Chapter 4

PROBLEM 4.1

> **Show that in the limit $\Delta x \to 0$, the difference equation for one-dimensional steady conduction with heat generation, Equation (4.2), is equivalent to the differential equation, Equation (2.24).**

GIVEN

- One dimensional steady conduction with heat generation

SHOW

(a) In the limit of small Δx, the difference equation is equivalent to the differential equation

SOLUTION

From Equation (4.2)

$$T_{i+1} - 2T_i + T_{i-1} = -\frac{\Delta x^2}{k}\dot{q}_{G,i}$$

By definition

$$T_{i-1} = T(x - \Delta x)$$
$$T_i = T(x)$$
$$T_{i+1} = T(x + \Delta x)$$

so we can rewrite Equation (4.2) as follows

$$\frac{T(x+\Delta x) - 2T(x) + T(x-\Delta x)}{\Delta x^2} = -\frac{\dot{q}_G(x)}{k}$$

Now, in the limit $\Delta x \to 0$, from calculus, the left hand side of the above equation becomes $\frac{d^2T}{dx^2}$ so we have

$$k\,\frac{d^2T}{dx^2} = -\dot{q}_G(x)$$

which is equivalent to Equation (2.24).

PROBLEM 4.2

"What is the physical significance of the statement that the temperature of each node is just the average of its neighbors if there is no heat generation" [with reference to Equation (4.3)]?

SOLUTION

The significance is that in regions without heat generation, the temperature profile must be linear.

Compare the subject equation with the solution of the differential equation

$$\frac{d^2T}{dx^2} = 0$$

which is $T(x) = a + bx$, which is also linear.

PROBLEM 4.3

Give an example of a practical problem in which the variation of thermal conductivity with temperature is significant and for which a numerical solution is therefore the only viable solution method.

SOLUTION

From Figure 1.6, the thermal conductivity of stainless steel (either 304 or 316) is a fairly strong function of temperature. For example

$$k_{ss\ 316}\ (100°\text{C}) = 14.2\ (\text{W/m K})$$

$$k_{ss\ 316}\ (500°\text{C}) = 19.6\ (\text{W/m K})$$

which is about a 38% difference.

Suppose a stainless steel sheet is to receive a heat treatment that involves heating the sheet to 500°C and then plunging it into a water bath. The water near the sheet would probably boil producing a sheet surface temperature near 100°C while the interior of the sheet would be at 500°C, at least for a short time. One would expect the large variation in thermal conductivity to be important in this type of problem.

PROBLEM 4.4

Discuss advantages and disadvantages of using a large control volume.

SOLUTION

The advantages of a large control volume are

(1) the numerical solution can be carried out quickly
(2) manual calculation for all control volumes are feasible for the purpose of verifying the numerical calculation
(3) energy will be conserved

Disadvantages are

(1) large temperature gradients cannot be accurately represented with large control volumes
(2) it is difficult to accommodate all but rectangular geometries.

PROBLEM 4.5

For one-dimensional conduction, why are the boundary control volumes half the size of interior control volumes?

GIVEN

- One-dimensional conduction

EXPLAIN

(a) Why the boundary control volume is half the size of internal control volumes

SOLUTION

There is a node on the boundary as well as one a distance Δx to the interior of the boundary. Since the interior nodes are centered within a control volume of width Δx, the control volume associated with the first non-boundary node comes within $\Delta x/2$ of the boundary. So, there is a volume of only $\Delta x/2$ left over for the boundary node.

PROBLEM 4.6

Discuss advantages and disadvantages of two methods for solving one-dimensional steady conduction problems.

SOLUTION

The two methods for solving one-dimensional steady conduction problems are matrix inversion and iteration.

Matrix inversion requires that we have some method (usually software) for inverting the matrix or for solving a tridiagonal system of equations. The method is difficult to apply to problems with variable thermal conductivity. If we have access to the required software, the method is simple and fast.

Iteration can handle variable thermal conductivity and does not require software for the inversion of a matrix. In practice, we will likely need to write a program or use a spreadsheet to carry out iteration and it may converge slowly.

PROBLEM 4.7

Solve the system of equations

$$2T_1 + T_2 - T_3 = 30$$

$$T_1 - T_2 + 7T_3 = 270$$

$$T_1 + 6T_2 - T_3 = 160$$

by Jacobi and Gauss-Seidel iteration. Use as a convergence criterion $|T_2^{(p)} - T_2^{(p-1)}| < 0.001$. Compare the rate of convergence for the two methods.

GIVEN

- A system of three equations

FIND

(a) The solution of the system of equation using Jacobi and Gauss-Seidel iteration

SOLUTION

Since we do not know the physical problem these equations originated from, it is difficult to make a good first guess. Let's use 0 for all three temperatures as an initial guess.

If we solve the equations in the order given for T_1, T_2, and T_3 and solve by iteration, we find that the solution is not stable. Let's solve the first equation for T_1, the second for T_3, and the third for T_2. For Jacobi iteration we have

$$T_1^{(p+1)} = \frac{1}{2}(30 - T_2^{(p)} + T_3^{(p)})$$

$$T_3^{(p+1)} = \frac{1}{7}(270 - T_1^{(p)} + T_2^{(p)})$$

$$T_2^{(p+1)} = \frac{1}{6}(160 - T_1^{(p)} + T_3^{(p)})$$

and for Gauss-Seidel iteration we have

$$T_1^{(p+1)} = \frac{1}{2}(30 - T_2^{(p)} + T_3^{(p)})$$

$$T_3^{(p+1)} = \frac{1}{7}(270 - T_1^{(p+1)} + T_2^{(p+1)})$$

$$T_2^{(p+1)} = \frac{1}{6}(160 - T_1^{(p+1)} + T_3^{(p)})$$

The solution was carried out using a spreadsheet as shown on the next page

Problem 4.7 Filename 4_7.WQ1

	============= Jacobi	==========		======= Gauss-Seidel	=====	
iteration	T1	T2	T3	T1	T2	T3
0	0.000	0.000	0.000	0.000	0.000	0.000
1	15.000	26.667	38.571	15.000	24.167	39.881
2	20.952	30.595	40.238	22.857	29.504	39.521
3	19.821	29.881	39.949	20.009	29.919	39.987
4	20.034	30.021	40.009	20.034	29.992	39.994
5	19.994	29.996	39.998	20.001	29.999	40.000

6	20.001	30.001	40.000	20.000	30.000	40.000
7	20.000	30.000	40.000	20.000	30.000	40.000
8	20.000	30.000	40.000	20.000	30.000	40.000
9	20.000	30.000	40.000	20.000	30.000	40.000

Applying the criterion that the temperature change per iteration should be less than 0.001, we see that Jacobi iteration requires 7 iterations while Gauss-Seidel iteration requires 6 iterations.

PROBLEM 4.8

Develop the control volume difference equation for one-dimensional steady conduction in a fin with variable cross-sectional area *A*(*x*) and perimeter *P*(*x*). The heat transfer coefficient from the fin to ambient is a constant $\bar{h}_o$ and the fin tip is adiabatic.

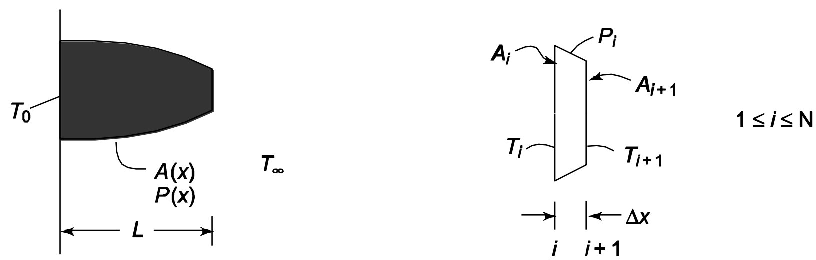
GIVEN

- Fin with variable cross-sectional area and perimeter
- Convection coefficient to ambient is constant, h_o

FIND

(a) Control volume difference equation

SKETCH



SOLUTION

Consider a control volume as shown above

An energy balance on this control volume is expressed by

heat conducted into left face =

heat convected out perimeter + heat conducted out right face

or

$$-kA_i\left(\frac{T_i - T_{i-1}}{\Delta x}\right) = \Delta x\, P_i\, h_o\,(T_i - T_\infty) - kA_{i+1}\left(\frac{T_{i+1} - T_i}{\Delta x}\right)$$

which can be rearranged to give

$$T_{i-1}\,A_i + T_i\left(-A_i - A_{i+1} - \frac{\Delta x^2}{k}P_i h_o\right) + T_{i+1}\,A_{i+1} = -\frac{\Delta x^2}{k}P_i\, h_o\, T_\infty$$

The boundary conditions can be written as

$$T_1 = T_o$$
$$T_N = T_{N-1}$$

This can be written in the form of a tridiagonal matrix, per Equation (3.11) where the coefficients of the matrix are

$$a_1 = 1 \quad b_1 = 0 \quad d_1 = T_o$$

$$c_i = -A_i \quad a_i = A_i + A_{i+1} + \frac{\Delta x^2}{k}P_i\, h_o \quad b_i = -A_{i+1} \quad d_i = \frac{\Delta x^2}{k}P_i\, h_o\, T_\infty \quad 1 < i < N$$

$$c_N = -1 \quad a_N = 1 \quad d_N = 0$$

PROBLEM 4.9

Using your results from Problem 4.8, find the heat flow at the base of the fin for the following conditions:

$$\boldsymbol{k} = \mathbf{34\ W/(m\ K)}$$

$$\boldsymbol{L} = \mathbf{5\ cm}$$

$$\boldsymbol{A(x)} = \mathbf{3.23*10^{-4}*} \left(\mathbf{1}-\frac{\mathbf{1}}{\mathbf{3}}\mathbf{sinh}\left(\frac{\boldsymbol{x}}{\boldsymbol{L}}\right)\right)\ \mathbf{m^2}$$

$$\boldsymbol{P(x)} = (\boldsymbol{A}(\boldsymbol{x}))^{\frac{1}{2}}$$

$$\boldsymbol{h_o} = \mathbf{110\ W/(m^2\ K)}$$

$$\boldsymbol{T_o} = \mathbf{93°C}$$

$$\boldsymbol{T_\infty} = \mathbf{27°C}$$

Use a grid spacing of 0.5 cm.

GIVEN

- A fin with variable cross-sectional area and perimeter

FIND

(a) Heat flow rate for conditions given above

SOLUTION

The number of nodes is $N = 1 + \frac{L}{\Delta x} = 11$. The cross-sectional area at any node is

$$A_i = 3.23*10^{-4}* \left(1-\frac{1}{3}\sinh\left(\frac{i-1\ \Delta x}{L}\right)\right)\ \text{m}^2$$

and the perimeter at any node is

$$P_i = A_i^{\frac{1}{2}}$$

Heat transfer at the fin root is

$$q_{\text{fin}} = \frac{k}{12\Delta x} A_1 (T_1 - T_2)$$

The difference equation as derived in Problem 4.8 is

$$T_{i-1} A_i + T_i \left(-A_i - A_{i+1} - \frac{\Delta x^2}{k} P_i h_o\right) + T_{i+1} A_{i+1} = -\frac{\Delta x^2}{k} P_i\, h_o\, T_\infty$$

The boundary conditions can be written as

$$T_1 = T_o$$

$$T_N = T_{N-1}$$

This can be written in the form of a tridiagonal matrix, per Equation (4.10) where the coefficients of the matrix are

$$a_1 = 1 \quad b_1 = 0 \quad d_1 = T_o$$

$$c_i = -A_i \qquad a_i = A_i + A_{i+1} + \frac{\Delta x^2}{k} P_i h_o \qquad b_i = -A_{i+1} \qquad d_i = \frac{\Delta x^2}{k} P_i h_o T_\infty \qquad 1 < i < N$$

$$c_N = -1 \qquad a_N = 1 \qquad d_N = 0$$

This set of equations can be easily solved using the matrix inversion function of a spreadsheet

Ai	Pi	..................Matrix										....
0.00032	0.01797	1.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.00031	0.01767	-0.00031	0.00061	-0.00030	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.00030	0.01736	0.00000	-0.00030	0.00059	-0.00029	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.00029	0.01704	0.00000	0.00000	-0.00029	0.00057	-0.00028	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.00028	0.01670	0.00000	0.00000	0.00000	-0.00028	0.00055	-0.00027	0.00000	0.00000	0.00000	0.00000	0.00000
0.00027	0.01634	0.00000	0.00000	0.00000	0.00000	-0.00027	0.00052	-0.00025	0.00000	0.00000	0.00000	0.00000
0.00025	0.01595	0.00000	0.00000	0.00000	0.00000	0.00000	-0.00025	0.00050	-0.00024	0.00000	0.00000	0.00000
0.00024	0.01553	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	-0.00024	0.00047	-0.00023	0.00000	0.00000
0.00023	0.01508	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	-0.00023	0.00044	-0.00021	0.00000
0.00021	0.01458	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	-0.00021	0.00041	-0.00020
0.00020	0.01402	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	-1.00000	1.00000

..................Inverse Matrix										....	D	T
1	0	0	0	0	0	0	0	0	0	0	93	93
0.97	3095.94	2999.79	2914.46	2840.04	2776.67	2724.63	2684.32	2656.38	2641.72	0.52	0.00	90.80
0.94	2999.79	6122.24	5948.10	5796.21	5666.89	5560.68	5478.42	5421.38	5391.47	1.06	0.00	88.81
0.91	2914.46	5948.10	9126.64	8893.59	8695.17	8532.19	8405.97	8318.46	8272.56	1.63	0.00	87.06
0.89	2840.04	5796.21	8893.59	12162.01	11890.66	11667.80	11495.19	11375.51	11312.74	2.22	0.00	85.52
0.87	2776.67	5666.89	8695.17	11890.66	15288.56	15002.01	14780.07	14626.20	14545.49	2.86	0.00	84.22
0.85	2724.63	5560.68	8532.19	11667.80	15002.01	18577.16	18302.33	18111.79	18011.85	3.54	0.00	83.14
0.84	2684.32	5478.42	8405.97	11495.19	14780.07	18302.33	22114.05	21883.82	21763.07	4.28	0.00	82.31
0.83	2656.38	5421.38	8318.46	11375.51	14626.20	18111.79	21883.82	26008.12	25864.60	5.08	0.00	81.74
0.82	2641.72	5391.47	8272.56	11312.74	14545.49	18011.85	21763.07	25864.60	30402.27	5.97	0.00	81.44
0.82	2641.72	5391.47	8272.56	11312.74	14545.49	18011.85	21763.07	25864.60	30402.27	6.97	0.00	81.44

$$q_{\text{fin}} = \frac{k}{\Delta x} A_1 (T_1 - T_2) = q_{\text{fin}} = \frac{34}{0.005} * 3.23 * 10^{-4} * (93 - 90.8) = 4.83 \text{ W}$$

The above problem is also solved using matlab. The matlab code for the problem is given below

```
%Problem 4.9 matlab solution
L=0.05; % in m
N=11;
delx=L/(N-1); % in m
h=110; % W/(m^2 K)
k=34; % W/(m K)
T(1)=93; % in celsius
T(2:N)=0; % in celsius
Tinf=27; % in celsius
for l=1:1:N
A(l)=3.23*10^-4*(1-sinh((l-1)*delx/L)/3);
P(l)=A(l)^0.5;
```

```
end
for j=1:1:3000
for i=N-1:-1:2
    Tf=T;
T(i)=(((delx^2)*P(i)*h*Tinf/k)+(T(i+1)*A(i+1))+(T(i-
1)*A(i)))/(A(i)+A(i+1)+((delx^2)*P(i)*h/k));
T(N)=T(N-1);
end;
count=0;
for i=1:N

    if abs(Tf(i)-T(i))<=10^-4
        count=count+1;
    end
end
    if count==N
 break
    end

end

q=(T(1)-T(2))*A(1)*k/delx % in Watt
```

From the above MATLAB program we get q=4.884 Watt

PROBLEM 4.10

Consider a pin fin with variable conductivity $k(T)$, constant cross sectional area A_c and constant perimeter, P. Develop the difference equations for steady one-dimensional conduction in the fin and suggest a method for solving the equations. The fin is exposed to ambient temperature T_a through a heat transfer coefficient h. The fin tip is insulated and the fin root is at temperature T_o.

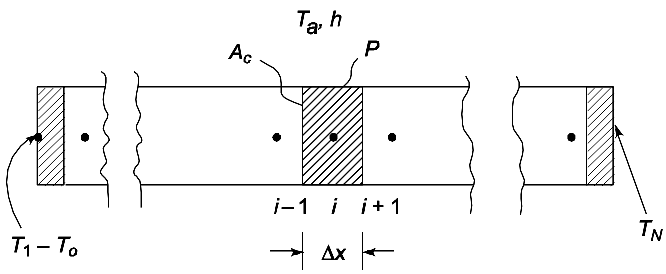
GIVEN

- Fin with variable thermal conductivity, $k(T)$

FIND

(a) Difference equation
(b) Solution method

SKETCH



SOLUTION

For the control volume centered over the interior node i, an energy balance gives

$$A_c \left(\frac{T_{i-1} - T_i}{\Delta x} k_{\text{left}} + \frac{T_{i+1} - T_i}{\Delta x} k_{\text{right}} \right) = h_o\, P\, (T_i - T_a)$$

The thermal conductivities are given in Section 3.2.1

$$k_{\text{left}} = \frac{2k_i\, k_{i-1}}{k_i + k_{i-1}} = \frac{2k\ \ T_i\ \ k\ \ T_{i-1}}{k\ \ T_i\ + k\ \ T_{i-1}}$$

and

$$k_{\text{right}} = \frac{2k_i\, k_{i+1}}{k_i + k_{i+1}} = \frac{2k\ \ T_i\ \ k\ \ T_{i+1}}{k\ \ T_i\ + k\ \ T_{i+1}}$$

For the node at the root $T_1 = T_o$.
At the tip, an energy balance gives

$$A_c\, k_N\, \frac{T_{N-1} - T_N}{\Delta x} = h_o\, P\, (T_N - T_a)$$

where

$$k_N = \frac{2k\ \ T_N\ \ k\ \ T_{N-1}}{k\ \ T_N\ + k\ \ T_{N-1}}$$

These equations can be written in tridiagonal form, Equation (4.10)

$$a_i\, T_i = b_i\, T_{i+1} + c_i\, T_{i-1} + d_i$$

where

$$a_1 = 1 \quad b_1 = 0 \quad c_1 = 0 \quad d_1 = T_o$$

For $1 < i < N$

$$a_i = h_o P + \frac{A_c}{\Delta x} (k_{\text{left}} + k_{\text{right}})$$

$$b_i = \frac{A_c}{\Delta x} k_{\text{right}}$$

$$c_i = \frac{A_c}{\Delta x} k_{\text{left}}$$

$$d_i = h_o P T_a$$

and

$$a_N = h_o P + \frac{A_c k_N}{\Delta x}$$

$$b_N = 0$$

$$c_N = \frac{A_c k_N}{\Delta x}$$

$$d_N = h_o P T_a$$

Note that k_{right}, k_{left}, and k_N depend on the nodal temperatures. To solve the system of equations, it will be necessary to

(1) Guess at the nodal temperatures
(2) Calculate the values for k_{right}, k_{left}, and k_N
(3) Calculate the matrix coefficients a_i, b_i, c_i and d_i, $1 \leq i \leq N$
(4) Solve for the nodal temperatures by inverting the matrix as in Equation (4.11)
(5) Repeat steps 2 through 4 until the nodal temperatures cease to change

PROBLEM 4.11

How would you treat a radiation heat transfer boundary condition for a one-dimensional steady problem? Develop the difference equation for a control volume near the boundary and explain how to solve the entire system of difference equations. Assume that the heat flux at the surface is $q = \varepsilon \sigma (T_s^4 - T_e^4)$ where T_s is the surface temperature and T_e is the temperature of an enclosure surrounding the surface.

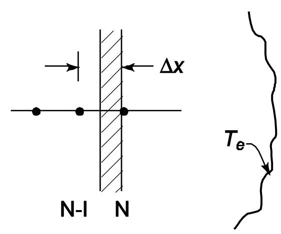
GIVEN

- Radiation boundary condition
- One-dimensional steady conduction

FIND

(a) Difference equation for control volume near surface
(b) Solution method

SKETCH



SOLUTION

An energy balance on the half control volume surrounding the surface node is

$$k \frac{T_{N-1} - T_N}{\Delta x} = \varepsilon \sigma (T_N^4 - T_e^4)$$

The right side of the above equation can be written

$$(T_N - T_e)\, h_r$$

where

$$h_r = \varepsilon \sigma (T_N^2 + T_e^2)(T_N + T_e)$$

The difference equation can be written in the tridiagonal form like Equation (4.10) as follows

$$a_i T_i = b_i T_{i+1} + c_i T_{i-1} + d_i$$

The coefficients for $1 < i < N$ are given just before Equation (4.11). For $i = 1$, the coefficients will depend on the boundary condition at the left boundary. For $i = N$, the coefficients are

$$a_i = h_r + \frac{k}{\Delta x} \quad b_i = 0 \quad c_i = \frac{k}{\Delta x} \quad d_i = h_r T_e$$

To solve the set of difference equations, an initial temperature distribution guess will be made. This will allow a determination of all of the coefficients. The tridiagonal matrix can then be solved to get an updated temperature distribution. This distribution will be used to update the coefficients, and the procedure will be repeated to convergence.

PROBLEM 4.12

How should the control volume method be implemented at an interface between two materials with different thermal conductivities? Illustrate with a steady, one-dimensional example. Neglect contact resistance.

GIVEN

- Interface between two different materials with different thermal conductivities

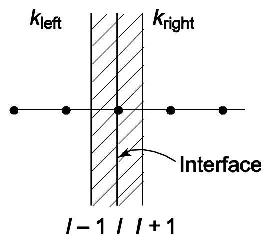
FIND

(a) Difference equation at the interface

ASSUMPTIONS

- No heat generation

SKETCH



SOLUTION

As shown in the sketch, the node at the interface is $i = I$. The thermal conductivity to the left of the interface is k_{left} and on the right side of the interface it is k_{right}. Since there is no contact resistance or heat generation, an energy balance for the control volume that straddles the interface is

$$k_{left} \frac{T_{I-1} - T_I}{\Delta x} = k_{right} \frac{T_I - T_{I+1}}{\Delta x}$$

Simplifying and writing this in the tridiagonal form

$$T_1 (k_{left} + k_{right}) = T_{1+1} k_{right} + T_{I-1} k_{left}$$

The above coefficients would be used to write the Ith row of the tridiagonal matrix. The remaining rows for internal nodes would be written as before and those for the boundaries would depend on specified boundary conditions.

PROBLEM 4.13

How would you include contact resistance between the two materials in Problem 4.12? Derive the appropriate difference equations.

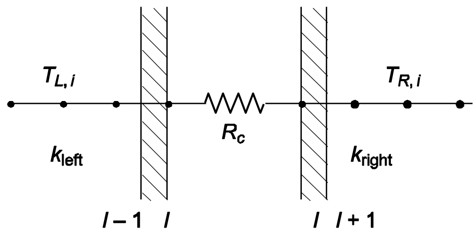
GIVEN

- Interface between two materials with different thermal conductivities and contact resistance at the interface

FIND

(a) The appropriate difference equations

SKETCH



SOLUTION

Let the contact resistance be R_c. The interface is located at node $i = I$. Represent temperatures to the left of the interface with $T_{L,i}$ and to the right of the interface with $T_{R,i}$. Thermal conductivity to the left of the interface is k_{left} and to the right of the interface is k_{right}. We have drawn two half control volumes, one just to the left of the interface and one just to the right of the interface.

An energy balance on the left control volume is

$$k_{\text{left}} \frac{T_{L,I-1} - T_{L,I}}{\Delta x} = \frac{T_{L,I} - T_{R,I}}{R_c}$$

and for the right control volume

$$k_{\text{right}} \frac{T_{R,I+1} - T_{R,I}}{\Delta x} = \frac{T_{R,I} - T_{L,I}}{R_c}$$

Writing these equations in the tridiagonal form we have

$$T_{L,I}\left(\frac{1}{R_c} + \frac{k_{\text{left}}}{\Delta x}\right) = T_{L,I-1}\frac{k_{\text{left}}}{\Delta x} + T_{R,I}\frac{1}{R_c}$$

$$T_{R,I}\left(\frac{1}{R_c} + \frac{k_{\text{right}}}{\Delta x}\right) = T_{L,I}\frac{1}{R_c} + T_{R,I+1}\frac{k_{\text{right}}}{\Delta x}$$

From these equations, the coefficients for the tridiagonal matrix can be defined

$$a_{L,I} = \frac{1}{R_c} + \frac{k_{\text{left}}}{\Delta x} \quad b_{L,I} = \frac{1}{R_c} \quad c_{L,I} = \frac{k_{\text{left}}}{\Delta x} \quad d_{L,I} = 0$$

$$a_{R,I} = \frac{1}{R_c} + \frac{k_{\text{right}}}{\Delta x} \quad b_{R,I} = \frac{k_{\text{right}}}{\Delta x} \quad c_{R,I} = \frac{1}{R_c} \quad d_{R,I} = 0$$

The vector of nodal temperatures in Equation (4.11) would be modified to look like

$$\begin{bmatrix} \cdot \\ \cdot \\ T_{L,I-1} \\ T_{L,I} \\ T_{R,I} \\ T_{R,I+1} \\ \cdot \\ \cdot \end{bmatrix}$$

The coefficients with subscripts L, I would appear in the row corresponding to $T_{L,\ I}$ and those with subscripts R, I would appear in the row corresponding to $T_{R,\ I}$. Remaining coefficients would be determined as for any other one-dimensional steady problems including those determined by the boundary conditions.

PROBLEM 4.14

A turbine blade 5-cm-long, with cross-sectional area A = 4.5 cm^2 and perimeter P = 12 cm, is made of a high-alloy steel [k = 25 W/(m K)]. The temperature of the blade attachment point is 500°C and the blade is exposed to combustion gases at 900°C. The heat transfer coefficient between the blade surface and the combustion gases is 500 W/(m^{2}K). Using the nodal network shown in the accompanying sketch, (a) determine the temperature distribution in the blade, the rate of heat transfer to the blade and the fin efficiency of the blade and, (b) compare the fin efficiency calculated numerically with that calculated by the exact method.

