CHAPTER 24

24.1 (a) The solution can be assumed to be $T = e^{\lambda x}$. This, along with the second derivative $T'' = \lambda^2 e^{\lambda x}$, can be substituted into the differential equation to give

$$\lambda^2 e^{\lambda x} - 0.15 e^{\lambda x} = 0$$

which can be used to solve for

$$\lambda^2 - 0.15 = 0$$
$$\lambda = \pm\sqrt{0.15}$$

Therefore, the general solution is

$$T = Ae^{\sqrt{0.15}\,x} + Be^{-\sqrt{0.15}\,x}$$

The constants can be evaluated by substituting each of the boundary conditions to generate two equations with two unknowns,

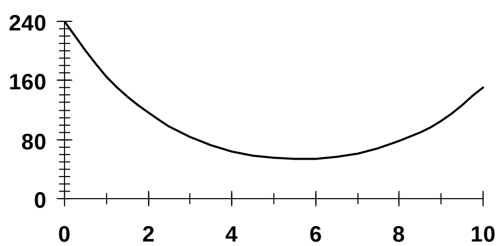
$$240 = A + B$$
$$150 = 48.08563A + 0.020796B$$

which can be solved for A = 3.016944 and B = 236.9831. The final solution is, therefore,

$$T = 3.016944e^{\sqrt{0.15}\,x} + 236.9831e^{-\sqrt{0.15}\,x}$$

which can be used to generate the values below:

x	*T*
0	240
1	165.329
2	115.7689
3	83.79237
4	64.54254
5	55.09572
6	54.01709
7	61.1428
8	77.55515
9	105.7469
10	150



(b) Reexpress the second-order equation as a pair of ODEs:

$$\frac{dT}{dx} = z \qquad\qquad \frac{dz}{dx} = 0.15T$$

These can be stored in a function

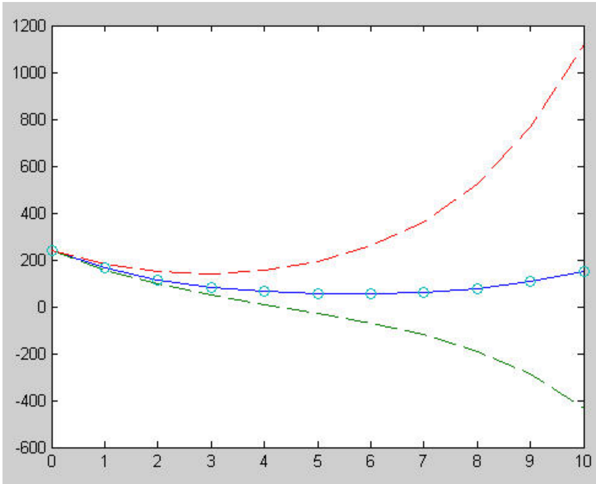
```
function dy=prob2401sys(x,y)
dy=[y(2);0.15*y(1)];
```

The solution was then generated with the following script. Note that we have generated a plot of all the shots as well as the analytical solution.

```
xi=0;xf=10;
xe=[xi:xf];Te=3.016944*exp(sqrt(0.15)*xe)+236.9831*exp(-sqrt(0.15)*xe);
za1=-100;za2=-75;Ta=240;Tb=150;
[x1,T1]=ode45(@prob2401sys,xe,[Ta za1]);
Tb1=T1(length(T1));
[x2,T2]=ode45(@prob2401sys,xe,[Ta za2]);
Tb2=T2(length(T2));
za=za1+(za2-za1)/(Tb2-Tb1)*(Tb-Tb1);
[x,T]=ode45(@prob2401sys,xe,[Ta za]);
plot(x,T(:,1),x1,T1(:,1),'--',x2,T2(:,1),'--',xe,Te,'o')
disp('results:')
fprintf('1st shot:  za1 = %8.4g Tb1 = %8.4g\n',za1,Tb1)
fprintf('2nd shot:  za2 = %8.4g Tb2 = %8.4g\n',za2,Tb2)
fprintf('Final shot: za = %8.4g   T = %8.4g\n',za,T(length(T)))
fprintf('\n     x         T        dT/dx\n')
disp([x T])
```

The results are

```
results:
1st shot:  za1 =     -100 Tb1 =   -432.4
2nd shot:  za2 =      -75 Tb2 =     1119
Final shot: za =   -90.61   T =      150

     x         T        dT/dx
         0  240.0000  -90.6147
    1.0000  165.3278  -60.5889
    2.0000  115.7683  -39.7664
    3.0000   83.7921  -24.9838
    4.0000   64.5424  -13.9958
    5.0000   55.0957   -5.1334
    6.0000   54.0171    2.9492
    7.0000   61.1429   11.4799
    8.0000   77.5553   21.7541
    9.0000  105.7471   35.3325
   10.0000  150.0000   54.2772
```



(c) A centered finite difference can be substituted for the second derivative to give,

$$\frac{T_{i-1} - 2T_i + T_{i+1}}{h^2} - 0.15T_i = 0$$

or for $h = 1$,

$$-T_{i-1} + 2.15T_i - T_{i+1} = 0$$

The first interior node would be

$$2.15T_1 - T_2 = 240$$

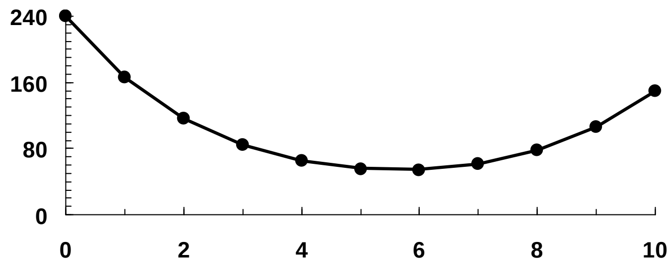
and the last interior node would be

$$-T_8 + 2.15T_9 = 150$$

The tridiagonal system can be solved with the Thomas algorithm or Gauss-Seidel for (the analytical solution is also included)

x	*T*	Analytical
0	240	240
1	165.7573	165.3290
2	116.3782	115.7689
3	84.4558	83.7924
4	65.2018	64.5425
5	55.7281	55.0957
6	54.6136	54.0171
7	61.6911	61.1428
8	78.0223	77.5552
9	106.0569	105.7469
10	150	150

The following plot of the results (with the analytical shown as filled circles) indicates close agreement.



24.2 (a) The solution can be assumed to be $T = e^{\lambda x}$. This, along with the second derivative $T'' = \lambda^2 e^{\lambda x}$, can be substituted into the differential equation to give

$$\lambda^2 e^{\lambda x} - 0.15 e^{\lambda x} = 0$$

which can be used to solve for

$$\lambda^2 - 0.15 = 0$$
$$\lambda = \pm\sqrt{0.15}$$

Therefore, the general solution is

$$T = Ae^{\sqrt{0.15}\,x} + Be^{-\sqrt{0.15}\,x}$$

To evaluate the second condition, we can differentiate the solution and evaluate the result at x = 10,

$$\frac{dT}{dx} = 18.62348A - 0.00805B$$

Therefore, the boundary conditions are expressed as two equations with two unknowns,

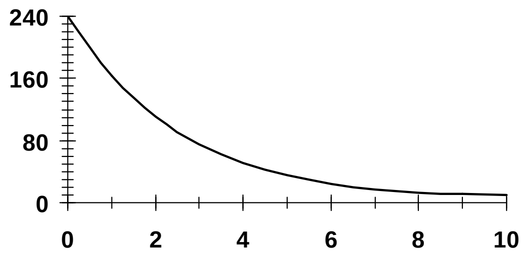
$$240 = A + B$$
$$0 = 48.08563A + 0.020796B$$

which can be solved for A = 0.10375 and B = 239.896. The final solution is, therefore,

$$T = 0.10375e^{\sqrt{0.15}\,x} + 239.89625e^{-\sqrt{0.15}\,x}$$

which can be used to generate the values below:

x	*T*
0	240.000
1	163.016
2	110.791
3	75.393
4	51.447
5	35.315
6	24.546
7	17.506
8	13.124
9	10.736
10	9.978



(b) Reexpress the second-order equation as a pair of ODEs:

$$\frac{dT}{dx} = z \qquad \frac{dz}{dx} = 0.15T$$

These can be stored in a function

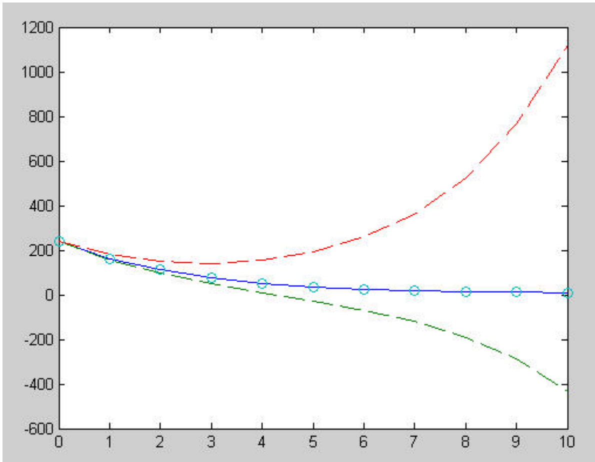
```
function dy=prob2402sys(x,y)
dy=[y(2);0.15*y(1)];
```

