CHAPTER 13

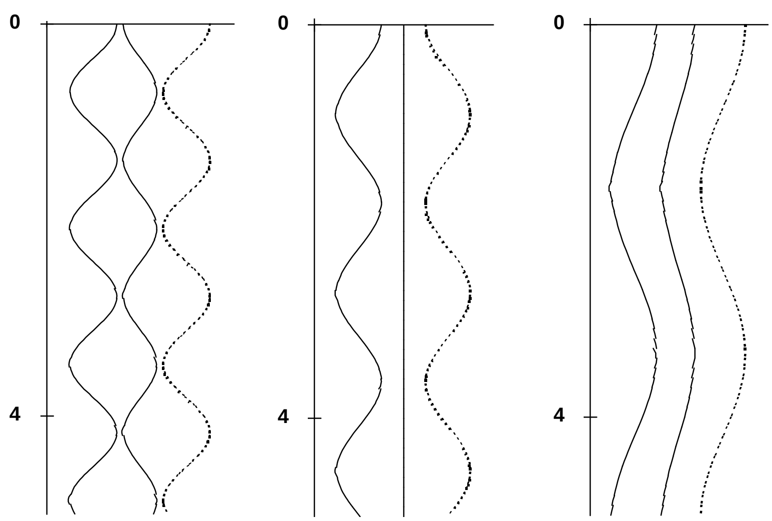
13.1 For three springs

$$\begin{array}{llll}
\left(\frac{2k_1}{m_1}-\omega^2\right)A_1 & -\frac{k_1}{m_1}A_2 & & = 0 \\
-\frac{k_1}{m_1}A_1 & +\left(\frac{2k_1}{m_1}-\omega^2\right)A_2 & -\frac{k_1}{m_1}A_3 & = 0 \\
 & -\frac{k_1}{m_1}A_2 & +\left(\frac{2k_1}{m_1}-\omega^2\right)A_3 & = 0
\end{array}$$

Substituting m = 40 kg and k = 240 gives

$$\begin{array}{llll}
\left(12-\omega^2\right)A_1 & -6A_2 & & = 0 \\
-6A_1 & +\left(12-\omega^2\right)A_2 & -6A_3 & = 0 \\
 & -6A_2 & +\left(12-\omega^2\right)A_3 & = 0
\end{array}$$

The determinant is $-\omega^6 + 36\omega^4 - 360\omega^2 + 864 = 0$, which can be solved for ω^2 = 20.4853, 12, and 3.5147 s^{-2}. Therefore the frequencies are ω = 4.526, 3.464, and 1.875 s^{-1}. Substituting these values into the original equations yields for ω^2 = 20.4853, $A_1 = -0.707$ and $A_2 = A_3$. For ω^2 = 12, $A_1 = -A_3$ and $A_2 = 0$. For ω^2 = 3.5147, $A_1 = 0.707$ and $A_2 = A_3$. Based on these results, the following plots can be developed:



13.2 The system and the initial guesses can be set up as

```
>> format short g
>> a=[2 8 10;8 4 5;10 5 7];
>> x=[1 1 1]';
```

<u>First iteration:</u>

```
>> x=a*x
x =
    20
    17
    22
```

```
>> e=max(x)
e =
    22
>> x=x/e
x =
    0.90909
    0.77273
    1.0000
```

Second iteration:

```
>> x=a*x
x =
   18.0000
   15.364
   19.955
>> e=max(x)
e =
   19.955
>> x=x/e
x =
    0.90205
    0.76993
    1.0000
```

Third iteration:

```
>> x=a*x
x =
   17.964
   15.2964
   19.870
>> e=max(x)
e =
   19.870
>> x=x/e
x =
    0.90405
    0.7698
    1.0000
```

Fourth iteration:

```
>> x=a*x
x =
   17.967
   15.312
   19.889
>> e=max(x)
e =
   19.889
>> x=x/e
x =
    0.90332
    0.76983
    1.0000
```

Thus, after four iterations, the result is converging on the highest eigenvalue. After several more iterations, it will converge on an eigenvalue of 19.884 with a corresponding eigenvector of [0.90351 0.76983 1].

