Vector Search **Class #XXXXX Fall 2023**

Lecture number XX: Probability Recap

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1 Preliminaries

A variable X is a random variable if it assumes different values according to a probability distribution. For example, X can denote the outcome of a three sided die throw. The variable X takes the values $x = 1, 2, 3$ with equal probabilities.

The expectation of X is the sum over the possible values times the probability of the events.

$$\mathbb{E}[X] = \sum_{x=1}^{3} x \Pr(X = x) = 1\frac{1}{3} + 2\frac{1}{3} + 3\frac{1}{3} = 2 \tag{1}$$

Continuous random variable are described by their distribution function φ.

$$\Pr[Y \in [a, b]] = \int_a^b \varphi(t)dt.$$

For a function φ to be a valid distribution we must have:

$$\forall\, t,\ \ \varphi(t) \ \ \geq \ \ 0 \quad \text{(where it is defined)} \tag{2}$$

$$\int_a^b \varphi(t)dt \qquad \text{is well defined for all } a \text{ and } b \tag{3}$$

$$\int_{-\infty}^{-\infty} \varphi(t)dt \ \ = \ \ 1 \tag{4}$$

For example consider the continuous variable Y taking values in $[0, 1]$ uniformly. That means $\varphi(t) = 1$ if $t \in [0, 1]$ and zero else. This means that the probability of Y being in the interval $[t, t + dt]$ is exactly dt. And so the expectation of Y is:

$$\mathbb{E}[Y] = \int_{t=0}^{1} t\varphi(t)dt = \int_{t=0}^{1} t \cdot dt = \frac{1}{2}t^2|_0^1 = 1/2 \tag{5}$$

Remark 1.1. *Strictly speaking, distributions are not necessarily continuous or bounded functions. In fact, they can even no be a function at all. For example, the distribution of X above includes three Dirac-δ functions which are not valid functions.*

1.1 Dependence and Independence

A variable X is said to be *dependent* on Y if the distribution of X given Y is different than the distribution of X. For example. Assume the variable X takes the value 1 if Y takes a value of less than 1/3 and the values 2 or 3 with equal probably otherwise (1/2 each). Clearly, the probability of X assuming each of its values is still 1/3. however, if we know that Y is 0.7234 the probability of X assuming the value 1 is zero. Let us calculate the expectation of X given Y as an exercise.

$$\mathbb{E}(X|Y) = \sum_{x=1}^{3} x \Pr(X = x|Y \leq 1/3) = 1 \cdot 1 \tag{6}$$

$$\mathbb{E}(X|Y) = \sum_{x=1}^{3} x \Pr(X = x|Y > 1/3) = 1 \cdot 0 + 2\frac{1}{2} + 3\frac{1}{2} = 2.5 \tag{7}$$

$E(X|Y) = 1$ for $y \in [0, 1/3]$ and $E(X|Y) = 2.5$ for $y \in (1/3, 1]$.
Remember: $\mathbb{E}(X|Y)$ is a function of y!

Definition 1.1 (Independence)**.** *Two variables are said to be Independent if:*

$$\forall y, \ \ \Pr[X = x|Y = y] = \Pr[X = x].$$

They are dependent otherwise.

Fact 1.1. *If two variables X and Y are Independent the so are $f(X)$ and $g(Y)$ for any functions f and g.*

Fact 1.2 (Linearity of expectation 1)**.** *For any random variable and any constant α:*

$$\mathbb{E}[\alpha X] = \alpha \mathbb{E}[X] \tag{8}$$

Fact 1.3 (Linearity of expectation 2)**.** *For any two random variables*

$$\mathbb{E}_{X,Y}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y] \tag{9}$$

even if they are dependent.

Fact 1.4 (Multiplication of random variables)**.** *For any two* **independent** *random variables*

$$\mathbb{E}_{X,Y}[XY] = \mathbb{E}[X]\mathbb{E}[Y] \tag{10}$$

This does not necessarily hold if they are dependent.

Definition 1.2 (Variance)**.** *For a random variable X we have*

$$\mathrm{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \tag{11}$$

The standard deviation σ of X is defined to be $\sigma(X) \equiv \sqrt{\mathrm{Var}[X]}$.

Definition 1.3 (Additivity of variances)**.** *For any two* **independent** *variables X and Y we have*

$$\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] \tag{12}$$

Fact 1.5 (Markov's inequality)**.** *For any positive random variable X:*

$$\Pr(X > t) \leq \frac{E[X]}{t} \tag{13}$$

Fact 1.6 (Chebyshev's inequality)**.** *For any random variable X*

$$\Pr[|X - E[X]| > t] \leq \frac{\sigma^2(X)}{t^2} \tag{14}$$