Age-Structured Model for Bobcats

Question and Background

As stated in Project 1, most species of wild cats are endangered, including bobcats. So for this project, we will be analyzing the same bobcat population in Florida (from Cox, et. al, *Closing the Gaps in Florida's Wildlife Habitat Conservation System* [1]). Given 16 bobcat age classes, a survival best and worst case scenario for each bobcat age class, and a reproduction worst and best case scenario for each bobcat age class, we wanted to understand how this bobcat population could change over time. More specifically, how different are the population distributions of the worst and best-case scenario outcomes? What rate is the bobcat population likely going to grow? How does the relative bobcat population distribution change in ten years? If catastrophes lower the reproduction rate by 30% every 25 years, then what would the bobcat population distribution look like after 10 years?

In order to construct an aged-structure model for bobcats, we had to make a couple of fundamental assumptions. First, we are excluding male bobcats from this analysis as the rates provided as the demographic data only capture females born to female rates. Furthermore, we are assuming there are only 16 age classes of bobcats and that the survival and reproductive best and worst-case scenarios are fixed rates that do not change over time. We are also assuming that both the demographic rates and transition probabilities for ecological success are representative of the Florida bobcat population we are looking at. Since the average bobcat lives for 7 years, and the oldest wild bobcat lived for 16 years, we assumed that the time step for this problem would be 1 year (10-time steps to get to 10 years). For the simulation, we are making the assumption that 100 trials over a 10-year period are sufficient to accurately capture how catastrophes affect this bobcat population over time. We also assumed that a catastrophe is a stationary process - each year is equally likely to be a catastrophe year. Thus, the random behavior is simulated with a uniform distribution from 0 to 1, where anything less than or equal to 1/25 is treated as a catastrophe year. Since we have stated our assumptions, we will now look at the state diagram we constructed for the bobcats, which will be helpful for understanding our Leslie matrix and simulation later on in the analysis.

Model Construction

This problem was posed in three parts: first, how does a bobcat population grow over time under the best of conditions; second, how does it grow over time under the worst of conditions; and finally, how does it grow over time under occasionally catastrophic conditions? Here, "conditions" refers to the survival and reproduction rates reported by Cox et al. [1]. These authors report the maximum and minimum reproduction and survival rates for each of the 16 bobcat age classes; for the best-case scenario, each year sees the maximum reproduction and survival rate; for the worst-case scenario, each year sees the minimum reproduction and survival rate; for the occasional catastrophe scenario, each year sees the maximum survival rate and either the maximum reproduction rate or 70% of the maximum reproduction rate. These three scenarios are summarized in **Table 1**.

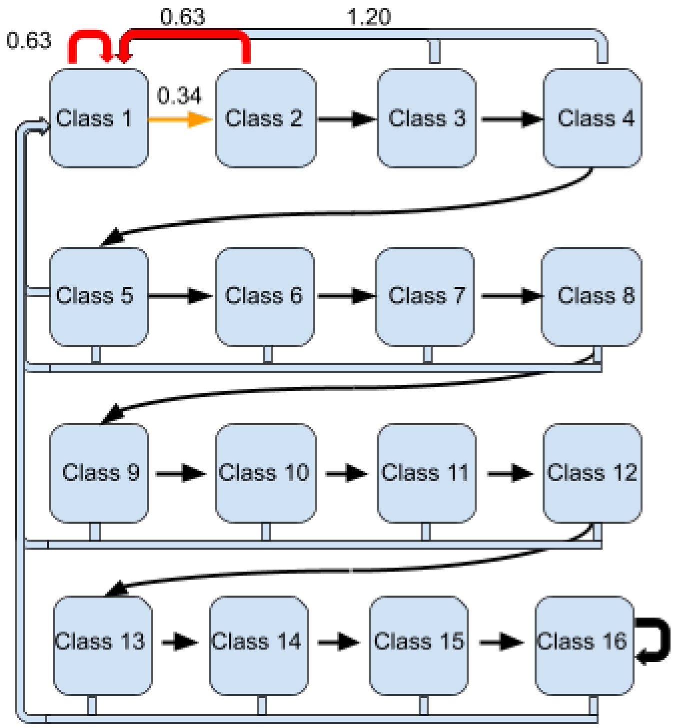
Table 1 - Summary of the age-structured reproduction and survival rates used for each scenario.