13.3 The following script can be developed to determine the smallest eigenvalue with the power method. The script is set up to compute 4 iterations:

```
clear, clc, format short g
a=[2 8 10;8 4 5;10 5 7];
ai = inv(a)
x=[1 1 1]';
for i = 1:4
  disp('Iteration:')
  x=ai*x
  e=max(x)
  x=x/e
end
xn=x/norm(x)
```

The results are

```
ai =
    -0.071429      0.14286 -8.3267e-017
      0.14286       2.0476      -1.6667
            0      -1.6667       1.3333

Iteration:
x =
     0.071429
      0.52381
     -0.33333
e =
      0.52381
x =
      0.13636
            1
     -0.63636

Iteration:
x =
      0.13312
       3.1277
      -2.5152
e =
       3.1277
x =
     0.042561
            1
     -0.80415

Iteration:
x =
      0.13982
        3.394
      -2.7389
e =
        3.394
x =
     0.041196
            1
     -0.80699

Iteration:
x =
      0.13991
       3.3985
      -2.7426
e =
```

```
       3.3985
x =
      0.04117
            1
     -0.80702

xn =
     0.032022
       0.7778
      -0.6277
```

Thus, after four iterations, the estimate of the lowest eigenvalue is 1/(3.3985) = 0.29424 with a normalized eigenvector of [0.032022 0.7778 –0.6277]. This result can be compared with the lowest eigenvalue computed with the `eig` function,

```
>>[v,d]=eig(a)

v =
     -0.81247    -0.032022      0.58213
      0.38603      -0.7778      0.49599
      0.43689       0.6277       0.6443
d =
      -7.1785            0            0
            0      0.29424            0
            0            0       19.884
```

13.4 By summing forces on each mass and equating that to the mass times acceleration, the resu1ting differential equations can be written

$$\ddot{x}_1 + \left(\frac{k_1 + k_2}{m_1}\right)x_1 - \left(\frac{k_2}{m_1}\right)x_2 = 0$$

$$\ddot{x}_2 - \left(\frac{k_2}{m_2}\right)x_1 + \left(\frac{k_2 + k_3}{m_2}\right)x_2 - \left(\frac{k_3}{m_2}\right)x_3 = 0$$

$$\ddot{x}_3 - \left(\frac{k_3}{m_3}\right)x_2 + \left(\frac{k_3 + k_4}{m_3}\right)x_3 = 0$$

In matrix form

$$\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} \frac{k_1 + k_2}{m_1} & -\frac{k_2}{m_1} & 0 \\ -\frac{k_2}{m_2} & \frac{k_2 + k_3}{m_2} & -\frac{k_3}{m_2} \\ 0 & -\frac{k_3}{m_3} & \frac{k_3 + k_4}{m_3} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

The k/m matrix becomes with: $k_1 = k_4 = 15$ N/m, $k_2 = k_3 = 35$ N/m, and $m_1 = m_2 = m_3 = 1.5$ kg

$$\left[\frac{k}{m}\right] = \begin{bmatrix} 33.33333 & -23.33333 & 0 \\ -23.33333 & 46.66667 & -23.33333 \\ 0 & -23.33333 & 33.33333 \end{bmatrix}$$

Solve for the eigenva1ues/natural frequencies using MATLAB:

```
>> k1=15;k4=15;k2=35;k3=35;
>> m1=1.5;m2=1.5;m3=1.5;
>> a=[(k1+k2)/m1 -k2/m1 0;-k2/m2 (k2+k3)/m2 -k3/m2;0 -k3/m3 (k3+k4)/m3];
>> w2=eig(a)
w2 =
    6.3350
   33.3333
   73.6650
>> w=sqrt(w2)
w =
    2.5169
    5.7735
    8.5828
```

13.5 Here is a MATLAB session that uses `eig` to determine the eigenvalues and the natural frequencies:

```
>> k=2;
>> kmw2=[2*k,-k,-k;-k,2*k,-k;-k,-k,2*k];
>> [v,d]=eig(kmw2)
v =
    0.5774    0.2673    0.7715
    0.5774   -0.8018   -0.1543
    0.5774    0.5345   -0.6172
d =
   -0.0000         0         0
         0    6.0000         0
         0         0    6.0000
```

Therefore, the eigenvalues are 0, 6, and 6. Setting these eigenvalues equal to $m\omega^2$, the three frequencies can be obtained.

$$m\omega_1^{\,2} = 0 \Rightarrow \omega_1 = 0 \ \text{ (Hz) } \ 1^{\text{st}} \text{ mode of oscillation}$$
$$m\omega_2^{\,2} = 6 \Rightarrow \omega_2 = \sqrt{6} \text{ (Hz) } \ 2^{\text{nd}} \text{ mode}$$
$$m\omega_3^{\,2} = 6 \Rightarrow \omega_3 = \sqrt{6} \text{ (Hz) } \ 3^{\text{rd}} \text{ mode}$$

13.6 The solution along with its second derivative can be substituted into the simultaneous ODEs. After simplification, the result is

$$\begin{array}{llll}
\left(\dfrac{1}{C_1} - L_1\omega^2\right)A_1 & -\dfrac{1}{C_2}A_2 & & = 0 \\
-\dfrac{1}{C_1}A_1 & \left(\dfrac{1}{C_1} + \dfrac{1}{C_2} - L_1\omega^2\right)A_2 & -\dfrac{1}{C_2}A_3 & = 0 \\
 & -\dfrac{1}{C_2}A_2 & \left(\dfrac{1}{C_2} + \dfrac{1}{C_3} - L_3\omega^2\right)A_3 & = 0
\end{array}$$

