1.	Prove that $\sum_{k=0}^{n} C_k^n = 2^n$

From Binomial Thm,

$$(x+y)^n = \sum_{k=0}^{n} C_k^n x^k y^{n-k}$$

let, $x = y = 1$

$$\therefore 2^n = \sum_{k=0}^{n} C_k^n 1^k \cdot 1^{n-k} = \sum_{k=0}^{n} C_k^n \quad \checkmark$$

2.	You take a random sample from a population and form a 95% confidence interval for the population mean, μ. What quantity is guaranteed to fall within the confidence interval that you construct? *(Circle and write down the letter of your selection. Include brief computations or short writeup (1-2 sentences) on work sheet below to justify your selection. NOTE – the correct selection without any justification will receive zero score)* *a)* σ *b)* μ *c)* $\bar{x}$ *d)* $z_{\alpha=0.025}$

By definition of std unit normal variable

$$Z = \frac{\bar{X} - \mu_X}{\sigma/\sqrt{n}}$$

$$\therefore\ l \le \mu \le u \rightarrow \bar{X} - Z_{\alpha/2}\,\sigma/\sqrt{n} \le \mu \le \bar{X} + Z_{\alpha/2}\,\sigma/\sqrt{n}$$

$\rightarrow$ $\bar{X}$ will be at center of CI (see slide # 8 in lecture slides on CI).

3.	Let $x_1, x_2, \dots, x_{100}$ be independent random variables that all follow the same probability distribution (that is, same pdf, mean, variance). The population mean, $\mu = E\{x_i\} = 12.5$, and variance $\sigma^2 = Var\{x_i\} = 9$. Find the (approximate) probability that $P(x_1 + x_2 + \cdots + x_{100} > \cancel{729})$ using the central limit theorem $n = 100$

1229

let $y = \frac{1}{100}(X_1 + X_2 + \dots + X_{100}) = \bar{X}$

$$\therefore P(X_1 + X_2 + \dots + X_{100} > \overset{1229}{\cancel{729}}) = P\left(y > \frac{\overset{1229}{\cancel{729}}}{100}\right) = P\left(\bar{X} > \frac{\overset{1229}{\cancel{729}}}{100}\right)$$

let $Z = \dfrac{\bar{X} - \mu}{\sigma / \sqrt{n}}$

$$\therefore P(\bar{X} > \overset{12.29}{\cancel{7.29}}) = P\left(Z > \underbrace{\frac{\overset{12.29}{\cancel{7.29}} - 12.5}{10/\sqrt{100}}}_{-0.21}\right) = 1 - 0.4168$$

$$= \underline{0.5832}$$

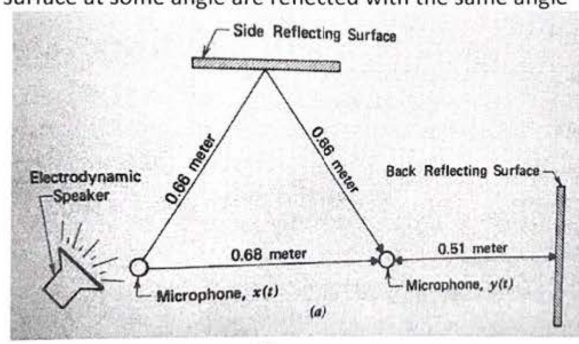
4. Consider the experimental arrangement shown in the figure below. Assume that the speaker is driven by random noise and that the sound waves coming from the speaker behave as spherical waves that travel with constant speed, $c = 340\ m/sec$ (speed of sound in air)

a) Consider the cross-correlation, $R_{xy}(\tau)$. At what values of the lag index, τ will microphone measurements x and y be highly correlated?

b) Consider the cross-correlation, $R_{yx}(\tau)$. Again, at what values of the lag index, τ will microphone measurements x and y be highly correlated?

c) Comment on the lag index values that you obtain from parts (a) and (b). Do they make physical sense? If you only had the xy microphone data and the cross-correlation results and did not know the experimental arrangement, could you (approximately) determine the location of the speaker?

