评分标准及答案

重庆大学《线性代数》课程试卷

⊙A卷
○B卷

2019—2020 学年第 1 学期

开课学院：数统学院 课程编号：MATH30084 考试日期：2019.12.24
考试方式：开卷、闭卷、其它 考试时间：120 分钟

题号	1	2	3	4	5	6	总分
得分							

考试提示

1. 严禁随身携带通讯工具等电子设备参加考试；
2. 考试作弊，留校察看，毕业当年不授学位；请人代考、替他人考试、两次以上作弊等，属严重作弊，开除学籍.

考试教室 姓名 学号 年级 专业、班 学院
公平竞争、诚实守信、严肃考纪、拒绝作弊
线 订 装

1. Find the least squares solutions of the following equation system. Determine whether or not the least square solutions are the solutions of the system. Justify your answer (15 points).

$$x_1 - x_2 + 3x_3 + 2x_4 = 1$$
$$-x_1 + x_2 - 2x_3 + x_4 = -2$$
$$2x_1 - 2x_2 + 5x_3 + x_4 = 1$$

Solution:

The system's matrix form is

$$Ax = b$$

with $A = \begin{bmatrix} 1 & -1 & 3 & 2 \\ -1 & 1 & -2 & 1 \\ 2 & -2 & 5 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. (4 Points)

Solve $A^TAx = A^Tb$. (2 Points)

$$A^T = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 3 & -2 & 5 \\ 2 & 1 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 4 & -6 & 15 & 3 \\ -6 & 6 & -15 & -3 \\ 15 & -15 & 38 & 9 \\ 3 & -3 & 9 & 6 \end{bmatrix} \qquad A^Tb = \begin{bmatrix} 5 \\ -5 \\ 12 \\ 1 \end{bmatrix}$$ ($\cancel{4}$ 2 Points)

There are infinitely many ~~The~~ least square solutions. (2 points).

$\cancel{x = (A^TA)^{-1}A^T}b$ ~~(2 Points)~~

$$\begin{bmatrix} 1 & -1 & 3 & 2 & 1 \\ -1 & 1 & -2 & 1 & -2 \\ 2 & -2 & 5 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 & 2 & 1 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & -1 & -3 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 & 2 & 1 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

No solution! (5 points).

The least squares solution

命题（组题）人：黄辉斥
审题人：赵显锋
命题时间：19年12月9日
教务处制

2. For $A = \begin{bmatrix} 2 & 5 & 7 \\ 0 & 1 & -2 \\ 0 & -3 & 6 \end{bmatrix}$, find eigenvalues and the corresponding eigenspaces of A, and compute e^A (20 points).

Solution:

Solve $\det(\lambda I - A) = \det \begin{pmatrix} \lambda-2 & -5 & -7 \\ 0 & \lambda-1 & 2 \\ 0 & 3 & \lambda-6 \end{pmatrix} = 0$

We get eigenvalues of A are 0, 2, 7. (6 points)

$\lambda = 0$

Solve $AX=0$. we get eigenvectors of the form $x_3 \begin{pmatrix} -\frac{17}{2} \\ 2 \\ 1 \end{pmatrix}$. ($\cancel{8}$ 3 points)

$\lambda=2$ Solve $(A-2I)X=0$. We get eigenvectors of the form $x_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ (3 points)

$\lambda=7$ Solve $(A-7I)X=0$. We get eigenvectors of the form $x_2 \begin{pmatrix} -\frac{16}{5} \\ 1 \\ -3 \end{pmatrix}$. (3 points)

$A = XDX^{-1}$

with $X = \begin{pmatrix} -\frac{17}{2} & 1 & -\frac{16}{5} \\ 2 & 0 & 1 \\ 1 & 0 & -3 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{pmatrix}$.

$e^A = X \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^7 \end{pmatrix} X^{-1}$. (5 points)

3. For $A = \begin{bmatrix} 1 & 1 & 2 & 6 & -2 \\ 1 & 0 & -1 & 3 & 1 \\ 2 & 1 & 0 & 3 & 2 \\ 0 & -2 & -1 & 5 & 7 \end{bmatrix}$, find an orthonormal basis of the column space of A. Here the inner product on $\mathbb{R}^n$ is given by the scalar product $x^T y$ for all x, y in $\mathbb{R}^n$ (15 points).

