# Test of ODE

1. (1) $(2x\sin y+3x^2y)\,dx+(x^3+x^2\cos y+y^2)\,dy=0.$ [Exact Equation]

$P(x,y)=2x\sin y+3x^2y \qquad Q(x,y)=x^3+x^2\cos y+y^2$

$\cancel{\partial\Phi}\ \frac{\partial\Phi}{\partial x}=2x\sin y+3x^2y \qquad \frac{\partial\Phi}{\partial y}=x^3+x^2\cos y+y^2.$

$\Phi(x,y)=\int 2x\sin y+3x^2y\,dx=x^2\sin y+x^3y+\varphi(y)$

$\frac{\partial\Phi}{\partial y}=x^2\cos y+x^3+\varphi'(y)=x^3+x^2\cos y+y^2.$

$\therefore \varphi'(y)=y^2 \quad\Rightarrow \varphi(y)=\frac{1}{3}y^3+C$

$\therefore \Phi(x,y)=x^2\sin y+x^3y+\frac{1}{3}y^3+C$ , where $C$ is an arbitrary constant.

(2). $\frac{dy}{dx}=\frac{x+y}{x-y}$ [variable substitution; separable equations]

Let $y=ux$.

$\frac{dy}{dx}=x\frac{du}{dx}+u=\frac{1+u}{1-u} \quad\Rightarrow\quad x\frac{du}{dx}=\frac{1+u}{1-u}-u=\frac{1+u^2}{1-u}$

$\frac{dx}{x}=\frac{1-u}{1+u^2}\,du \quad\Rightarrow \ln|x|=\arctan u-\ln(1+u^2)^{\frac{1}{2}}+\ln C.$

$\therefore x\cdot\sqrt{1+u^2}=Ce^{\arctan u}$

$\therefore \sqrt{x^2+y^2}=Ce^{\arctan\frac{y}{x}}.$

let $x=r\cos\theta$, $y=r\sin\theta$ $\Rightarrow r=Ce^{\theta}.$

(3). $\frac{dy}{dx}+\frac{1}{x}y=x^3 \quad (x\neq 0).$ [first order linear equation]

$y=e^{-\int\frac{1}{x}dx}\left[C+\int x^3e^{\int\frac{1}{x}dx}\,dx\right]$

$=\frac{1}{|x|}\left(C+\int x^4\,dx\right)$

$=\frac{1}{|x|}\left(C+\frac{1}{5}x^5\right)=\frac{x^4}{5}+\frac{C}{x} \quad (x\neq 0)$

4. $(3x^2+y)dx+(2x^2y-x)dy=0$. [integrating factor]

$$\frac{\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}}{Q}=\frac{1-4xy+1}{2x^2y-x}=\frac{-2}{x}$$

$$u=e^{\int\frac{-2}{x}dx}=\frac{1}{x^2}$$

$$3dx+2y\,dy+\frac{ydx-xdy}{x^2}=0$$

$$3x+y^2-\frac{y}{x}=C.$$

(5) $(x^2+1)(y^2-1)dx+xy\,dy=0$. [separable equation]

when $x(y^2-1)\neq 0$; $\frac{x^2+1}{x}dx+\frac{y}{y^2-1}dy=0$.

$$x^2+\ln x^2+\ln|y^2-1|=C_1$$

$$x^2e^{x^2}|y^2-1|=e^{C_1}$$

$$y^2=1+C\frac{e^{-x^2}}{x^2}.$$

when $x=0$, is a particular solutions.

2. $y''+ay=0$

1) when $a=0$. $y''=0$. $\lambda^2=0$. $\lambda_1=\lambda_2=0$ $y=C_1+C_2x$.

2). when $a>0$. $\lambda^2+a=0$ $\lambda^2=-a$ $\lambda=\pm\sqrt{a}\,i$ $y=C_1\cos\sqrt{a}x+C_2\sin\sqrt{a}x$.

3) when $a<0$, $\therefore\lambda=\pm\sqrt{-a}$ $y=C_1e^{\sqrt{-a}x}+C_2e^{-\sqrt{-a}x}$.

