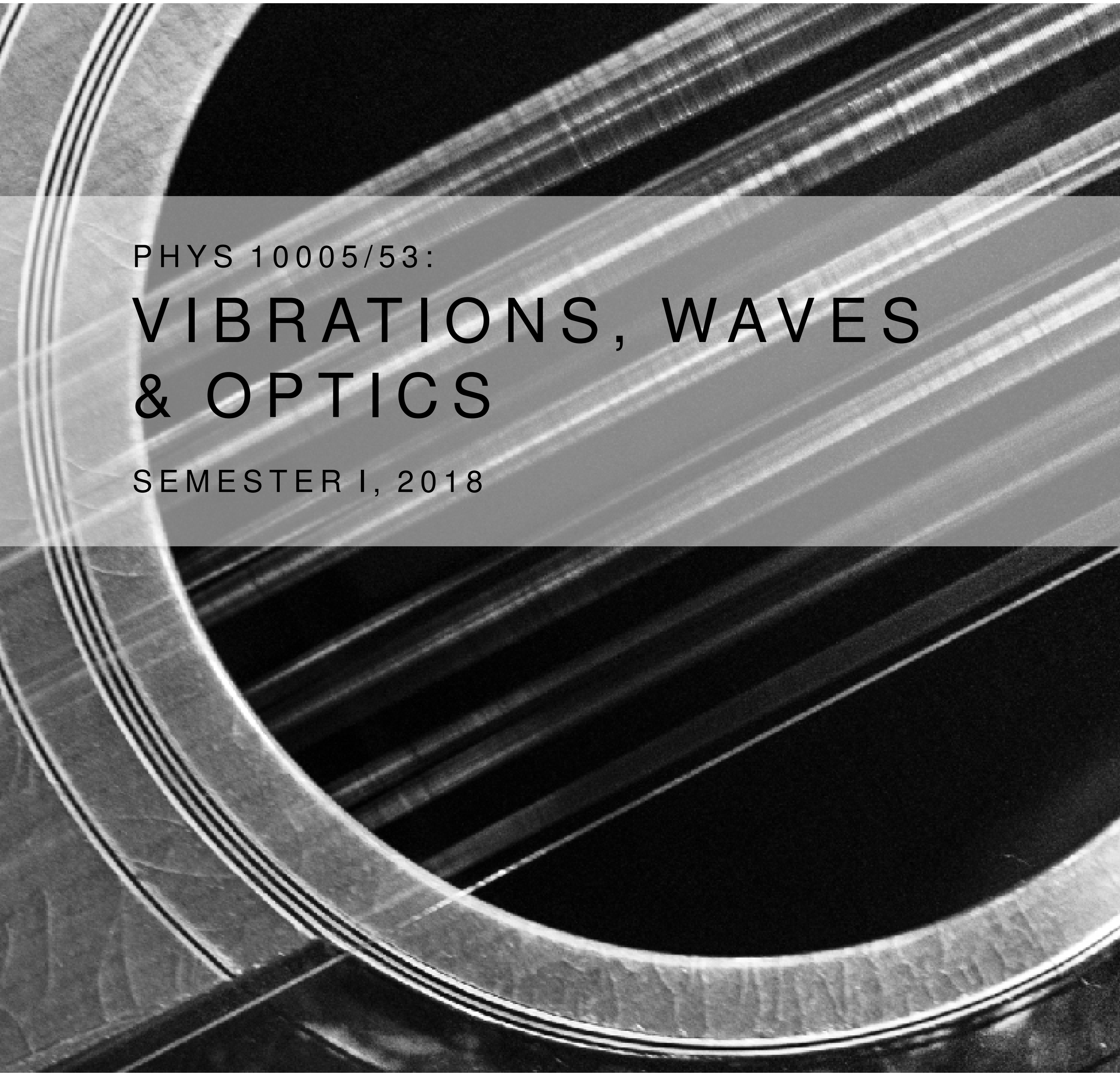


PHYS 10005/53:

VIBRATIONS, WAVES & OPTICS

SEMESTER I, 2018

DEPARTMENT OF PHYSICS, UNIVERSITY OF BATH

1

Simple Harmonic Motion

Introduction

Lecture 1
§Suggested Reading:

- ***Pain***, Chapter 1, Simple Harmonic Oscillators
- ***Tipler***, Chapter 14, Oscillations

Periodic motion is motion of an object that regularly repeats, i.e. the object returns to a given position after a fixed time interval.