Solution:

Change A to an echelon form:

$$A \longrightarrow \begin{bmatrix} 1 & 1 & 2 & 6 & -2 \\ 0 & -1 & -3 & -3 & 6 \\ 0 & -3 & -4 & -9 & 2 \\ 0 & -2 & -1 & 5 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & 6 & -2 \\ 0 & 1 & 3 & 3 & -6 \\ 0 & 0 & 5 & 0 & -16 \\ 0 & 0 & 7 & 11 & -5 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & 2 & 6 & -2 \\ 0 & 1 & 3 & 3 & -6 \\ 0 & 0 & 5 & 0 & -16 \\ 0 & 0 & 0 & 11 & -174 \end{bmatrix}$$

A basis of $\operatorname{col} A = \left\{ v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix}, v_4 = \begin{bmatrix} 6 \\ 3 \\ 3 \\ 5 \end{bmatrix} \right\}$ (6 points)

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$ (2 points)

$$u_2 = \frac{v_2 - \langle v_2, u_1 \rangle u_1}{\|v_2 - \langle v_2, u_1 \rangle u_1\|} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} \Big/ \|\cdot\| = \frac{1}{\sqrt{186}} \begin{pmatrix} 5 \\ 1 \\ -4 \\ -12 \end{pmatrix}.$$ (2 points)

$$u_3 = \frac{v_3 - \langle v_3, u_1 \rangle u_1 - \langle v_3, u_2 \rangle u_2}{\|v_3 - \langle v_3, u_1 \rangle u_1 - \langle v_3, u_2 \rangle u_2\|} =$$

$$\langle v_3, u_1 \rangle u_1 = \frac{1}{6} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} \qquad \langle v_3, u_2 \rangle u_2 = \frac{-3}{186} \begin{pmatrix} 5 \\ 1 \\ -4 \\ -12 \end{pmatrix} = -\frac{1}{62} \begin{pmatrix} 5 \\ 1 \\ -4 \\ -12 \end{pmatrix}.$$

$$v_3 - \langle v_3, u_1 \rangle u_1 - \langle v_3, u_2 \rangle u_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} + \frac{1}{62} \begin{bmatrix} 5 \\ 1 \\ -4 \\ -12 \end{bmatrix}$$

$$= \frac{1}{93} \begin{bmatrix} 178 \\ -189 \\ -37 \\ -111 \end{bmatrix}$$

$$u_3 = \frac{1}{\sqrt{(178)^2 + (189)^2 + (37)^2 + (111)^2}} \begin{bmatrix} 178 \\ -189 \\ -37 \\ -111 \end{bmatrix}$$ (2 points).

$$u_4 = \frac{v_4 - \langle v_4, u_1 \rangle u_1 - \langle v_4, u_2 \rangle u_2 - \langle v_4, u_1 \rangle u_1}{\|v_4 - \langle v_4, u_1 \rangle u_1 - \langle v_4, u_2 \rangle u_2 - \langle v_4, u_1 \rangle u_1\|}$$ (1 point).

4. Compute det $\begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 2 & -1 & 3 \\ 1 & 5 & 0 & -2 \\ -1 & 1 & 1 & 0 \end{bmatrix}$ and find its inverse (15 points).

Solution:

$$\det\begin{bmatrix} -1 & 0 & 1 & 4 \\ 0 & 2 & -1 & 3 \\ 1 & 5 & 0 & -2 \\ -1 & 1 & 1 & 0 \end{bmatrix} = \det\begin{bmatrix} -1 & 0 & 1 & 4 \\ 0 & 2 & -1 & 3 \\ 0 & 5 & 1 & 2 \\ 0 & 1 & 0 & -4 \end{bmatrix}$$

$$= -\det\begin{pmatrix} 2 & -1 & 3 \\ 5 & 1 & 2 \\ 1 & 0 & -4 \end{pmatrix} = -\det\begin{pmatrix} 2 & -1 & 3 \\ 7 & 0 & 5 \\ 1 & 0 & -4 \end{pmatrix}$$

$$= -\det\begin{pmatrix} 7 & 5 \\ 1 & -4 \end{pmatrix} = 33.$$

(10 points)

$$\begin{bmatrix} -1 & 0 & 1 & 4 \\ 0 & 2 & -1 & 3 \\ 1 & 5 & 0 & -2 \\ -1 & 1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} \ \end{bmatrix}$$