Thus, we have formulated an eigenvalue problem. Further simplification results for the special case where the C's and L's are constant. For this situation, the system can be expressed in matrix form as

$$\begin{bmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} = \{0\} \qquad \text{(P13.6}a\text{)}$$

where $\lambda = LC\omega^2$. MATLAB can be employed to determine values for the eigenvalues and eigenvectors
:

```
>>a=[1 –1 0; –1 2 –1; 0 –1 2];
>>[v,d]=eig(a)
```

```
v =
0.7370 0.5910 0.3280
0.5910 -0.3280 -0.7370
0.3280 -0.7370 0.5910
d =
0.1981 0 0
0 1.5550 0
0 0 3.2470
```

The matrix v consists of the system's three eigenvectors (arranged as columns), and d is a matrix with the corresponding eigenvalues on the diagonal. Thus, the package computes that the eigenvalues are λ = 0.1981, 1.555, and 3.247. These values in turn can be used to compute the natural circular frequencies of the system

$$\omega = \begin{cases} 0.4450/\sqrt{LC} \\ 1.2450/\sqrt{LC} \\ 1.8019/\sqrt{LC} \end{cases}$$

Aside from providing the natural frequencies, the eigenvalues can be substituted into the Eq. (P13.6*a*) to gain further insight into the circuit's physical behavior. For example, substituting $\lambda = 0.1981$ yields

$$\begin{bmatrix} 0.8019 & -1 & 0 \\ -1 & 1.8019 & -1 \\ 0 & -1 & 1.8019 \end{bmatrix} \begin{Bmatrix} i_1 \\ i_2 \\ i_3 \end{Bmatrix} = \{0\}$$

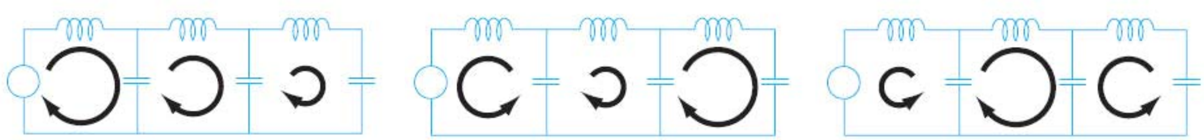
Although this system does not have a unique solution, it will be satisfied if the currents are in fixed ratios, as in

$$0.8019 i_1 = i_2 = 1.8019 i_3 \qquad \text{(P13.6}b\text{)}$$

Thus, as depicted in (*a*) in the figure below, they oscillate in the same direction with different magnitudes. Observe that if we assume that $i_1 = 0.737$, we can use Eq. (P13.6*b*) to compute the other currents with the result

$$\{i\} = \begin{Bmatrix} 0.737 \\ 0.591 \\ 0.328 \end{Bmatrix}$$

which is the first column of the v matrix calculated with MATLAB.



(*a*) $\omega = \dfrac{0.4451}{\sqrt{LC}}$ (*b*) $\omega = \dfrac{1.2470}{\sqrt{LC}}$ (*c*) $\omega = \dfrac{1.8019}{\sqrt{LC}}$

In a similar fashion, the second eigenvalue of $\lambda = 1.555$ can be substituted and the result evaluated to yield $-1.8018 i_1 = i_2 = 2.247 i_3$. As depicted in the above figure (*b*), the first loop oscillates in the opposite direction from the second and third.

Finally, the third mode can be determined as $-0.445i_1 = i_2 = -0.8718i_3$. Consequently, as in the above figure (*c*), the first and third loops oscillate in the opposite direction from the second.

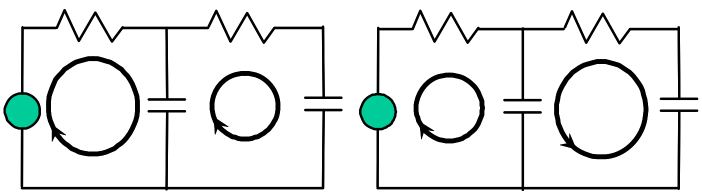
13.7 Using an approach similar to Prob. 13.6, the system can be expressed in matrix form as

$$\begin{bmatrix} 1-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} \begin{Bmatrix} i_1 \\ i_2 \end{Bmatrix} = \{0\}$$

A package like MATLAB can be used to evaluate the eigenvalues and eigenvectors as in

```
>> a=[1 -1;-1 2];
>> [v,d]=eig(a)
v =
    0.8507   -0.5257
    0.5257    0.8507
d =
    0.3820         0
         0    2.6180
```

Thus, we can see that the eigenvalues are $\lambda = 0.382$ and 2.618 or natural frequencies of $\omega = 0.618/\sqrt{LC}$ and $1.618/\sqrt{LC}$. The eigenvectors tell us that these correspond to oscillations that coincide (0.8507 0.5257) and which run counter to each other (–0.5257 0.8507).