GIVEN

- Turbine blade exposed to combustion gases

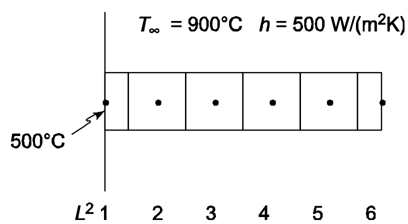
FIND

(a) Blade temperature distribution, heat gain, and fin efficiency
(b) Fin efficiency calculated exactly

ASSUMPTIONS

- The convection coefficient applies at the blade tip

SKETCH



SOLUTION

For the node and control volume arrangement shown in the sketch, we have

$$x_i = \Delta x(i-1) \qquad i = 1, 2, \ldots, N = 6 \qquad \Delta x = \frac{L}{N-1}$$

For the control volume at i = 1, we have a specified temperature, therefore

$$T_1 = T_{\text{root}}$$

For the interior control volumes, i = 2, 3, 4, 5, an energy balance gives

$$kA\left\{\frac{T_{i+1}-T_i}{\Delta x}+\frac{T_{i-1}-T_i}{\Delta x}\right\} + P\Delta x h\,(T_\infty - T_i) = 0$$

Writing this in the tridiagonal form

$$T_i\left(2+\frac{P\Delta x^2 h}{kA}\right) = T_{i+1} + T_{i-1} + \frac{P\Delta x^2 h}{kA}\;T_\infty$$

For the control volume at node $i = N$, an energy balance gives

$$kA\,\frac{T_{N-1}-T_N}{\Delta x} + h\,(T_\infty - T_N)\left(P\frac{\Delta x}{2}+A\right) = 0$$

In the tridiagonal form this becomes

$$T_N \left(1+\frac{h\Delta x}{kA}\left(P\frac{\Delta x}{2}+A\right)\right) = T_{N-1} + \frac{h\Delta x}{kA}\left(P\frac{\Delta x}{2}+A\right) T_\infty$$

Filling in the matrix *A* coefficients in Equation (4.11) we have

$$a_1 = 1 \quad b_1 = 0 \quad c_1 = 0 \quad d_1 = T_{\text{root}}$$

$$a_i = 2 + \frac{P\Delta x^2 h}{kA} \quad b_i = 1 \quad c_i = 1 \quad d_i = \frac{P\Delta x^2 h}{kA}\, T_\infty \qquad i = 2, 3, 4, 5$$

$$a_N = 1 + \frac{h\Delta x}{kA}\left(P\frac{\Delta x}{2}+A\right) \quad b_N = 0 \quad c_N = 1 \; d_N = \frac{h\Delta x}{kA}\left(P\frac{\Delta x}{2}+A\right)\, T_\infty$$

The matrix can be inverted using a spreadsheet and then the inverse matrix is multiplied by the vector *D* to give the solution vector *T* of temperatures.
Heat transfer from the fin is given by the heat loss from the first control volume

$$Q_{\text{fin}} = h\,\frac{\Delta x}{2}\,P\,(T_1 - T_\infty) + \frac{kA}{\Delta x}\,(T_1 - T_2)$$

The spreadsheet is shown below

```
Problem 4.14 Filename: 4_14.WQ1
PROBLM PARAMETERS
=================================
  Ac    =   0.00045  (fin cross sectional area, m^2)
  P     =      0.12  (fin perimeter, m)
  L     =      0.05  (fin length, m)
  h     =       500  (heat transfer coefficient, W/m^2K)
  k     =        25  (fin thermal conductivity, W/mK)
  Troot =       500  (root temperature, deg C)
  Tgas  =       900  (gas temperature, deg C)
  N     =         6  (number of nodes)
  dx    =      0.01  (length of control volume, m)
  K1    =  0.533333  (-)
  K2    =  444.4444  (m^ -2)
  K3    =   0.00105  (m^2)
```

COEFFICIENT MATRIX

1	0	0	0	0	0
-1	2.533333	-1	0	0	0
0	-1	2.533333	-1	0	0
0	0	-1	2.533333	-1	0
0	0	0	-1	2.533333	-1
0	0	0	0	-1	1.466667

INVERSE MATRIX						VECTOR D	VECTOR PRODUCT T
1	0	0	0	0	0	500	500
0.489927	0.489927	0.241149	0.120984	0.065343	0.044552	480	704.0291
0.241149	0.241149	0.610911	0.306492	0.165536	0.112865	480	803.5404
0.120984	0.120984	0.306492	0.655463	0.354014	0.241374	480	851.6065
0.065343	0.065343	0.165536	0.354014	0.731301	0.498614	480	873.8628
0.044552	0.044552	0.112865	0.241374	0.498614	1.021782	420	882.1792

FIN HEAT LOSS --> -349.533 watts

The heat loss from the blade is –349.533 watts, i.e., the fin gains 349.533 watts from the combustion gases.

To determine the fin efficiency of the blade, consider that if the entire blade were at the root temperature, the heat loss would be

$$Q_{I,\max} = (PL + A)\,(T_{\text{root}} - T_\infty)$$

$$Q_{I,\max} = 500\,\text{W/(m}^2\text{K)}\ \ ((0.12\text{ m})\,(0.05\text{ m}) + 0.00045\text{ m}^2)\,(900 - 500)\text{ K}$$

$$= 1290.0\text{ watt}$$

The fin efficiency is therefore $\eta_{\text{fin}} = \dfrac{Q_I}{Q_{I,\max}} = \dfrac{349.5}{1290.0} = 0.271$

For the exact solution, use Table 2.1, entry 4 with

$$m = \sqrt{\frac{hP}{kA}} = \sqrt{\frac{500\,\text{W/(m}^2\text{K)}\ \ 0.12\,\text{m}}{25\,\text{W/(m K)}\ \ 0.00045\,\text{m}^2}} = 73.0297\text{ m}^{-1}$$

$$m\,L = (73.0297\text{ m})\,(0.05\text{ m}) = 3.6514$$

$$M = \sqrt{hPkA}\,(T_{\text{root}} - T_\infty) = \sqrt{500\,\text{W/(m}^2\text{K)}\ \ 0.12\,\text{m}\ \ 25(\text{W/m K}\ \ 0.00045\,\text{m}^2}$$

$$= 328.633\text{ watt}$$

giving

$$Q_{\text{fin}} = 328.381\text{ watt}$$

which is about 6% less than our numerical solution. Presumably, as we increase N, the accuracy would improve.

PROBLEM 4.15

Light-emitting diodes or LEDs are currently perhaps the most energy-efficient lighting systems. Finned surface heat sinks are used to cool high-intensity LED lighting that are used for spot and/or track lighting systems. A typical circular pin-fin heat sink is shown in the figure, and it is desirable that the fin-base temperature be less than 115°C to ensure efficient performance and longer life of the LED lamp. The fins are made of cold-forged, high-conductivity aluminum (*k* = 210 W/m K). Each pin has a diameter of 4 mm and a length of 40 mm. If the surrounding air is at 22°C, and it has a convection heat transfer coefficient of $\overline{h}_c$ = 10 W/(m^2 K), determine the temperature distribution in the fin, considering convection from the fin tip and the heat transfer rate from the fin. Model it as a one-dimensional system, use a minimum of 9 nodes (including the ones at the base and tip) or more in your numerical scheme, and determine the effect of extra nodes. Compare the results with the one-dimensional fin analysis of Chapter 2. Also, if 90 such fins are evenly distributed on a 50-mm-diameter circular base, what is the maximum heat transfer rate that is dissipated by this heat sink so that the base temperature is less than 115°C? In a typical LED lamp, approximately 70% of the electric power (or wattage) is dissipated as heat (the remainder 30% is useful light). Determine the maximum power of the lamp, in watts, for which this heat sink is used.

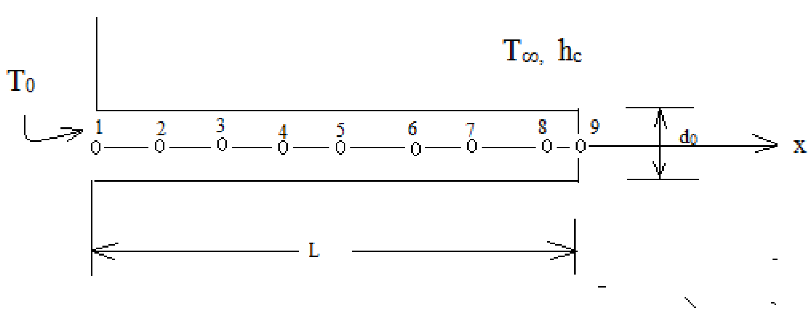
GIVEN

- Light emitting diodes
- Fin base temperature (T_0)= 115^{0}C
- Surrounding air temperature (T_∞)= 22^{0}C
- Convection heat transfer coefficient ($\overline{h}_c$)= 10 W/(m^2 K)
- Thermal conductivity (k)=210 W/(m K)
- Fin diameter (d)= 4 mm = 0.004 m
- Fin length (L)= 40 mm =0.04 m
- Number of nodes (N)=9

FIND

(a) Temperature distribution in the fin considering convection from fin tip
(b) Heat transfer rate from the fin.
(c) Maximum heat transfer rate dissipated by 90 such evenly distributed fins
(d) Maximum power of lamp in watts.

SKETCH



SOLUTION

For the node and control volume arrangement shown in the sketch, we have

$$x_i = \Delta x(i-1) \quad i = 1, 2, \ldots, N = 9 \quad \Delta x = \frac{L}{N-1}$$

For the control volume at i = 1, we have a specified temperature, therefore

$$T_1 = T_{\text{root}}$$

For the interior control volumes, i = 2, 3, 4, 5,… , 8 an energy balance gives

$$kA\left\{\frac{T_{i+1}-T_i}{\Delta x}+\frac{T_{i-1}-T_i}{\Delta x}\right\} + P\Delta x h\,(T_\infty - T_i) = 0$$

Writing this in the tridiagonal form

$$T_i\left(2+\frac{P\Delta x^2 h}{kA}\right) = T_{i+1} + T_{i-1} + \frac{P\Delta x^2 h}{kA}\,T_\infty$$

=>

$$T_i = \frac{T_{i+1} + T_{i-1} + \dfrac{P\Delta x^2 h}{kA}T_\infty}{\left(2+\dfrac{P\Delta x^2 h}{kA}\right)}$$

For the control volume at node $i = N$, an energy balance gives

$$kA\,\frac{T_{N-1}-T_N}{\Delta x} + h\,(T_\infty - T_N)\left(P\frac{\Delta x}{2}+A\right) = 0$$

In the tridiagonal form this becomes

$$T_N\left(1+\frac{h\Delta x}{kA}\left(P\frac{\Delta x}{2}+A\right)\right) = T_{N-1} + \frac{h\Delta x}{kA}\left(P\frac{\Delta x}{2}+A\right)T_\infty$$

$$T_N = \frac{T_{N-1} + \dfrac{h\Delta x}{kA}\left(P\dfrac{\Delta x}{2}+A\right)T_\infty}{\left(1+\dfrac{h\Delta x}{kA}\left(P\dfrac{\Delta x}{2}+A\right)\right)}$$

The heat transferred through the fin is given by

$$Q_{\text{fin}} = h\,\frac{\Delta x}{2}\,P\,(T_1 - T_\infty) + \frac{kA}{\Delta x}\,(T_1 - T_2)$$

We can solve the above problem by MATLAB by discretization. The boundary conditions for the discretization are

$T_1=T_0$

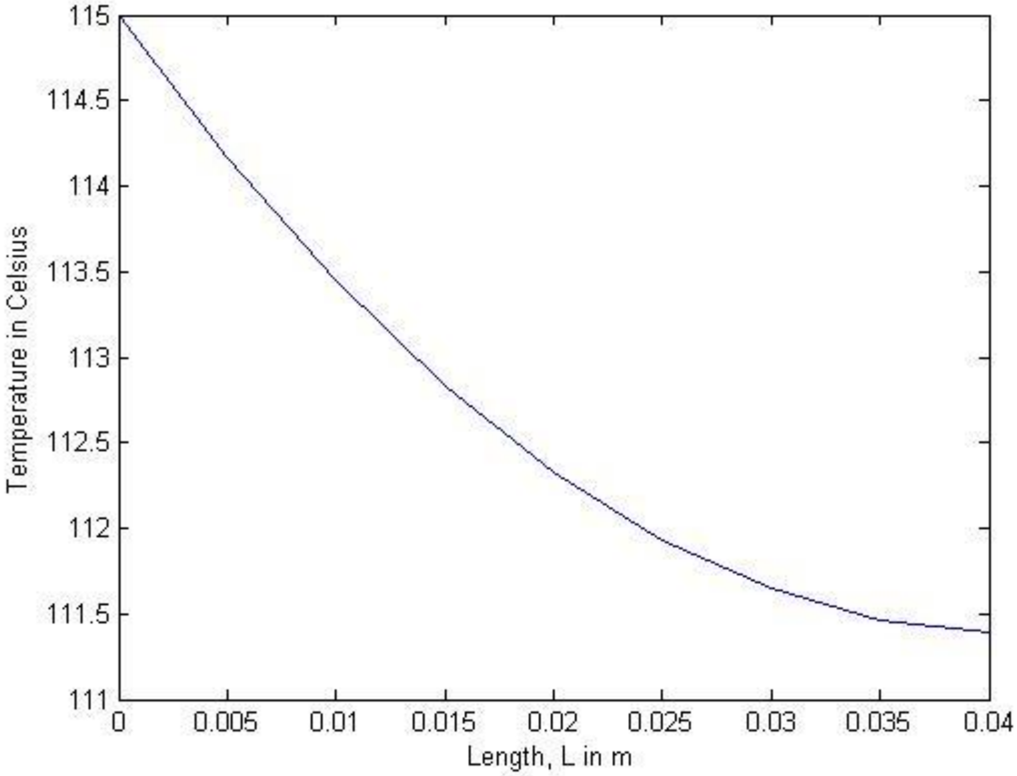
$$T_N = \frac{T_{N-1} + \dfrac{h\Delta x}{kA}\left(P\dfrac{\Delta x}{2}+A\right)T_\infty}{\left(1+\dfrac{h\Delta x}{kA}\left(P\dfrac{\Delta x}{2}+A\right)\right)}$$

Following is the MATLAB code for executing the program.

%Matlab code Problem_4_15

```
L=0.04;  % in m
N=9;
delx=L/(N-1);     % in m
d=0.004;            % in m
h=10;           % W/(m^2 K)
k=210;           % W/(m K)
T(1)=115;         % Celsius
T(2:N)=0;          % Celsius
Tinf=22;           % Celsius
A=pi*d^2/4;        % m^2
P=pi*d;            % m

for j=1:1:3000
for i=N-1:-1:2
    Tf=T;
T(i)=(((delx^2)*P*h*Tinf/(k*A))+T(i+1)+T(i-1))/(2+((delx^2)*P*h/(k*A)));
T(N)=(T(N-
1)+(h*delx/(k*A)*(0.5*P*delx+A)*Tinf))/(1+(h*delx/(k*A)*(0.5*P*delx+A)))
;
end;
count=0;
for i=1:N
    if abs(Tf(i)-T(i))<10^-4
        count=count+1;
    end

end
    if count==N
        break
    end
end

for i=1:1:N
   x(i)=(i-1)*delx;
end
plot(x,T)



q=(T(1)-T(2))*A*k/delx+0.5*h*delx*P*(T(1)-Tinf)   % in Watts
```

(a) The following graph is obtained for temperature distribution along the length of fin.



following graph is obtained for temperature distribution along the length of fin.

(b) The heat transfer through the fin is calculated from above program using the expression

$$Q_{\text{fin}} = h\,\frac{\Delta x}{2}\,P\,(T_1 - T_\infty) + \frac{kA}{\Delta x}\,(T_1 - T_2) \text{ as}$$

$$Q_{\text{fin}} = 0.4681 \text{ Watt}$$

For the exact solution, use Table 2.1, entry 4 with

For the exact solution, use Table 2.1, entry 4 with

$$\text{m} = \sqrt{\frac{hP}{kA}} = \sqrt{\frac{\left(10\,\text{W}/(\text{m}^2\text{K})\right)(\pi * 0.004\,\text{m})}{(210\,\text{W}/(\text{m}\,\text{K}))\left(\pi * 0.004^2 / 4\,\text{m}^2\right)}} = 6.9\ \text{m}^{-1}$$

$$\text{m}\,L = (6.9\ \text{m}^{-1})\,(0.04\ \text{m}) = 0.276$$

$$Q = \sqrt{hPkA}\,(T_{\text{root}} - T_\infty)\,\frac{\sinh(mL) + \left(h_c/mk\right)\cosh(mL)}{\cosh(mL) + \left(h_c/mk\right)\sinh(mL)}$$

$= \sqrt{\left(10\,\mathrm{W/(m^2K)}\right)(\pi * 0.004\,\mathrm{m})(210\,(\mathrm{W/m\,K})\left(\pi * 0.004^2 \ / 4\,\mathrm{m}^2\right)}$ (115-22)^{0}C* $\dfrac{\sinh(mL) + \left(h_c / mk\right)\cosh(mL)}{\cosh(mL) + \left(h_c / mk\right)\sinh(mL)}$

= 1.693*0.27558

giving

Q_{fin} = 0.4665 watt

Which is very close to the heat transfer rate obtained by discretization.

(c) If 90 such fins are evenly distributed the heat transfer through 90 fins is given by

$Q_{\mathrm{total}} = Q_{\mathrm{fin}}*90 = 42.13$ Watt

(d) If P is the total wattage of the LED lamp, maximum heat dissipated by the lamp is 0.7 P. Thus

0.7*P=42.13 Watt

P= 60 Watt

Thus the wattage of the lamp is 60 Watt.

PROBLEM 4.16

A heat sink made of an array of fins that have a straight, rectangular cross section is used to cool an electronic micro-chip module, as schematically shown in the figure. The heat sink (base and fins) is made of copper, and each fin is 3 mm thick and 30 mm long. If the fin-base temperature is 85°C and the cooling air is at 25°C with an average convective heat transfer coefficient of $\bar{h}_c$ =100 W/(m^2 K), determine the temperature distribution in one fin and the rate of heat transferred per unit width of the fin. If the fins are made of aluminum instead, what is the change in the rate of heat dissipated by the aluminum fin of same thickness? For the numerical analysis, in both the cases, consider a minimum of 11 nodes from the base to tip (i.e., 3 mm apart) and convective heat transfer from the fin tip. Compare the results with the one-dimensional fin analysis of Chapter 2.

GIVEN

- Heat sink made of an arry of straight rectangular cross section fins.
- Fin base temperature (T_0)= 85^{0}C
- Surrounding air temperature (T_∞)= 25^{0}C
- Convection heat transfer coefficient ($\bar{h}_c$)= 100 W/(m^2 K)
- Fin thickness(t)= 3 mm = 0.003 m
- Fin length (L)= 30 mm =0.03 m
- Number of nodes (N)=11

FIND

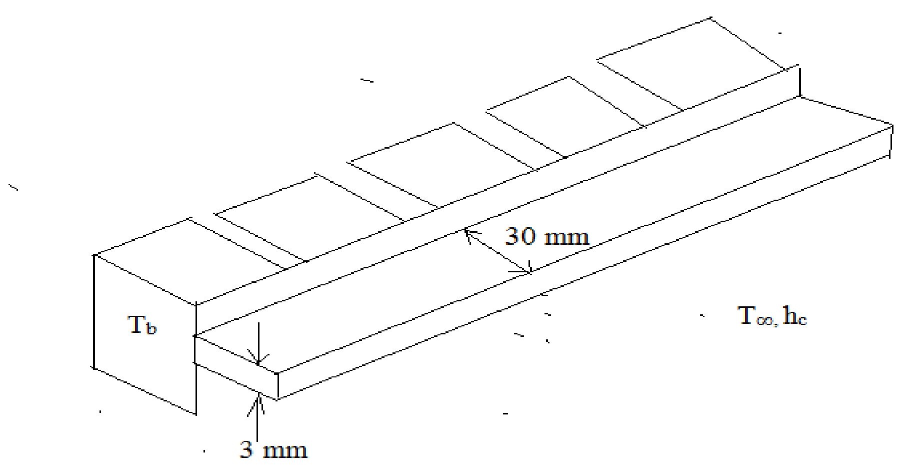
(a) Temperature distribution in one fin considering convection from fin tip
(b) Heat transfer rate per unit width of fin
(c) Rate of change in heat dissipated if Aluminum of same thickness is used.

PROPERTIES

From Appendix 2 Table

- Thermal conductivity of copper(k_{Cu})= 396 W/(m K)
- Thermal conductivity of Aluminum (k_{Al})=238 W/(m K)

SKETCH



SOLUTION

For the node and control volume arrangement shown in the sketch, we have

$$x_i = \Delta x(i-1) \quad i = 1, 2, \ldots, N = 9 \quad \Delta x = \frac{L}{N-1}$$

For the control volume at i = 1, we have a specified temperature, therefore

$$T_1 = T_{\text{root}}$$

For the interior control volumes, i = 2, 3, 4, 5,… , 8 an energy balance gives

$$kA\left\{\frac{T_{i+1}-T_i}{\Delta x}+\frac{T_{i-1}-T_i}{\Delta x}\right\} + P\Delta x h\,(T_\infty - T_i) = 0$$

Writing this in the tridiagonal form

$$T_i\left(2+\frac{P\Delta x^2 h}{kA}\right) = T_{i+1} + T_{i-1} + \frac{P\Delta x^2 h}{kA}\,T_\infty$$

=>

$$T_i = \frac{T_{i+1} + T_{i-1} + \dfrac{P\Delta x^2 h}{kA}T_\infty}{\left(2+\dfrac{P\Delta x^2 h}{kA}\right)}$$

Since W>>t

P=2(W+t)$\simeq$2W

Cross sectional Area (A)= W*t

$$T_i = \frac{T_{i+1} + T_{i-1} + \dfrac{2W\Delta x^2 h}{k*W*t}T_\infty}{\left(2+\dfrac{2*W*\Delta x^2 h}{k*W*t}\right)}$$

$$T_i = \frac{T_{i+1} + T_{i-1} + \dfrac{2\Delta x^2 h}{kt}T_\infty}{\left(2+\dfrac{2\Delta x^2 h}{kt}\right)}$$

For the control volume at node $i = N$, an energy balance gives

$$kA\,\frac{T_{N-1}-T_N}{\Delta x} + h\,(T_\infty - T_N)\left(P\frac{\Delta x}{2}+A\right) = 0$$

In the tridiagonal form this becomes

$$T_N\left(1+\frac{h\Delta x}{kA}\left(P\frac{\Delta x}{2}+A\right)\right) = T_{N-1} + \frac{h\Delta x}{kA}\left(P\frac{\Delta x}{2}+A\right)T_\infty$$

$$T_N = \frac{T_{N-1} + \frac{h\Delta x}{kA}\left(P\frac{\Delta x}{2} + A\right)T_\infty}{\left(1 + \frac{h\Delta x}{kA}\left(P\frac{\Delta x}{2} + A\right)\right)}$$

$$T_N = \frac{T_{N-1} + \frac{h\Delta x}{kWt}\left(2W\frac{\Delta x}{2} + Wt\right)T_\infty}{\left(1 + \frac{h\Delta x}{kWt}\left(2W\frac{\Delta x}{2} + Wt\right)\right)}$$

$$T_N = \frac{T_{N-1} + \frac{h\Delta x}{kt}\left(\Delta x + t\right)T_\infty}{\left(1 + \frac{h\Delta x}{kt}\left(\Delta x + t\right)\right)}$$

The heat transferred through the fin per unit width is given by

$$Q_{\text{fin}} = h\ \frac{\Delta x}{2}\ 2*W(T_1 - T_\infty) + \frac{kWt}{\Delta x}\ (T_1 - T_2)$$

$$\frac{Q_{\text{fin}}}{W} = h\ \ \Delta x\ \ (T_1 - T_\infty) + \frac{kt}{\Delta x}\ \ (T_1 - T_2)$$

We can solve the above problem by MATLAB by discretization. The boundary conditions for the discretization are

$T_1=T_0$

$$T_N = \frac{T_{N-1} + \frac{h\Delta x}{kt}\left(\Delta x + t\right)T_\infty}{\left(1 + \frac{h\Delta x}{kt}\left(\Delta x + t\right)\right)}$$

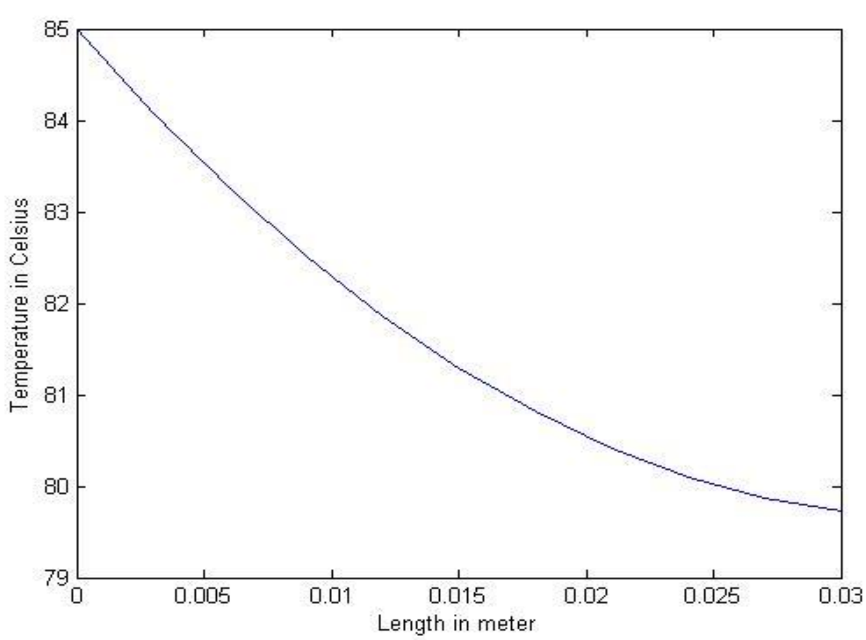
Following is the MATLAB code for executing the program.