3. $y'''+3y''+3y'+y=e^{-x}(x-5)$.

$$\lambda^3+3\lambda^2+3\lambda+1=(\lambda+1)^3=0.$$

general solution: $y=C_1e^{-x}+C_2xe^{-x}+C_3x^2e^{-x}$.

let $y^*=x^3(ax+b)e^{-x}$.

$$y^{*\prime}=(-ax^4+4ax^3-bx^3+3bx^2)e^{-x}$$

$$y^{*\prime\prime}=(ax^4-8ax^3+bx^3+12ax^2-6bx^2+6bx)e^{-x}.$$

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$Y''' = (-ax^4+12ax^3-bx^3-36ax^2+9bx^2+24ax-18bx+6b)e^{-x}$

将 $Y''', Y'', Y', Y$ 代入 $y'''+3y''+3y'+y=e^{-x}(x-5)$.

$(24ax+6b)e^{-x}=(x-5)e^{-x}$.

$\therefore a=\frac{1}{24} \quad b=-\frac{5}{6}$.

$\therefore y=(C_1+C_2x+C_3x^2-\frac{5}{6}x^3+\frac{1}{24}x^4)e^{-x}$.

4. $y''-5y'+6y=xe^{2x}$.

$\lambda^2-5\lambda+6=0 \quad \lambda_1=2 \quad \lambda_2=3$.

general solution $y=C_1e^{2x}+C_2e^{3x}$.

$Y=x(bx+c)e^{2x}=(bx^2+cx)e^{2x}$.

$Y'=(2bx^2+2bx+2cx+c)e^{2x}$.

$Y''=(4bx^2+8bx+4cx+2b+4c)e^{2x}$.

$-2bx+2b-c=x$

$\therefore b=-\frac{1}{2} \quad c=-1$

$\therefore y=C_1e^{2x}+C_2e^{3x}-(\frac{1}{2}x^2+x)e^{2x}$.

5. $y''+y'-2y=3e^x-\frac{1}{2}\sin x$.

$\lambda^2+\lambda-2=0 \quad \lambda_1=1,\ \lambda_2=-2$.

general solution $y=C_1e^x+C_2e^{-2x}$.

① $y''+y'-2y=3e^x$

$Y_1=Axe^x$.

$Y_1'=A(e^x+xe^x) \qquad Y_1''=A(2e^x+xe^x)$

$3Ae^x=3e^x \quad A=1$.

$\therefore Y_1=xe^x$

② $y''+y'-2y=-\frac{1}{2}\sin x$.

$Y_2=B\cos x+C\sin x$

$Y_2'=-B\sin x+C\cos x \quad Y_2''=-B\cos x-C\sin x$.

$(-3C-B)\sin x+(C-3B)\cos x=-\frac{1}{2}\sin x$

$\therefore \begin{cases}-3C-B=-\frac{1}{2}\\ C-3B=0\end{cases} \quad \therefore \begin{cases}B=\frac{1}{20}\\ C=\frac{3}{20}\end{cases}$

$\therefore Y_2=\frac{1}{20}\cos x+\frac{3}{20}\sin x$.

$\therefore y = C_1e^{x} + C_2e^{-2x} + xe^{x} + \frac{1}{20}(\cos x + 3\sin x)$.

6. $r_1 = r_2 = 1$.

$r_3 = r_4 = \pm 3i$.

$(r-1)^2 \cdot (r^2+9) = 0$.

$(r^2-2r+1)(r^2+9) = 0$.

$r^4 - 2r^3 + 10r^2 - 18r + 9 = 0$.

$y^{(4)} - 2y^{(3)} + 10y'' - 18y' + 9y = 0$.

$y = (C_1 + C_2x)e^{x} + C_3\sin 3x + C_4\cos 3x$.

7. $y' + P(x)y = q(x)y^{n}$

1) when $n=0$. $y' + P(x)y = q(x)$.

$$y = e^{-\int P(x)dx}\left(C + \int q(x)e^{\int P(x)dx}dx\right).$$

2) when $n=1$. $y' + P(x)y = q(x)y$.

$y' = (q(x) - P(x))y$.

$\frac{dy}{dx} = (q(x) - P(x))y$.

$\frac{1}{y}dy = (q(x) - P(x))dx$.

$\ln|y| = \int q(x) - P(x)\,dx$.

$y = e^{\int q(x) - P(x)dx}$

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