$$\omega = \frac{0.618}{\sqrt{LC}} \qquad \omega = \frac{1.618}{\sqrt{LC}}$$

13.8 Force balances can be written as

$$m_1 \frac{d^2x_1}{dt^2} = -k_1x_1 + k_2(x_2 - x_1)$$
$$m_2 \frac{d^2x_2}{dt^2} = -k_2(x_2 - x_1)$$

Assume solutions

$x_i = X_i \sin(\omega t)$ $\qquad$ $x_i'' = -X_i\, \omega^2 \sin(\omega t)$

Substitute

$$-m_1X_1\omega^2 \sin(\omega t) = -k_1X_1 \sin(\omega t) + k_2(X_2 \sin(\omega t) - X_1 \sin(\omega t))$$
$$-m_2X_2\omega^2 \sin(\omega t) = -k_2(X_2 \sin(\omega t) - X_1 \sin(\omega t))$$

Cancel $\sin(\omega t)$ and collect terms

$$\left(\frac{k_1+k_2}{m_1}-\omega^2\right)X_1-\frac{k_2}{m_1}X_2=0$$
$$-\frac{k_2}{m_2}X_1+\left(\frac{k_2}{m_2}-\omega^2\right)X_2=0$$

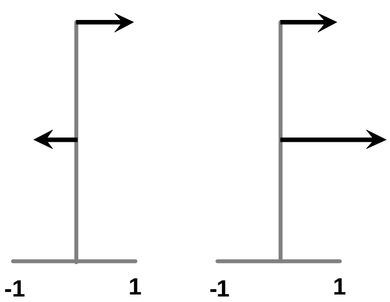
Substitute parameters

$$\left(450-\omega^2\right)X_1-200X_2=0$$
$$-240X_1+\left(240-\omega^2\right)X_2=0$$

MATLAB solution:

```
A=[450 -200 0;-240 240];
[v,d]=eig(A)
wn=sqrt(diag(d))'/2/pi

v =
    0.8232    0.4983
   -0.5678    0.8670
d =
  587.9506         0
         0  102.0494
wn =
    3.8591    1.6078
```



13.9 Force balances can be written as

$$m_1\frac{d^2x_1}{dt^2}=-k_1x_1+k_2(x_2-x_1)$$
$$m_2\frac{d^2x_2}{dt^2}=-k_2(x_2-x_1)-k_3(x_2-x_3)$$
$$m_3\frac{d^2x_3}{dt^2}=-k_3(x_3-x_2)-k_4(x_3-x_4)$$
$$m_4\frac{d^2x_4}{dt^2}=k_4(x_3-x_4)$$

Assume solutions

$x_i = X_i \sin(\omega t)$ $\quad\quad\quad\quad$ $x_i'' = -X_i\, \omega^2 \sin(\omega t)$

Substitute

$$
\begin{aligned}
-m_1 X_1 \omega^2 \sin(\omega t) &= -k_1 X_1 \sin(\omega t) + k_2 (X_2 \sin(\omega t) - X_1 \sin(\omega t)) \\
-m_2 X_2 \omega^2 \sin(\omega t) &= -k_2 (X_2 \sin(\omega t) - X_1 \sin(\omega t)) - k_3 (X_2 \sin(\omega t) - X_3 \sin(\omega t)) \\
-m_3 X_3 \omega^2 \sin(\omega t) &= -k_3 (X_3 \sin(\omega t) - X_2 \sin(\omega t)) - k_4 (X_3 \sin(\omega t) - X_2 \sin(\omega t)) \\
-m_4 X_4 \omega^2 \sin(\omega t) &= k_4 (X_3 \sin(\omega t) - X_4 \sin(\omega t))
\end{aligned}
$$

Collect terms

$$
\begin{aligned}
\left(\frac{k_1 + k_2}{m_1} - \omega^2\right) X_1 \quad - \frac{k_2}{m_1} X_2 &= 0 \\
-\frac{k_2}{m_2} X_1 + \left(\frac{k_2 + k_3}{m_2} - \omega^2\right) X_2 \quad - \frac{k_3}{m_2} X_3 &= 0 \\
-\frac{k_3}{m_3} X_2 + \left(\frac{k_3 + k_4}{m_3} - \omega^2\right) X_3 \quad - \frac{k_4}{m_3} X_4 &= 0 \\
-\frac{k_4}{m_4} X_3 + \left(\frac{k_4}{m_4} - \omega^2\right) X_4 &= 0
\end{aligned}
$$

