Problem 5.1

[Difficulty: 2]

5.1 Which of the following sets of equations represent possible three-dimensional incompressible flow cases?

(a) $u = 2y^2 + 2xz;\ v = -2yz + 6x^2yz;\ w = 3x^2z^2 + x^3y^4$

(b) $u = xyzt;\ v = -xyzt^2;\ w = z^2(xt^2 - yt)$

(c) $u = x^2 + 2y + z^2;\ v = x - 2y + z;\ w = -2xz + y^2 + 2z$

Given: Velocity fields

Find: Which are 3D incompressible

Solution: We will check these flow fields against the continuity equation

Governing Equation:

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)+\frac{\partial \rho}{\partial t}=0 \quad \text{(Continuity equation)}$$

Assumption: Incompressible flow (ρ is constant)

Based on the assumption, the continuity equation reduces to: $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = 0$

This is the criterion against which we will check all of the flow fields.

a) $u(x,y,z,t) = 2\cdot y^2 + 2\cdot x\cdot z$ $\quad v(x,y,z,t) = -2\cdot y\cdot z + 6\cdot x^2\cdot y\cdot z$ $\quad w(x,y,z,t) = 3\cdot x^2\cdot z^2 + x^3\cdot y^4$

$\frac{\partial}{\partial x}u(x,y,z,t) = 2\cdot z$ $\quad \frac{\partial}{\partial y}v(x,y,z,t) = 6\cdot x^2\cdot z - 2\cdot z$ $\quad \frac{\partial}{\partial z}w(x,y,z,t) = 6\cdot x^2\cdot z$

Hence $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w \neq 0$ NOT INCOMPRESSIBLE

b) $u(x,y,z,t) = x\cdot y\cdot z\cdot t$ $\quad v(x,y,z,t) = -x\cdot y\cdot z\cdot t^2$ $\quad w(x,y,z,t) = z^2\cdot\left(x\cdot t^2 - y\cdot t\right)$

$\frac{\partial}{\partial x}u(x,y,z,t) = t\cdot y\cdot z$ $\quad \frac{\partial}{\partial y}v(x,y,z,t) = -t^2\cdot x\cdot z$ $\quad \frac{\partial}{\partial z}w(x,y,z,t) = 2\cdot z\cdot\left(t^2\cdot x - t\cdot y\right)$

Hence $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w \neq 0$ NOT INCOMPRESSIBLE

c) $u(x,y,z,t) = x^2 + 2\cdot y + z^2$ $\quad v(x,y,z,t) = x - 2\cdot y + z$ $\quad w(x,y,z,t) = -2\cdot x\cdot z + y^2 + 2\cdot z$

$\frac{\partial}{\partial x}u(x,y,z,t) = 2\cdot x$ $\quad \frac{\partial}{\partial y}v(x,y,z,t) = -2$ $\quad \frac{\partial}{\partial z}w(x,y,z,t) = 2 - 2\cdot x$

Hence $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = 0$ INCOMPRESSIBLE

Problem 5.2

(Difficulty 1)

5.2 Which of the following sets of equations represent possible two-dimensional incompressible flow cases?

(a) $u = 2xy; v = -x^2 y$

(b) $u = y - x + x^2; v = x + y - 2xy$

(c) $u = x^2 t + 2y; v = 2x - yt^2$

(d) $u = -x^2 - y^2 - xyt; v = x^2 + y^2 + xyt$

Find: The sets of equations represent possible two-dimensional incompressible flow.

Solution:

From the continuity equation for two-dimensional incompressible flow, we have:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

(a)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2y - x^2 \neq 0$$

This is not incompressible flow.

(b)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -1 + 2x + 1 - 2x = 0$$

This represents the incompressible flow.

(c)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2xt - t^2 \neq 0$$

This is not incompressible flow.

(d)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -2x - yt + 2y + xt \neq 0$$

This is not incompressible flow.

Problem 5.3

(Difficulty 1)

5.3 In an incompressible three-dimensional flow field, the velocity components are given by $u = ax + byz$; $v = cy + dxz$. Determine the form of the z component of velocity. If the z component were not a function of x or y, what would the form be?

Find: The z component of velocity.

Assumptions: The flow is steady and incompressible

Solution: Use the continuity equation for incompressible flow is:

$$\nabla \cdot \vec{V} = 0$$

or

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Thus

$$\frac{\partial u}{\partial x} = a$$

$$\frac{\partial v}{\partial y} = c$$

$$\frac{\partial w}{\partial z} = -a - c$$

As w is not a function of x and y we have:

$$w = -(a + c)z + constant$$

Problem 5.4

(Difficulty 1)

5.4 In a two-dimensional incompressible flow field, the x component of velocity is given by $u = 2x$. Determine the equation for y component of velocity if $v = 0$ along the x-axis.

Find: The equation for y component of velocity.

Assumptions: The flow is steady and incompressible

Solution: Use the continuity equation for incompressible two-dimensional flow:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

We have:

$$\frac{\partial u}{\partial x} = 2$$

Thus

$$\frac{\partial v}{\partial y} = -2$$

$$v = -2y + f(x)$$

We know:

At $y = 0$, $v = 0$, so we have:

$$f(x) = 0$$

The equation for y component of velocity is:

$$v = -2y$$

Problem 5.5

[Difficulty: 1]

5.5 The three components of velocity in a velocity field are given by $u = Ax + By + Cz$, $v = Dx + Ey + Fz$, and $w = Gx + Hy + Jz$. Determine the relationship among the coefficients A through J that is necessary if this is to be a possible incompressible flow field.

Given: The velocity field provided above

Find: The conditions under which this fields could represent incompressible flow

Solution: We will check this flow field against the continuity equation

Governing Equations:

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)+\frac{\partial \rho}{\partial t}=0 \quad \text{(Continuity equation)}$$

Assumptions: (1) Incompressible flow (ρ is constant)

Based on the assumption listed, the continuity equation reduces to: $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$

Calculating the partial derivatives of the velocity components: $\frac{\partial u}{\partial x}=A \qquad \frac{\partial v}{\partial y}=E \qquad \frac{\partial w}{\partial z}=J$

Applying this information to the continuity equation we get the necessary condition for incompressible flow:

$$A + E + J = 0$$

(B, C, D, F, G, and H are arbitrary)

Problem 5.6 [Difficulty: 2]

5.6 The x component of velocity in a steady, incompressible flow field in the xy plane is $u = A/x$, where $A = 2\ \mathrm{m^2/s}$, and x is measured in meters. Find the simplest y component of velocity for this flow field.

Given: The x-component of velocity in a steady, incompressible flow field

Find: The simplest y-component of velocity for this flow field

Solution: We will check this flow field against the continuity equation

Governing Equations:

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)+\frac{\partial \rho}{\partial t}=0 \quad \text{(Continuity equation)}$$

Assumptions:
(1) Incompressible flow (ρ is constant)
(2) Two dimensional flow (velocity is not a function of z)

Based on the two assumptions listed above, the continuity equation reduces to: $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$

The partial of u with respect to x is: $\frac{\partial u}{\partial x}=-\frac{A}{x^2}$ Therefore from continuity, we have $\frac{\partial v}{\partial y}=-\frac{\partial u}{\partial x}=\frac{A}{x^2}$

Integrating this expression will yield the y-component of velocity: $v=\int \frac{A}{x^2}\,dy+f(x)=\frac{A\cdot y}{x^2}+f(x)$

The simplest version of this velocity component would result when f(x) = 0: $v=\frac{A\cdot y}{x^2}$

Problem 5.7

[Difficulty: 3]

5.7 The y component of velocity in a steady incompressible flow field in the xy plane is

$$v = \frac{2xy}{(x^2+y^2)^2}$$

Show that the simplest expression for the x component of velocity is

$$u = \frac{1}{(x^2+y^2)} - \frac{2y^2}{(x^2+y^2)^2}$$

Given: y component of velocity

Find: x component for incompressible flow; Simplest x component

Solution:

Basic equation: $\frac{\partial}{\partial x}(\rho\cdot u) + \frac{\partial}{\partial y}(\rho\cdot v) + \frac{\partial}{\partial z}(\rho\cdot w) + \frac{\partial}{\partial t}\rho = 0$

Assumption: Incompressible flow; flow in x-y plane

Hence $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$ or $\frac{\partial}{\partial x}u = -\frac{\partial}{\partial y}v = -\frac{\partial}{\partial y}\left[\frac{2\cdot x\cdot y}{\left(x^2+y^2\right)^2}\right] = -\left[\frac{2\cdot x\cdot\left(x^2-3\cdot y^2\right)}{\left(x^2+y^2\right)^3}\right]$

Integrating $u(x,y) = -\int\left[\frac{2\cdot x\cdot\left(x^2-3\cdot y^2\right)}{\left(x^2+y^2\right)^3}\right]dx = \frac{x^2-y^2}{\left(x^2+y^2\right)^2} + f(y) = \frac{x^2+y^2-2\cdot y^2}{\left(x^2+y^2\right)^2} + f(y)$

$$u(x,y) = \frac{1}{x^2+y^2} - \frac{2\cdot y^2}{\left(x^2+y^2\right)^2} + f(y)$$

The simplest form is $u(x,y) = \frac{1}{x^2+y^2} - \frac{2\cdot y^2}{\left(x^2+y^2\right)^2}$

Note: Instead of this approach we could have verified that u and v satisfy continuity

$$\frac{\partial}{\partial x}\left[\frac{1}{x^2+y^2} - \frac{2\cdot y^2}{\left(x^2+y^2\right)^2}\right] + \frac{\partial}{\partial y}\left[\frac{2\cdot x\cdot y}{\left(x^2+y^2\right)^2}\right] = 0$$

However, this does not verify the solution is the simplest.

Problem 5.8

(Difficulty 1)

5.8 The velocity components for an incompressible steady flow field are $u = a(x^2 + z^2)$ and $v = b(xy + yz)$. Determine the general expression for the z component of velocity. If the flow were unsteady, what would be the expression for z component ?

Find: The expression for z component velocity.

Assumptions: The flow is steady and incompressible

Solution: Use the continuity equation:

$$\nabla \cdot \rho\vec{V} + \frac{\partial \rho}{\partial t} = 0$$

For incompressible flow:

$$\rho = constant$$

$$\nabla \cdot \vec{V} = 0$$

Thus

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} = 2ax$$

$$\frac{\partial v}{\partial y} = b(x + z)$$

$$\frac{\partial w}{\partial z} = -2ax - bx - bz = (-2a - b)x - bz$$

$$w = (-2a - b)xz - \frac{b}{2}z^2 + f(x, y)$$

$f(x, y)$ is a general function of x and y.

If the flow were non-steady, the expression for z component will be the same because ρ is constant. The term respect to time in the continuity equation is always zero for incompressible flow.

Problem 5.9

(Difficulty 2)

5.9 The radial component of velocity in an incompressible two-dimensional flow is given by $V_r = 3r - 2r^2\cos(\theta)$. Determine the general expression for the θ component of velocity. If the flow were non-steady, what would be the expression for the θ component?

Find: The expression for θ component velocity.

Assumptions: The flow is steady and incompressible

Solution: Use the continuity equation:

$$\nabla \cdot \rho\vec{V} + \frac{\partial \rho}{\partial t} = 0$$

For incompressible flow:

$$\rho = constant$$

$$\nabla \cdot \vec{V} = 0$$

Thus

$$\frac{\partial (rV_r)}{\partial r} + \frac{\partial V_\theta}{\partial \theta} = 0$$

$$\frac{\partial (rV_r)}{\partial r} = \frac{\partial (3r^2 - 2r^3\cos(\theta))}{\partial r} = 6r - 6r^2\cos(\theta)$$

We can get:

$$\frac{\partial V_\theta}{\partial \theta} = 6r^2\cos(\theta) - 6r$$

$$V_\theta = 6r^2\sin(\theta) - 6r\theta + f(r)$$

If the flow were non-steady, the expression for θ component will be the same because ρ is constant. The term respect to time in the continuity equation is always zero for incompressible flow.

Problem 5.10

[Difficulty: 2]

5.10 A crude approximation for the x component of velocity in an incompressible laminar boundary layer is a linear variation from $u = 0$ at the surface ($y = 0$) to the freestream velocity, U, at the boundary-layer edge ($y = \delta$). The equation for the profile is $u = Uy/\delta$, where $\delta = cx^{1/2}$ and c is a constant. Show that the simplest expression for the y component of velocity is $v = uy/4x$. Evaluate the maximum value of the ratio v/U, at a location where $x = 0.5$ m and $\delta = 5$ mm.

Given: Approximate profile for a laminar boundary layer:

$$u = \frac{U \cdot y}{\delta} \qquad \delta = c \cdot \sqrt{x} \qquad \text{(c is constant)}$$

Find: (a) Show that the simplest form of v is

$$v = \frac{u}{4} \cdot \frac{y}{x}$$

(b) Evaluate maximum value of v/u where δ = 5 mm and x = 0.5 m

Solution: We will check this flow field using the continuity equation

Governing Equations:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) + \frac{\partial \rho}{\partial t} = 0 \qquad \text{(Continuity equation)}$$

Assumptions:
(1) Incompressible flow (ρ is constant)
(2) Two dimensional flow (velocity is not a function of z)

Based on the two assumptions listed above, the continuity equation reduces to: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

The partial of u with respect to x is: $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \delta}\frac{d\delta}{dx} = -\frac{Uy}{\delta^2} \times \frac{1}{2} cx^{-\frac{1}{2}} = -\frac{Uy}{2cx^{\frac{3}{2}}}$ Therefore from continuity: $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = \frac{Uy}{2cx^{\frac{3}{2}}}$

Integrating this expression will yield the y-component of velocity:

$$v = \int \frac{U \cdot y}{2 \cdot c \cdot x^{\frac{3}{2}}}\, dy + f(x) = \frac{U \cdot y^2}{4 \cdot c \cdot x^{\frac{3}{2}}} + f(x)$$

Now due to the no-slip condition at the wall (y = 0) we get f(x) = 0. Thus: $v = \frac{U \cdot y^2}{4 \cdot c \cdot x^{\frac{3}{2}}} = \frac{U \cdot y}{c \cdot x^{\frac{1}{2}}} \cdot \frac{y}{4 \cdot x} = \frac{u \cdot y}{4 \cdot x}$ (Q.E.D.)

$$v = \frac{u}{4} \cdot \frac{y}{x}$$

The maximum value of v/U is where y = δ: $v_{ratmax} = \frac{v}{u} = \frac{\delta}{4 \cdot x}$ $\qquad v_{ratmax} = \frac{5 \times 10^{-3} \cdot m}{4 \times 0.5 \cdot m}$

$$v_{ratmax} = 0.0025$$

Problem 5.11

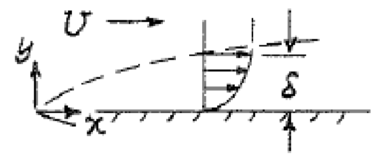
[Difficulty: 3]

5.11 A useful approximation for the x component of velocity in an incompressible laminar boundary layer is a parabolic variation from $u = 0$ at the surface ($y = 0$) to the freestream velocity, U, at the edge of the boundary layer ($y = \delta$). The equation for the profile is $u/U = 2(y/\delta) - (y/\delta)^2$, where $\delta = cx^{1/2}$ and c is a constant. Show that the simplest expression for the y component of velocity is

$$\frac{v}{U} = \frac{\delta}{x}\left[\frac{1}{2}\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right]$$

Plot v/U versus y/δ to find the location of the maximum value of the ratio v/U. Evaluate the ratio where $\delta = 5$ mm and $x = 0.5$ m.

Given: Approximate (parabolic) profile for a laminar boundary layer:



$$\frac{u}{U} = 2\cdot\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \qquad \delta = c\cdot\sqrt{x} \quad \text{(c is constant)}$$

Find: (a) Show that the simplest form of v for incompressible flow is

$$\frac{v}{U} = \frac{\delta}{x}\cdot\left[\frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\cdot\left(\frac{y}{\delta}\right)^3\right]$$

(b) Plot v/U versus y/δ

(c) Evaluate maximum value of v/U where δ = 5 mm and x = 0.5 m

Solution: We will check this flow field using the continuity equation

Governing Equations:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) + \frac{\partial \rho}{\partial t} = 0 \quad \text{(Continuity equation)}$$

Assumptions:

(1) Incompressible flow (ρ is constant)

(2) Two dimensional flow (velocity is not a function of z)

Based on the two assumptions listed above, the continuity equation reduces to: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

The partial of u with respect to x is: $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \delta}\frac{d\delta}{dx} = U\left[-\frac{2y}{\delta^2} + \frac{2y^2}{\delta^3}\right] \times \frac{1}{2}cx^{-\frac{1}{2}}$ Now since $\delta = c\cdot x^{\frac{1}{2}}$ $x^{-\frac{1}{2}} = \frac{c}{\delta}$ and thus

$$\frac{\partial u}{\partial x} = \frac{Uc^2}{\delta}\left[-\frac{y}{\delta^2} + \frac{y^2}{\delta^3}\right] = \frac{Uc^2}{\delta^2}\left[-\left(\frac{y}{\delta}\right) + \left(\frac{y}{\delta}\right)^2\right]$$ Therefore from continuity: $$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = \frac{Uc^2}{\delta^2}\left[\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right]$$

Integrating this expression will yield the y-component of velocity: $v = \int \frac{U\cdot c^2}{\delta}\cdot\left[\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right] dy + f(x)$ Evaluating:

$$v = \frac{U\cdot c^2}{\delta^2}\cdot\left(\frac{y^2}{2\cdot\delta} - \frac{y^3}{3\cdot\delta^2}\right) + f(x) = \frac{U\cdot c^2}{\delta}\cdot\left[\frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\cdot\left(\frac{y}{\delta}\right)^3\right] + f(x)$$ Since $\delta = c\cdot x^{\frac{1}{2}}$ $c^2 = \frac{\delta^2}{x}$ Thus:

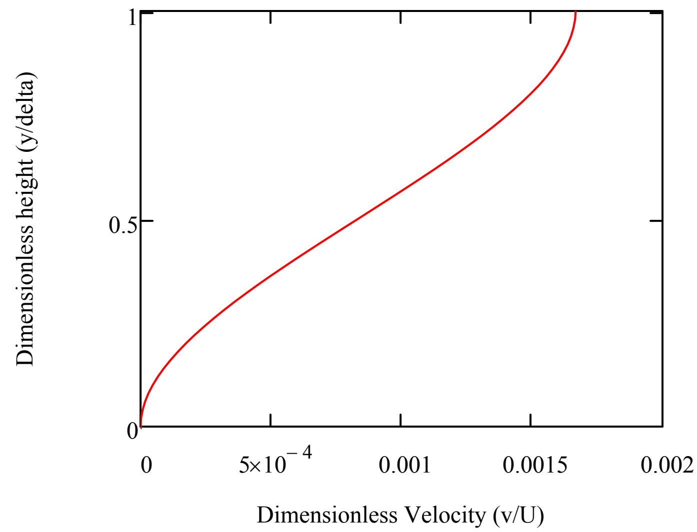
$$v = U\cdot\frac{\delta}{x}\cdot\left[\frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\cdot\left(\frac{y}{\delta}\right)^3\right] + f(x)$$

Now due to the no-slip condition at the wall (y = 0) we get f(x) = 0. Therefore:

$$\frac{v}{U} = \frac{\delta}{x}\cdot\left[\frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\cdot\left(\frac{y}{\delta}\right)^3\right] \quad \text{(Q.E.D.)} \qquad \frac{v}{U} = \frac{\delta}{x}\cdot\left[\frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\cdot\left(\frac{y}{\delta}\right)^3\right]$$

Plotting this relationship shows:

Assuming x = 0.5 m and δ = 5 mm



The maximum value of v/U is where y = δ: $v_{ratmax} = \frac{v}{U} = \frac{\delta}{x}\cdot\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\delta}{6\cdot x}$ $\quad v_{ratmax} = \frac{5 \times 10^{-3}\cdot m}{6 \times 0.5\cdot m}$ $\quad v_{ratmax} = 0.00167$

Problem 5.12 [Difficulty: 3]

5.12 A useful approximation for the x component of velocity in an incompressible laminar boundary layer is a cubic variation from $u=0$ at the surface ($y=0$) to the freestream velocity, U, at the edge of the boundary layer ($y=\delta$). The equation for the profile is $u/U=\frac{3}{2}(y/\delta)-\frac{1}{2}(y/\delta)^3$, where $\delta=cx^{1/2}$ and c is a constant. Derive the simplest expression for v/U, the y component of velocity ratio. Plot u/U and v/U versus y/δ, and find the location of the maximum value of the ratio v/U. Evaluate the ratio where $\delta=5$ mm and $x=0.5$ m.

Given: Data on boundary layer

Find: *y* component of velocity ratio; location of maximum value; plot velocity profiles; evaluate at particular point

Solution:

$$u(x,y) = U\cdot\left[\frac{3}{2}\cdot\left(\frac{y}{\delta(x)}\right) - \frac{1}{2}\cdot\left(\frac{y}{\delta(x)}\right)^3\right] \quad \text{and} \quad \delta(x) = c\cdot\sqrt{x}$$

so

$$u(x,y) = U\cdot\left[\frac{3}{2}\cdot\left(\frac{y}{c\cdot\sqrt{x}}\right) - \frac{1}{2}\cdot\left(\frac{y}{c\cdot\sqrt{x}}\right)^3\right]$$

For incompressible flow $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$

Hence

$$v(x,y) = -\int \frac{d}{dx}u(x,y)\,dy \quad \text{and} \quad \frac{du}{dx} = \frac{3}{4}\cdot U\cdot\left(\frac{y^3}{c^3\cdot x^{\frac{5}{2}}} - \frac{y}{c\cdot x^{\frac{3}{2}}}\right)$$

so

$$v(x,y) = -\int \frac{3}{4}\cdot U\cdot\left(\frac{y^3}{c^3}\cdot\frac{x^5}{2} - \frac{y}{c}\cdot\frac{x^3}{2}\right)dy$$

$$v(x,y) = \frac{3}{8}\cdot U\cdot\left(\frac{y^2}{c\cdot x^{\frac{3}{2}}} - \frac{y^4}{2\cdot c^3\cdot x^{\frac{5}{2}}}\right) \qquad v(x,y) = \frac{3}{8}\cdot U\cdot\frac{\delta}{x}\cdot\left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^4\right]$$

The maximum occurs at $y = \delta$ as seen in the *Excel* work shown below.

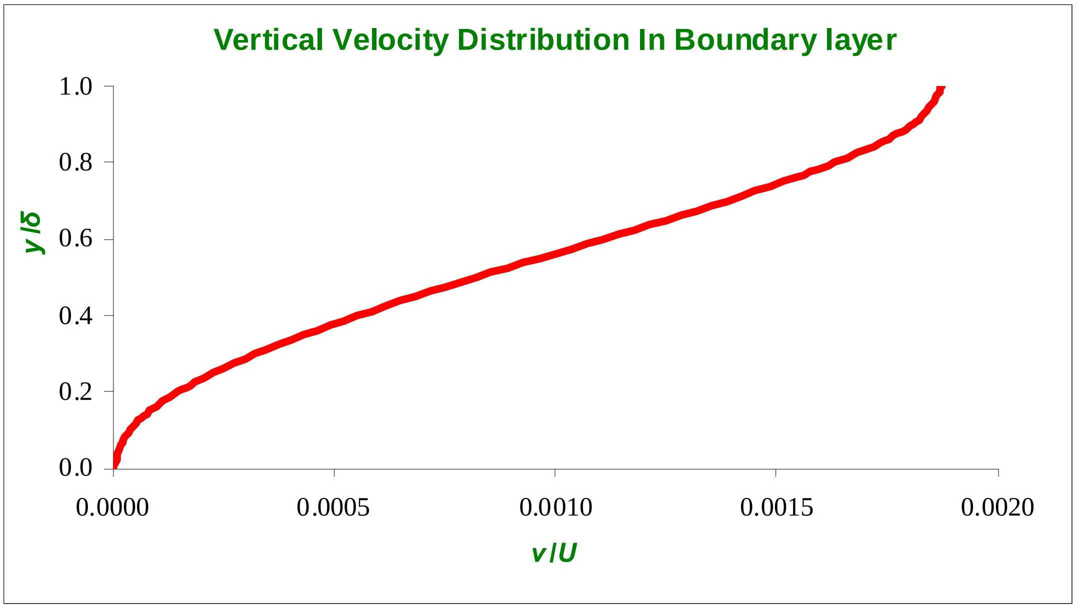
$$v_{max} = \frac{3}{8}\cdot U\cdot\frac{\delta}{x}\cdot\left(1 - \frac{1}{2}\cdot 1\right)$$

At $\delta = 5\cdot\text{mm}$ and $x = 0.5\cdot\text{m}$, the maximum vertical velocity is $\frac{v_{max}}{U} = 0.00188$

To find when v/U is maximum, use *Solver* in *Excel*

v/U	y/δ
0.00188	1.0

v/U	y/δ
0.000000	0.0
0.000037	0.1
0.000147	0.2
0.000322	0.3
0.000552	0.4
0.00082	0.5
0.00111	0.6
0.00139	0.7
0.00163	0.8
0.00181	0.9
0.00188	1.0



Problem 5.13

[Difficulty: 3]

5.13 For a flow in the xy plane, the x component of velocity is given by $u = Ax^2y^2$, where $A = 0.3\ \mathrm{m^{-3}\cdot s^{-1}}$, and x and y are measured in meters. Find a possible y component for steady, incompressible flow. Is it also valid for unsteady, incompressible flow? Why? How many possible y components are there? Determine the equation of the streamline for the simplest y component of velocity. Plot the streamlines through points (1, 4) and (2, 4).

Given: Steady, incompressible flow in x-y plane:

$u = A \cdot x^2 \cdot y^2 \qquad A = 0.3 \cdot m^{-3} \cdot s^{-1}$

Find:
(a) a possible y component of velocity for this flow field
(b) if the result is valid for unsteady, incompressible flow
(c) number of possible y components for velocity
(d) equation of the streamlines for the flow
(e) plot streamlines through points (1,4) and (2,4)

Solution: We will check this flow field using the continuity equation

Governing Equations:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) + \frac{\partial \rho}{\partial t} = 0 \quad \text{(Continuity equation)}$$

Assumptions:
(1) Incompressible flow (ρ is constant)
(2) Two dimensional flow (velocity is not a function of z)

Based on the two assumptions listed above, the continuity equation reduces to: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

The partial of u with respect to x is: $\frac{\partial u}{\partial x} = 2Axy^2$ Therefore from continuity: $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -2Axy^2$

Integrating this expression will yield the y-component of velocity: $v = \int -2 \cdot A \cdot x \cdot y^2\, dx + f(x)$ $\qquad v = -\frac{2}{3} \cdot A \cdot x^2 \cdot y^3 + f(x)$

The basic equation reduces for the same form for unsteady flow. Hence The result is valid for unsteady, incompressible flow.

Since f(x) is arbitrary: There are an infinite number of possible y-components of velocity.

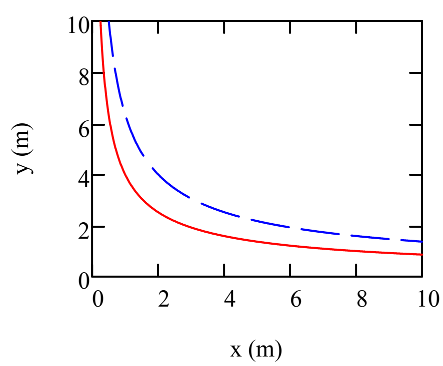
The simplest version of v is when f(x) = 0. Therefore, the equation of the corresponding streamline is:

$$\frac{dy}{dx} = \frac{v}{u} = \frac{-\frac{2}{3} A \cdot x^2 \cdot y^3}{A \cdot x^2 \cdot y^2} = -\frac{2}{3} \cdot \frac{y}{x}$$ Separating variables and integrating: $\frac{dy}{y} = -\frac{2}{3} \cdot \frac{dx}{x}$ $\quad \ln(y) = -\frac{2}{3} \cdot \ln(x)$ Thus: $x \cdot y^{\frac{3}{2}} = \text{constant}$

are the equations of the streamlines of this flow field.

Plotting streamline for point (1, 4): $1 \times 4^{\frac{3}{2}} = 8 \qquad x \cdot y^{\frac{3}{2}} = 8$

Plotting streamline for point (2, 4): $2 \times 4^{\frac{3}{2}} = 16 \qquad x \cdot y^{\frac{3}{2}} = 16$



The two streamlines are plotted here in red (1,4) and blue (2,4):

Problem 5.14

[Difficulty: 5]

5.14 Consider a water stream from a jet of an oscillating lawn sprinkler. Describe the corresponding pathline and streakline.

Discussion: Refer back to the discussion of streamlines, pathlines, and streaklines in Section 2-2.

Because the sprinkler jet oscillates, this is an unsteady flow. Therefore pathlines and streaklines need not coincide.

A *pathline* is a line tracing the path of an individual fluid particle. The path of each particle is determined by the jet angle and the speed at which the particle leaves the jet.

Once a particle leaves the jet it is subject to gravity and drag forces. If aerodynamic drag were negligible, the path of each particle would be parabolic. The horizontal speed of the particle would remain constant throughout its trajectory. The vertical speed would be slowed by gravity until reaching peak height, and then it would become increasingly negative until the particle strikes the ground. The effect of aerodynamic drag is to reduce the particle speed. With drag the particle will not rise as high vertically nor travel as far horizontally. At each instant the particle trajectory will be lower and closer to the jet compared to the no-friction case. The trajectory after the particle reaches its peak height will be steeper than in the no-friction case.

A *streamline* is a line drawn in the flow that is tangent everywhere to the velocity vectors of the fluid motion. It is difficult to visualize the streamlines for an unsteady flow field because they move laterally. However, the streamline pattern may be drawn at an instant.

A *streakline* is the locus of the present locations of fluid particles that passed a reference point at previous times. As an example, choose the exit of a jet as the reference point. Imagine marking particles that pass the jet exit at a given instant and at uniform time intervals later. The first particle will travel farthest from the jet exit and on the lowest trajectory; the last particle will be located right at the jet exit. The curve joining the present positions of the particles will resemble a spiral whose radius increases with distance from the jet opening.

Problem 5.15

[Difficulty: 2]

5.15 Which of the following sets of equations represent possible incompressible flow cases?

(a) $V_r = U\cos\theta$; $V_\theta = -U\sin\theta$

(b) $V_r = -q/2\pi r$; $V_\theta = K/2\pi r$

(c) $V_r = U\cos\theta\,[1-(a/r)^2]$; $V_\theta = -U\sin\theta[1+(a/r)^2]$

Given: The list of velocity fields provided above

Find: Which of these fields possibly represent incompressible flow

Solution: We will check these flow fields against the continuity equation

Governing Equations:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho V_r)+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho V_\theta)+\frac{\partial}{\partial z}(\rho V_z)+\frac{\partial\rho}{\partial t}=0 \qquad \text{(Continuity equation)}$$

Assumptions:
(1) Incompressible flow (ρ is constant)
(2) Two dimensional flow (velocity is not a function of z)

Based on the two assumptions listed above, the continuity equation reduces to: $\frac{\partial(rV_r)}{\partial r}+\frac{\partial V_\theta}{\partial\theta}=0$

This is the criterion against which we will check all of the flow fields.

(a) $V_r = U\cdot\cos(\theta)$

$V_\theta = -U\cdot\sin(\theta)$

$$\frac{\partial(rV_r)}{\partial r}+\frac{\partial V_\theta}{\partial\theta}=(U\cos\theta)+(-U\cos\theta)=0$$

This could be an incompressible flow field.

(b) $V_r = -\frac{q}{2\cdot\pi\cdot r}$

$V_\theta = \frac{K}{2\cdot\pi\cdot r}$

$$\frac{\partial(rV_r)}{\partial r}+\frac{\partial V_\theta}{\partial\theta}=0+0=0$$

This could be an incompressible flow field.

(c) $V_r = U\cdot\cos(\theta)\cdot\left[1-\left(\frac{a}{r}\right)^2\right]$

$V_\theta = -U\cdot\sin(\theta)\cdot\left[1+\left(\frac{a}{r}\right)^2\right]$

$$\frac{\partial(rV_r)}{\partial r}+\frac{\partial V_\theta}{\partial\theta}=U\cos\theta\left[1+\left(\frac{a}{r}\right)^2\right]-U\cos\theta\left[1+\left(\frac{a}{r}\right)^2\right]=0$$

This could be an incompressible flow field.

Problem 5.16 [Difficulty: 3]

5.16 For an incompressible flow in the $r\theta$ plane, the r component of velocity is given as $V_r = U \cos\theta$.
(a) Determine a possible θ component of velocity.
(b) How many possible θ components are there?