Example 1.1: Examples of Periodic Motion

- Orbital motion of the Earth;
- Orbital motion of the moon;
- Vibrations of Molecules in a solid;
- Electromagnetic Waves (light);
- Loudspeakers;
- Oscillations of Bridges;
- Swaying of Tall Buildings.

A very important kind of periodic motion is ***Simple Harmonic Motion (SHM)***. In a certain sense, SHM describes the simplest possible periodic motion. Understanding SHM well can lead to a physical understanding of all kinds of dynamical systems, such as those in the examples above.

1.1 Mechanics of Simple Harmonic Motion

In oscillating mechanical systems there is always a force that acts to reduce displacement of an object and return the system to equilibrium. This force is called the ***restoring force***, F_r.

Simple Harmonic Motion

SHM occurs whenever the restoring force is linearly proportional to the displacement from equilibrium and opposite to the direction of the displacement, i.e.

$$\vec{F}_r \propto -\vec{x}. \tag{1.1}$$

Any (stable) system that oscillates, ***if the oscillation is small enough***, will appear to be undergoing simple harmonic motion.

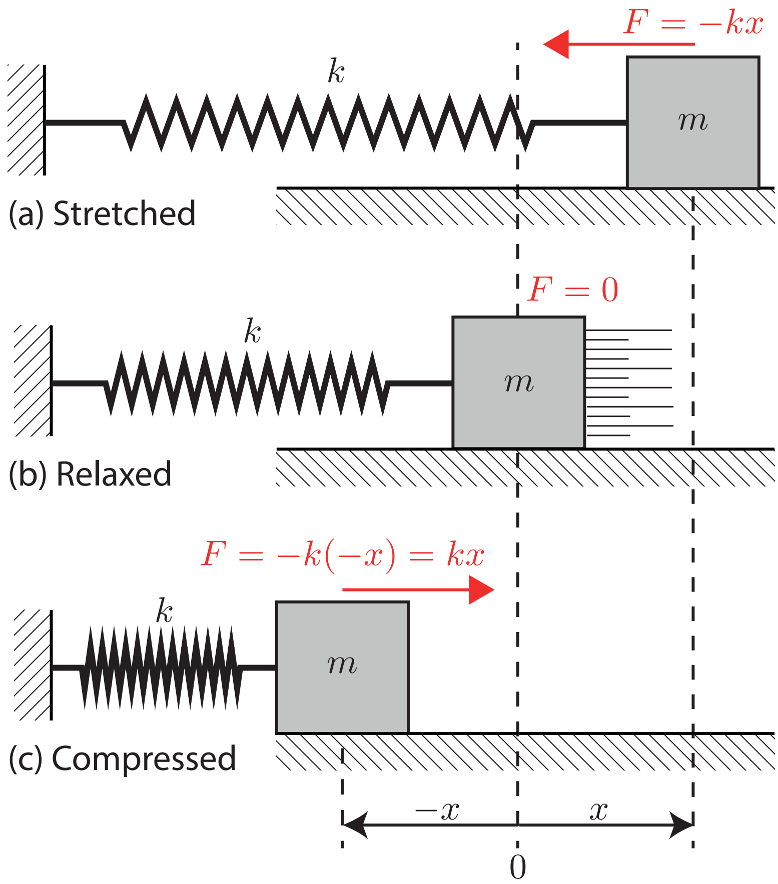


Figure 1.1: In this model our assumptions are: ***(1)*** There is no friction; ***(2)*** The mass of spring is negligibly small in comparison with the mass of the block; ***(3)*** The spring only responds linearly.

Consider the mass m attached to a spring, shown in Figure 1.1. When the mass is ***in equilibrium*** there is no (net) force on it. When the mass is displaced, the spring exerts a ***restoring force*** $-kx$, as given by ***Hooke's Law***

Hooke's Law

$$F_r = -kx \tag{1.2}$$

where k is the constant of proportionality between the displacement and the restoring force, called the ***spring constant*** (or ***force constant***).

