University of Cincinnati
College of Engineering & Applied Science
2021 Fall Semester

20MET6031/5131 Intro to Robotics
Homework #8, Homogenous Transformations (95 pts total)

Student Name: _____Solution keys (courtesy of several students)___Score: _________

Prob 1. (15 pts) Consider the diagram of Figure 2.14. A robot is set up *1* meter from a table. The table top is *1* meter high and *1* meter square. A cube measuring *20 cm* on a side is placed in the center of the table with frame $o_2x_2y_2z_2$ established at the center of the cube. The same frame also represents the gripper picking up the cube at the moment. Once the gripper picks up the cube, it will place the cube at the back corner of the table with both sides of the cube flush with two sides of the table. The new gripper position and orientation are represented by frame $o_3x_3y_3z_3$. A camera is situated directly above the center of the cube 2 meters above the table top with frame $o_4x_4y_4z_4$ attached as shown.

a) Find the frame $o_2x_2y_2z_2$ transformation matrix relative to robot base frame 0.
b) Find the frame $o_3x_3y_3z_3$ transformation matrix relative to robot base frame 0.
c) Find the frame $o_4x_4y_4z_4$ transformation matrix relative to robot base frame 0.

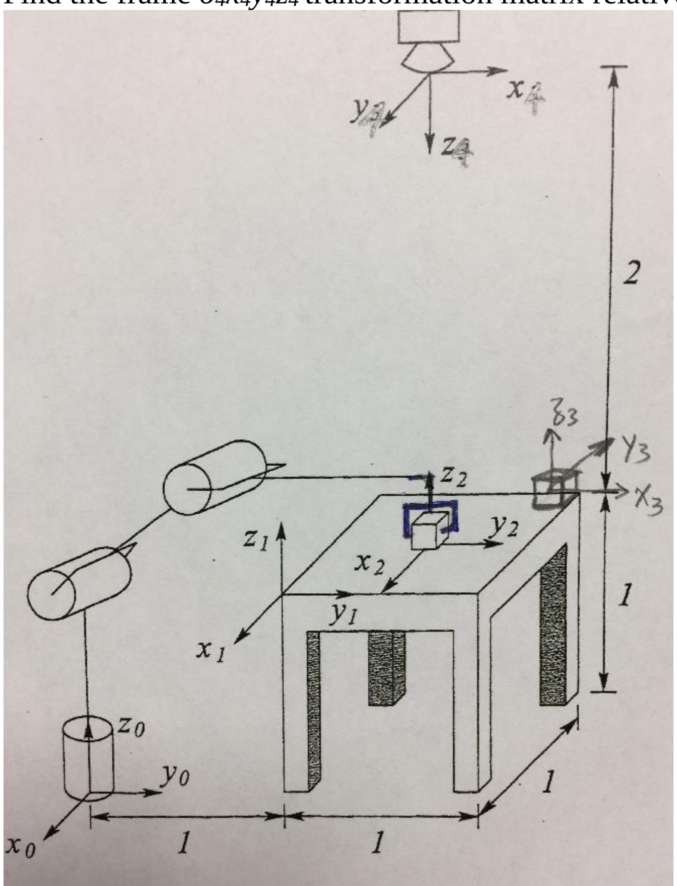


Figure 2.14: Diagram for Problem 2-39.

1a.

$$F = H_2^0 = \begin{array}{c} \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \end{bmatrix} \\ \begin{matrix} x_3 & y_3 & z_3 & O_3 \end{matrix} \end{array}$$

$$F = H_2^0 = \begin{bmatrix} 1 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 1.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b.

$$F = H_3^0 = \begin{bmatrix} 0 & -1 & 0 & -0.9 \\ 1 & 0 & 0 & 1.9 \\ 0 & 0 & 1 & 1.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$F = H_4^0 \begin{bmatrix} 0 & 1 & 0 & -0.5 \\ 1 & 0 & 0 & 1.5 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Prob 2. (Prob 2.6) (10 pts)

For frame F, find the values of the missing elements and complete the matrix representation of the frame.