Given: r component of velocity

Find: θ component for incompressible flow; How many θ components

Solution:

Basic equation: $$\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(\rho\cdot r\cdot V_r\right) + \frac{1}{r}\cdot\frac{\partial}{\partial \theta}\left(\rho\cdot V_\theta\right) + \frac{\partial}{\partial z}\left(\rho\cdot V_z\right) + \frac{\partial}{\partial t}\rho = 0$$

Assumptions: Incompressible flow
Flow in r-θ plane

Hence $$\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\cdot V_r\right) + \frac{1}{r}\cdot\frac{\partial}{\partial \theta}\left(V_\theta\right) = 0$$ or $$\frac{\partial}{\partial \theta}V_\theta = -\frac{\partial}{\partial r}\left(r\cdot V_r\right) = -\frac{\partial}{\partial r}(r\cdot U\cdot\cos(\theta)) = -U\cdot\cos(\theta)$$

Integrating $$V_\theta(r,\theta) = -\int U\cdot\cos(\theta)\,d\theta = -U\cdot\sin(\theta) + f(r)$$

$$V_\theta(r,\theta) = -U\cdot\sin(\theta) + f(r)$$

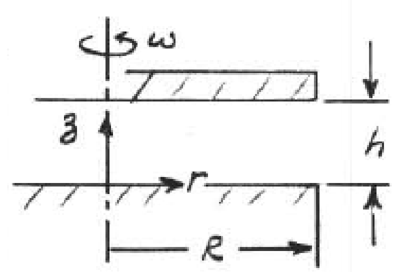
There are an infinite number of solutions as f(r) can be any function of r

The simplest form is $V_\theta(r,\theta) = -U\cdot\sin(\theta)$

Problem 5.17 [Difficulty: 2]

5.17 A viscous liquid is sheared between two parallel disks of radius R, one of which rotates while the other is fixed. The velocity field is purely tangential, and the velocity varies linearly with z from $V_\theta = 0$ at $z = 0$ (the fixed disk) to the velocity of the rotating disk at its surface ($z = h$). Derive an expression for the velocity field between the disks.

Given: Flow between parallel disks as shown. Velocity is purely tangential. No-slip condition is satisfied, so velocity varies linearly with z.



Find: An expression for the velocity field

Solution: We will apply the continuity equation to this system.

Governing Equations:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho V_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho V_\theta) + \frac{\partial}{\partial z}(\rho V_z) + \frac{\partial \rho}{\partial t} = 0 \quad \text{(Continuity equation)}$$

$$\vec{V} = V_r \hat{e}_r + V_\theta \hat{e}_\theta + V_z \hat{k} \quad \text{(Velocity flow field)}$$

Assumptions:
(1) Incompressible flow (ρ is constant)
(2) Purely tangential flow
(3) Linear velocity variation with z

Based on the first two assumptions, the continuity equation reduces to: $\frac{\partial V_\theta}{\partial \theta} = 0$ thus: $V_\theta = V_\theta(r,z)$

Since the velocity is linear with z, we may write: $V_\theta(r,z) = z \cdot f(r) + C$ Now we apply known boundary conditions:

1: $V_\theta(r,0) = 0$ $\quad 0 \cdot f(r) + C = 0$ $\quad C = 0$ $\qquad$ 2: $V_\theta(r,h) = r\cdot\omega$ $\quad h \cdot f(r) = r \cdot \omega$ $\quad f(r) = \frac{r\cdot\omega}{h}$

Therefore the tangential velocity is: $V_\theta = \omega \cdot r \cdot \frac{z}{h}$ Thus, the velocity field is:

$$\vec{V} = \omega r \frac{z}{h} \hat{e}_\theta$$

Problem 5.18

[Difficulty: 3]

5.18 A velocity field in cylindrical coordinates is given as $\vec{V} = \hat{e}_r A/r + \hat{e}_\theta B/r$, where A and B are constants with dimensions of m²/s. Does this represent a possible incompressible flow? Sketch the streamline that passes through the point $r_0 = 1$ m, $\theta = 90°$ if $A = B = 1$ m²/s, if $A = 1$ m²/s and $B = 0$, and if $B = 1$ m²/s and $A = 0$.

Given: The velocity field

Find: Whether or not it is a incompressible flow; sketch various streamlines

Solution:

$$V_r = \frac{A}{r} \qquad V_\theta = \frac{B}{r}$$

For incompressible flow $\frac{1}{r}\cdot\frac{d}{dr}(r\cdot V_r) + \frac{1}{r}\cdot\frac{d}{d\theta}V_\theta = 0$ $\qquad \frac{1}{r}\cdot\frac{d}{dr}(r\cdot V_r) = 0 \qquad \frac{1}{r}\cdot\frac{d}{d\theta}V_\theta = 0$

Hence $\frac{1}{r}\cdot\frac{d}{dr}(r\cdot V_r) + \frac{1}{r}\cdot\frac{d}{d\theta}V_\theta = 0$ $\qquad$ Flow is incompressible

For the streamlines $\frac{dr}{V_r} = \frac{r\cdot d\theta}{V_\theta} \qquad \frac{r\cdot dr}{A} = \frac{r^2\cdot d\theta}{B}$

so $\int \frac{1}{r}\,dr = \int \frac{A}{B}\,d\theta$ $\qquad$ Integrating $\qquad \ln(r) = \frac{A}{B}\cdot\theta + \text{const}$

Equation of streamlines is $r = C\cdot e^{\frac{A}{B}\cdot\theta}$

(a) For $A = B = 1$ m²/s, passing through point (1m, $\pi/2$)

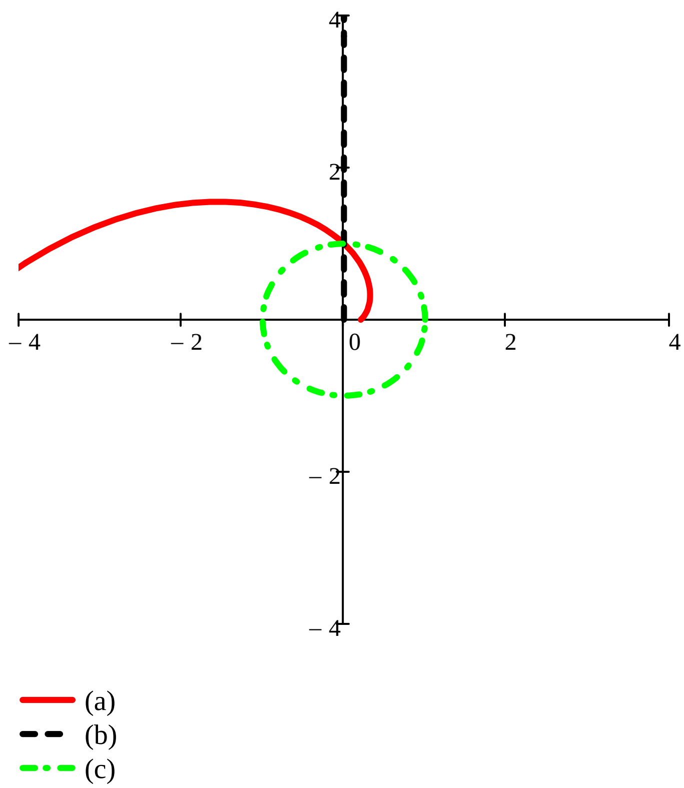
$$r = e^{\theta - \frac{\pi}{2}}$$

(b) For $A = 1$ m²/s, $B = 0$ m²/s, passing through point (1m, $\pi/2$)

$$\theta = \frac{\pi}{2}$$

(c) For $A = 0$ m²/s, $B = 1$ m²/s, passing through point (1m, $\pi/2$)

$$r = 1\cdot m$$



Problem 5.19

[Difficulty: 3]

5.19 Determine the family of stream functions ψ that will yield the velocity field $\vec{V} = 2y(2x+1)\hat{i} + [x(x+1) - 2y^2]\hat{j}$.

Given: Velocity field

Find: Stream function ψ

Solution:

Basic equations: $\frac{\partial}{\partial x}(\rho \cdot u) + \frac{\partial}{\partial y}(\rho \cdot v) + \frac{\partial}{\partial z}(\rho \cdot w) + \frac{\partial}{\partial t}\rho = 0$ $\quad u = \frac{\partial}{\partial y}\psi \quad v = -\frac{\partial}{\partial x}\psi$

Assumptions: Incompressible flow
Flow in x-y plane

Hence $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$ or $\frac{\partial}{\partial x}[2 \cdot y \cdot (2x + 1)] + \frac{\partial}{\partial y}\left[x \cdot (x+1) - 2 \cdot y^2\right] = 0$

Hence $u = 2 \cdot y \cdot (2 \cdot x + 1) = \frac{\partial}{\partial y}\psi$ $\quad \psi(x,y) = \int 2 \cdot y \cdot (2 \cdot x + 1)\, dy = 2 \cdot x \cdot y^2 + y^2 + f(x)$

and $v = x \cdot (x+1) - 2 \cdot y^2 = -\frac{\partial}{\partial x}\psi$ $\quad \psi(x,y) = -\int \left[x \cdot (x+1) - 2 \cdot y^2\right] dx = -\frac{x^3}{3} - \frac{x^2}{2} + 2 \cdot x \cdot y^2 + g(y$

Comparing these $f(x) = -\frac{x^3}{3} - \frac{x^2}{2}$ and $g(y) = y^2$

The stream function is $\psi(x,y) = y^2 + 2 \cdot x \cdot y^2 - \frac{x^2}{2} - \frac{x^3}{3}$

Checking $u(x,y) = \frac{\partial}{\partial y}\left(y^2 + 2 \cdot x \cdot y^2 - \frac{x^2}{2} - \frac{x^3}{3}\right) \rightarrow u(x,y) = 2 \cdot y + 4 \cdot x \cdot y$

$v(x,y) = -\frac{\partial}{\partial x}\left(y^2 + 2 \cdot x \cdot y^2 - \frac{x^2}{2} - \frac{x^3}{3}\right) \rightarrow v(x,y) = x^2 + x - 2 \cdot y^2$

Problem 5.20

[Difficulty: 2]

5.20 The stream function for a certain incompressible flow field is given by the expression $\psi = -Ur\sin\theta + q\theta/2\pi$. Obtain an expression for the velocity field. Find the stagnation point(s) where $|\vec{V}| = 0$, and show that $\psi = 0$ there.

Given: Stream function for an incompressible flow field:

$$\psi = -U \cdot r \cdot \sin(\theta) + \frac{q}{2 \cdot \pi} \cdot \theta$$

Find: (a) Expression for the velocity field
(b) Location of stagnation points
(c) Show that the stream function is equal to zero at the stagnation points.

Solution: We will generate the velocity field from the stream function.

Governing Equations:

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad V_\theta = -\frac{\partial \psi}{\partial r} \qquad \text{(Definition of stream function)}$$

Taking the derivatives of the stream function: $V_r = -U \cdot \cos(\theta) + \frac{q}{2 \cdot \pi \cdot r}$ $\quad V_\theta = U \cdot \sin(\theta)$

So the velocity field is:

$$\vec{V} = \left(-U\cos\theta + \frac{q}{2\pi R}\right)\hat{e}_r + U\sin\theta\,\hat{e}_\theta$$

To find the stagnation points we must find the places where both velocity components are zero. When $V_r = 0$ $\quad r = \frac{q}{2 \cdot \pi \cdot U \cdot \cos(\theta)}$

When $V_\theta = 0$ $\quad \sin(\theta) = 0$ therefore: $\theta = 0, \pi$ Now we can apply these values of θ to the above relation to find r:

For θ = 0: $r = \frac{q}{2 \cdot \pi \cdot U \cdot \cos(0)} = \frac{q}{2 \cdot \pi \cdot U}$ For θ = π: $r = \frac{q}{2 \cdot \pi \cdot \cos(\pi)} = -\frac{q}{2 \cdot \pi \cdot U}$ These represent the same point:

Stagnation point at:

$$(r, \theta) = \left(\frac{q}{2 \cdot \pi \cdot U}, 0\right)$$

At the stagnation point: $\psi_{stagnation} = -U \cdot \frac{q}{2 \cdot \pi \cdot U} \cdot \sin(0) + \frac{q}{2 \cdot \pi} \cdot 0 = 0$

$$\psi_{stagnation} = 0$$

Problem 5.21

(Difficulty 2)

5.21 Determine the stream functions for the following flow fields. For the value of $\Psi = 2$, plot the streamline in the region between $x = -1$ and $x = 1$.

(a) $u = 4; v = 3$

(b) $u = 4y, v = 0$

(c) $u = 4y, v = 4x$

(d) $u = 4y, v = -4x$

Find: Determine the stream functions for the flow fields.

Assumptions: The flow is steady and incompressible

Solution: Use the definitions of stream function:

$$u = \frac{\partial \Psi}{\partial y}$$

$$v = -\frac{\partial \Psi}{\partial x}$$

(a) In this case, we have:

$$u = \frac{\partial \Psi}{\partial y} = 4$$

$$\Psi = 4y + f(x)$$

$$v = -\frac{\partial \Psi}{\partial x} = -\frac{\partial f}{\partial x} = 3$$

$$\frac{\partial f}{\partial x} = -3$$

$$f(x) = -3x + c$$

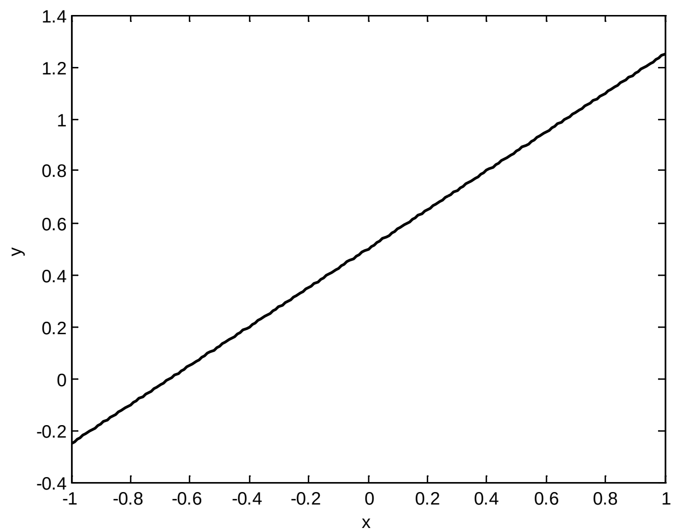
So the stream function is:

$$\Psi = 4y - 3x + c$$

The plot for the streamline is shown by ($c = 0$):

$$\Psi = 4y - 3x = 2$$

$$y = 0.75x + 0.5$$



(b) In this case we have:

$$u = \frac{\partial \Psi}{\partial y} = 4y$$

$$\Psi = 2y^2 + f(x)$$

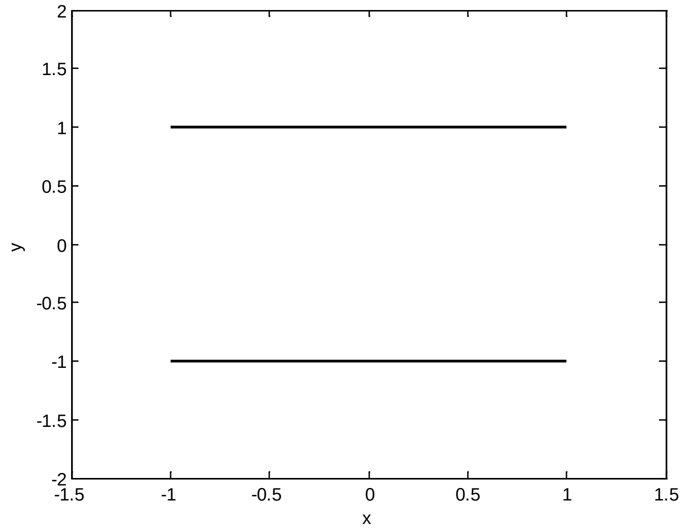
$$v = -\frac{\partial \Psi}{\partial x} = -\frac{\partial f}{\partial x} = 0$$

$$f(x) = c$$

$$\Psi = 2y^2 + c$$

The streamline is showing by ($c = 0$):

$$2y^2 = 2$$
$$y^2 = 1$$
$$y = \pm 1$$



(c) In this case we have:

$$u = \frac{\partial \Psi}{\partial y} = 4y$$

$$\Psi = 2y^2 + f(x)$$

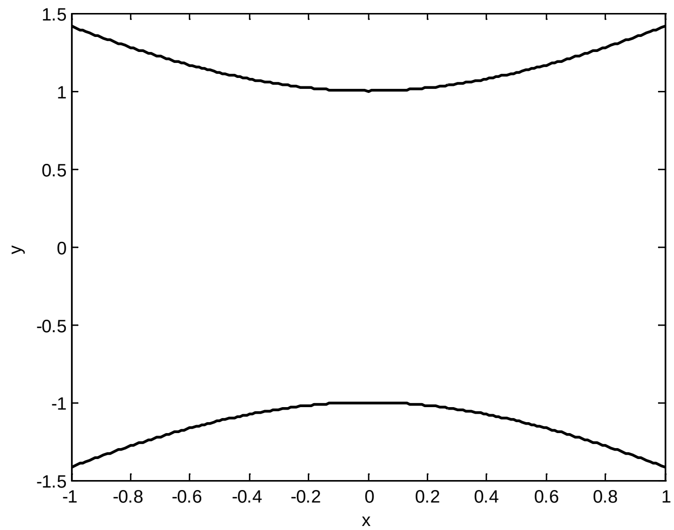
$$v = -\frac{\partial \Psi}{\partial x} = -\frac{\partial f}{\partial x} = 4x$$

$$f(x) = -2x^2$$

$$\Psi = 2y^2 - 2x^2 + c$$

The streamline is showing by ($c = 0$):

$$2y^2 - 2x^2 = 2$$
$$y^2 = x^2 + 1$$
$$y = \pm\sqrt{x^2 + 1}$$



(d) In this case we have:

$$u = \frac{\partial \Psi}{\partial y} = 4$$

$$\Psi = 4y + f(x)$$

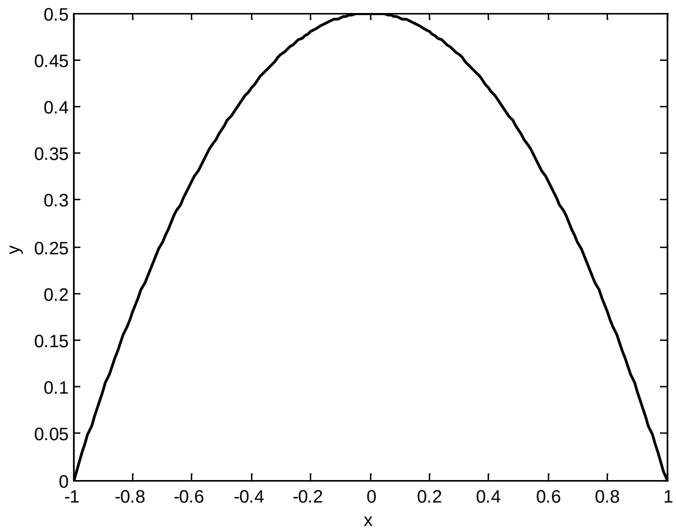
$$v = -\frac{\partial \Psi}{\partial x} = -\frac{\partial f}{\partial x} = -4x$$

$$f(x) = 2x^2$$

$$\Psi = 4y + 2x^2 + c$$

The streamline is showing by ($c = 0$):

$$4y + 2x^2 = 2$$
$$y = \frac{-x^2 + 1}{2}$$



0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
y
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
x

Problem 5.22

(Difficulty 1)

5.22 Determine the stream function for the steady incompressible flow between parallel plates. The velocity profile is parabolic and given by $u = u_c + ay^2$, where u_c is the centerline velocity and y is the distance measured from the centerline. The plate spacing is $2b$ and the velocity is zero at each plate. Explain why the stream function is not a function of x.

Find: Determine the stream function Ψ and explain why it is not a function of x.

Assumptions: The flow is steady and incompressible

Solution: Use the definition of stream function

For this flow we have the velocity as:

$$u = u_c + ay^2$$

$$v = 0$$

For the stream function, we have:

$$u = \frac{\partial \Psi}{\partial y}$$

$$v = -\frac{\partial \Psi}{\partial x}$$

Thus

$$u = \frac{\partial \Psi}{\partial y} = u_c + ay^2$$

$$\Psi = u_c y + \frac{1}{3} a y^3 + f(x)$$

$$v = -\frac{\partial \Psi}{\partial x} = -\frac{\partial f(x)}{\partial x} = 0$$

$$f(x) = c$$

c is a constant. So we get:

$$\Psi = u_c y + \frac{1}{3} a y^3 + c$$

The reason that stream function is independent of x is because this is steady flow between two plates. This is unidirectional flow and the velocity profile is the same at all x locations and the y component of velocity v is zero.

Problem 5.23

[Difficulty: 3]

5.23 An incompressible frictionless flow field is specified by the stream function $\psi = -5Ax - 2Ay$, where $A = 2$ m/s, and x and y are coordinates in meters.
(a) Sketch the streamlines $\psi = 0$ and $\psi = 5$, and indicate the direction of the velocity vector at the point (0, 0) on the sketch.
(b) Determine the magnitude of the flow rate between the streamlines passing through (2, 2) and (4, 1).

Given: Stream function for an incompressible flow field:

$$\psi = -5 \cdot A \cdot x - 2 \cdot A \cdot y \qquad A = 2 \cdot \frac{m}{s}$$

Find:
(a) Sketch streamlines $\psi = 0$ and $\psi = 5$
(b) Velocity vector at (0, 0)
(c) Flow rate between streamlines passing through points (2, 2) and (4, 1)

Solution: We will generate the velocity field from the stream function.

Governing Equations:

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

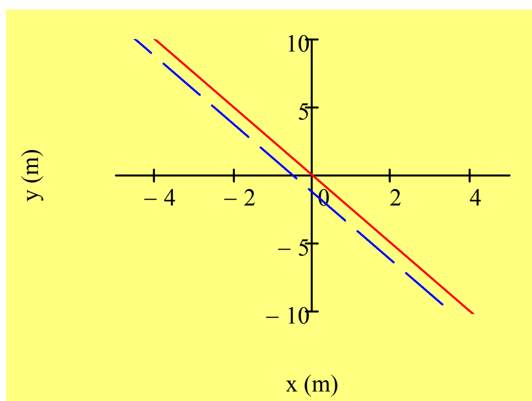
(Definition of stream function)

Assumptions: Incompressible flow (ρ is constant)
Flow is only in the x-y plane

For $\psi = 0$: $\quad 0 = -5 \cdot A \cdot x - 2 \cdot A \cdot y \quad$ Solving for y: $\quad y = -\frac{5}{2} \cdot x$

For $\psi = 5$: $\quad 5 = -5 \cdot A \cdot x - 2 \cdot A \cdot y \quad$ Solving for y: $\quad y = -\frac{5}{2} \cdot x - \frac{5}{2} \cdot \frac{m^2}{s} \times \frac{s}{2 \cdot m} = -\frac{5}{2} \cdot x - \frac{5}{2} \cdot m$

Here is the plot of the two streamlines: $\psi = 0$ is in red; $\psi = 5$ is in blue



Generating the velocity components from the stream function derivatives:

$u = -2 \cdot A \qquad v = 5 \cdot A \qquad$ Therefore, the velocity vector at (0, 0) is:

$$\vec{V} = -4\hat{i} + 10\hat{j}$$

At the point (2, 2) the stream function value is: $\quad \psi_a = -5 \times 2 \cdot \frac{m}{s} \times 2 \cdot m - 2 \times 2 \cdot \frac{m}{s} \times 2 \cdot m \quad \psi_a = -28 \frac{m^2}{s}$

At the point (4, 1) the stream function value is: $\quad \psi_b = -5 \times 2 \cdot \frac{m}{s} \times 4 \cdot m - 2 \times 2 \cdot \frac{m}{s} \times 1 \cdot m \quad \psi_b = -44 \frac{m^2}{s}$

The flow rate between these two streamlines is: $\quad Q = \psi_b - \psi_a \quad Q = \left(-44 \cdot \frac{m^2}{s}\right) - \left(-28 \cdot \frac{m^2}{s}\right) \quad Q = -16 \cdot \frac{m^3}{s \cdot m}$

Flow rate is 16 m^3/s per meter of depth

Problem 5.24

5.24 A parabolic velocity profile was used to model flow in a laminar incompressible boundary layer in Problem 5.11. Derive the stream function for this flow field. Locate streamlines at one-quarter and one-half the total volume flow rate in the boundary layer.

Given: Approximate profile for a laminar boundary layer:

$$\frac{u}{U} = 2\cdot\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \qquad \delta = c\cdot\sqrt{x} \quad \text{(c is constant)}$$

Find: (a) Stream function for the flow field
(b) Location of streamlines at one-quarter and one-half the total flow rate in the boundary layer.

Solution: We will generate the stream function from the velocity field.

Governing Equations:

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x} \qquad \text{(Definition of stream function)}$$

Integrating the x-component of velocity yields the stream function:

$$\psi = \int U\cdot\left[2\cdot\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2\right] dy + f(x) = U\cdot\delta\cdot\left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\cdot\left(\frac{y}{\delta}\right)^3\right] + f(x)$$

If we set $\psi = 0$ at $y = 0$ the stream function would be:

$$\psi = U\cdot\delta\cdot\left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\cdot\left(\frac{y}{\delta}\right)^3\right]$$

The total flow rate per unit depth within the boundary layer is: $Q = \psi(\delta) - \psi(0) = U\cdot\delta\cdot\left[\left(\frac{\delta}{\delta}\right)^2 - \frac{1}{3}\cdot\left(\frac{\delta}{\delta}\right)^3\right] - 0 = \frac{2}{3}\cdot U\cdot\delta$

At one-quarter of the flow rate in the boundary layer: $Q = \frac{1}{4}\cdot\frac{2}{3}\cdot U\cdot\delta = \frac{1}{6}\cdot U\cdot\delta$ Therefore, the streamline would be located at:

$\frac{1}{6}\cdot U\cdot\delta = U\cdot\delta\cdot\left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\cdot\left(\frac{y}{\delta}\right)^3\right]$ or $2\cdot\left(\frac{y}{\delta}\right)^3 - 6\cdot\left(\frac{y}{\delta}\right)^2 + 1 = 0$ We may solve this cubic for y/δ using several methods, including Goal Seek in Excel or polyroots in Mathcad. Once the roots are determined, only one root would make physical sense.

So at one-quarter of the flow rate: $\frac{y}{\delta} = 0.442$

At one-half of the flow rate in the boundary layer: $Q = \frac{1}{2}\cdot\frac{2}{3}\cdot U\cdot\delta = \frac{1}{3}\cdot U\cdot\delta$ Therefore, the streamline would be located at:

$\frac{1}{3}\cdot U\cdot\delta = U\cdot\delta\cdot\left[\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\cdot\left(\frac{y}{\delta}\right)^3\right]$ or $\left(\frac{y}{\delta}\right)^3 - 3\cdot\left(\frac{y}{\delta}\right)^2 + 1 = 0$ We solve this cubic as we solved the previous one.

So at one-half of the flow rate: $\frac{y}{\delta} = 0.653$

Problem 5.25

(Difficulty 2)

5.25 A flow field is characterized by the stream function $\Psi = 3x^2y - y^3$. Demonstrate that the flow field represents a two-dimensional incompressible flow. Show that the magnitude of the velocity depends only on the distance from the origin of the coordinates. Plot the stream line $\Psi = 2$.

Find: Demonstrate two-dimensional incompressible flow and that the magnitude only depends on distance from the origin. Plot stream line $\Psi = 2$.

Assumptions: The flow is steady and incompressible

Solution: Use the definition of stream function

For this flow, the stream function is:

$$\Psi = 3x^2y - y^3$$

The velocity field is given by:

$$u = \frac{\partial \Psi}{\partial y} = 3x^2 - 3y^2$$

$$v = -\frac{\partial \Psi}{\partial x} = -6xy$$

For the two-dimensional incompressible flow, we should satisfy the continuity equation as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

So we have:

$$\frac{\partial u}{\partial x} = 6x$$

$$\frac{\partial v}{\partial y} = -6x$$

Thus

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 6x - 6x = 0$$

So this is two-dimensional incompressible flow.

The magnitude of the velocity is:

$$V = \sqrt{u^2 + v^2} = \sqrt{(3x^2 - 3y^2)^2 + (-6xy)^2}$$

$$V = \sqrt{9x^4 - 18x^2y^2 + 9y^4 + 36x^2y^2} = \sqrt{9x^4 + 18x^2y^2 + 9y^4} = \sqrt{9(x^2 + y^2)^2} = 3(x^2 + y^2)$$

As we know the distance from the origin is:

$$r^2 = x^2 + y^2$$

Thus

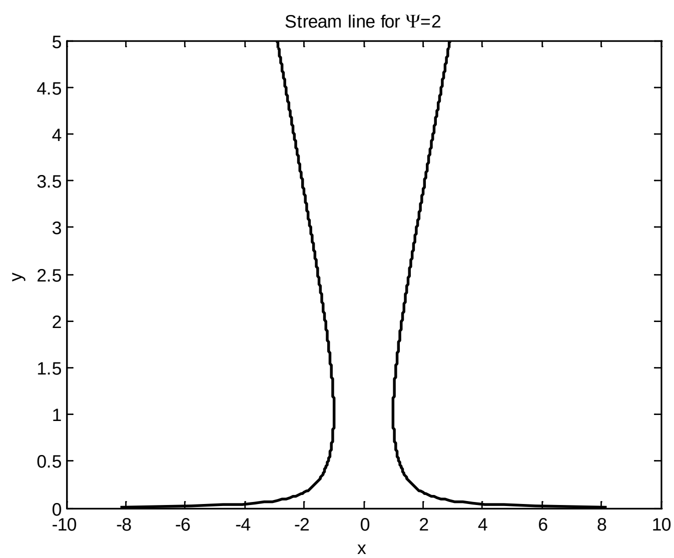
$$V = 3r^2$$

So the magnitude of the velocity depends only on the distance from the origin.

The stream line for $\Psi = 2$ is shown by:

$$x^2 = \frac{2 + y^3}{3y}$$

$$x = \pm\sqrt{\frac{2 + y^3}{3y}}$$



Problem 5.26

(Difficulty 1)

5.26 A flow field is characterized by the stream function $\Psi = xy$. Plot sufficient streamlines to represent the flow field. Determine the location of any stagnation points. Give at least two possible physical interpretations of this flow.

Find: Plot sufficient streamlines to represent the flow field. Determine the stagnation points.

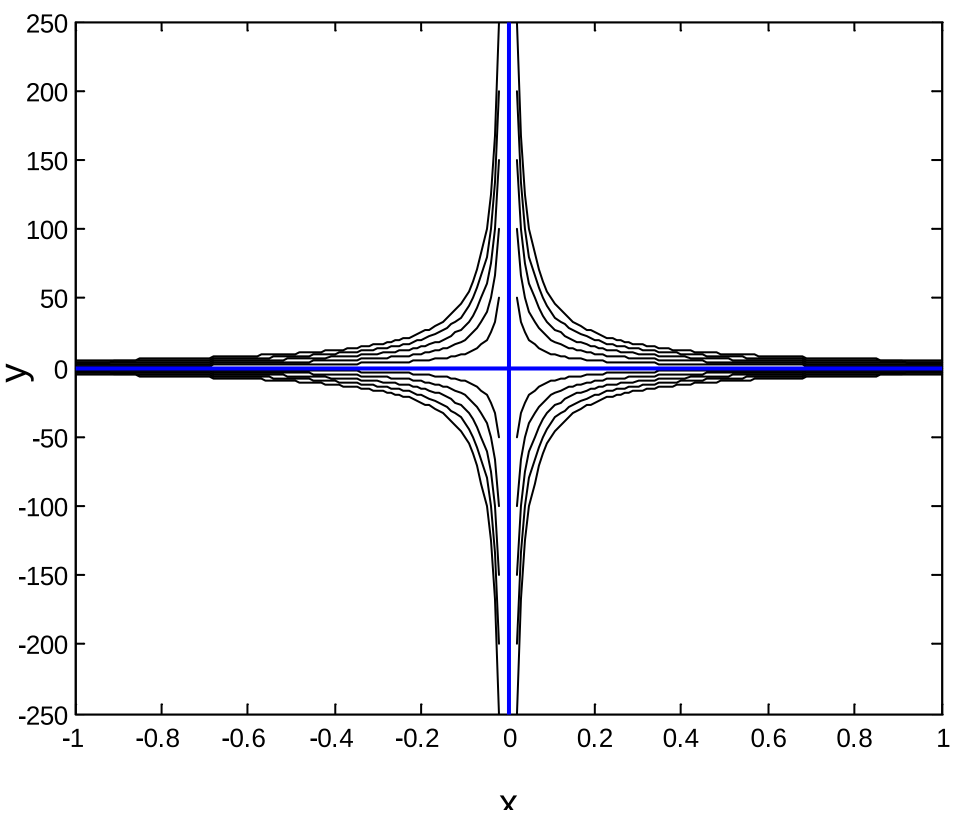
Assumptions: The flow is steady and incompressible

Solution: Use the definition of stream function

For this flow, the stream function is:

$$\Psi = xy$$

The plot of the streamlines is then



The velocity field is:

$$u = \frac{\partial \Psi}{\partial y} = x$$

$$v = -\frac{\partial \Psi}{\partial x} = -y$$

For the stagnation points, we have:

$$u = v = 0$$

Thus

$$u = 0 \ at \ x = 0 \ and \ v = 0 \ at \ y = 0$$

The stagnation point is the origin.

This flow can represent

(1) a jet hitting a wall;

(2) flow in a corner.

Problem 5.27

[Difficulty: 3]

5.27 A cubic velocity profile was used to model flow in a laminar incompressible boundary layer in Problem 5.12 . Derive the stream function for this flow field. Locate streamlines at one-quarter and one-half the total volume flow rate in the boundary layer.