%Solution to problem_4_15

```
L=0.03;  % in m
N=11;
delx=L/(N-1);     % in m
t=0.003;            % in m
h=100;            % W/(m^2 K)
k=396;            % W/(m K)
T(1)=85;          % Celsius
T(2:N)=0;           % Celsius
Tinf=25;            % Celsius


for j=1:1:3000
for i=N-1:-1:2
    Tf=T;
T(i)=((2*(delx^2)*h*Tinf/(k*t))+T(i+1)+T(i-1))/(2+(2*(delx^2)*h/(k*t)));
```

```
T(N)=(T(N-
1)+(h*delx/(k*t)*(delx+t)*Tinf))/(1+(h*delx/(k*t)*(2*delx+t)));
end;
count=0;
for i=1:N
    if abs(Tf(i)-T(i))<10^-4
        count=count+1;
    end

end
    if count==N
        break
    end
end

for i=1:1:N
   x(i)=(i-1)*delx;
end
plot(x,T)



q=(T(1)-T(2))*t*k/delx+h*delx*(T(1)-Tinf)   % in Watts
```

The following temperature distribution is obtained along the length of fin from above program.



The heat transfer per fin is calculated as per equation

$$\frac{Q_{fin}}{W} = h\ \Delta x\ (T_1 - T_\infty) + \frac{kt}{\Delta x}\ (T_1 - T_2) = 381.81 \text{ Watts}$$

When Aluminum fin is used instead of Copper

Replacing thermal conductivity (k) in above program we get

$$\frac{Q_{\text{fin}}}{W} = h\ \Delta x\ (T_1 - T_\infty) + \frac{kt}{\Delta x}\ (T_1 - T_2) = 367.5 \text{ Watts}$$

For the exact solution, use Table 2.1, entry 4 with

For the exact solution, use Table 2.1, entry 4 with

$$\text{m} = \sqrt{\frac{hP}{kA}} = \sqrt{\frac{2h}{kt}} = \sqrt{\frac{2*\left(100\,\text{W/(m}^2\text{K)}\right)}{(396\,\text{W/(m K)})(0.003m)}} = 12.975 \text{ m}^{-1}$$

$$\text{m}\,L = (73.0297 \text{ m})\ (0.03 \text{ m}) = 0.38925$$

$$\text{Q/W} = \sqrt{2hkt}\ (T_{\text{root}} - T_\infty)\frac{\sinh(mL) + \left(h_c/mk\right)\cosh(mL)}{\cosh(mL) + \left(h_c/mk\right)\sinh(mL)}$$

$$= \sqrt{2*\left(100\,\text{W/(m}^2\text{K)}\right)(396\,(\text{W/m K})(0.003\text{m})}\ (85\text{-}25)^0\text{C}*\frac{\sinh(mL) + \left(h_c/mk\right)\cosh(mL)}{\cosh(mL) + \left(h_c/mk\right)\sinh(mL)}$$

$$= 924.8*0.38738$$

giving

$$Q_{\text{fin}} = 358.27 \text{ watt}$$

Which is close to the heat transfer rate obtained by discretization.

PROBLEM 4.17

Show that in the limit $\Delta x \to 0$ and $\Delta t \to 0$, the difference Equation (4.13) is equivalent to the differential Equation (2.5).

GIVEN

- The difference equation for one-dimensional transient conduction

SHOW

(a) As Δx and $\Delta t \to 0$, the difference equation is equivalent to the differential equation, Equation (2.5)

SOLUTION

Equation (4.13) is

$$T_{i,\,m+1} = T_{i,\,m} + \frac{\Delta t}{\rho c \Delta x}\left(\frac{k}{\Delta x}\; T_{i+1,m} - 2T_{i,m} + T_{i-1,m} \;\; + \dot{q}_{G\,i,m}\,\Delta x\right)$$

By definition

$$T_{i,\,m} = T(x,\,t)$$

$$T_{i+1,\,m} = T(x + \Delta x,\,t)$$

$$T_{i-1,\,m} = T(x - \Delta x,\,t)$$

$$T_{i,\,m+1} = T(x,\,t + \Delta t)$$

So, the difference equation is equivalent to

$$\rho\, c\;\frac{T(x,t+\Delta t) - T(x,t)}{\Delta t} = k\;\frac{T(x+\Delta x,t) - 2T(x,t) + T(x-\Delta x,t)}{\Delta x^2} + \dot{q}_G\,(x,\,t)$$

In the limit as $\Delta t \to 0$, from calculus, the left hand side of the above equation becomes

$$\rho\, c\;\frac{\partial T}{\partial t}$$

and in the limit as $\Delta x \to 0$, from calculus, the first term on the right hand side of the equation becomes

$$k\;\frac{\partial^2 T}{\partial x^2}$$

So the equation is equivalent to

$$\rho\, c\;\frac{\partial T}{\partial t} = k\;\frac{\partial^2 T}{\partial x^2} + \dot{q}_G\,(x,\,t)$$

which is the same as Equation (2.5).

PROBLEM 4.18

Determine the largest permissible time step for a one-dimensional transient conduction problem to be solved by an explicit method if the node spacing is 1 mm and the material is (a) carbon steel 1C, and (b) window glass. Explain the difference in the two results.

GIVEN

- One-dimensional transient conduction in a 1-mm-thickness of carbon steel and window glass

FIND

(a) Largest permissible time step for each material

SOLUTION

(a) From Table 10 in Appendix 2, the thermal diffusivity for carbon steel is $\alpha = 1.172 \times 10^{-5}\ \text{m}^2/\text{s}$. The largest permissible time step is given by Equation (3.15)

$$\Delta t_{\text{MAX}} = \frac{\Delta x^2}{2\alpha} = \frac{(10^{-3}\ \text{m})^2}{(2)\ 1.172\times10^{-5}\ \text{m}^2/\text{s}} = 0.0427\ \text{s}$$

(b) From Table 11 in Appendix 2 the thermal diffusivity for window glass is $\alpha = 0.034\times10^{-5}\ \text{m}^2/\text{s}$. The largest permissible time step is given by Equation (3.15)

$$\Delta t_{\text{MAX}} = \frac{\Delta x^2}{2\alpha} = \frac{(10^{-3}\ \text{m})^2}{(2)\ 0.034\times10^{-5}\ \text{m}^2/\text{s}} = 1.47\ \text{s}$$

Since the heat diffuses much more slowly through the window glass, much larger time steps are allowed.

PROBLEM 4.19

Consider one-dimensional transient conduction with a convection boundary condition in which the ambient temperature near the surface is a function of time. Determine the energy balance equation for the boundary control volume. How would the solution method need to be modified to accommodate this complexity?

GIVEN

- One-dimensional transient conduction where the ambient temperature near the surface is a function of time

FIND

(a) The difference equation for the boundary control volume and explain how to solve the problem

SOLUTION

The difference equation would be derived exactly as Equation (4.18). Assuming we are the boundary in question is the left boundary we would have:

$$T_{1,\,m+1} = T_{1,\,m} = \frac{2\Delta t}{\rho c \Delta x}\left\{h \;\; T_{\infty,m} - T_{1,m} \;\; + \dot{q}_{G1,\,m}\frac{\Delta x}{2} + k\,\frac{T_{2,m} - T_{1,m}}{\Delta x}\right\}$$

Here, the term T_∞ will depend on the time step m. Since this function of time is presumably known, a marching procedure can be used to solve the set of equations for the whole problem.

PROBLEM 4.20

What are the advantages and disadvantages of using explicit and implicit difference equations?

EXPLAIN

(a) Advantages and disadvantages of explicit and implicit methods

SOLUTION

The explicit method can be solved by marching, which is very simple to implement but the maximum time step is limited by stability considerations. The implicit method forces the use of matrix inversion software to find the solution, but the size of the time step is not limited by stability considerations. (It is limited by accuracy considerations just as it is for any method.)

PROBLEM 4.21

Equation (4.16) is often called the fully-implicit form of the one-dimensional transient conduction difference equation because all quantities in the equation, except for the temperatures in the energy storage term, are evaluated at the new time step, *m* + 1. In an alternate form called Crank-Nicholson, these quantities are evaluated at both time step *m* and *m* + 1 and then averaged. This averaging significantly improves the accuracy of the numerical solution relative to the fully-implicit form without increasing the complexity of the solution method. Derive the one-dimensional transient conduction difference equation in the Crank-Nicholson form.

GIVEN

- One-dimensional transient conduction difference equation in the implicit form

FIND

(a) The Crank-Nicholson form of the difference equation

SOLUTION

We have the explicit difference equation, Equation (4.14)

$$T_{i,\,m+1} = T_{i,\,m} + \frac{\Delta t}{\rho c \Delta x}\left(\frac{k}{\Delta x}\; T_{i+1,m} - 2T_{i,m} + T_{i-1,m} \;+ \dot{q}_{G\,i,m}\,\Delta x\right)$$

and the implicit difference equation, Equation (3.16)

$$T_{i,\,m+1} = T_{i,\,m} + \frac{\Delta t}{\rho c \Delta x}\left(\frac{k}{\Delta x}\; T_{i+1,m+1} - 2T_{i,m+1} + T_{i-1,m+1} \;+ \dot{q}_{G\,i,m+1}\,\Delta x\right)$$

Adding these two equations and dividing by 2 gives the desired Crank-Nicholson form of the one-dimensional transient difference equation

$$T_{i,\,m+1} = T_{i,\,m} + \frac{\Delta t}{2\rho c \Delta x}$$

$$\left(\frac{k}{\Delta x}\; T_{i+1,m} - 2T_{i,m} + T_{i-1,m} + T_{i+1,m+1} - 2T_{i,m+1} + T_{i-1,m+1} \;+\; \dot{q}_{G\,i,m} + \dot{q}_{G,i,m+1} \;\; \Delta x\right)$$

PROBLEM 4.22

A 3-m-long steel rod (k = 43 W/(mK), α = 1.17 × 10^{-5} m²/s) is initially at 20°C and insulated completely except for its end faces. One end is suddenly exposed to the flow of combustion gases at 1000°C through a heat transfer coefficient of 250 W/(m² K) and the other end is held at 20°C. How long does it take for the exposed end to reach 700°C? How much energy will the rod have absorbed if it is circular in cross section and has a diameter of 3 cm?

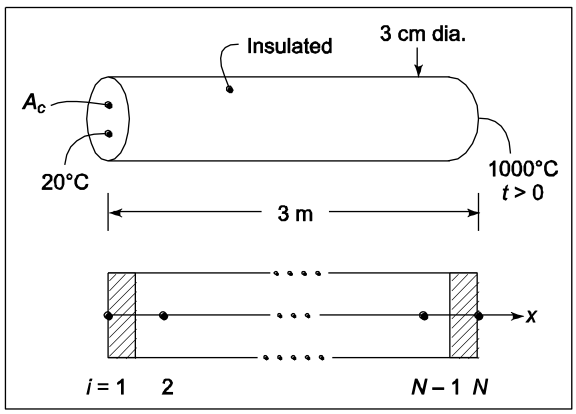
GIVEN

- Steel rod with one end at fixed temperature and the other end exposed to combustion gases

FIND

(a) Time required for the exposed face to reach 700°C
(b) Heat input to the rod

SOLUTION



Control Volume and Node Layout

See the figure to the right for the arrangement of control volumes and nodes and symbol definitions. The nodes are located at

$$x_i = (i - 1)\,\Delta x \qquad \Delta x = \frac{L}{(N-1)} \qquad i = 1, 2, \ldots, N$$

and the time steps are given by

$$t_m = m\Delta t \qquad m = 0, 1, 2, \ldots$$

For the half control volume at i = 1, the temperature is constant so

$$T_{1,\,m} = T_{\text{initial}} \qquad m \geq 0$$

For the half control volume at i = N, the explicit form of the energy balance is

$$k\,\frac{T_{N-1,m} - T_{N,m}}{\Delta x} + h(T_\infty - T_{N,\,m}) = \rho\, c\left(\frac{\Delta x}{2}\right)\frac{T_{N,m+1} - T_{N,m}}{\Delta t}$$

Solving for $T_{N,\,m+1}$

$$T_{N,\,m+1} = T_{N,\,m} + \frac{2\Delta t}{\rho c \Delta x}\;\frac{k}{\Delta x}\;T_{N-1,m} - T_{N,m} \;+ h\;T_\infty - T_{N.m}$$

For all the interior nodes, i = 2, 3, 4, … N – 1, the energy balance is

$$\frac{k}{\Delta x}\,\{(T_{i-1,\,m} - T_{i,\,m}) + (T_{i+1,\,m} - T_{i,\,m})\} = \rho\, c\Delta x\;\frac{T_{i,m+1} - T_{i,m}}{\Delta t}$$

Solving for $T_{i,\,m+1}$

$$T_{i,\,m+1} = T_{i,\,m} + \frac{\alpha\Delta t}{\Delta x^2}\{T_{i-1,\,m} - 2T_{i,\,m} + T_{i+1,\,m}\} \quad i = 2, 3, \ldots N-1$$

The heat input to the rod after any time step m is given by

$$Q_{\text{input},\,m} = A_c\,\rho\, c\Delta x\left\{\sum_{i=2}^{N-1} T_{i,m} - T_{i,m=0} + \frac{1}{2}\, T_{N,m} - T_{N,m=0}\right\}$$

The factor of 1/2 is because the control volume at $i = N$ is $\Delta x/2$ in width.

Since we have chosen an explicit method, we can use the marching procedure as described in Section 4.3.1. Also, the time step Δt is restricted via Equation (4.15). After setting up the computer program to step through the time steps, the energy balance on nodes $i = 1$, 2, and N were checked by hand to insure that the code was correct. The several runs were made with various values of N and Δt to find how large N and how small Δt must be to get an accurate solution. The table below summarizes these runs

N	Δt_{max} (s)	Δt (s)	t_{final} (s)	Q_{input} (J/m^2)
11	3846	10.0	5990	503.72
11	3846	1.0	5993	503.79
21	962	10.0	6350	490.33
41	240	10.0	6440	487.50
81	60	10.0	6460	486.69
81	60	5.0	6455	486.37

Since there is little change between the last 3 runs, the solution is that 6455 seconds are required for the exposed face to reach 700°C and the heat input to the rod is 486.4 joules.

PROBLEM 4.23

A Trombe wall is a masonry wall often used in passive solar homes to store solar energy. Suppose that such a wall, fabricated from 20-cm-thick solid concrete blocks (k = 0.13 W/(mK), $\alpha = 0.05 \times 10^{-5}$ m²/s) is initially at 15°C in equilibrium with the room in which it is located. It is suddenly exposed to sunlight and absorbs 500 W/m² on the exposed face. The exposed face loses heat by radiation and convection to the outside ambient temperature of –15°C through a combined heat transfer coefficient of 10 W/(m² K). The other face of the wall is exposed to the room air through a heat transfer coefficient of 10 W/(m² K). Assuming that the room air temperature does not change, determine the maximum temperature in the wall after 4 hours of exposure and the net heat transferred to
the room.

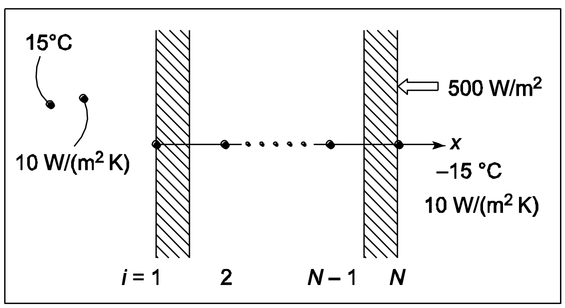
GIVEN

- Trombe wall suddenly exposed to sunlight

FIND

(a) Maximum temperature in the wall after 4 hours
(b) Heat input to the room

SOLUTION



Control Volume and Node Layout

See the figure to the right for the arrangement of control volumes and nodes and symbol definitions. The nodes are located at

$$x_i = (i-1)\Delta x \quad \Delta x = \frac{L}{(N-1)} \quad i = 1, 2, \ldots, N$$

and the time steps are given by

$$t_m = m\Delta t \qquad m = 0, 1, 2, \ldots$$

For the half control volume at i = 1, the explicit form of the energy balance is

$$k\,\frac{T_{2,m} - T_{1,m}}{\Delta x} + h\,(T_\infty - T_{1,m}) = \rho\, c\,\frac{\Delta x}{2}\,\frac{T_{1,m+1} - T_{1,m}}{\Delta t}$$

Solving for $T_{1,\,m+1}$

$$T_{1,\,m+1} = T_{1,\,m} + \frac{2\Delta t}{\rho c \Delta x}\;\frac{k}{\Delta x}\;T_{2,m} - T_{1,m}\;+ h\;T_\infty - T_{1,m}$$

where h is the heat transfer coefficient on the room-side of the wall. For the half control volume at $i = N$, the explicit form of the energy balance is

$$k\,\frac{T_{N-1,m}-T_{N,m}}{\Delta x} + q_{abs} = \rho\, c\,\frac{\Delta x}{2}\,\frac{T_{N,m+1}-T_{N,m}}{\Delta t} + U_o\,(T_{N,m} - T_{\text{out}})$$

where U_o is the combined heat transfer coefficient to outside ambient.

Solving for $T_{N,\,m+1}$

$$T_{N,\,m+1} = T_{N,\,m} + \frac{2\Delta t}{\rho c \Delta x}\left[\frac{k}{\Delta x}\left(T_{N-1,m} - T_{N,m}\right) + q_{abs} - U_o\left(T_{N,m} - T_{\text{out}}\right)\right]$$

For all the interior nodes, $i = 2, 3, 4, \ldots N - 1$, the energy balance is

$$\frac{k}{\Delta x}\,\{(T_{i-1,\,m} - T_{i,\,m}) + (T_{i+1,\,m} - T_{i,\,m})\} = \rho\, c\Delta x\,\frac{T_{i,m+1}-T_{i,m}}{\Delta t}$$

Solving for $T_{i,\,m+1}$

$$T_{i,\,m+1} = T_{i,\,m} + \frac{\alpha\Delta t}{\Delta x^2}\,\{T_{i-1,\,m} - 2T_{i,\,m} + T_{i+1,\,m}\} \qquad i = 2, 3, \ldots N - 1$$

The maximum temperature in the wall at any time step m must be $T_{N,\,m}$.

The heat input to the room after any time step m is given by

$$Q_{\text{input},\,m} = h\,\Delta t \sum_{m=1}^{m_{\text{final}}} (T_{1,m} - T_\infty)\ \ \text{J/m}^2$$

Since we have chosen an explicit method, we can use the marching procedure as described in Section 4.3.1. Also, the time step Δt is restricted via Equation (4.15). After setting up the computer program to step through the time steps, the energy balance on nodes i = 1, 2, and N were checked by hand to insure that the code was correct.

Then several runs were made with various values of N and Δt to find how large N and how small Δt must be to get an accurate solution. The table below summarizes these runs

N	Δt_{max} (s)	Δt (s)	Q_{input} (J/m^2)	TN (°C)
11	400	100	31838	
11	400	50	31713	
11	400	25	31650	
21	100	25	30892	
31	44	25	30751	
41	25	20	30689	33.29
41	25	5	30666	33.29
41	25	5	30653	33.29
61	11	10	30630	33.29

Since there is little change between the last 4 runs, the solution is that after 4 hours the heat input to the room is 30630 joules per m^2 of wall area and the maximum temperature in the wall is 33.29°C.

PROBLEM 4.24

Repeat the numerical calculations of Problem 4.23 (Trombe wall) in order to obtain results to graph the temperature distribution in the Trombe wall after 2, 4, 6, and 8 hours of exposure.

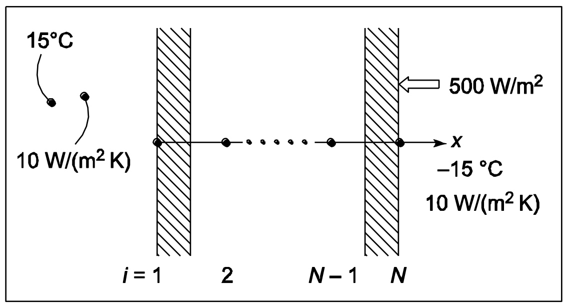
GIVEN

- Trombe wall suddenly exposed to sunlight

FIND

(a) Graph the temperature distribution in Trombe wall after 2, 4, 6 and 8 hours of exposure.

SOLUTION



Control Volume and Node Layout

See the figure to the right for the arrangement of control volumes and nodes and symbol definitions. The nodes are located at

$$x_i = (i-1)\Delta x \qquad \Delta x = \frac{L}{(N-1)} \qquad i = 1, 2, \ldots, N$$

and the time steps are given by

$$t_m = m\Delta t \qquad m = 0, 1, 2, \ldots$$

For the half control volume at $i = 1$, the explicit form of the energy balance is

$$k\,\frac{T_{2,m} - T_{1,m}}{\Delta x} + h\,(T_\infty - T_{1,m}) = \rho\, c\,\frac{\Delta x}{2}\,\frac{T_{1,m+1} - T_{1,m}}{\Delta t}$$

Solving for $T_{1,\,m+1}$

$$T_{1,\,m+1} = T_{1,\,m} + \frac{2\Delta t}{\rho c \Delta x}\;\frac{k}{\Delta x}\;T_{2,m} - T_{1,m} \;+ h\;T_\infty - T_{1,m}$$

where h is the heat transfer coefficient on the room-side of the wall. For the half control volume at $i = N$, the explicit form of the energy balance is

$$k\,\frac{T_{N-1,m} - T_{N,m}}{\Delta x} + q_{abs} = \rho\, c\,\frac{\Delta x}{2}\,\frac{T_{N,m+1} - T_{N,m}}{\Delta t} + U_o\,(T_{N,\,m} - T_{\text{out}})$$

where U_o is the combined heat transfer coefficient to outside ambient.
Solving for $T_{N,\,m+1}$

$$T_{N,\,m+1} = T_{N,\,m} + \frac{2\Delta t}{\rho c\Delta x}\ \frac{k}{\Delta x}\ T_{N-1,m} - T_{N,m} \ + q_{abs} - U_o\ \ T_{N,m} - T_{\text{out}}$$

For all the interior nodes, $i = 2, 3, 4, \ldots N - 1$, the energy balance is

$$\frac{k}{\Delta x}\{(T_{i-1,\,m} - T_{i,\,m}) + (T_{i+1,\,m} - T_{i,\,m})\} = \rho\, c\Delta x\ \frac{T_{i,m+1} - T_{i,m}}{\Delta t}$$

Solving for $T_{i,\,m+1}$

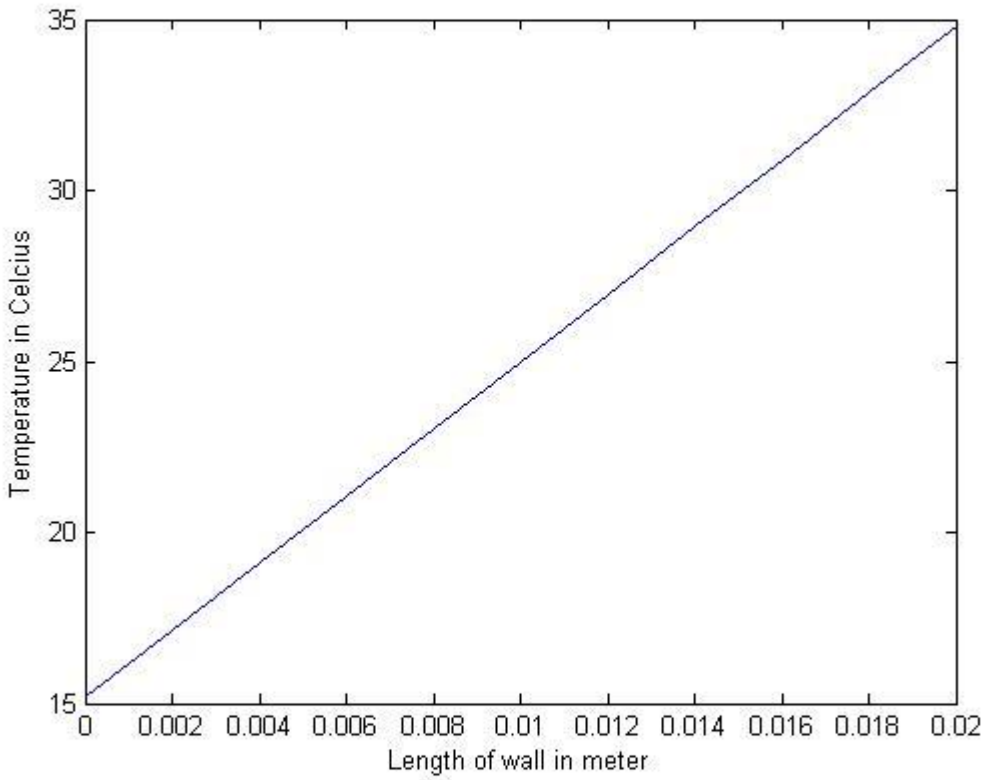
$$T_{i,\,m+1} = T_{i,\,m} + \frac{\alpha\Delta t}{\Delta x^2}\{T_{i-1,\,m} - 2T_{i,\,m} + T_{i+1,\,m}\} \qquad i = 2, 3, \ldots N - 1$$

The solution of the above problem is obtained by discretization in matlab.
Following is the matlab code for problem.

```
% Discretization solution for trombe wall problem 4_24

L=0.02;  % in m
N=11;
t=2; % in hour
delt=0.5 ;   % in seconds
delx=L/(N-1);    % in m
qabs=500;     % in W/m^2
h=10;% W/(m^2 K)
crho= 2.6*10^5;
alpha=5*10^-7;   % in m^2/s
k=0.13;          % W/(m K)
Tinf=15;        % Celsius
T(1:N,20)=10;         % Celsius
Tout=-15;          % Celsius


for m=1:1:(3600*t/delt)
for i=N-1:-1:2
    Tf=T;

T(i,m+1)=T(i,m)+(alpha*delt/delx^2*(T(i-1,m)-2*T(i,m)+T(i+1,m)));
end;

T(1,m+1)=T(1,m)+(2*delt/(crho*delx)*(k/delx*(T(2,m)-T(1,m)+h*(Tinf-
T(1,m)))));
T(N,m+1)=T(N,m)+(2*delt/(crho*delx)*(k/delx*(T(N-1,m)-T(N,m)+qabs-
h*(T(N,m)-Tout))));
% count=0;
% for i=1:N
%     if abs(Tf(i)-T(i))<10^-4
%         count=count+1;
%     end
%
% end
%     if count==N
%         break
%     end
end

for i=1:1:N
```

```
   x(i)=(i-1)*delx;
end
plot(x,T(:,14401))
```

The temperature distribution for 2 hrs is obtained is shown below.