NOTE: as implied in the figure, sound waves that are incident to the side reflecting surface at some angle are reflected with the same angle



$R_{xy}(\tau)$

$\tau_1 = 2$ msec direct

$\tau_2 = 3.9$ msec side

$\tau_3 = 5$ msec back

$R_{yx}(\tau)$

$\tau_1 = -2$ msec

$\tau_2 = -3.9$ msec

$\tau_3 = -5$ msec

$R_{xy}(\tau) = \frac{1}{T}\int_0^T x(t)y(t+\tau)dt$ – since $\tau_i > 0$ indicates waves reach x first, then y

$R_{yx}(\tau') = \frac{1}{T}\int_0^T y(t)x(t+\tau')dt = R_{xy}(-\tau) \rightarrow$ For $R_{yx}(\tau)$, ~~correlation values~~ will show high correlation at $\tau_i' = -\tau_i < 0$

$\Rightarrow$ This indicates that again, waves first reach x-microphone before reaching y-microphone

Can deduce that waves travel from $x \rightarrow y$. Hence, speaker must be to left of x.

5.	The continuous random variable x, has a cumulative distribution function (cdf) modeled as $$F(x)=\begin{cases}0 & x\le -4\\ \dfrac{x+4}{9} & -4\le x\le 5\\ 1 & x\ge 5\end{cases}$$ a) Determine the probability density function (pdf), $f(x)$ b) Compute the **characteristic function ($\phi(k)$)** of $f(x)$, defined as $\phi(k)=E\{e^{jkx}\}=\int_{-\infty}^{\infty} e^{jkx}f(x)dx$, where $k\equiv$ wavenumber, rad/m (assumes x has units of meters) and $j=\sqrt{-1}$. Observe that the characteristic function, $\phi(k)$ is the Fourier Transform of $f(x)$. Apply integral definition above to compute $\phi(k)$. Use Euler's identity: $e^{j\theta}=cos\theta+jsin\theta$ to express the characteristic function in terms of its Real and Imaginary parts c) ~~From the equation, $\phi(k)=E\{e^{jkx}\}$, derive expressions for both the mean $[\mu_x=E\{x\}]$, and mean square $[\psi_x^2=E\{x^2\}]$ values in terms of $\phi(k=0)$~~ $\phi(k=0)$?

$$f(x)=\frac{dF}{dx}=\begin{cases}0 & x\le -4\\ 1/9 & -4<x<5\\ 0 & x>5\end{cases}$$

a) see above

b) $\phi(k)=\int_{-\infty}^{\infty} e^{jkx} f(x)\,dx=\frac{1}{9}\int_{-4}^{5} e^{jkx}\,dx=\frac{1}{j9k}e^{jkx}\Big|_{-4}^{5}$

$$=\frac{-j}{9k}e^{jkx}\Big|_{-4}^{5}=\frac{j}{9k}e^{jkx}\Big|_{5}^{-4}=\frac{j}{9k}\left[e^{-j4k}-e^{j5k}\right]$$

$$=\frac{j}{9k}\left\{\cos 4k-j\sin 4k-\cos 5k-j\sin 5k\right\}$$

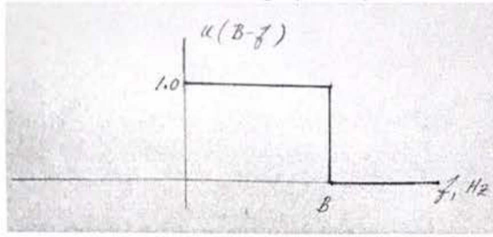
$$=\frac{j}{9k}\left\{\cos 4k-\cos 5k-j(\sin 4k+\sin 5k)\right\}$$