Given: Data on boundary layer

Find: Stream function; locate streamlines at 1/4 and 1/2 of total flow rate

Solution:

$$u(x,y) = U\cdot\left[\frac{3}{2}\cdot\left(\frac{y}{\delta}\right) - \frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^3\right] \qquad \text{and} \qquad \delta(x) = c\cdot\sqrt{x}$$

For the stream function $u = \frac{\partial}{\partial y}\psi = U\cdot\left[\frac{3}{2}\cdot\left(\frac{y}{\delta}\right) - \frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^3\right]$

Hence $$\psi = \int U\cdot\left[\frac{3}{2}\cdot\left(\frac{y}{\delta}\right) - \frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^3\right] dy \qquad \psi = U\cdot\left(\frac{3}{4}\cdot\frac{y^2}{\delta} - \frac{1}{8}\cdot\frac{y^4}{\delta^3}\right) + f(x)$$

Let $\psi = 0 = 0$ along $y = 0$, so $f(x) = 0$, so $$\psi = U\cdot\delta\cdot\left[\frac{3}{4}\cdot\left(\frac{y}{\delta}\right)^2 - \frac{1}{8}\cdot\left(\frac{y}{\delta}\right)^4\right]$$

The total flow rate in the boundary layer is

$$\frac{Q}{W} = \psi(\delta) - \psi(0) = U\cdot\delta\cdot\left(\frac{3}{4} - \frac{1}{8}\right) = \frac{5}{8}\cdot U\cdot\delta$$

At 1/4 of the total $$\psi - \psi_0 = U\cdot\delta\cdot\left[\frac{3}{4}\cdot\left(\frac{y}{\delta}\right)^2 - \frac{1}{8}\cdot\left(\frac{y}{\delta}\right)^4\right] = \frac{1}{4}\cdot\left(\frac{5}{8}\cdot U\cdot\delta\right)$$

$$24\cdot\left(\frac{y}{\delta}\right)^2 - 4\cdot\left(\frac{y}{\delta}\right)^4 = 5 \qquad \text{or} \qquad 4\cdot X^2 - 24\cdot X + 5 = 0 \qquad \text{where} \qquad X^2 = \frac{y}{\delta}$$

The solution to the quadratic is $$X = \frac{24 - \sqrt{24^2 - 4\cdot 4\cdot 5}}{2\cdot 4} \qquad X = 0.216 \qquad \text{Note that the other root is} \quad \frac{24 + \sqrt{24^2 - 4\cdot 4\cdot 5}}{2\cdot 4} = 5.784$$

Hence $$\frac{y}{\delta} = \sqrt{X} = 0.465$$

At 1/2 of the total flow $$\psi - \psi_0 = U\cdot\delta\cdot\left[\frac{3}{4}\cdot\left(\frac{y}{\delta}\right)^2 - \frac{1}{8}\cdot\left(\frac{y}{\delta}\right)^4\right] = \frac{1}{2}\cdot\left(\frac{5}{8}\cdot U\cdot\delta\right)$$

$$12\cdot\left(\frac{y}{\delta}\right)^2 - 2\cdot\left(\frac{y}{\delta}\right)^4 = 5 \qquad \text{or} \qquad 2\cdot X^2 - 12\cdot X + 5 = 0 \qquad \text{where} \qquad X^2 = \frac{y}{\delta}$$

The solution to the quadratic is $$X = \frac{12 - \sqrt{12^2 - 4\cdot 2\cdot 5.}}{2\cdot 2} \qquad X = 0.450 \qquad \text{Note that the other root is} \quad \frac{12 + \sqrt{12^2 - 4\cdot 2\cdot 5}}{2\cdot 2} = 5.55$$

Hence $$\frac{y}{\delta} = \sqrt{X} = 0.671$$

Problem 5.28

(Difficulty 2)

5.28 A flow field is characterized by the stream function

$$\Psi = \frac{1}{2\pi}\left(tan^{-1}\frac{y-a}{x} - tan^{-1}\frac{y+a}{x}\right) - \frac{1}{2\pi}\ln\sqrt{x^2+y^2}$$

Locate the stagnation points and sketch the flow field. Derive an expression for the velocity at $(a, 0)$.

Find: Locate stagnation points and sketch the flow. Determine the velocity at $(a, 0)$.

Assumptions: The flow is steady and incompressible

Solution: Use the definition of stream function

The stream function for this flow is given by:

$$\Psi = \frac{1}{2\pi}\left(tan^{-1}\frac{y-a}{x} - tan^{-1}\frac{y+a}{x}\right) - \frac{1}{2\pi}\ln\sqrt{x^2+y^2}$$

The velocity field is related to the stream function by:

$$u = \frac{\partial \Psi}{\partial y} = -\frac{1}{2\pi}\frac{y}{x^2+y^2} + \frac{1}{2\pi}\left\{\frac{x}{x^2+(y-a)^2} - \frac{x}{x^2+(y+a)^2}\right\}$$

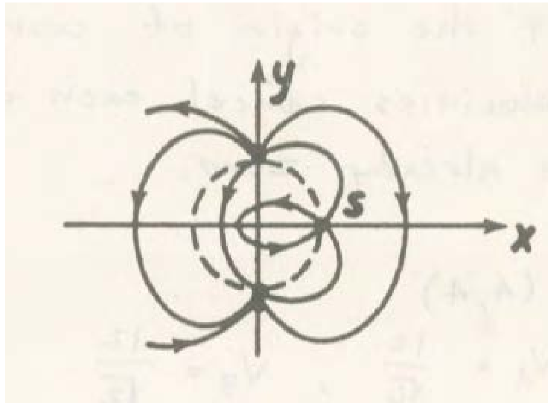
$$v = -\frac{\partial \Psi}{\partial x} = \frac{1}{2\pi}\frac{x}{x^2+y^2} - \frac{1}{2\pi}\left\{\frac{a-y}{x^2+(y-a)^2} + \frac{a+y}{x^2+(y+a)^2}\right\}$$

The velocity expression at $(a, 0)$ is:

$$u = \frac{\partial \Psi}{\partial y} = \frac{1}{2\pi}\left\{\frac{a}{a^2+(0-a)^2} - \frac{a}{a^2+(a)^2}\right\} = 0$$

$$v = -\frac{\partial \Psi}{\partial x} = \frac{1}{2\pi}\frac{1}{a} - \frac{1}{2\pi}\left\{\frac{a}{a^2+(-a)^2} + \frac{a}{a^2+(a)^2}\right\} = \frac{1}{2\pi}\frac{1}{a} - \frac{1}{2\pi}\frac{2a}{2a^2} = \frac{1}{2\pi}\frac{1}{a} - \frac{1}{2\pi}\frac{1}{a} = 0$$

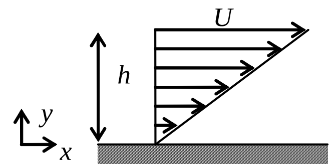
So the stagnation point is $(a, 0)$. The fluid flow is sketched as:


y
s
x

Problem 5.29

[Difficulty: 3]

5.29 In a parallel one-dimensional flow in the positive x direction, the velocity varies linearly from zero at $y = 0$ to 30 m/s at $y = 1.5$ m. Determine an expression for the stream function, ψ. Also determine the y coordinate above which the volume flow rate is half the total between $y = 0$ and $y = 1.5$ m.



Given: Linear velocity profile

Find: Stream function ψ; y coordinate for half of flow

Solution:

Basic equations: $u = \frac{\partial}{\partial y}\psi \qquad v = -\frac{\partial}{\partial x}\psi$ and we have $u = U\cdot\left(\frac{y}{h}\right) \qquad v = 0$

Assumption: Incompressible flow; flow in x-y plane

Check for incompressible $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$

$\frac{\partial}{\partial x}\left(U\cdot\frac{y}{h}\right) \rightarrow 0 \qquad \frac{\partial}{\partial y}0 \rightarrow 0$

Hence $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$ Flow is INCOMPRESSIBLE

Hence $u = U\cdot\frac{y}{h} = \frac{\partial}{\partial y}\psi \qquad \psi(x,y) = \int U\cdot\frac{y}{h}\,dy = \frac{U\cdot y^2}{2\cdot h} + f(x)$

and $v = 0 = -\frac{\partial}{\partial x}\psi \qquad \psi(x,y) = -\int 0\,dx = g(y)$

Comparing these $f(x) = 0$ and $g(y) = \frac{U\cdot y^2}{2\cdot h}$

The stream function is $\psi(x,y) = \frac{U\cdot y^2}{2\cdot h}$

For the flow $(0 < y < h)$ $Q = \int_0^h u\,dy = \frac{U}{h}\cdot\int_0^h y\,dy = \frac{U\cdot h}{2}$

For half the flow rate $\frac{Q}{2} = \int_0^{h_{half}} u\,dy = \frac{U}{h}\cdot\int_0^{h_{half}} y\,dy = \frac{U\cdot h_{half}^2}{2\cdot h} = \frac{1}{2}\cdot\left(\frac{U\cdot h}{2}\right) = \frac{U\cdot h}{4}$

Hence $h_{half}^2 = \frac{1}{2}\cdot h^2 \qquad h_{half} = \frac{1}{\sqrt{2}}\cdot h = \frac{1.5\cdot m}{\sqrt{2}} = 1.06\cdot m$

Problem 5.30

[Difficulty: 2]

5.30 Consider the flow field given by $\vec{V} = xy^2\hat{i} - \frac{1}{3}y^3\hat{j} + xy\hat{k}$. Determine (a) the number of dimensions of the flow, (b) if it is a possible incompressible flow, and (c) the acceleration of a fluid particle at point $(x, y, z) = (1, 2, 3)$.

Given: The velocity field provided above

Find: (a) the number of dimensions of the flow
(b) if this describes a possible incompressible flow
(c) the acceleration of a fluid particle at point (1,2,3)

Solution: We will check this flow field against the continuity equation, and then apply the definition of acceleration

Governing Equations:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) + \frac{\partial \rho}{\partial t} = 0 \quad \text{(Continuity equation)}$$

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t} \quad \text{(Particle acceleration)}$$

Assumptions: (1) Incompressible flow (ρ is constant)
(2) Two dimensional flow (velocity is not a function of z)
(3) Steady flow (velocity is not a function of t)

Based on assumption (2), we may state that: The flow is two dimensional.

Based on assumptions (1) and (3), the continuity equation reduces to: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

This is the criterion against which we will check the flow field.

$$u = x \cdot y^2$$

$$v = -\frac{1}{3} \cdot y^3$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = y^2 - y^2 = 0$$

This could be an incompressible flow field.

Based on assumptions (2) and (3), the acceleration reduces to: $\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y}$ and the partial derivatives of velocity are:

$\frac{\partial \vec{V}}{\partial x} = y^2\hat{i} + y\hat{k}$ and $\frac{\partial \vec{V}}{\partial y} = 2xy\hat{i} - y^2\hat{j} + x\hat{k}$ Therefore the acceleration vector is equal to:

$$\vec{a}_p = xy^2\left(y^2\hat{i} + y\hat{k}\right) - \frac{1}{3}y^3\left(2xy\hat{i} - y^2\hat{j} + x\hat{k}\right) = \frac{1}{3}xy^4\hat{i} + \frac{1}{3}y^5\hat{j} + \frac{2}{3}xy^3\hat{k}$$

At point (1,2,3), the acceleration is:

$$\vec{a}_p = \left(\frac{1}{3}\times 1\times 2^4\right)\hat{i} + \left(\frac{1}{3}\times 2^5\right)\hat{j} + \left(\frac{2}{3}\times 1\times 2^3\right)\hat{k} = \frac{16}{3}\hat{i} + \frac{32}{3}\hat{j} + \frac{16}{3}\hat{k}$$

$$\vec{a}_p = \frac{16}{3}\hat{i} + \frac{32}{3}\hat{j} + \frac{16}{3}\hat{k}$$

Problem 5.31

[Difficulty: 2]

5.31 Consider the flow field given by $\vec{V} = ax^2y\hat{i} - by\hat{j} + cz^2\hat{k}$, where $a = 2\ \text{m}^{-2}\cdot\text{s}^{-1}$, $b = 2\ \text{s}^{-1}$, and $c = 1\ \text{m}^{-1}\cdot\text{s}^{-1}$. Determine (a) the number of dimensions of the flow, (b) if it is a possible incompressible flow, and (c) the acceleration of a fluid particle at point $(x, y, z) = (2, 1, 3)$.

Given: The velocity field provided above

Find: (a) the number of dimensions of the flow
(b) if this describes a possible incompressible flow
(c) the acceleration of a fluid particle at point (2,1,3)

Solution: We will check this flow field against the continuity equation, and then apply the definition of acceleration

Governing Equations:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) + \frac{\partial \rho}{\partial t} = 0 \qquad \text{(Continuity equation)}$$

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t} \qquad \text{(Particle acceleration)}$$

Assumptions: (1) Incompressible flow (ρ is constant)
(2) Steady flow (velocity is not a function of t)

Since the velocity is a function of x, y, and z, we may state that: The flow is three dimensional.

Based on assumptions (1) and (2), the continuity equation reduces to: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

This is the criterion against which we will check the flow field.

$u = a\cdot x^2\cdot y$

$v = -b\cdot y$

$w = c\cdot z^2$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 2axy - b + 2cz \neq 0$$

This can not be incompressible.

Based on assumption (2), the acceleration reduces to: $\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$ and the partial derivatives of velocity are:

$\frac{\partial \vec{V}}{\partial x} = 2axy\hat{i}$ $\quad \frac{\partial \vec{V}}{\partial y} = ax^2\hat{i} - b\hat{j}$ and $\frac{\partial \vec{V}}{\partial z} = 2cz\hat{k}$ Therefore the acceleration vector is equal to:

$$\vec{a}_p = ax^2y(2axy\hat{i}) - by(ax^2\hat{i} - b\hat{j}) + cz^2(2cz\hat{k}) = (2a^2x^3y^2 - abx^2y)\hat{i} + (b^2y)\hat{j} + (2c^2z^3)\hat{k}$$

At point (2,1,3):

$$\vec{a}_p = \left[2\times\left(\frac{2}{\text{m}^2\cdot\text{s}}\right)^2\times(2\ \text{m})^3\times(1\ \text{m})^2 - \frac{2}{\text{m}^2\cdot\text{s}}\times\frac{2}{\text{s}}\times(2\ \text{m})^2\times 1\ \text{m}\right]\hat{i} + \left[\left(\frac{2}{s}\right)^2\times 1\ \text{m}\right]\hat{j} + \left[2\times\left(\frac{1}{\text{m}\cdot\text{s}}\right)^2\times(3\ \text{m})^3\right]\hat{k}$$

$$= 48\hat{i} + 4\hat{j} + 54\hat{k}\ \frac{\text{m}}{\text{s}^2}$$

$$\vec{a}_p = 48\hat{i} + 4\hat{j} + 54\hat{k}\ \frac{\text{m}}{\text{s}^2}$$

Problem 5.32 [Difficulty: 2]

5.32 The velocity field within a laminar boundary layer is approximated by the expression

$$\vec{V} = \frac{AUy}{x^{1/2}}\hat{i} + \frac{AUy^2}{4x^{3/2}}\hat{j}$$

In this expression, $A = 141\ \text{m}^{-1/2}$, and $U = 0.240$ m/s is the freestream velocity. Show that this velocity field represents a possible incompressible flow. Calculate the acceleration of a fluid particle at point $(x, y) = (0.5\ \text{m}, 5\ \text{mm})$. Determine the slope of the streamline through the point.

Given: The velocity field provided above

Find: (a) if this describes a possible incompressible flow
(b) the acceleration of a fluid particle at point (x,y) = (0.5 m, 5 mm)
(c) the slope of the streamline through that point

Solution: We will check this flow field against the continuity equation, and then apply the definition of acceleration

Governing Equations:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) + \frac{\partial \rho}{\partial t} = 0 \quad \text{(Continuity equation)}$$

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t} \quad \text{(Particle acceleration)}$$

Assumptions:
(1) Incompressible flow (ρ is constant)
(2) Two-dimensional flow (velocity is not a function of z)
(3) Steady flow (velocity is not a function of t)

Based on the assumptions above, the continuity equation reduces to: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ This is the criterion against which we will check the flow field.

$$u = \frac{A\cdot U\cdot y}{x^{\frac{1}{2}}} \qquad v = \frac{A\cdot U\cdot y^2}{4\cdot x^{\frac{3}{2}}} \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{1}{2}\frac{AUy}{x^{\frac{3}{2}}} + 2\frac{AUy}{4x^{\frac{3}{2}}} = 0$$

This represents a possible incompressible flow field.

Based on assumptions (2) and (3), the acceleration reduces to: $\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y}$ and the partial derivatives of velocity are:

$$\frac{\partial \vec{V}}{\partial x} = -\frac{AUy}{2x^{3/2}}\hat{i} - \frac{3AUy^2}{8x^{5/2}}\hat{j} \quad \text{and} \quad \frac{\partial \vec{V}}{\partial y} = \frac{AU}{x^{1/2}}\hat{i} + \frac{AUy}{2x^{3/2}}\hat{j}$$

Therefore the acceleration vector is equal to:

$$\vec{a}_p = \frac{AUy}{x^{1/2}}\left(-\frac{AUy}{2x^{3/2}}\hat{i} - \frac{3AUy^2}{8x^{5/2}}\hat{j}\right) + \frac{AUy^2}{4x^{3/2}}\left(\frac{AU}{x^{1/2}}\hat{i} + \frac{AUy}{2x^{3/2}}\hat{j}\right) = -\frac{A^2U^2y^2}{4x^2}\hat{i} - \frac{A^2U^2y^3}{4x^3}\hat{j}$$

At (5 m, 5 mm):

$$\vec{a}_p = -\left[\frac{1}{4}\times\left(\frac{141}{\text{m}^{1/2}}\right)^2\times\left(0.240\frac{\text{m}}{\text{s}}\right)^2\times\left(\frac{0.005}{0.5}\right)^2\right]\hat{i} - \left[\frac{1}{4}\times\left(\frac{141}{\text{m}^{1/2}}\right)^2\times\left(0.240\frac{\text{m}}{\text{s}}\right)^2\times\left(\frac{0.005}{0.5}\right)^3\right]\hat{j}$$

$$\vec{a}_p = -2.86\left(10^{-2}\,\hat{i} + 10^{-4}\,\hat{j}\right)\frac{m}{s^2}$$

The slope of the streamline is given by: $\text{slope} = \dfrac{v}{u} = \dfrac{A \cdot U \cdot y^2}{4 \cdot x^{\frac{3}{2}}} \cdot \dfrac{x^{\frac{1}{2}}}{A \cdot U \cdot y} = \dfrac{y}{4 \cdot x}$

Therefore, $\text{slope} = \dfrac{0.005}{4 \times 0.5}$

$$\text{slope} = 2.50 \times 10^{-3}$$

Problem 5.33

(Difficulty 1)

5.33 A velocity field is given by $\vec{V} = 10t\hat{\imath} - \frac{10}{t^3}\hat{\jmath}$. Show that the flow field is a two-dimensional flow and determine the acceleration as a function of time.

Find: Show that this is two-dimensional flow. Determine the acceleration.

Assumptions: The flow is steady and incompressible

Solution: Use the expression for acceleration

$$\frac{d\vec{V}}{dt} = \vec{a}_p = \frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y}$$

The velocity is given by:

$$\vec{V} = 10t\hat{\imath} - \frac{10}{t^3}\hat{\jmath}.$$

Thus

$$u = 10t$$

$$v = -\frac{10}{t^3}$$

$$w = 0$$

This is two-dimensional flow with u and v depending only on the time t. The x- and y-accelerations are given by

$$a_x = \frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial u}{\partial t} = \frac{\partial u}{\partial t} = 10$$

$$a_y = \frac{Dv}{Dt} = v\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial v}{\partial t} = \frac{\partial v}{\partial t} = \frac{30}{t^4}$$

So the acceleration is:

$$\vec{a} = 10\hat{\imath} + \frac{30}{t^4}\hat{\jmath}$$

Problem 5.34

[Difficulty: 2]

5.34 The y component of velocity in a two-dimensional, incompressible flow field is given by $v = -Axy$, where v is in m/s, x and y are in meters, and A is a dimensional constant. There is no velocity component or variation in the z direction. Determine the dimensions of the constant, A. Find the simplest x component of velocity in this flow field. Calculate the acceleration of a fluid particle at point $(x, y) = (1, 2)$.

Given: The 2-dimensional, incompressible velocity field provided above

Find:
(a) dimensions of the constant A
(b) simplest x-component of the velocity
(c) acceleration of a particle at (1,2)

Solution: We will check the dimensions against the function definition, check the flow field against the continuity equation, and then apply the definition of acceleration.

Governing Equations:

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)+\frac{\partial \rho}{\partial t}=0 \quad \text{(Continuity equation)}$$

$$\vec{a}_p=\frac{D\vec{V}}{Dt}=u\frac{\partial \vec{V}}{\partial x}+v\frac{\partial \vec{V}}{\partial y}+w\frac{\partial \vec{V}}{\partial z}+\frac{\partial \vec{V}}{\partial t} \quad \text{(Particle acceleration)}$$

Assumptions:
(1) Incompressible flow (ρ is constant)
(2) Two-dimensional flow (velocity is not a function of z)
(3) Steady flow (velocity is not a function of t)

Since $v = -A\cdot x\cdot y$ it follows that $A = -\frac{v}{x\cdot y}$ and the dimensions of A are given by: $[A]=\left[\frac{v}{xy}\right]=\frac{L}{t}\cdot\frac{1}{L}\cdot\frac{1}{L}$ $\quad [A]=\frac{1}{Lt}$

Based on the assumptions above, the continuity equation reduces to: $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$ Therefore: $\frac{\partial v}{\partial y}=-Ax=-\frac{\partial u}{\partial x}$

Integrating with respect to x will yield the x-component of velocity: $u = \int A\cdot x\,dx + f(y) = \frac{1}{2}\cdot A\cdot x^2 + f(y)$

The simplest x-component of velocity is obtained for f(y) = 0: $\quad u = \frac{1}{2}\cdot A\cdot x^2$

Based on assumptions (2) and (3), the acceleration reduces to: $\vec{a}_p=u\frac{\partial \vec{V}}{\partial x}+v\frac{\partial \vec{V}}{\partial y}$ and the partial derivatives of velocity are:

$\frac{\partial \vec{V}}{\partial x}=Ax\hat{i}-Ay\hat{j}$ and $\frac{\partial \vec{V}}{\partial y}=-Ax\hat{j}$ Therefore the acceleration vector is equal to:

$$\vec{a}_p=\frac{1}{2}Ax^2\left(Ax\hat{i}-Ay\hat{j}\right)-Axy\left(-Ax\hat{j}\right)=\frac{1}{2}A^2x^3\hat{i}+\frac{1}{2}A^2x^2y\hat{j} \quad \text{At } (1,2):$$

$$\vec{a}_p=\left(\frac{1}{2}\times A^2\times 1^3\right)\hat{i}+\left(\frac{1}{2}\times A^2\times 1^2\times 2\right)\hat{j}$$

$$\vec{a}_p=A^2\left(\frac{1}{2}\hat{i}+\hat{j}\right)$$

Problem 5.35

(Difficulty 1)

5.35 A 4 m diameter tank is filled with water and then rotated at a rate of $\omega = 2\pi(1 - e^{-t})\ \frac{rad}{s}$. At the tank walls, viscosity prevents relative motion between the fluid and the wall. Determine the speed and acceleration of the fluid particles next to the tank walls as a function of time.

Find: The speed and acceleration of the fluid particles next to tank walls.

Assumptions: The flow is steady and incompressible

Solution: Use the expression for acceleration

$$\frac{d\vec{V}}{dt} = \vec{a}_p = \frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y}$$

The fluid particles velocity next to the tank wall are the same as the tank because of the viscosity. The particle velocity is

$$V = \omega r = \frac{1}{2}\omega d = \frac{1}{2} \times 2\pi(1 - e^{-t})\ \frac{rad}{s} \times 4\ m = 4\pi(1 - e^{-t})\ \frac{m}{s}$$

The tangential acceleration is given by:

$$a_t = \frac{dV}{dt} = 4\pi e^{-t}\ \frac{m}{s^2}$$

The normal acceleration is given by:

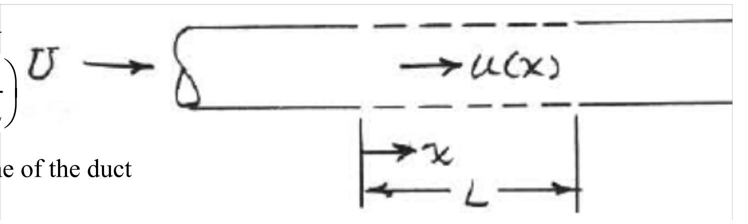
$$a_n = -\frac{V^2}{r} = -\frac{16\pi^2(1 - e^{-t})^2\ \frac{m^2}{s^2}}{2\ m} = -8\pi^2(1 - e^{-t})^2\ \frac{m}{s^2}$$

Problem 5.36 [Difficulty: 2]

5.36 An incompressible liquid with negligible viscosity flows steadily through a horizontal pipe of constant diameter. In a porous section of length $L = 0.3$ m, liquid is removed at a constant rate per unit length, so the uniform axial velocity in the pipe is $u(x) = U(1 - x/2L)$, where $U = 5$ m/s. Develop an expression for the acceleration of a fluid particle along the centerline of the porous section.

Given: Duct flow with incompressible, inviscid liquid

$$U = 5\cdot\frac{m}{s} \quad L = 0.3\cdot m \quad u(x) = U\cdot\left(1 - \frac{x}{2\cdot L}\right)$$



Find: Expression for acceleration along the centerline of the duct

Solution: We will apply the definition of acceleration to the velocity.

Governing Equation:

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t} \quad \text{(Particle acceleration)}$$

Assumptions:

(1) Incompressible flow (ρ is constant)
(2) One-dimensional flow along centerline (u = u(x) only)
(3) Steady flow (velocity is not a function of t)

Based on assumptions (2) and (3), the acceleration reduces to:

$$a_{px} = u\cdot\frac{\partial}{\partial x}u = \left[U\cdot\left(1 - \frac{x}{2\cdot L}\right)\right]\cdot\left(-\frac{U}{2\cdot L}\right) = -\frac{U^2}{2\cdot L}\cdot\left(1 - \frac{x}{2\cdot L}\right)$$

$$a_{px} = -\frac{U^2}{2\cdot L}\cdot\left(1 - \frac{x}{2\cdot L}\right)$$

Problem 5.37

(Difficulty 2)

5.37 Sketch the following flow fields and derive general expressions for the acceleration.

a) $u = 2xy; v = -x^2 y.$
b) $u = y - x + x^2; v = x + y - 2xy.$
c) $u = x^2 t + 2y; v = 2x - yt^2.$
d) $u = -x^2 - y^2 - xyt; v = x^2 + y^2 + xyt$

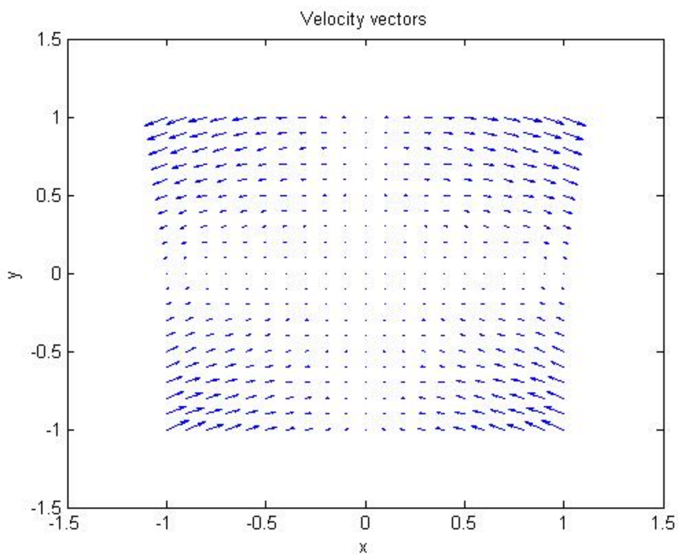
Find: Sketch the flow fields and derive the general expressions for acceleration.

Assumptions: The flow is steady and incompressible

Solution: Use the expression for acceleration

$$\frac{d\vec{V}}{dt} = \vec{a}_p = \frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y}$$

(a) The flow field is shown in the figure:

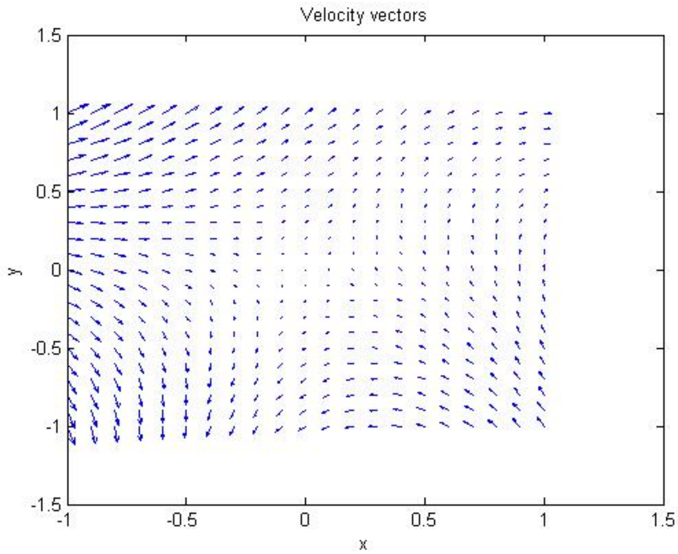


The acceleration can be calculated as:

$$a_x = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = 2xy(2y) + (-x^2 y)(2x) = 4xy^2 - 2x^3 y$$

$$a_y = \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = 2xy(-2xy) + (-x^2 y)(-x^2) = -4x^2 y^2 + x^4 y$$

(b) The flow field is shown in the figure:



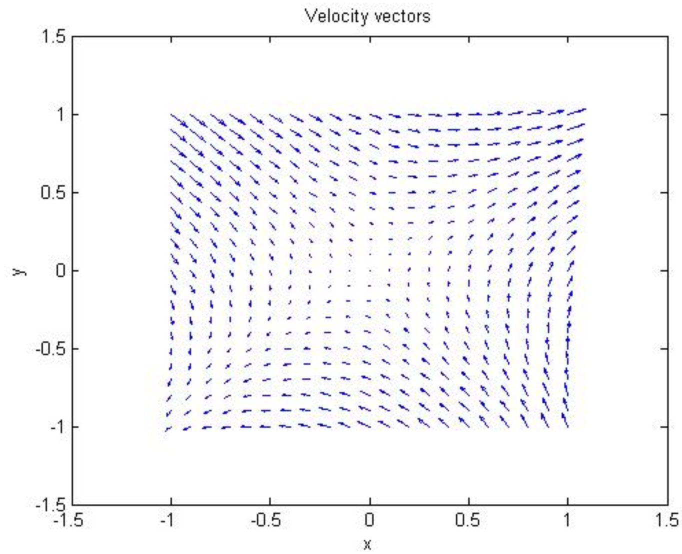
The acceleration can be calculated as:

$$a_x = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = (y - x + x^2)(-1 + 2x) + (x + y - 2xy)$$

$$a_x = 2x - 3x^2 + 2x^3$$

$$a_y = \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = (y - x + x^2)(1 - 2y) + (x + y - 2xy)(1 - 2x)$$

(c) The flow field at $t = 1s$ is shown in the figure:

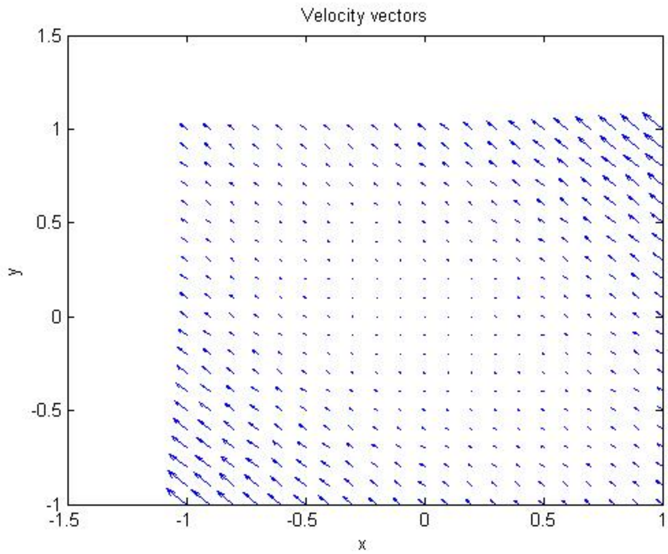


The acceleration can be calculated as:

$$a_x = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = x^2 + (x^2t + 2y)(2xt) + (4x - 2yt^2)$$

$$a_y = \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = (-2yt) + (2x^2t + 4y) + (2x - yt^2)(-t^2)$$

(d) The flow field at $t = 1s$ is shown in the figure:



The acceleration can be calculated as:

$$a_x = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}$$
$$= (-xy) + (-x^2 - y^2 - xyt)(-2x - yt) + (x^2 + y^2 + xyt)(-2y - xt)$$

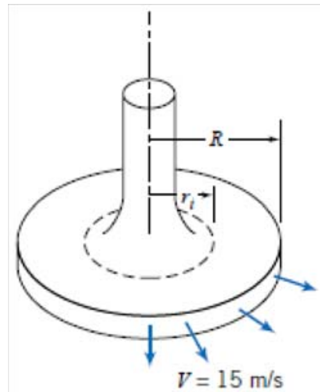
$$a_x = (-xy) + (x^2 + y^2 + xyt)(-2y - xt + 2x + yt)$$

$$a_y = \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}$$
$$= (xy) + (-x^2 - y^2 - xyt)(2x + yt) + (x^2 + y^2 + xyt)(2y + xt)$$

$$a_y = (xy) + (x^2 + y^2 + xyt)(2y + xt - 2x - yt)$$

Problem 5.38 [Difficulty: 2]

5.38 Consider the low-speed flow of air between parallel disks as shown. Assume that the flow is incompressible and inviscid, and that the velocity is purely radial and uniform at any section. The flow speed is $V = 15$ m/s at $R = 75$ mm. Simplify the continuity equation to a form applicable to this flow field. Show that a general expression for the velocity field is $\vec{V} = V(R/r)\hat{e}_r$ for $r_i \le r \le R$. Calculate the acceleration of a fluid particle at the locations $r = r_i$ and $r = R$.