$$F = \begin{bmatrix} n_x & 0 & -1 & 5 \\ n_y & 0 & 0 & 3 \\ n_z & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.

$$F = \begin{bmatrix} n_x & 0 & -1 & 5 \\ n_y & 0 & 0 & 3 \\ n_z & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$n_x^2 + n_y^2 + n_z^2 = 1$

$n \cdot o = 0$

$n_x o_x + n_y o_y + n_z o_z = 0$

$\vec{n} \times \vec{o} = \vec{a}$

$$\begin{bmatrix} 0 & 0 & -1 & 5 \\ 1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$n_x o_x + n_y o_y + n_z o_z = 0$

$n_x 0 + n_y \cdot 0 + n_z - 1 = 0$

$n_z = 0$

$n_x a_x + n_y a_y + n_z a_z = 0$

$n_x \cdot -1 + n_y \cdot 0 + 0 \cdot 0 = 0$

$n_x = 0$

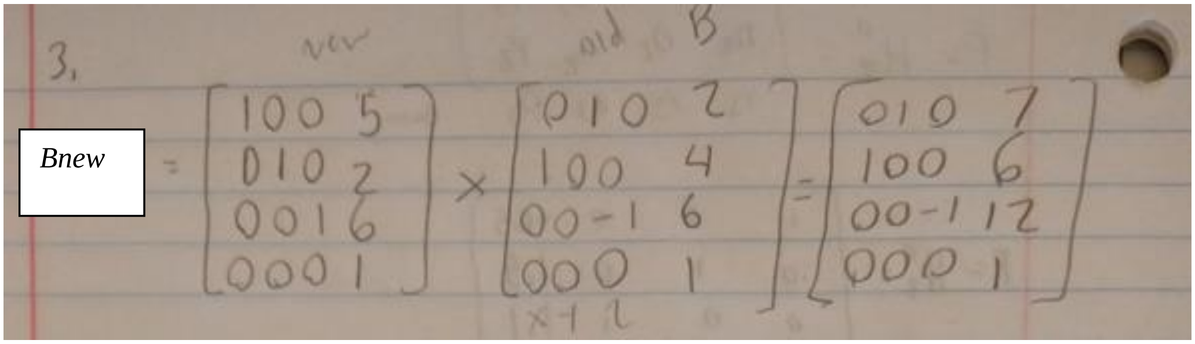
$0^2 + n_y^2 + 0^2 = 1$

$n_y = 1$

Prob 3. (Prob 2.5) (10 pts)

The following frame B was moved a distance of $d = (5, 2, 6)^T$. Find the new location of the frame relative to the reference frame. And verify graphically using the grid provided at the end of this assignment.

$$B = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

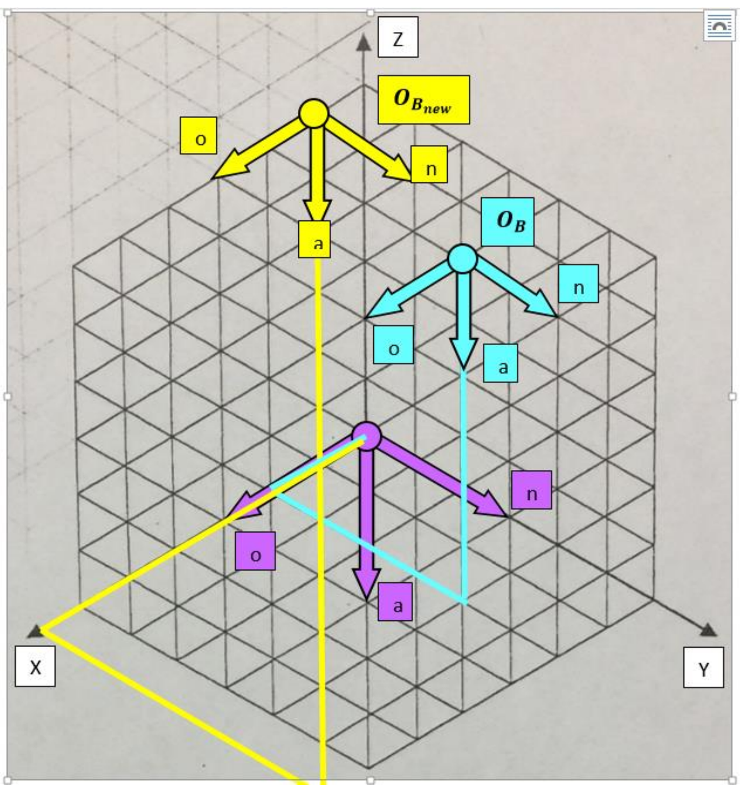


3.