The solution was then generated with the following script. Note that we have generated a plot of all the shots as well as the analytical solution.

```
xi=0;xf=10;
xe=[xi:xf];Te=0.10375*exp(sqrt(0.15)*xe)+239.89625*exp(-sqrt(0.15)*xe);
za1=-100;za2=-75;Ta=240;zb=0;
[x1,T1]=ode45(@prob2402sys,xe,[Ta za1]);
zb1=T1(length(T1),2);
[x2,T2]=ode45(@prob2402sys,xe,[Ta za2]);
zb2=T2(length(T2),2);
za=za1+(za2-za1)/(zb2-zb1)*(zb-zb1);
[x,T]=ode45(@prob2402sys,xe,[Ta za]);
plot(x,T(:,1),x1,T1(:,1),'--',x2,T2(:,1),'--',xe,Te,'o')
disp('results:')
fprintf('1st shot:  za1 = %8.4g zb1 = %8.4g\n',za1,zb1)
fprintf('2nd shot:  za2 = %8.4g zb2 = %8.4g\n',za2,zb2)
fprintf('Final shot: za = %8.4g\n',za)
fprintf('\n     x         T        dT/dx\n')
disp([x T])
```

The results are

```
1st shot:  za1 =     -100 zb1 =   -171.5
2nd shot:  za2 =      -75 zb2 =    429.9
Final shot: za =   -92.87

     x         T        dT/dx
         0  240.0000  -92.8712
    1.0000  163.0144  -63.0168
    2.0000  110.7902  -42.7345
    3.0000   75.3931  -28.9428
    4.0000   51.4469  -19.5470
    5.0000   35.3146  -13.1200
    6.0000   24.5461   -8.6858
    7.0000   17.5056   -5.5707
    8.0000   13.1240   -3.3018
    9.0000   10.7357   -1.5344
   10.0000    9.9780   -0.0000
```



(c) A centered finite difference can be substituted for the second derivative to give,

$$\frac{T_{i-1} - 2T_i + T_{i+1}}{h^2} - 0.15T_i = 0$$

or for $h = 1$,

$$-T_{i-1} + 2.15T_i - T_{i+1} = 0$$

The first interior node would be

$$2.15T_1 - T_2 = 240$$

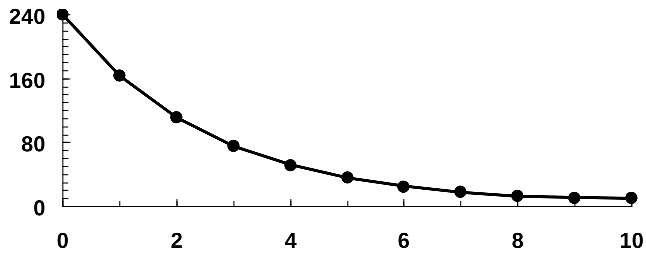
For the last node, we double the weight on the interior node to give

$$-2T_9 + 2.15T_{10} = 0$$

The tridiagonal system can be solved with the Thomas algorithm or Gauss-Seidel for (the analytical solution is also included)

x	*T*	Analytical
0	240	240
1	163.4080	163.0156
2	111.3270	110.7908
3	75.9448	75.3934
4	51.9541	51.4470
5	35.7561	35.3146
6	24.9212	24.5460
7	17.8241	17.5056
8	13.4003	13.1239
9	10.9862	10.7356
10	10.2197	9.9778

The following plot of the results (with the analytical shown as filled circles) indicates close agreement.



24.3 The second-order ODE can be expressed as the following pair of first-order ODEs,

$$\frac{dy}{dx} = z$$
$$\frac{dz}{dx} = \frac{2z + y - x}{7}$$

These can be solved for two guesses for the initial condition of *z*. For our cases we used –1 and –0.5. We solved the ODEs with the Heun method without iteration using a step size of 0.125. The results are

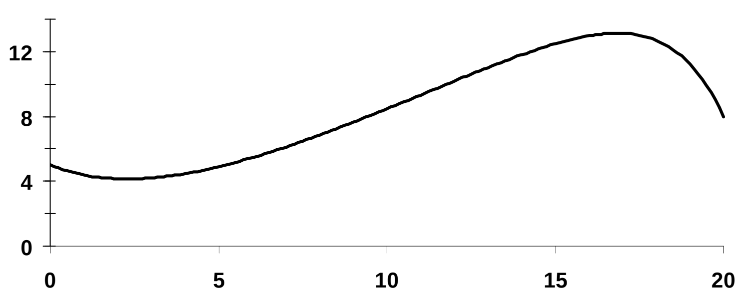
$z(0)$	–1	–0.5
$y(20)$	–11,837.64486	22,712.34615

Clearly, the solution is quite sensitive to the initial conditions. These values can then be used to derive the correct initial condition,

$$z(0) = -1 + \frac{-0.5 + 1}{22712.34615 - (-11837.64486)}(8 - (-11837.64486)) = -0.82857239$$

The resulting fit is displayed below:

x	*y*
0	5
2	4.151601
4	4.461229
6	5.456047
8	6.852243
10	8.471474
12	10.17813
14	11.80277
16	12.97942
18	12.69896
20	8



24.4 Centered finite differences can be substituted for the second and first derivatives to give,

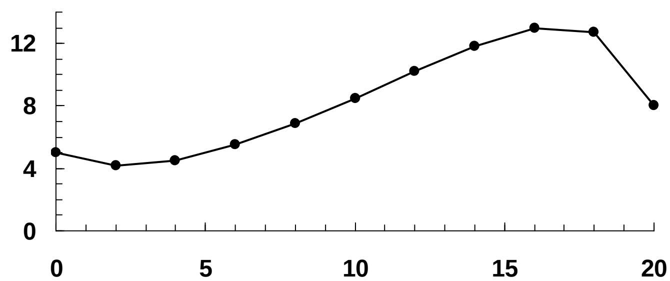
$$7\frac{y_{i+1}-2y_i+y_{i-1}}{\Delta x^2}-2\frac{y_{i+1}-y_{i-1}}{2\Delta x}-y_i+x_i=0$$

or substituting Δx = 2 and collecting terms yields

$$-2.25y_{i-1}+4.5y_i-1.25y_{i+1}=x_i$$

This equation can be written for each node and solved with methods such as the Tridiagonal solver, the Gauss-Seidel method or LU Decomposition. The following solution can be computed:

x	*y*
0	5
2	4.199592
4	4.518531
6	5.507445
8	6.893447
10	8.503007
12	10.20262
14	11.82402
16	13.00176
18	12.7231
20	8



24.5 The linearized term can be substituted into Eq. (P24.5) to give

$$0=\frac{d^2T}{dx^2}+0.05\left(200-T\right)+2.7\times10^{-9}(200)^4-2.7\times10^{-9}(300)^4-4(2.7\times10^{-9})(300)^3(T-300)$$

Collecting terms and substituting the parameter values gives

$$0=\frac{d^2T}{dx^2}+79.93-0.3416T$$

A centered-difference can be substituted for the derivative to give

$$0=\frac{T_{i+1}-2T_i+T_{i-1}}{\Delta x^2}+79.93-0.3416T_i$$

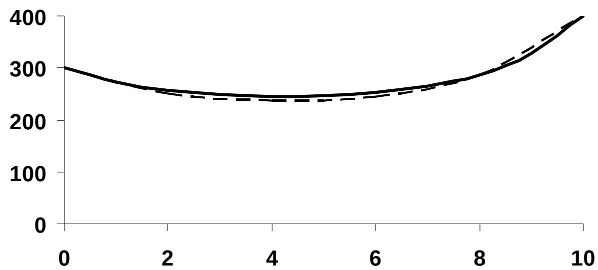
Collecting terms yields,

$$-T_{i+1}+2.3416T_i-T_{i-1}=79.93$$

This equation can be applied to each interior node and the resulting tridiagonal system solved for

x	*T*
0	300
1	271.7125
2	256.3119
3	248.5374
4	245.7334
5	246.9419
6	252.5757
7	264.5593
8	286.9864
9	327.5181
10	400

The comparison with the solution of the nonlinear version (dashed) is depicted in this plot:



24.6

```
function [x,T]=bvshoot(func,tspan,bc,tout,varargin)
% bvshoot: shooting method boundary value ODEs
%   [x,T]=bvshoot(func,tspan,bc,tout,p1,p2,...):
%      uses the shooting method to solve a linear 2nd-order ODE
% input:
%   func = name of the M-file that evaluates the ODEs
%   tspan = [ti , tf]; initial and final times
%   bc = boundary values of Dirichlet conditions
%   tout = desired times for output
%   p1,p2,... = additional parameters used by func
% output:
%   x = vector of independent variable
%   T = vector of solution for dependent variables

if nargin<3,error('at least 3 input arguments required'),end
if nargin<4|isempty(tout),tout=tspan;end
%first shot
za1=1;Ta=bc(:,1);Tb=bc(:,2);
[x1,T1]=ode45(func,tspan,[Ta za1]);
Tb1=T1(length(T1));
%second shot
za2=za1*1.1;
[x2,T2]=ode45(func,tspan,[Ta za2]);
Tb2=T2(length(T2));
%final shot
za=za1+(za2-za1)/(Tb2-Tb1)*(Tb-Tb1);
[x,T]=ode45(func,tout,[Ta za]);
plot(x,T(:,1))
disp('results:')
fprintf('Final shot: za = %8.4g   T = %8.4g\n',za,T(length(T)))
fprintf('\n     x         T        dT/dx\n')
```