$$\left[\begin{array}{cccc|cccc} -1 & 0 & 1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 3 & 0 & 1 & 0 & 0 \\ 1 & 5 & 0 & -2 & 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & -1 & -4 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 5 & 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -4 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & -1 & -4 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -4 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 22 & 6 & 0 & 1 & -5 \\ 0 & 0 & -1 & 11 & 2 & 1 & 0 & -2 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 18 & 5 & 0 & 1 & -5 \\ 0 & 1 & 0 & -4 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 22 & 6 & 0 & 1 & -5 \\ 0 & 0 & 0 & 33 & 8 & 1 & 1 & -7 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{21}{33} & \frac{-18}{33} & \frac{15}{33} & \frac{-39}{33} \\ 0 & 1 & 0 & 0 & -\frac{1}{33} & \frac{4}{33} & \frac{4}{33} & \frac{5}{33} \\ 0 & 0 & 1 & 0 & \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 1 & \frac{8}{33} & \frac{1}{33} & \frac{1}{33} & \frac{-7}{33} \end{array}\right]$$

$$A^{-1} = \frac{1}{33}\begin{bmatrix} 21 & -18 & 15 & -39 \\ -1 & 4 & 4 & 5 \\ 22 & -22 & 11 & -11 \\ 8 & 1 & 1 & -7 \end{bmatrix}$$

(5 points).

5. Find the matrix representation of the linear transformation $T : P_4 \to P_4$ given by $T(p) = p' + p$ under the ordered bases $[1 - x, 2x + 5, x^2 + 1, x^3 - x^2 - x]$ and $[1, x, x^2, x^3]$ of P_4 and P_4 respectively. Here p' stands for the derivatives of p (10 points).

Solution: Let $E = [1-x, 2x+5, x^2+1, x^3-x^2-x]$

$F = [1, x, x^2, x^3]$

Then the matrix A representing T w.r.t. E and F satisfies

$$A[p]_E = [T(p)]_F.$$

Hence $A = \left[[T(1-x)]_F \; [T(2x+5)]_F \; [T(x^2+1)]_F \; [T(x^3-x^2-x)]_F \right]$ (2 points)

$T(1-x) = -1+1-x = -x$ $\quad [T(1-x)]_F = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ (2 points)

$T(2x+5) = 2+2x+5 = 2x+7$ $\quad [T(2x+5)]_F = \begin{bmatrix} 7 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ (2 points)

$T(x^2+1) = 2x+x^2+1$ $\quad [T(x^2+1)]_F = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$ (2 points)

$T(x^3-x^2-x) = 3x^2-2x-1 + x^3-x^2-x$

$= x^3+2x^2-3x-1$ $\quad [T(x^3-x^2-x)]_F = \begin{bmatrix} -1 \\ -3 \\ 2 \\ 1 \end{bmatrix}$ (2 points)

$$A = \begin{bmatrix} 0 & 7 & 1 & -1 \\ -1 & 2 & 2 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6. Determine whether or not the following is true. If true, prove it. If not true, give a counter-example (25 points).

(1) If one adds a linear equation into a consistent linear equation system, then the new equation system is inconsistent.

(2) Each eigenvalue of a Hermitian matrix H is a real number;

(3) The transpose of a unitary matrix is Hermitian;

(4) The product of two invertible matrices is still invertible;

(5) The union of two subspaces of a vector space is a vector space.

(1) False (3 points) Add $x_1 - x_2 = 0$ into $\begin{cases} x_1 - 2x_2 = 0 \\ x_1 + x_2 = 0 \end{cases}$

New system is also consistent. (2 points)

(2) True. (3 points)

$Ax = \lambda x$ for $x \neq 0$

$\Rightarrow x^H A x = \lambda x^H x$

$\Rightarrow x^H A^H x = \bar{\lambda} x^H x$

$\Rightarrow \lambda x^H x = \bar{\lambda} x^H x$

$\Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda$ is real (2 points)

(3) False (3 points)

$A = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ $A^T = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$

A^T is not Hermitian, but unitary. (2 points)

(4) True. (3 points)

$(AB)^{-1} = B^{-1}A^{-1}$ (2 points).

(5). False (3 points)

$V_1 = \{(x, 0)^T \mid x \in \mathbb{R}\}$. $V_2 = \{(0, y)^T \mid y \in \mathbb{R}\}$.

$V_1 \cup V_2$ is not a subspace of $\mathbb{R}^2$. (2 points).