Given: Incompressible, inviscid flow of air between parallel disks

Find: (a) simplified version of continuity equation valid in this flow field
(b) show that the velocity is described by: $\vec{V} = V(R/r)\hat{e}_r$
(c) acceleration of a particle at $r = r_i$, $r = R$

Solution: We will apply the conservation of mass and the definition of acceleration to the velocity.

Governing Equations:

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho r V_r) + \frac{1}{r}\frac{\partial \vec{V}}{\partial \theta}(\rho V_\theta) + \frac{\partial}{\partial z}(\rho V_z) + \frac{\partial \rho}{\partial t} = 0 \quad \text{(Continuity Equation)}$$

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = (\vec{V}\cdot\nabla)\vec{V} + \frac{\partial \vec{V}}{\partial t} \quad \text{(Particle acceleration)}$$

Assumptions:
(1) Incompressible flow (ρ is constant)
(2) One-dimensional flow (velocity not a function of θ or z)
(3) Flow is only in the r-direction
(4) Steady flow (velocity is not a function of t)

Based on the above assumptions, the continuity equation reduces to: $\frac{1}{r}\cdot\frac{\partial}{\partial r}(r\cdot V_r) = 0$ or $r\cdot V_r = C$

Thus: $V_r = \frac{C}{r}$ should be the form of the solution. Now since at $r = R$: $R\cdot V = C$ it follows that: $V_r = \frac{R}{r}\cdot V$ or:

$$\vec{V} = V(R/r)\hat{e}_r$$

(Q.E.D.)

Based on assumptions (2) - (4), acceleration is radial only, and that acceleration is equal to: $a_{pr} = V_r\cdot\frac{\partial}{\partial r}V_r$

$$a_{pr} = \left(V\cdot\frac{R}{r}\right)\cdot\left(-V\cdot\frac{R}{r^2}\right) = -\frac{V^2}{R}\cdot\left(\frac{R}{r}\right)^3$$

Therefore, at r = ri: $a_{pr} = -\left(15\cdot\frac{m}{s}\right)^2 \times \frac{1}{0.075\cdot m} \times \left(\frac{75}{25}\right)^3$ $\quad a_{pr} = -8.1\times 10^4\,\frac{m}{s^2}$

Therefore, at r = R: $a_{pr} = -\left(15\cdot\frac{m}{s}\right)^2 \times \frac{1}{0.075\cdot m} \times \left(\frac{75}{75}\right)^3$ $\quad a_{pr} = -3\times 10^3\,\frac{m}{s^2}$

Problem 5.39 [Difficulty: 4]

5.39 As part of a pollution study, a model concentration c as a function of position x has been developed,

$$c(x) = A(e^{-x/2a} - e^{-x/a})$$

where $A = 3 \times 10^{-5}$ ppm (parts per million) and $a = 3$ ft. Plot this concentration from $x = 0$ to $x = 30$ ft. If a vehicle with a pollution sensor travels through the area at $u = U = 70$ ft/s, develop an expression for the measured concentration rate of change of c with time, and plot using the given data.
(a) At what location will the sensor indicate the most rapid rate of change?
(b) What is the value of this rate of change?

Given: Data on pollution concentration

Find: Plot of concentration; Plot of concentration over time for moving vehicle; Location and value of maximum rate change

Solution:

Basic equation:
$$\frac{D}{Dt} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} + \frac{\partial}{\partial t} \qquad \text{(Material Derivative)}$$

Assumption: Concentration of pollution is a function of x only
Sensor travels in x-direction only

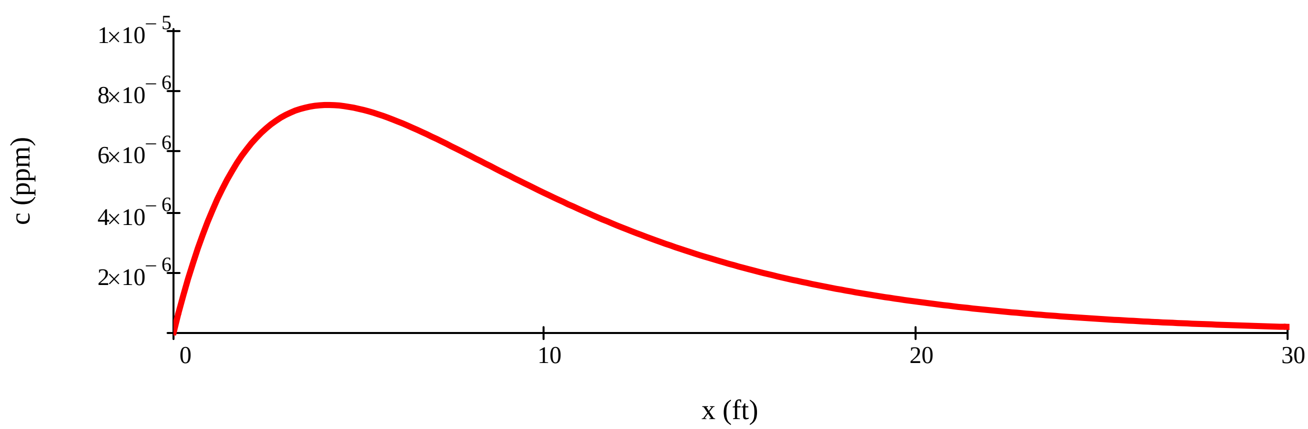
For this case we have
$$u = U \qquad v = 0 \qquad w = 0 \qquad c(x) = A\cdot\left(e^{-\frac{x}{2\cdot a}} - e^{-\frac{x}{a}}\right)$$

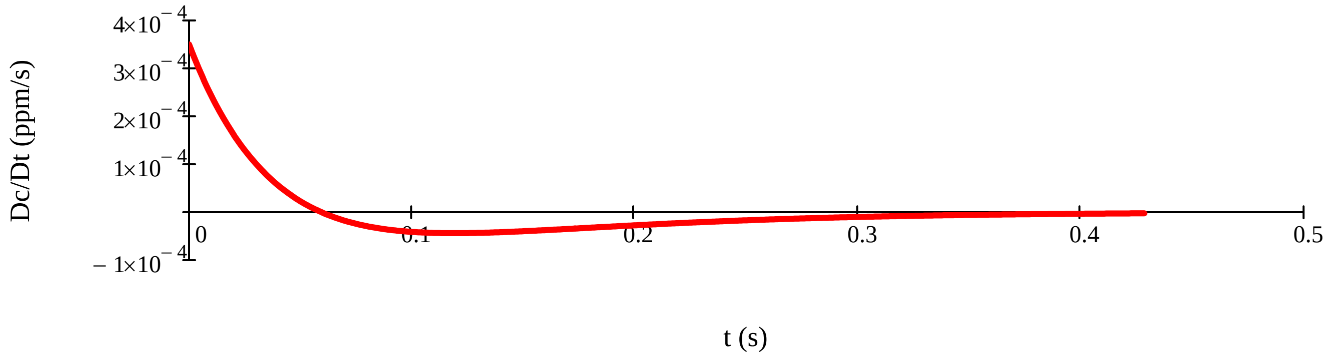
Hence
$$\frac{Dc}{Dt} = u\cdot\frac{dc}{dx} = U\cdot\frac{d}{dx}\left[A\cdot\left(e^{-\frac{x}{2\cdot a}} - e^{-\frac{x}{a}}\right)\right] = \frac{U\cdot A}{a}\cdot\left(e^{-\frac{x}{a}} - \frac{1}{2}\cdot e^{-\frac{x}{2\cdot a}}\right)$$

We need to convert this to a function of time. For this motion u = U so $x = U\cdot t$

$$\frac{Dc}{Dt} = \frac{U\cdot A}{a}\cdot\left(e^{-\frac{U\cdot t}{a}} - \frac{1}{2}\cdot e^{-\frac{U\cdot t}{2\cdot a}}\right)$$

The following plots can be done in *Excel*





The magnitude of the rate of change is maximized when

$$\frac{d}{dx}\left(\frac{Dc}{Dt}\right) = \frac{d}{dx}\cdot\left[\frac{U\cdot A}{a}\cdot\left(e^{-\frac{x}{a}} - \frac{1}{2}\cdot e^{-\frac{x}{2\cdot a}}\right)\right] = 0$$

$$\frac{U\cdot A}{a^2}\cdot\left(\frac{1}{4}\cdot e^{-\frac{x}{2\cdot a}} - e^{-\frac{x}{a}}\right) = 0 \qquad \text{or} \qquad e^{\frac{x}{2\cdot a}} = 4$$

$$x_{max} = 2\cdot a\cdot \ln(4) = 2 \times 3\cdot ft \times \ln(4) \qquad x_{max} = 8.32\cdot ft$$

$$t_{max} = \frac{x_{max}}{U} = 8.32\cdot ft \times \frac{s}{70\cdot ft} \qquad t_{max} = 0.119\cdot s$$

$$\frac{Dc_{max}}{Dt} = \frac{U\cdot A}{a}\cdot\left(e^{-\frac{x_{max}}{a}} - \frac{1}{2}\cdot e^{-\frac{x_{max}}{2\cdot a}}\right)$$

$$\frac{Dc_{max}}{Dt} = 70\cdot\frac{ft}{s} \times 3 \times 10^{-5}\cdot ppm \times \frac{1}{3\cdot ft} \times \left(e^{-\frac{8.32}{3}} - \frac{1}{2} \times e^{-\frac{8.32}{2\cdot 3}}\right) \qquad \frac{Dc_{max}}{Dt} = -4.38 \times 10^{-5}\cdot\frac{ppm}{s}$$

Note that there is another maximum rate, at t = 0 (x = 0)

$$\frac{Dc_{max}}{Dt} = 70\cdot\frac{ft}{s} \times 3 \times 10^{-5}\cdot ppm \times \frac{1}{3\cdot ft}\cdot\left(1 - \frac{1}{2}\right) \qquad \frac{Dc_{max}}{Dt} = 3.50 \times 10^{-4}\cdot\frac{ppm}{s}$$

Problem 5.40

[Difficulty: 2]

5.40 As an aircraft flies through a cold front, an onboard instrument indicates that ambient temperature drops at the rate of 0.7°F/min. Other instruments show an air speed of 400 knots and a 2500 ft/min rate of climb. The front is stationary and vertically uniform. Compute the rate of change of temperature with respect to horizontal distance through the cold front.

Given: Instruments on board an aircraft flying through a cold front show ambient temperature dropping at 0.7 °F/min, air speed of 400 knots and 2500 ft/min rate of climb.

Find: Rate of temperature change with respect to horizontal distance through cold front.

Solution: We will apply the concept of substantial derivative

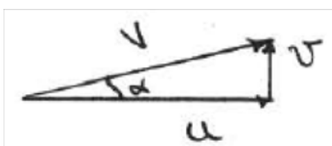
Governing Equation:

$$\frac{DT}{Dt} = u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} + \frac{\partial T}{\partial t} \qquad \text{(Substantial Derivative)}$$

Assumptions:
(1) Two-dimensional motion (velocity not a function of z)
(2) Steady flow (velocity is not a function of t)
(3) Temperature is constant in y direction

Based on the above assumptions, the substantial derivative reduces to: $\frac{DT}{Dt} = u\frac{\partial T}{\partial x}$

Finding the velocity components: $V = 400\cdot\frac{\text{nmi}}{\text{hr}} \times \frac{6080\cdot\text{ft}}{\text{nmi}} \times \frac{\text{hr}}{3600\cdot\text{s}} \quad V = 675.56\cdot\frac{\text{ft}}{\text{s}} \qquad v = 2500\cdot\frac{\text{ft}}{\text{min}} \times \frac{\text{min}}{60\cdot\text{s}} \quad v = 41.67\cdot\frac{\text{ft}}{\text{s}}$



Therefore: $u = \sqrt{\left(675.56\cdot\frac{\text{ft}}{\text{s}}\right)^2 - \left(41.67\cdot\frac{\text{ft}}{\text{s}}\right)^2} \qquad u = 674.27\cdot\frac{\text{ft}}{\text{s}}$

So the rate of change of temperature through the cold front is: $\delta T_x = \frac{-0.7\cdot\Delta°\text{F}}{\text{min}} \times \frac{\text{s}}{674.27\cdot\text{ft}} \times \frac{\text{min}}{60\cdot\text{s}} \times \frac{5280\cdot\text{ft}}{\text{mi}}$

$$\delta T_x = -0.0914\cdot\frac{\Delta°\text{F}}{\text{mi}}$$

Problem 5.41

[Difficulty: 4]

5.41 Wave flow of an incompressible fluid into a solid surface follows a sinusoidal pattern. Flow is axisymmetric about the z axis, which is normal to the surface. The z component of the flow follows the pattern

$$V_z = Az\sin\left(\frac{2\pi t}{T}\right)$$

Determine (a) the radial component of flow (V_r) and (b) the convective and local components of the acceleration vector.

Given: Z component of an axisymmetric transient flow.

Find: Radial component of flow and total acceleration.

Solution:

Governing Equations:

$$\frac{1}{r}\frac{\partial(rV_r)}{\partial r}+\frac{1}{r}\frac{\partial V_\theta}{\partial \theta}+\frac{\partial V_z}{\partial z}=0 \quad \text{(Continuity Equation for an Incompressible Fluid)}$$

$$a_{r,p}=V_r\frac{\partial V_r}{\partial r}+\frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta}-\frac{V_\theta^2}{r}+V_z\frac{\partial V_r}{\partial z}+\frac{\partial V_r}{\partial t}$$

(Particle acceleration)

$$a_{z,p}=V_r\frac{\partial V_z}{\partial r}+\frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta}+V_z\frac{\partial V_z}{\partial z}+\frac{\partial V_z}{\partial t}$$

Assumptions: Incompressible fluid
No motion along the wall ($z = 0$) limited to two dimensions ($V_\theta = 0$ and all partials with respect to θ are zero).

The given or available data is: $V_Z = Az\cdot\sin\left(\frac{2\pi t}{T}\right)$ $\quad V_\theta = 0 \quad \frac{\partial(\ )}{\partial\theta}=0$

Simplify the continuity equation to find V_r: $\frac{1}{r}\frac{\partial(rV_r)}{\partial r}=-\frac{\partial V_z}{\partial z}\Rightarrow\frac{\partial(rV_r)}{\partial r}=r\times -A\cdot\sin\left(\frac{2\pi t}{T}\right)$

Solve using separation of variables: $rV_r=-\frac{r^2A}{2}\cdot\sin\left(\frac{2\pi t}{T}\right)+C$

Use the boundary condition of no flow at the origin to solve for the constant of integration

$$V_r=-\frac{rA}{2}\cdot\sin\left(\frac{2\pi t}{T}\right)$$

Find the convective terms of acceleration. $a_{r,conv}=V_r\frac{\partial V_r}{\partial r}+V_z\frac{\partial V_r}{\partial z}=-\frac{rA}{2}\sin\left(\frac{2\pi t}{T}\right)\times -\frac{A}{2}\sin\left(\frac{2\pi t}{T}\right)+Az\sin\left(\frac{2\pi t}{T}\right)\times 0$

$$a_{r,conv}=\frac{rA^2}{4}\sin^2\left(\frac{2\pi t}{T}\right)$$

$$a_{z,conv}=V_r\frac{\partial V_z}{\partial r}+V_z\frac{\partial V_z}{\partial z}=-\frac{rA}{2}\sin\left(\frac{2\pi t}{T}\right)\times 0+Az\cdot\sin\left(\frac{2\pi t}{T}\right)\times A\cdot\sin\left(\frac{2\pi t}{T}\right)$$

$$a_{z,conv}=zA^2\sin^2\left(\frac{2\pi t}{T}\right)$$

Find the local terms: $$a_{r,local} = \frac{\partial V_r}{\partial t} = -\frac{2\pi}{T} \times \frac{rA}{2} \cos\left(\frac{2\pi t}{T}\right)$$

$$a_{r,local} = \frac{-\pi rA}{T} \cos\left(\frac{2\pi t}{T}\right)$$

$$a_{z,local} = \frac{\partial V_z}{\partial t} = -\frac{2\pi}{T} \times Az \cdot \cos\left(\frac{2\pi t}{T}\right)$$

$$a_{z,local} = \frac{2\pi zA}{T} \cos\left(\frac{2\pi t}{T}\right)$$

Problem 5.42

[Difficulty: 3]

5.42 A steady, two-dimensional velocity field is given by $\vec{V} = Ax\hat{i} - Ay\hat{j}$, where $A = 1\ \mathrm{s}^{-1}$. Show that the streamlines for this flow are rectangular hyperbolas, $xy = C$. Obtain a general expression for the acceleration of a fluid particle in this velocity field. Calculate the acceleration of fluid particles at the points $(x, y) = (\frac{1}{2}, 2)$, $(1, 1)$, and $(2, \frac{1}{2})$, where x and y are measured in meters. Plot streamlines that correspond to $C = 0$, 1, and 2 m^2 and show the acceleration vectors on the streamline plot.

Given: Steady, two-dimensional velocity field represented above

Find:
(a) proof that streamlines are hyperbolas (xy = C)
(b) acceleration of a particle in this field
(c) acceleration of particles at (x,y) = (1/2m, 2m), (1m,1m), and (2m, 1/2m)
(d) plot streamlines corresponding to C = 0, 1, and 2 m^2 and show accelerations

Solution: We will apply the acceleration definition, and determine the streamline slope.

Governing Equations:

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t} \quad \text{(Particle acceleration)}$$

Assumptions:
(1) Two-dimensional flow (velocity is not a function of z)
(2) Incompressible flow

Streamlines along the x-y plane are defined by $\frac{dy}{dx} = \frac{v}{u} = \frac{-A\cdot y}{A\cdot x}$ Thus: $\frac{dx}{x} + \frac{dy}{y} = 0$

After integrating: $\ln(x) + \ln(y) = \ln(C)$ which yields: $x\cdot y = C$ (Q.E.D.)

Based on the above assumptions the particle acceleration reduces to: $\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y}$ Substituting in the field:

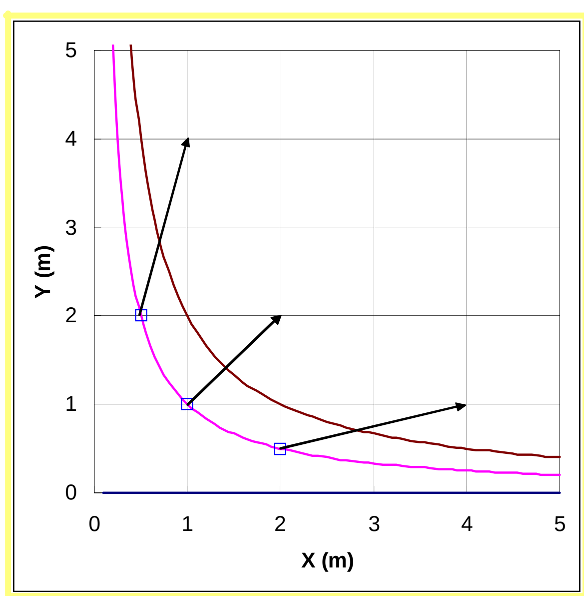
$$\vec{a}_p = (Ax)A\hat{i} + (-Ay)(-A)\hat{j} = A^2\left(x\hat{i} + y\hat{j}\right)$$

which simplifies to $\vec{a}_p = A^2\left(x\hat{i} + y\hat{j}\right)$

At (x,y) = (0.5m, 2m) $\vec{a}_p = \left(0.5\hat{i} + 2\hat{j}\right)\frac{\mathrm{m}}{\mathrm{s}^2}$ At (x,y) = (1m, 1m) $\vec{a}_p = \left(\hat{i} + \hat{j}\right)\frac{\mathrm{m}}{\mathrm{s}^2}$ At (x,y) = (2m, 0.5m) $\vec{a}_p = \left(2\hat{i} + 0.5\hat{j}\right)\frac{\mathrm{m}}{\mathrm{s}^2}$

Here is the plot of the streamlines:

(When C = 0 the streamline is on the x- and y-axes.)



Problem 5.43

[Difficulty: 3]

5.43 A velocity field is represented by the expression $\vec{V} = (Ax - B)\hat{i} + Cy\hat{j} + Dt\hat{k}$, where $A = 2\ \text{s}^{-1}$, $B = 4\ \text{m}\cdot\text{s}^{-1}$, $D = 5\ \text{m}\cdot\text{s}^{-2}$, and the coordinates are measured in meters. Determine the proper value for C if the flow field is to be incompressible. Calculate the acceleration of a fluid particle located at point $(x, y) = (3, 2)$. Plot a few flow streamlines in the xy plane.

Given: Velocity field represented above

Find:
(a) the proper value for C if the flow field is incompressible
(b) acceleration of a particle at (x,y) = (3m,2m)
(c) sketch the streamlines in the x-y plane

Solution: We will check the velocity field against the continuity equation, apply the acceleration definition, and determine the streamline slope.

Governing Equations:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) + \frac{\partial \rho}{\partial t} = 0 \quad \text{(Continuity equation)}$$

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t} \quad \text{(Particle acceleration)}$$

Assumptions:
(1) Two-dimensional flow (velocity is not a function of z)
(2) Incompressible flow

Based on the above assumptions the continuity equation reduces to: $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$ This is the criterion to check the velocity.

The partial derivatives are: $\frac{\partial}{\partial x}u = A$ and $\frac{\partial}{\partial y}v = C$ Thus from continuity: $A + C = 0$ or $C = -A$ $\quad C = -2\cdot\text{s}^{-1}$

Based on the above assumptions the particle acceleration reduces to: $\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + \frac{\partial \vec{V}}{\partial t}$ Substituting in the field:

$$\vec{a}_p = (Ax - B)A\hat{i} + (Cy)C\hat{j} + D\hat{k} = (A^2 x - AB)\hat{i} + C^2 y\hat{j} + D\hat{k}$$

At (x,y) = (3m, 2m)

$$\vec{a}_p = \left[\left(\frac{2}{\text{s}}\right)^2 \times 3\ \text{m} - \frac{2}{\text{s}} \times 4\frac{\text{m}}{\text{s}}\right]\hat{i} + \left(-\frac{2}{\text{s}}\right)^2 \times 2\ \text{m}\hat{j} + 5\frac{\text{m}}{\text{s}^2}\hat{k}$$

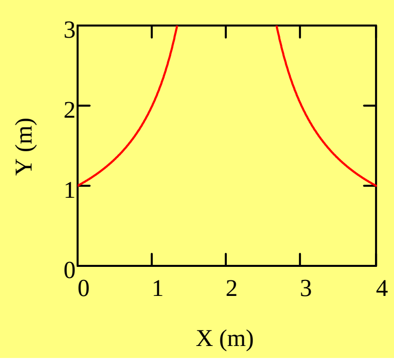
$$\vec{a}_p = \left(4\hat{i} + 8\hat{j} + 5\hat{k}\right)\frac{\text{m}}{\text{s}^2}$$

Streamlines along the x-y plane are defined by $\frac{dy}{dx} = \frac{v}{u} = \frac{C\cdot y}{A\cdot x - B}$ Thus: $-\frac{1}{A}\cdot\frac{dy}{y} = \frac{dx}{A\cdot x - B}$ or $\frac{dx}{x - \frac{B}{A}} + \frac{dy}{y} = 0$

Solving this ODE by integrating: $\ln\left(x - \frac{B}{A}\right) + \ln(y) = \text{const}$

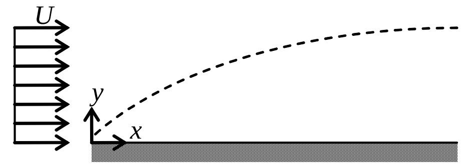
Therefore: $y\cdot\left(x - \frac{B}{A}\right) = \text{constant}$

Here is a plot of the streamlines passing through (3, 2):



Problem 5.44 [Difficulty: 4]

5.44 A parabolic approximate velocity profile was used in Problem 5.11 to model flow in a laminar incompressible boundary layer on a flat plate. For this profile, find the x component of acceleration, a_x, of a fluid particle within the boundary layer. Plot a_x at location $x = 0.8$ m, where $\delta = 1.2$ mm, for a flow with $U = 6$ m/s. Find the maximum value of a_x at this x location.



Given:
Flow in boundary layer

Find:
Expression for particle acceleration a_x; Plot acceleration and find maximum at x = 0.8 m

Solution:

Basic equations

$$\frac{u}{U} = 2\cdot\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \qquad \frac{v}{U} = \frac{\delta}{x}\cdot\left[\frac{1}{2}\cdot\left(\frac{y}{\delta}\right) - \frac{1}{3}\cdot\left(\frac{y}{\delta}\right)^3\right] \qquad \delta = c\cdot\sqrt{x}$$

$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\substack{\text{total}\\ \text{acceleration}\\ \text{of a particle}}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\ \text{acceleration}}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\substack{\text{local}\\ \text{acceleration}}}$$

We need to evaluate

$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u$$

First, substitute

$$\lambda(x,y) = \frac{y}{\delta(x)} \qquad \text{so} \qquad \frac{u}{U} = 2\cdot\lambda - \lambda^2 \qquad \frac{v}{U} = \frac{\delta}{x}\cdot\left(\frac{1}{2}\cdot\lambda - \frac{1}{3}\cdot\lambda^3\right)$$

Then

$$\frac{\partial}{\partial x}u = \frac{du}{d\lambda}\cdot\frac{d\lambda}{dx} = U\cdot(2 - 2\cdot\lambda)\cdot\left(-\frac{y}{\delta^2}\right)\cdot\frac{d\delta}{dx} \qquad \frac{d\delta}{dx} = \frac{1}{2}\cdot c\cdot x^{-\frac{1}{2}}$$

$$\frac{\partial}{\partial x}u = U\cdot(2 - 2\cdot\lambda)\cdot\left(-\frac{\lambda}{\delta}\right)\cdot\frac{1}{2}\cdot c\cdot x^{-\frac{1}{2}} = U\cdot(2 - 2\cdot\lambda)\cdot\left(-\frac{\lambda}{c\cdot x^{\frac{1}{2}}}\right)\cdot\frac{1}{2}\cdot c\cdot x^{-\frac{1}{2}}$$

$$\frac{\partial}{\partial x}u = -U\cdot(2 - 2\cdot\lambda)\cdot\frac{\lambda}{2\cdot x} = -\frac{U\cdot\left(\lambda - \lambda^2\right)}{x}$$

$$\frac{\partial}{\partial y}u = U\cdot\left(\frac{2}{\delta} - 2\cdot\frac{y}{\delta^2}\right) = \frac{2\cdot U}{\delta}\cdot\left[\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2\right] = \frac{2\cdot U\cdot\left(\lambda - \lambda^2\right)}{y}$$

Hence

$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u = U\cdot\left(2\cdot\lambda - \lambda^2\right)\cdot\left[-\frac{U\cdot\left(\lambda - \lambda^2\right)}{x}\right] + U\cdot\frac{\delta}{x}\cdot\left(\frac{1}{2}\cdot\lambda - \frac{1}{3}\cdot\lambda^3\right)\cdot\left[\frac{2\cdot U\cdot\left(\lambda - \lambda^2\right)}{y}\right]$$

Collecting terms

$$a_x = \frac{U^2}{x}\cdot\left(-\lambda^2 + \frac{4}{3}\cdot\lambda^3 - \frac{1}{3}\cdot\lambda^4\right) = \frac{U^2}{x}\cdot\left[-\left(\frac{y}{\delta}\right)^2 + \frac{4}{3}\cdot\left(\frac{y}{\delta}\right)^3 - \frac{1}{3}\cdot\left(\frac{y}{\delta}\right)^4\right]$$

To find the maximum

$$\frac{da_x}{d\lambda} = 0 = \frac{U^2}{x}\cdot\left(-2\cdot\lambda + 4\cdot\lambda^2 - \frac{4}{3}\cdot\lambda^3\right) \qquad \text{or} \qquad -1 + 2\cdot\lambda - \frac{2}{3}\cdot\lambda^2 = 0$$

The solution of this quadratic ($\lambda < 1$) is

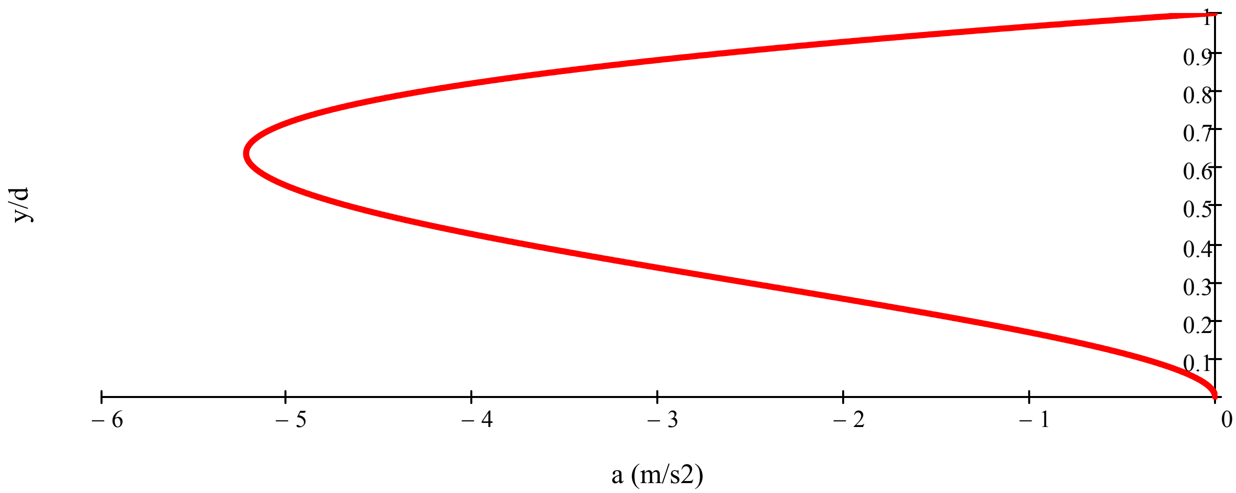
$$\lambda = \frac{3 - \sqrt{3}}{2} \qquad \lambda = 0.634 \qquad \frac{y}{\delta} = 0.634$$

At $\lambda = 0.634$

$$a_x = \frac{U^2}{x}\cdot\left(-0.634^2 + \frac{4}{3}\cdot 0.634^3 - \frac{1}{3}\cdot 0.634^4\right) = -0.116\cdot\frac{U^2}{x}$$

$$a_x = -0.116 \times \left(6\cdot\frac{m}{s}\right)^2 \times \frac{1}{0.8\cdot m} \qquad a_x = -5.22\,\frac{m}{s^2}$$

The following plot can be done in *Excel*



Problem 5.45

(Difficulty: 3)

5.45 A cubic approximate velocity profile was used in problem 5.12 to model flow in a laminar incompressible boundary layer on a flat plate. For this profile, obtain an expression for the x and y components of acceleration of a fluid particle in the boundary layer. Plot a_x and a_y at location $x = 3\ ft$, where $\delta = 0.04\ in.$, for a flow with $U = 20\ ft/s$. Find the maxima of a_x at this x location.

Given: Cubic profile for two-dimensional boundary layer

Find: (1) x and y components of acceleration of a fluid particle

(2) plot components as functions of y/δ for $U = 20\ ft/s$, $x = 3\ ft$, $\delta = 0.04\ in$

(3) maximum values of acceleration at this x location

Assumptions: (1) two dimensional flow (velocity is not a function of z)

(2) incompressible flow

(3) steady flow

Solution: We will apply the acceleration definition.