$$Bnew = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 7 \\ 1 & 0 & 0 & 6 \\ 0 & 0 & -1 & 12 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$ teal frame in the grid.

$$B_{new} = \begin{bmatrix} 0 & 1 & 0 & 7 \\ 1 & 0 & 0 & 6 \\ 0 & 0 & -1 & 12 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$ yellow frame in the grid.

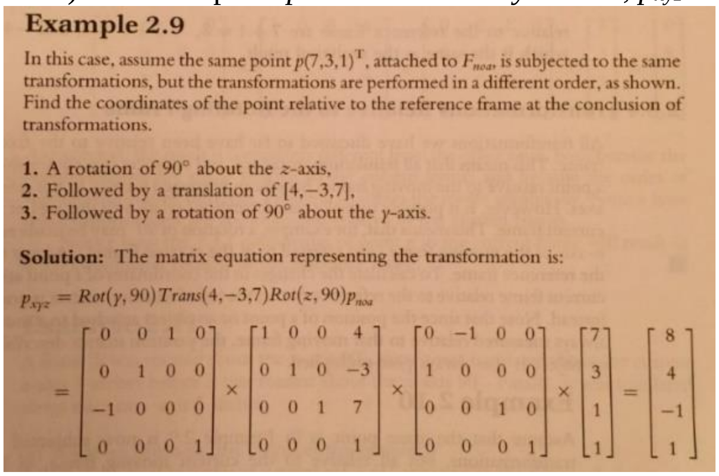


Prob 4. (10 pts)

As in Ex 2.9 in lecture slides or below.

a) Find the new frame representation in xyz fixed frame at the conclusion of three transformations. Use the grid at the end of this assignment to illustrate the frame.

b) Find the point p coordinates in xyz frame, p_{xyz} and illustrate it in the same grid.



Example 2.9

In this case, assume the same point $p(7,3,1)^T$, attached to F_{noa}, is subjected to the same transformations, but the transformations are performed in a different order, as shown. Find the coordinates of the point relative to the reference frame at the conclusion of transformations.

1. A rotation of 90° about the z-axis,
2. Followed by a translation of [4,−3,7],
3. Followed by a rotation of 90° about the y-axis.

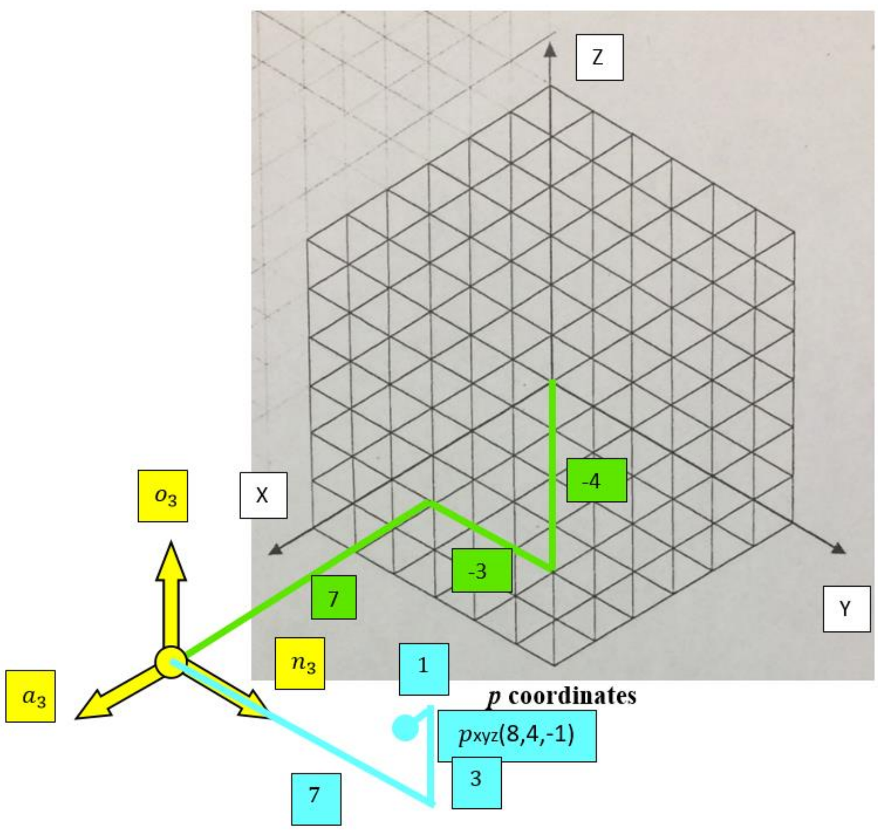
Solution: The matrix equation representing the transformation is:

$$p_{xyz} = Rot(y,90)\,Trans(4,-3,7)\,Rot(z,90)\,p_{noa}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 7 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ -1 \\ 1 \end{bmatrix}$$

$a)\ B_{new} = Rot(y,90) * Trans(4,-3,7) * Rot(x,90) = \begin{bmatrix} 0 & 0 & 1 & 7 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, see yellow frame in grid.

b) $BPxyz = Bnew * Pnoa = \begin{bmatrix} 0 & 0 & 1 & 7 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ -1 \\ 1 \end{bmatrix}$



Prob 5. (Prob 2.13) (10 pts)
Find the new location of point $P(1, 2, 3)^T$ relative to the reference frame after a rotation of 30 degrees about the z-axis followed by a rotation of 60 degrees about the y-axis.

$$Rot(z,30)=\begin{bmatrix} \cos(30) & -\sin(30) & 0 \\ \sin(30) & \cos(30) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad Rot(y,60)=\begin{bmatrix} \cos(60) & 0 & \sin(60) \\ 0 & 1 & 0 \\ -\sin(60) & 0 & \cos(60) \end{bmatrix}$$

If we apply the two rotations to the frame, the new coordinates in the fixe frame will be:

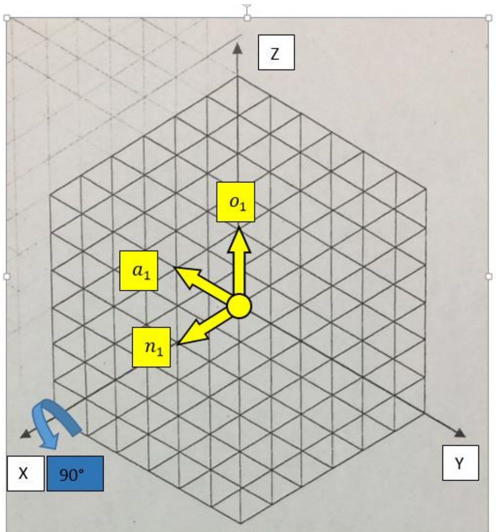
$$P_{xyz}=Rot(y,60)\,Rot(z,30)\,P_{noa}=\begin{bmatrix} \cos(60) & 0 & \sin(60) \\ 0 & 1 & 0 \\ -\sin(60) & 0 & \cos(60) \end{bmatrix}\begin{bmatrix} \cos(30) & -\sin(30) & 0 \\ \sin(30) & \cos(30) & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}=\begin{bmatrix} \sqrt{3}/4 & -1/4 & \sqrt{3}/2 \\ 1/2 & \sqrt{3}/2 & 0 \\ -3/4 & \sqrt{3}/4 & 1/2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}=\begin{bmatrix} 2{,}53 \\ 2{,}23 \\ 1{,}62 \end{bmatrix}$$

Prob 6. (Prob 2.15) (20 pts)