Scenario	**Reproduction Rates [Class 1, … , Class 16]**	**Survival Rates [Class 1, … , Class 16]**
Best Case Growth	[0.63, 0.63, 1.20, … , 1.20]	[0.34, 0.71, … , 0.71]
Worst Case Growth	[0.60, 0.60, 1.15, … , 1.15]	[0.32, 0.68, … , 0.68]
Stochastic Catastrophe	[0.63, 0.63, 1.20, … , 1.20] or [0.441, 0.441, 0.84, … , 0.84]	[0.34, 0.71, … , 0.71]

In the best and worst case scenarios, the rates shown in **Table 1** are applied to each year, but in the stochastic catastrophe case, there is a random variable that controls which of the two reproduction rates apply to any given year. Cox et al. report that catastrophe occurs, on average, once every 25 years, which means that each simulated time step has a 4% chance of having the lower of the two reproduction rates. Assuming that each year has the same chance of catastrophe, a uniformly-distributed random number between 0 and 1 is generated for each simulated year, and if it is less than or equal to 0.04, then the catastrophic reproduction rate is applied; otherwise, the best-case reproduction rate is applied.

To model these scenarios, the first step is to develop a state diagram for each case, which ensures that the proper relationships between age classes are modeled. Given these state diagrams, a Leslie matrix for each case is developed, as summarized in **Fig. 1**.

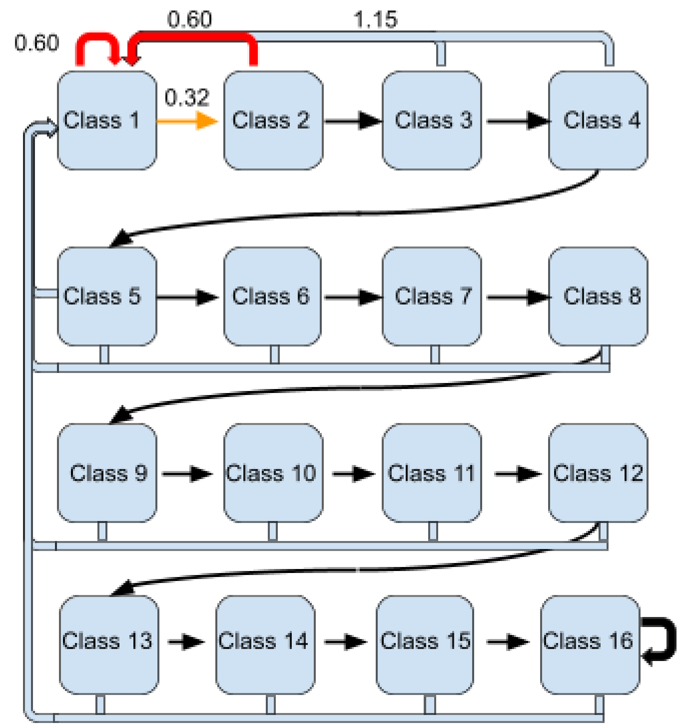
Best Case:



Blue arrows/lines: 1.20
Black arrows: 0.71

0.63	0.63	1.20	…	1.20
0.34	0	0	…	0
0	0.71	0	…	…
…	0	…	0	0
0	…	0	0.71	0.71

Worst Case:



Blue arrows/lines: 1.15
Black arrows: 0.68

0.60	0.60	1.15	…	1.15
0.32	0	0	…	0
0	0.68	0	…	…
…	0	…	0	0
0	…	0	0.68	0.68

These diagrams show that most of the age classes only interact with previous, subsequent, and first classes, so most of the entries in the 16-by-16 Leslie matrices are zero. Similarly, for the stochastic portion of the model, a Leslie matrix is constructed, as shown in **Fig. 2**. One matrix was created for the case of no catastrophe (identical to the best case growth) and another for the catastrophic case.

0.63	0.63	1.20	…	1.20
0.34	0	0	…	0
0	0.71	0	…	…
…	0	…	0	0
0	…	0	0.71	0.71

Stochastic Leslie Matrix
(non-catastrophe year)

0.441	0.441	0.84	…	0.84
0.34	0	0	…	0
0	0.71	0	…	…
…	0	…	0	0
0	…	0	0.71	0.71

Stochastic Leslie Matrix
(catastrophe year)

Fig. 2. The two Leslie matrices used in the stochastic growth portion of the model. All were 16x16 matrices.