Based on the above assumptions the particle acceleration reduces to:

$$\vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y}$$

According to problem 5.12 we have for the velocity profile:

$$\frac{u}{U} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

$$\delta = cx^{\frac{1}{2}}$$

To make the analysis easier, define η:

$$\eta = \frac{y}{\delta} = \eta(x, y)$$

$$\frac{d\delta}{dx} = \frac{c}{2}x^{\frac{-1}{2}} = \frac{\delta}{2x}$$

$$\frac{\partial \eta}{\partial x} = \frac{\partial \eta}{\partial \delta}\cdot\frac{d\delta}{dx} = -\frac{y}{\delta^2}\cdot\frac{\delta}{2x} = -\frac{y}{2x\delta}$$

The velocities are given as:

$$u = U\left(\frac{3}{2}\eta - \frac{1}{2}\eta^3\right)$$

From the continuity equation we have:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = U\left(\frac{3}{2} - \frac{3}{2}\eta^2\right)\cdot\left(-\frac{y}{2x\delta}\right)$$

So we have:

$$\frac{\partial v}{\partial y} = U\left(\frac{3}{2} - \frac{3}{2}\eta^2\right)\cdot\left(\frac{y}{2x\delta}\right) = U\left(\frac{3}{2} - \frac{3}{2}\left(\frac{y}{\delta}\right)^2\right)\cdot\left(\frac{y}{2x\delta}\right)$$

Integrating

$$v = U\left[\frac{3y^2}{8x\delta} - \frac{3}{16}\frac{y^4}{x\delta^2}\right] + f(x)$$

Apply the boundary condition for $v = 0\ at\ y = 0$ then

$$f(x) = 0$$

$$u = U\left(\frac{3}{2}\eta - \frac{1}{2}\eta^3\right)$$

$$v = U\left[\frac{3y^2}{8x\delta} - \frac{3}{16}\frac{y^4}{x\delta^2}\right] = U\left[\frac{3y^2}{8x^{\frac{3}{2}}c} - \frac{3}{16}\frac{y^4}{cx^2}\right]$$

$$\frac{\partial u}{\partial x} = U\left(\frac{3}{2} - \frac{3}{2}\eta^2\right)\cdot\left(-\frac{\eta}{2x}\right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial y} = U\left(\frac{3}{2} - \frac{3}{2}\eta^2\right)\cdot\frac{1}{\delta}$$

$$\frac{\partial v}{\partial x} = U\left[-\frac{9}{16}\frac{y^2}{x^{\frac{5}{2}}c} + \frac{3}{8}\frac{y^4}{cx^3}\right]$$

$$\frac{\partial v}{\partial y} = U\left[\frac{3y}{4x^{\frac{3}{2}}c} - \frac{3}{4}\frac{y^3}{cx^2}\right]$$

So the accelerations are:

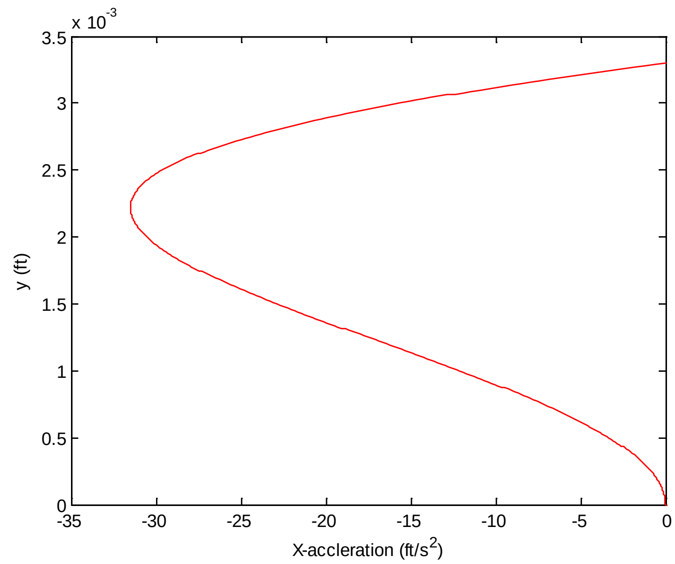
$$a_{px} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\left(\frac{3}{2}\eta - \frac{1}{2}\eta^3\right)U\left(\frac{3}{2} - \frac{3}{2}\eta^2\right)\cdot\left(-\frac{\eta}{2x}\right) + U\left[\frac{3y^2}{8x^{\frac{3}{2}}c} - \frac{3}{16}\frac{y^4}{cx^2}\right]U\left(\frac{3}{2} - \frac{3}{2}\eta^2\right)\cdot\frac{1}{\delta}$$

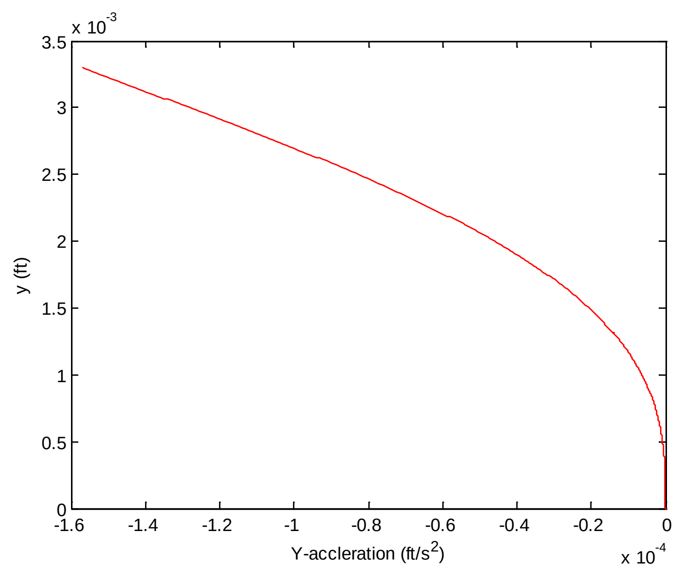
$$a_{px} = U^2\left(\frac{3}{2}-\frac{3}{2}\eta^2\right)\left\{\left(\frac{3}{2}\eta-\frac{1}{2}\eta^3\right)\cdot\left(-\frac{\eta}{2x}\right)+\left[\frac{3y^2}{8x^{\frac{3}{2}}c}-\frac{3}{16}\frac{y^4}{cx^2}\right]\cdot\frac{1}{\delta}\right\}$$

$$a_{py} = u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} = U\left(\frac{3}{2}\eta-\frac{1}{2}\eta^3\right)\cdot U\left[-\frac{9}{16}\frac{y^2}{x^{\frac{5}{2}}c}+\frac{3}{8}\frac{y^4}{cx^3}\right]+U\left[\frac{3y^2}{8x^{\frac{3}{2}}c}-\frac{3}{16}\frac{y^4}{cx^2}\right]\cdot U\left[\frac{3y}{4x^{\frac{3}{2}}c}-\frac{3}{4}\frac{y^3}{cx^2}\right]$$

$$a_{py} = U^2\left\{\left(\frac{3}{2}\eta-\frac{1}{2}\eta^3\right)\cdot\left[-\frac{9}{16}\frac{y^2}{x^{\frac{5}{2}}c}+\frac{3}{8}\frac{y^4}{cx^3}\right]+\left[\frac{3y^2}{8x^{\frac{3}{2}}c}-\frac{3}{16}\frac{y^4}{cx^2}\right]\cdot\left[\frac{3y}{4x^{\frac{3}{2}}c}-\frac{3}{4}\frac{y^3}{cx^2}\right]\right\}$$

When $x = 3\ ft, \delta = 0.04\ in = 0.0033\ ft, U = 20\ \ ft/s$, we have the folllowing plot:





The maximum $a_{px} = -31.5\ \frac{ft}{s^2}$ at $y = 0.00222\ ft$

Problem 5.46 [Difficulty: 3]

5.46 The velocity field for steady inviscid flow from left to right over a circular cylinder, of radius R, is given by

$$\vec{V} = U\cos\theta\left[1-\left(\frac{R}{r}\right)^2\right]\hat{e}_r - U\sin\theta\left[1+\left(\frac{R}{r}\right)^2\right]\hat{e}_\theta$$

Obtain expressions for the acceleration of a fluid particle moving along the stagnation streamline ($\theta = \pi$) and for the acceleration along the cylinder surface ($r = R$). Plot a_r as a function of r/R for $\theta = \pi$, and as a function of θ for $r = R$; plot a_θ as a function of θ for $r = R$. Comment on the plots. Determine the locations at which these accelerations reach maximum and minimum values.

Given: Steady inviscid flow over a circular cylinder of radius R

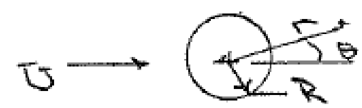
Find:
(a) Expression for acceleration of particle moving along $\theta = \pi$
(b) Expression for accleeration of particle moving along $r = R$
(c) Locations at which accelerations in r- and θ- directions reach maximum and minimum values
(d) Plot a_r as a function of R/r for $\theta = \pi$ and as a function of θ for $r = R$
(e) Plot a_θ as a function of θ for $r = R$

Solution: We will apply the particle acceleration definition to the velocity field

Governing Equation:

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = \left(\vec{V}\cdot\nabla\right)\vec{V} + \frac{\partial\vec{V}}{\partial t} \qquad \text{(Particle Accleration)}$$

Assumptions:
(1) Steady flow
(2) Inviscid flow
(3) No flow in z-direction, velocity is not a function of z



Based on the above assumptions the particle acceleration reduces to: $\vec{a}_p = V_r\dfrac{\partial\vec{V}}{\partial r} + \dfrac{V_\theta}{r}\dfrac{\partial\vec{V}}{\partial\theta}$ and the components are:

$$a_{pr} = V_r\cdot\frac{\partial}{\partial r}V_r + \frac{V_\theta}{r}\cdot\frac{\partial}{\partial\theta}V_r - \frac{{V_\theta}^2}{r} \qquad a_{p\theta} = V_r\cdot\frac{\partial}{\partial r}V_\theta + \frac{V_\theta}{r}\cdot\frac{\partial}{\partial\theta}V_\theta + \frac{V_r\cdot V_\theta}{r}$$

When $\theta = \pi$:

$$V_r = -U\cdot\left[1-\left(\frac{R}{r}\right)^2\right] \quad V_\theta = 0 \quad \frac{\partial}{\partial r}V_r = -U\cdot -2\cdot -\frac{R^2}{r^3} = -2\cdot U\cdot\frac{R^2}{r^3} \quad \frac{\partial}{\partial\theta}V_r = 0 \quad \frac{\partial}{\partial r}V_\theta = 0 \quad \frac{\partial}{\partial\theta}V_\theta = 0$$

So the accelerations are:

$$a_{pr} = -U\cdot\left[1-\left(\frac{R}{r}\right)^2\right]\cdot -2\cdot U\cdot\frac{R^2}{r^3} = \frac{2\cdot U^2}{R}\cdot\left(\frac{R}{r}\right)^3\cdot\left[1-\left(\frac{R}{r}\right)^2\right] \qquad a_{pr} = \frac{2\cdot U^2}{R}\cdot\left(\frac{R}{r}\right)^3\cdot\left[1-\left(\frac{R}{r}\right)^2\right]$$

$$a_{p\theta} = 0$$

To find the maximum acceleration, we take the derivative of the accleration and set it to zero: Let $\eta = \dfrac{R}{r}$

$$\frac{d}{d\eta}a_{pr} = \frac{2\cdot U^2}{R}\cdot\left[3\cdot\eta^2\cdot\left(1-\eta^2\right) - \eta^3\cdot 2\cdot\eta\right] = \frac{2\cdot U^2}{R}\left(-5\cdot\eta^4 + 3\eta^2\right) = 0 \quad \text{Therefore:} \quad \eta = \sqrt{\frac{3}{5}} \quad \text{or} \quad r = 1.291\cdot R$$

The maximum acceleration would then be:

$$a_{prmax} = \frac{2\cdot U^2}{R}\cdot\left(\frac{1}{1.291}\right)^3\cdot\left[1-\left(\frac{1}{1.291}\right)^2\right] \qquad a_{prmax} = 0.372\cdot\frac{U^2}{R}$$

When r = R: $V_r = 0$ $V_\theta = -2\cdot U\cdot\sin(\theta)$ $\frac{\partial}{\partial r}V_r = 0$ $\frac{\partial}{\partial\theta}V_r = 0$ $\frac{\partial}{\partial r}V_\theta = 0$ $\frac{\partial}{\partial\theta}V_\theta = -2\cdot U\cdot\cos(\theta)$

So the accelerations are: $a_{pr} = -\frac{(-2\cdot U\cdot\cos(\theta))^2}{R} = -\frac{4\cdot U^2}{R}\cdot(\sin(\theta))^2$

$$a_{p\theta} = \frac{-2\cdot U\cdot\sin(\theta)}{R}\cdot -2\cdot U\cdot\cos(\theta) = \frac{4\cdot U^2}{R}\cdot\sin(\theta)\cdot\cos(\theta)$$

$$a_{pr} = -\frac{4\cdot U^2}{R}\cdot(\sin(\theta))^2$$

$$a_{p\theta} = \frac{4\cdot U^2}{R}\cdot\sin(\theta)\cdot\cos(\theta)$$

Radial acceleration is minimum at $\theta = 180\cdot deg$

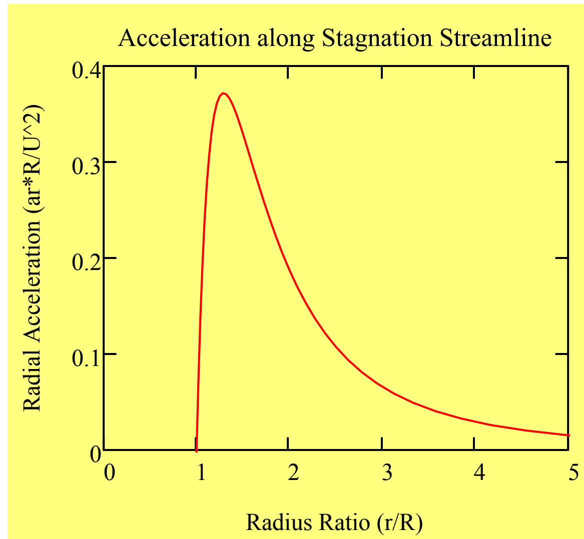
Accelerations at this angle are: $a_{rmin} = -4\cdot\frac{U^2}{R}$ $a_\theta = 0$

Azimuthal acceleration is maximum at $\theta = 45\cdot deg$

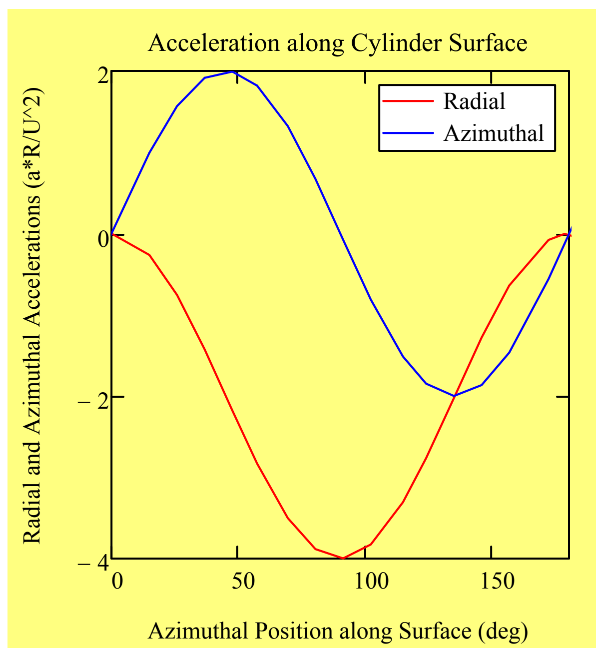
Accelerations at this angle are: $a_r = -2\cdot\frac{U^2}{R}$ $a_{\theta max} = 2\cdot\frac{U^2}{R}$

Azimuthal acceleration is minimum at $\theta = 135\cdot deg$

Accelerations at this angle are: $a_r = -2\cdot\frac{U^2}{R}$ $a_{\theta min} = -2\cdot\frac{U^2}{R}$

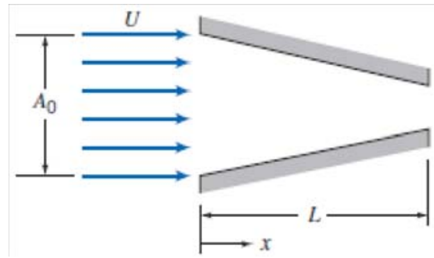


The plots of acceleration along the stagnation streamline and the cylinder surface are shown here. In all cases the accelerations have been normalized by U^2/R



Problem 5.47 [Difficulty: 3]

5.47 Consider the incompressible flow of a fluid through a nozzle as shown. The area of the nozzle is given by $A = A_0(1 - bx)$ and the inlet velocity varies according to $U = U_0(0.5 + 0.5\cos\omega t)$ where $A_0 = 5\text{ ft}^2$, $L = 20$ ft, $b = 0.02\text{ ft}^{-1}$, $\omega = 0.16$ rad/s, and $U_0 = 20$ ft/s. Find and plot the acceleration on the centerline, with time as a parameter.



Given: Velocity field and nozzle geometry

Find: Acceleration along centerline; plot

Solution:

Assumption: Incompressible flow

The given data is $A_0 = 5\cdot\text{ft}^2$ $\quad L = 20\cdot\text{ft}$ $\quad b = 0.2\cdot\text{ft}^{-1}$ $\quad U_0 = 20\cdot\frac{\text{ft}}{\text{s}}$ $\quad \omega = 0.16\frac{\text{rad}}{\text{s}}$ $\quad A(x) = A_0\cdot(1 - b\cdot x)$

The velocity on the centerline is obtained from continuity $\quad u(x)\cdot A(x) = U_0\cdot A_o$

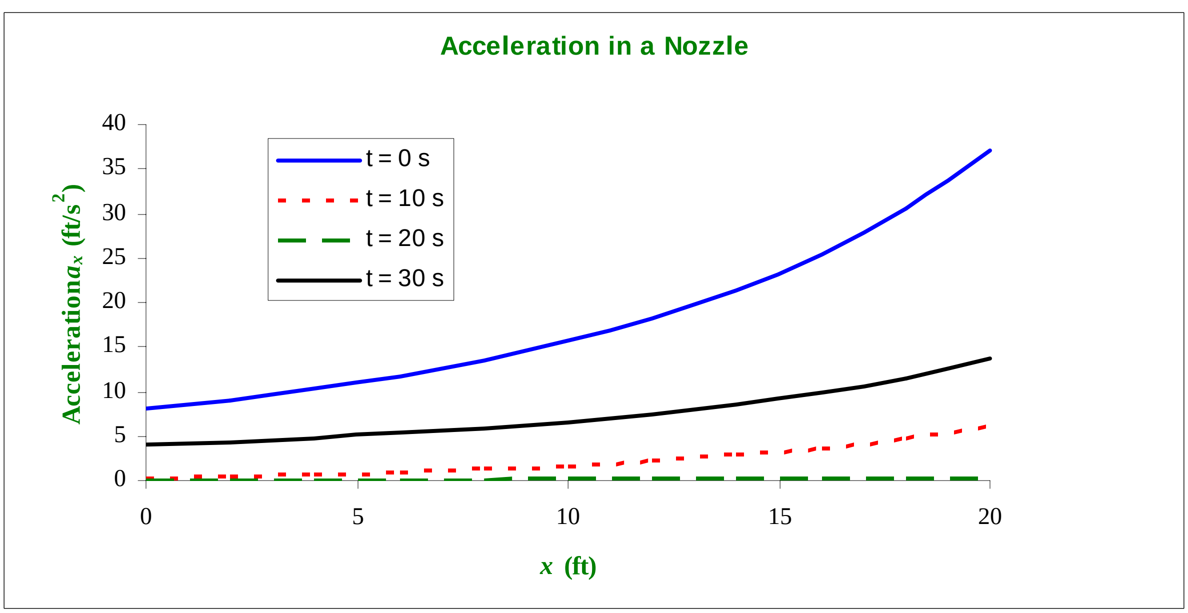
so
$$u(x,t) = \frac{A_0}{A(x)}\cdot U_0\cdot(0.5 + 0.5\cdot\cos(\omega\cdot t)) = \frac{U_0}{(1 - b\cdot x)}\cdot(0.5 + 0.5\cdot\cos(\omega\cdot t))$$

The acceleration is given by
$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\substack{\text{total}\\ \text{acceleration}\\ \text{of a particle}}} = \underbrace{u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y} + w\frac{\partial\vec{V}}{\partial z}}_{\substack{\text{convective}\\ \text{acceleration}}} + \underbrace{\frac{\partial\vec{V}}{\partial t}}_{\substack{\text{local}\\ \text{acceleration}}}$$

For the present 1D flow
$$a_x = \frac{\partial}{\partial t}u + u\cdot\frac{\partial}{\partial x}u = -\frac{0.5\cdot U_0\cdot\omega\cdot\sin(\omega\cdot t)}{1 - b\cdot x} + \frac{U_0}{(1 - b\cdot x)}\cdot(0.5 + 0.5\cdot\cos(\omega\cdot t))\cdot\left[\frac{U_0\cdot b\cdot(0.5\cdot\cos(\omega\cdot t) + 0.5)}{(1 - b\cdot x)^2}\right]$$

$$a_x = \frac{U_0}{(1 - b\cdot x)}\cdot\left[-(0.5\cdot\omega\cdot\sin(\omega\cdot t)) + (0.5 + 0.5\cdot\cos(\omega\cdot t))\cdot\left[\frac{U_0\cdot b\cdot(0.5\cdot\cos(\omega\cdot t) + 0.5)}{(1 - b\cdot x)^2}\right]\right]$$

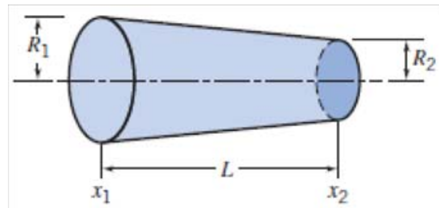
The plot is shown here:



Problem 5.48

[Difficulty: 3]

5.48 Consider the one-dimensional, incompressible flow through the circular channel shown. The velocity at section ① is given by $U = U_0 + U_1 \sin \omega t$, where $U_0 = 20$ m/s, $U_1 = 2$ m/s, and $\omega = 0.3$ rad/s. The channel dimensions are $L = 1$ m, $R_1 = 0.2$ m, and $R_2 = 0.1$ m. Determine the particle acceleration at the channel exit. Plot the results as a function of time over a complete cycle. On the same plot, show the acceleration at the channel exit if the channel is constant area, rather than convergent, and explain the difference between the curves.



Given: One-dimensional, incompressible flow through circular channel.

Find:
(a) the acceleration of a particle at the channel exit
(b) plot as a function of time for a compleye cycle.
(c) plot acceleration if channel is constant area
(d) explain difference between the two acceleration cases

Solution: We will apply the particle acceleration definition to the velocity field

Governing Equations:

$$\vec{a}_p = \frac{D\vec{V}}{Dt} = \left(\vec{V} \cdot \nabla\right)\vec{V} + \frac{\partial \vec{V}}{\partial t} \qquad \text{(Particle Accleration)}$$

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho dV\!\!\!\!- + \int_{CS} \rho \vec{V} \cdot d\vec{A} \qquad \text{(Continuity equation)}$$

Assumptions:
(1) Incompressible flow
(2) One-dimensional flow

Based on the above assumptions the continuity equation can provide the velocity at any location: $u = U \cdot \frac{A_1}{A} = \left(\frac{R_1}{r}\right)^2$

Now based on the geometry of the channel we can write $r = R_1 - \left(R_1 - R_2\right) \cdot \frac{x}{L} = R_1 - \Delta R \cdot \frac{x}{L}$ Therefore the flow speed is:

$$u = U \cdot \frac{{R_1}^2}{\left(R_1 - \Delta R \cdot \frac{x}{L}\right)^2} = \frac{\left(U_0 + U_1 \cdot \sin(\omega \cdot t)\right)}{\left[1 - \frac{\Delta R}{R_1} \cdot \left(\frac{x}{L}\right)\right]^2}$$

Based on the above assumptions the particle acceleration reduces to:

$$\vec{a}_p = \left(u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}\right)\hat{i}$$

Substituting the velocity and derivatives into this expression we can get the acceleration in the x-direction:

$$a_x = \frac{\left(U_0 + U_1 \cdot \sin(\omega \cdot t)\right)}{\left[1 - \frac{\Delta R}{R_1} \cdot \left(\frac{x}{L}\right)\right]^2} \cdot \frac{\left(U_0 + U_1 \cdot \sin(\omega \cdot t)\right)}{\left[1 - \frac{\Delta R}{R_1} \cdot \left(\frac{x}{L}\right)\right]^3} \cdot (-2) \cdot \left(-\frac{\Delta R}{R_1 \cdot L}\right) + \frac{\omega \cdot U_1 \cdot \cos(\omega \cdot t)}{\left[1 - \frac{\Delta R}{R_1} \cdot \left(\frac{x}{L}\right)\right]^2}$$

When we simplify this expression we get:

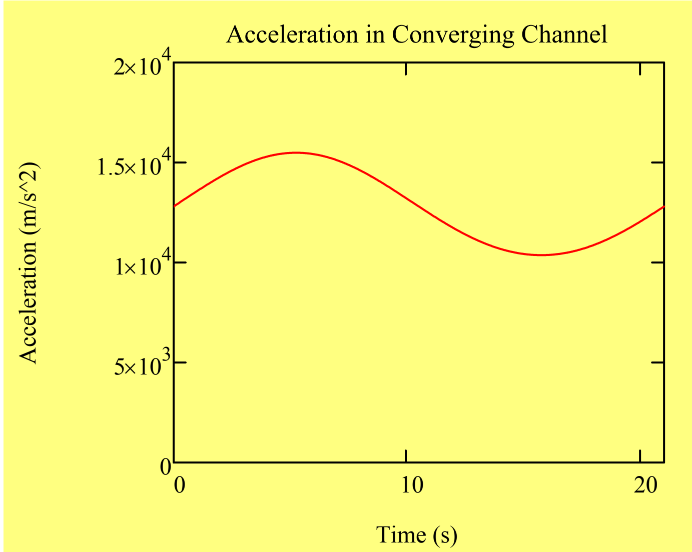
$$a_x = \frac{2 \cdot \Delta R}{R_1 \cdot L} \cdot \frac{\left(U_0 + U_1 \cdot \sin(\omega \cdot t)\right)^2}{\left[1 - \frac{\Delta R}{R_1} \cdot \left(\frac{x}{L}\right)\right]^5} + \frac{\omega \cdot U_1 \cdot \cos(\omega \cdot t)}{\left[1 - \frac{\Delta R}{R_1} \cdot \left(\frac{x}{L}\right)\right]^2}$$

Now we substitute the given values into this expression we get:

$$a_x = 2 \times 0.1 \cdot m \times \frac{1}{0.2 \cdot m} \times \frac{1}{1 \cdot m} \times (20 + 2 \cdot \sin(\omega \cdot t))^2 \cdot \frac{m^2}{s^2} \times \frac{1}{\left(1 - \frac{0.1 \cdot m}{0.2 \cdot m} \times 1\right)^5} + 0.3 \cdot \frac{rad}{s} \times 2 \cdot \frac{m}{s} \times \cos(\omega \cdot t) \times \frac{1}{\left(1 - \frac{0.1 \cdot m}{0.2 \cdot m} \times 1\right)^2}$$

$$a_x = \left[32 \cdot \left(20 + 2 \cdot \sin\left(0.3 \cdot \frac{rad}{s} \cdot t\right)\right)^2 + 2.4 \cdot \cos\left(0.3 \cdot \frac{rad}{s} \cdot t\right)\right] \cdot \frac{m}{s^2}$$

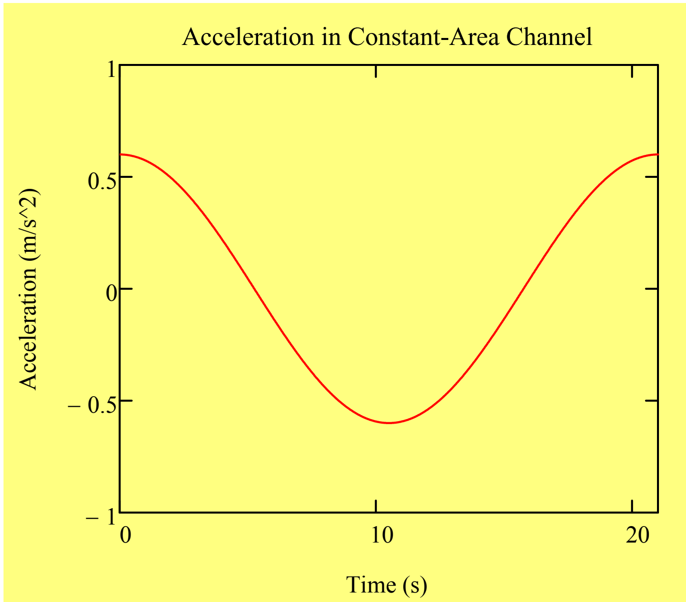
Here is a plot of the acceleration versus time.



For a constant area channel, $\Delta R = 0$ and the acceleration becomes:

$$a_x = \left(0.6 \cdot \cos\left(0.3 \cdot \frac{rad}{s} \cdot t\right)\right) \cdot \frac{m}{s^2}$$

The plot of that acceleration is shown below.
The acceleration is so much larger for the converging channel than in the constant area channel because the convective acceleration is generated by the converging channel - the constant area channel has only local acceleration.



Problem 5.49 [Difficulty: 4]

5.49 Expand $(\vec{V}\cdot\nabla)\vec{V}$ in cylindrical coordinates by direct substitution of the velocity vector to obtain the convective acceleration of a fluid particle. (Recall the hint in footnote 1 on page 150.) Verify the results given in Eqs. 5.12a.

Given: Definition of "del" operator in cylindrical coordinates, velocity vector

Find: (a) An expression for $(\vec{V}\cdot\vec{\nabla})\vec{V}$ in cylindrical coordinates.
(b) Show result is identical to Equations 5.12a.

Solution: We will apply the velocity field to the del operator and simplify.

Governing Equations:

$$\vec{\nabla} = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z} \quad \text{(Definition of "del" operator)}$$

$$\vec{V} = V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k} \quad \text{(Velocity flow field)}$$

$$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta \qquad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r \quad \text{(Hints from footnote)}$$

Substituting $(\vec{V}\cdot\vec{\nabla})\vec{V}$ using the governing equations yields:

$$(\vec{V}\cdot\vec{\nabla})\vec{V} = \left[\left(V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k}\right)\cdot\left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z}\right)\right]\left(V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k}\right)$$

$$= \left(V_r\frac{\partial}{\partial r} + \frac{V_\theta}{r}\frac{\partial}{\partial \theta} + V_z\frac{\partial}{\partial z}\right)\left(V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k}\right)$$

$$= V_r\frac{\partial}{\partial r}\left(V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k}\right) + \frac{V_\theta}{r}\frac{\partial}{\partial \theta}\left(V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k}\right) + V_z\frac{\partial}{\partial z}\left(V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{k}\right)$$

$$= V_r\frac{\partial}{\partial r}V_r\hat{e}_r + V_r\frac{\partial}{\partial r}V_\theta\hat{e}_\theta + V_r\frac{\partial}{\partial r}V_z\hat{k} + \frac{V_\theta}{r}\frac{\partial}{\partial \theta}(V_r\hat{e}_r) + \frac{V_\theta}{r}\frac{\partial}{\partial \theta}(V_\theta\hat{e}_\theta) + \frac{V_\theta}{r}\frac{\partial}{\partial \theta}V_z\hat{k} + V_z\frac{\partial}{\partial z}V_r\hat{e}_r$$
$$+ V_z\frac{\partial}{\partial z}V_\theta\hat{e}_\theta + V_z\frac{\partial}{\partial z}V_z\hat{k}$$

Applying the product rule to isolate derivatives of the unit vectors:

$$(\vec{V}\cdot\vec{\nabla})\vec{V} = V_r\frac{\partial V_r}{\partial r}\hat{e}_r + V_r\frac{\partial V_\theta}{\partial r}\hat{e}_\theta + V_r\frac{\partial V_z}{\partial r}\hat{k} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta}\hat{e}_r + \frac{V_\theta}{r}\frac{\partial \hat{e}_r}{\partial \theta}V_r + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta}\hat{e}_\theta + \frac{V_\theta}{r}\frac{\partial \hat{e}_\theta}{\partial \theta}V_\theta + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta}\hat{k}$$
$$+ V_z\frac{\partial V_r}{\partial z}\hat{e}_r + V_z\frac{\partial V_\theta}{\partial z}\hat{e}_\theta + V_z\frac{\partial V_z}{\partial z}\hat{k}$$

Collecting terms:

$$(\vec{V}\cdot\vec{\nabla})\vec{V} = \left(V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta^2}{r} + V_z\frac{\partial V_r}{\partial z}\right)\hat{e}_r + \left(V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_rV_\theta}{r} + V_z\frac{\partial V_\theta}{\partial z}\right)\hat{e}_\theta$$
$$+ \left(V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z}\right)\hat{k}$$

The three terms in parentheses are the three components of convective acceleration given in Equations 5.12a.

Problem 5.50

(Difficulty 1)

5.50 Determine the velocity potential for:

(a) a flow field characterized by the stream function $\Psi = 3x^2y - y^3$.

(b) a flow field characterized by the stream function $\Psi = xy$.

Find: The velocity potential ϕ

Assumptions: The flow is steady and incompressible

Solution: Use the definitions of stream function and velocity potential

(a) The stream function is given by:

$$\Psi = 3x^2y - y^3$$

We have for the x-component of velocity:

$$u = \frac{\partial \Psi}{\partial y} = -\frac{\partial \phi}{\partial x} = 3x^2 - 3y^2$$

$$\frac{\partial \phi}{\partial x} = -3x^2 + 3y^2$$

$$\phi = -x^3 + 3xy^2 + f(y)$$

Also we have for the y-component of velocity:

$$v = -\frac{\partial \Psi}{\partial x} = -\frac{\partial \phi}{\partial y}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \Psi}{\partial x} = 6xy = 6xy + \frac{df(y)}{dy}$$

$$\frac{df(y)}{dy} = 0$$

$$f(y) = c$$

Where c is a constant.

So the velocity potential can be given by:

$$\phi = -x^3 + 3xy^2 + c$$

(b) The stream function is:

$$\Psi = xy$$

We have for the x-component of velocity:

$$u = \frac{\partial \Psi}{\partial y} = -\frac{\partial \phi}{\partial x} = x$$

$$\frac{\partial \phi}{\partial x} = -x$$

$$\phi = -\frac{1}{2}x^2 + f(y)$$

Also we have for the y-component of velocity:

$$v = -\frac{\partial \Psi}{\partial x} = -\frac{\partial \phi}{\partial y}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \Psi}{\partial x} = y$$

$$\frac{df(y)}{dy} = y$$

$$f(y) = \frac{1}{2}y^2 + c$$

Where c is a constant.