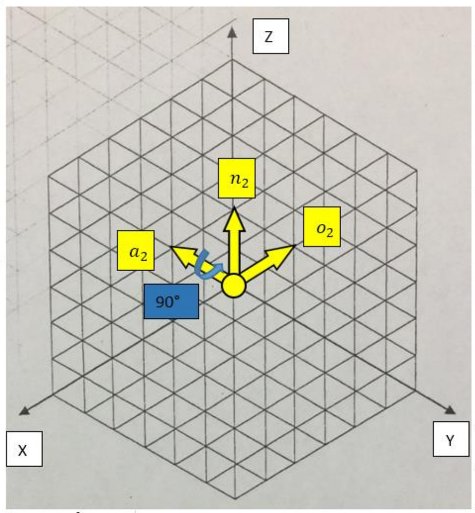
A point p in space is defined as ${}^{B}p = (2, 2, 5)^{T}$ relative to frame B which is attached to the origin of the reference frame A and is parallel to it. Apply the following transformations to frame B and find ${}^{A}p$. Using the 3D grid, plot each transformation and the result to verify it.

- Rotate 90 degrees about x-axis, then
- Rotate 90 degrees about local a-axis, then
- Translate 3 units about y-, 6 units about z-, and 5 units about x-axes.

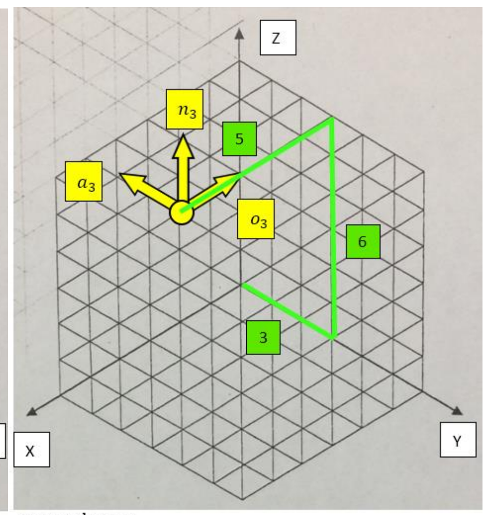
$$^{A}p = Trans(5,3,6)\,Rot(x,90)\,B * Rot(a,90)\,^{B}p = \begin{bmatrix} 1&0&0&5\\ 0&1&0&3\\ 0&0&1&6\\ 0&0&0&1 \end{bmatrix}\begin{bmatrix} 1&0&0&0\\ 0&0&-1&0\\ 0&1&0&0\\ 0&0&0&1 \end{bmatrix}\begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix}\begin{bmatrix} 0&-1&0&0\\ 1&0&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix}\begin{bmatrix} 2\\ 2\\ 5\\ 1 \end{bmatrix} = \begin{bmatrix} 3\\ -2\\ 8\\ 1 \end{bmatrix}$$



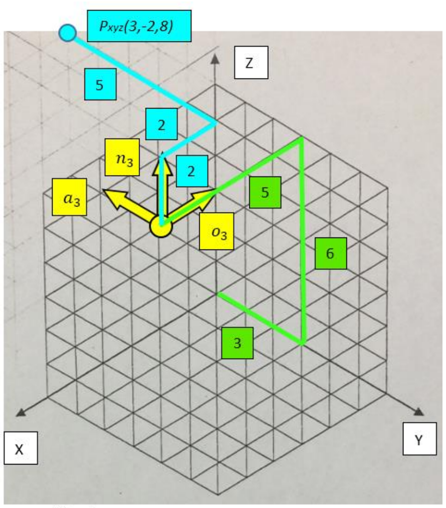
After 1st trans.



After 2nd trans.



After 3rd trans.