These Leslie matrices describe how each class in the bobcat population develops from one-time step to the next. This is implemented iteratively:

$$(1) \qquad x(n+1) = L(n)x(n)$$

where *x* is the column vector of the population (one row per age class) and *L(n)* is the Leslie matrix describing how this population develops from year *n* to year *n + 1* (*L* is allowed to vary from year to year). For the best and worst case scenarios, the Leslie matrix was the same for each transition, so the population distribution at any time step *n* could be given in terms of the initial population distribution (*n* = 0):

$$(2) \qquad x(n) = L^n x(0)$$

where *x(n)* is the column vector of the population at year *n* and *x(0)* is the column vector of the population at year 0. For the stochastic case, the Leslie matrix had the potential to vary from year to year, so Eq. 1 was used for this case; Eq. 2 was used for the others.

With the Leslie matrices specified for each scenario, the model then determines the long-term growth rate of the population and its long-term distribution among the 16 age classes by finding the dominant eigenvalue and corresponding eigenvector of each Leslie matrix. These are found in MATLAB using the “eig” command. The dominant eigenvalue represents the growth rate to which the population approaches as time goes to infinity; the eigenvector when normalized so that the entries sum to 1, represents the age distribution that corresponds to this long-term growth. This assumes that the population continues to grow at the rate described by the Leslie matrix, and therefore is only valid for the best-case and worst-case scenarios - the stochastic model allowed for a different Leslie matrix from one year to the next, so the dominant eigenvector would be different for different simulations. With the long-term behavior defined, the model then uses the initial population distribution given in the problem statement and projects it out to *n = 10* years, using Eq. 2 for the best and worst case scenarios and Eq. 1 for the stochastic scenario.

Finally, the model predicts the distribution of the bobcat population among the 16 age classes after 10 years by modeling the possibility of catastrophe. Given a certain probability of catastrophe, a random number between 0 and 1 is generated for each simulated year (using the “rand” command in MATLAB), and if this number is less than or equal to the given probability, then the year is considered a catastrophe year, and the catastrophic Leslie matrix is applied. Otherwise, the non-catastrophic Leslie matrix is applied. The random number was generated with a uniform distribution; each year is assumed to have an equal chance of being a catastrophe year. This procedure is simulated many times for each time step so that a spread of possible population sizes is generated for each year.

Analysis and Results

The first application of the model was to determine how the female bobcat population would develop as time was allowed to approach infinity. For the best and worst cases, in which the same transition matrix was applied from year to year, this meant finding the dominant eigenvalue of each Leslie matrix, and then determining the corresponding eigenvector, normalized so that the sum of its entries was one. These results are shown in **Table 2** for both scenarios. The entries in the normalized dominant eigenvector for each case are expressed in percentages (shown to two significant figures), which represents the portion of the total population that each age class will constitute as time goes to infinity.

To give a sense of how long the population takes to reach this “long-term” behavior, **Table 2** also shows the population distribution, in both percentages and counts, after 10 years, given an initial population of $\overline{x_0}$ = [0; 100; 50; 50; 25; 10; 0; …; 0] individuals.

Table 2 - The female bobcat population, distributed among the 16 age classes, for both the best and worst cases, evaluated as time goes to infinity and at *t = 10* years.

	Best Case Growth Model			**Worst Case Growth Model**		
	Dominant Eigenvalue:	1.24 (24% growth rate)		**Dominant Eigenvalue:**	1.18 (18% growth rate)	
	Dominant Eigenvector (Long-Term)	**Population Distribution (After 10 Years)**		**Dominant Eigenvector (Long-Term)**	**Population Distribution (After 10 Years)**	
		(Percentage)	(Count)		(Percentage)	(Count)
Class 1	61 %	61 %	1818	61 %	61 %	1130
Class 2	17 %	17 %	498	17 %	17 %	305
Class 3	9.6 %	9.6 %	284	9.5 %	9.5 %	175
Class 4	5.5 %	5.5 %	162	5.5 %	5.5 %	101
Class 5	3.1 %	3.1 %	93	3.1 %	3.1 %	58
Class 6	1.8 %	1.8 %	53	1.8 %	1.8 %	33
Class 7	1.0 %	1.0 %	30	1.0 %	1.0 %	19
Class 8	0.58 %	0.59 %	17	0.60 %	0.61 %	11
Class 9	0.33 %	0.35 %	10	0.34 %	0.36 %	6
Class 10	0.19 %	0.17 %	4	0.20 %	0.17 %	3
Class 11	0.11 %	0.00 %	0	0.11 %	0.00 %	0
Class 12	0.063 %	0.11 %	3	0.065 %	0.11 %	2
Class 13	0.036 %	0.055 %	1	0.037 %	0.057 %	1
Class 14	0.020 %	0.055 %	1	0.022 %	0.057 %	1
Class 15	0.012 %	0.027 %	0	0.012 %	0.029 %	0
Class 16	0.016 %	0.011 %	0	0.017 %	0.011 %	0
		Total:	2,974		Total:	1,845