So the velocity potential can be given by:

$$\phi = -\frac{1}{2}x^2 + \frac{1}{2}y^2 + c$$

Problem 5.51

(Difficulty 1)

5.51 Determine whether the following flow fields are irrotational.

(a) $u = 2xy$; $v = -x^2 y$

(b) $u = y - x + x^2$; $v = x + y - 2xy$

(c) $u = x^2 t + 2y$; $v = 2x - yt^2$

(d) $u = -x^2 - y^2 - xyt$; $v = x^2 + y^2 + xyt$

Find: Determine whether the flow fields are irrotational.

Assumptions: The flows are steady and incompressible

Solution: Use the definition for irrotational flow :

$$\vec{\omega} = \frac{1}{2}\nabla \times \vec{V} = 0$$

$$\nabla \times \vec{V} = 0$$

(a) The velocity field is:

$$u = 2xy$$
$$v = -x^2 y$$

$$\nabla \times \vec{V} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k} = (-2xy - 2x)\hat{k} \neq 0$$

This flow is not irrotational.

(b) The velocity field is:

$$u = y - x + x^2$$

$$v = x + y - 2xy$$

$$\nabla \times \vec{V} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k} = (1 - 2y - 1)\hat{k} \neq 0$$

This flow is not irrotational.

(c) The velocity field is:

$$u = x^2 t + 2y$$

$$v = 2x - yt^2$$
$$\nabla \times \vec{V} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k} = (2 - 2)\hat{k} = 0$$

This flow is irrotational.

(d) The velocity field is:

$$u = -x^2 - y^2 - xyt$$
$$v = x^2 + y^2 + xyt$$
$$\nabla \times \vec{V} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k} = (2x + yt + 2y + xt)\hat{k} \neq 0$$

This flow is not irrotational.

Problem 5.52

(Difficulty 1)

5.52 The velocity profile for steady flow between parallel is parabolic and given by $u = u_c + ay^2$, where u_c is the centerline velocity and y is the distance measured from the centerline. The plate spacing is $2b$ and the velocity is zero at each plate. Demonstrate that the flow is rotational. Explain why your answer is correct even though the fluid does not rotate but moves in straight parallel paths.

Find: Demonstrate the flow is rotational and explain why it is rotational

Assumptions: The flow is steady and incompressible

Solution: Use the definition of the rotation vector to evaluate whether the flow is irrotational:

$$\vec{\omega} = \frac{1}{2}\nabla \times \vec{V}$$

The rotation vector for this velocity profile is then

$$\vec{\omega} = \frac{1}{2}\nabla \times \vec{V} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k} = (0\ - 2ay)\hat{k} = -2ay\hat{k} \neq 0$$

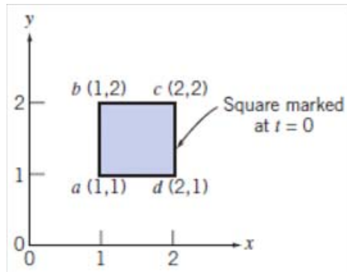
So the fluid flow is rotational.

The answer is correct because the rotation of fluid particles and circular streamlines are two different concepts. We should not be confused about them. An irrotational flow can also have circular streamlines. Similarly, the rotational flow can also only have straight streamlines. In this case the flow is rotational because the velocity profile represents viscous flow between parallel plates, and the effect of viscosity is to introduce rotation into the flow.

Problem 5.53

[Difficulty: 2]

5.53 Consider the velocity field for flow in a rectangular "corner," $\vec{V} = Ax\hat{i} - Ay\hat{j}$, with $A = 0.3\,\text{s}^{-1}$, as in Example 5.8. Evaluate the circulation about the unit square of Example 5.8.



Given: Velocity field for flow in a rectangular corner as in Example 5.8.

Find: Circulation about the unit square shown above.

Solution: We will apply the definition of circulation to the given velocity field.

Governing Equation: $\Gamma = \oint \vec{V} \cdot d\vec{s}$ (Definition of circulation)

From the definition of circulation we break up the integral: $\Gamma = \int_{ab} \vec{V} \cdot d\vec{s} + \int_{bc} \vec{V} \cdot d\vec{s} + \int_{cd} \vec{V} \cdot d\vec{s} + \int_{da} \vec{V} \cdot d\vec{s}$

The integrand is equal to: $\vec{V} \cdot d\vec{s} = \left(Ax\hat{i} - Ay\hat{j}\right)\cdot\left(dx\hat{i} + dy\hat{j}\right) = Axdx - Aydy$ Therefore, the circulation is equal to:

$$\Gamma = \int_{x_a}^{x_d} A\cdot x\,dx + \int_{y_d}^{y_c} -A\cdot y\,dy + \int_{x_c}^{x_b} A\cdot x\,dx + \int_{y_b}^{y_a} -A\cdot y\,dy = \frac{A}{2}\cdot\left[\left(x_d^{\,2} - x_a^{\,2}\right) - \left(y_c^{\,2} - y_d^{\,2}\right) + \left(x_b^{\,2} - x_c^{\,2}\right) - \left(y_a^{\,2} - y_b^{\,2}\right)\right]$$

$$\Gamma = \frac{1}{2} \times 0.3\cdot\frac{1}{s} \times \left[\left(2^2 - 1^2\right) - \left(2^2 - 1^2\right) + \left(1^2 - 2^2\right) - \left(1^2 - 2^2\right)\right]\cdot m^2$$

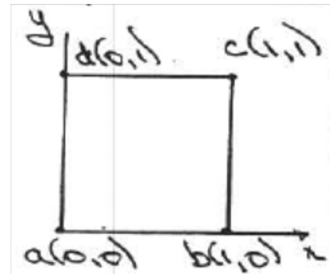
$$\boxed{\Gamma = 0\,\frac{m^2}{s}}$$

This result is to be expected since the flow is irrotational and by Stokes' theorem, the circulation is equal to the curl of the velocity over the bounded area (Eqn. 5.18).

Problem 5.54

[Difficulty: 2]

5.54 Consider the two-dimensional flow field in which $u = Ax^2$ and $v = Bxy$, where $A = 1/2\ \text{ft}^{-1}\cdot\text{s}^{-1}$, $B = -1\ \text{ft}^{-1}\cdot\text{s}^{-1}$, and the coordinates are measured in feet. Show that the velocity field represents a possible incompressible flow. Determine the rotation at point $(x, y) = (1, 1)$. Evaluate the circulation about the "curve" bounded by $y = 0$, $x = 1$, $y = 1$, and $x = 0$.



Given: Two-dimensional flow field

Find:
(a) show that the velocity field represents a possible incompressible flow
(b) Rotation at (x, y) = (1, 1)
(c) Circulation about the unit square shown above

Solution: We will apply the definition of circulation to the given velocity field.

Governing Equations:

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)+\frac{\partial \rho}{\partial t}=0 \quad \text{(Continuity equation)}$$

$$\vec{\omega}=\frac{1}{2}\nabla\times\vec{V} \quad \text{(Definition of rotation)}$$

$$\Gamma=\oint \vec{V}\cdot d\vec{s} \quad \text{(Definition of circulation)}$$

Assumptions:
(1) Steady flow
(2) Incompressible flow
(3) Two dimensional flow (velocity is not a function of z)

Based on the assumptions listed above, the continuity equation reduces to: $\frac{\partial}{\partial x}u+\frac{\partial}{\partial y}v=0$

This is the criterion against which we will check the flow field.

$$\frac{\partial}{\partial x}u+\frac{\partial}{\partial y}v=2A\cdot x+B\cdot x=2\times\frac{1}{2\cdot\text{ft}\cdot\text{s}}\cdot x+\frac{-1}{\text{ft}\cdot\text{s}}\cdot x=0$$

This could be an incompressible flow field.

From the definition of rotation:
$$\vec{\omega}=\frac{1}{2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ Ax^2 & Bxy & 0\end{vmatrix}=\frac{1}{2}By\hat{k}$$

At (x, y) = (1, 1)
$$\vec{\omega}=-0.5\hat{k}\,\frac{\text{rad}}{\text{s}}$$

From the definition of circulation we break up the integral: $\Gamma=\int_{ab}\vec{V}\cdot d\vec{s}+\int_{bc}\vec{V}\cdot d\vec{s}+\int_{cd}\vec{V}\cdot d\vec{s}+\int_{da}\vec{V}\cdot d\vec{s}$

The integrand is equal to: $\vec{V}\cdot d\vec{s}=\left(Ax^2\hat{i}+Bxy\hat{j}\right)\cdot\left(dx\hat{i}+dy\hat{j}\right)=Ax^2dx+Bxydy$ Therefore, the circulation is equal to:

$$\Gamma=\int_{x_a}^{x_b}A\cdot x^2\,dx+\int_{y_b}^{y_c}B\cdot x\cdot y\,dy+\int_{x_c}^{x_d}A\cdot x^2\,dx+\int_{y_d}^{y_a}B\cdot x\cdot y\,dy$$ Evaluating the integrals:

$$\Gamma=\frac{A}{3}\cdot\left(x_b^{\,3}-x_a^{\,3}+x_d^{\,3}-x_c^{\,3}\right)+\frac{B}{2}\left[x_c\cdot\left(y_c^{\,2}-y_b^{\,2}\right)+x_a\cdot\left(y_a^{\,2}-y_d^{\,2}\right)\right]$$ Since $x_a=x_d=0$ and $x_b=x_c$ we can simplify:

$$\Gamma=\frac{B}{2}\cdot x_c\left(y_c^{\,2}-y_b^{\,2}\right)$$ Substituting given values: $$\Gamma=\frac{1}{2}\times\left(-\frac{1}{\text{ft}\cdot\text{s}}\right)\times 1\cdot\text{ft}\times\left(1^2-0^2\right)\cdot\text{ft}^2$$

$$\Gamma=-0.500\cdot\frac{\text{ft}^2}{\text{s}}$$

Problem 5.55

[Difficulty: 3]

5.55 Consider a flow field represented by the stream function $\psi = 3x^5y - 10x^3y^3 + 3xy^5$. Is this a possible two-dimensional incompressible flow? Is the flow irrotational?

Given: Stream function

Find: If the flow is incompressible and irrotational

Solution:

Basic equations: Incompressibility $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$ Irrotationality $\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$

Note: The fact that ψ exists means the flow is incompressible, but we check anyway

$$\psi(x,y) = 3\cdot x^5\cdot y - 10\cdot x^3\cdot y^3 + 3\cdot x\cdot y^5$$

Hence $u(x,y) = \frac{\partial}{\partial y}\psi(x,y) = 3\cdot x^5 - 30\cdot x^3\cdot y^2 + 15\cdot x\cdot y^4$ $v(x,y) = -\frac{\partial}{\partial x}\psi(x,y) = 30\cdot x^2\cdot y^3 - 15\cdot x^4\cdot y - 3\cdot y^5$

For incompressibility

$$\frac{\partial}{\partial x}u(x,y) = 15\cdot x^4 - 90\cdot x^2\cdot y^2 + 15\cdot y^4 \qquad \frac{\partial}{\partial y}v(x,y) = 90\cdot x^2\cdot y^2 - 15\cdot x^4 - 15\cdot y^4$$

Hence $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$ INCOMPRESSIBLE

For irrotationality

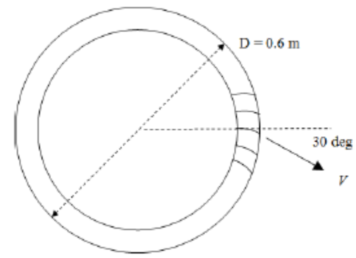
$$\frac{\partial}{\partial x}v(x,y) = 60\cdot x\cdot y^3 - 60\cdot x^3\cdot y \qquad \frac{\partial}{\partial y}u(x,y) = 60\cdot x^3\cdot y - 60\cdot x\cdot y^3$$

Hence $\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$ IRROTATIONAL

Problem 5.56

(Difficulty 2)

5.56 Fluid passes through the set of thin closely space blades at a velocity of 3 $\frac{m}{s}$. Determine the circulation for the flow.



P 5.56

Find: The circulation Γ.

Assumptions: The flow is steady and incompressible

Solution: Use the definition of circulation:

$$\Gamma = \oint_c \vec{V} \cdot d\vec{s} = \int_A \left(\nabla \times \vec{V}\right)_z dA$$

The angular velocity is given as:

$$\omega = \frac{V \sin 30°}{\frac{1}{2}D} = \frac{3\ \frac{m}{s} \times 0.5}{0.5 \times 0.6\ m} = 5\ \frac{1}{s}$$

The angular velocity equals that of the blades at the outer radius:

$$V_\theta = \omega r$$

And the radial component of velocity is

$$V_r = V \cos 30°$$

Thus the cross-product in the integral for circulation is:

$$\left(\nabla \times \vec{V}\right)_z = \left(\frac{1}{r}\frac{\partial r V_\theta}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta}\right) = 2\omega$$

The circulation is then

$$\Gamma = 2\omega\pi r^2 = 2 \times 5\ \frac{1}{s} \times \pi \times (0.3\ m)^2 = 2.83\ \frac{m^2}{s}$$

Problem 5.57

(Difficulty 2)

5.57 A two-dimensional flow field is characterized as $u = Ax^2$ and $v = Bxy$ where $A = \frac{1}{2}\frac{1}{m\cdot s}$ and $B = -1\,\frac{1}{m\cdot s}$, and x and y are in meters. Demonstrate that the velocity field represents a possible incompressible flow field. Determine the rotation at the location $(1,1)$. Evaluate the circulation about the "curve" bounded by $y = 0$, $x = 1$, $y = 1$, and $x = 0$.

Find: Demonstrate the flow is possibly incompressible.

Determine the rotation $\vec{\omega}$ at $(1,1)$.

Evaluate the circulation Γ bounded by the curve.

Assumptions: The flow is steady and incompressible

Solution: Use the continuity equation and the relations for circulation

From the two-dimensional continuity equation, for incompressible flow we have:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

For this velocity profile we have

$$\frac{\partial u}{\partial x} = 2Ax$$

$$\frac{\partial v}{\partial y} = Bx$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 2 \times \frac{1}{2}\frac{1}{m\cdot s}x - 1\,\frac{1}{m\cdot s}x = 0$$

So this is a possible two-dimensional incompressible flow.

The rotation is calculated for this velocity profile as:

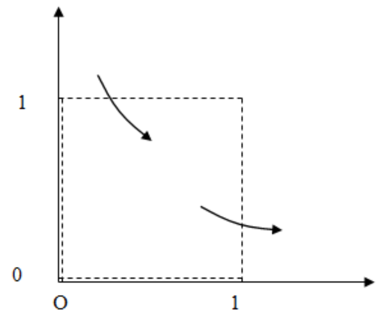
$$\vec{\omega} = \frac{1}{2}\nabla\times\vec{V} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k} = \frac{1}{2}(By - 0)\hat{k} = \frac{1}{2}By\hat{k}$$

So the rotation at the location $(1,1)$ will be:

$$\vec{\omega} = \frac{1}{2}\times\left(-1\,\frac{1}{m\cdot s}\right)\times 1\,m\,\hat{k} = -0.5\,\frac{1}{s}\,\hat{k}$$

$$\omega_z = -0.5\ \frac{1}{s}$$

The flow and the curve y=0, x=1, y=1, and x = 0 is shown in the figure:



The circulation about the curve is:

$$\Gamma = \oint_c \vec{V} \cdot d\vec{s} = \int_A \left(\nabla \times \vec{V}\right)_z dA = 2\omega_z A = 2 \times \left(-0.5\ \frac{1}{s}\right) \times 1\ m^2 = -1\ \frac{m^2}{s}$$

Problem 5.58

(Difficulty 1)

5.58 A flow field is represented by the stream function $\Psi = x^4 - 2x^3y + 2xy^3 - y^4$. Is this a possible two-dimensional flow? Is this flow irrotational?

Find: Whether the flow field is two-dimensional and whether the flow is irrotational.

Assumptions: The flow is steady and incompressible

Solution: For the flow to be two-dimensional flow, it needs to satisfy the continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

The velocities u and v are

$$u = \frac{\partial \Psi}{\partial y} = -2x^3 + 6xy^2 - 4y^3$$

$$\frac{\partial u}{\partial x} = -6x^2 + 6y^2$$

$$v = -\frac{\partial \Psi}{\partial x} = -4x^3 + 6x^2y - 2y^3$$

$$\frac{\partial v}{\partial y} = 6x^2 - 6y^2$$

Using the continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -6x^2 + 6y^2 + 6x^2 - 6y^2 = 0$$

So this flow is possible two-dimensional flow.

For the vorticity we have:

$$\vec{\xi} = \nabla \times \vec{V}$$

$$\nabla \times \vec{V} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k} = -12x^2 + 12xy - 12xy + 12y^2\hat{k} = (-12x^2 + 12y^2)\hat{k} \neq 0$$

So this flow is not irrotational.

Problem 5.59

[Difficulty: 2]

5.59 Consider a velocity field for motion parallel to the x axis with constant shear. The shear rate is $du/dy = A$, where $A = 0.1\ \text{s}^{-1}$. Obtain an expression for the velocity field, $\vec{V}$. Calculate the rate of rotation. Evaluate the stream function for this flow field.

Given: Velocity field for motion in the x-direction with constant shear

Find: (a) Expression for the velocity field
(b) Rate of rotation
(c) Stream function

Solution: We will apply the definition of circulation to the given velocity field.

Governing Equations:

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)+\frac{\partial \rho}{\partial t}=0 \quad \text{(Continuity equation)}$$

$$\vec{\omega}=\frac{1}{2}\nabla\times\vec{V} \quad \text{(Definition of rotation)}$$

Assumptions: (1) Steady flow
(2) Incompressible flow

The x-component of velocity is: $u = \int A\,dy + f(x) = Ay + f(x)$ Since flow is parallel to the x-axis: $\vec{V}=\left[Ay+f(x)\right]\hat{i}$

From the definition of rotation: $$\vec{\omega}=\frac{1}{2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ Ay+f(x) & 0 & 0\end{vmatrix}=-\frac{1}{2}A\hat{k}$$ $$\vec{\omega}=-0.05\hat{k}\,\frac{\text{rad}}{\text{s}}$$

From the definition of the stream function $\psi = \int u\,dy + g(x) = \int (A\cdot y + f(x))\,dy + g(x) = \frac{1}{2}\cdot A\cdot y^2 + f(x)\cdot y + g(x)$

$v = -\frac{\partial}{\partial x}\psi = -\frac{d}{dx}f(x)\cdot y - \frac{d}{dx}g(x) = 0$ Therefore, the derivatives of both f and g are zero, and thus f and g are constants:

$$\psi = \frac{1}{2}\cdot A\cdot y^2 + c_1\cdot y + c_2$$

Problem 5.60 [Difficulty: 2]

5.60 Consider the flow field represented by the stream function $\psi = Axy + Ay^2$, where $A = 1\ \text{s}^{-1}$. Show that this represents a possible incompressible flow field. Evaluate the rotation of the flow. Plot a few streamlines in the upper half plane.

Given: Flow field represented by a stream function.

Find: (a) Show that this represents an incompressible velocity field
(b) the rotation of the flow
(c) Plot several streamlines in the upper half plane

Solution: We will apply the definition of rotation to the given velocity field.

Governing Equation:
$$\vec{\omega} = \frac{1}{2}\nabla \times \vec{V}$$ (Definition of rotation)

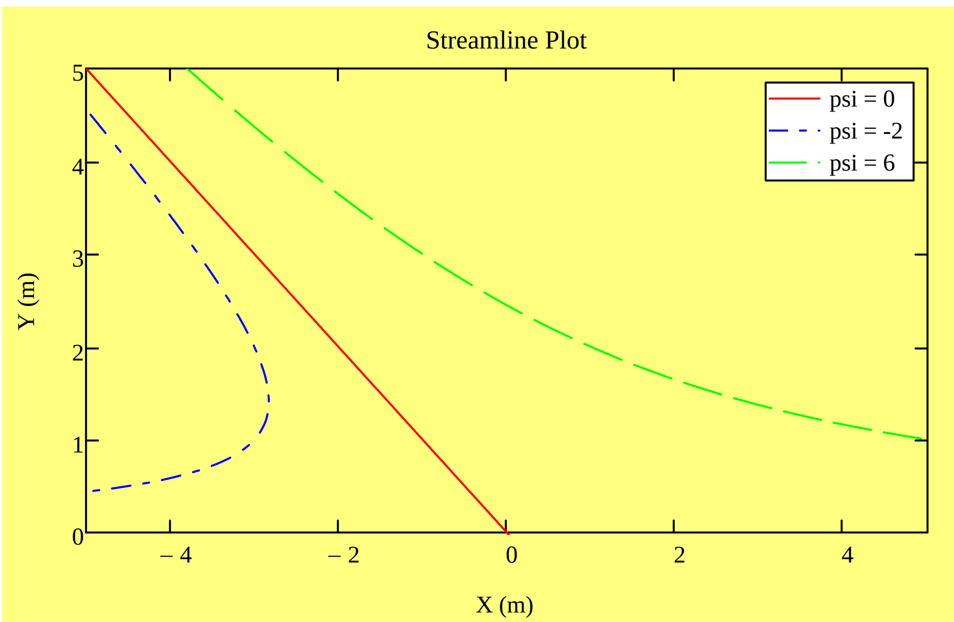
Assumptions: (1) Steady flow
(2) Incompressible flow

From the definition of the stream function: $u = \frac{\partial}{\partial y}\psi = A \cdot x + 2 \cdot A \cdot y$ $\quad v = -\frac{\partial}{\partial x}\psi = -A \cdot y$ Applying the continuity equation:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = A - A = 0$$ This could be an incompressible flow field

From the definition of rotation:
$$\vec{\omega} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A(x+2y) & -Ay & 0 \end{vmatrix} = \frac{1}{2}(-2A)\hat{k} = -A\hat{k}$$
$$\vec{\omega} = -A\hat{k}$$

The streamlines are curves where the stream function is constant, i.e., $\psi = \text{constant}$ Here is a plot of streamlines:



Problem 5.61

[Difficulty: 2]

5.61 Consider the velocity field given by $\vec{V} = Ax^2\hat{i} + Bxy\hat{j}$, where $A = 1\ \text{ft}^{-1}\cdot\text{s}^{-1}$, $B = -2\ \text{ft}^{-1}\cdot\text{s}^{-1}$, and the coordinates are measured in feet.

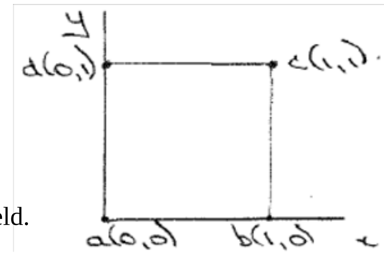
(a) Determine the fluid rotation.

(b) Evaluate the circulation about the "curve" bounded by $y = 0$, $x = 1$, $y = 1$, and $x = 0$.

(c) Obtain an expression for the stream function.

(d) Plot several streamlines in the first quadrant.

Given: Flow field represented by a velocity function.



Find:
(a) Fluid rotation
(b) Circulation about the curve shown
(c) Stream function
(d) Plot several streamlines in first quadrant

Solution: We will apply the definition of rotation and circulation to the given velocity field.

Governing Equation:

$$\vec{\omega} = \frac{1}{2}\nabla\times\vec{V} \qquad \text{(Definition of rotation)}$$

$$\Gamma = \oint \vec{V}\cdot d\vec{s} \qquad \text{(Definition of circulation)}$$

Assumption: Steady flow

From the definition of rotation:
$$\vec{\omega} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ Ax^2 & Bxy & 0 \end{vmatrix} = \frac{1}{2}(By)\hat{k}$$

$$\vec{\omega} = -y\hat{k}\,\frac{\text{rad}}{\text{ft}\cdot\text{s}}$$

From the definition of circulation we break up the integral: $\Gamma = \int_{ab}\vec{V}\cdot d\vec{s} + \int_{bc}\vec{V}\cdot d\vec{s} + \int_{cd}\vec{V}\cdot d\vec{s} + \int_{da}\vec{V}\cdot d\vec{s}$

The integrand is equal to: $\vec{V}\cdot d\vec{s} = \left(Ax^2\hat{i} + Bxy\hat{j}\right)\cdot\left(dx\hat{i} + dy\hat{j}\right) = Ax^2dx + Bxydy$ Therefore, the circulation is equal to:

$$\Gamma = \int_{x_a}^{x_b} A\cdot x^2\,dx + \int_{y_b}^{y_c} B\cdot x\cdot y\,dy + \int_{x_c}^{x_d} A\cdot x^2\,dx + \int_{y_d}^{y_a} B\cdot x\cdot y\,dy$$ Evaluating the integral:

$$\Gamma = \frac{A}{3}\cdot\left(x_b^{\,3} - x_a^{\,3} + x_d^{\,3} - x_c^{\,3}\right) + \frac{B}{2}\left[x_c\cdot\left(y_c^{\,2} - y_b^{\,2}\right) + x_a\cdot\left(y_a^{\,2} - y_d^{\,2}\right)\right]$$ Since $x_a = x_d = 0$ and $x_b = x_c$ we can simplify:

$$\Gamma = \frac{B}{2}\cdot x_c\cdot\left(y_c^{\,2} - y_b^{\,2}\right)$$ Substituting given values: $\Gamma = \frac{1}{2}\times -\frac{2}{\text{ft}\cdot\text{s}}\times 1\cdot\text{ft}\times\left(1^2 - 0^2\right)\cdot\text{ft}^2$

$$\Gamma = -1.000\cdot\frac{\text{ft}^2}{\text{s}}$$

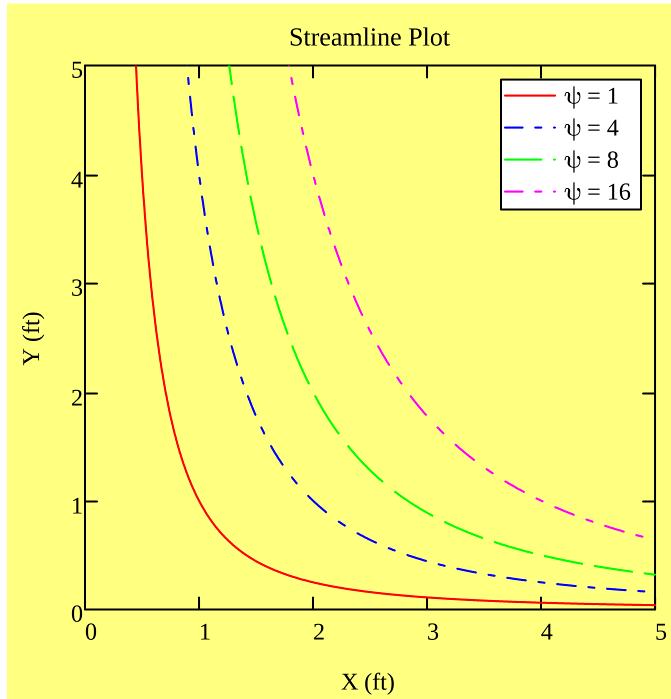
From the definition of the stream function: $u = \frac{\partial}{\partial y}\psi \qquad \psi = \int u\,dy + f(x) = \int A\cdot x^2\,dy + f(x) = A\cdot x^2\cdot y + f(x)$

In addition, $v = -\frac{\partial}{\partial x}\psi \qquad \psi = -\int v\,dx + g(y) = -\int B\cdot x\cdot y\,dx + g(y) = -\frac{B}{2}\cdot x^2\cdot y + g(y)$ Comparing the two stream functions:

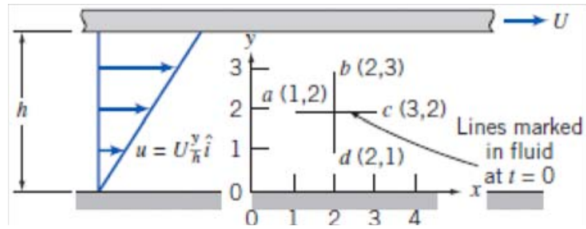
$\frac{1}{\text{ft}\cdot\text{s}}\cdot x^2\cdot y + f(x) = \frac{1}{\text{ft}\cdot\text{s}}\cdot x^2\cdot y + g(y)$ Thus, $f = g = \text{constant}$ Taking f(x) = 0:

$$\psi = A\cdot x^2\cdot y$$

The streamlines are curves where the stream function is constant, i.e., $\psi = \text{constant}$ Here is a plot of streamlines:



5.62 Consider again the viscometric flow of Example 5.7. Evaluate the average rate of rotation of a pair of perpendicular line segments oriented at ±45° from the x axis. Show that this is the same as in the example.

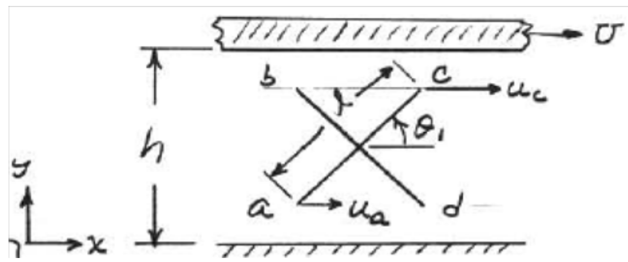


Given: Viscometric flow of Example 5.7, V = U(y/h)i, where U = 4 mm/s and h = 4 mm

Find: (a) Average rate of rotation of two line segments at +/- 45 degrees
(b) Show that this is the same as in the example

Solution: We will apply the definition of rotation to the given velocity field.

Governing Equation: $\vec{\omega} = \frac{1}{2}\nabla \times \vec{V}$ (Definition of rotation)

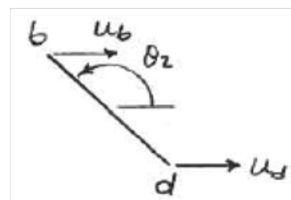


Assumptions: (1) Steady flow
(2) Incompressible flow

Considering the lines shown: $u_c = u_a + \frac{\partial}{\partial y}u\cdot\left(l\cdot\sin(\theta_1)\right)$

$-\omega_{ac} = \frac{(u_c - u_a)\cdot\sin(\theta_1)}{l}$ since the component normal to l is $u\cdot\sin(\theta_1)$

$-\omega_{ac} = \frac{\partial}{\partial y}u\cdot\frac{\left(l\cdot\sin(\theta_1)\right)\cdot\sin(\theta_1)}{l} = \frac{\partial}{\partial y}u\cdot\left(\sin(\theta_1)\right)^2 = \frac{U}{h}\cdot\left(\sin(\theta_1)\right)^2$ Now consider this sketch:



$u_b = u_d + \frac{\partial}{\partial y}u\cdot\left(l\cdot\sin(\theta_2)\right)$ $\quad -\omega_{bd} = \frac{(u_d - u_b)\cdot\sin(\theta_2)}{l}$ since the component normal to l is $u\cdot\sin(\theta_2)$

$-\omega_{bd} = \frac{\partial}{\partial y}u\cdot\frac{\left(l\cdot\sin(\theta_2)\right)\cdot\sin(\theta_2)}{l} = \frac{\partial}{\partial y}u\cdot\left(\sin(\theta_2)\right)^2 = \frac{U}{h}\cdot\left(\sin(\theta_2)\right)^2$ Now we sum these terms:

$\omega = \frac{1}{2}\cdot(\omega_{ac} + \omega_{bd}) = -\frac{1}{2}\cdot\frac{U}{h}\cdot\left[\left(\sin(\theta_1)\right)^2 + \left(\sin(\theta_2)\right)^2\right]$ When $\theta_1 = 45\cdot\text{deg}$ and $\theta_2 = 135\cdot\text{deg}$

$$\omega = -\frac{1}{2}\cdot\frac{U}{h}\cdot\left[\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2\right]$$

$$\omega = -\frac{1}{2}\cdot\frac{U}{h}$$

Substituting for U and h: $\omega = -\frac{1}{2} \times 4\cdot\frac{mm}{s} \times \frac{1}{4\cdot mm}$

$$\omega = -0.5\frac{1}{s}$$

Problem 5.63 [Difficulty: 3]

5.63 The velocity field near the core of a tornado can be approximated as

$$\vec{V} = -\frac{q}{2\pi r}\hat{e}_r + \frac{K}{2\pi r}\hat{e}_\theta$$

Is this an irrotational flow field? Obtain the stream function for this flow.

Given: Velocity field approximation for the core of a tornado

Find: (a) Whether or not this is an irrotational flow
(b) Stream function for the flow

Solution: We will apply the definition of rotation to the given velocity field.