Since the heat flow becomes steady state in less than 2 hours, the temperature distribution for 4, 6 and 8 hrs are same as above.

PROBLEM 4.25

To more accurately model the energy input from the sun, suppose the absorbed flux in Problem 4.23 is given by

$$q_{abs}(t) = t\,(375 - 46.875\,t)$$

where t is in hours and q_{abs} is in W/m^2. (This time variation of q_{abs} gives the same total heat input to the wall as in Problem 3.23, i.e., 2000 W hr/m^2). Repeat Problem 4.23 with the above equation for q_{abs} in place of the constant value of 500 W/m^2. Explain your results.

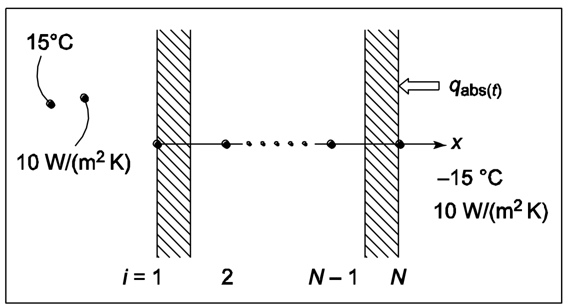
GIVEN

- Trombe wall with specified absorbed solar flux as a function of time

FIND

(a) Maximum temperature in the wall after 4 hours
(b) Heat input to the room

SOLUTION



Control Volume and Node Layout

See the accompanying figure for the arrangement of control volumes and nodes and symbol definitions. The nodes are located as

$$x_i = (i-1)\,\Delta x$$

$$\Delta x = \frac{L}{(N-1)} \qquad i = 1, 2, \ldots, N$$

and the time steps are given by

$$t_m = m\,\Delta t \qquad m = 0, 1, 2, \ldots$$

For the half control volume at $i = 1$, the explicit form of the energy balance is

$$k\,\frac{T_{2,m} - T_{1,m}}{\Delta x} + h(T_\infty - T_{1,m}) = \rho\, c\,\frac{\Delta x}{2}\,\frac{T_{1,m+1} - T_{1,m}}{\Delta t}$$

Solving for $T_{1,m+1}$

$$T_{1,m+1} = T_{1,m} + \frac{2\Delta t}{\rho c \Delta x}\;\frac{k}{\Delta x}\; T_{2,m} - T_{1,m} \;+ h\;\, T_\infty - T_{1,m}$$

For the half control volume at $i = N$, the explicit from of the energy balance is

$$k\ \frac{T_{N-1,m} - T_{N,m}}{\Delta x}\ + q_{\text{abs},m} = \rho\, c\ \frac{\Delta x}{2}\frac{T_{N,m+1} - T_{N,m}}{\Delta x}\ + U_o\,(T_{N,\,m} - T_{\text{out}})$$

Solving for $T_{N,\,m+1}$

$$T_{N,\,m+1}\ = T_{N,\,m} + \ \frac{2\Delta t}{\rho c \Delta x}\ \ \frac{k}{\Delta x}\ \ T_{N-1,m} - T_{N,m}\ \ + q_{\text{abs},m} - U_o\ \ T_{N,m} - T_{\text{out}}$$

For all the interior nodes, $i = 2, 3, 4, \ldots N - 1$, the energy balance is

$$\frac{k}{\Delta x}\ \{(T_{i-1,\,m} - T_{i,\,m}) + (T_{i+1,\,m} - T_{i,\,m}) = \rho\, c \Delta x\ \frac{T_{i,m+1} - T_{i,m}}{\Delta t}$$

Solving for $T_{i,\,m+1}$

$$T_{i,\,m+1}\ = T_{i,\,m} + \ \frac{\alpha \Delta t}{\Delta x^2}\ \{T_{i-1,\,m} - 2T_{i,\,m} + T_{i+1,\,m}\} \qquad i = 2, 3, \ldots N - 1$$

The maximum temperature in the wall at any time step m must be $T_{N,\,m}$ and the heat input to the room after any time step m is given by

$$Q_{\text{input},\,m}\ = h\ \Delta t\ \sum_{m=1}^{m} (T_{1,m} - T_{\infty})\left(\frac{\text{J}}{\text{m}^2}\right)$$

Since we have chosen an explicit method, we can use the marching procedure as described in Section 4.3.1. Also, the time step Δt is restricted via Equation (4.15). After setting up the computer program to step through the time steps, the energy balance on nodes $i = 1$, 2, and N were checked by hand to insure that the code was correct. A run was then made with $N = 41$, $\Delta t = 5$ seconds. The results indicate that the heat input to the room is –1834 joules per m^2 of wall area and the maximum wall temperature is 54.16°C. In comparison with the results from Problem 3.23 where 30630 J/m^2 was delivered to the room, here the room has lost 1834 J/m^2 to the wall. The reason is that for early times, before the absorbed solar flux becomes significant, the wall is losing heat to the outside and is rapidly cooling. The room-side face of the wall dips below the air temperature of 15°C and begins to remove heat from the room. Only at later times does the wall heat up sufficiently to begin transferring heat back to the room. For the short 4 hour run, the net effect is a loss of heat from the room to the wall.

PROBLEM 4.26

An interior wall of a cold furnace, initially at 0°C, is suddenly exposed to a radiant flux of 15 kW/m^2 when the furnace is brought on line. The outer surface of the wall is exposed to ambient air at 20°C through a heat transfer coefficient of 10 W/(m^2 K). The wall is 20 cm thick and is made of expanded perlite (k = 0.10 W/(mK), α = 0.03 × 10^{-5} m^2/s) sandwiched between two sheets of oxidized steel. Determine how long after startup will the inner (hot) sheet metal surface get hot enough so that reradiation becomes significant.

GIVEN

- Furnace wall suddenly exposed to radiant heat flux

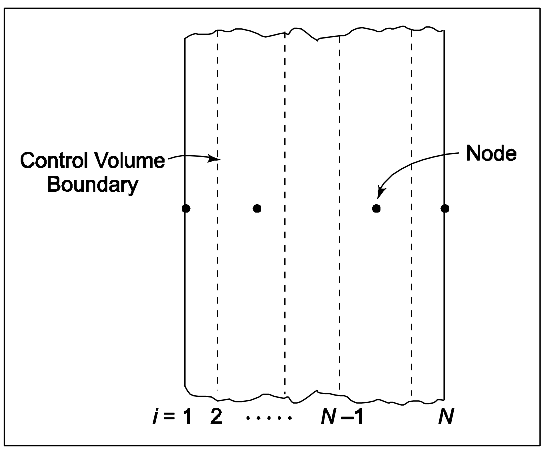
FIND

(a) How long before reradiation from the heated wall becomes significant.

ASSUMPTIONS

(a) Reradiation becomes significant when the reradiated flux from the exposed wall exceeds 10% of the incident radiant flux.
(b) The oxidized surface of the exposed wall is black.

SOLUTION



Control Volume and Node Layout

See the figure to the right for the arrangement of control volumes and nodes and symbol definitions. The nodes are located at

$$x_i = (i - 1)\,\Delta x \quad \Delta x = \frac{L}{(N-1)} \qquad i = 1, 2, \ldots, N$$

and the time steps are given by

$$t_m = m\,\Delta t \qquad m = 0, 1, 2, \ldots$$

For the half control volume at i = 1, the explicit form of the energy balance is

$$k\,\frac{T_{2,m} - T_{1,m}}{\Delta x} + h(T_\infty - T_{1,m}) = \rho\, c\,\frac{\Delta x}{2}\frac{T_{1,m+1} - T_{1,m}}{\Delta t}$$

Solving for $T_{1,\,m+1}$

$$T_{1,\,m+1} = T_{1,\,m} + \frac{2\Delta t}{\rho c \Delta x}\;\frac{k}{\Delta x}\;T_{2,m} - T_{1,m} \;+ h\;T_\infty - T_{1,m}$$

For the half control volume at $i = N$, the explicit form of the energy balance is

$$k \frac{T_{N-1,m} - T_{N,m}}{\Delta x} + q_{\text{abs}} = \rho\, c \frac{\Delta x}{2} \frac{T_{N,m+1} - T_{N,m}}{\Delta t}$$

Solving for $T_{N,\, m+1}$

$$T_{N,\, m+1} = T_{N,\, m} + \frac{2\Delta t}{\rho c \Delta x} \frac{k}{\Delta x} T_{N-1,m} - T_{N,m} + q_{\text{abs}}$$

For all the interior nodes, $i = 2, 3, 4, \ldots N - 1$, the energy balance is

$$\frac{k}{\Delta x} \{(T_{i-1,\, m} - T_{i,\, m}) + (T_{i+1,\, m} - T_{i,\, m})\} = \rho\, c \Delta x \frac{T_{i,m+1} - T_{i,m}}{\Delta t}$$

Solving for $T_{i,\, m+1}$

$$T_{i,\, m+1} = T_{i,\, m} + \frac{\alpha \Delta t}{\Delta x^2} \{T_{i-1,\, m} - 2T_{i,\, m} + T_{i+1,\, m}\} \qquad i = 2, 3, \ldots N - 1$$

Since the exposed wall is black, the reradiated flux from the hot wall is σT_N^4 and the criterion we seek is

$$\sigma T_N^4 \geq 0.1\, q_{\text{abs}}$$

For the given values of problem parameters, this equates to T_N = 130.3°C.

Since we have chosen an explicit method, we can use the marching procedure as described in Section 4.3.1. Also, the time step Δt is restricted via Equation (4.15). After setting up the computer program to step through the time steps, the energy balance on nodes i = 1, 2, and N were checked by hand to insure that the code was correct. Then several runs were made with various values of N and Δt to find how large N and how small Δt must be to get an accurate solution. The table below summarizes these runs

N	Δt_{max} (s)	Δt (s)	t_{final} (s)	T_{max} (C)
21	16.7	0.1	15.25	131.5
41	41.7	0.1	8.0	130.9
61	18.5	0.1	5.6	131.7
81	10.4	0.1	4.4	131.4
161	2.6	0.1	2.7	130.4
321	0.65	0.1	2.2	133.0
641	0.163	0.1	2.0	130.6
1000	0.07	0.05	2.0	131.0

Note that a very large number of nodes is needed because the suddenly imposed flux causes very large temperature gradients in the furnace door. This requires a large number of nodes to accurately depict the temperature profile. The solution is that 2.0 seconds is required before reradiation must be considered.

COMMENTS

The answer given above is conservative because the emissivity of the exposed door surface will be less than 1 and the door will therefore heat up more quickly.

PROBLEM 4.27

A long cylindrical rod, 8 cm in diameter, is initially at a uniform temperature of 20°C. At time t = 0, the rod is exposed to an ambient temperature of 400°C through a heat transfer coefficient of 20 W/(m^2 K). The thermal conductivity of the rod is 0.8 W/(mK) and the thermal diffusivity is 3×10^{-6} m^2/s. Determine how much time will be required for the temperature change at the centerline of the rod to reach 93.68% of its maximum value. Use an explicit difference equation and compare your numerical results with a chart solution from Chapter 2.

GIVEN

- Cylindrical rod suddenly exposed to increased ambient temperature

FIND

(a) Time required for the centerline temperature change to reach 93.68% of its maximum value

SOLUTION

Since the rod will eventually reach 400°C, the maximum possible temperature change for any part of the rod is 400 – 20 = 380°C. Taking 93.68% of this temperature difference, we need to find the time such that the centerline temperature is 20 + (0.9368 × 380) = 376°C.

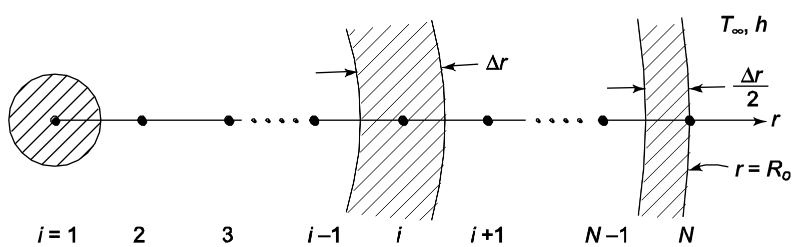
As in Figure 4.22 and Section 4.5, the radius is given by

$$r = (i-1)\,\Delta r \qquad i = 1, 2, \ldots N \qquad \Delta r = \frac{R_o}{N-1}$$

and the time is given by

$$t_m = m\,\Delta t, \qquad m = 0, 1, 2, \ldots$$

Note that since all gradients with respect to the circumferential direction, θ, are zero, the index j is not needed. Let the convection coefficient be h and ambient temperature be T_∞. The following sketch shows the control volumes necessary to solve the problem numerically



Referring to the above sketch, the inner surface area per unit length of the shaded control volume at node $i = N$ is

$$2\pi\left(R_o - \frac{\Delta r}{2}\right)$$

and the outer surface area is

$$2\pi R_o$$

The volume of the control volume per unit length is

$$\pi\left(R_o^2 - \left(R_o - \frac{\Delta r}{2}\right)^2\right) = \pi\left(R_o \Delta r - \frac{\Delta r^2}{4}\right) \equiv V_N$$

The explicit form of the energy balance on the control volume at $i = N$ gives

$$\rho\, c\, V_N\ \frac{T_{N,m+1}-T_{N,m}}{\Delta t} = k\ \frac{T_{N-1,m}-T_{N,m}}{\Delta r}\ 2\pi\left(R_o-\frac{\Delta r}{2}\right) + 2\pi R_o\, h\,(T_\infty - T_{N,\,m})$$

Solving for $T_{N,\,m+1}$,

$$T_{N,\,m+1} = T_{N,\,m}\left\{1-\frac{\alpha\,\Delta t 2\pi}{V_N}\left(\frac{R_o}{\Delta r}-\frac{1}{2}\right)-\frac{\Delta t 2\pi R_o h}{\rho c V_N}\right\} + T_{N-1,\,m}\left\{\frac{\alpha\,\Delta t 2\pi}{V_N}\left(\frac{R_o}{\Delta r}-\frac{1}{2}\right)\right\} + \frac{\Delta t 2\pi R_o h T_\infty}{\rho c V_N}$$

For the control volume at the centerline node, $i = 1$, the volume per unit length is

$$\pi\left(\frac{\Delta r}{2}\right)^2 \equiv V_1$$

and the surface area per unit length is

$$2\pi\ \frac{\Delta r}{2}\ = \pi\,\Delta r$$

The energy balance on this node is

$$\rho\, c\, V_1\ \frac{T_{1,m+1}-T_{1,m}}{\Delta t}\ = k\ \frac{T_{2,m}-T_{1,m}}{\Delta r}\ \pi\,\Delta r$$

Solving for $T_{1,\,m+1}$

$$T_{1,\,m+1}\ = T_{1,\,m}\left(1-\frac{\alpha\,\Delta t\pi}{V_1}\right)\ + T_{2,\,m}\ \frac{\alpha\,\Delta t\pi}{V_1}$$

For nodes $1 < i < N$, set the $\dfrac{\partial}{\partial\theta}$ terms to zero and set $\Delta\,\theta = 2\pi$ in Equation (4.31),

$$\rho\, c\, r\, 2\pi\Delta\, r\frac{T_{i,m+1}-T_{i,m}}{\Delta t}\ =\ \frac{k2\pi r}{\Delta r}\left[T_{i-1,m}-2T_{i,m}+T_{i+1,m}+\frac{\Delta r}{2r}\ T_{i+1,m}-T_{i-1,m}\ \right]$$

Solving for $T_{i,\,m+1}$

$$T_{i,\,m+1}\ = T_{i,\,m}\left(1-\frac{2\alpha\,\Delta t}{\Delta r^2}\right)\ + T_{i+1,\,m}\frac{\alpha\,\Delta t}{\Delta r^2}\left(1+\frac{\Delta r}{2r}\right)\ + T_{i-1,m}\ \frac{\alpha\,\Delta t}{\Delta r^2}\left(1-\frac{\Delta r}{2r}\right)$$

Note that

$$\frac{\Delta r}{2r}\ =\ \frac{1}{2(i-1)}$$

and

$$\frac{R_o}{\Delta r}\ = N-1$$

Using a time step, Δt, such that

$$\Delta t\ = \Gamma\ \frac{\Delta r^2}{2\alpha}\ \text{ where } \Gamma < 1$$

then the explicit solution can be solved by marching and it should be stable. For N = 10 and Γ = 0.5, the centerline temperature is found to exceed 376°C at 994 seconds.

For the chart solution, we refer to Figure 2.43. The Biot number is

$$B_i = \frac{hr_o}{k} = \frac{20\,\mathrm{W/(m^2K)} \quad 0.04\,\mathrm{m}}{0.8\,\mathrm{W/(m\,K)}} = 1.0$$

We need to find the abscissa in the figure such that

$$\frac{T(0,t)-T_\infty}{T_i - T_\infty} = \frac{376-400}{20-400} = 0.063$$

For the Biot number calculated above, the abscissa is

$$\frac{\alpha\, t}{r_o{}^2} = 1.78$$

Solving for the time, we find t = 949 seconds, approximately 5% less than the numerical method predicts. Most likely, the difference is due to the precision with which the charts can be read.

PROBLEM 4.28

Develop a reasonable layout of nodes and control volumes for the geometry shown in the sketch. Provide a scale drawing showing the problem geometry overlaid with the nodes and control volumes.

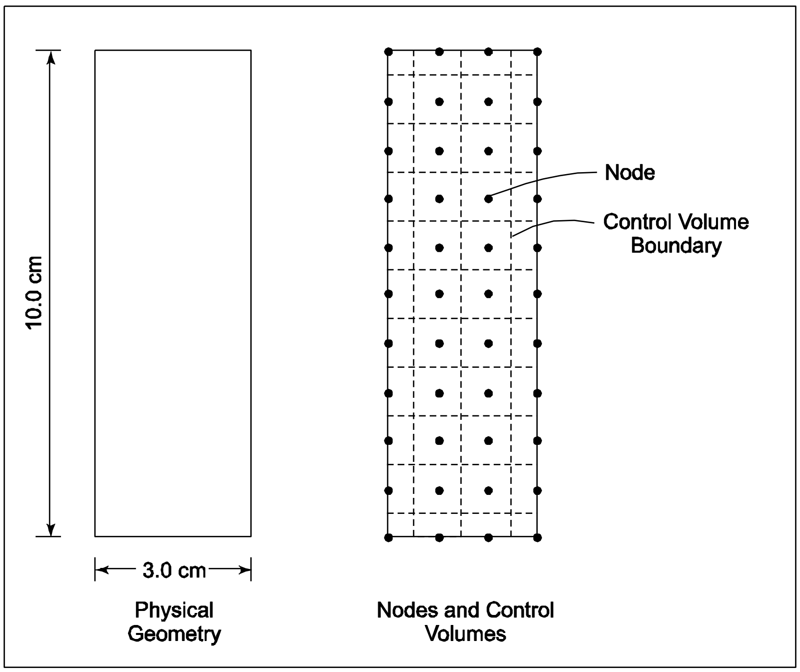
GIVEN

- Rectangular problem geometry

FIND

(a) A reasonable layout of nodes and control volumes

SKETCH



SOLUTION

The largest node spacing divisible into both 3 and 10 is 1 cm. So let's use $\Delta x = \Delta y = 1$ cm. The sketch on the right above shows the resulting placement of nodes and control volume boundaries.

PROBLEM 4.29

Develop a reasonable layout of nodes and control volumes for the geometry shown in the sketch below. Provide a scale drawing showing the problem geometry overlaid with the nodes and control volumes. Identify each type of control volume used.

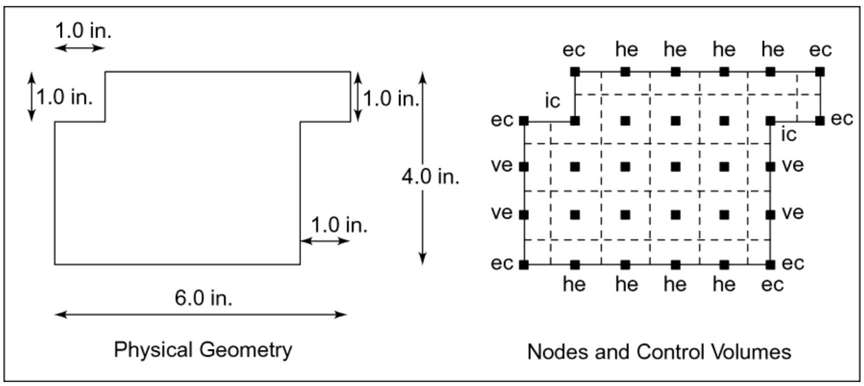
GIVEN

- Rectangular problem geometry with corners removed

FIND

(a) Reasonable layout of nodes and control volumes.
(b) Identify each type of control volume.

SKETCH



SOLUTION

The largest grid spacing for this problem is $\Delta x = \Delta y = 1$ cm. If we used a larger node spacing, we could not adequately represent the cutout corners. The right side of the figure shows the resulting placement of nodes and control volumes. The notation for the type of control volumes is: ec = exterior corner, ic = interior corner, he = horizontal edge, ve = vertical edge.

PROBLEM 4.30

Determine the temperature at the four nodes shown in the sketch. Assume steady conditions and two-dimensional heat conduction. The four faces of the square shape are each at different temperatures as shown.

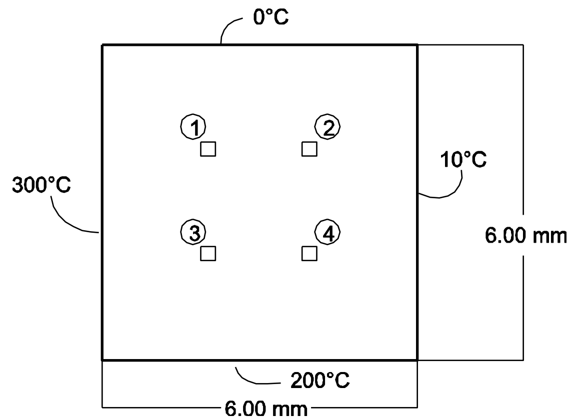
GIVEN

- Square shape with four different face temperatures

FIND

(a) Temperature at four interior nodes

SKETCH



SOLUTION

If the shape is divided into square control volumes then according to Section 4.4.1, the temperature at each node is the average of its four neighbors. The equation for each node is therefore

$$T_1 = \frac{1}{4}(0 + 300 + T_3 + T_2)$$

$$T_2 = \frac{1}{4}(0 + 10 + T_4 + T_1)$$

$$T_3 = \frac{1}{4}(300 + T_1 + T_4 + 200)$$

$$T_4 = \frac{1}{4}(200 + 10 + T_2 + T_3)$$

The equations can be solved by the iterative method. A table showing the calculation for the first 10 iterations is given below. The zero iteration is the initial guess of the temperature at the four nodes.

SOLUTION TO PROBLEM 4.30				
iteration	T_1 (°C)	T_2 (°C)	T_3 (°C)	T_4 (°C)
0	150	5	250	100
1	138.75	65.00	187.50	116.25
2	138.13	66.25	188.75	115.63
3	138.75	65.94	188.44	116.25
4	138.59	66.25	188.75	116.09

SOLUTION TO PROBLEM 4.30				
5	138.75	66.17	188.67	116.25
6	138.71	66.25	188.75	116.21
7	138.75	66.23	188.73	116.25
8	138.74	66.25	188.75	116.24
9	138.75	66.25	188.75	116.25
10	138.75	66.25	188.75	116.25

PROBLEM 4.31

The horizontal cross section of an industrial chimney is shown in the accompanying sketch. Flue gases maintain the interior surface of the chimney at 300°C and the outside is exposed to ambient temperature of 0°C through a heat transfer coefficient of 5 W/(m² K). The thermal conductivity of the chimney is *k* = 0.5 W/(mK). For a grid spacing of 0.2 m, determine the temperature distribution in the chimney and the rate of heat loss from the flue gases per unit length of the chimney.

GIVEN

- Chimney with hot flue gases inside, ambient temperature outside

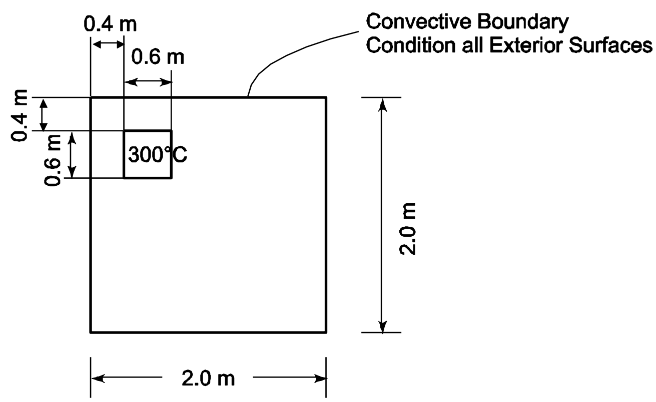
FIND

(a) Temperature distribution in the chimney
(b) Rate of heat loss from the flue gases per unit length

ASSUMPTIONS

- Steady state conditions
- Neglect radiation heat transfer

SKETCH



SOLUTION

Due a symmetry, only half of the problem geometry needs to be considered. The layout of control volumes and nodes is shown in the figure on the next page. There are a total of 63 control volumes although the temperature at the nodes for 7 of these is specified. So, we need to develop energy balance equations for the remainder.