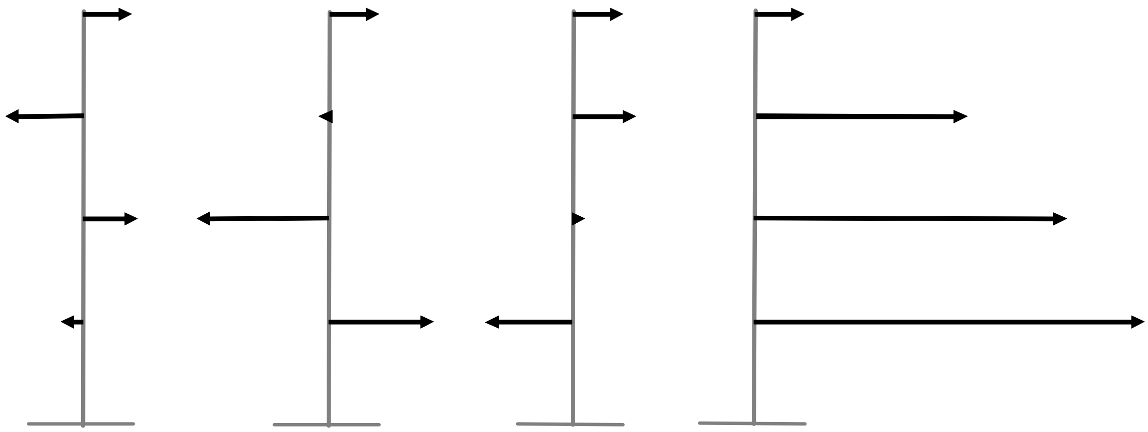
Substitute parameters

$$
\begin{aligned}
\left(450 - \omega^2\right) X_1 \quad - 200 X_2 &= 0 \\
-240 X_1 + \left(240 - \omega^2\right) X_2 \quad - 180 X_3 &= 0 \\
-225 X_2 + \left(375 - \omega^2\right) X_3 \quad - 150 X_4 &= 0 \\
-200 X_3 + \left(200 - \omega^2\right) X_4 &= 0
\end{aligned}
$$

MATLAB solution:

```
A=[450 -200 0 0;-240 420 -180 0;0 -225 375 -150;0 0 -200 200];
[v,d]=eig(A)
wn=sqrt(diag(d))'/2/pi

v =
   -0.4870   -0.5230   -0.4297    0.1870
    0.6786    0.0510   -0.5176    0.3855
   -0.5142    0.6832   -0.0336    0.5693
    0.1945   -0.5070    0.7392    0.7017
d =
  728.6584         0         0         0
         0  469.5115         0         0
         0         0  209.0839         0
         0         0         0   37.7462
wn =
    4.2962    3.4486    2.3013    0.9778
```



13.10 (a) For four interior points (h = 3⁄5), the resulting system of equations is

$$\begin{bmatrix} (2-0.36p^2) & -1 & 0 & 0 \\ -1 & (2-0.36p^2) & -1 & 0 \\ 0 & -1 & (2-0.36p^2) & -1 \\ 0 & 0 & -1 & (2-0.36p^2) \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = 0$$

Setting the determinant equal to zero and expanding it gives $(2 - 0.36p^2)^4 - 3(2 - 0.36p^2)^2 + 1 = 0$ which can be solved for the first four eigenvalues

$p = \pm 1.0301$
$p = \pm 1.9593$
$p = \pm 2.6967$
$p = \pm 3.1702$

(b) The system can be normalized so that the coefficient of the eigenvalue is unity:

$$\begin{bmatrix} 5.5556 & -2.7778 & 0 & 0 \\ -2.7778 & 5.5556 & -2.7778 & 0 \\ 0 & -2.7778 & 5.5556 & -2.7778 \\ 0 & 0 & -2.7778 & 5.5556 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = 0$$

MATLAB solution:

```
A=[5.5556 -2.7778 0 0;-2.7778 5.5556 -2.7778 0; ...
   0 -2.7778 5.5556 -2.7778;0 0 -2.7778 5.5556];
[v,d]=eig(A)
wn=sqrt(diag(d))'

v =
    0.3717   -0.6015   -0.6015   -0.3717
    0.6015   -0.3717    0.3717    0.6015
    0.6015    0.3717    0.3717   -0.6015
    0.3717    0.6015   -0.6015    0.3717
d =
    1.0610         0         0         0
```

```
         0    3.8388         0         0
         0         0    7.2723         0
         0         0         0   10.0501
wn =
    1.0301    1.9593    2.6967    3.1702
```

(c) Power method:

```
clear, clc, format short g
A=[5.5556 -2.7778 0 0;-2.7778 5.5556 -2.7778 0; ...
   0 -2.7778 5.5556 -2.7778;0 0 -2.7778 5.5556];
x=[1 1 1 1]';
for i = 1:4
  disp('Iteration:')
  x=a*x
  e=max(x)
  x=x/e
end
xn=x/norm(x)
```