Governing Equation:

$$\vec{\omega} = \frac{1}{2}\nabla \times \vec{V} \qquad \text{(Definition of rotation)}$$

$$\vec{\nabla} = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z} \qquad \text{(Definition of "del" operator)}$$

$$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta \qquad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r \qquad \text{(Hints from text)}$$

Assumptions: (1) Steady flow
(2) Two-dimensional flow (no z velocity, velocity is not a function of θ or z)

From the definition of rotation: $\vec{\omega} = \frac{1}{2}\left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(V_r \hat{e}_r + V_\theta \hat{e}_\theta\right)$ Employing assumption (2) yields:

$$\vec{\omega} = \frac{1}{2}\left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\right) \times \left(V_r \hat{e}_r + V_\theta \hat{e}_\theta\right) = \frac{1}{2}\left[\hat{e}_r \times \left(\hat{e}_r \frac{\partial V_r}{\partial r} + \hat{e}_\theta \frac{\partial V_\theta}{\partial r}\right) + \hat{e}_\theta \frac{1}{r} \times \frac{\partial}{\partial \theta}\left(V_r \hat{e}_r + V_\theta \hat{e}_\theta\right)\right]$$ From product rule:

$$\vec{\omega} = \frac{1}{2}\left[\left(\hat{e}_r \times \hat{e}_r\right)\frac{\partial V_r}{\partial r} + \left(\hat{e}_r \times \hat{e}_\theta\right)\frac{\partial V_\theta}{\partial r} + \hat{e}_\theta \frac{1}{r} \times \left(\hat{e}_r \frac{\partial V_r}{\partial \theta} + V_r \frac{\partial \hat{e}_r}{\partial \theta} + \hat{e}_\theta \frac{\partial V_\theta}{\partial \theta} + V_\theta \frac{\partial \hat{e}_\theta}{\partial \theta}\right)\right]$$ Using the hints from the text:

$$\vec{\omega} = \frac{1}{2}\left[\left(\hat{e}_r \times \hat{e}_r\right)\frac{\partial V_r}{\partial r} + \left(\hat{e}_r \times \hat{e}_\theta\right)\left(\frac{\partial V_\theta}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta} + \frac{V_\theta}{r}\right) + \left(\hat{e}_\theta \times \hat{e}_\theta\right)\left(\frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r}{r}\right)\right] = \frac{1}{2}\left(\frac{\partial V_\theta}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta} + \frac{V_\theta}{r}\right)\hat{k}$$

Since V is only a function of r: $\vec{\omega} = \frac{1}{2}\left(\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\right)\hat{k} = \frac{1}{2}\left(-\frac{K}{2\pi r^2} + \frac{K}{2\pi r^2}\right)\hat{k} = \vec{0}$

Flow is irrotational.

To build the stream function: $V_r = \frac{1}{r}\cdot\frac{\partial}{\partial \theta}\psi \qquad \psi = \int r \cdot V_r \, d\theta + f(r) = -\int \frac{q}{2\cdot\pi} \, d\theta + f(r) = -\frac{q\cdot\theta}{2\cdot\pi} + f(r)$

$$V_\theta = -\frac{\partial}{\partial r}\psi \qquad \psi = -\int V_\theta \, dr + g(\theta) = -\int \frac{K}{2\cdot\pi\cdot r} \, dr + g(\theta) = -\frac{K}{2\cdot\pi}\cdot\ln(r) + g(\theta)$$ Comparing these two expressions:

$$-\frac{q\cdot\theta}{2\cdot\pi} + f(r) = -\frac{K}{2\cdot\pi}\cdot\ln(r) + g(\theta) \qquad f(r) = -\frac{K}{2\cdot\pi}\cdot\ln(r)$$

$$\psi = -\frac{K}{2\cdot\pi}\cdot\ln(r) - \frac{q\cdot\theta}{2\cdot\pi}$$

Problem 5.64

(Difficulty 1)

5.64 A velocity field is given by $\vec{V} = 2\hat{\imath} - 4x\hat{\jmath}\ \frac{m}{s}$. Determine an equation for the streamline. Calculate the vorticity of the flow.

Find: The streamline Ψ and vorticity $\vec{\xi}$.

Assumptions: The flow is steady and incompressible

Solution: Apply the relations for streamlines and vorticity

The velocity field is:

$$u = 2\ \frac{m}{s}$$

$$v = 4x\ \frac{m}{s}$$

The x-component of the velocity is given in terms of the stream function as:

$$u = \frac{\partial \Psi}{\partial y} = 2$$

Integrating both sides we get:

$$\Psi = 2y + f(x)$$

The y-component of velocity is given by

$$v = -\frac{\partial \Psi}{\partial x} = -\frac{df(x)}{dx} = -4x$$

$$\frac{df(x)}{dx} = 4x$$

$$f(x) = 2x^2 + c$$

So the stream function is:

$$\Psi = 2y + 2x^2 + c$$

where c is a constant.

The vorticity of the flow is calculated as:

$$\vec{\xi} = \nabla \times \vec{V} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k} = (-4 - 0)\ \hat{k} = -4\ \hat{k}$$

Problem 5.65

[Difficulty: 3]

5.65 Consider the pressure-driven flow between stationary parallel plates separated by distance $2b$. Coordinate y is measured from the channel centerline. The velocity field is given by $u = u_{max}[1 - (y/b)^2]$. Evaluate the rates of linear and angular deformation. Obtain an expression for the vorticity vector, $\vec{\zeta}$. Find the location where the vorticity is a maximum.

Given: Velocity field for pressure-driven flow between stationary parallel plates

Find: (a) Rates of linear and angjular deformation for this flow
(b) Expression for the vorticity vector
(c) Location of maximum vorticity

Solution: We will apply the definition of vorticity to the given velocity field.

Governing Equation: $\vec{\zeta} = \nabla \times \vec{V}$ (Definition of vorticity)

Assumptions: (1) Steady flow

The volume dilation rate of the flow is: $\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

Rates of linear deformation in all three directions is zero.

The angular deformations are: x-y plane: $\frac{\partial}{\partial x}v + \frac{\partial}{\partial y}u = -u_{max} \cdot \frac{2 \cdot y}{b^2}$

y-z plane: $\frac{\partial}{\partial y}w + \frac{\partial}{\partial z}v = 0$

z-x plane: $\frac{\partial}{\partial z}u + \frac{\partial}{\partial x}w = 0$

$$angdef = -u_{max} \cdot \frac{2 \cdot y}{b^2}$$

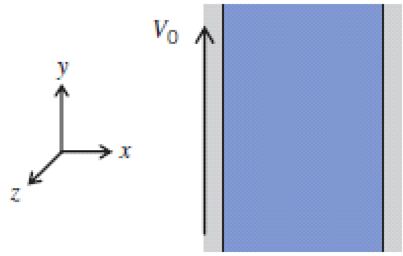
The vorticity is: $$\vec{\zeta} = \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_{max}\left[1 - \left(\frac{y}{b}\right)^2\right] & 0 & 0 \end{vmatrix} = u_{max}\frac{2y}{b^2}\hat{k}$$

$$\vec{\zeta} = u_{max}\frac{2y}{b^2}\hat{k}$$

The vorticity is a maximum at y=b and y=-b

Problem 5.66 [Difficulty: 2]

5.66 Consider a steady, laminar, fully developed, incompressible flow between two infinite plates, as shown. The flow is due to the motion of the left plate as well a pressure gradient that is applied in the y direction. Given the conditions that $\vec{V} \neq \vec{V}(z)$, $w = 0$, and that gravity points in the negative y direction, prove that $u = 0$ and that the pressure gradient in the y direction must be constant.



Given: Flow between infinite plates

Find: Prove that $u = 0$, dP/dy = constant

Solution:

Governing Equations:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad \text{(Continuity Equation)}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial P}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \qquad \text{(Navier-Stokes Equations)}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

Assumptions: Incompressible fluid
No motion along the wall ($x = 0$) limited to two dimensions ($w = 0$).

Prove that u = 0:

Given that $\vec{V} \neq \vec{V}(z)$ this means that $\dfrac{\partial u}{\partial z}=\dfrac{\partial v}{\partial z}=\dfrac{\partial w}{\partial z}=0$

Also given that the flow is fully developed which means that $\vec{V} \neq \vec{V}(y)$ so that $\dfrac{\partial u}{\partial y}=\dfrac{\partial v}{\partial y}=\dfrac{\partial w}{\partial y}=0$

And steady flow implies that $\vec{V} \neq \vec{V}(t)$

The continuity equation becomes $\dfrac{\partial u}{\partial x}=0$, but because $u \neq u(y,z,t)$ then $u = u(x)$ meaning that the partial derivative here becomes an ordinary derivative: $\dfrac{du}{dx}=0$

Integrating the ordinary derivative gives: $u = \text{constant}$

By the no-slip boundary condition u = 0 at the surface of either plate meaning the constant must be zero.

Hence: $u = 0$

Prove that $\frac{\partial P}{\partial y} = \text{constant}$ **:**

Due to the fact that u = 0, and gravity is in the negative y-direction the x-component of the Navier-Stokes Equation becomes:

$$\frac{\partial P}{\partial x} = 0 \quad \text{hence} \quad P \neq P(x)$$

Due to the fact that w = 0, and gravity is in the negative y-direction the z-component of the Navier-Stokes Equation becomes:

$$\frac{\partial P}{\partial z} = 0 \quad \text{hence} \quad P \neq P(z)$$

The y-component of the Navier-Stokes Equation reduces to:

$$0 = -\frac{\partial P}{\partial y} - \rho g + \mu\left(\frac{\partial^2 v}{\partial x^2}\right)$$

So then

$$\frac{\partial P}{\partial y} = -\rho g + \mu\left(\frac{\partial^2 v}{\partial x^2}\right) \quad [1]$$

It has been shown that $P \neq P(x,z)$ and because the flow is steady $P \neq P(t)$ meaning that $P = P(y)$. This means that the left hand side of [1] can only be a function of y or a constant. Additionally, by the fully developed, steady flow, and $\vec{V} \neq \vec{V}(z)$ conditions it is shown that $v = v(x)$. For this reason the right hand side of [1] can only be a function or x or a constant.

Mathematically speaking it is impossible for: $f(y) = g(x)$ so each side of [1] must be a constant.

Hence, $\frac{\partial P}{\partial y} = \text{constant}$

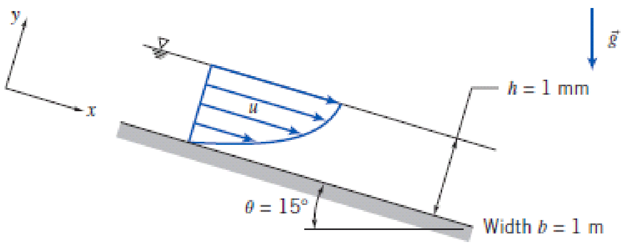
5.67 Assume the liquid film in Example 5.9 is not isothermal, but instead has the following distribution:

$$T(y) = T_0 + (T_w - T_0)\left(1 - \frac{y}{h}\right)$$

where T_0 and T_w are, respectively, the ambient temperature and the wall temperature. The fluid viscosity decreases with increasing temperature and is assumed to be described by

$$\mu = \frac{\mu_0}{1 + a(T - T_0)}$$

with $a > 0$. In a manner similar to Example 5.9, derive an expression for the velocity profile.



Given: temperature profile and temperature-dependent viscosity expression

Find: Velocity Profile

Solution:

Governing Equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{(Continuity Equation)}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho g_y - \frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \qquad \text{(Navier-Stokes Equations)}$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho g_z - \frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

Assumptions: Incompressible fluid

Similar to the Example 5.9, the x-component momentum equation can be simplified to

$$\frac{d\tau_{yx}}{dy} = -\rho g \sin\theta \qquad (1)$$

Integrating once, one has

$$\tau_{yx} = -\rho g y \sin\theta + C_1 \qquad (2)$$

Using the boundary condition: $\tau_{yx}(y = h) = 0$

$$c_1 = \rho g h \sin\theta \qquad (3)$$

Substituting c_1 into eq. (2),

$$\tau_{yx} = \mu\frac{du}{dy} = \rho g(h - y)\sin\theta \qquad (4)$$

Here, the fluid viscosity depends on the temperature,

$$\mu = \frac{\mu_0}{1 + a(T_w - T_0)(1 - y/h)} \qquad (5)$$

Substituting equation (5) into equation (4), we have

$$\frac{du}{dy} = \frac{\rho g h(1 - y/h)\sin\theta}{\mu_0}(1 + a(T_w - T_0)(1 - y/h)) \qquad (6)$$

Integrating equation (6) once

$$u = \frac{\rho g h \sin\theta}{\mu_0}(y(1 - \frac{y}{2h}) + a(T_w - T_0)y(1 - \frac{y}{h} + \frac{y^2}{3h^2})) + C_2 \qquad (7)$$

At y=0, u=0: c_2=0.

Substituting c_2=0 into eq. (7), one obtains

$$u = \frac{\rho g h \sin\theta}{\mu_0}(y(1 - \frac{y}{2h}) + a(T_w - T_0)y(1 - \frac{y}{h} + \frac{y^2}{3h^2})) \qquad (8)$$

When *a*=0, eq. (8) can be simplified to

$u = \frac{\rho g h \sin\theta}{\mu_0} y(1 - \frac{y}{2h})$, and it is exactly the same velocity profile in Example 5.9.

Problem 5.68

(Difficulty 2)

5.68 Consider a steady, laminar, fully developed incompressible flow between two infinite parallel plates as shown. The flow is due to a pressure gradient applied in the x-direction. Given that $\vec{V} \neq \vec{V}(z)$, $w = 0$ and that gravity points in the negative y direction, prove that $v = 0$ and that the pressure gradients in the x- and y-directions are constant.

P 5.68

Find: Prove that $v = 0$ and pressure gradient in the x- and y-directions are constant.

Assumptions: The flow is steady and incompressible

Solution:

For 2D incompressible steady flow we have the following governing equations:

Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Momentum equation for the x, y, and z directions

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\rho g - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

For this steady fully developed flow, we have:

$$w = 0$$

$$\vec{V} \neq \vec{V}(z)$$

$$\frac{\partial u}{\partial x} = 0$$

From the continuity equation:

$$\frac{\partial v}{\partial y} = 0$$

$$v = c$$

We know that $v = 0$ at the plate, so

$$c = 0$$

So we get:

$$v = 0$$

From the z component momentum equation, because the velocity w and its derivatives are zero:

$$\frac{\partial p}{\partial z} = 0$$

Thus from the y component momentum equation:

$$\rho(u \times 0 + 0 \times 0 + 0 \times 0) = -\rho g - \frac{\partial p}{\partial y} + \mu(0 + 0 + 0)$$

$$0 = -\rho g - \frac{\partial p}{\partial y}$$

$$\frac{\partial p}{\partial y} = -\rho g$$

For incompressible the density ρ is constant, so

$$\frac{\partial p}{\partial y} = constant$$

From the x component momentum equation:

$$\rho\left(u \times 0 + 0 \times \frac{\partial u}{\partial y} + 0 \times 0\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + 0 + 0\right)$$

$$0 = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial p}{\partial x} = \mu\left(\frac{\partial^2 u}{\partial y^2}\right)$$

Because u is not a function of x or z, so $\frac{\partial p}{\partial x}$ is also not a function of x and z. However, $\frac{\partial p}{\partial x}$ could be a function of y. Also we already have:

$$\frac{\partial p}{\partial y} = constant$$

Differentiating $\frac{\partial p}{\partial x}$ with respect to y and interchanging the order of differentiation

$$\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right) = 0$$

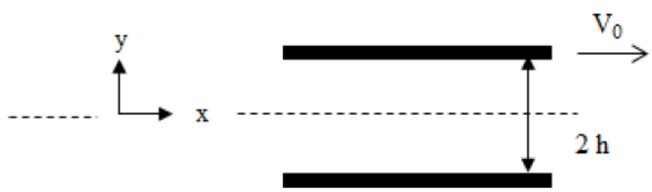
Therefore $\frac{\partial p}{\partial x}$ does not vary with y. Then $\frac{\partial p}{\partial x}$ is not a function of $x, y,\ or\ z$. So

$$\frac{\partial p}{\partial x} = constant.$$

Problem 5.69

(Difficulty 3)

5.69 Consider a steady, laminar, fully developed incompressible flow between two infinite parallel plates separated by a distance $2h$ as shown below. The top plate moves with a velocity V_0. Derive an expression for the velocity profile. Determine the pressure gradient for which the flow rate is zero. Plot the profile for that condition.



P 5.69

Find: The expression for the velocity profile. Determine the pressure gradient for which the flow rate is zero and plot the velocity profile.

Assumptions: The flow is steady and incompressible. The effect of gravity is neglected or gravity acts in the z-direction

Solution:

For 2D incompressible steady flow we have the following governing equations:

Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Momentum equations in the x and y directions

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \rho g - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

This flow is steady and fully developed and the vertical velocity v is zero at the walls. Thus we have:

$$v = 0$$

$$\frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} = 0$$

Neglecting the body force due to gravity in the y-direction, we have from the y-momentum equation:

$$\rho(u \times 0 + 0 \times 0) = 0 - \frac{\partial p}{\partial y} + \mu(0 + 0)$$

$$\frac{\partial p}{\partial y} = 0$$

The pressure p is not a function of y and can only be a function of y.

We have from the x-momentum equation:

$$\rho\left(u \times 0 + 0 \times \frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + 0\right)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{dp}{dx}$$

Where we can use the total differential *d* since the velocity and pressure only vary with x.

$$\frac{du}{dy} = \frac{1}{\mu}\frac{dp}{dx}y + C_1$$

Integrating with respect to x

$$u = \frac{1}{2\mu}\frac{dp}{dx}y^2 + C_1 y + C_2$$

Applying the boundary condition:

$$u = 0 \text{ at } y = -h$$

$$\frac{1}{2\mu}\frac{dp}{dx}h^2 - C_1 h + C_2 = 0$$

$$u = V_0 \text{ at } y = h$$

$$\frac{1}{2\mu}\frac{dp}{dx}h^2 + C_1 h + C_2 = V_0$$

So we have:

$$2C_1 h = V_0$$

$$C_1 = \frac{V_0}{2h}$$

$$2C_2 = V_0 - \frac{1}{\mu}\frac{dp}{dx}h^2$$

$$C_2 = \frac{V_0}{2} - \frac{1}{2\mu}\frac{dp}{dx}h^2$$

The velocity profile is then

$$u = \frac{1}{2\mu}\frac{dp}{dx}y^2 + \frac{V_0}{2h}y + \frac{V_0}{2} - \frac{1}{2\mu}\frac{dp}{dx}h^2$$

or

$$u = \frac{h^2}{2\mu}\frac{dp}{dx}\left(\left(\frac{y}{h}\right)^2 - 1\right) + \frac{V_0}{2}\left(\frac{y}{h} + 1\right)$$

The flow rate per unit depth is:

$$Q = \int_{-h}^{h} u dy = \int_{-1}^{1} u\, h\, d\left(\frac{y}{h}\right) = \int_{-1}^{1} h\left[\frac{h^2}{2\mu}\frac{dp}{dx}(z^2 - 1) + \frac{V_0}{h}(z+1)\right] dz \quad \text{where } z = \frac{y}{h}$$

$$Q = \left[\left(\frac{h^3}{2\mu}\frac{dp}{dx}\right)\left(\frac{z^3}{3} - z\right) + \frac{V_0}{2}\left(\frac{z^2}{2} + z\right)\right]_{-1}^{1}$$

$$Q = \left[\left(\frac{h^3}{2\mu}\frac{dp}{dx}\right)\left(\frac{z^3}{3} - z\right) + \frac{V_0}{2}\left(\frac{z^2}{2} + z\right)\right]_{-1}^{1}$$

$$Q = \frac{2h^3}{3\mu}\left(-\frac{dp}{dx}\right) + V_0 h = 0$$

Now, the flow will be zero when

$$\frac{2h^3}{3\mu}\frac{dp}{dx} = V_0 h$$

or

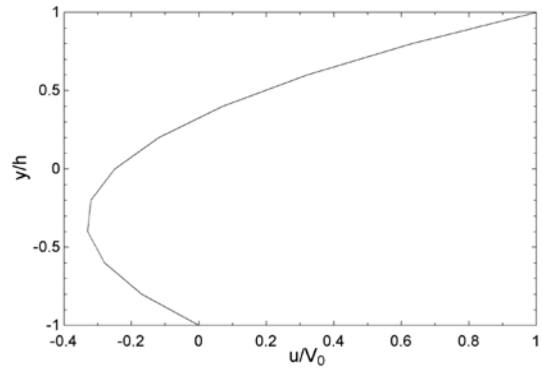
$$\frac{dp}{dx} = \frac{3\mu V_0}{2h^2}$$

Substituting this value of $\frac{dp}{dx}$ into the velocity equation, we have for the velocity profile:

$$u = \frac{h^2}{2\mu}\frac{3\mu V_0}{2h^2}\left(\left(\frac{y}{h}\right)^2 - 1\right) + \frac{V_0}{2}\left(\frac{y}{h} + 1\right)$$

$$u = \frac{V_0}{4}\left(3\left(\frac{y}{h}\right)^2 + 2\left(\frac{y}{h}\right) - 1\right)$$

We can plot the velocity profile as

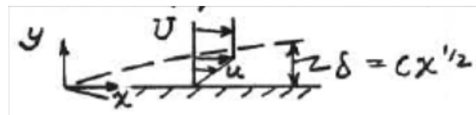


Note the velocity is u = 0 at h = -h and u = V_0 at y = h, which are the boundary conditions.

Problem 5.70 [Difficulty: 3]

5.70 A linear velocity profile was used to model flow in a laminar incompressible boundary layer in Problem 5.10. Express the rotation of a fluid particle. Locate the maximum rate of rotation. Express the rate of angular deformation for a fluid particle. Locate the maximum rate of angular deformation. Express the rates of linear deformation for a fluid particle. Locate the maximum rates of linear deformation. Express the shear force per unit volume in the x direction. Locate the maximum shear force per unit volume; interpret this result.

Given: Linear approximation for velocity profile in laminar boundary layer



Find:
(a) Express rotation, find maximum
(b) Express angular deformation, locate maximum
(c) Express linear deformation, locate maximum
(d) Express shear force per unit volume, locate maximum

Solution: We will apply the definition of rotation to the given velocity field.

Governing Equation: $\vec{\omega} = \frac{1}{2}\nabla \times \vec{V}$ (Definition of rotation)

Assumptions: (1) Steady flow

The rotation is: $\vec{\omega} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ U\frac{y}{\delta} & \frac{U}{4}\frac{y}{x}\frac{y}{\delta} & 0 \end{vmatrix} = \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{U}{4}\frac{y}{x}\frac{y}{\delta}\right) - \frac{\partial}{\partial y}\left(U\frac{y}{\delta}\right)\right]\hat{k}$ Computing the partial derivatives:

$$\omega_z = -\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{2}\cdot\frac{U\cdot y^2}{c\cdot x^{\frac{5}{2}}} - \frac{1}{2}\cdot\frac{U}{c\cdot x^{\frac{1}{2}}} = -\frac{1}{2}\cdot\left(\frac{3}{8}\cdot\frac{U\cdot y^2}{c\cdot x^{\frac{5}{2}}} + \frac{U}{c\cdot x^{\frac{1}{2}}}\right) = -\frac{U}{2\cdot c\cdot x^{\frac{1}{2}}}\cdot\left[1 + \frac{3}{8}\cdot\left(\frac{y}{x}\right)^2\right]$$

$$\omega_z = -\frac{U}{2\cdot\delta}\cdot\left[1 + \frac{3}{8}\cdot\left(\frac{y}{x}\right)^2\right]$$

Maximum value at $y = \delta$

The angular deformation is: $\frac{\partial}{\partial x}v + \frac{\partial}{\partial y}u = \frac{\partial}{\partial x}\left(\frac{U}{4}\cdot\frac{y^2}{c\cdot x^{\frac{3}{2}}}\right) + \frac{\partial}{\partial y}\left(U\cdot\frac{y}{c\cdot x^{\frac{1}{2}}}\right) = \frac{U}{c}\left(-\frac{1}{4}\cdot\frac{3}{2}\cdot\frac{y^2}{x^{\frac{5}{2}}} + \frac{1}{x^{\frac{1}{2}}}\right) = \frac{U}{c\cdot x^{\frac{1}{2}}}\cdot\left[1 - \frac{3}{8}\cdot\left(\frac{y}{x}\right)^2\right]$

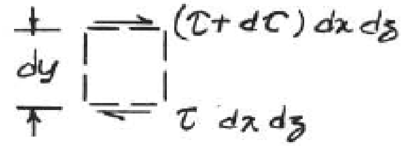
$$angdef = \frac{U}{\delta}\cdot\left[1 - \frac{3}{8}\cdot\left(\frac{y}{x}\right)^2\right]$$

Maximum value at $y = 0$

Linear deformation: $\frac{\partial}{\partial x}u = \frac{\partial}{\partial x}\left(U\cdot\frac{y}{c\cdot x^{\frac{1}{2}}}\right) = -\frac{1}{2}\cdot\frac{U\cdot y}{c\cdot x^{\frac{3}{2}}}$ $\quad \frac{\partial}{\partial x}u = -\frac{U}{2\delta}\cdot\frac{y}{x}$ Maximum value at $y = \delta$

$\frac{\partial}{\partial y}v = \frac{\partial}{\partial y}\left(\frac{U}{4}\cdot\frac{y^2}{c\cdot x^{\frac{3}{2}}}\right) = \frac{2}{4}\cdot\frac{U\cdot y}{c\cdot x^{\frac{3}{2}}}$ $\quad \frac{\partial}{\partial y}v = \frac{U}{2\delta}\cdot\frac{y}{x}$ Maximum value at $y = \delta$

The shear stress is $\tau_{yx} = \mu \cdot \left(\frac{\partial}{\partial x} v + \frac{\partial}{\partial y} u \right) = \frac{\mu \cdot U}{\delta} \cdot \left[1 - \frac{3}{8} \cdot \left(\frac{y}{x} \right)^2 \right]$



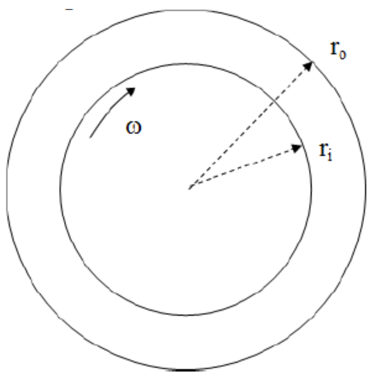
The net shear force on a fluid element is dτ dx dz: $d\tau = \frac{\partial}{\partial y} \tau \cdot dy = \frac{\mu \cdot U}{\delta} \cdot \left(-\frac{3}{8} \cdot \frac{2 \cdot y}{x^2} \right) \cdot dy = -\frac{3 \cdot \mu \cdot U \cdot y}{4 \cdot \delta \cdot x^2} \cdot dy$

Therefore the shear stress per unit volume is: $\frac{d}{dV} F = -\frac{3 \cdot \mu \cdot U}{4 \cdot \delta \cdot x} \cdot \frac{y}{x}$ Maximum value at $y = \delta$

Problem 5.71

(Difficulty 3)

5.71 A cylinder of radius r_i rotates at a speed ω coaxially inside a fixed cylinder of radius r_0. A viscous fluid fills the space between the two cylinders. Determine the velocity profile in the space between the cylinders and the shear stress on the surface of each cylinder. Explain why the shear stresses are not equal.



P 5.71

Find: The velocity profile and stress on each cylinder.

Assumptions: The flow is steady and incompressible

Solution:

For this two dimensional steady incompressible flow with circular streamlines we have the following governing equations:

Continuity:

$$\frac{1}{r}\frac{\partial(rV_r)}{\partial r}+\frac{1}{r}\frac{\partial V_\theta}{\partial \theta}=0$$

Momentum equation:

$$\rho(\vec{u}\cdot\nabla\vec{u})=-\nabla p+\mu\nabla^2\vec{u}$$

In this particular case, we have:

$$V_r=0$$

$$\frac{\partial V_\theta}{\partial \theta}=0$$

Thus

$$\vec{u}\cdot\nabla\vec{u} = V_\theta \hat{e}_\theta \cdot \nabla\vec{u} = \frac{V_\theta}{r}\frac{\partial(V_\theta \hat{e}_\theta)}{\partial\theta} = \frac{V_\theta}{r}\left[\frac{\partial V_\theta}{\partial\theta}\hat{e}_\theta + V_\theta\frac{\partial\hat{e}_\theta}{\partial\theta}\right] = -\frac{V_\theta^2}{r}\hat{e}_r$$

$$\nabla^2\vec{u} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_\theta \hat{e}_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2(V_\theta\hat{e}_\theta)}{\partial\theta^2}$$

As we have:

$$\frac{\partial\hat{e}_\theta}{\partial r} = 0$$

$$\frac{\partial^2\hat{e}_\theta}{\partial\theta^2} = \frac{\partial(-\hat{e}_r)}{\partial\theta} = -\hat{e}_\theta$$

Thus

$$\nabla^2\vec{u} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_\theta}{\partial r}\right)\hat{e}_\theta - \frac{V_\theta}{r^2}\hat{e}_\theta$$

The momentum equation becomes:

$$-\rho\frac{V_\theta^2}{r}\hat{e}_r = -\frac{\partial p}{\partial r}\hat{e}_r - \frac{1}{r}\frac{\partial p}{\partial\theta}\hat{e}_\theta + \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_\theta}{\partial r}\right)\hat{e}_\theta - \mu\frac{V_\theta}{r^2}\hat{e}_\theta$$

So we have:

$$-\frac{1}{r}\frac{\partial p}{\partial\theta}\hat{e}_\theta + \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_\theta}{\partial r}\right)\hat{e}_\theta - \mu\frac{V_\theta}{r^2}\hat{e}_\theta = 0$$

As the pressure at $\theta = 0$ and $\theta = 2\pi$ are the same.

$$\frac{\partial p}{\partial\theta} = 0$$

Thus

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_\theta}{\partial r}\right) - \frac{V_\theta}{r^2} = 0$$

$$\frac{\partial^2 V_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial V_\theta}{\partial r} - \frac{V_\theta}{r^2} = 0$$

This differential equation is easily solved with:

$$V_\theta = C_1 r + C_2\frac{1}{r}$$

Apply the boundary conditions:

$$V_\theta = V = \omega r_i \text{ at } r = r_i$$

$$V_\theta = 0 \text{ at } r = r_0$$

The constants are evaluated as

$$C_1 r_i + C_2 \frac{1}{r_i} = V$$

$$C_1 r_0 + C_2 \frac{1}{r_0} = 0$$

The velocity profile becomes

$$V_\theta = \frac{V r_0 r_i}{r_i^2 - r_0^2}\left(\frac{r}{r_0} - \frac{r_0}{r}\right) = \frac{\omega r_0 {r_i}^2}{r_i^2 - r_0^2}\left(\frac{r}{r_0} - \frac{r_0}{r}\right)$$

For the shear stress we have:

$$\tau_{r\theta} = \mu r \frac{d}{dr}\left(\frac{V_\theta}{r}\right) = \mu r \frac{d}{dr}\left[\frac{\omega r_0 {r_i}^2}{r_i^2 - r_0^2}\left(\frac{1}{r_0} - \frac{r_0}{r^2}\right)\right]$$

$$\tau_{r\theta} = \frac{\mu r \omega {r_0}^2 {r_i}^2}{r_i^2 - r_0^2}\frac{d}{dr}\left(-\frac{1}{r^2}\right) = \frac{\mu r \omega {r_0}^2 {r_i}^2}{r_i^2 - r_0^2}\frac{2}{r^3} = \frac{\mu \omega {r_0}^2 {r_i}^2}{r_i^2 - r_0^2}\frac{2}{r^2}$$

$$\tau_{r\theta} = \frac{2\mu \omega {r_0}^2 {r_i}^2}{r_i^2 - r_0^2}\frac{1}{r^2}$$

For the rotating surface:

$$\tau_{r\theta i} = \frac{2\mu \omega {r_0}^2 {r_i}^2}{r_i^2 - r_0^2}\frac{1}{{r_i}^2} = \frac{2\mu \omega {r_0}^2}{r_i^2 - r_0^2}$$

For the fixed surface:

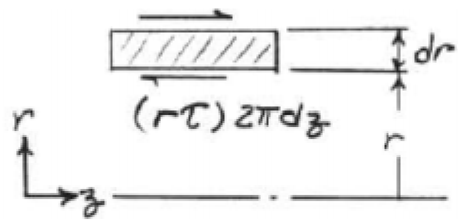
$$\tau_{r\theta 0} = \frac{2\mu \omega {r_0}^2 {r_i}^2}{r_i^2 - r_0^2}\frac{1}{{r_0}^2} = \frac{2\mu \omega {r_i}^2}{r_i^2 - r_0^2}$$

We see that the shear stress is a function of radius r, and that the shear stresses are not equal. Because the torque is the same on both cylinders, the larger radius r cylinder will have more surface area and a smaller shear stress.

Problem 5.72

(Difficulty: 2)

5.72 The velocity profile for fully developed laminar flow in a circular tube is $u = u_{max}[1-(r/R)^2]$. Obtain an expression for the shear force per unit volume in the x direction for this flow. Evaluate its maximum value for a pipe radius of 75 mm and a maximum velocity of 3 m/s. The fluid is water.