***p* coordinates**

Prob 7. Prob 2.16. (20 pts)

A frame B is rotated 90^0 about the z-axis, then translated 3 and 5 units relative to the n and o-axes respectively, then rotated another 90^0 about the n-axis, and finally, 90^0 about the y-axis.

a) Find the new location and orientation of the frame B_{new}.
b) Illustrate the new frame after each transformation, ie. illustrate $n_1o_1a_1$, $n_2o_2a_2$, $n_3o_3a_3$, $n_4o_4a_4$.
c) Find the coordinates of the origins O_1, O_2, O_3 and O_4 of above four transformations and illustrate the origins.

$$B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

a)

$$B_{new} = Rot(y,90)\,Rot(z,90)\,B * Trans(3,5,0)\,Rot(n_3,90)$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 6 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B_{new} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 6 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

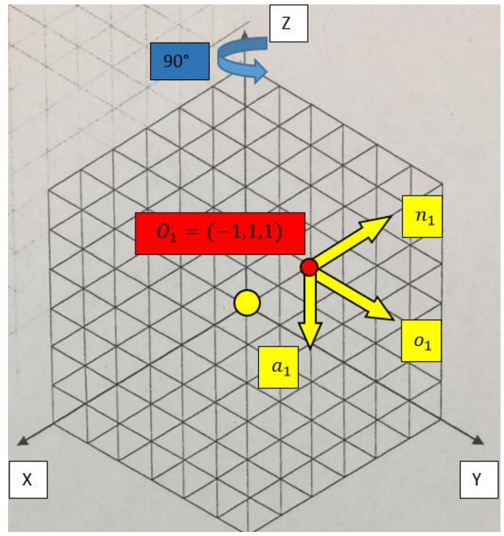
b) And c)

$$B1 = Rot(z, 90)B_{old}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

O1=(-1, 1 1)

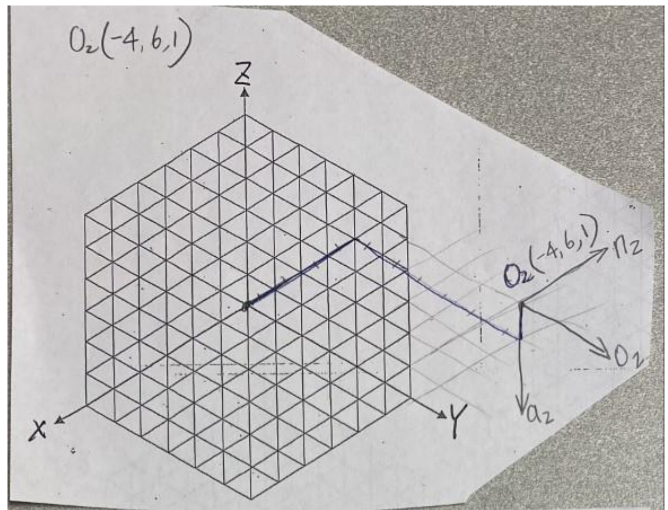


$$B2 = B_1 Trans(3,5,0)$$

$$\begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

O2=(-4,6,1)

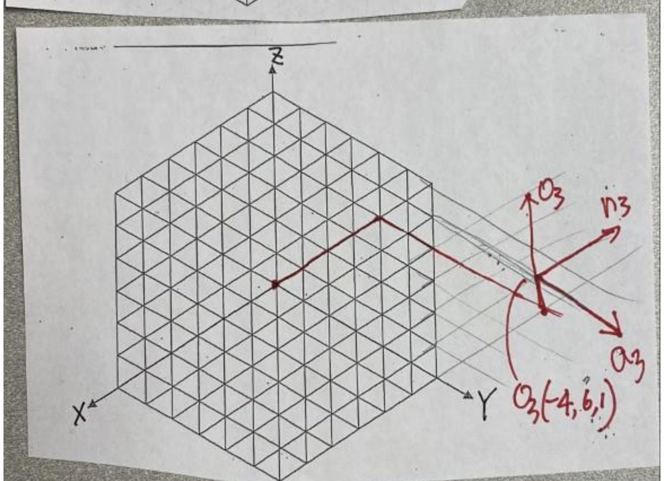


B3=$B_2 Rot(n, 90)$

$$\begin{bmatrix} -1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 & -4 \\ 0 & 0 & -1 & 6 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

O3=(-4, 6, 1)



$$B4 = Rot(y, 90)B_3$$

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & -4 \\ 0 & 0 & -1 & 6 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 6 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

O4=(1, 6, 4)