For shorthand, let's define

$$T \equiv T_{i,j} \quad T_1 \equiv T_{i-1,j} \quad T_r \equiv T_{i+1,j} \quad T_u \equiv T_{i,j+1} \quad T_d \equiv T_{i,j-1}$$

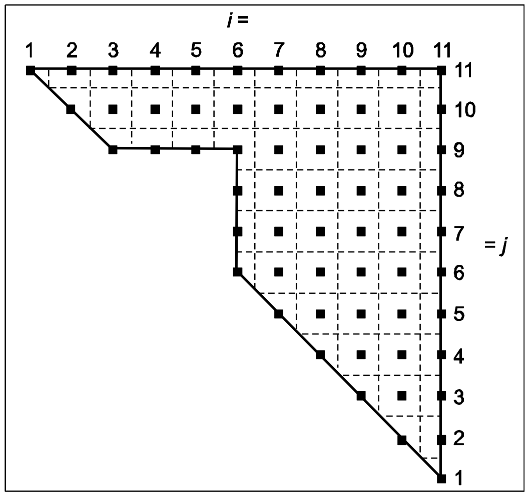
The subscripts in the previous equation stand for left, right, up, and down.

Interior nodes are given by the following indices

$i = 3, 4, \ldots 10;$ $\quad j = 10$
$i = 7, 8, 9, 10;$ $\quad j = 9$
$i = 7, 8, 9, 10;$ $\quad j = 8$
$i = 7, 8, 9, 10;$ $\quad j = 7$
$i = 7, 8, 9, 10;$ $\quad j = 6$
$i = 8, 9, 10;$ $\quad j = 5$
$i = 9, 10;$ $\quad j = 4$
$i = 10;$ $\quad j = 3$

and for these control volumes the energy balance equations are

$$T = \frac{1}{4}\,(T_u + T_d + T_1 + T_r)$$



Control Volume and Node Layout

For the nodes along the top edge (except for the corner nodes) the energy balance gives

$$k\left[\frac{T_1 - T}{\Delta x}\frac{\Delta y}{2} + \frac{T_r - T}{\Delta x}\frac{\Delta y}{2} + \frac{T_d - T}{\Delta y}\Delta x\right] + h\,\Delta x\,(T_\infty - T) = 0$$

or

$$T = \frac{\frac{1}{2}\; T_1 + T_r \;\; + T_d + \frac{h\Delta x}{k} T_\infty}{2 + \frac{h\Delta x}{k}} \qquad j = 11 \quad i = 2, 3, \ldots 10$$

For the nodes along the right edge (except for the corner nodes)

$$k\left[\frac{T_1 - T}{\Delta x}\Delta y + \frac{T_u - T}{\Delta y}\frac{\Delta x}{2} + \frac{T_d - T}{\Delta y}\frac{\Delta x}{2}\right] + h\,\Delta y\,(T_\infty - T) = 0$$

or

$$T = \frac{\frac{1}{2}\; T_u + T_d \;\; + T_1 + \frac{h\Delta y}{k} T_\infty}{2 + \frac{h\Delta y}{k}} \qquad i = 11 \quad j = 2, 3, \ldots 10$$

Nodes along the diagonal are identified by the following i, j pairs

i =	2	7	8	9	10
j =	10	5	4	3	2

and for these control volumes we have

$$k\left[\frac{T_r - T}{\Delta x}\Delta y + \frac{T_u - T}{\Delta y}\Delta x\right] = 0$$

or

$$T = \frac{1}{2}\,(T_u + T_r)$$

Nodes along he chimney inner surface are identified by the indices

$j = 9$; $i = 3, 4, 5, 6$ and

$i = 6$; $j = 6, 7, 8$

and for these control volumes

$$T = 300°\text{C}$$

For the corners

$i = 1$; $j = M$

$$k\left[\frac{T_r - T}{\Delta x}\frac{\Delta y}{2}\right] + h\,\frac{\Delta x}{2}\,(T_\infty - T) = 0$$

or

$$T = \frac{T_r + \dfrac{h\Delta x}{k}T_\infty}{1 + \dfrac{h\Delta x}{k}} \qquad j = 11\ i = 1$$

$i = 11$; $j = 11$

$$k\left[\frac{T_1 - T}{\Delta x}\frac{\Delta y}{2} + \frac{T_d - T}{\Delta y}\frac{\Delta x}{2}\right](\Delta x + \Delta y)\,(T_\infty - T) = 0$$

or

$$T = \frac{T_1 + T_d + \dfrac{h}{k}\ \Delta x + \Delta y\ T_\infty}{2 + \dfrac{h}{k}\ \Delta x + \Delta y} \qquad j = 11,\ i = 11$$

$i = N$; $j = 1$

$$k\left[\frac{Tu - T}{\Delta y}\frac{\Delta x}{2}\right] + h\,\frac{\Delta y}{2}\,(T_\infty - T) = 0$$

or

$$T = \frac{T_u + \dfrac{h}{k}\Delta y\,T_\infty}{1 + \dfrac{h}{k}\Delta y} \qquad j = 1,\ i = 11$$

This set of difference equations can be solved by iteration. An initial guess of

$$T_{i,j} = \frac{1}{2}\,(0 + 300)°\text{C}$$

for all nodes gives rapid convergence.

The rate of heat loss, q, from the flue gas is equal to the convective loss from the outside surface of the chimney. From symmetry we have

$$q = 2h\quad T_{1,11} - T_\infty\ \frac{\Delta x}{2}\ T_{11,11} - T_\infty\ \frac{\Delta x + \Delta y}{2}\ T_{1,1} - T_\infty\ \frac{\Delta y}{2}$$

$$+\sum_{i=2}^{10} T_{i,11} - T_\infty\ \Delta x + \sum_{j=2}^{10} T_{11,j} - T_\infty\ \Delta y\ \Big\}$$

Results are given in the following table.

```
Heat loss from chimney = 923.937002 W/m
```

Node temperatures, °C

j= \ i=	1	2	3	4	5	6	7	8	9	10	11
11	10.0705	30.2115	48.8663	55.6317	56.3262	51.8848	38.8923	26.6768	17.0475	9.4740	3.0928
10		91.3775	152.5436	169.9305	171.5467	159.9300	116.2884	78.7375	50.1145	27.8258	9.0827
9			300.0000	300.0000	300.0000	300.0000	187.5939	121.8701	76.8473	42.6318	13.9174
8						300.0000	212.2172	144.3014	92.7730	51.9366	16.9928
7						300.0000	216.9733	150.3454	98.0064	55.3487	18.1517
6						300.0000	205.3306	142.1003	93.5585	53.3000	17.5236
5							162.2487	119.1666	80.8271	46.7692	15.4367
4								91.4904	63.8140	37.5129	12.4314
3									45.4255	27.0368	8.9886
2										16.2204	5.4038
1											1.8013

As a check on the above heat loss calculation, we can also calculate the heat loss by determining the heat transferred out of the control volumes at the chimney inner surface. The appropriate equation is

$$q = 2k\left\{\frac{T_{3,9}-T_{3,10}}{\Delta y}\Delta x+\frac{T_{4,9}-T_{4,10}}{\Delta y}\Delta x+\frac{T_{5,9}-T_{5,10}}{\Delta y}\Delta x+\frac{T_{6,9}-T_{6,10}}{\Delta y}\Delta x\right.$$

$$\left.+\frac{T_{6,9}-T_{7,9}}{\Delta y}\Delta y+\frac{T_{6,8}-T_{7,8}}{\Delta x}\Delta y+\frac{T_{6,7}-T_{7,7}}{\Delta x}\Delta y+\frac{T_{6,6}-T_{7,6}}{\Delta x}\Delta y\right\}$$

The result of this calculation gives

$$q = 923.934 \text{ W/m}$$

which is very close to the value determined via convection at the outer surface.

PROBLEM 4.32

In a long, 30-cm square bar shown in the accompanying sketch, the left face is maintained at 40°C and the top face is maintained at 250°C. The right face is in contact with a fluid at 40°C through a heat transfer coefficient of 60 W/(m² K) and the bottom face is in contact with a fluid at 250°C through a heat transfer coefficient of 100 W/(m² K). If the thermal conductivity of the bar is 20 W/(mK), calculate the temperature at the 9 nodes shown in the sketch.

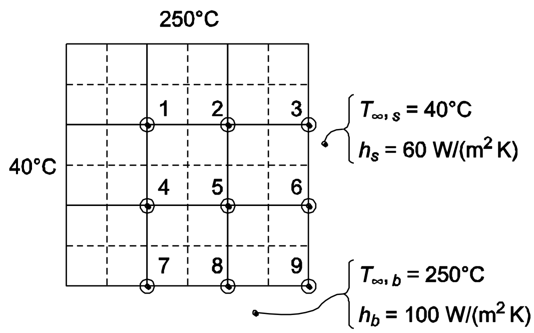
GIVEN

- Square bar with two surfaces at fixed temperature and two surfaces with convective boundary conditions

FIND

(a) Temperature at 9 shown nodes

SKETCH



SOLUTION

Define the following symbols

k	—	thermal conductivity = 20 W/(m K)
$\Delta x = \Delta y$	—	node spacing = 0.1 m
T_T	—	top edge temperature = 250°C
T_L	—	left edge temperature = 40°C
h_s	—	right edge heat transfer coefficient = 60 W/(m² K)
$T_{\infty s}$	—	right edge ambient temperature = 40°C
h_b	—	bottom edge heat transfer coefficient = 100 W/(m² K)
$T_{\infty b}$	—	bottom edge ambient temperature = 250°C

From Equation (4.24), the temperature at nodes 1, 2, 4, and 5 is just the average of the temperature at the neighbor nodes

$$T_1 = \frac{1}{4}\,(T_L + T_T + T_2 + T_4)$$

$$T_2 = \frac{1}{4}\,(T_T + T_1 + T_3 + T_5)$$

$$T_4 = \frac{1}{4}\,(T_L + T_1 + T_5 + T_7)$$

$$T_5 = \frac{1}{4}\,(T_2 + T_4 + T_6 + T_8)$$

The remaining control volumes have convective boundary conditions and we need to develop individual energy balance equations for each.

For the control volume surrounding node 3

$$k\left\{\frac{T_r - T_3}{\Delta x}\frac{\Delta x}{2} + \frac{T_2 - T_3}{\Delta x}\Delta x + \frac{T_6 - T_3}{\Delta x}\frac{\Delta x}{2}\right\} + h_s\,(T_{\infty s} - T_3)\,\Delta x = 0$$

which can be solved for T_3 as follows

$$T_3 = \frac{T_T + 2T_2 + T_6 + 2\dfrac{h_s \Delta x}{k} T_{\infty s}}{4 + 2\dfrac{h_s \Delta x}{k}}$$

For the control volume at node 6

$$k\left\{\frac{T_5 - T_6}{\Delta x}\Delta x + \frac{T_3 - T_6}{\Delta x}\frac{\Delta x}{2} + \frac{T_9 - T_6}{\Delta x}\frac{\Delta x}{2}\right\} + h_s\,(T_{\infty s} - T_c)\,\Delta x = 0$$

or

$$T_6 = \frac{2T_5 + T_3 + T_9 + 2\dfrac{h_s \Delta x}{k} T_{\infty s}}{4 + 2\dfrac{h_s \Delta x}{k}}$$

For the control volume at node 7

$$k\left\{\frac{T_L - T_7}{\Delta x}\frac{\Delta x}{2} + \frac{T_4 - T_7}{\Delta x}\Delta x + \frac{T_8 - T_7}{\Delta x}\frac{\Delta x}{2}\right\} + h_b\,(T_{\infty b} - T_7)\,\Delta x = 0$$

or

$$T_7 = \frac{T_L + 2T_4 + T_8 + 2\dfrac{h_b \Delta x}{k} T_{\infty b}}{4 + 2\dfrac{h_b \Delta x}{k}}$$

For the control volume at node 8

$$k\left\{\frac{T_7 - T_8}{\Delta x}\frac{\Delta x}{2} + \frac{T_5 - T_8}{\Delta x}\Delta x + \frac{T_9 - T_8}{\Delta x}\frac{\Delta x}{2}\right\} + h_b\,(T_{\infty b} - T_8)\,\Delta x = 0$$

or

$$T_8 = \frac{T_7 + 2T_5 + T_9 + 2\dfrac{h_b \Delta x}{k} T_{\infty b}}{4 + 2\dfrac{h_b \Delta x}{k}}$$

$$k\left\{\frac{T_6 - T_9}{\Delta x}\frac{\Delta x}{2} + \frac{T_8 - T_9}{\Delta x}\frac{\Delta x}{2}\right\} + h_s(T_{\infty s} - T_9)\frac{\Delta x}{2} + h_b(T_{\infty b} - T_9)\frac{\Delta x}{2} = 0$$

or

$$T_9 = \frac{T_6 + T_8 + \dfrac{\Delta x}{k}\ \ h_s T_{\infty b} + h_b T_{\infty b}}{2 + \dfrac{\Delta x}{k}\ \ h_s + h_b}$$

This set of equations can be solved iteratively. The table below shows the results of the first 25 iterations after which the calculation appears to converge. Values for the 9 nodal temperatures at the zero iteration are the first guess.

				Temperature, °C					
iteration	T1	T2	T3	T4	T5	T6	T7	T8	T9
======	=====	=====	=====	=====	=====	=====	=====	=====	=====
0	40.000	40.000	40.000	40.000	40.000	40.000	40.000	40.000	40.000
1	92.500	105.625	114.185	53.125	59.688	64.687	87.250	99.325	107.504
2	112.188	134.015	131.895	74.781	93.202	97.783	107.778	130.337	130.400
3	124.699	149.949	146.018	91.420	117.372	116.340	120.635	147.156	143.034
4	132.842	161.558	155.099	102.712	131.942	127.395	128.516	157.087	150.529
5	138.568	1678.902	160.695	109.756	140.785	134.086	133.320	163.084	155.061
6	142.165	173.411	164.110	114.067	146.162	138.151	136.244	166.726	157.813
.									
.									
19	147.794	180.423	169.412	120.765	154.500	144.454	140.779	172.371	162.080
20	147.797	180.427	169.415	120.769	154.505	144.458	140.782	172.375	162.083
21	147.799	180.430	169.417	120.772	154.508	144.460	140.784	172.377	162.085
22	147.800	180.431	169.418	120.773	154.510	144.462	140.785	172.378	162.086
23	147.801	180.432	169.419	120.774	154.511	144.462	140.785	172.379	162.086
24	147.802	180.433	169.419	120.775	154.512	144.463	140.786	172.379	162.086
25	147.802	180.433	169.419	120.775	154.513	144.463	140.786	172.380	162.087

PROBLEM 4.33

Repeat Problem 4.32 if the temperature distribution on the top surface of the bar varies sinusoidally from 40°C at the left edge to a maximum of 250°C in the center and back to 40°C at the right edge.

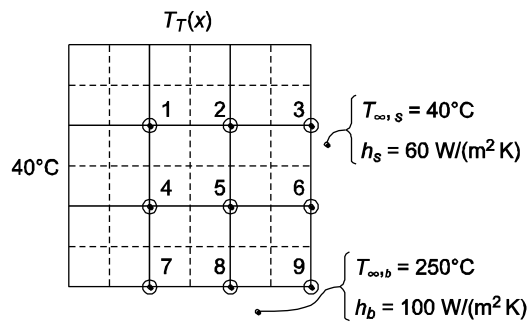
GIVEN

- Square bar with one surface at fixed temperature, one surface with a specified temperature distribution, and two surfaces with convective boundary conditions

FIND

(a) Temperature at 9 shown nodes

SKETCH



SOLUTION

Define the following symbols

k	—	thermal conductivity = 20 W/(mK)
L	—	bar width = 30 cm
$\Delta x = \Delta y$	—	node spacing = 0.1 m
T_L	—	left edge temperature = 40°C
h_s	—	right edge heat transfer coefficient = 60 W/(m^2 K)
$T_{\infty s}$	—	right edge ambient temperature = 40°C
h_b	—	bottom edge heat transfer coefficient = 100 W/(m^2 K)
$T_{\infty b}$	—	bottom edge ambient temperature = 250°C
$T_{\max}$	—	maximum temperature on the top edge = 250°C
$T_{\min}$	—	minimum temperature on the top edge = 40°C

Since the temperature varies sinusoidally across the top surface we have

$$T_T(x) = a + b \sin \frac{x}{L} \pi$$

Since $T(0) = T_{\min}$, $T\left(\frac{L}{2}\right) = T_{\max}$, $T(L) = T_{\min}$ we can solve for a and b giving

$$T_T(x) = T_{\min} + (T_{\max} - T_{\min}) \sin \frac{x}{L} \pi$$

Now, define the temperature at the four nodes on the top edge as

$$T_{T0} \equiv T_T(0) = 40°\text{C} \qquad T_{T1} \equiv T_T\left(\frac{L}{3}\right) = 221.866°\text{C}$$

$$T_{T2} \equiv T_T\left(\frac{2L}{3}\right) = 221.866°\text{C} \qquad T_{T3} \equiv T_T(L) = 40°\text{C}$$

From Equation (4.24), the temperature at nodes 1, 2, 4, and 5 is just the average of the temperature at the neighbor nodes

$$T_1 = \frac{1}{4}\ (T_L + T_{T1} + T_2 + T_4)$$

$$T_2 = \frac{1}{4}\ (T_{T2} + T_1 + T_3 + T_5)$$

$$T_4 = \frac{1}{4}\ (T_L + T_1 + T_5 + T_7)$$

$$T_5 = \frac{1}{4}\ (T_2 + T_4 + T_6 + T_8)$$

The remaining control volumes have convective boundary conditions and we need to develop individual energy balance equations for each.

For the control volume surrounding node 3

$$k\left\{\frac{T_{T3}-T_3}{\Delta x}\frac{\Delta x}{2}+\frac{T_2-T_3}{\Delta x}\Delta x+\frac{T_6-T_3}{\Delta x}\frac{\Delta x}{2}\right\} + h_s\,(T_{\infty s} - T_3)\,\Delta x = 0$$

which can be solved for T_3 as follows

$$T_3 = \frac{T_{T3}+2T_2+T_6+2\dfrac{h_s\Delta x}{k}T_{\infty s}}{4+2\dfrac{h_s\Delta x}{k}}$$

For the control volume at node 6

$$k\left\{\frac{T_5-T_6}{\Delta x}\Delta x+\frac{T_3-T_6}{\Delta x}\frac{\Delta x}{2}+\frac{T_9-T_6}{\Delta x}\frac{\Delta x}{2}\right\} + h_s\,(T_{\infty s} - T_6)\,\Delta x = 0$$

or

$$T_6 = \frac{2T_5+T_3+T_9+2\dfrac{h_s\Delta x}{k}T_{\infty s}}{4+2\dfrac{h_s\Delta x}{k}}$$

For the control volume at node 7

$$k\left\{\frac{T_L-T_7}{\Delta x}\frac{\Delta x}{2}+\frac{T_4-T_7}{\Delta x}\Delta x+\frac{T_8-T_7}{\Delta x}\frac{\Delta x}{2}\right\} + h_b\,(T_{\infty b} - T_7)\,\Delta x = 0$$

or

$$T_7 = \frac{T_L+2T_4+T_8+2\dfrac{h_b\Delta x}{k}T_{\infty b}}{4+2\dfrac{h_b\Delta x}{k}}$$

For the control volume at node 8

$$k\quad\frac{T_7-T_8}{\Delta x}\frac{\Delta x}{2}+\frac{T_5-T_8}{\Delta x}\Delta x+\frac{T_9-T_8}{\Delta x}\frac{\Delta x}{2}\quad + h_b\,(T_{\infty b} - T_8)\,\Delta x = 0$$

or

$$T_8 = \frac{T_7+2T_5+T_9+2\dfrac{h_b\Delta x}{k}T_{\infty b}}{4+2\dfrac{h_b\Delta x}{k}}$$

Finally, for the control volume at node 9

$$k\left\{\frac{T_6 - T_9}{\Delta x}\frac{\Delta x}{2} + \frac{T_8 - T_9}{\Delta x}\frac{\Delta x}{2}\right\} + h_s(T_{\infty s} - T_9)\frac{\Delta x}{2} + h_b(T_{\infty b} - T_9)\frac{\Delta x}{2} = 0$$

or

$$T_9 = \frac{T_6 + T_8 + \frac{\Delta x}{k}\ \ h_s T_{\infty s} + h_b T_{\infty b}}{2 + \frac{\Delta x}{k}\ \ h_s + h_b}$$

This set of equations can be solved iteratively. Here we used a spreadsheet to employ the Gauss-Seidel iteration method. The table below shows the results of the first 25 iterations after which the calculation appears to converge. Values for the 9 nodal temperatures at the zero iteration are the first guess.

				Temperature, °C					
iteration	T1	T2	T3	T4	T5	T6	T7	T8	T9
0	40.000	40.000	40.000	40.000	40.000	40.000	40.000	40.000	40.000
1	85.467	96.833	64.710	51.367	57.050	52.785	86.547	98.129	102.826
2	102.516	111.536	73.882	71.528	83.494	79.934	106.237	125.211	122.195
3	111.232	122.619	84.603	85.241	103.251	95.065	117.139	139.167	132.583
4	117.431	131.788	91.878	94.455	115.119	104.065	123.616	147.287	138.697
5	122.027	137.723	96.415	100.190	122.316	109.510	127.534	152.173	142.387
6	124.945	141.386	99.192	103.699	126.692	112.818	129.914	155.137	144.627
.									
.									
.									
.									
.									
19	129.523	147.090	103.504	109.148	133.476	117.946	133.604	159.730	148.099
20	129.526	147.093	103.507	109.152	133.480	117.949	133.607	159.733	148.101
21	129.528	147.095	103.509	109.154	133.483	117.951	133.608	159.735	148.102
22	129.529	147.097	103.510	109.155	133.484	117.952	133.609	159.736	148.103
23	129.529	147.097	103.510	109.156	133.485	117.953	133.609	159.737	148.103
24	129.530	147.098	103.511	109.156	133.486	117.953	133.610	159.737	148.104
25	129.530	147.098	103.511	109.156	133.486	117.954	133.610	159.737	148.104

COMMENTS

Comparing the results with those from Problem 4.32, we see that the bottom row of temperatures, nodes 7, 8, 9 show that the effect of lower temperatures near the top corners has propagated down through the bar.

PROBLEM 4.34

A 1-cm-thick, 1-m-square steel plate is exposed to sunlight and absorbs a solar flux of 800 W/m². The bottom of the plate is insulated, the edges are maintained at 20°C by water-cooled clamps, and the exposed face is cooled by a convection coefficient of 10 W/(m² K) to an ambient temperature of 10°C. The plate is polished to minimize reradiation. Determine the temperature distribution in the plate using a node spacing of 20 cm. The thermal conductivity of the steel is 40 W/(m K).

GIVEN

- Square plate with water-cooled edges exposed to solar flux

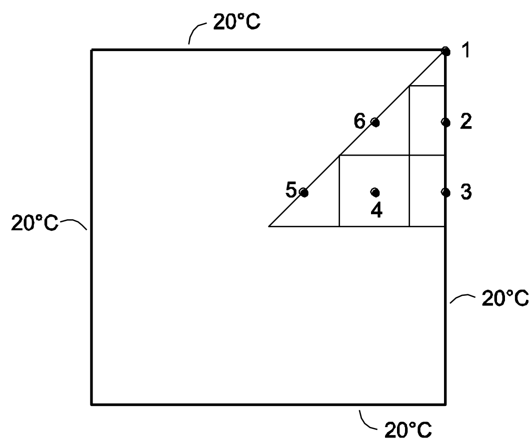
FIND

(a) Temperature distribution in the plate

ASSUMPTIONS

(a) Neglect temperature gradients through the plate thickness

SKETCH



SOLUTION

Because of problem symmetry, we need only consider the 6 nodes in 1/8 th of the square plate as shown in the above sketch. The boundary condition gives us the temperature of nodes 1, 2, and 3 so we only need to perform a heat balance on nodes 4, 5, and 6. Define the following symbols

k	= plate thermal conductivity = 40 W/(mK)
h	= convection coefficient = 10 W/(m² K)
T_∞	= ambient temperature = 10°C
Δx	= node spacing = 20 cm = 0.2 m
t	= plate thickness = 1 cm = 0.01 m
q''	= absorbed solar flux = 800 W/m²
T_{edge}	= specified edge temperature = 20°C

Node 4 transfers heat by conduction with nodes 3, 5, and 6, by convection to ambient, and absorbs the specified solar flux. The energy balance on node 4 is therefore

$$k\,\frac{T_3 - T_4}{\Delta x}\,t\,\Delta x + k\,\frac{T_6 - T_4}{\Delta x}\,t\,\Delta x + k\,\frac{T_5 - T_4}{\Delta x}\,t\,\Delta x + h\,\Delta x^2\,(T_\infty - T_4) + q''\,\Delta x^2 = 0$$

Solving for T_4

$$T_4 = \frac{kt\;\; T_3 + T_5 + T_6 \;\; + h\Delta x^2 T_\infty + q''\,\Delta x^2}{3kt + h\Delta x^2}$$

Node 5 transfers heat by conduction with node 4, by convection to ambient, and absorbs the specified flux. The energy balance on node 5 is therefore

$$k\;\frac{T_4 - T_5}{\Delta x}\;\; t\,\Delta x + h\;\frac{\Delta x^2}{2}\;\;(T_\infty - T_5) + \;q''\;\;\frac{\Delta x^2}{2}\;\; = 0$$

Solving for T_5

$$T_5 = \frac{2kt\,T_4 + h\Delta x^2 T_\infty + q''\,\Delta x^2}{2kt + h\Delta x^2}$$

Node 6 transfers heat by conduction with nodes 4 and 2, by convection to ambient, and absorbs the specified flux. The energy balance on node 6 is therefore

$$k\;\frac{T_4 - T_6}{\Delta x}\;\; t\,\Delta x + k\;\frac{T_2 - T_6}{\Delta x}\;\; t\,\Delta x + h\;\frac{\Delta x^2}{2}\;\;(T_\infty - T_6) + \;q''\;\frac{\Delta x^2}{2}\;\; = 0$$

Solving for T_6

$$T_6 = \frac{2kt\;\; T_2 + T_4 \;\; + h\Delta x^2 T_\infty + q''\,\Delta x^2}{4kt + h\Delta x^2}$$

This set of equations can be solved by iteration. The table below shows the results of Gauss-Seidel iteration. Iteration 0 is the first guess for the temperature at nodes 4, 5, and 6.