$$\phi(k)=\frac{\sin 4k+\sin 5k+j(\cos 4k-\cos 5k)}{9k}$$

c) ~~From $\phi(k)=E\{e^{jkx}\}$ $\left\{\frac{d\phi}{dk}\right\}_{k=0}=\left\{E\{jxe^{jkx}\}\right\}=j\cdot E\{x\}$~~

$$\phi(k=0)=E\{e^{j0}\}=E\{1\}=1.=\int_{-\infty}^{\infty} f(x)\,dx$$

6. Let, $n(t)$ represent a zero-mean, bandlimited, random noise process. Its power spectrum is given by $G_{nn}(f) = Ku(B-f), V^2/Hz$, where, $K \rightarrow$ const and $u(B-f) \rightarrow$ unit step function shown graphically as



Determine the expression for the auto-correlation function, $R_{nn}(\tau)$ for $n(t)$

By definition

$$R_{nn}(\tau) = \int_{-\infty}^{\infty} \overset{\text{even}}{S_{nn}(f)}\, e^{j2\pi f\tau}\, df$$

$$= 2\int_{0}^{\infty} S_{nn}(f) \cos 2\pi f\tau\, df$$

$$= \int_{0}^{\infty} G_{nn}(f) \cos 2\pi f\tau\, df$$

$$= K\int_{0}^{\infty} u(B-f) \cos 2\pi f\tau\, df = K\int_{0}^{B} \cos 2\pi f\tau\, df$$

$$= K \left. \frac{\sin 2\pi f\tau}{2\pi\tau} \right|_{0}^{B} = K\frac{\sin 2\pi B\tau}{2\pi\tau}$$

$$= KB\frac{\sin \pi(2B\tau)}{\pi(2B\tau)} = KB\, \mathrm{sinc}(2B\tau)$$

"normalized" sinc function

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

For $\mathrm{sinc}(2B\tau)$, zero crossings at

$$2B\tau = n \quad (n = 1, 2, 3, \ldots$$

$$\tau = \frac{n}{2B}$$

7.	Show that the following are Fourier Transform pairs ($T \to$ const) a) $\cos(2\pi fT)\,X(f) \Leftrightarrow 1/2[x(t-T)+x(t+T)]$ b) $\sin(2\pi fT)\,X(f) \Leftrightarrow 1/2[x(t-T)-x(t+T)]$	5.54

a) By definition

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$

$$\therefore \cos(2\pi f T)\, X(f) = \int_{-\infty}^{\infty} \cos(2\pi f T)\, x(t)\, e^{-j2\pi f t}\, dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \left[e^{j2\pi f T} + e^{-j2\pi f T}\right] x(t)\, e^{-j2\pi f t}\, dt$$

$$= \frac{1}{2}\underbrace{\int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f(t-T)}\, dt}_{\text{let, } t' = t-T,\ dt' = dt} + \frac{1}{2}\underbrace{\int_{-\infty}^{\infty} e^{-j2\pi f(t+T)}\, x(t)\, dt}_{\text{let, } t'' = t+T,\ dt'' = dt}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} x(t'+T)\, e^{-j2\pi f t'}\, dt' + \frac{1}{2}\int_{-\infty}^{\infty} x(t''-T)\, e^{-j2\pi f t''}\, dt''$$

$$= \frac{1}{2}\mathcal{F}\{x(t+T)\} + \frac{1}{2}\mathcal{F}\{x(t-T)\} \leftarrow \text{proves (a)}$$

b) $$\sin(2\pi f T)\, X(f) = \frac{1}{2j}\int_{-\infty}^{\infty} \left[e^{j2\pi f T} - e^{-j2\pi f T}\right] x(t)\, e^{-j2\pi f t}\, dt$$

$$= \frac{1}{2j}\int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f(t-T)}\, dt - \frac{1}{2j}\int_{-\infty}^{\infty} e^{-j2\pi f(t+T)}\, x(t)\, dt$$

$$= \frac{1}{2j}\mathcal{F}\{x(t+T)\} - \frac{1}{2j}\mathcal{F}\{x(t-T)\} \leftarrow \text{proves (b)}$$