Both scenarios had positive growth rates, so even under the worst growth conditions, the female bobcat population would still be expected to grow. Moreover, the distribution of the population after 10 years, as predicted by each model, appeared to be similar to each other and to their respective long-term values. This is summarized graphically by the bar graphs in **Fig. 3**, which show the percentage of the total population (vertical axis) that each age class (horizontal axis) was predicted to constitute after ten 1-year time steps, when starting from the initial population distribution $\overline{x_0}$ = [0; 100; 50; 50; 25; 10; 0; …; 0] . The population was distributed almost identically in both cases, but the population count was significantly different; in the worst case, the total population was 62% of the total population predicted by the best case model.

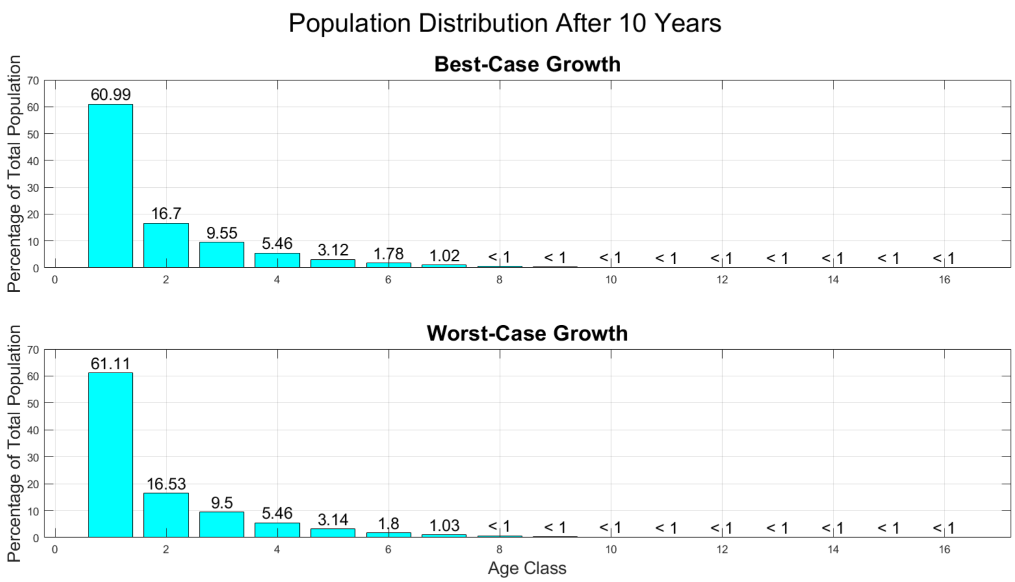


Figure 3: Comparison of the worst case scenario and best case scenario for population growth of the female bobcat population over 10 years.

The final portion of the model was intended to simulate the growth of the female bobcat population when catastrophic events are taken into account. The initial population, $\overline{x_0}$ = [0; 100; 50; 50; 25; 10; 0; …; 0] individuals, is projected to 10 years, with a 1/25 (4%) chance of catastrophe each year. To capture the impact of this stochasticity, this simulation was performed 100 times, as shown in **Fig. 4**, which shows the total female bobcat population as a function of time, with one curve for each of the 100 simulations.