	Temperature °C					
iteration	T1	T2	T3	T4	T5	T6
0	20	20	20	50	50	50
1	20.000	20.000	20.000	52.500	65.000	47.000
2	20.000	20.000	20.000	55.500	67.000	48.200
3	20.000	20.000	20.000	56.300	67.533	48.520
4	20.000	20.000	20.000	56.513	67.676	48.605
5	20.000	20.000	20.000	56.570	67.713	48.628
6	20.000	20.000	20.000	56.585	67.724	48.634
7	20.000	20.000	20.000	56.589	67.726	48.636
8	20.000	20.000	20.000	56.591	67.727	48.636
9	20.000	20.000	20.000	56.591	67.727	48.636
10	20.000	20.000	20.000	56.591	67.727	48.636

The solution converges after about 8 iterations giving a peak temperature of 67.727°C at node 5.

PROBLEM 4.35

The plate in Problem 4.34 gradually oxidizes over time so that the surface emissivity increases to 0.5. Calculate the resulting temperature in the plate including radiation heat transfer to the surroundings at the same temperature as the ambient temperature.

GIVEN

- Plate in Problem 4.34 oxidizes

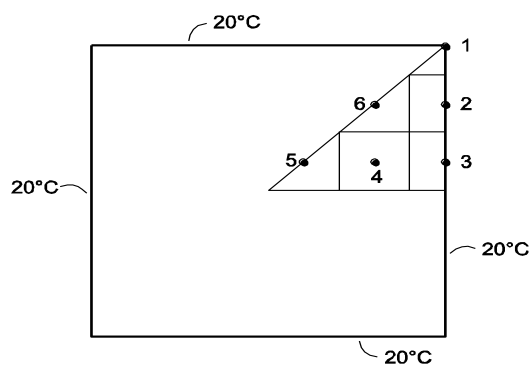
FIND

(a) New temperature distribution considering radiation heat transfer

ASSUMPTIONS

(a) Neglect temperature gradients through the plate thickness

SKETCH



SOLUTION

Addition of radiative heat transfer from the plate can be most easily handled by computing the radiative heat transfer coefficient for each node (using the temperature for the node calculated from the previous iteration) and by then adding this radiative heat transfer coefficient to the convective heat transfer coefficient for the present iteration. The radiative heat transfer coefficient for node *i* is

$h_{ri} = \varepsilon\,\sigma\,(T_i^2 + T_\infty^2)\,(T_i + T_\infty)$. The following table gives the results for the Gauss-Seidel iteration. (Recall that temperature must be expressed in Kelvins.)

Iteration	Temperature (K)			Temperature (K)/radiative heat transfer coefficient (W/(m² K))		
	T1	T2	T3	T4/hr4	T5/hr5	T6/hr6
0	323	323	323	323	323	323
				3.168	3.168	3.168
1	293.300	293.000	293.000	322.381	330.865	316.622
				3.158	3.299	3.066
2	293.000	293.000	293.000	322.735	330.890	316.820
				3.164	3.299	3.069
3	293.000	293.000	293.000	322.781	330.917	316.836
				3.165	3.300	3.069
4	293.000	293.000	293.000	322.790	330.922	316.839
				3.165	3.300	3.069
5	293.000	293.000	293.000	322.792	330.923	316.840
				3.165	3.300	3.069
6	293.000	293.000	293.000	322.792	330.923	316.840
	Node temperature in degrees C					
	20.000	20.000	20.000	49.792	57.923	43.840

The peak temperature has been reduced by about 9.8 K due to radiation.

PROBLEM 4.36

Determine (a) the temperature at the 16 equally spaced points shown in the accompanying sketch to an accuracy of three significant figures and (b) the rate of heat flow per meter thickness. Assume two-dimensional heat flow and *k* = 1 W/(mK).

GIVEN

- A two-dimensional object with specified surface temperatures

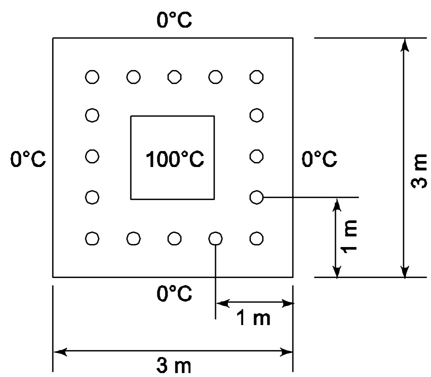
FIND

(a) The temperature at the 16 specified locations
(b) The heat flow per meter thickness

ASSUMPTIONS

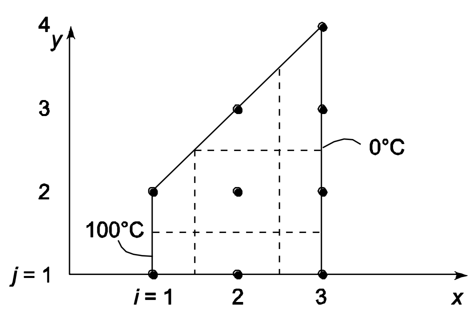
- Steady state

SKETCH



SOLUTION

Because of symmetry, it is only necessary to consider 1/8 th of the figure as shown below



(a) Temperature Distribution

There are three nodes remaining for which we must determine the temperature. For these nodes, we need energy balance equations for the control volumes. The control volumes are shown as dashed lines surrounding each node.

For the node at $i = 2$, $j = 1$

The x axis is a line of symmetry so no heat flows into the control volume across it

$$\frac{\Delta y}{2}\,(T_{1,1} - T_{2,1}) + \Delta x\,(T_{2,2} - T_{2,1}) + \frac{\Delta y}{2}\,(T_{3,1} - T_{2,1}) = 0$$

Since we have chosen $\Delta x = \Delta y$, this equation simplifies to

$$T_{2,1} = \frac{1}{4}(T_{1,1} + 2T_{2,2} + T_{3,1})$$

For the node at $i = 2$, $j = 2$, we use Equation (3.24)

$$T_{2,2} = \frac{1}{4}(T_{1,2} + T_{3,2} + T_{2,1} + T_{2,3})$$

And for the node at $i = 2$, $j = 3$, we have for energy balance

$$\Delta x\,(T_{2,2} - T_{2,3}) + \Delta y\,(T_{3,3} - T_{2,3}) = 0$$

or

$$T_{2,3} = \frac{1}{2}(T_{2,2} + T_{3,3})$$

Substituting the known boundary temperatures, these equations simplify to

$$4T_{2,1} = 2T_{2,2} + 100 \qquad (1)$$

$$4T_{2,2} = T_{2,1} + T_{2,3} + 100 \qquad (2)$$

$$2T_{2,3} = T_{2,2} \qquad (3)$$

These three equations can be solved by elimination. Substitute Equations (1) and (3) into Equation (2) to give

$$T_{2,2} = 41.666°\text{C}$$

Substitute this result into Equation (3) to get

$$T_{2,3} = 20.833°\text{C}$$

and then from Equation (1) we find

$$T_{2,1} = 45.833°\text{C}$$

(b) Heat flow

The total heat flow for the object can be calculated from

$$Q = 8\left\{k\,\frac{T_{1,1} - T_{2,1}}{\Delta x}\,\frac{\Delta y}{2} + k\,\frac{T_{1,2} - T_{2,2}}{\Delta x}\,\Delta y\right\}$$

which simplifies to

$$Q = 8k\ \ \frac{1}{2}(T_{1,1} - T_{2,1}) + T_{1,2} - T_{2,2}$$

$$= 8\ \ 1\,\text{W/(mK)}\ \ \frac{1}{2}(100 - 45.833) + 100 - 41.666\ \ (\text{K}) = 683.2\ \text{W/m}$$

PROBLEM 4.37

A long steel beam with rectangular cross section of 40 cm by 60 cm is mounted on an insulating wall as shown in the following sketch. The rod is heated by radiant heaters that maintain the top and bottom surfaces at 300°C. A stream of air at 130°C cools the exposed face through a heat transfer coefficient of 20 W/(m²K). Using a node spacing of 1 cm, determine the temperature distribution in the rod and the rate of heat input to the rod. The thermal conductivity of the steel is 40 W/(m K).

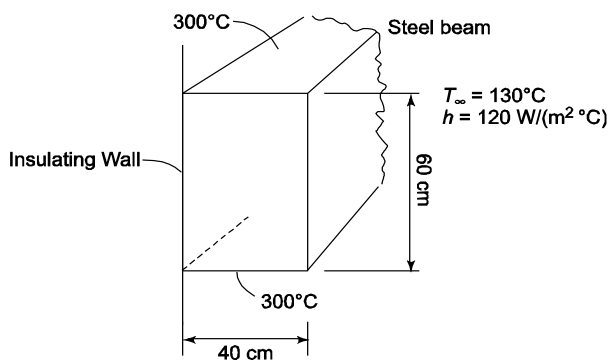
GIVEN

- Rectangular steel beam mounted on an insulating surface, heated top and bottom, with exposed face cooled by an air flow

FIND

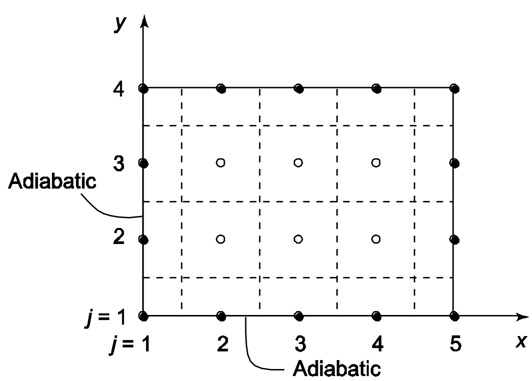
(a) Temperature distribution in the rod
(b) Rate of heat input to the rod

SKETCH



SOLUTION

Since the rod is long, we can consider a two-dimensional solution. By symmetry, the rod can be divided along its horizontal midplane. The sketch below shows the resulting geometry along with the node and control volume locations.



Define the top surface temperature as T_{top} = 300°C, and the ambient temperature as T_∞ = 130°C.

We now need to determine an energy balance for each control volume.

Along the top edge we have

$$T_{i,4} = T_{top} \qquad i = 1, 2, 3, 4, 5$$

For the central nodes we have from Equation (3.24)

$$4\,T_{i,j} = T_{i-1,j} + T_{i+1,j} + T_{i,j+1} + T_{i,j-1} \qquad i = 2, 3, 4 \quad j = 2, 3$$

Along the left edge, an energy balance on the two nodes at j = 2 and 3 gives

$$k\left\{\frac{T_{2,j}-T_{1,j}}{\Delta x}\Delta x+\frac{T_{1,j+1}-T_{1,j}}{\Delta x}\frac{\Delta x}{2}+\frac{T_{1,j-1}-T_{1,j}}{\Delta x}\frac{\Delta x}{2}\right\}=0 \qquad j=2,3$$

or $\quad 4\,T_{i,j} = 2\,T_{2,j} + T_{1,j+1} + T_{1,j-1} \quad j = 2, 3$

Along the bottom edge, an energy balance on the three nodes at i = 2, 3, and 4 gives

$$k\left\{\frac{T_{i,2}-T_{i,1}}{\Delta x}\Delta x+\frac{T_{i-1,1}-T_{i,1}}{\Delta x}\frac{\Delta x}{2}+\frac{T_{i+1,1}-T_{i,1}}{\Delta x}\frac{\Delta x}{2}\right\}=0 \qquad i=2,3,4$$

or $\quad 4\,T_{i,1} = 2\,T_{i,2} + T_{i-1,1} + T_{i+1,1} \quad i = 2, 3, 4$

For the node at i = 1, j = 1, an energy balance gives

$$k\left\{\frac{T_{1,2}-T_{1,1}}{\Delta x}\frac{\Delta x}{2}+\frac{T_{2,1}-T_{1,1}}{\Delta x}\frac{\Delta x}{2}\right\}=0$$

or $\quad 2\,T_{1,1} = T_{1,2} + T_{2,1}$

For the node at i = 5, j = 1, an energy balance gives

$$k\left\{\frac{T_{4,1}-T_{5,1}}{\Delta x}\frac{\Delta x}{2}+\frac{T_{5,2}-T_{5,1}}{\Delta x}\frac{\Delta x}{2}\right\}+h\,\frac{\Delta x}{2}\,(T_\infty-T_{5,1})=0$$

or $\quad \left(2+\dfrac{h\Delta x}{k}\right)T_{5,1} = (T_{4,1} + T_{5,2}) + \dfrac{h\Delta x}{k}\,T_\infty$

Finally, along the right edge, an energy balance on the two nodes at j = 2 and 3 gives

$$k\left\{\frac{T_{4,j}-T_{5,j}}{\Delta x}\Delta x+\frac{T_{5,j-1}-T_{5,j}}{\Delta x}\frac{\Delta x}{2}+\frac{T_{5,j+1}-T_{5,j}}{\Delta x}\frac{\Delta x}{2}\right\}+h\,\Delta x\,(T_\infty-T_{5,j})=0 \qquad j=2,3$$

or $\quad \left(4+\dfrac{2h\Delta x}{k}\right)T_{5,j} = 2\,T_{4,j} + T_{5,j-1} + T_{5,j+1} + \dfrac{2h\Delta x}{k}\,T_\infty \qquad j = 2, 3$

This set of difference equations can be written as a matrix equation as follows

$$AT = C$$

where the coefficient matrix A is given by

–2	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	–4	1	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	–4	1	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	–4	1	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	–2.05	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	–4	2	0	0	0	1	0	0	0	0	0	0	0	0	0
0	1	0	0	0	1	–4	1	0	0	0	1	0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	–4	1	0	0	0	1	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1	–4	1	0	0	0	1	0	0	0	0	0	0
0	0	0	0	1	0	0	0	2	–4.1	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	–4	2	0	0	0	1	0	0	0	0
0	0	0	0	0	0	1	0	0	0	1	–4	1	0	0	0	1	0	0	0
0	0	0	0	0	0	0	1	0	0	0	1	–4	1	0	0	0	1	0	0
0	0	0	0	0	0	0	0	1	0	0	0	–1	4	1	0	0	0	1	0
0	0	0	0	0	0	0	0	0	1	0	0	0	2	–4.1	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	–1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	–1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	–1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	–1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	–1

and the temperature vector T and constant vector C are given by

$T\,(1,1) \qquad 0$

$$
T = \begin{array}{l}
T(2,1)\\
T(3,1)\\
T(4,1)\\
T(5,1)\\
T(1,2)\\
T(2,2)\\
T(3,2)\\
T(4,2)\\
T(5,2)\\
T(1,3)\\
T(2,3)\\
T(3,3)\\
T(4,3)\\
T(5,3)\\
T(1,4)\\
T(2,4)\\
T(3,4)\\
T(4,4)\\
T(5,4)
\end{array}
\qquad
C = \begin{array}{l}
0\\
0\\
0\\
-6.5\\
0\\
0\\
0\\
0\\
-13\\
0\\
0\\
0\\
0\\
-13\\
-300\\
-300\\
-300\\
-300\\
-300
\end{array}
$$

Inverting the matrix A with a spreadsheet program and multiplying the constant vector C by the inverted matrix, we get the vector of nodal temperatures

$$
T = \begin{array}{l}
295.2854\\
294.667\\
292.6705\\
288.8777\\
282.6697\\
295.9039\\
295.356\\
293.5687\\
290.0853\\
284.0951\\
297.6181\\
297.2843\\
296.1632\\
293.7996\\
288.9498\\
300\\
300\\
300\\
300\\
300
\end{array}
\quad {}^{\circ}\mathrm{C}
$$

To determine the heat flow to the rod, consider the surface of the exposed control volumes. The rate of convective heat transfer from these surfaces must equal the rate of heat input to the rod. Remembering to double the value of account for the symmetry, we have

$$q_{\text{input}} = 2h\Delta x\left(\frac{1}{2}\ T_{5,4} - T_{\infty} \ + \frac{1}{2}\ T_{5,1} - T_{\infty} \ + \ T_{5,2} - T_{\infty} \ + \ T_{5,3} - T_{\infty}\right)$$

Inserting the nodal temperatures from the solution vector T given above, we find

$$q_{\text{input}} = 1897 \text{ watts}$$

PROBLEM 4.38

Consider a band-saw blade that is to cut steel bar stock. The blade thickness is 2 mm, its height is 20 mm, and it has penetrated the steel workpiece to a depth of 5 mm (see the accompanying sketch). Exposed surfaces of the blade are cooled by an ambient temperature of 20°C through a convection coefficient of 40 W/(m^{2}K). Thermal conductivity of the blade steel is 30 W/(mK). Energy dissipated by the cutting process supplies a heat flux of 10^4 W/m^2 to the surfaces of the blade that are in contact with the workpiece. Assuming two-dimensional, steady conduction, determine the maximum and minimum temperature in the blade cross section. Use a node spacing of 0.5 mm horizontally and 2 mm vertically.

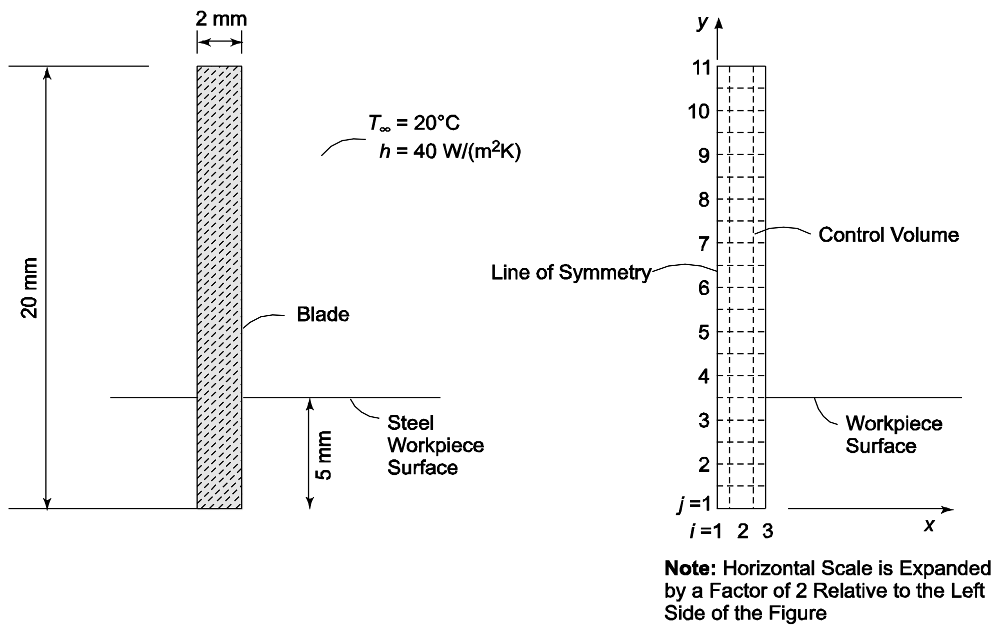
GIVEN

- Band saw blade cutting steel bar stock

FIND

(a) Maximum and minimum temperatures in the blade cross section

SKETCH



SOLUTION

Because of symmetry, we only need to consider half of the geometry as shown in the right side of the sketch. With a node spacing of Δx = 0.5 mm and Δy = 2.0 mm, we have for the number of horizontal and vertical nodes

$$M = \frac{\frac{t}{2}}{\Delta x} + 1 = 3$$

$$N = \frac{H}{\Delta y} + 1 = 11$$

We have 33 control volumes and need to develop an energy balance equation for each. For all the interior nodes

$$\frac{T_{i+1,j} - T_{i,j}}{\Delta x}\Delta y + \frac{T_{i-1,j} - T_{i,j}}{\Delta x}\Delta y + \frac{T_{i,j+1} - T_{i,j}}{\Delta y}\Delta x + \frac{T_{i,j-1} - T_{i,j}}{\Delta x}\Delta x = 0$$

or

$$T_{i,j} = \frac{\frac{\Delta y}{\Delta x}\left(T_{i+1,j} - T_{i-1,j}\right) + \frac{\Delta x}{\Delta y}\left(T_{i,j+1} - T_{i,j-1}\right)}{2\left(\frac{\Delta y}{\Delta x} + \frac{\Delta x}{\Delta y}\right)} \qquad i = 2 \quad j = 2, 3 \ldots N-1$$

For all nodes (except the corner nodes) on the left edge

$$\frac{T_{i+1,j} - T_{i,j}}{\Delta x}\,\Delta y + \frac{T_{i,j+1} - T_{i,j}}{\Delta x}\frac{\Delta x}{2} + \frac{T_{i,j-1} - T_{i,j}}{\Delta x}\frac{\Delta x}{2} = 0$$

or

$$T_{i,j} = \frac{\frac{\Delta y}{\Delta x}T_{i+1,j} + \frac{\Delta x}{2\Delta y}\left(T_{i,j+1} - T_{i,j-1}\right)}{\frac{\Delta y}{\Delta x} + \frac{\Delta x}{\Delta y}} \qquad i = 1 \quad j = 2, 3 \ldots \mathrm{N} - 1$$

For $i = 1$, $j = 1$

$$k\left(\frac{T_{i+1,j} - T_{i,j}}{\Delta x}\frac{\Delta x}{2} + \frac{T_{i,j+1} - T_{i,j}}{\Delta y}\frac{\Delta x}{2}\right) + q''\frac{\Delta x}{2} = 0$$

or

$$T_{i,j} = \frac{\frac{\Delta y}{\Delta x}T_{i+1,j} + \frac{\Delta x}{\Delta y}T_{i,j+1} + q''\frac{\Delta x}{k}}{\frac{\Delta y}{\Delta x} + \frac{\Delta x}{\Delta y}} \qquad i = 1 \qquad j = 1$$

For $i = 1$, $j = N$

$$k\left(\frac{T_{i+1,j} - T_{i,j}}{\Delta x}\frac{\Delta y}{2} + \frac{T_{i,j-1} - T_{i,j}}{\Delta y}\frac{\Delta x}{2}\right) + h\,\frac{\Delta x}{2}\,(T_\infty - T_{i,j}) = 0$$

or

$$T_{i,j} = \frac{\frac{\Delta y}{\Delta x}T_{i+1,j} + \frac{\Delta x}{\Delta y}T_{i,j-1} + h\frac{\Delta x}{k}T_\infty}{\frac{\Delta y}{\Delta x} + \frac{\Delta x}{\Delta y} + \frac{h\Delta x}{k}} \qquad i = 1 \qquad j = N$$

For $i = 2$, $j = N$

$$k\left(\frac{T_{i+1,j} - T_{i,j}}{\Delta x}\frac{\Delta y}{2} + \frac{T_{i-1,j} - T_{i,j}}{\Delta x}\frac{\Delta y}{2} + \frac{T_{i,j-1} - T_{i,j}}{\Delta y}\Delta x\right) + h\,\Delta x\,(T_\infty - T_{i,j}) = 0$$

or

$$T_{i,j} = \frac{\frac{\Delta y}{2\Delta x}\left(T_{i+1,j} + T_{i-1,j}\right) + \frac{\Delta x}{\Delta y}T_{i,j-1} + \frac{h\Delta x}{k}T_\infty}{\frac{\Delta y}{\Delta x} + \frac{\Delta x}{\Delta y} + \frac{h\Delta x}{k}} \qquad i = 2 \qquad j = N$$

For $i = 2$, $j = 1$

$$k\left(\frac{T_{i+1,j} - T_{i,j}}{\Delta x}\frac{\Delta y}{2} + \frac{T_{i-1,j} - T_{i,j}}{\Delta x}\frac{\Delta y}{2} + \frac{T_{i,j+1} - T_{i,j}}{\Delta y}\Delta x\right) + q''\,\Delta x = 0$$

or

$$T_{i,j} = \frac{\frac{\Delta y}{2\Delta x}\left(T_{i+1,j}+T_{i-1,j}\right)+\frac{\Delta x}{\Delta y}T_{i,j+1}+q''\frac{\Delta x}{k}}{\frac{\Delta y}{\Delta x}+\frac{\Delta x}{\Delta y}} \qquad i = 2 \qquad j = 1$$

For $i = M$, $j = N$

$$k\left(\frac{T_{i-1,j}-T_{i,j}}{\Delta x}\frac{\Delta y}{2}+\frac{T_{i,j-1}-T_{i,j}}{\Delta y}\frac{\Delta x}{2}\right)+\frac{h}{2}(\Delta x+\Delta y)(T_\infty - T_{i,j}) = 0$$

or

$$T_{i,j} = \frac{\frac{\Delta y}{\Delta x}T_{i-1,j}+\frac{\Delta x}{\Delta y}T_{i,j-1}+\frac{h}{k}\left(\Delta x+\Delta y\right)T_\infty}{\frac{\Delta y}{\Delta x}+\frac{\Delta x}{\Delta y}+\frac{h}{k}\left(\Delta x+\Delta y\right)} \qquad i = M \qquad j = N$$

For $i = M$, $j = 1$

$$k\left(\frac{T_{i-1,j}-T_{i,j}}{\Delta x}\frac{\Delta y}{2}+\frac{T_{i,j+1}-T_{i,j}}{\Delta y}\frac{\Delta x}{2}\right)+q''\,\frac{1}{2}(\Delta x+\Delta y) = 0$$

or

$$T_{i,j} = \frac{\frac{\Delta y}{\Delta x}T_{i-1,j}+\frac{\Delta x}{\Delta y}T_{i,j+1}+\frac{q''}{k}\left(\Delta x+\Delta y\right)}{\frac{\Delta y}{\Delta x}+\frac{\Delta x}{\Delta y}} \qquad i = M \qquad j = 1$$

For $i = M$, $j = 2, 3$

$$k\left(\frac{T_{i-1,j}-T_{i,j}}{\Delta x}\Delta y+\frac{T_{i,j+1}-T_{i,j}}{\Delta y}\frac{\Delta x}{2}+\frac{T_{i,j-1}-T_{i,j}}{\Delta y}\frac{\Delta x}{2}\right)+q''\,\Delta y = 0$$

or

$$T_{i,j} = \frac{\frac{\Delta y}{\Delta x}T_{i-1,j}+\frac{\Delta x}{2\Delta y}\left(T_{i,j+1}+T_{i,j-1}\right)+\frac{q''}{k}\Delta y}{\frac{\Delta y}{\Delta x}+\frac{\Delta x}{\Delta y}} \qquad i = M \qquad j = 2, 3$$

Finally, for $i = M$, $j = 4, 5, \ldots N-1$

$$k\left(\frac{T_{i-1,j}-T_{i,j}}{\Delta x}\Delta y+\frac{T_{i,j+1}-T_{i,j}}{\Delta y}\frac{\Delta x}{2}+\frac{T_{i,j-1}-T_{i,j}}{\Delta y}\frac{\Delta x}{2}\right)+h\,\Delta y\,(T_\infty - T_{i,j}) = 0$$

or

$$T_{i,j} = \frac{\frac{\Delta y}{\Delta x}T_{i-1,j}+\frac{\Delta x}{2\Delta y}\left(T_{i,j+1}+T_{i,j-1}\right)+\frac{h}{k}\Delta y\,T_\infty}{\frac{\Delta y}{\Delta x}+\frac{\Delta x}{\Delta y}+\frac{h}{k}\Delta y} \qquad i = M \quad J = 4, 5\ldots N-1$$

This set of equations can be solved iteratively as described in Section 3.4. An initial guess for the temperature distribution is inserted into the right hand side of all the above equations to produce an improved value for $T[i, j]$ for all i and j. These improved values are inserted into the right hand side of the same equations for the next update on $T[i, j]$ and so on. We carried out the iteration until the temperature at $i = 2$, $j = 1$ changed by less than 10^{-6} °C. The results indicate a maximum temperature of 130.401°C at $i = 3$, $j = 1$, and a minimum temperature of 108.693°C at $i = 3$, $j = M$.