8. Suppose that against a certain opponent the number of points that the Chongqing University basketball team scores follows a normal distribution with unknown mean, μ and unknown variance, σ^2. Also suppose that over the course of the last *10* games between the two teams, Chongqing University scored the following point totals: *59, 62, 59, 74, 70, 61, 62, 66, 62, 75*

a) Compute a *95%* t–confidence interval for μ

b) Now suppose that you learn that $\sigma^2 = 36$. Compute a *95%* z–confidence interval for μ. How does this compare to the interval in (a)?

$n = 10$

$\bar{x} = 65$

a) $S_x^2 = 35.76$, $S_x = 5.98$

CI: $\bar{x} - t_{\frac{\alpha}{2},9} \frac{S_x}{\sqrt{n}} \le \mu \le \bar{x} + t_{\frac{\alpha}{2},9}$ $\quad t_{0.025,9} = 2.262$

$\therefore$ CI: $65 - (2.262)\left(\frac{5.98}{\sqrt{10}}\right) \le \mu \le 65 + (2.262)\left(\frac{5.98}{\sqrt{10}}\right)$

$\boxed{60.72 \le \mu \le 69.28}$

b) let, $\sigma^2 = 36$, $\sigma = 6$

$\bar{X} - Z_{\alpha/2} \sigma/\sqrt{n} \le \mu \le \bar{X} + Z_{\frac{\alpha}{2}} \sigma/\sqrt{n}$ $\quad Z_{0.025} = 1.96$

$\therefore 65 - (1.96)\frac{6}{\sqrt{10}} \le \mu \le 65 + (1.96)\left(\frac{6}{\sqrt{10}}\right)$

$\boxed{61.28 \le \mu \le 68.72}$ ← little smaller than (a)

9.	In the lecture slides, we studied the simple asset price model for modeling stock price fluctuations. A similar model is used for simulating commodity prices (*com-mod-i-ty* – 4 English syllables). In the marketplace, commodities include things such as oil, metals, farm and agricultural products. The model is given by $dC = \mu dt + \sigma dX, \qquad C(0) = C_o$ where, $C(X,t)$ is the commodity price. Similar to the simple asset price model (but not exactly the same), the (constant) coefficients, μ and σ are related to drift and diffusion price movements. $X(t)$ is a Wiener process. a) Apply the Ito Formula to derive the exact solution for this stochastic ODE b) Determine expressions for $E\{C\}$ and $Var\{C\}$

a) From Ito

$$dC = \left[\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial X^2}\right]dt + \frac{\partial C}{\partial X}dX$$

$\therefore$ (1) $\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial X^2} = \mu$ (2) $\frac{\partial C}{\partial X} = \sigma \rightarrow C(X,t) = \sigma X + g(t)$

(3) I.C. $C(X(0), 0) = \sigma X(0)^{0} + g(0) = C_0$

require $g(0) = C_0$

$\therefore$ from (3)

(4) $C(X,t) = \sigma X + g(t), \; g(0) = C_0$

use (4) $\rightarrow$ (1)

(5) $\frac{\partial C}{\partial t} = \frac{dg}{dt}, \quad \frac{\partial C}{\partial X} = \sigma, \quad \frac{\partial^2 C}{\partial X^2} = 0$

$\therefore$ Obtain –

(6) $\frac{dg}{dt} = \mu \rightarrow g(t) = \mu t + C_0$

Combine (4), (6) $\rightarrow$

$$\boxed{(7)\; C(X,t) = \mu t + \sigma X + C_0}$$

b) $E\{C\} = E\{\mu t + \sigma X + C_0\} = \mu t + C_0$

~~$Var\{C\} = E\{C^2\} - [E\{C\}]^2 = \sigma^2 E\{X^2\} \ldots \mu^2 t^2 \ldots 2\mu C_0 t \ldots C_0^2$~~

(see back)