The minus sign in Hooke's law arises because the force is in the opposite direction of the displacement.

Using Newton's Second Law of Motion we get

$$\vec{F}(x) = m\vec{a}(x) \qquad \text{or} \qquad -k\vec{x} = m\vec{a} \tag{1.3}$$

This can be written as

$$\frac{d^2\vec{x}}{dt^2} = -\frac{k}{m}\vec{x} \tag{1.4}$$

The spring constant has units of [Mass][Time]$^{-2}$, therefore $\omega^2 \equiv k/m$ has units of $1/[\text{Time}]^2$

If we calculate the dimensions of k/m then we find it has the dimensions of $[\text{Time}]^{-2}$ so we can write,

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{1.5}$$

$$\ddot{x} + \omega^2 x = 0$$

The differential equation for SHM

where $\omega = \sqrt{k/m}$ is an ***angular frequency***.

Equation (1.5) is a second order differential equation. It is often referred to as the ***differential equation of Simple Harmonic Motion***.

Its solution can always be written in the form,

$$x = \underbrace{A}_{\text{Amplitude}} \cos\overbrace{(\omega t + \underbrace{\delta}_{\text{Initial Phase}})}^{\text{Phase}} \tag{1.6}$$

Note that if δ gives the initial phase for some SHM, so can $\delta + 2\pi N$ where N is some integer. δ is not unique because the solution is periodic!

where A and δ are determined by the initial conditions (i.e. the displacement and velocity at $t = 0$). According to Eq. (1.6), the function $x(t)$ has the form:

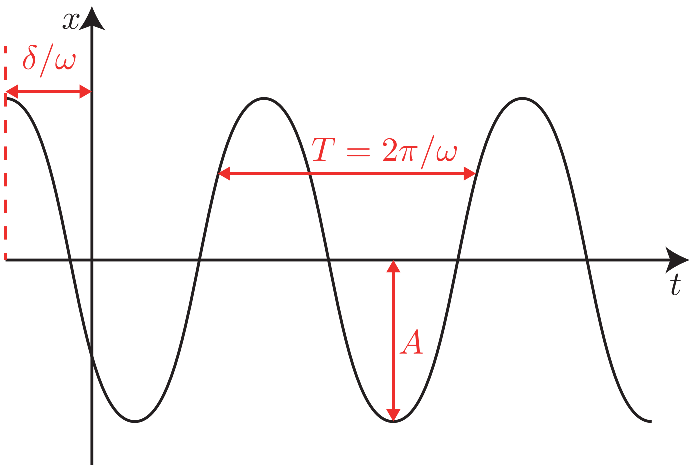


Figure 1.2: ***Units:*** Period T is in 'seconds'; Frequency $f = 1/T$ is in 'Hertz'; Angular frequency $\omega = 2\pi f$ is in 'radians per second'; and initial phase δ is in 'radians'.

The period of the oscillation T is given by

$$T = \frac{2\pi}{\omega}. \tag{1.7}$$

Period in terms of angular frequency

1.2 *Displacement, Velocity, and Acceleration*

If we have the solution $x = A\cos(\omega t + \delta)$, then the velocity is

$$v = \frac{dx}{dt} = -A\omega \sin(\omega t + \delta) \tag{1.8}$$

SHM solutions for x, v, and a.

Remember the chain rule:

$$\frac{df(g(t))}{dt} = \frac{df}{dg} \cdot \frac{dg}{dt}$$

$$\frac{dx}{dt} = A \underbrace{\frac{d[\cos(\omega t + \delta)]}{d(\omega t + \delta)}}_{-\sin(\omega t + \delta)} \underbrace{\frac{d(\omega t + \delta)}{dt}}_{\omega}$$

and the acceleration is

$$a = -A\omega^2 \cos(\omega t + \delta) = -\omega^2 x \tag{1.9}$$

For initial phase $\delta = 0$, the equations simplify to:

$$x = A\cos\omega t \tag{1.10a}$$
$$v = -\omega A \sin\omega t \tag{1.10b}$$
$$a = -\omega^2 A \cos\omega t = -\omega^2 x. \tag{1.10c}$$