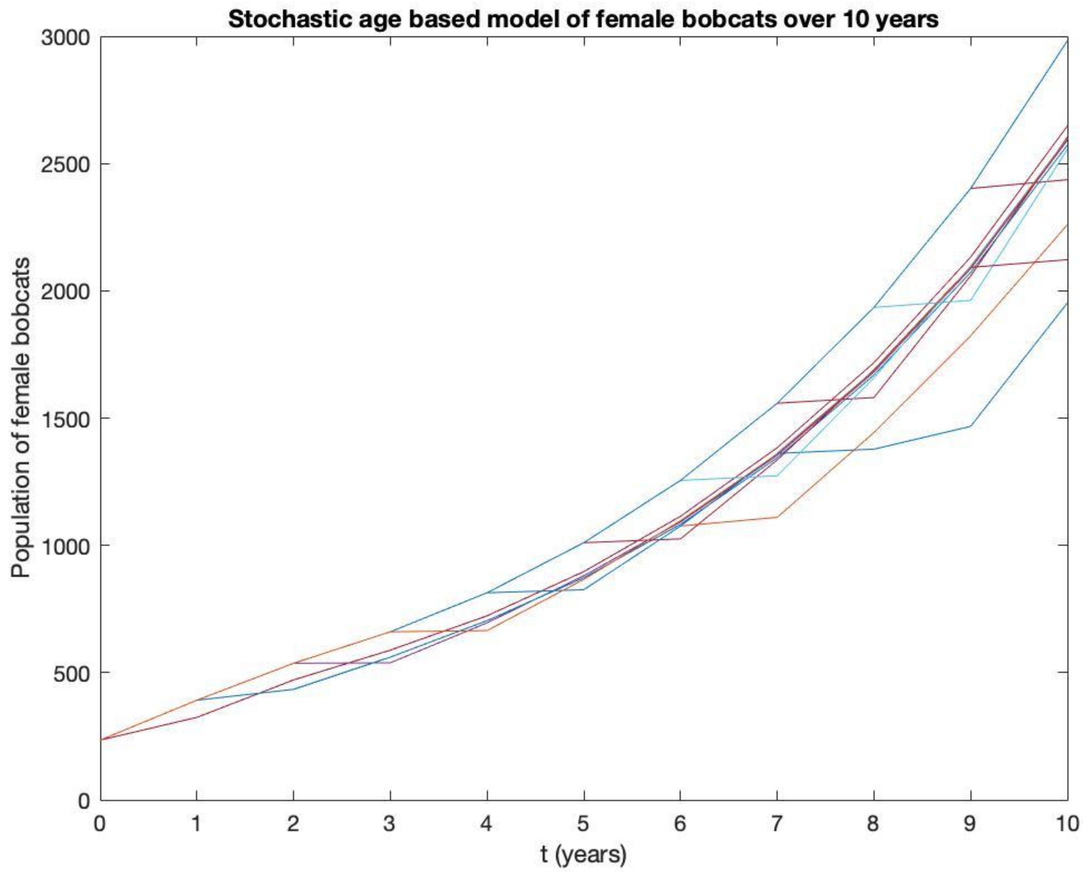


Figure 4: Plot of the population of female bobcats in a stochastic catastrophe-based model

According to the Leslie matrices for the model, the population would grow exponentially under the circumstance that no catastrophic events occur. When a catastrophe does occur, the growth rates and reproductive rates for each age group decline, but the dominant eigenvalue was still greater than 1 for this case (1.08), so even if a catastrophe occurred every year, the population would still be expected to grow (8% growth rate). With only a 4% chance of catastrophe, most of the simulated growth curves experienced one or two catastrophes during their 10-year run; the simulation with the greatest population at the end of the 10 years had no catastrophic events, and the simulation with the lowest population had 3 catastrophic events. These results are only specific to our plot of 100 runs; given that the model is stochastic, one should expect to get slightly different results every time we run this model. Moreover, the results would change if the chance of catastrophe were increased; since our simulation specified a 4% chance of catastrophe each year, 10 years did not offer many chances for a catastrophe to occur. Either increasing the probability of a catastrophic event, running more trials, or evaluating each curve to more than 10 time steps would produce some more runs with a higher count of catastrophes.

Catastrophe turned out to impact different age groups to different extents, as shown by the box plot in **Fig. 5**. This box plot shows the spread of the population (vertical axis) for each of the 16 age groups (horizontal axis) at the end of the 10-year period across 100 simulations. There was far greater spread in the first age class than any of the subsequent age classes, which suggests that catastrophe impacts the younger age classes more than any of the others, which is likely due to the fact that catastrophe impacts reproduction rates, not survival rates.