PROBLEM 4.39

A novel method to cool high-powered microelectronic chips and chip modules is to etch the liquid coolant flow channels in silicon (Si) substrate itself [9–12]. A typical such arrangement with channels of trapezoidal cross section is shown in the figure. The channel and silicon substrate dimensions are as follows: *a* =481.4 µm, *b*=200 µm, *d* = 200 µm, and *p*=250 µm. The thermophysical properties of silicon are ρ =2300 kg/m^3 , *c* =750 J/(kg K), and *k*=3.6 W/(m K). The liquid coolant flowing the trapezoidal channels is at T_f=20°C with a convection heat transfer coefficient $\overline{h_c}$ =61,250 W/(m^2 K). If the maximum temperature at the chip interface has to be $T_{cs} \le$ 75°C, determine (a) the temperature distribution in the silicon substrate due to conduction and the transfer of heat to the coolant in the microchannel and (b) the maximum heat flux dissipation that is accommodated at the chip interface. For the numerical solution and the associated nodal network that needs to be constructed due to symmetry in the geometrical arrangement, consider only the segment indicated by the planes "A–A" in the figure. Also for the heat flux calculation, consider the chip area to be (*p* + *a)* * (*p* + *a*).

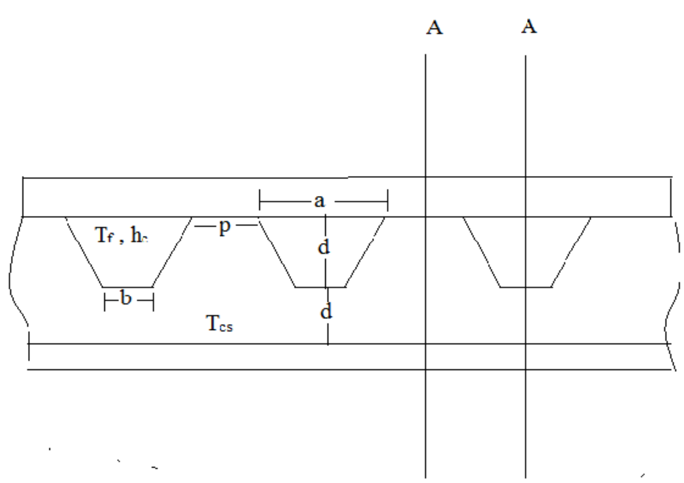
GIVEN

- Coling of high powered microelectronic chips with channels of trapezoidal cross section.
- Silicon substrate dimensions: *a* =481.4 µm, *b*=200 µm, *d* =200 µm, and *p*=250 µm.
- Thermophysical properties of Silicon: ρ =2300 kg/m^3 , c =750 J/(kg K), and k=3.6 W/(m K)
- Convection heat transefer coefficient ($\overline{h_c}$) =61,250 W/(m^2 K)
- Maximum chip temperature at interface $T_{cs} \le 75$°C

FIND

(a) Temperature distribution in Silicon substrate due to conduction and transfer of heat to coolant in microchannel.
(b) Maximum heat flux dissipation that is accommodated at the chip interface.

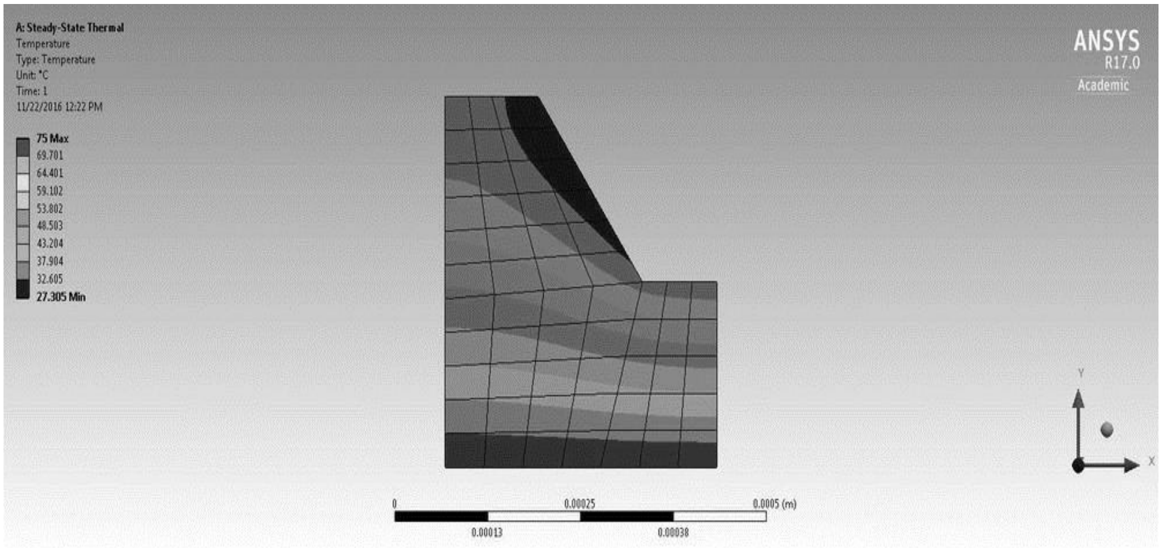
SKETCH



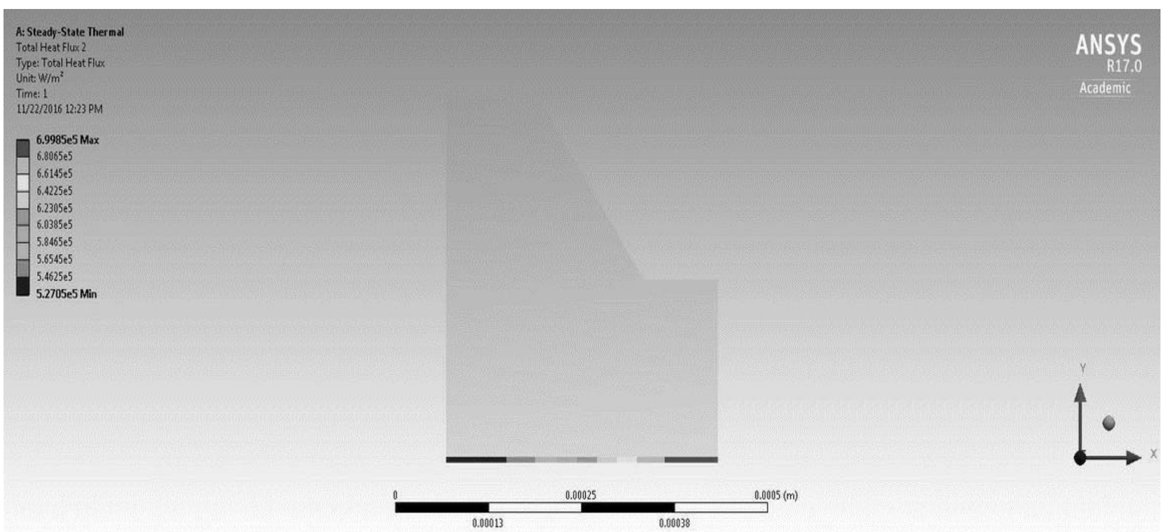
SOLUTION

The above problem is solved by using ANSYS Steady State Thermal Analysis. The material is added in Engineering data as Silicon and its properties are added. The A-A cross section geometry is drawn with given dimensions, face meshing is done and the suitable boundary conditions are provided to the specimen. The setup is then run to find the solution. The following results are obtained from the Results section of ANSYS.

(a) The temperature distribution is given in ANSYS as follows



(b) The heat flux dissipation at the chip surface is shown below



The maximum heat flux dissipated from chip surface is $6.9985*10^5$ W/m^2 as per the results.

PROBLEM 4.40

Repeat Problem 4.39 by considering, instead of the trapezoidal microchannel, the two geometries shown in the following figure (a) rectangular channel of dimension *a*=*b*=200 µm, *d*= 200 µm, and *p*=150 µm and the coolant flow heat transfer coefficient $\bar{h}_c$ given by Nu=3.608 and (b) triangular channel of dimension *a* =200 µm, *d*=200 µm, and *p*=150 µm and the coolant flow heat transfer coefficient $\bar{h}_c$ given by Nu=3.102. Compare and contrast the results for both the temperature distribution in the silicon and the maximum chip heat flux dissipation so that the chip interface temperature is $T_{cs} \leq$75°C. The coolant flow temperature in both cases is the same, or T_f=20°C. What microchannel geometry do you recommend for optimal cooling?

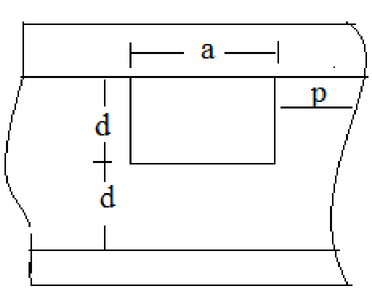
GIVEN

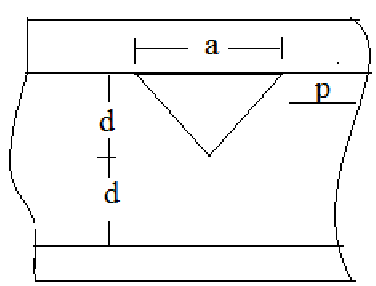
- Cooling of high powered microelectronic chips with channels of trapezoidal cross section.
- Rectangular channel dimension: *a*=*b*=200 µm, *d*= 200 µm, and *p*=150 µm
- Triangular channel dimension: *a* =200 µm, *d*=200 µm, and *p*=150 µm
- Thermophysical properties of Silicon: ρ =2300 kg/m^3 , c =750 J/(kg K), and k=3.6 W/(m K)
- Nu for rectangular channel =3.608
- Nu for triangular channel=3.102
- Maximum chip temperature at interface $T_{cs} \leq$75°C

FIND

(a) Temperature distribution in Silicon substrate due to conduction and transfer of heat to coolant in microchannel.
(b) Maximum heat flux dissipation that is accommodated at the chip interface.

SKETCH





SOLUTION

The above problem is solved by using ANSYS Steady State Thermal Analysis. The material is added in Engineering data as Silicon and its properties are added. The A-A cross section geometry is drawn with given dimensions, face meshing is done and the suitable boundary conditions are provided to the specimen. The setup is then run to find the solution.

Nusselt Number is provided in each of the above case.
For channel of non-circular cross section hydraulic diameter is calculated as D_H=4*A/P
Where A is cross sectional area and P is the perimeter.

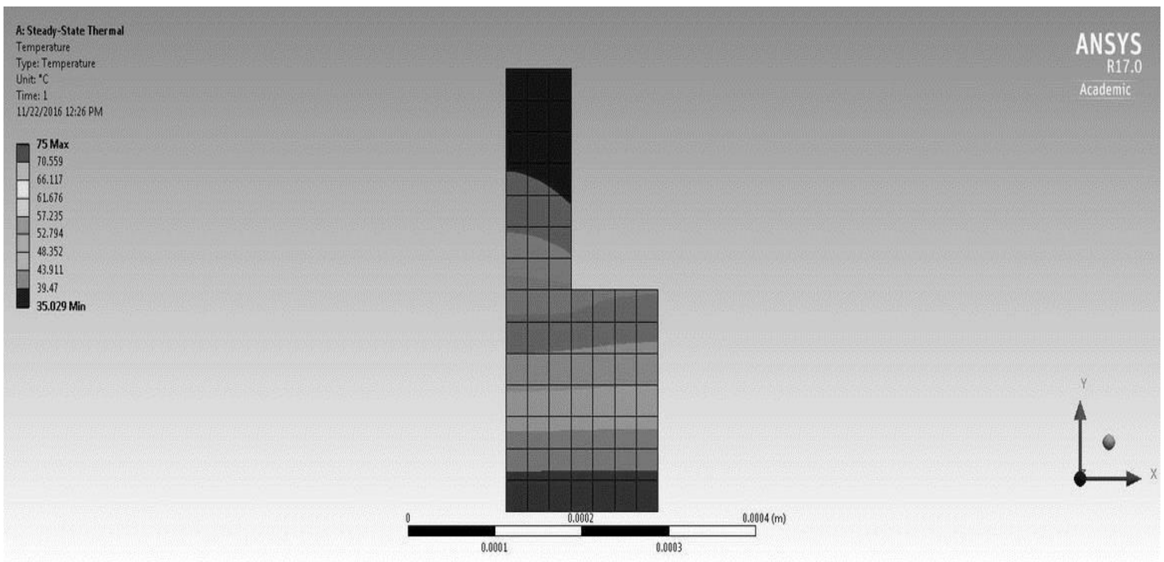
For rectangular cross section D_H= 200 μm
For triangular cross section D_H=179 μm

The heat transfer coefficient is calculated considering the coolant as water.

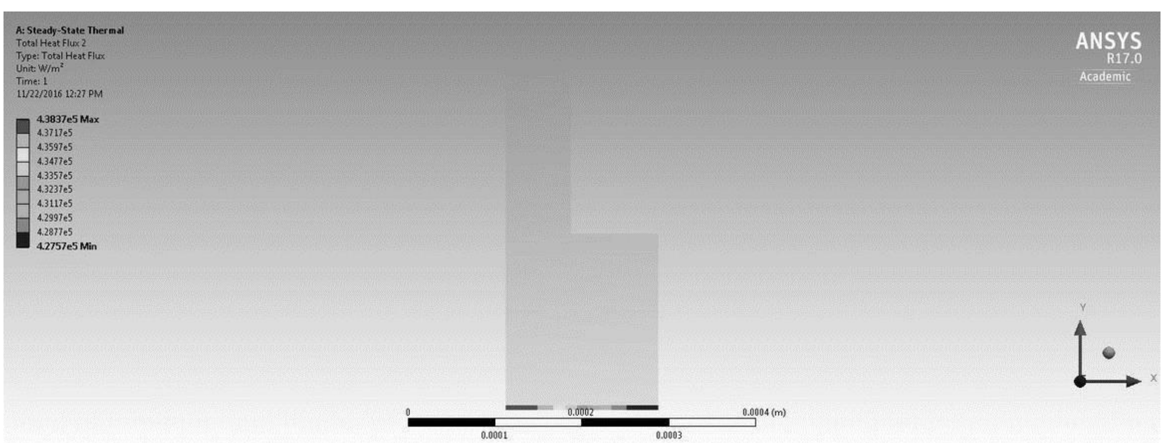
- For rectangular cross section convection heat transefer coefficient ($\overline{h_c}$) =10,788 W/(m^2 K)
- For triangular cross section Convection heat transefer coefficient ($\overline{h_c}$) =10363 W/(m^2 K)

The following results are obtained from the Results section of ANSYS.

(a) The temperature distribution for rectangular cross section channel is given in ANSYS as follows

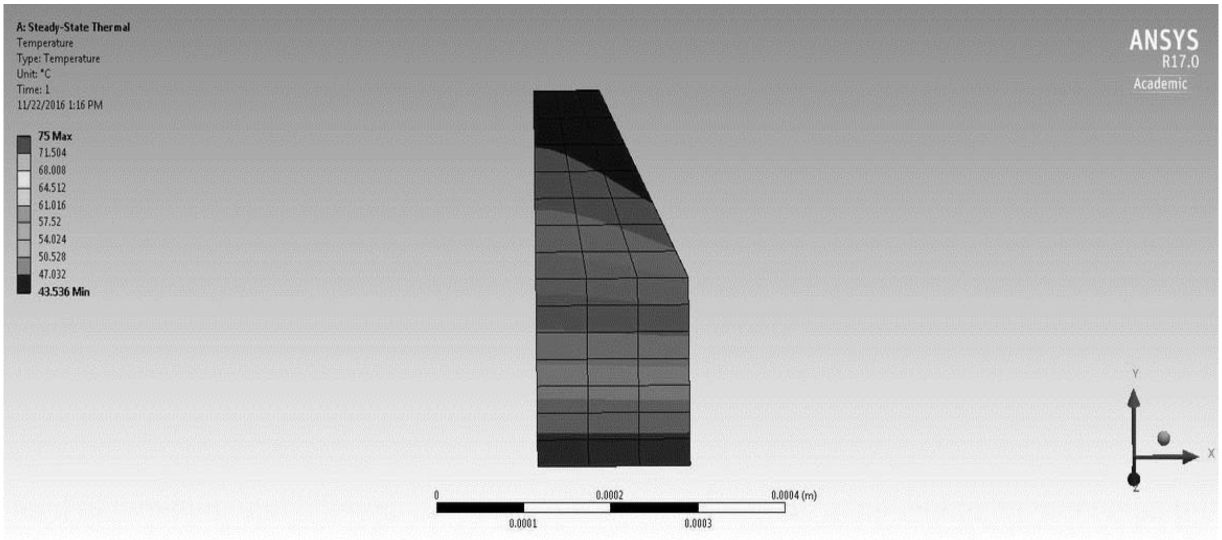


The heat flux dissipation at the chip surface is shown below

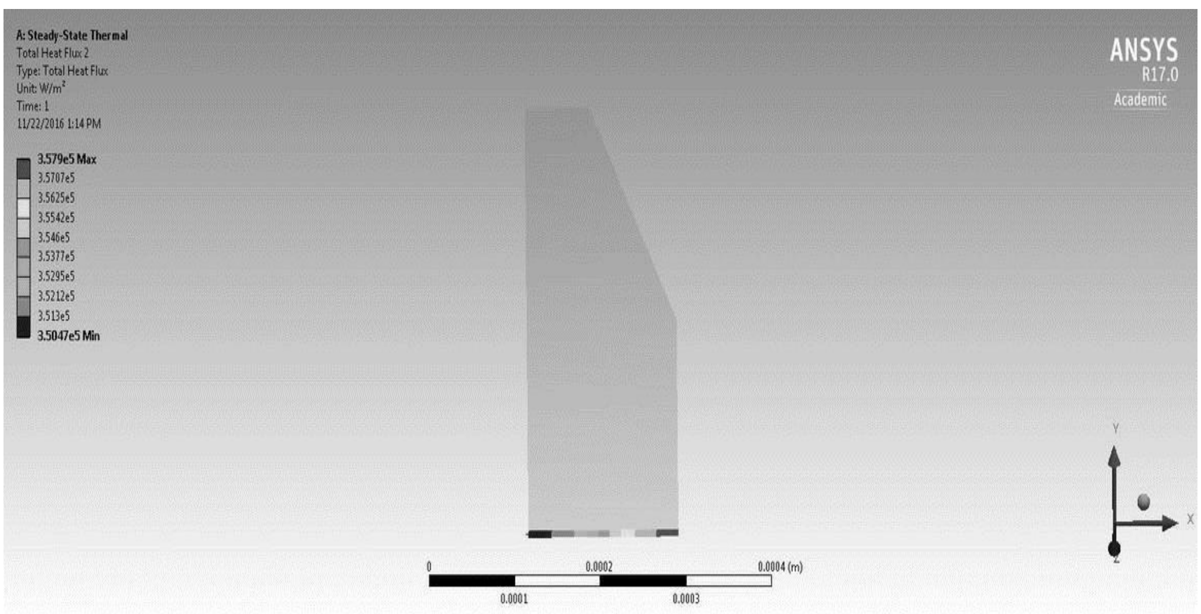


The maximum heat flux dissipated from chip surface is $4.387*10^5$ W/m^2 as per the results.

(b) The temperature distribution for triangular cross section channel is given in ANSYS as follows



The heat flux dissipation at the chip surface is shown below



The maximum heat flux dissipated from chip surface is $3.579*10^5$ W/m^2 as per the results.

Since, the maximum heat flux and average heat flux dissipated from chip surface is highest for the trapezoidal case, trapezoidal microchannel geometry is recommended for optimum cooling.

PROBLEM 4.41

Show that in the limit $\Delta x \to 0$, $\Delta y \to 0$, $\Delta t \to 0$, the difference Eq. (4.23) is equivalent to the two-dimensional version of the differential Eq. (2.6).

GIVEN

- Difference equation, Equation (4.23)

SHOW

(a) In the limit Δx, Δy, and $\Delta t \to 0$, the difference equation is equivalent to the two-dimensional version of the differential equation, Equation (2.6).

SOLUTION

Equation (4.23) is

$$\frac{T_{i+1,j,m} - 2T_{i,j,m} + T_{i-1,j,m}}{\Delta x^2} + \frac{T_{i,j+1,m} - 2T_{i,j,m} + T_{i,j-1,m}}{\Delta y^2} + \frac{\dot{q}_{G,i,j,m}}{k} = \frac{\rho c}{k}\frac{T_{i,j,m+1} - T_{i,j,m}}{\Delta t}$$

We have by definition

$$T_{i,j,m} = T(x, y, t)$$
$$T_{i+1,j,m} = T(x + \Delta x, y, t)$$
$$T_{i-1,j,m} = T(x - \Delta x, y, t)$$
$$T_{i,j+1,m} = T(x, y + \Delta y, t)$$
$$T_{i,j-1,m} = T(x, y - \Delta y, t)$$
$$T_{i,j,m+1} = T(x, y, t + y\Delta t)$$

so Equation (4.23) is equivalent to

$$\frac{T(x+\Delta x, y, t) - 2T(x, y, t) + T(x-\Delta x, y, t)}{\Delta x^2}$$
$$+ \frac{T(x, y+\Delta y, t) - 2T(x, y, t) + T(x, y-\Delta y, t)}{\Delta y^2}$$
$$+ \frac{\dot{q}_G(x, y, t)}{k} = \frac{\rho c}{k}\frac{T(x, y, t+\Delta t) - T(x, y, t)}{\Delta t}$$

In the limit $\Delta x \to 0$, from calculus, the first term becomes $\dfrac{\partial^2 T}{\partial x^2}$

In the limit $\Delta y \to 0$, the second term becomes $\dfrac{\partial^2 T}{\partial y^2}$

In the limit $\Delta t \to 0$, the right side of the equation becomes

$$\frac{\rho c}{k}\frac{\partial T}{\partial t}$$

so the difference equation becomes

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\dot{q}_G}{k} = \frac{\rho c}{k}\frac{\partial T}{\partial t}$$

which is equivalent to the two-dimensional version of Equation (2.6) as required.

PROBLEM 4.42

Derive the stability criterion for the explicit solution of two-dimensional transient conduction.

GIVEN

- Two-dimensional transient conduction

FIND

(a) The stability criterion for an explicit solution

SOLUTION

From Equation (4.22), the coefficient on the $T_{i,\,j,\,m}$ term is

$$\frac{1}{\alpha \Delta t} - \frac{2}{\Delta x^2} - \frac{2}{\Delta y^2}$$

which must be greater than zero to ensure stability. Therefore

$$\frac{1}{\alpha \Delta t} > 2\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)$$

or

$$\Delta t < \frac{1}{2\alpha}\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)^{-1}$$

which is the stability criterion as required.

PROBLEM 4.43

Derive Equation (4.28).

GIVEN

- Two-dimensional transient conduction at an inside corner with specified-flux boundary condition

FIND

(a) The control volume energy balance equation, Equation (4.28)

SOLUTION

Referring to Figure 4.14, heat conducted into the control volume is given by

$$k\ \frac{T_{i-1,j,m} - T_{i,j,m}}{\Delta x}\ \Delta y + k\ \frac{T_{i,j+1,m} - T_{i,j,m}}{\Delta y}\ \Delta x$$

$$+\ k\ \frac{T_{i,j-1,m} - T_{i,j,m}}{\Delta y}\ \frac{\Delta x}{2} + k\ \frac{T_{i+1,j,m} - T_{i,j,m}}{\Delta x}\ \frac{\Delta y}{2}$$

The rate of heat generation in the control volume is given by

$$\dot{q}_{G,i,j,m}\,\frac{3}{4}\ \Delta x \Delta y$$

Heat transferred out of the boundaries by the specified fluxes is

$$q''_x\ \frac{\Delta y}{2} - q''_y\ \frac{\Delta x}{2}$$

The rate at which energy is stored in the control volume is given by

$$\rho c\ \frac{T_{i,j,m+1} - T_{i,j,m}}{\Delta t}\,\frac{3}{4}\Delta x \Delta y$$

The sum of the first two equations above must equal the sum of the last two equations above. The coefficients on the individual terms is

$$T_{i,j,m} : 1 - 2\alpha\Delta t\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right) \quad T_{i-1,j,m} : \frac{4}{3}\frac{\alpha\Delta t}{\Delta x^2} \quad T_{i+1,j,m} : \frac{2}{3}\frac{\alpha\Delta t}{\Delta x^2} \quad T_{i,j+1,m} : \frac{4}{3}\frac{\alpha\Delta t}{\Delta y^2}$$

$$T_{i,j-1,m} : \quad \dot{q}_G : \frac{\Delta t}{\rho c} \quad q''_x : -\frac{2}{3}\frac{\alpha\Delta t}{k\Delta x} \quad q''_y : \frac{2}{3}\frac{\alpha\Delta t}{k\Delta y}$$

which is identical to those in Equation (4.28).

PROBLEM 4.44

Derive the stability criterion for an inside-corner boundary control volume for two-dimensional steady conduction when a convection boundary condition exists.