We can check to see that the solution's acceleration and displacement indeed satisfy the original SHM differential equation

$$\ddot{x} + \omega^2 x = 0$$
$$-A\omega^2 \cos(\omega t + \delta) + \omega^2 A \cos(\omega t + \delta) = 0$$

Let's plot out the above solutions for $x(t)$ assuming $\delta = 0$:

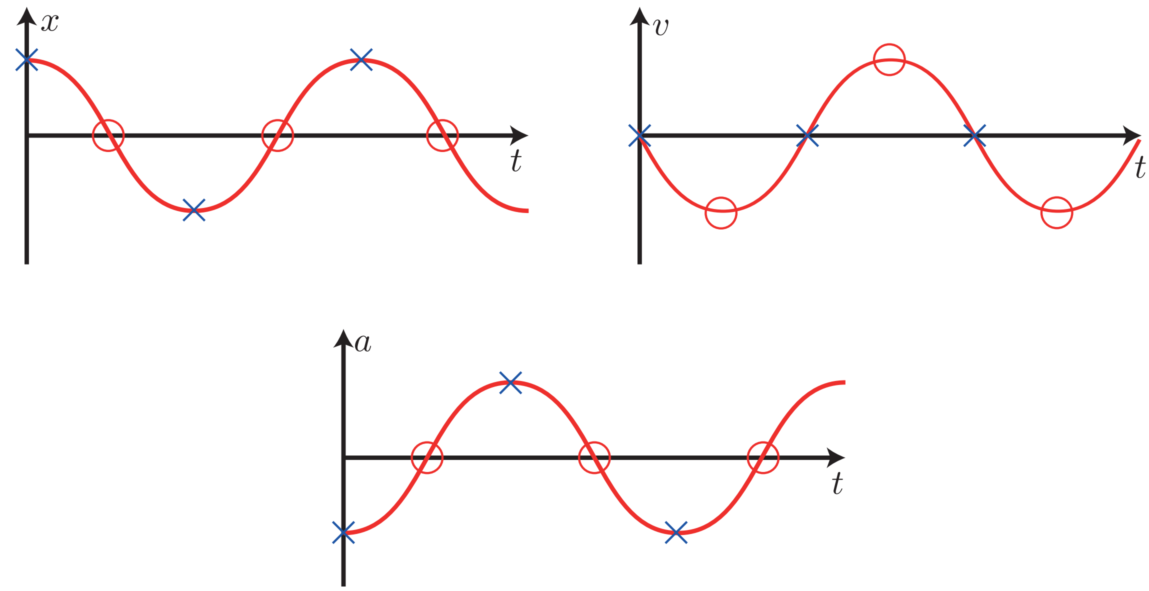


Figure 1.3: Position, Velocity and Acceleration for SHM with $\delta = 0$.

- Velocity is $\pi/2$ out of phase with displacement.
- o - velocity is maximum/minimum when displacement is zero.
- x - Velocity is zero when displacement is maximum/minimum.
- Acceleration is proportional to displacement and acts in the opposite direction to the displacement.

Note that the ***minimum*** isn't the zero point. The displacement, velocity, and acceleration oscillate about zero, ***the equilibrium point***.

The ***frequency*** and ***period*** are related to the ***stiffness*** k of the spring and the ***mass*** of the particle

f and T in terms of k and m for a SHM

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad \text{and} \qquad T = 2\pi\sqrt{\frac{m}{k}} \tag{1.11}$$

Note that the ***spring consant*** k is only a constant for small displacements.

Large displacements could cause (for example) permanent deformation of the spring and the system would not oscillate with SHM. This reminds us that the SHM is only found for small displacements.

Example 1.2: An air-track glider

An air-track glider attached to a spring oscillates with a period of 1.5s. At $t = 0$ the glider is 5cm left of the equilibrium position and moving to the right at 36.3 cm/s.

(a) What are the amplitude and initial phase of the oscillations?

(b) Write down an expression that describes the position of the glider as a function of time

(c) What is the glider's position at $t = 0.5$s?