~~$= \sigma^2 t + \mu^2 t^2 \ldots 2\mu C_0 t \ldots C_0^2$~~

$$C(x,t) = \mu t + C_0 + \sigma x$$

$$E\{C\} = \mu t + C_0$$

$$\therefore C - E\{C\} = \sigma x$$

$$\boxed{\begin{aligned} \mathrm{var}\{C\} &= E\{(C - E\{C\})^2\} = E\{(\sigma x)^2\} \\ &= \sigma^2 E\{x^2\} = \underline{\sigma^2 t} \end{aligned}}$$

10. A random sample of size $n_1 = 16$ is selected from a normal population with a $\mu_1 = 75$ and $\sigma_1 = 8$. A second random sample of size $n_2 = 9$ is taken from another, independent normal population with $\mu_2 = 70$ and standard deviation $\sigma_2 = 12$. Let $\bar{x}_1$ and $\bar{x}_2$ be the two sample mean values. Compute the (approximate) probabilities in parts (a) and (b).

a) The probability that $(\bar{x}_1 - \bar{x}_2) > 4$

b) The probability that $3.5 \le (\bar{x}_1 - \bar{x}_2) \le 5.5$

a) Since $X_1, X_2 \to$ normal, $\bar{X}_1, \bar{X}_2 \to$ normal

Now define

(1) $Z_1 = \dfrac{\bar{X}_1 - \mu_1}{\sigma_1/\sqrt{n_1}}$ (2) $Z_2 = \dfrac{\bar{X}_2 - \mu_2}{\sigma_2/\sqrt{n_2}}$

(3) $\bar{X}_1 = \mu_1 + Z_1 \dfrac{\sigma_1}{\sqrt{n_1}}$ (4) $\bar{X}_2 = \mu_2 + Z_2 \dfrac{\sigma_2}{\sqrt{n_2}}$

$\therefore \bar{X}_1 - \bar{X}_2 = (\mu_1 - \mu_2) + Z_1 \dfrac{\sigma_1}{\sqrt{n_1}} - Z_2 \dfrac{\sigma_2}{\sqrt{n_2}}$

$= (75 - 70) + Z_1 \dfrac{8}{\sqrt{16}} - Z_2 \dfrac{12}{\sqrt{9}}$

let, $Y = \bar{X}_1 - \bar{X}_2 = 5 + 2Z_1 - 4Z_2$

Since $Y \to$ linear combination of $Z_1 Z_2 \to$ NORMAL

$\mu_Y = E\{Y\} = E\{5 + 2Z_1 - 4Z_2\} = 5$

$\sigma_Y^2 = Var\{Y\} = E\{(Y - \mu_Y)^2\} = E\{4Z_1^2 - 16 Z_1 Z_2 + 16 Z_2^2\}$

$\sigma_Y^2 = 4\sigma_1^2 + 16\sigma_2^2 = 4 \cdot 8^2 + 16 \cdot 12^2 = \underline{2560}$

$\therefore P(\bar{X}_1 - \bar{X}_2 > 4) = P(Y > 4) = P(Z_Y > -0.0198) =$

$Z_Y = \dfrac{Y - \mu_Y}{\sigma_Y} = \dfrac{4 - 5}{\sqrt{2560}} = -0.0198$

$P(Z_Y > -0.0198) \doteq 1 - 0.4920 = \underline{\underline{0.508}}$

(over)

b) $P(3.5 \le \bar{X}_1 - \bar{X}_2 \le 5.5) = P(3.5 \le y \le 5.5)$

$$Z_{y_1} = \frac{3.5-5}{\sqrt{2560}} = -0.0296$$

$$Z_{y_2} = \frac{5.5-5}{\sqrt{2560}} = 0.0099$$

$$P(3.5 \le y \le 5.5) = P(-0.0296 \le Z_y \le 0.0099)$$

$$\cong (-0.4880 + 0.5040) = \underline{\underline{0.016}}$$