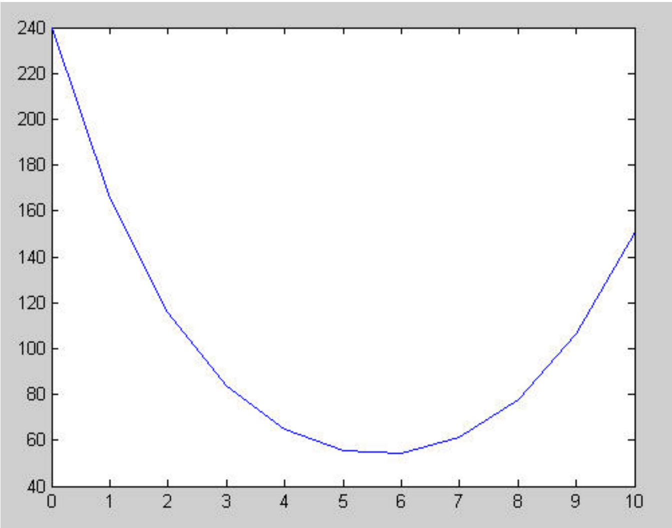
```
disp([x T])
```

Mfile to hold ODEs to test for Example 24.2:

```
function dy=prob2406sys(y)
dy=[y(2);0.15*y(1)];
```

Test run:

```
>> [x,T]=bvshoot(@prob2406sys,[0 10],[240 150],[0:1:10]);
results:
Final shot: za =    -90.6   T =    150.8

     x          T         dT/dx
         0  240.0000  -90.6016
    1.0000  165.3413  -60.5748
    2.0000  115.7973  -39.7492
    3.0000   83.8409  -24.9608
    4.0000   64.6186  -13.9634
    5.0000   55.2108   -5.0869
    6.0000   54.1886    3.0169
    7.0000   61.3968   11.5791
    8.0000   77.9302   21.8999
    9.0000  106.3000   35.5470
   10.0000  150.8148   54.5931
```



24.7 A general formulation that describes Example 24.5 as well as Probs. 24.1 and 24.3 is

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy+f(x)=0$$

Finite difference approximations can be substituted for the derivatives:

$$a\frac{y_{i+1}-2y_i+y_{i-1}}{\Delta x^2}+b\frac{y_{i+1}-y_{i-1}}{2\Delta x}+cy_i+f(x_i)=0$$

Collecting terms

$$-(a-0.5b\Delta x)\,y_{i-1}+\left(2a-c\Delta x^2\right)y_i-(a+0.5b\Delta x)\,y_{i+1}=f(x_i)\Delta x^2$$

Dividing by Δx^2,

$$-\left(a/\Delta x^2-0.5b/\Delta x\right)y_{i-1}+\left(2a/\Delta x^2-c\right)y_i-\left(a/\Delta x^2+0.5b/\Delta x\right)y_{i+1}=f(x_i)$$

The following M-file is set up to solve this general system:

```
function [x,y] = bvFD(xspan,n,bc,a,b,c,fx)
% bvFD: finite-difference method for boundary value ODEs
%   [x,y] = bvFD(xspan,n,bc,a,b,c,fx):
%      uses the finite-difference method to solve a linear 2nd-order ODE
% input:
%   func = name of the M-file that evaluates the ODEs
%   xspan = [xi xf]; initial and final values of independent variable
%   n = number of segments
%   bc = boundary values of Dirichlet conditions
%   a,b,c,fx = parameters
% output:
%   x = vector of independent variable
%   y = vector of solution for dependent variable
m=n-1;
dx=(max(xspan)-min(xspan))/n;
x=[min(xspan):dx:max(xspan)];
%a=1;b=0;c=-0.05;Tinf=200;Ta=300;Tb=400;
e=(-(a/dx^2-b/(2*dx)))*ones(m,1);
g=-(a/dx^2+b/(2*dx))*ones(m,1);
f=(2*a/dx^2-c)*ones(m,1);
r=-fx*ones(m,1);
r(1)=r(1)-e(1)*bc(1);
r(m)=r(m)-g(m)*bc(2);
y = Tridiag(e,f,g,r);
y=[bc(1) y bc(2)];
```

For Example 24.5, $a = 1$, $b = 0$, $c = -h' = -0.05$ and $f(x) = h'T_a = -0.05(200)$. These parameters can be passed to the M-file to yield the solution:

```
>> [x,T]=bvFD([0 10],5,[300 400],1,0,-0.05,-0.05*200)

x =
     0     2     4     6     8    10
T =
  300.0000  283.2660  283.1853  299.7416  336.2462  400.0000
```

24.8 First, the 2nd-order ODE can be reexpressed as the following system of 1st-order ODE's

$$\frac{dT}{dx}=z \qquad\qquad \frac{dz}{dx}=-25$$

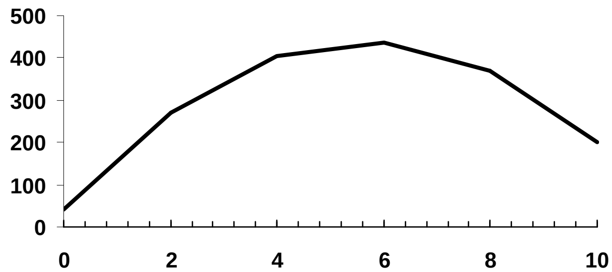
(a) Shooting method: These can be solved for two guesses for the initial condition of z. For our cases we used –1 and –0.5. We solved the ODEs with the 4th-order RK method using a step size of 0.125. The results are

$z(0)$	–1	–0.5
$T(10)$	–1220	–1215

These values can then be used to derive the correct initial condition,

$$z(0) = -1 + \frac{-0.5+1}{-1215-(-1220)}(200-(-1220)) = 141$$

The resulting fit is displayed below:



(b) Finite difference: Centered finite differences can be substituted for the second and first derivatives to give,

$$\frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2} + 25 = 0$$

or substituting Δx = 2 and collecting terms yields

$$-T_{i+1} + 2T_i - T_{i-1} = 100$$

This equation can be written for each node with the result

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 140 \\ 100 \\ 100 \\ 300 \end{Bmatrix}$$

These equations can be solved with MATLAB:

```
>> A=[2 -1 0 0;-1 2 -1 0;0 -1 2 -1;0 0 -1 2];
>> b=[140 100 100 300]';
>> T=A\b
T =
  272.0000
  404.0000
  436.0000
  368.0000
```

24.9 First, the 2nd-order ODE can be reexpressed as the following system of 1st-order ODE's

$$\frac{dT}{dx} = z$$
$$\frac{dz}{dx} = -(0.12x^3 - 2.4x^2 + 12x)$$

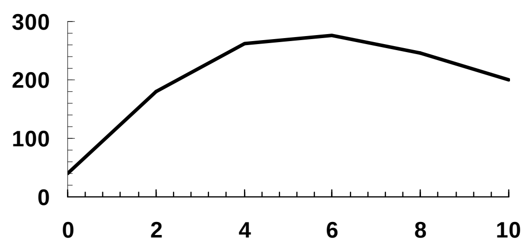
(a) Shooting method: These can be solved for two guesses for the initial condition of *z*. For our cases we used –1 and –0.5. We solved the ODEs with the 4[th]-order RK method using a step size of 0.125. The results are

$z(0)$	–1	–0.5
$T(10)$	–570	–565

These values can then be used to derive the correct initial condition,

$$z(0) = -1 + \frac{-0.5+1}{-565-(-570)}(200-(-570)) = 76$$

The resulting fit is displayed below:



(b) Finite difference: Centered finite differences can be substituted for the second and first derivatives to give,

$$\frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2} + 0.12x_i^3 - 2.4x_i^2 + 12x_i = 0$$

or substituting Δx = 2 and collecting terms yields

$$-T_{i+1} + 2T_i - T_{i-1} = \Delta x^2 (0.12x_i^3 - 2.4x_i^2 + 12x_i)$$

This equation can be written for each node with the result

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 101.44 \\ 69.12 \\ 46.08 \\ 215.36 \end{Bmatrix}$$