The 4 iterations are

```
Iteration:
x =
    2.7778
         0
         0
    2.7778
e =
    2.7778
x =
     1
     0
     0
     1

Iteration:
x =
    5.5556
   -2.7778
   -2.7778
    5.5556
e =
    5.5556
x =
    1.0000
   -0.5000
   -0.5000
    1.0000

Iteration:
x =
    6.9445
   -4.1667
   -4.1667
    6.9445
e =
    6.9445
x =
    1.0000
   -0.6000
```

```
   -0.6000
    1.0000

Iteration:
x =
    7.2223
   -4.4445
   -4.4445
    7.2223
e =
    7.2223
x =
    1.0000
   -0.6154
   -0.6154
    1.0000

xn =
    0.6022
   -0.3706
   -0.3706
    0.6022
```

If the process is continued, the result for the maximum eigenvalue is p = 7.2724 with an associated eigenvector [0.6015 –0.3717 –0.3717 0.6015].

13.11 (a) The assumed solutions and their derivatives can be substituted into the original differential equations to give,

$$c_1\lambda e^{\lambda t} = -5c_1e^{\lambda t} + 3c_2e^{\lambda t}$$
$$c_2\lambda e^{\lambda t} = 100c_1e^{\lambda t} - 301c_2e^{\lambda t}$$

Cancelling the exponentials and rearranging converts the system into an eigenvalue problem

$$(-5-\lambda)c_1 + 3c_2 = 0$$
$$100c_1 + (-301-\lambda)c_2 = 0$$

The characteristic equation is

$$\begin{vmatrix} -5-\lambda & 3 \\ 100 & -301-\lambda \end{vmatrix} = (-5-\lambda)(-301-\lambda) - 3(100) = \lambda^2 + 306\lambda + 1205$$

which can be solved for the eigenvalues

$$\begin{matrix}\lambda_1 \\ \lambda_2\end{matrix} = \frac{-306 \pm \sqrt{(-306)^2 - 4(1)(1205)}}{2} = 302.0101, -3.98993$$

Alternatively, we can use MATLAB to compute the same values. This can be done in two ways. First, we can directly form and solve the characteristic polynomial:

```
clear,clc,clf,format long g
A=[-5 3;100 -301];
p=poly(A)
d=roots(p)
```

```
p =
           1          306         1205
d =
         -302.010066774027
         -3.98993322597295
```

Therefore, the eigenvalues for this system are –302.0101 and –3.98993.

(b) The `eig` function can be used to determine the same result as (a) as well as obtaining the associated eigenvectors:

```
[v,d]=eig(A)
v =
         0.947724927231108        -0.010100152527012
         0.319088486637782         0.999948992158566
d =
         -3.98993322597295                         0
                         0         -302.010066774027
```

(c) The solution can then be written as

$$\{y\} = c_1 \begin{Bmatrix} 0.94772 \\ 0.31909 \end{Bmatrix} e^{-302.0101t} + c_2 \begin{Bmatrix} -0.0101 \\ 0.99995 \end{Bmatrix} e^{-3.98993t}$$

The unknown constants are evaluated by applying the initial conditions to the previous equation at $t = 0$,

$$\begin{bmatrix} 0.94772 & -0.0101 \\ 0.31909 & 0.99995 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 50 \\ 100 \end{Bmatrix}$$

which can be solved for

```
>> y0 = [50 100]';
>> c = v\y0
c =
          53.6412825029131
          82.8879122783221
```

which can be substituted back into the solution to give

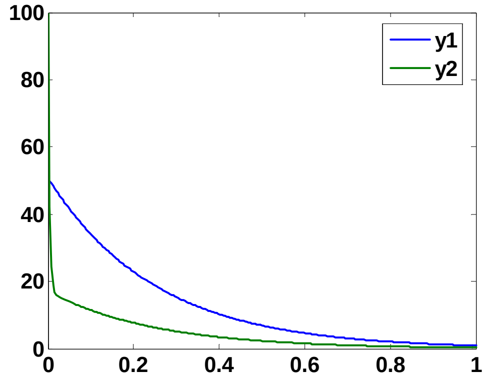
$$\{y\} = 53.64128 \begin{Bmatrix} 0.94772 \\ 0.31909 \end{Bmatrix} e^{-302.0101t} + 82.8879 \begin{Bmatrix} -0.0101 \\ 0.99995 \end{Bmatrix} e^{-3.98993t}$$

which yields the final solution

$$y_1 = 50.83718 e^{-3.98993t} - 0.83718 e^{-302.0101t}$$
$$y_2 = 17.11632 e^{-3.98993t} + 82.88368 e^{-302.0101t}$$

(d)

```
t=[0:1/256:1];
y1=50.83718*exp(-3.98993*t)-0.83718*exp(-302.0101*t);
y2=17.11632*exp(-3.98993*t)+82.88368*exp(-302.0101*t);
plot(t,y1,t,y2,'--')
ylim([0 100]),legend('y1','y2','location','Best')
```