We know that the glider undergoes SHM, with solution

$$x = A\cos(\omega t + \delta)$$

We need to know ω, A, and δ in order to completely describe the motion. For ω we have,

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{1.5\mathrm{s}} = \frac{4\pi}{3}\mathrm{rad/s} = 4.2\mathrm{rad/s}$$

(a) At $t = 0$ we have

$$x(0) = A\cos(0+\delta) = A\cos\delta = -5\mathrm{cm}$$
$$v(0) = -A\omega\sin(0+\delta) = -A\omega\sin\delta = 36.5\mathrm{cm/s}$$

Dividing $v(0)$ by$x(0)$, we can find δ,

$$\frac{v(0)}{x(0)} = -\frac{A\omega\sin\delta}{A\cos\delta} = -\omega\tan\delta = \frac{36.5\mathrm{cm/s}}{-5\mathrm{cm}}$$
$$\tan\delta = 1.734$$
$$\Rightarrow \delta = \pi/3 \text{ rad} = 60^\circ$$

Finally, from the expression for $x(0)$ above, we can find A

$$A = \frac{x(0)}{\cos\delta} = \frac{-5cm}{0.5}$$
$$\Rightarrow A = -10\mathrm{cm}$$

(b)

$$x(t) = A\cos(\omega t + \delta)$$
$$\Rightarrow x(t) = -10\cos\left(\frac{4\pi}{3}t + \frac{\pi}{3}\right)$$

(c)

$$x(0.5\mathrm{s}) = -10\cos(\underbrace{(4\pi/3)\cdot 0.5}_{2\pi/3\text{ rad}=120^\circ} + \pi/3) = -10\underbrace{\cos\pi}_{-1}$$
$$\Rightarrow x(0.5\mathrm{s}) = 10\mathrm{cm}$$

End of Lecture 1

1.3 *Further examples of Simple Harmonic Motion (SHM)*

Lecture 2
Kahoot! Quiz on SHM

FROM OUR BASIC DEFINITION, we observe SHM when the restoring force is linearly proportional to displacement. Conversely, if SHM is observed, then we must have

$$F_r = -kx$$

This is because we know that SHM obeys the SHM differential equation!

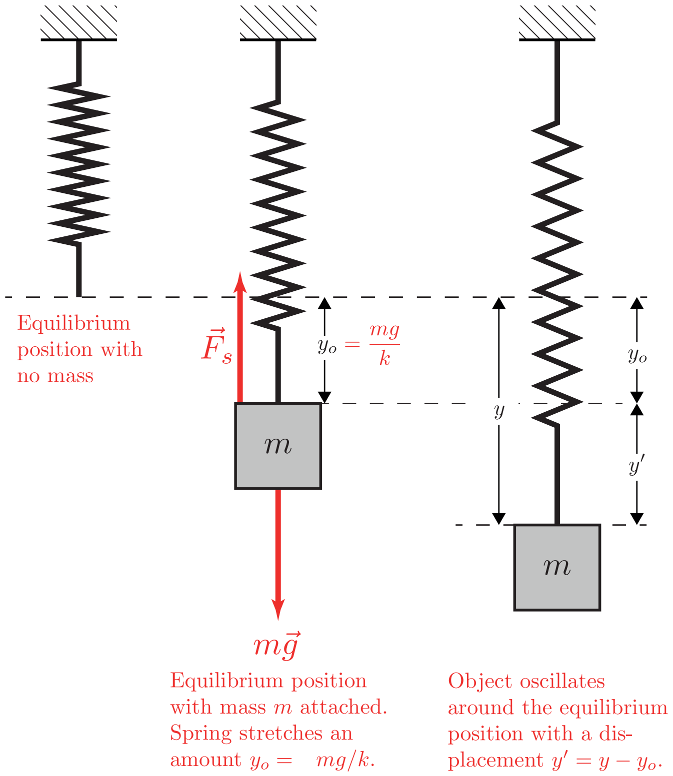
Then for any simple harmonic motion we can write out the expression for the ***force constant*** k,

$$k = -\frac{F_x(t)}{x(t)} \qquad \text{so that} \qquad \omega = \sqrt{\frac{k}{m}} \tag{1.12}$$

where $x(t)$ is measured from the ***equilibrium*** point.