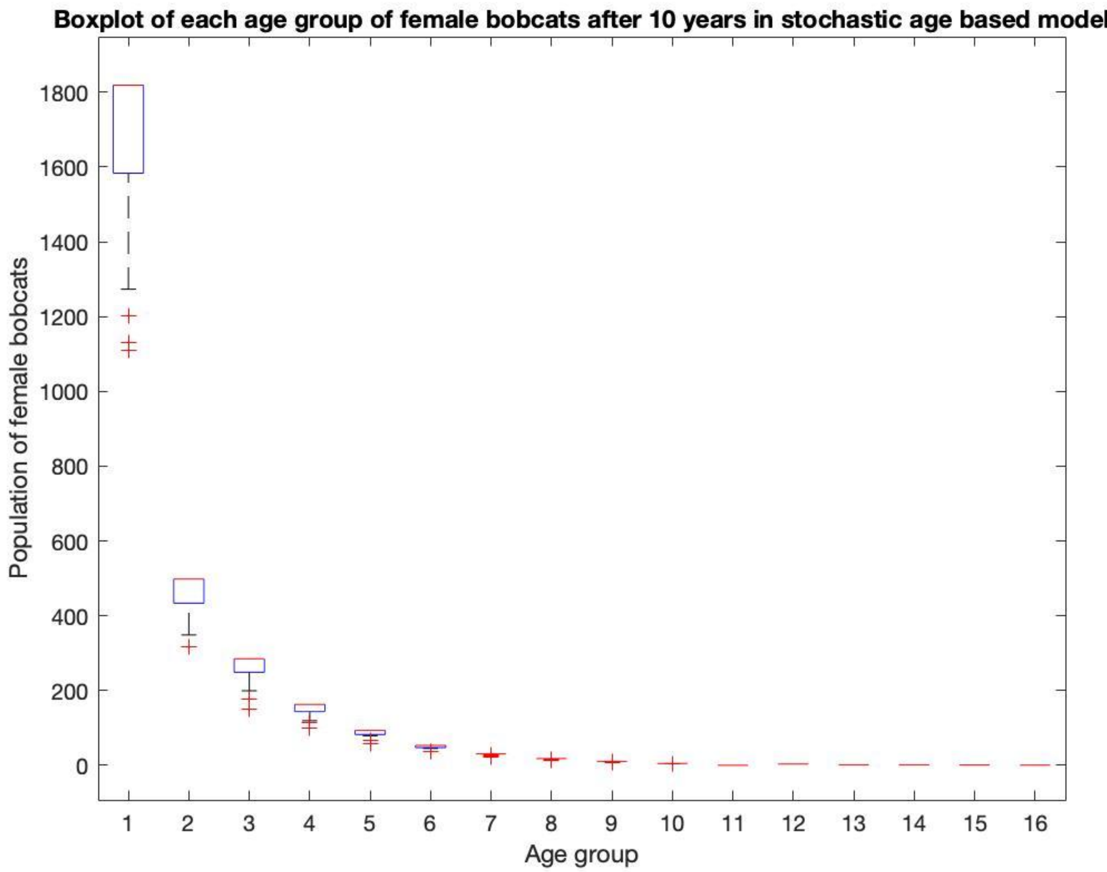


Figure 5: A box plot for each age group of female bobcats after 10 years of stochastic growth with catastrophes.

Figure 5 also indicates that the majority of the simulated population after 10 years is in the first age group. This result aligns with the population distribution seen in the best and worst case models; the dominant eigenvectors indicated that 60% of the population would be in the first age class, with a rapid decline in subsequent groups. **Figure 6** groups the simulated population by simulated year, rather than by age class, to demonstrate the cumulative effect of stochasticity.