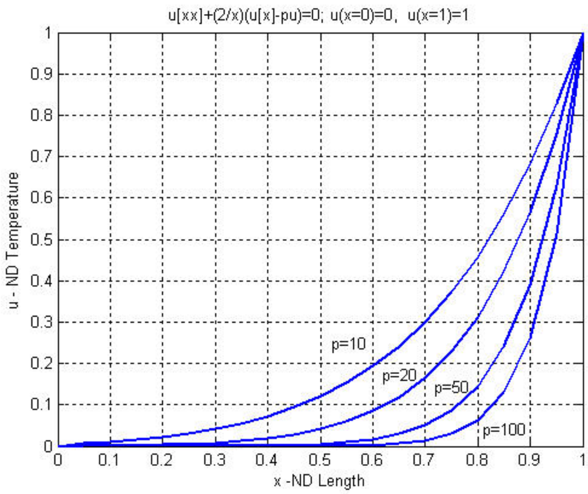
These equations can be solved with MATLAB:

```
>> A=[2 -1 0 0
-1 2 -1 0
0 -1 2 -1
0 0 -1 2];
>> b=[101.44 69.12 46.08 215.36]';
>> x=A\b

x =
  184.1280
  266.8160
  280.3840
```

```
247.8720
```

24.10 This problem can be solved with MATLAB:

```
%   ODE Boundary Value Problem
%   Tapered conical cooling fin
%      u[xx]+(2/x)(u[x]-pu)=0
%      BC. u(x=0)=0  u(x=1)=1
%   i=spatial index, from 1 to R
%       numbering for points is i=1 to i=R for (R-1) dx spaces
%   u(i=1)=0 and u(i=R)=1

R=21;
%Constants
dx=1/(R-1);
dx2=dx*dx;
%Parameters
p(1)=10; p(2)=20; p(3)=50; p(4)=100;
%sizing matrices
u=zeros(1,R); x=zeros(1,R);
a=zeros(1,R); b=zeros(1,R); c=zeros(1,R); d=zeros(1,R);
ba=zeros(1,R); ga=zeros(1,R);
%Independent Variable
x=0:dx:1;
%Boundary Conditions
u(1)=0; u(R)=1;

for k=1:4;
  %/Coefficients
  b(2)=-2-2*p(k)*dx2/dx;
  c(2)=2;
  for i=3:R-2,
    a(i)=1-dx/(dx*(i-1));
    b(i)=-2-2*p(k)*dx2/(dx*(i-1));
    c(i)=1+1/(i-1);
  end
  a(R-1)=1-dx/(dx*(R-2));
  b(R-1)=-2-2*p(k)*dx2/(dx*(R-2));
  d(R-1)=-(1+1/(R-2));
       %Solution by Thomas Algorithm
  ba(2)=b(2);
  ga(2)=d(2)/b(2);
  for i=3:R-1,
    ba(i)=b(i)-a(i)*c(i-1)/ba(i-1);
    ga(i)=(d(i)-a(i)*ga(i-1))/ba(i);
  end
  %back substitution step
  u(R-1)=ga(R-1);
  for i=R-2:-1:2,
    u(i)=ga(i)-c(i)*u(i+1)/ba(i);
  end
  %Plot
  plot(x,u)
  title('u[xx]+(2/x)(u[x]-pu)=0; u(x=0)=0,  u(x=1)=1')
  xlabel('x -ND Length')
  ylabel('u - ND Temperature')
  hold on
end
grid
hold off
gtext('p=10');gtext('p=20');gtext('p=50');gtext('p=100');
```



24.11 Several methods could be used to obtain a solution for this problem (e.g., finite-difference, shooting method, finite-element). The finite-difference approach is straightforward:

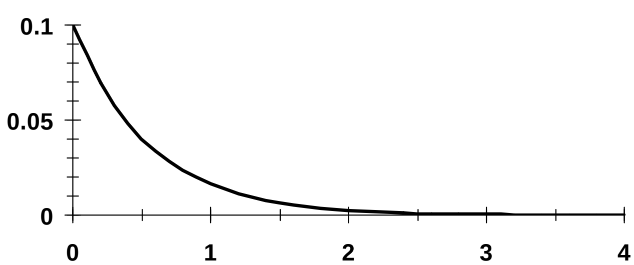
$$D\frac{A_{i-1} - 2A_i + A_{i+1}}{\Delta x^2} - kA_i = 0$$

Substituting parameter values and collecting terms gives

$$-1.5\times10^{-6} A_{i-1} + (3\times10^{-6} + 5\times10^{-6}\Delta x^2)A_i - 1.5\times10^{-6} A_{i+1} = 0$$

Using a Δx = 0.2 cm this equation can be written for all the interior nodes. The resulting linear system can be solved with an approach like the Gauss-Seidel method. The following table and graph summarize the results.

x	*A*	*x*	*A*	*x*	*A*	*x*	*A*
0	0.1						
0.2	0.069544	1.2	0.011267	2.2	0.001779	3.2	0.000257
0.4	0.048359	1.4	0.007814	2.4	0.001224	3.4	0.000166
0.6	0.033621	1.6	0.005415	2.6	0.000840	3.6	9.93E-05
0.8	0.023368	1.8	0.003748	2.8	0.000574	3.8	4.65E-05
1	0.016235	2	0.002591	3	0.000389	4	0



24.12 Substitute centered difference for the derivatives,

$$D\frac{c_{i+1} - 2c_i + c_{i-1}}{\Delta x^2} - U\frac{c_{i+1} - c_{i-1}}{2\Delta x} - kc_i = 0$$

Collecting terms gives

$$-\left(\frac{D}{\Delta x^2}+\frac{U}{2\Delta x}\right)c_{i-1}+\left(\frac{2D}{\Delta x^2}+k\right)c_i-\left(\frac{D}{\Delta x^2}-\frac{U}{2\Delta x}\right)c_{i+1}=0$$

Substituting the parameter values yields

$$-55c_{i-1}+102c_i-45c_{i+1}=0$$

For the inlet node (i = 1), we must use a finite difference approximation for the first derivative. We use a second-order version (recall Table 19.3) so that the accuracy is comparable to the centered differences we employ for the interior nodes,

$$Uc_{\text{in}}=Uc_1-D\frac{-c_3+4c_2-3c_1}{2\Delta x}$$

which can be solved for

$$\left(\frac{3D}{2\Delta x^2}+\frac{U}{\Delta x}\right)c_1-\left(\frac{2D}{\Delta x^2}\right)c_2+\left(\frac{D}{2\Delta x^2}\right)c_3=\frac{U}{\Delta x}c_{\text{in}}$$

Substituting the parameters gives

$$85c_1-100c_2+25c_3=1000$$

For the outlet node (i = 10), the zero derivative condition implies that $c_{11} = c_9$,

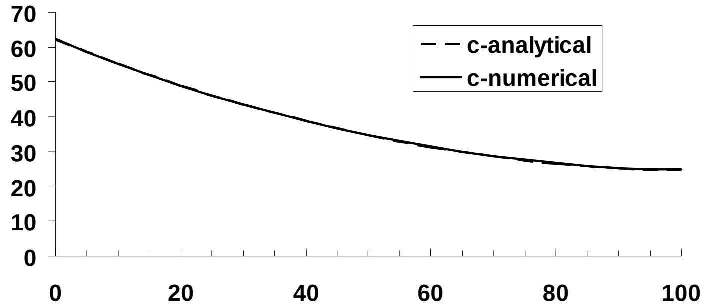
$$-\left(\frac{2D}{\Delta x^2}\right)c_9+\left(\frac{2D}{\Delta x^2}+k\right)c_{10}=0$$

Substituting the parameters gives

$$-100c_9+102c_{10}=0$$

The tridiagonal system can be solved. Here are the results together with the analytical solution. As can be seen, the results are quite close.

x	***c*-numerical**	***c*-analytical**
0	63.6967	62.1767
10	56.4361	55.0818
20	50.0702	48.8634
30	44.5150	43.4390
40	39.7038	38.7437
50	35.5881	34.7300
60	32.1394	31.3708
70	29.3528	28.6615
80	27.2516	26.6258
90	25.8945	25.3223
100	25.3868	24.8552



24.13 Centered differences can be substituted for the derivatives to give

$$D\frac{c_{a,i+1}-2c_{a,i}+c_{a,i-1}}{\Delta x^2}-U\frac{c_{a,i+1}-c_{a,i-1}}{2\Delta x}-k_1c_{a,i}=0$$
$$D\frac{c_{b,i+1}-2c_{b,i}+c_{b,i-1}}{\Delta x^2}-U\frac{c_{b,i+1}-c_{b,i-1}}{2\Delta x}+k_1c_{a,i}-k_2c_{b,i}=0$$
$$D\frac{c_{c,i+1}-2c_{c,i}+c_{c,i-1}}{\Delta x^2}-U\frac{c_{c,i+1}-c_{c,i-1}}{2\Delta x}+k_2c_{b,i}=0$$

Collecting terms gives

$$-50c_{a,i-1}+83c_{a,i}-30c_{a,i+1}=0$$
$$-50c_{b,i-1}+81c_{b,i}-30c_{b,i+1}=3c_{a,i}$$
$$-50c_{c,i-1}+80c_{c,i}-30c_{c,i+1}=c_{b,i}$$

For the inlet node (i = 1), we must use a finite difference approximation for the first derivative. We use a second-order version (recall Table 19.3) so that the accuracy is comparable to the centered differences we employ for the interior nodes. For example, for reactant A,

$$Uc_{a,\text{in}}=Uc_{a,1}-D\frac{-c_{a,3}+4c_{a,2}-3c_{a,1}}{2\Delta x}$$

which can be solved for

$$\left(\frac{3D}{2\Delta x^2}+\frac{U}{\Delta x}\right)c_{a,1}-\left(\frac{2D}{\Delta x^2}\right)c_{a,2}+\left(\frac{D}{2\Delta x^2}\right)c_{a,3}=\frac{U}{\Delta x}c_{a,\text{in}}$$