13.12 The decay rate can be calculated as 0.69315/28.8 = 0.0240676/yr. Substituting this value into the differential equations yields

$$\frac{dc_1}{dt} = -0.029668c_1$$

$$\frac{dc_2}{dt} = -0.034068c_2$$

$$\frac{dc_3}{dt} = 0.01902c_1 + 0.01387c_2 - 0.071068c_3$$

$$\frac{dc_4}{dt} = 0.33597c_3 - 0.40007c_4$$

$$\frac{dc_5}{dt} = 0.11364c_4 - 0.15707c_5$$

Assume solutions of the form: $c_i = c_i(0)e^{-\lambda t}$. Substitute the solutions and their derivatives into the differential equations converts the system into an eigenvalue problem

$$\begin{bmatrix} 0.029668-\lambda & 0 & 0 & 0 & 0 \\ 0 & 0.034068-\lambda & 0 & 0 & 0 \\ -0.01902 & -0.01387 & 0.071068-\lambda & 0 & 0 \\ 0 & 0 & -0.33597 & 0.40007-\lambda & 0 \\ 0 & 0 & 0 & -0.11364 & 0.15707-\lambda \end{bmatrix} \{c\} = \{0\}$$

The eigenvalues and eigenvectors can be determined with MATLAB:

```
clear,clc,clf
format short g
k=log(2)/28.8;
A=zeros(5);A(1,1)=-(0.0056+k);A(2,2)=-(0.01+k);
A(3,3)=-(0.047+k);A(4,4)=-(0.376+k);A(5,5)=-(0.133+k);
A(3,1)=0.01902;A(3,2)=0.01387;A(4,3)=0.33597;A(5,4)=0.11364;
[v,d]=eig(A)
```

```
v =
          0          0          0    0.81034          0
          0          0          0          0    0.85749
          0          0    0.50874    0.37229    0.32144
          0    0.90584    0.51952    0.33768    0.29507
          1   -0.42362    0.68649    0.30121    0.27261
d =

    0.15707          0          0          0          0
          0    0.40007          0          0          0
          0          0   0.071068          0          0
          0          0          0   0.029668          0
          0          0          0          0   0.034068
```

The solution can then be written as

$$\{c\} = c_1(0)\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} e^{-0.15707t} + c_2(0)\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.90584 \\ -0.42362 \end{Bmatrix} e^{-0.40007t} + c_3(0)\begin{Bmatrix} 0 \\ 0 \\ 0.50874 \\ 0.51952 \\ 0.68649 \end{Bmatrix} e^{-0.071068t}$$
$$+ c_4(0)\begin{Bmatrix} 0.81034 \\ 0 \\ 0.37229 \\ 0.33768 \\ 0.30121 \end{Bmatrix} e^{-0.029668t} + c_5(0)\begin{Bmatrix} 0 \\ 0.85749 \\ 0.32144 \\ 0.29507 \\ 0.27261 \end{Bmatrix} e^{-0.034068t}$$

The unknown constants can then be evaluated by applying the initial conditions

```
y0=[17.7 30.5 43.9 136.3 30.1]';
c=v\y0;
c'
ans =
     24.749       103.31       47.834       21.843       35.569
```

which can be substituted back into the solution to give

$$\{c\} = 24.749\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} e^{-0.15707t} + 103.31\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.90584 \\ -0.42362 \end{Bmatrix} e^{-0.40007t} + 47.834\begin{Bmatrix} 0 \\ 0 \\ 0.50874 \\ 0.51952 \\ 0.68649 \end{Bmatrix} e^{-0.071068t}$$
$$+ 21.843\begin{Bmatrix} 0.81034 \\ 0 \\ 0.37229 \\ 0.33768 \\ 0.30121 \end{Bmatrix} e^{-0.029668t} + 35.569\begin{Bmatrix} 0 \\ 0.85749 \\ 0.32144 \\ 0.29507 \\ 0.27261 \end{Bmatrix} e^{-0.034068t}$$