Example 1.3: An Object on a Vertical Spring

Consider a mass hanging on a vertical spring:



In general we must have

$$\sum F = ma$$

while at the equilibrium point we have

$$\sum F = 0.$$

We need to take into account the gravitational force mg in addition to the force of the spring $F_s = -ky$ where y is the vertical displacement. In equilibrium with the gravitational force, the spring is extended by a length y_o.
In order to find the equilibrium position, y_o, we need to apply Newton's Second law:

$$\sum F = mg - ky_o = 0 \tag{1.13}$$

So the equilibrium position must be,

$$y_o = mg/k = g/\omega^2 \tag{1.14}$$

Away from equilibrium, we have

$$\begin{aligned} \sum F &= ma \\ mg - ky &= ma \\ mg - ky &= m\ddot{y} \end{aligned} \tag{1.15}$$

This differs from Eq. (1.3) by a constant term mg.
We can handle this extra term by changing to a new variable

$$y' = y - y_o \tag{1.16}$$

such that we have

$$\begin{aligned} m\ddot{y} + ky &= mg \\ \ddot{y} + \underbrace{\omega^2}_{k/m} y &= g \\ \ddot{y}' + \omega^2 y' + \omega^2 \underbrace{y_o}_{g/\omega^2} &= g \end{aligned}$$

With this definition of y' we have

$$y = y' + y_o$$

Taking two time derivatives and remembering y_o is a constant we also have

$$\ddot{y} = \ddot{y}'$$

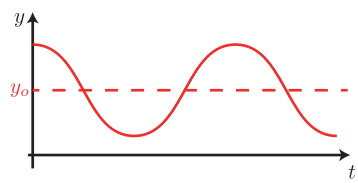
So that for y' Eq. (1.15) reduces to

$$\frac{d^2y'}{dt^2} + \omega^2 y' = 0 \tag{1.17}$$

and its solution is

$$\begin{aligned} y'(t) &= A\cos(\omega t + \delta) \\ \Rightarrow y(t) &= y_o + A\cos(\omega t + \delta) \end{aligned} \tag{1.18}$$

Sketch the solution $y(t)$



SHM around the equilibrium point y_o.

Example 1.4: The Simple Pendulum

Consider the simple pendulum, as shown in Figure 1.4. We model this by assuming that m is a ***point mass*** connected to the anchor point by a ***massless, inextensible string***.
The path of the mass traces out part of a circle. We will call the arc length along this path s.
Let's also call the tension force in the string $\vec{T}$, which pulls the mass towards the anchor point.
The restoring force along the arc of the circle is:

$$F_r = -mg\sin\phi \tag{1.19}$$

The tangential component of the acceleration is $\frac{d^2s}{dt^2}$, which lets us write the tangential component of Newton's second law as

$$m\frac{d^2s}{dt^2} = -mg\sin\phi \tag{1.20}$$

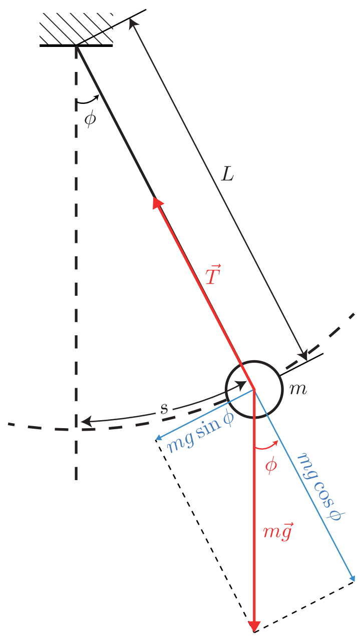


Figure 1.4: The simple pendulum of mass m and length L.

For small angles $\sin\phi \simeq s/L$:

$$\frac{d^2s}{dt^2} = -\underbrace{\frac{g}{L}}_{\omega^2} s \quad \text{or} \quad \frac{d^2s}{dt^2} + \omega^2 s = 0 \tag{1.21}$$

and $s = s_0 \cos(\omega t\delta)$ with $\omega = \sqrt{g/L}$.
Thus for ***small displacements***, the period of the pendulum is ***independent*** of mass of the bob, and depends only on the length of the pendulum.
Notice that the pendulum ***does not*** exhibit true SHM for ***any*** angle. If the angle is less than around $10°$, the motion is close to, and can be ***modelled*** as, simple harmonic.