Because the condition does not include reaction rates, similar equations can be written for the other nodes. Substituting the parameters gives

$$80c_{a,1}-80c_{a,2}+20c_{a,3}=200$$
$$80c_{b,1}-80c_{b,2}+20c_{b,3}=0$$
$$80c_{c,1}-80c_{c,2}+20c_{c,3}=0$$

For the outlet node (i = 10), the zero derivative condition implies that $c_{11} = c_9$,

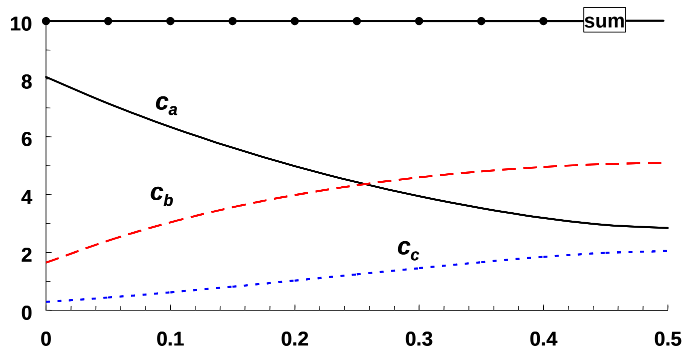
$$-\left(\frac{2D}{\Delta x^2}\right)c_9+\left(\frac{2D}{\Delta x^2}+k\right)c_{10}=0$$

Again, because the condition does not include reaction rates, similar equations can be written for the other nodes. Substituting the parameters gives

$$-80c_{a,9} + 83c_{a,10} = 0$$
$$-80c_{b,9} + 81c_{b,10} = 3c_{a,10}$$
$$-80c_{c,9} + 80c_{c,10} = c_{b,10}$$

Notice that because the reactions are in series, we can solve the systems for each reactant separately in sequence. The result is

x	c_a	c_b	c_c	sum
0	8.0646	1.6479	0.2875	10
0.05	7.1492	2.4078	0.4430	10
0.1	6.3385	3.0397	0.6218	10
0.15	5.6212	3.5603	0.8184	10
0.2	4.9878	3.9846	1.0276	10
0.25	4.4309	4.3258	1.2433	10
0.3	3.9459	4.5955	1.4587	10
0.35	3.5320	4.8035	1.6644	10
0.4	3.1955	4.9572	1.8473	10
0.45	2.9542	5.0591	1.9867	10
0.5	2.8474	5.1021	2.0505	10



24.14 Centered differences can be substituted for the derivatives to give

$$D\frac{c_{a,i+1} - 2c_{a,i} + c_{a,i-1}}{\Delta x^2} = 0 \qquad 0 \le x < L$$
$$D_f\frac{c_{a,i+1} - 2c_{a,i} + c_{a,i-1}}{\Delta x^2} - kc_{a,i} = 0 \qquad L \le x < L + L_f$$

Collecting terms

$$-\frac{D}{\Delta x^2}c_{a,i-1} + \frac{2D}{\Delta x^2}c_{a,i} - \frac{D}{\Delta x^2}c_{a,i+1} = 0 \qquad 0 \le x < L$$
$$-\frac{D_f}{\Delta x^2}c_{a,i-1} + \left(\frac{2D_f}{\Delta x^2} + k\right)c_{a,i} - \frac{D_f}{\Delta x^2}c_{a,i+1} = 0 \qquad L \le x < L + L_f$$

The boundary conditions can be developed. For the first node (i = 1),

$$\frac{2D}{\Delta x^2}c_{a,1} - \frac{D}{\Delta x^2}c_{a,2} = \frac{D}{\Delta x^2}c_{a0}$$

For the last,

$$-\frac{2D_f}{\Delta x^2}c_{a,n-1} + \left(\frac{2D_f}{\Delta x^2} + k\right)c_{a,n}$$

A special equation is also required at the interface between the diffusion layer and the biofilm ($x = L$). A flux balance can be written around this node as

$$D\frac{c_{a,i-1} - c_{a,i}}{\Delta x} + D_f\,\frac{c_{a,i+1} - c_{a,i}}{\Delta x} - k\frac{\Delta x}{2}c_{a,i} = 0$$

Collecting terms gives

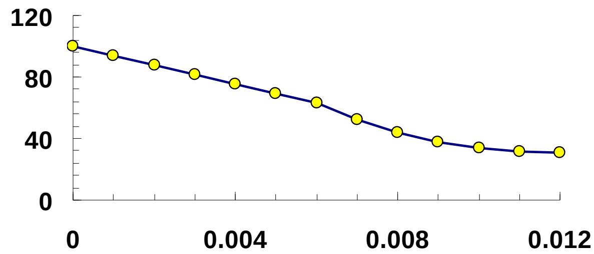
$$-\frac{D}{\Delta x}c_{a,i-1} + \left(\frac{D + D_F}{\Delta x} + k\frac{\Delta x}{2}\right)c_{a,i} - \frac{D_f}{\Delta x}c_{a,i+1} = 0$$

Substituting the parameters gives

first node:	$160,000c_{a,1} - 80,000c_{a,2} = 8{,}000{,}000$
interior nodes (diffusion layer):	$-80,000c_{a,i-1} + 160,000c_{a,i} - 80,000c_{a,i+1} = 0$
boundary node ($i = 6$):	$-80,000c_{a,5} + 121,000c_{a,6} - 40,000c_{a,7} = 0$
interior nodes (biofilm):	$-40,000c_{a,i-1} + 82,000c_{a,i} - 40,000c_{a,i+1} = 0$
last node:	$-80,000c_{a,n-1} + 82,000c_{a,n} = 0$

The solution is

x	$\boldsymbol{c_a}$
0	100.0000
0.001	93.8274
0.002	87.6549
0.003	81.4823
0.004	75.3097
0.005	69.1372
0.006	62.9646
0.007	52.1936
0.008	44.0322
0.009	38.0725
0.01	34.0164
0.011	31.6611
0.012	30.8889

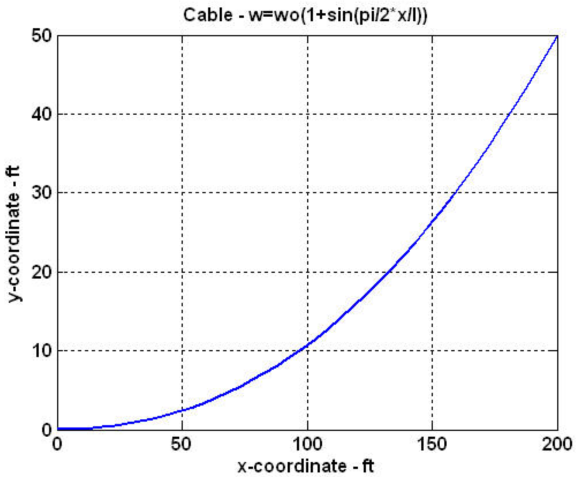


24.15 Main Program:

```
% Hanging static cable - w=w(x)
% Parabolic solution w=w(x)
% CUS Units (lb,ft,s)
% w = wo(1+sin(pi/2*x/l)
% Independent Variable x, xs=start x, xf=end x
% initial conditions [y(1)=cable y-coordinate, y(2)=cable slope];
es=0.5e-7;
xspan=[0,200];
ic=[0 0];
global wToP
wToP=0.0025;
[x,y]=ode45(@slp,xspan,ic);
yf(1)=y(length(x));
wTo(1)=wToP;
ea(1)=1;
wToP=0.002;
[x,y]=ode45(@slp,xspan,ic);
yf(2)=y(length(x));
wTo(2)=wToP;
ea(2)=abs( (yf(2)-yf(1))/yf(2) );
for k=3:10
  wTo(k)=wTo(k-1)+(wTo(k-1)-wTo(k-2))/(yf(k-1)-yf(k-2))*(50-yf(k-1));
  wToP=wTo(k);
  [x,y]=ode45(@slp,xspan,ic);
  yf(k)=y(length(x));
  ea(k)=abs( (yf(k)-yf(k-1))/yf(k) );
  if (ea(k)<=es)
    plot(x,y(:,1)); grid;
    xlabel('x-coordinate - ft'); ylabel('y-coordinate - ft');
    title('Cable - w=wo(1+sin(pi/2*x/l))');
    break
  end
end
```

Function ‘slp’:

```
function dxy=slp(x,y)
global wToP
dxy=[y(2);(wToP)*(1+sin(pi/2*x/200))];
```



24.16 This is a boundary-value problem because the values of only one of the variables are known at two separate points; i.e., at $x = 0$, $y = 0$ and at $x = L$, $y = 0$. We can either use finite differences or the shooting method to obtain results.