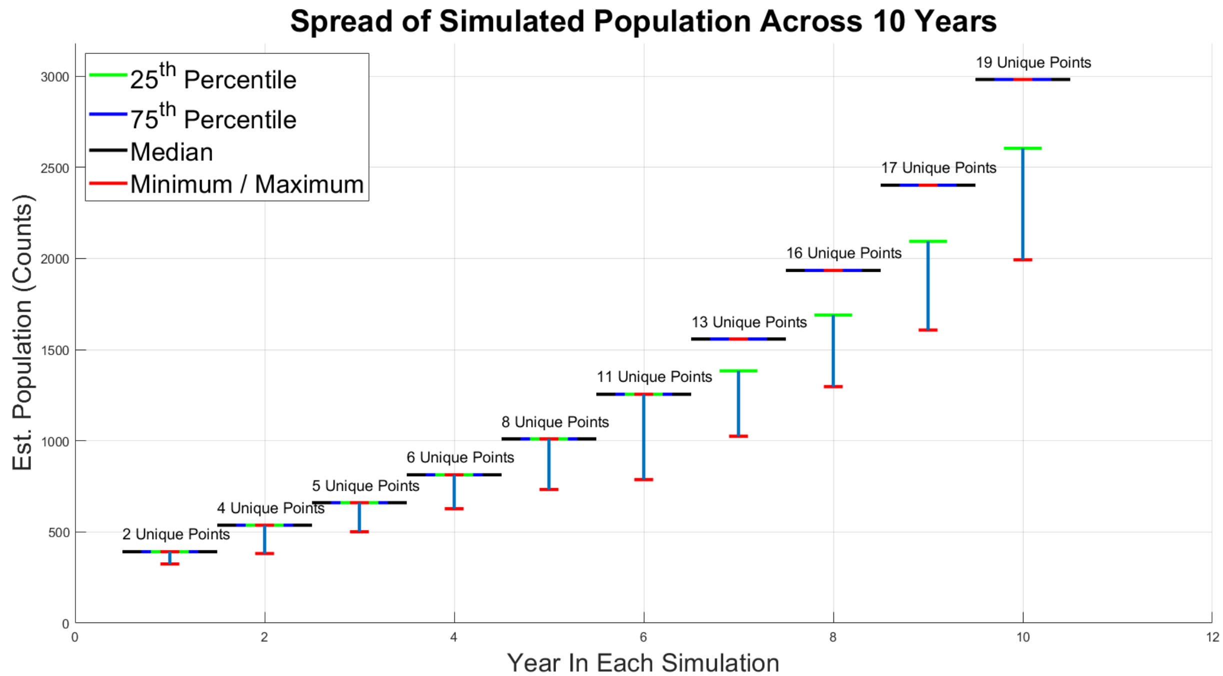


Figure 6: Box plot for total population of female bobcats at each time step.

When simulating a catastrophe, there were only two options for each year: either a catastrophe occurred, or no catastrophe occurred. Accordingly, the first time step only had two possible values, no matter how many simulations were run or how likely catastrophe was (assuming it was neither 0% or 100%). Each box is labeled with "unique points", which refers to the number of distinct results that were observed across the 100 simulations for each year. Since only two options were available for the first year, only two unique points were observed. If enough simulations are run, the number of unique points at year *n* would be expected to approach 2^n. The spread increases over time as the possibility of a disaster accumulates. This also shows that the population is consistently growing; even at the minimum estimated population count of each year there is growth (if a disaster occurred every year, the growth rate would approach the dominant eigenvalue of the catastrophe Leslie matrix, which would mean a growth rate of 8%).

Conclusions

From the Leslie matrices that we developed for the bobcat population,we demonstrated that the bobcat population is expected to grow. What was interesting was that even with a 4% chance of a catastrophe every year, the endangered bobcat population is still expected to grow. From our simulation, we also saw little variation between the 100 runs and saw similar age distributions across the 16 classes of bobcats. Furthermore, the best and worst case scenarios both showed the bobcat population growing over time, and had similar dominant eigenvalues, Leslie matrices, and bobcat age distribution over 10 years. These results are promising, but for this simulation and model to work, we had to make a variety of assumptions. First, we excluded male bobcats from our model to create a Leslie matrix, this is a major issue when trying to understand an age distribution of a population (both male and female). Furthermore, we assumed that a catastrophe was equally likely every year at 4%, which is both a low percentage and the probability of a catastrophe having a uniform distribution in real life is likely small. So, if we were to repeat this study, it might be good to re-run this simulation with more trials (more than 100) as 100 may not be a large enough sample size to capture the worst-case and best-case scenarios. Using more runs in our simulation may increase the variation in the population distributions across the 16 classes of bobcats. We also might want to try different probabilities of catastrophes every year and try different distributions other than a uniform distribution. Overall, with the assumptions we made, we feel confident in the construction of our Leslie matrices and the results of the simulation, but we also believe there are ways to improve this study to make the model more realistic.

Bibliography

[1] James Cox, Randy Kautz, Maureen MacLaughlin, and Terry Gilbert. *Closing the Gaps in Florida's Wildlife Conservation System*. Office of Environmental Services, Florida Game, and Fresh Water Fish Commission, Tallahassee, Florida, 1994.