1.4 *Energy in Simple Harmonic Motion*

WE REMEMBER FROM BASIC MECHANICS that for any ***conservative*** force that there is a direct link to the potential energy of the system, given by

$$F_x = -\frac{dU}{dx} \tag{1.22}$$

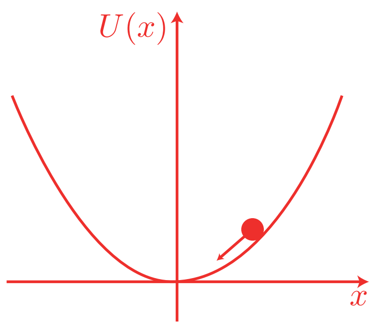
E.g. gravity near the Earth's surface:

$$U(x) = mgx$$
$$F = mg$$

For SHM we know that $F_x = -kx$, so we can write

$$U(x) = \int \underbrace{kx}_{-F_x} dx \tag{1.23}$$

$$U(x) = \frac{1}{2}kx^2 \tag{1.24}$$



This quadratic/parabolic dependence of potential energy on displacement is a ***general result*** for anything moving with SHM.
The kinetic energy of the oscillator is given by

$$E_{\text{kin}} = \frac{1}{2}mv^2 \tag{1.25}$$

and the total energy at any moment of time is

$$E_{\text{tot}} = U + E_{\text{kin}}$$
$$E_{\text{tot}} = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \tag{1.26}$$

For any oscillator we can write

$$x = A\cos(\omega t + \delta)$$
$$v = -A\omega\sin(\omega t + \delta)$$
$$k = m\omega^2$$

therefore

Total Energy of SHM
Remembering the trig identity

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

$$E_{\text{tot}} = \frac{1}{2}kA^2\cos^2(\omega t + \delta) + \frac{1}{2}\underbrace{m\omega^2}_{k}A^2\sin^2(\omega t + \delta)$$
$$= \frac{1}{2}kA^2 \tag{1.27}$$

From this we can conclude that ***energy in SHM is conserved***!

1.5 *The General Applicability of Simple Harmonic Motion*

IN SHM, THE RESTORING FORCE is zero ($F_r = 0$) at the equilibrium point. As $F_r = -\frac{dU}{dx}$, that means the potential energy at this point has either a local maximum or a minimum.

For ***small displacements***, we may expand $U(x)$ in a Taylor Series:

$f(x) = f(0) + \frac{1}{1!}f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(0)x^3 + \ldots$

$$U(x) = U(0) + \underbrace{x\frac{dU}{dx}(0)}_{0} + \frac{1}{2}x^2\frac{d^2U}{dx^2}(0) + \ldots \tag{1.28}$$

If we chose the zero point of the potential energy such that $U(0) = 0$, then the first two terms of the equation above are equal to zero. Thus we can write that, approximately,

$$U(x) \simeq \frac{1}{2}kx^2 \tag{1.29}$$

Potential Energy for SHM

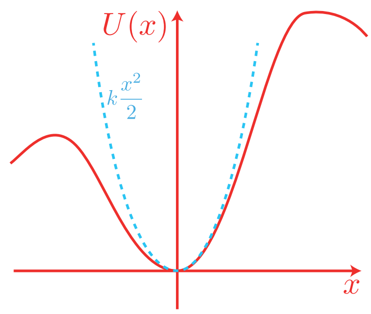
where $k = \frac{d^2U}{dx^2}(0)$.

What happens if $k = \frac{d^2U}{dx^2}(0) < 0$***? Clearly the SHM differential equation and solution no longer apply. Is this a stable equilibrium?***

Why is the Simple Harmonic Oscillator a good approximation for so many systems we encounter?

$$U = \frac{1}{2}kx^2$$
$$\frac{dU}{dx} = kx(= -F_x)$$
$$\frac{d^2U}{dx^2} = k$$

For the general case:



End of Lecture 2