For example, for the finite-difference approach, we can substitute centered differences for the second derivative gives

$$EI\,\frac{y_{i-1} - 2y_i + y_{i+1}}{(\Delta x)^2} = \frac{wLx_i}{2} - \frac{wx_i^2}{2}$$

which can be expressed as

$$y_{i-1} - 2y_i + y_{i+1} = \frac{(\Delta x)^2}{EI}\left(\frac{wLx_i}{2} - \frac{wx_i^2}{2}\right)$$

Substituting the parameter values with $\Delta x = 0.6$ m yields

$$y_{i-1} - 2y_i + y_{i+1} = 8.1\times10^{-5}\,x_i - 1.6\times10^{-5}\,x_i^2$$

Writing this equation for the four internal nodes gives

$$\begin{bmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & 1 & -2 & 1 \\ & & 1 & -2 \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = \begin{Bmatrix} 6.48\times10^{-5} \\ 9.72\times10^{-5} \\ 9.72\times10^{-5} \\ 6.48\times10^{-5} \end{Bmatrix}$$

These equations can be solved for the displacements. The results together with the analytical solution are tabulated as

x	*y* (FD)	*y* (analytical)
0	0	0
0.6	-0.000162	-0.0001566
1.2	-0.0002592	-0.0002511
1.8	-0.0002592	-0.0002511
2.4	-0.000162	-0.0001566

3	0	0

Thus, the results are pretty close. If a finer grid is used the results would be improved.

24.17 We can substitute centered differences for the fourth derivative to give

$$EI\,\frac{y_{i+2}-4y_{i+1}+6y_i-4y_{i-1}+y_{i-2}}{\Delta x^4}=-w$$

Substituting the parameters and collecting terms gives

$$y_{i-2}-4y_{i-1}+6y_i-4y_{i+1}+y_{i+2}=-3.24\times10^{-5}$$

Writing the finite-difference equation for the four internal nodes gives the following simultaneous equations

$$\begin{aligned}
y_{-1}-4y_0+6y_1-4y_2+y_3&=-3.24\times10^{-5}\\
y_0-4y_1+6y_2-4y_3+y_4&=-3.24\times10^{-5}\\
y_1-4y_2+6y_3-4y_4+y_5&=-3.24\times10^{-5}\\
y_2-4y_3+6y_4-4y_5+y_6&=-3.24\times10^{-5}
\end{aligned}$$

The boundary conditions can be used to reduce the number of unknowns. The zero end conditions mean that $y_0 = y_5 = 0$. The zero second derivative at node 1 can be represented in finite-difference form as

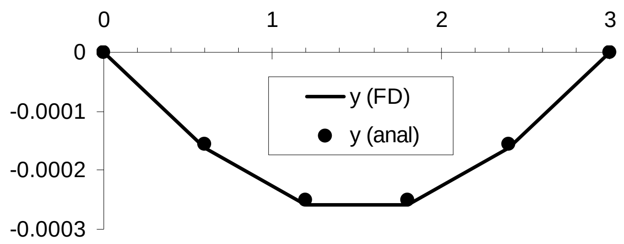
$$\frac{y_{-1}-2y_0+y_1}{(\Delta x)^2}=0$$

Because $y_0 = 0$, this condition implies that $y_{-1} = -y_1$. Similarly, the zero second derivative at node 5 yields $y_6 = -y_4$. Substituting these results into the system of equations gives,

$$\begin{aligned}
5y_1-4y_2+y_3\qquad\qquad&=-3.24\times10^{-5}\\
-4y_1+6y_2-4y_3+y_4\quad&=-3.24\times10^{-5}\\
y_1-4y_2+6y_3-4y_4&=-3.24\times10^{-5}\\
y_2-4y_3+5y_4&=-3.24\times10^{-5}
\end{aligned}$$

The equations can be solved for the displacements.

x	***y* (FD)**	***y* (anal)**
0	0	0
0.6	-0.000162	-0.000157
1.2	-0.000259	-0.000251
1.8	-0.000259	-0.000251
2.4	-0.000162	-0.000157
3	0	0



24.18 (a) The second-order equation can be reexpressed as a pair of first-order equations:

$$\frac{dh}{dx} = z$$

$$\frac{dz}{dx} = -\frac{N}{K\bar{h}} = -\frac{0.0001}{1(7.5)} = -0.0000133333$$

These can be stored in a function

```
function dy=prob2418sys(x,y)
hb=7.5;N=0.0001;K=1;
dy=[y(2);-N/K/hb];
```

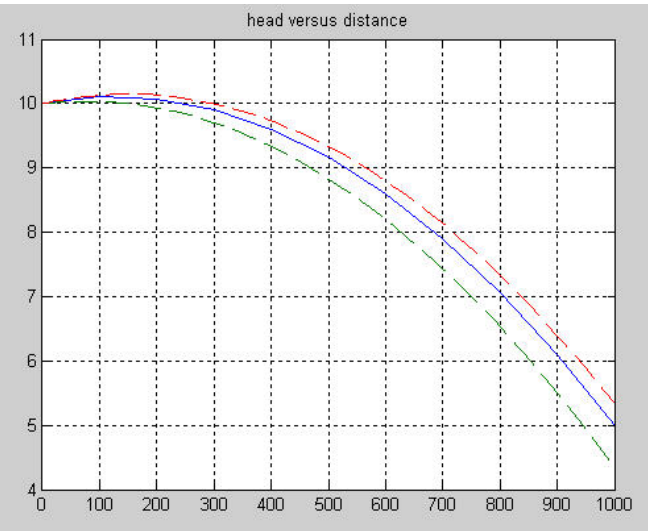
The solution was then generated with the following script. Note that we have generated a plot of all the shots:

```
xi=0;xf=1000;
za1=0.001;za2=0.002;ha=10;hb=5;
[x1,h1]=ode45(@prob2418sys,[xi xf],[ha za1]);
hb1=h1(length(h1));
[x2,h2]=ode45(@prob2418sys,[xi xf],[ha za2]);
hb2=h2(length(h2));
za=za1+(za2-za1)/(hb2-hb1)*(hb-hb1);
[x,h]=ode45(@prob2418sys,[xi:(xf-xi)/10:xf],[ha za]);
plot(x,h(:,1),x1,h1(:,1),'--',x2,h2(:,1),'--')
grid;title('head versus distance')
disp('results:')
fprintf('1st shot:  za1 = %8.4g hb1 = %8.4g\n',za1,hb1)
fprintf('2nd shot:  za2 = %8.4g hb2 = %8.4g\n',za2,hb2)
fprintf('Final shot: za = %8.4g   h = %8.4g\n',za,h(length(h)))
z=[x';h(:,1)'];
fprintf('\n     x          h\n');
fprintf('%6d   %10.5f\n',z);
```

The results are

```
results:
1st shot:  za1 =    0.001 hb1 =    4.333
2nd shot:  za2 =    0.002 hb2 =    5.333
Final shot: za = 0.001667   h =        5

     x         h
     0     10.00000
   100     10.10000
   200     10.06667
   300      9.90000
   400      9.60000
```

```
   500      9.16667
   600      8.60000
   700      7.90000
   800      7.06667
   900      6.10000
  1000      5.00000
```



(b) We can substitute a centered difference for the second derivative to give

$$K\bar{h}\frac{h_{i-1} - 2h_i + h_{i+1}}{(\Delta x)^2} + N = 0$$

Collecting terms,

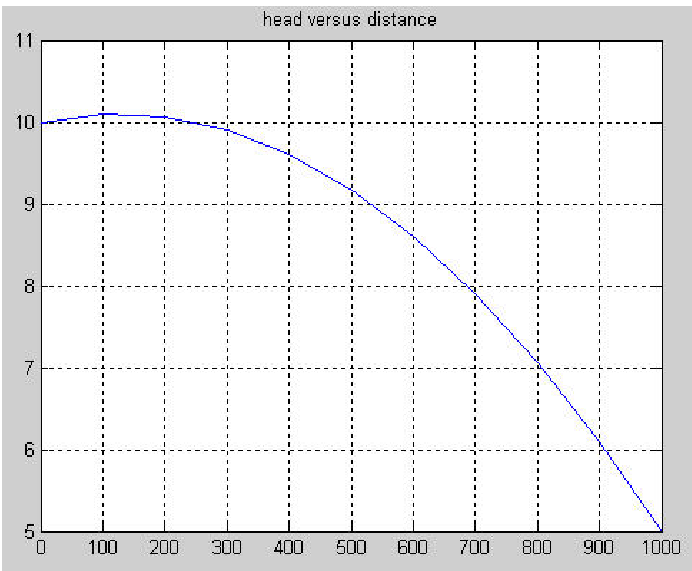
$$-h_{i-1} + 2h_i - h_{i+1} = \frac{N(\Delta x)^2}{K\bar{h}}$$

Substituting the parameter values gives a tridiagonal system of linear algebraic equations. These can be solved with MATLAB as in the following script,

```
h0=10;hn=5;hb=(h0+hn)/2;N=0.0001;K=1;dx=100;
L=1000;n=L/dx-1;
rhs=N*dx^2/K/hb;
a=-1;b=2;c=-1;
A = b*diag(ones(n,1)) + c*diag(ones(n-1,1),1) + a*diag(ones(n-1,1),-1);
b = rhs*ones(n,1);
b(1)=b(1)+h0;
b(n)=b(n)+hn;
h=A\b;
h=[h0 h' hn];
x=[0:dx:L];
z=[x;h];
disp('Results:')
fprintf('\n     x         h\n');
fprintf('%6d   %10.5f\n',z);
plot(x,h);grid;title('head versus distance')
```