which yields the final solution

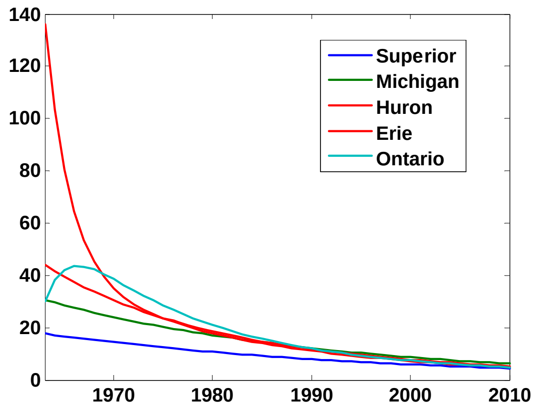
$$c_1 = 21.843(0.81034)e^{-0.029668t}$$
$$c_2 = 35.569(0.85749)e^{-0.034068t}$$
$$c_3 = 21.843(0.37229)e^{-0.029668t} + 35.569(0.32144)e^{-0.034068t} + 47.834(0.50874)e^{-0.071068t}$$
$$c_4 = 21.843(0.33768)e^{-0.029668t} + 35.569(0.29507)e^{-0.034068t} + 47.834(0.51952)e^{-0.071068t} + 103.31(0.90584)e^{-0.40007t}$$
$$c_5 = 21.843(0.30121)e^{-0.029668t} + 35.569(0.27261)e^{-0.034068t} + 47.834(0.68649)e^{-0.071068t} + 103.31(-0.42362)e^{-0.40007t} + 24.749(1)e^{-0.15707t}$$

or

$$c_1 = 17.7e^{-0.029668t}$$
$$c_2 = 30.5e^{-0.034068t}$$
$$c_3 = 8.132e^{-0.029668t} + 11.433e^{-0.034068t} + 24.335e^{-0.071068t}$$
$$c_4 = 7.376e^{-0.029668t} + 10.495e^{-0.034068t} + 24.85e^{-0.071068t} + 93.582e^{-0.40007t}$$
$$c_5 = 6.579e^{-0.029668t} + 9.696e^{-0.034068t} + 32.837e^{-0.071068t} - 43.764e^{-0.40007t} + 24.749e^{-0.15707t}$$

A plot of the final results can be generated as

```
t=[1963:2010];
c1=17.7*exp(-0.029668*(t-1963));
c2=30.5*exp(-0.034068*(t-1963));
c3=8.132*exp(-0.029668*(t-1963))+11.433*exp(-0.034068*(t-1963))...
   +24.335*exp(-0.071068*(t-1963));
c4=7.376*exp(-0.029668*(t-1963))+10.495*exp(-0.034068*(t-1963))...
   +24.85*exp(-0.071068*(t-1963))+93.582*exp(-0.40007*(t-1963));
c5=6.579*exp(-0.029668*(t-1963))+9.696*exp(-0.034068*(t-1963))...
   +32.837*exp(-0.071068*(t-1963))-43.764*exp(-0.40007*(t-1963))...
   +24.749*exp(-0.15707*(t-1963));
plot(t,c1,t,c2,'--',t,c3,':',t,c4,'r',t,c5,'-.','linewidth',2)
ylim([0 140]),xlim([1963 2010])
legend('Superior','Michigan','Huron','Erie','Ontario','location','Best')
```



13.13

```
function [eval, evect,ea,iter] = powereig(A,es,maxit)
%Power method for largest eigenvalue
% input:
% es = desired relative error (default = 0.0001%)
% maxit = maximum allowable iterations (default = 50)
% output:
% eval = largest eigenvalue
% evect = largest eigenvector
```

```
% ea = approximate relative error (%)
% iter = number of iterations

if nargin<1,error('at least 1 input argument required'),end
if nargin<2|isempty(es), es=0.0001;end
if nargin<3|isempty(maxit), maxit=50;end

format short g
n=length(A);
vect=ones(n,1);
val=0;iter=0;ea=100;
while(1)
  valold=val;
  vect=A*vect;
  val=max(abs(vect));
  vect=vect./val;
  iter=iter+1;
  if val~=0, ea = abs((val-valold)/val)*100; end
  if ea<=es | iter >= maxit,break,end
end
eval=val;evect=vect;
end
```

<u>Example 13.3:</u>

```
A=[40 -20 0;-20 40 -20;0 -20 40];
[eval,evect,ea,iter] = powereig(A)
evect=evect/norm(evect)
[v,d]=eig(A)

eval =
       68.284
evect =
      0.70711
           -1
      0.70711
ea =
  5.1534e-005
iter =
    11
evect =
          0.5
     -0.70711
          0.5
v =
          0.5     -0.70711         -0.5
      0.70711 -2.0817e-017      0.70711
          0.5      0.70711         -0.5
d =
       11.716            0            0
            0           40            0
            0            0       68.284
```

<u>Prob. 13.2:</u>

```
clear,clc
A=[2 8 10;8 4 5;10 5 7];
[eval, evect,ea,iter] = powereig(A)
evect=evect/norm(evect)
[v,d]=eig(A)

eval =
       19.884
```

```
evect =
      0.90351
      0.76983
            1
ea =
  7.9461e-005
iter =
    11
evect =
      0.58213
      0.49599
       0.6443
v =
     -0.81247    -0.032022      0.58213
      0.38603      -0.7778      0.49599
      0.43689       0.6277       0.6443
d =
      -7.1785            0            0
            0      0.29424            0
            0            0       19.884
```