Given: Velocity profile for fully developed flow in a tube

Find: (a) Express shear force per unit volume in the x direction

(b) Maximum value at these conditions

Assumptions: Steady incompressible flow

Solution:

The differential of shear force would be:

$$dF_{shear} = (\tau + d\tau)2\pi r dz dr - \tau 2\pi r dz dr = 2\pi r d\tau dz dr$$

In cylindrical coordinates:

$$\frac{dF_{sz}}{dV} = \frac{1}{2\pi r dr dz}\frac{d}{dr}(r\tau)2\pi dr dz = \frac{1}{r}\frac{d}{dr}\left(r\mu\frac{du}{dr}\right)$$

From the given profile:

$$\frac{du}{dr} = -u_{max}\frac{2r}{R^2}$$

$$\frac{d^2u}{dr^2} = -u_{max}\frac{2}{R^2}$$

Therefore:

$$\frac{dF_{sz}}{dV} = \mu\frac{d^2u}{dr^2} + \mu\frac{1}{r}\frac{du}{dr} = -\mu u_{max}\frac{2}{R^2} - \mu\frac{1}{r}u_{max}\frac{2r}{R^2} = -\frac{4\mu u_{max}}{R^2}$$

$$F_{Vmax} = -\frac{4\mu u_{max}}{R^2}$$

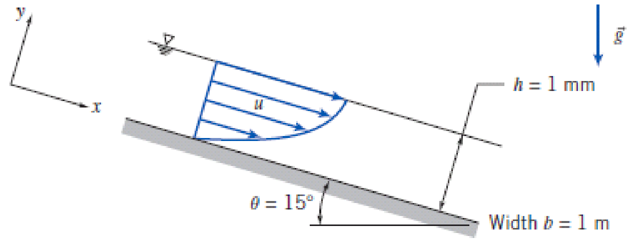
For water we have:

$$\mu = 1.002 \times 10^{-3}\, Pa \cdot s$$

$$F_{Vmax} = -\frac{4 \times 1.002 \times 10^{-3}\, Pa \cdot s \times 3\ \frac{m}{s}}{(0.075\, m)^2} = 2.14\ \frac{N}{m^3}$$

Problem 5.73 [Difficulty: 3]

5.73 Assume the liquid film in Example 5.9 is horizontal (i.e., $\theta = 0°$) and that the flow is driven by a constant shear stress on the top surface ($y = h$), $\tau_{yx} = C$. Assume that the liquid film is thin enough and flat and that the flow is fully developed with zero net flow rate (flow rate $Q = 0$). Determine the velocity profile $u(y)$ and the pressure gradient dp/dx.



Given: Horizontal, fully developed flow

Find: Velocity Profile and pressure gradient

Solution:

Governing Equations:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad \text{(Continuity Equation)}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial P}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \qquad \text{(Navier-Stokes Equations)}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

Assumptions:
(1) Incompressible fluid
(2) Zero net flow rate

For fully developed flow

$$\frac{d^2u}{dy^2}=\frac{1}{\mu}\frac{dp}{dx} \qquad (1)$$

The general solution for equation (1) is

$$u=\frac{y^2}{2\mu}\frac{dp}{dx}+C_1y+C_2 \qquad (2)$$

where C_1 and C_2 are constants.

Apply the boundary conditions

$$u=0 \text{ at } y=0$$

$$\mu\frac{du}{dy}=C \text{ at } y=h$$

Then, we can get $C_1=\frac{1}{\mu}(C-h\frac{dp}{dx})$ and $C_2=0$

$$u=-\frac{h^2}{\mu}\frac{dp}{dx}(y'-\frac{1}{2}y'^2)+\frac{Ch}{\mu}y', \text{ where } y'=\frac{y}{h}$$

The net flow or flow rate is zero:

$$0 = -\frac{h^2}{\mu}\frac{dp}{dx}\int_0^1 (y' - \frac{1}{2}y'^2)dy + \frac{Ch}{\mu}\int_0^1 y' dy$$

Thus, $\frac{dp}{dx} = \frac{3}{2}\frac{C}{h}$

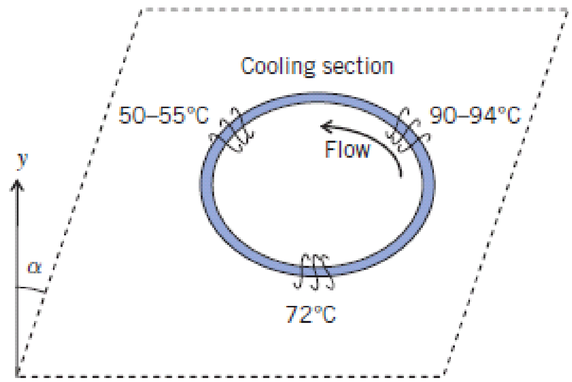
Problem 5.74

[Difficulty: 2]

5.74 The common thermal polymerase chain reaction (PCR) process requires the cycling of reagents through three distinct temperatures for denaturation (90–94°C), annealing (50–55°C), and extension (72°C). In continuous-flow PCR reactors, the temperatures of the three thermal zones are maintained as fixed while the reagents are cycled continuously through these zones. These temperature variations induce significant variations in the fluid density, which under appropriate conditions can be used to generate fluid motion. The figure depicts a thermosiphon-based PCR device (Chen et al., 2004, *Analytical Chemistry*, 76, 3707–3715). The closed loop is filled with PCR reagents. The plan of the loop is inclined at an angle α with respect to the vertical. The loop is surrounded by three heaters and coolers that maintain different temperatures.

(a) Explain why the fluid automatically circulates in the closed loop along the counterclockwise direction.

(b) What is the effect of the angle α on the fluid velocity?



Given: Temperature-dependent fluid density and the Navier-Stokes equations

Find: Explanation for the buoyancy-driven flow; effect of angle on fluid velocity

Solution:

Governing Equations:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad \text{(Continuity Equation)}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial P}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \qquad \text{(Navier-Stokes Equations)}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

Assumption: Incompressible fluid

(1) The first term in the right-hand-side of the momentum equations (5.27a)-(5.27c) represents the gravitational body force, which is proportional to the local fluid density. The fluid density in the region at temperature 72°C is higher than that in the region at temperature 90-94 °C, and meanwhile is lower than that in the region at temperature 50-55 °C. Thus, the net gravitational force induces counter-clockwise fluid circulation within the loop.

(2) Since the fluid circulation is driven by buoyancy force which is proportional to $g\times\cos\alpha$ where g is the gravitational acceleration, one can control the flow rate in the loop by adjusting the inclination angle α. When the angle α=90°, there is no fluid motion. When α=0, the flow rate is the maximum.

Problem 5.75

[Difficulty: 3]

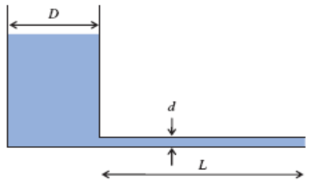
5.75 A tank contains water (20°C) at an initial depth $y_0 = 1$ m. The tank diameter is $D = 250$ mm, and a tube of diameter $d = 3$ mm and length $L = 4$ m is attached to the bottom of the tank. For laminar flow a reasonable model for the water level over time is

$$\frac{dy}{dt} = -\frac{d^4 \rho g}{32 D^2 \mu L} y \qquad y(0) = y_0$$

Using Euler methods with time steps of 12 min and 6 min:
(a) Estimate the water depth after 120 min, and compute the errors compared to the exact solution

$$y_{\text{exact}}(t) = y_0 e^{-\frac{d^4 \rho g}{32 D^2 \mu L} t}$$

(b) Plot the Euler and exact results.



$d =$ 3 mm
$D =$ 250 mm
$y_0 =$ 1 m
$L =$ 4 m

$\rho =$ 999 kg/m^3
$\mu =$ 0.001 N·s/m^2

$h =$ 12 min $\qquad h =$ 6 min

$$\frac{dy}{dt} = -\frac{d^4 \rho g}{32 D^2 \mu L} y \qquad y(0) = y_0$$

$$y_{\text{Exact}}(t) = y_0 e^{-\frac{d^4 \rho g}{32 D^2 \mu L} t}$$

$$y_{n+1} = y_n + h k y_n \qquad k = -\frac{d^4 \rho g}{32 D^2 \mu L} \qquad t_{n+1} = t_n + h$$

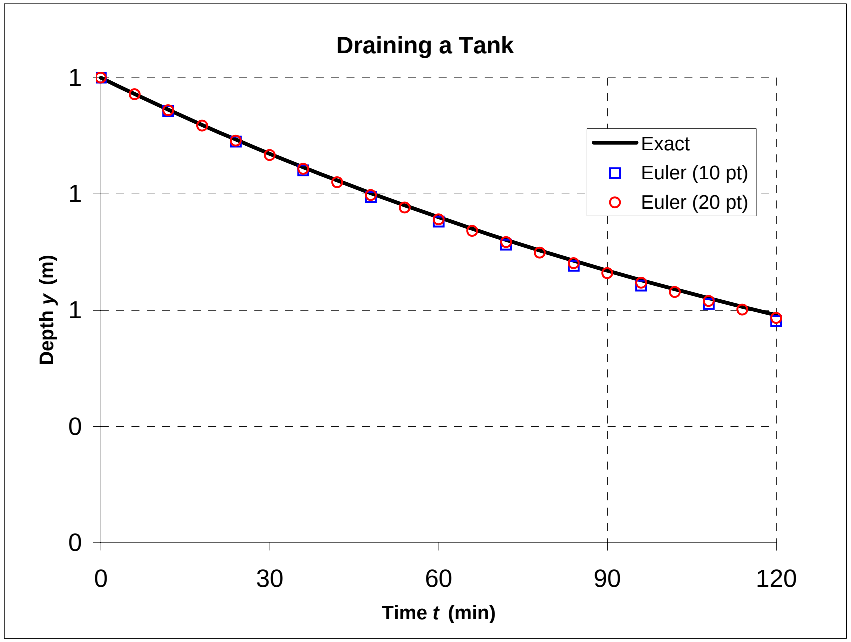
$k =$ 0.000099 s^{-1}

n	t_n (min)	y_n (m)
0	0	1
1	12	0.929
2	24	0.862
3	36	0.801
4	48	0.743
5	60	0.690
6	72	0.641
7	84	0.595
8	96	0.553
9	108	0.513
10	120	0.477

Error: 3%

n	t_n (min)	y_n (m)	y_{Exact} (m)
0	0.0	1	1
1	6.0	0.964	0.965
2	12.0	0.930	0.931
3	18.0	0.897	0.898
4	24.0	0.865	0.867
5	30.0	0.834	0.836
6	36.0	0.804	0.807
7	42.0	0.775	0.779
8	48.0	0.748	0.751
9	54.0	0.721	0.725
10	60.0	0.695	0.700
11	66.0	0.670	0.675
12	72.0	0.646	0.651
13	78.0	0.623	0.629
14	84.0	0.601	0.606
15	90.0	0.579	0.585
16	96.0	0.559	0.565
17	102.0	0.539	0.545
18	108.0	0.520	0.526
19	114.0	0.501	0.507
20	120.0	0.483	0.489

Error: 1%



Problem 5.76

[Difficulty: 3]

5.76 Use *Excel* to generate the solution of Eq. 5.31 for $m = 1$ shown in Fig. 5.18. To do so, you need to learn how to perform linear algebra in *Excel*. For example, for $N = 4$ you will end up with the matrix equation of Eq. 5.37. To solve this equation for the u values, you will have to compute the inverse of the 4×4 matrix, and then multiply this inverse into the 4×1 matrix on the right of the equation. In *Excel*, to do *array operations*, you must use the following rules: Pre-select the cells that will contain the result; use the appropriate *Excel array function* (look at *Excel's* Help for details); press Ctrl+Shift+Enter, not just Enter. For example, to invert the 4×4 matrix you would: Pre-select a blank 4×4 array that will contain the inverse matrix; type = *minverse*([array containing matrix to be inverted]); press Ctrl+Shift+Enter. To multiply a 4×4 matrix into a 4×1 matrix you would: Pre-select a blank 4×1 array that will contain the result; type = *mmult*([array containing 4×4 matrix], [array containing 4×1 matrix]); press Ctrl+Shift+Enter.

$$\frac{du}{dx} + u^m = 0; \qquad 0 \le x \le 1; \qquad u(0) = 1$$

N = **4**
Δx = **0.333**

	Eq. 5.34 (LHS)				**(RHS)**
	1.000	0.000	0.000	0.000	1
	-1.000	1.333	0.000	0.000	0
	0.000	-1.000	1.333	0.000	0
	0.000	0.000	-1.000	1.333	0

x	**Inverse Matrix**				**Result**	**Exact**	**Error**
0.000	1.000	0.000	0.000	0.000	1.000	1.000	0.000
0.333	0.750	0.750	0.000	0.000	0.750	0.717	0.000
0.667	0.563	0.563	0.750	0.000	0.563	0.513	0.001
1.000	0.422	0.422	0.563	0.750	0.422	0.368	0.001
							0.040

N = **8**

Δx = **0.143**

Eq. 5.34 (LHS)								**(RHS)**
1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1
-1.000	1.143	0.000	0.000	0.000	0.000	0.000	0.000	0
0.000	-1.000	1.143	0.000	0.000	0.000	0.000	0.000	0
0.000	0.000	-1.000	1.143	0.000	0.000	0.000	0.000	0
0.000	0.000	0.000	-1.000	1.143	0.000	0.000	0.000	0
0.000	0.000	0.000	0.000	-1.000	1.143	0.000	0.000	0
0.000	0.000	0.000	0.000	0.000	-1.000	1.143	0.000	0
0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.143	0

Inverse Matrix

x	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**Result**	**Exact**	**Error**
0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.000	1.000	0.000
0.143	0.875	0.875	0.000	0.000	0.000	0.000	0.000	0.000	0.875	0.867	0.000
0.286	0.766	0.766	0.875	0.000	0.000	0.000	0.000	0.000	0.766	0.751	0.000
0.429	0.670	0.670	0.766	0.875	0.000	0.000	0.000	0.000	0.670	0.651	0.000
0.571	0.586	0.586	0.670	0.766	0.875	0.000	0.000	0.000	0.586	0.565	0.000
0.714	0.513	0.513	0.586	0.670	0.766	0.875	0.000	0.000	0.513	0.490	0.000
0.857	0.449	0.449	0.513	0.586	0.670	0.766	0.875	0.000	0.449	0.424	0.000
1.000	0.393	0.393	0.449	0.513	0.586	0.670	0.766	0.875	0.393	0.368	0.000
											0.019

N = **16**

Δx = **0.067** **Eq. 5.34 (LHS)**

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**(RHS)**
1	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1
2	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
3	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
4	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
5	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
6	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
7	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
8	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
9	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0
10	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0
11	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0
12	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0
13	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0
14	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0
15	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0
16	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0

x	**Inverse Matrix**																**Result**	**Exact**	**Error**
0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.000	1.000	0.000
0.067	0.938	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.938	0.936	0.000
0.133	0.879	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.879	0.875	0.000
0.200	0.824	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.824	0.819	0.000
0.267	0.772	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.772	0.766	0.000
0.333	0.724	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.724	0.717	0.000
0.400	0.679	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.679	0.670	0.000
0.467	0.637	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.637	0.627	0.000
0.533	0.597	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.597	0.587	0.000
0.600	0.559	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.559	0.549	0.000
0.667	0.524	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.524	0.513	0.000
0.733	0.492	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.492	0.480	0.000
0.800	0.461	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.461	0.449	0.000
0.867	0.432	0.432	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.432	0.420	0.000
0.933	0.405	0.405	0.432	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.405	0.393	0.000
1.000	0.380	0.380	0.405	0.432	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.380	0.368	0.000
																			0.009

N	**Δ*x***	**Error**
4	0.333	0.040
8	0.143	0.019
16	0.067	0.009

Problem 5.77

(Difficulty 2)

5.77 For a small spherical particle of styrofoam $density = 16\ \frac{kg}{m^3}$ with a diameter of 5 mm falling in air, the drag is given by $F_D = 3\pi\mu V d$, where μ is the air viscosity and V is the sphere velocity. Derive the differential equation that describes the motion. Using the Euler method, find the maximum speed starting from rest, and the time it takes to reach 95 percent of this speed. Plot the speed as function of time.

Find: Derive the differential equation that describes the motion. Maximum velocity: V_{max}. Time to reach 95 percent of maximum velocity: t.

Assumptions: The air is quiescent and the only drag is due to viscous friction.

Solution: Use Newton's second law of motion. For motion in the y-direction

$$\sum F_y = m\, a$$

The forces are the body force F_B acting downward and the drag force F_D acting upward. For positive y in the direction of motion (downward) we have:

$$F_B - F_D = m\, a$$

Or, the acceleration is

$$a = \frac{F_B - F_D}{m}$$

The mass can be calculated by:

$$m = \rho \forall$$

The volume of the particle is:

$$\forall = \frac{\pi}{6} d^3 = \frac{\pi}{6} \times (0.005\ m)^3 = 6.55\ \times 10^{-8}\ m^3$$

The body force is :

$$F_B = mg = \rho g \forall$$

This motion is along the vertical direction.

So the differential equation for the motion ($Euler\ method$) is:

$$a = \frac{dV}{dt} = \frac{\rho g \forall - 3\pi\mu V d}{\rho \forall}$$

When the acceleration $a = 0$, the particle reaches the maximum velocity, so we have:

$$\rho g \forall - 3\pi\mu V_{max} d = 0$$

The viscosity of air is:

$$\mu = 1.827 \times 10^{-5} \ \frac{kg}{m \cdot s}$$

The density of the particle is:

$$\rho = 16 \ \frac{kg}{m^3}$$

Thus

$$V_{max} = \frac{\rho g \forall}{3\pi\mu d} = \frac{16 \ \frac{kg}{m^3} \times 9.81 \ \frac{m}{s^2} \times 6.55 \ \times 10^{-8} \ m^3}{3 \times \pi \times 1.827 \times 10^{-5} \ \frac{kg}{m \cdot s} \times 0.005 \ m} = 11.93 \ \frac{m}{s}$$

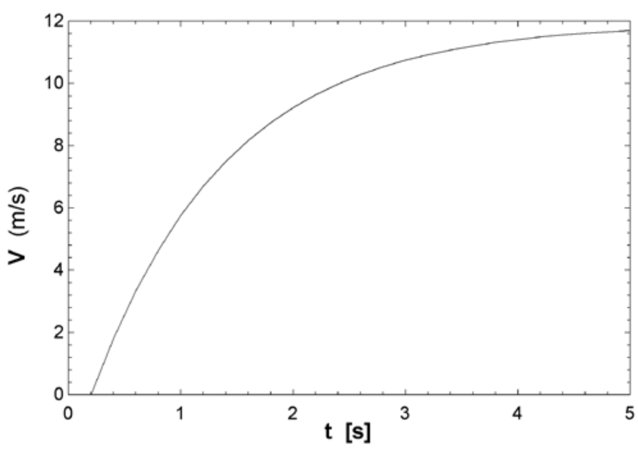
From the differential equation of motion, we have:

$$\frac{dV}{dt} = \frac{\rho g \forall - 3\pi\mu V d}{\rho \forall}$$

The finite difference equation for the Euler method is

$$\Delta V = \left(\frac{\rho g \forall - 3\pi\mu V d}{\rho \forall}\right) \Delta t$$

Using the Euler method in an equation solver, the time it takes reach 95 percent of the maximum velocity, which is 11.4 m/s, is 3.8 s. The velocity versus time is plotted as:



Problem 5.78

[Difficulty: 3]

5.78 Following the steps to convert the differential equation Eq. 5.31 (for $m = 1$) into a difference equation (for example, Eq. 5.37 for $N = 4$), solve

$$\frac{du}{dx} + u = 2x^2 + x \qquad 0 \le x \le 1 \quad u(0) = 3$$

for $N = 4$, 8, and 16 and compare to the exact solution

$$u_{exact} = 2x^2 - 3x + 3$$

Hint: Follow the hints provided in Problem 5.101.

New Eq. 5.37: $-u_{i-1} + (1+\Delta x)u_i = \Delta x \cdot \left(2x_i^2 + x_i\right)$

N = **4**
Δx = **0.333**

	Eq. 5.34 (LHS)				(RHS)
	1.000	0.000	0.000	0.000	3
	-1.000	1.333	0.000	0.000	0.18519
	0.000	-1.000	1.333	0.000	0.51852
	0.000	0.000	-1.000	1.333	1

x	Inverse Matrix				Result	Exact	Error
0.000	1.000	0.000	0.000	0.000	3.000	3.000	0.000
0.333	0.750	0.750	0.000	0.000	2.389	2.222	0.007
0.667	0.563	0.563	0.750	0.000	2.181	1.889	0.021
1.000	0.422	0.422	0.563	0.750	2.385	2.000	0.037
							0.256

N = **8**
Δx = **0.143**

Eq. 5.34 (LHS)								(RHS)
1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	3
-1.000	1.143	0.000	0.000	0.000	0.000	0.000	0.000	0.02624
0.000	-1.000	1.143	0.000	0.000	0.000	0.000	0.000	0.06414
0.000	0.000	-1.000	1.143	0.000	0.000	0.000	0.000	0.1137
0.000	0.000	0.000	-1.000	1.143	0.000	0.000	0.000	0.17493
0.000	0.000	0.000	0.000	-1.000	1.143	0.000	0.000	0.24781
0.000	0.000	0.000	0.000	0.000	-1.000	1.143	0.000	0.33236
0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.143	0.42857

x	Inverse Matrix 1	2	3	4	5	6	7	8	Result	Exact	Error
0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	3.000	3.000	0.000
0.143	0.875	0.875	0.000	0.000	0.000	0.000	0.000	0.000	2.648	2.612	0.000
0.286	0.766	0.766	0.875	0.000	0.000	0.000	0.000	0.000	2.373	2.306	0.001
0.429	0.670	0.670	0.766	0.875	0.000	0.000	0.000	0.000	2.176	2.082	0.001
0.571	0.586	0.586	0.670	0.766	0.875	0.000	0.000	0.000	2.057	1.939	0.002
0.714	0.513	0.513	0.586	0.670	0.766	0.875	0.000	0.000	2.017	1.878	0.002
0.857	0.449	0.449	0.513	0.586	0.670	0.766	0.875	0.000	2.055	1.898	0.003
1.000	0.393	0.393	0.449	0.513	0.586	0.670	0.766	0.875	2.174	2.000	0.004
											0.113

N = **16**

Δx = **0.067** **Eq. 5.34 (LHS)**

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	(RHS)
1	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	3
2	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.00504
3	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.01126
4	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.01867
5	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.02726
6	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.03704
7	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.048
8	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.06015
9	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.07348
10	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.000	0.088
11	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.000	0.1037
12	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.000	0.12059
13	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.000	0.13867
14	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.000	0.15793
15	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.000	0.17837
16	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	-1.000	1.067	0.2

x	**Inverse Matrix**																**Result**	**Exact**	**Error**
0.000	1.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	3.000	3.000	0.000
0.067	0.938	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	2.817	2.809	0.000
0.133	0.879	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	2.652	2.636	0.000
0.200	0.824	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	2.503	2.480	0.000
0.267	0.772	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	2.373	2.342	0.000
0.333	0.724	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	2.259	2.222	0.000
0.400	0.679	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	2.163	2.120	0.000
0.467	0.637	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	2.084	2.036	0.000
0.533	0.597	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	0.000	2.023	1.969	0.000
0.600	0.559	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	0.000	1.979	1.920	0.000
0.667	0.524	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	0.000	1.952	1.889	0.000
0.733	0.492	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	0.000	1.943	1.876	0.000
0.800	0.461	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	0.000	1.952	1.880	0.000
0.867	0.432	0.432	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	0.000	1.978	1.902	0.000
0.933	0.405	0.405	0.432	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	0.000	2.022	1.942	0.000
1.000	0.380	0.380	0.405	0.432	0.461	0.492	0.524	0.559	0.597	0.637	0.679	0.724	0.772	0.824	0.879	0.938	2.083	2.000	0.000
																			0.054

N	**Δ*x***	**Error**
4	0.333	0.256
8	0.143	0.113
16	0.067	0.054

Problem 5.79

[Difficulty: 3]

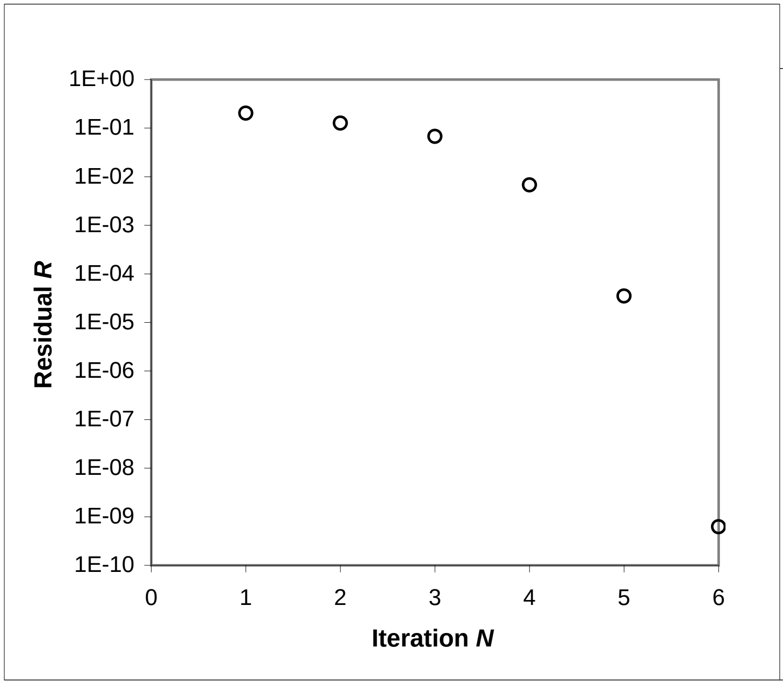
5.79 Use *Excel* to generate the solutions of Eq. 5.31 for $m = 2$, as shown in Fig. 5.21.

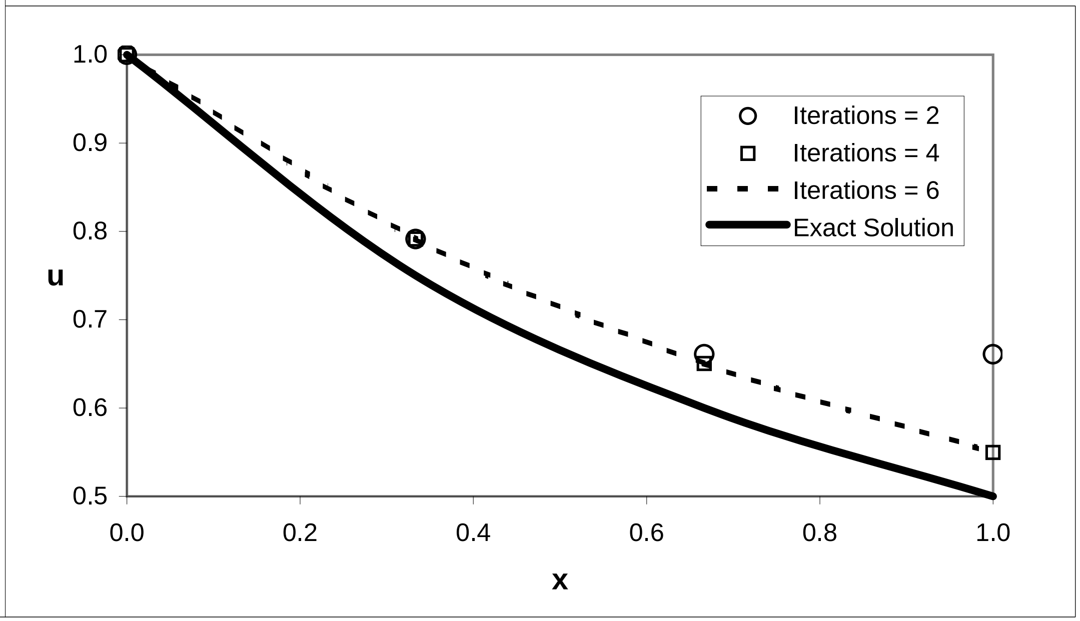
$$u_i = \frac{u_{g_{i-1}} + \Delta x\, u_{g_i}^2}{1 + 2\Delta x\, u_{g_i}}$$

$\Delta x =$ 0.333

		x		
Iteration	0.000	0.333	0.667	1.000
0	1.000	1.000	1.000	1.000
1	1.000	0.800	0.800	0.800
2	1.000	0.791	0.661	0.661
3	1.000	0.791	0.650	0.560
4	1.000	0.791	0.650	0.550
5	1.000	0.791	0.650	0.550
6	1.000	0.791	0.650	0.550
Exact	**1.000**	**0.750**	**0.600**	**0.500**

Residuals
0.204
0.127
0.068
0.007
0.000
0.000





Problem 5.80

[Difficulty: 3]

5.80 Use *Excel* to generate the solutions of Eq. 5.31 for $m = -1$, with $u(0) = 3$, using 4 and 16 points over the interval from $x = 0$ to $x = 3$, with sufficient iterations, and compare to the exact solution

$$u_{\text{exact}} = \sqrt{9 - 2x}$$

To do so, follow the steps described in "Dealing with Non-linearity" section.

$$\Delta u_i = u_i - u_{g_i}$$

$$\frac{1}{u_i} = \frac{1}{u_{g_i} + \Delta u_i} \approx \frac{1}{u_{g_i}}\left(1 - \frac{\Delta u_i}{u_{g_i}}\right)$$

$$\frac{u_i - u_{i-1}}{\Delta x} + \frac{1}{u_i} = 0$$

$$\frac{u_i - u_{i-1}}{\Delta x} + \frac{1}{u_{g_i}}\left(1 - \frac{u_i - u_{g_i}}{u_{g_i}}\right) = 0$$

$$\frac{u_i - u_{i-1}}{\Delta x} + \frac{1}{u_{g_i}}\left(2 - \frac{u_i}{u_{g_i}}\right) = 0$$

$$u_i\left(1 - \frac{\Delta x}{u_{g_i}^2}\right) = u_{i-1} - \frac{2\Delta x}{u_{g_i}}$$

$$u_i = \frac{u_{i-1} - \dfrac{2\Delta x}{u_{g_i}}}{1 - \dfrac{\Delta x}{u_{g_i}^2}}$$

$\Delta x =$ 1.500

Iteration	*x* = 0.000	1.500	3.000	4.500
0	3.000	3.000	3.000	3.000
1	3.000	2.400	2.400	2.400
2	3.000	2.366	1.555	1.555
3	3.000	2.366	1.151	-0.986
4	3.000	2.366	1.816	-7.737
5	3.000	2.366	1.310	2.260
6	3.000	2.366	0.601	-0.025
Exact	**3.000**	**2.449**	**1.732**	**0.000**

$\Delta x =$ 0.300

Iteration	*x* = 0.000	0.300	0.600	0.900	1.200	1.500	1.800	2.100	2.400	2.700	3.000	3.300	3.600	3.900	4.200	4.500
0	3.000	3.000	3.000	3.000	3.000	3.000	3.000	3.000	3.000	3.000	3.000	3.000	3.000	3.000	3.000	3.000
1	3.000	2.897	2.897	2.897	2.897	2.897	2.897	2.897	2.897	2.897	2.897	2.897	2.897	2.897	2.897	2.897
2	3.000	2.896	2.789	2.789	2.789	2.789	2.789	2.789	2.789	2.789	2.789	2.789	2.789	2.789	2.789	2.789
3	3.000	2.896	2.789	2.677	2.677	2.677	2.677	2.677	2.677	2.677	2.677	2.677	2.677	2.677	2.677	2.677
4	3.000	2.896	2.789	2.677	2.560	2.560	2.560	2.560	2.560	2.560	2.560	2.560	2.560	2.560	2.560	2.560
5	3.000	2.896	2.789	2.677	2.560	2.438	2.438	2.438	2.438	2.438	2.438	2.438	2.438	2.438	2.438	2.438
6	3.000	2.896	2.789	2.677	2.560	2.436	2.308	2.308	2.308	2.308	2.308	2.308	2.308	2.308	2.308	2.308
7	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.170	2.170	2.170	2.170	2.170	2.170	2.170	2.170	2.170
8	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.023	2.023	2.023	2.023	2.023	2.023	2.023	2.023
9	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.862	1.862	1.862	1.862	1.862	1.862	1.862
10	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.686	1.686	1.686	1.686	1.686	1.686
11	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.487	1.487	1.487	1.487	1.487
12	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.254	1.254	1.254	1.254
13	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.958	0.958	0.958
14	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.901	0.493	0.493
15	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	1.349	3.091
16	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.544	1.192
17	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	14.403	0.051
18	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.859	-0.024
19	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.338	-0.051

20	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.538	-0.105
21	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	5.953	-0.239
22	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.805	-1.998
23	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.286	1.195
24	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.450	-0.273
25	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.900	-0.876
26	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.369	2.601
27	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.605	0.145
28	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-0.517	0.266
29	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-17.059	0.858
30	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.935	-29.971
31	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.392	0.955
32	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.663	-0.352
33	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-0.020	-1.662
34	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-0.041	0.383
35	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-0.088	1.534
36	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-0.204	-0.549
37	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-0.621	198.629
38	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	8.435	-0.624
39	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.831	41.087
40	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.313	0.817
41	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.494	-0.765
42	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	1.379	2.623
43	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.551	1.203
44	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-16.722	0.066
45	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.936	0.377
46	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.392	0.591
47	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.664	-4.391
48	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-0.014	0.813
49	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-0.029	-1.376
50	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-0.061	0.483
51	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-0.135	4.578
52	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-0.347	-0.270
53	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-1.765	-0.603
54	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	1.371	-4.389
55	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.549	1.532
56	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-40.363	0.180
57	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.914	5.316
58	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.379	0.810
59	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	0.627	-0.668
60	3.000	2.896	2.789	2.677	2.560	2.436	2.306	2.168	2.019	1.858	1.679	1.476	1.233	0.899	-0.243	4.652
Exact	**3.000**	**2.898**	**2.793**	**2.683**	**2.569**	**2.449**	**2.324**	**2.191**	**2.049**	**1.897**	**1.732**	**1.549**	**1.342**	**1.095**	**0.775**	**0.000**

Here are graphs comparing the numerical and exact solutions.