The output is

```
Results:

      x          h
      0     10.00000
    100     10.10000
    200     10.06667
    300      9.90000
    400      9.60000
    500      9.16667
    600      8.60000
    700      7.90000
    800      7.06667
    900      6.10000
   1000      5.00000
```



24.19 (a) The product rule can be used to evaluate the derivative

$$Kh\frac{d^2h}{dx^2} + K\left(\frac{dh}{dx}\right)^2 + N = 0$$

We can then define a new variable, $z = dh/dx$ and express the second-order ODE as a pair of first-order ODEs,

$$\frac{dh}{dx} = z$$

$$\frac{dz}{dx} = -\frac{1}{h}z^2 - \frac{N}{Kh}$$

These equations can be solved iteratively using an approach similar to Example 24.4. First,an M-file can be developed to compute the right-hand sides of these equations,

```
function dy=dydxGW(x,y)
dy=[y(2);-1/y(1)*y(2)^2-0.0001/y(1)];
```

Next, we can build a function to hold the residual that we will try to drive to zero as

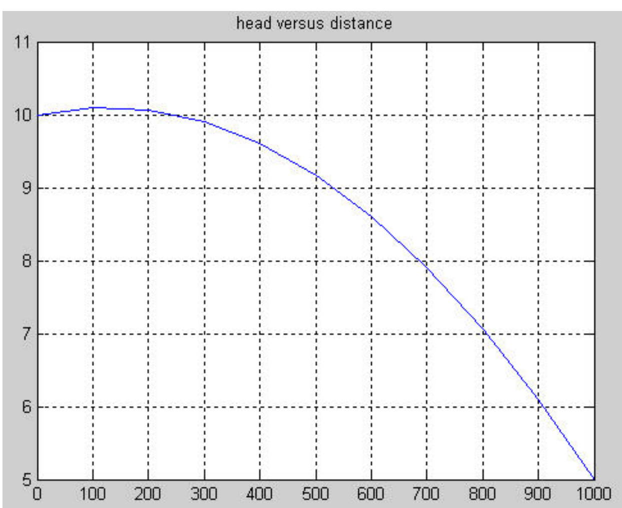
```
function r=resGW(za)
[x,y]=ode45(@dydxGW,[0 1000],[10 za]);
r=y(length(x),1)-5;
```

We can then find the root with the `fzero` function (note that we obtain the initial guess from the linear solution obtained in Prob. 24.18),

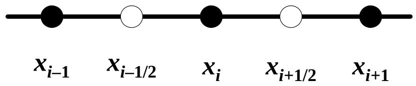
```
>> format long
>> za=fzero(@resGW, 0.0001667)

za =
   0.00125000235608
```

Thus, we see that if we set the initial trajectory $z(0) = 0.00125$, the residual function will be driven to zero and the head boundary condition, $h(1000) = 5$ should be satisfied. This can be implemented by developing a script to generate the entire solution and plotting head versus distance,

```
[x,h]=ode45(@dydxGW,[0:100:1000],[10 za]);
z=[x';h(:,1)'];
fprintf('\n     x         h\n');
fprintf('%6d   %10.5f\n',z);
plot(x,y(:,1))

     x         h
     0     10.00000
   100     10.07472
   200     10.04988
   300      9.92472
   400      9.69536
   500      9.35414
   600      8.88820
   700      8.27648
   800      7.48332
   900      6.44206
  1000      5.00000
```



(b) A nodal scheme for this problem is shown below:



We can substitute a centered difference for the outer first derivative to give

$$\frac{\left(Kh\frac{dh}{dx}\right)_{i+1/2} - \left(Kh\frac{dh}{dx}\right)_{i-1/2}}{\Delta x} + N = 0 \qquad (1)$$

We can then apply centered differences to evaluate the remaining derivatives

$$\left(Kh\frac{dh}{dx}\right)_{i+1/2} = Kh_{i+1/2}\frac{h_{i+1} - h_i}{\Delta x}$$

$$\left(Kh\frac{dh}{dx}\right)_{i-1/2} = Kh_{i-1/2}\frac{h_i - h_{i-1}}{\Delta x}$$

which can be substituted into (1) to give
‘

$$K\frac{\frac{h_i + h_{i+1}}{2}\frac{h_{i+1} - h_i}{\Delta x} - \frac{h_i + h_{i-1}}{2}\frac{h_i - h_{i-1}}{\Delta x}}{\Delta x} + N = 0$$

Collecting terms yields

$$(h_i + h_{i+1})(h_{i+1} - h_i) - (h_i + h_{i-1})(h_i - h_{i-1}) = -\frac{2\Delta x^2}{K}N$$

The terms on the left-hand side can be multiplied out to give

$$h_i h_{i+1} - h_{i+1}h_i - h_i h_i + h_{i+1}h_{i+1} - h_i h_i + h_i h_{i-1} - h_{i-1}h_i + h_{i-1}h_{i-1} = -\frac{2\Delta x^2}{K}N$$

Cancelling and collecting terms gives

$$h_{i-1}^2 - 2h_i^2 + h_{i+1}^2 = -\frac{2\Delta x^2}{K}N$$

An iterative solution similar to Gauss-Seidel can then be developed as

$$h_i = \sqrt{\frac{h_{i-1}^2 + h_{i+1}^2}{2} + \frac{\Delta x^2}{K}N}$$

The results are

x	***h***
0	10
100	10.07472
200	10.04988
300	9.924717
400	9.69536

500	9.354143
600	8.888194
700	8.276473
800	7.483315
900	6.442049
1000	5

Note that these values are quite similar to those obtained with the shooting method in part **(a)**.

24.20 A centered differences can be substituted for the derivative to give

$$\frac{V_{i+1} - 2V_i + V_{i-1}}{\Delta x^2} = -\frac{\rho_v}{\varepsilon}$$

Collecting variables and substituting parameters gives

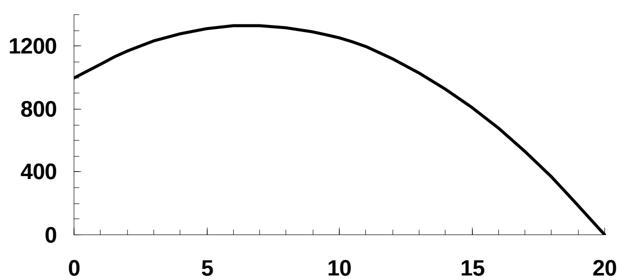
$$-V_{i+1} + 2V_i - V_{i-1} = 60$$

The equations for the first and last node reflect the boundary conditions

$$2V_1 - V_2 = 1060$$
$$-V_8 + 2V_9 = 60$$

The equations can be solved for

x	*V*
0	1000
2	1170
4	1280
6	1330
8	1320
10	1250
12	1120
14	930
16	680
18	370
20	0



24.21 The second-order equation can be reexpressed as a pair of first-order equations:

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = g - \frac{c}{m}v$$

A function can be developed to hold these equations,

```
function dy=prob2421sys(x,y)
g=9.81;c=12.5;m=70;
dy=[y(2);g-c/m*y(2)];
```

The solution was then generated with the following script. Note that we have generated a plot of all the shots as well as the analytical solution.

```
ti=0;tf=12;
za1=0;za2=50;xa=0;xb=500;
[t1,x1]=ode45(@prob2421sys,[ti tf],[xa za1]);
xb1=x1(length(x1));
[t2,x2]=ode45(@prob2421sys,[ti tf],[xa za2]);
xb2=x2(length(x2));
za=za1+(za2-za1)/(xb2-xb1)*(xb-xb1);
[t,x]=ode45(@prob2421sys,[ti tf],[xa za]);
plot(t,x(:,1),t1,x1(:,1),'--',t2,x2(:,1),'--')
disp('results:')
fprintf('1st shot:  za1 = %8.4g xb1 = %8.4g\n',za1,xb1)
fprintf('2nd shot:  za2 = %8.4g xb2 = %8.4g\n',za2,xb2)
fprintf('Final shot: za = %8.4g   x = %8.4g\n',za,x(length(x)))
fprintf('\n     t         x        dx/dt\n')
disp([t x])
```

The results are

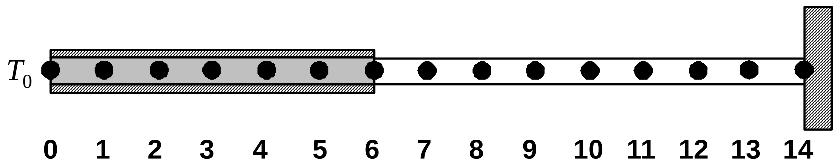
```
results:
1st shot:  za1 =        0 xb1 =    387.7
2nd shot:  za2 =       50 xb2 =    634.8
Final shot: za =    22.72   x =      500

     t         x        dx/dt
         0         0   22.7224
    1.2000   31.1280   28.9359
    2.4000   68.9674   33.9508
    3.6000  112.2237   37.9985
    4.8000  159.8520   41.2654
    6.0000  211.0091   43.9023
    7.2000  265.0143   46.0305
    8.4000  321.3183   47.7482
    9.6000  379.4776   49.1346
   10.8000  439.1345   50.2536
   12.0000  500.0000   51.1567
```

24.22 Heat balances for the two sections can be written as

rod: $$\frac{d^2T}{dx^2} = 0$$

tube: $$\frac{d^2T}{dx^2} + \frac{2h}{rk}(T_\infty - T) = 0$$

The nodes can be set up as shown:



Substituting finite differences for the interior nodes gives

Rod nodes i = 1 through 5: $\frac{T_{i-1} - 2T_i + T_{i+1}}{\Delta x^2} = 0$

Tube nodes i = 7 through 13: $\frac{T_{i-1} - 2T_i + T_{i+1}}{\Delta x^2} + \frac{2h}{rk_2}(T_\infty - T_i) = 0$

At the left end (node 0), $T = T_{0,}$ so the heat balance for node 1 is

$$2T_1 - T_2 = T_0$$

For nodes 2 through 5:

$$-T_1 + 2T_2 - T_3 = 0$$
$$-T_2 + 2T_3 - T_4 = 0$$
$$-T_3 + 2T_4 - T_5 = 0$$
$$-T_4 + 2T_5 - T_6 = 0$$

For nodes 7 through 13

$$-T_{i+1} + \left(2 + \frac{2h\Delta x^2}{rk_2}\right)T_i - T_{i-1} = \frac{2h\Delta x^2}{rk_2}T_\infty$$

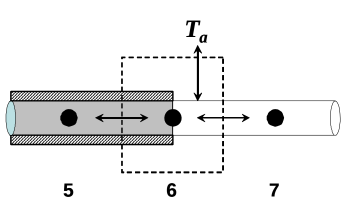
or substituting the parameters,

$$-T_{i-1} + \left(2 + \frac{2\times 3000 \text{ J/(s m}^2\text{K)}(0.05 \text{ m})^2}{0.03 \text{ m}\times 0.615 \text{ J/(s m K)}}\right)T_i + T_{i+1} = \frac{2\times 3000 \text{ J/(s m}^2\text{K)}(0.05 \text{ m})^2}{0.03 \text{ m}(0.615 \text{ J/(s m K)}}T_\infty$$

which gives

$$-T_{i-1} + 3254T_i - T_{i+1} = 975{,}610$$

To write a heat balance at the node between the rod and the tube, a heat balance can be written for the element enclosed by the dashed line depicted below:



The steady-state heat balance for this element is

$$0 = -k_1 \frac{T_6 - T_5}{\Delta x} A_c - k_2 \frac{T_6 - T_7}{\Delta x} A_c + h(T_\infty - T_6) A_s$$

where A_c = the cross-sectional area (m^2) = πr^2, and A_s = the surface area (m^2) of the tube within the element boundary = $\pi r \Delta x$. Substituting these areas gives

$$0 = -k_1 \frac{T_6 - T_5}{\Delta x} \pi r^2 - k_2 \frac{T_6 - T_7}{\Delta x} \pi r^2 + h(T_\infty - T_6) \pi r \Delta x$$

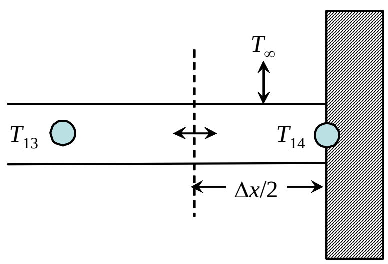
or collecting terms

$$-k_1 T_5 + \left(k_1 + k_2 + \frac{h}{r} \Delta x^2 \right) T_6 - k_2 T_7 = \frac{h}{r} \Delta x^2 T_\infty$$

Substituting the parameters gives

$$-80.2T_5 + 1080.8T_6 - 0.615T_7 = 300,000$$

At the right end (i = 14), a heat balance is written as



$$0 = -k_2 \frac{T_{14} - T_{13}}{\Delta x} A_c + h(T_\infty - T_{14}) A_s$$

Substituting the areas and collecting terms gives

$$-T_{13} + \left(1 + \frac{h \Delta x^2}{k_2 r} \right) T_{14} = \frac{h \Delta x^2}{k_2 r} T_\infty$$

Substituting parameters yields

$$-T_{13} + 1627T_{14} = 487805$$

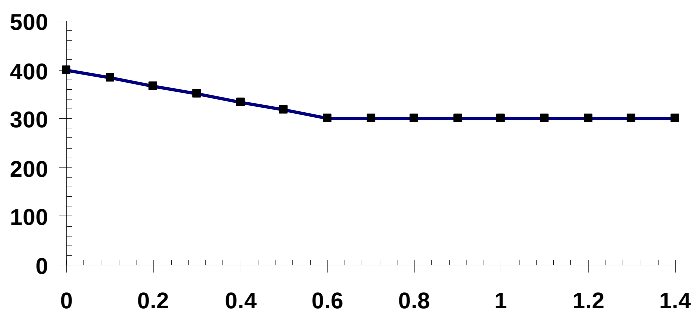
The foregoing equations can be assembled in matrix form as

$$
\begin{bmatrix}
2 & -1 & & & & & & & & & & & & \\
-1 & 2 & -1 & & & & & & & & & & & \\
 & -1 & 2 & -1 & & & & & & & & & & \\
 & & -1 & 2 & -1 & & & & & & & & & \\
 & & & -1 & 2 & -1 & & & & & & & & \\
 & & & & -80.2 & 1080.8 & -0.615 & & & & & & & \\
 & & & & & -1 & 3254 & -1 & & & & & & \\
 & & & & & & -1 & 3254 & -1 & & & & & \\
 & & & & & & & -1 & 3254 & -1 & & & & \\
 & & & & & & & & -1 & 3254 & -1 & & & \\
 & & & & & & & & & -1 & 3254 & -1 & & \\
 & & & & & & & & & & -1 & 3254 & -1 & \\
 & & & & & & & & & & & -1 & 3254 & -1 \\
 & & & & & & & & & & & & -1 & 1627
\end{bmatrix}
\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \\ T_{10} \\ T_{11} \\ T_{12} \\ T_{13} \\ T_{14} \end{Bmatrix}
=
\begin{Bmatrix} 400 \\ 0 \\ 0 \\ 0 \\ 0 \\ 300000 \\ 975610 \\ 975610 \\ 975610 \\ 975610 \\ 975610 \\ 975610 \\ 975610 \\ 487805 \end{Bmatrix}
$$

These can be solved for

$T_1 = 383.553$	$T_2 = 367.106$	$T_3 = 350.659$	$T_1 = 334.212$	$T_1 = 317.765$
$T_5 = 301.318$	$T_6 = 300.0004$	$T_7 = 300$	$T_8 = 300$	$T_9 = 300$
$T_{10} = 300$	$T_{11} = 300$	$T_{12} = 300$	$T_{14} = 300$	

A plot of the results can be developed as



24.23 For this case, with the exception of the node at the boundary between the rod and the tube, all interior nodes are represented by

$$-T_{i-1} + 2T_i - T_{i+1} = 0$$

The first interior node is

$$2T_1 - T_2 = 400$$

and the last interior node is

$$-T_{12} + 2T_{13} = 300$$

A heat balance can be written for the node at the boundary between the rod and the tube as

$$0 = -k_1 \frac{T_6 - T_5}{\Delta x} - k_2 \frac{T_6 - T_7}{\Delta x}$$

or multiplying by Δx and substituting the parameters

$$-80.2T_5 + 80.815T_6 - 0.615T_7 = 0$$

The foregoing equations can be assembled in matrix form as

$$\begin{bmatrix} 2 & -1 & & & & & & & & & & & \\ -1 & 2 & -1 & & & & & & & & & & \\ & -1 & 2 & -1 & & & & & & & & & \\ & & -1 & 2 & -1 & & & & & & & & \\ & & & -1 & 2 & -1 & & & & & & & \\ & & & & -80.2 & 80.815 & -0.615 & & & & & & \\ & & & & & -1 & 2 & -1 & & & & & \\ & & & & & & -1 & 2 & -1 & & & & \\ & & & & & & & -1 & 2 & -1 & & & \\ & & & & & & & & -1 & 2 & -1 & & \\ & & & & & & & & & -1 & 2 & -1 & \\ & & & & & & & & & & -1 & 2 & -1 \\ & & & & & & & & & & & -1 & 2 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \\ T_{10} \\ T_{11} \\ T_{12} \\ T_{13} \end{Bmatrix} = \begin{Bmatrix} 400 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 300 \end{Bmatrix}$$

These can be solved for

$T_1 = 399.9047$ $T_2 = 399.8094$ $T_3 = 399.7141$ $T_4 = 399.6188$ $T_5 = 399.5235$
$T_6 = 399.4282$ $T_7 = 386.9996$ $T_8 = 374.5711$ $T_9 = 362.1426$ $T_{10} = 349.7141$
$T_{11} = 337.857$ $T_{12} = 324.857$ $T_{13} = 312.4285$

A plot of the results can be developed as