GIVEN

- Two-dimensional steady conduction at an inside corner with a convection boundary condition

FIND

(a) The stability criterion

SOLUTION

In Equation (4.28) write

$$q''_{x,i,j,m} = h\,(T_{i,j,m} - T_\infty)$$

$$q''_{y,i,j,m} = h\,(T_\infty - T_{i,j,m})$$

to account for the convection boundary condition.

The coefficient on $T_{i,j,m}$ is now

$$1 - 2\alpha\Delta t\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right) + \frac{2}{3}\,\frac{\alpha\Delta t}{k}\left(-\frac{1}{\Delta x}-\frac{1}{\Delta y}\right)$$

$$= 1 - 2a\Delta t\left(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}\right) - \frac{2}{3}\,\frac{\alpha h\Delta t}{k}\left(\frac{1}{\Delta x}+\frac{1}{\Delta y}\right)$$

This coefficient must be greater than zero for stability, therefore

$$\Delta\mathrm{t} < \frac{1}{2\alpha\left(\dfrac{1}{\Delta x^2}+\dfrac{1}{\Delta y^2}\right)+\dfrac{2}{3}\dfrac{\alpha h}{k}\left(\dfrac{1}{\Delta x}+\dfrac{1}{\Delta y}\right)}$$

Note that as we have seen before, the criterion for a convective boundary condition is more restrictive than other boundary condition stability criteria.

PROBLEM 4.45

A long concrete beam is to undergo a thermal test to determine its loss of strength in the event of a building fire. The beam cross section is triangular as shown in the sketch. Initially, the beam is at a uniform temperature of 20°C. At the start of the test, one of the short faces and the long face are exposed to hot gases at 400°C through a heat transfer coefficient of 10 W/(m² K) and the remaining short face is adiabatic. Produce a graph showing the highest and lowest temperatures in the beam as a function of time for the first 1 hour of exposure. For the concrete properties, use k = 0.5 W/(mK) and $n = 5 \times 10^{-7}$ m²/s. Use a node spacing of 4 cm. and use an explicit difference scheme.

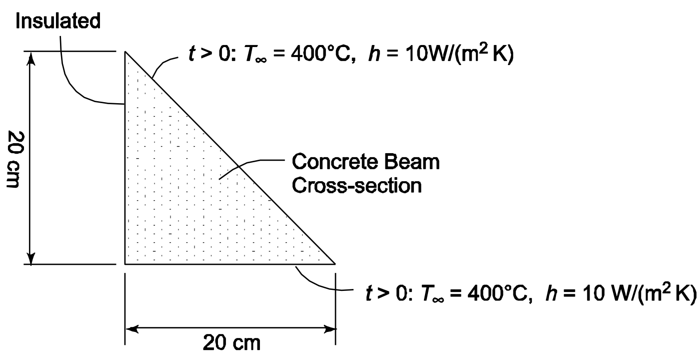
GIVEN

- Concrete beam suddenly exposed to hot gases

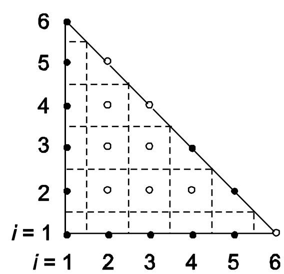
FIND

(a) Highest and lowest temperatures in the beam as a function of time

SKETCH



Problem Definition Sketch



Node and Control Volume Layout

SOLUTION

The arrangement of nodes and control volumes is shown in the figure to right. Examination of this figure reveals that we have 7 unique control volumes. We need to develop an energy balance for each type. To simplify the notation, use the following

$T_o \equiv T_{i,j,k}$ $\quad$ $T_1 \equiv T_{i-1,j,k}$ (left) $\quad$ $T_r \equiv T_{i+1,j,k}$ (right) $\quad$ $T_u \equiv T_{i,j+1,k}$ (up) $\quad$ $T_d \equiv T_{i,j-1,k}$ (down)

and

$$K1 \equiv \frac{\alpha \Delta t}{\Delta x^2} \qquad K2 \equiv \frac{h \Delta t}{\rho c \Delta x}$$

For al interior control volumes: i = 2, j = 2, 3, 4; i = 3, j = 2, 3; and i = 4, j = 2, we have for the energy balance

$$k\{T_1 + T_r + T_u + T_d - 4T_o\} = \rho c\, \Delta x^2 \, \frac{T_{i,j,k+1} - T_o}{\Delta t}$$

Solving for $T_{i,j,k+1}$

$$T_{i,j,k+1} = T_o + K1\,(T_1 + T_r + T_u + T_d - 4\,T_o)$$

For the bottom row of control volumes (not corners), j = 1, i = 2, 3, 4, 5, we have

$$k\left[\frac{T_1-T_o}{2}+\frac{T_r-T_o}{2}+T_u-T_o\right]+h\,\Delta x\,(T_\infty-T_o)=\rho c\,\frac{\Delta x^2}{2}\,\frac{T_{i,j,k+1}-T_o}{\Delta t}$$

Solving for $T_{i,\,j,\,k+1}$

$$T_{i,\,j,\,k+1}=T_o+2\,K1\left(\frac{1}{2}(T_1+T_r)+T_u-2T_o\right)+2K2\,(T_\infty-T_o)$$

For the left edge (not corners) $i = 1$, $j = 2, 3, 4, 5$

$$k\left[\frac{T_d-T_o}{2}+\frac{T_u-T_o}{2}+T_r-T_o\right]=\rho c\,\frac{\Delta x^2}{2}\,\frac{T_{i,j,k+1}-T_o}{\Delta t}$$

Solving for $T_{i,\,j,\,k+1}$

$$T_{i,\,j,\,k+1}=T_o+2\,K1\left(\frac{1}{2}(T_u+T_d)+T_r-2T_o\right)$$

For control volumes on the diagonal $(i, j) = (2, 5), (3, 4), (4, 3), (5, 2)$ we have

$$k\,\{T_1-T_o+T_d-T_o\}+h\sqrt{2}\,\Delta x\,(T_\infty-T_o)=\rho c\,\frac{\Delta x^2}{2}\,\frac{T_{i,j,k+1}-T_o}{\Delta t}$$

Solving for $T_{i,\,j,\,k+1}$

$$T_{i,\,j,\,k+1}=T_o+2\,K1\,(T_1+T_d-2\,T_o)+2\,\sqrt{2}\,K2\,(T_\infty-T_o)$$

Finally, for the corners

$i = 1$, $j = 1$

$$k\left\{\frac{T_u-T_o}{2}+\frac{T_r-T_o}{2}\right\}+h\,\frac{\Delta x}{2}\,(T_\infty-T_o)=\rho\,c\,\frac{\Delta x^2}{4}\,\frac{T_{i,j,k+1}-T_o}{\Delta t}$$

Solving for $T_{i,\,j,\,k+1}$

$$T_{i,\,j,\,k+1}=T_o+4\,K1\left(\frac{1}{2}(T_u+T_r)-T_o\right)+2\,K2\,(T_\infty-T_o)$$

$i = 6$, $j = 1$

$$k\left\{\frac{T_i-T_o}{2}\right\}+h\,\frac{\Delta x}{2}\,(1+\sqrt{2})\,(T_\infty-T_o)=\rho\,\mathrm{c}\,\frac{\Delta x^2}{8}\,\frac{T_{i,j,k+1}-T_o}{\Delta t}$$

Solving for $T_{i,\,j,\,k+1}$

$$T_{i,\,j,\,k+1}=T_o+8\,K1\left(\frac{1}{2}(T_i-T_o)\right)+4\,K2\,(1+\sqrt{2})\,(T_\infty-T_o)$$

$i = 1$, $j = 6$

$$k\left\{\frac{T_d-T_o}{2}\right\}+h\,\Delta x\,\frac{\sqrt{2}}{2}\,(T_\infty-T_o)=\rho\,c\,\frac{\Delta x^2}{8}\,\frac{T_{i,j,k+1}-T_o}{\Delta t}$$

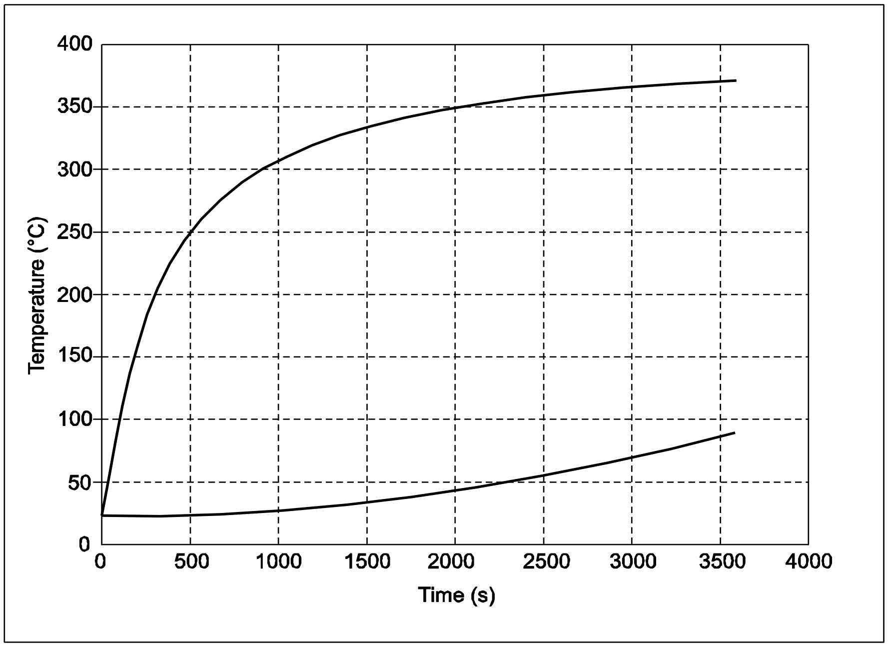
Solving for $T_{i,\,j,\,k+1}$

$$T_{i,\,j,\,k+1}=T_o+8\,K1\left(\frac{1}{2}(T_d-T_o)\right)+8\,\sqrt{2}\,K2\,(T_\infty-T_o)$$

The system of equations can be solved by the marching procedure. We must keep in mind the limitation in Δt given by Equation (4.15) which gives

$$\Delta t_{max} = 800 \text{ seconds}$$

The equations were solved using Δt = 10 seconds. A check was performed by hand on each of the seven unique control volume energy balances. The maximum temperature occurs at i = 6, j = 1, and the minimum temperature occurs at i = 1, j = 3. The resulting temperature as a function of time is given below.



PROBLEM 4.46

A steel billet is to be heat treated by immersion in a molten salt bath. The billet is 5-cm-square and 1-m-long. Prior to immersion in the bath, the billet is at a uniform temperature of 20°C. The bath is at 600°C and the heat transfer coefficient at the billet surface is 20 W/(m²K). Plot the temperature at the center of the billet as a function of time. How much time is needed to heat the billet center to 500°C? Use an implicit difference scheme with node spacing of 1 cm. The thermal conductivity of the steel is 40 W/(m K) and the thermal diffusivity is 1×10^{-5} m²/s.

GIVEN

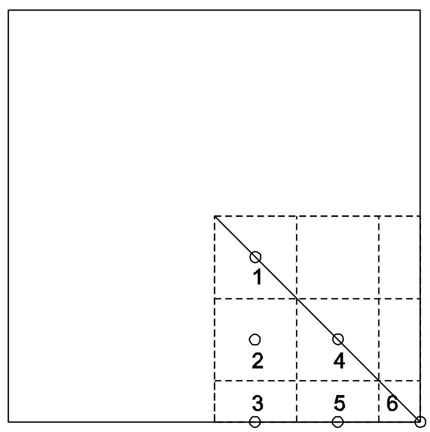
- Steel billet undergoing heat treatment

FIND

(a) Temperature at the center of the billet as a function of time
(b) How much time is needed to heat the center of the billet to 500°C

SOLUTION

The billet can be considered two-dimensional since it is very long. The accompanying sketch shows the geometry.



Allowing for symmetry, we need only consider 6 nodes and control volumes. These are also shown in the sketch. We need to develop a heat balance on each of the these control volumes. In the implicit form

Node (1)

$$k \, \frac{T_{2,k+1} - T_{1,k+1}}{\Delta x} \Delta x = \rho \, c \, \frac{\Delta x^2}{2} \, \frac{T_{1,k+1} - T_{1,k}}{\Delta t}$$

or

$$T_{1,\,k+1} \left(1 + \frac{2\alpha\Delta t}{\Delta x^2}\right) - T_{2,\,k+1} \left(\frac{2\alpha\Delta t}{\Delta x^2}\right) = T_{1,\,k}$$

Node (2)

$$k \left[\frac{T_{1,k+1} - T_{2,k+1}}{\Delta x} \Delta x + \frac{T_{4,k+1} - T_{2,k+1}}{\Delta x} \Delta x + \frac{T_{3,k+1} - T_{2,k+1}}{\Delta x} \Delta x\right]$$

$$= \rho \, c \, \Delta x^2 \, \frac{T_{2,k+1} - T_{2,k}}{\Delta t}$$

or

$$T_{2,\,k+1} \left(1 + \frac{3\alpha\Delta t}{\Delta x^2}\right) - (T_{1,\,k+1} + T_{3,\,k+1} + T_{4,\,k+1}) \left(\frac{\alpha\Delta t}{\Delta x^2}\right) = T_{2,\,k}$$

Node (3)

$$k\left[\frac{T_{2,k+1}-T_{3,k+1}}{\Delta x}\Delta x+\frac{T_{5,k+1}-T_{3,k+1}}{\Delta x}\frac{\Delta x}{2}\right]+h\,\Delta x\,(T_\infty-T_{3,\,k+1})=\rho\,c\;\frac{\Delta x^2}{2}\;\frac{T_{3,k+1}-T_{3,k}}{\Delta t}$$

or

$$T_{3,\,k+1}\left(1+\frac{3\alpha\Delta t}{\Delta x^2}+\frac{2h\Delta t}{\rho c\Delta x}\right)-T_{2,\,k+1}\left(\frac{2\alpha\Delta t}{\Delta x^2}\right)-T_{5,\,k+1}\left(\frac{\alpha\Delta t}{\Delta x^2}\right)=T_{3,\,k}+\frac{2h\Delta t T_\infty}{\rho c\Delta x}$$

Node (4)

$$k\left[\frac{T_{2,k+1}-T_{4,k+1}}{\Delta x}\Delta x+\frac{T_{5,k+1}-T_{4,k+1}}{\Delta x}\right]=\rho\,c\;\frac{\Delta x^2}{2}\;\frac{T_{4,k+1}-T_{4,k}}{\Delta t}$$

or

$$T_{4,\,k+1}\left(1+\frac{4\alpha\Delta t}{\Delta x^2}\right)-T_{2,\,k+1}\left(\frac{2\alpha\Delta t}{\Delta x^2}\right)-T_{5,\,k+1}\left(\frac{2\alpha\Delta t}{\Delta x^2}\right)=T_{4,\,k}$$

Node (5)

$$k\left[\frac{T_{4,k+1}-T_{5,k+1}}{\Delta x}\Delta x+\frac{T_{3,k+1}-T_{5,k+1}}{\Delta x}\frac{\Delta x}{2}+\frac{T_{6,k+1}-T_{5,k+1}}{\Delta x}\frac{\Delta x}{2}\right]$$

$$+\,h\,\Delta x\,(T_\infty-T_{5,\,k+1})=\rho\,c\;\frac{\Delta x^2}{2}\;\frac{T_{5,k+1}-T_{5,k}}{\Delta t}$$

or

$$T_{5,\,k+1}\left(1+\frac{4\alpha\Delta t}{\Delta x^2}+\frac{2h\Delta t}{\rho c\Delta x}\right)-T_{3,\,k+1}\left(\frac{\alpha\Delta t}{\Delta x^2}\right)-T_{4,\,k+1}\left(\frac{2\alpha\Delta t}{\Delta x^2}\right)$$

$$-\,T_{6,\,k+1}\,\frac{\alpha\Delta t}{\Delta x^2}=T_{5,\,k}+\frac{2h\Delta t T_\infty}{\rho c\Delta x}$$

Node (6)

$$k\;\frac{T_{5,k+1}-T_{6,k+1}}{\Delta x}\frac{\Delta x}{2}+\frac{h\Delta x}{2}\;(T_\infty-T_{6,\,k+1})=\rho\,c\;\frac{\Delta x^2}{8}\;\frac{T_{6,k+1}-T_{6,k}}{\Delta t}$$

or

$$T_{6,\,k+1}\left(1+\frac{4\alpha\Delta t}{\Delta x^2}+\frac{4h\Delta t}{\rho c\Delta x}\right)-T_{5,\,k+1}\left(\frac{4\alpha\Delta t}{\Delta x^2}\right)=T_{6,\,k}+\frac{4h\Delta t T_\infty}{\rho c\Delta x}$$

The 6 equations for the 6 control volumes can be written in matrix form as follows

$$\begin{bmatrix} 1+2K_1 & -2K_1 & 0 & 0 & 0 & 0 \\ K_1 & 1+3K_1 & -K_1 & -K_1 & 0 & 0 \\ 0 & -2K_1 & 1+3K_1+2K_2 & 0 & -K_1 & 0 \\ 0 & -2K_1 & 0 & 1+4K_1 & -2K_1 & 0 \\ 0 & 0 & -K_1 & -2K_1 & 1+4K_1+2K_2 & -K_1 \\ 0 & 0 & 0 & 0 & -4K_1 & 1+4K_1+4K_2 \end{bmatrix}$$

$$\begin{bmatrix} T_{1,k+1} \\ T_{2,k+1} \\ T_{3,k+1} \\ T_{4,k+1} \\ T_{5,k+1} \\ T_{6,k+1} \end{bmatrix}=\begin{bmatrix} T_{1,k} \\ T_{2,k} \\ T_{3,k} \\ T_{4,k} \\ T_{5,k} \\ T_{6,k} \end{bmatrix}+\begin{bmatrix} \\ \\ 2K \\ \\ 2K \\ 2K \end{bmatrix}$$

In the above matrix, we have used the following notation

$$K_1 = \frac{\alpha \Delta t}{\Delta x^2}$$

and

$$K_2 = \frac{h \Delta t}{\rho c \Delta x}$$

The matrix equation can be written as

$$AT_{k+1} = T_{k+} C$$

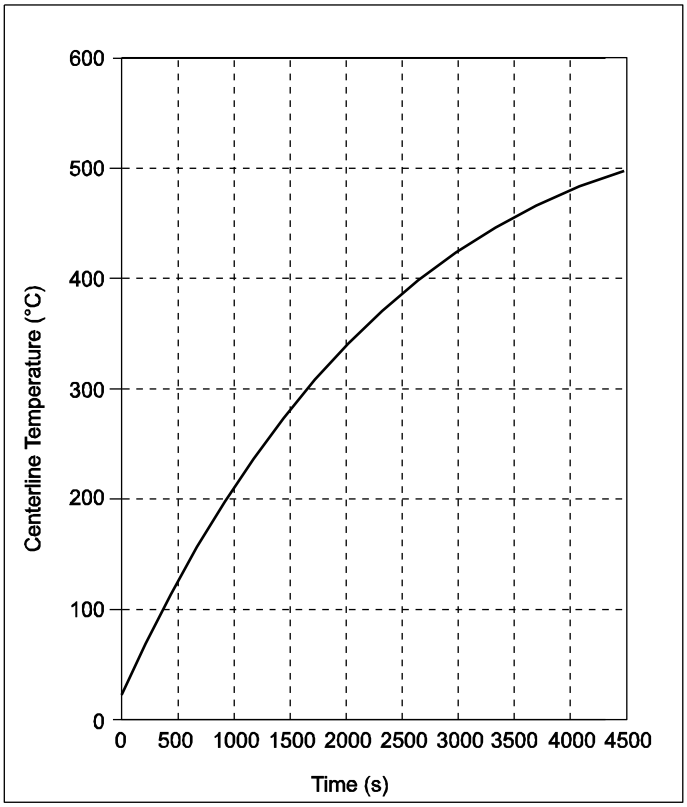
For $k = 0$, we know the vector T_k from the initial conditions. Therefore, we know the right-hand side of the above equation. Inverting the matrix A and multiplying by both sides of the matrix equation, we have the solution for T_{k+1}

$$T_{k+1} = A^{-1}(T_k + C)$$

Incrementing k to $k = 1$, we can then insert the solution for T_1 into the right-hand side of the above equation to find T_2 and so forth. This can be implemented fairly easily with a spreadsheet program in two steps. First, the coefficients of the matrix A are determined from the problem parameters. The matrix is then inverted. In the second step, the inverted matrix is repeatedly multiplied by the sum of the two vectors T_k and C. Each time it is multiplied by the sum of these two vectors, the vector T_k is updated with the results. The temperature at node 1 is nearest the center, so it is saved for later plotting. The spreadsheet is shown below.

```
Problem 4_46 Filename: 4_46.WQ1
PROBLEM PARAMETERS
alpha  =  1 K-0.5 m^2/s
dx     =  0.01 m
dt     =  10 sec
h      =  20 W/m^2K
k      =  40 W/mK
rho C  =  4000000 Ws/m^3K
T inf  =  600 C
K1     =  1 (-)
K2     =  0.005 (-)
                                    Coefficient Matrix
       ===============================
       3                     -2    0     0     0      0
       -1                     4   -1    -1     0      0
       0                     -2 4.01     0    -1      0
       0                     -2    0     5    -2      0
       0                      0   -1    -2  5.01     -1
       0                      0    0     0    -4   5.02
                                                                      T(K+1) =
                                                                      INVERSE
                                                                      MATRIX
               VECTOR                                   VECTOR  VECTOR VECTOR
       INVERSE MATRIX T(K)                                C       SUM  SUM
============================================            ======== ==  ====== == =======
       0.4358340.3075020.0924230.0867460.063115 0.012573  500.356  0   500.3     500.35
       0.1537510.4612530.1386350.13012 0.094673 0.018859 500.5545  0  500.55  5 500.554
       0.0924230.277270.3523690.109747 0.135732 0.027038 500.9511  6  506.95  1 500.951
       0.0867460.2602390.1097470.3239860.179844 0.035826 500.7526  0  500.75  6 500.752
       0.0631150.1893450.1357320.1798440.354938 0.070705 501.1484  6  507.14  4 501.148
       0.0502910.1508730.1081530.1433020.282819 0.255542 501.5426 12  513.54  6 501.542
       The macro below automatically multiplies
       the inverse matrix by the "vector sum" and puts the
       result for T(1,k+1) into the table to the left for plotting
Temperature of a function of time.
       Iterationtime          { / Math: MultiplyMatrix}-
       k(sec)T (1, k){GOTO}
       00      20       c39-
       11021.04797    {END}
```

```
  2      20   22.75763   {DOWN 2}
  3      30   24.78853   {/ Block; Copy}
  .       .          .   $1$26-
  .       .          .   {IF L26 < 500} {BRANCH E37}
  .       .          .   {BEEP 1}
441    4410   499.1603
442    4420   499.3605
443    4430    499.959
444    4440   500.3564
```



Centerline Temperature vs. Time for Problems 3.45

(b) The temperature at the billet centerline exceeds 500°C at 4440 seconds.

PROBLEM 4.47

Determine the difference equations applicable to the centerline and at the surface of an axisymmetric cylindrical geometry with volumetric heat generation and convection boundary condition. Assume steady-state conditions.

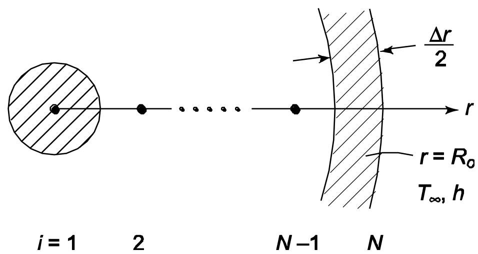
GIVEN

- Axisymmetric, steady, cylindrical geometry with volumetric heat generation and surface convection boundary condition

FIND

(a) Difference equations for the centerline and surface

SKETCH



SOLUTION

The solution to this problem completes the formulation of the cylindrical geometry presented in Section 3.5, with the added constraints of steady state conditions and symmetry.

As in Figure 4.22 and Section 4.5, the radius is given by

$$r = (i-1)\,\Delta r \qquad i = 1, 2, \ldots N \qquad \Delta r = \frac{R_o}{N-1}$$

Let the convection coefficient be h and ambient temperature be T_∞. The inner surface area per unit length of the shaded control volume is

$$2\pi\left(R_o - \frac{\Delta r}{2}\right)$$

and the outer surface area is

$$2\pi R_o$$

The volume of the control volume per unit length is

$$\pi\left(R_o^2 - \left(R_o - \frac{\Delta r}{2}\right)^2\right) = \pi\left(R_o\Delta r - \frac{\Delta r^2}{4}\right)$$

The energy balance on the control volume gives

$$k\,\frac{T_{N-1} - T_N}{\Delta r}\;2\pi\left(R_o - \frac{\Delta r}{2}\right) + 2\pi R_o h\,(T_\infty - T_N) + \dot{q}_G\,\frac{\Delta r}{2}\left(R_o - \frac{\Delta r}{4}\right) = 0$$

Simplifying and putting into the tridiagonal form

$$T_N\left(\frac{k}{\Delta r}\left(R_o - \frac{\Delta r}{2}\right) + R_o h\right) = T_{N-1}\left(\frac{k}{\Delta r}\left(R_o - \frac{\Delta r}{2}\right)\right) + \left(R_o h T_\infty + \dot{q}_G\,\frac{\Delta r}{2}\left(R_o - \frac{\Delta r}{4}\right)\right)$$

For the control volume for the centerline node, $i = 1$, the volume per unit length is

$$\pi \left(\frac{\Delta r}{2}\right)^2$$

and the surface area per unit length is

$$2\pi \frac{\Delta r}{2} = \pi \Delta r$$

The energy balance gives

$$k \frac{T_2 - T_1}{\Delta r} \pi \Delta r + \dot{q}_G \pi \left(\frac{\Delta r}{2}\right)^2 = 0$$

Simplifying and putting into the tridiagonal form

$$T_2 - T_{\mathrm{I}} + \dot{q}_G \frac{\Delta r^2}{4k} = 0$$

COMMENTS

The above two difference equations can be combined with Equation (4.31) to produce the full set of difference equations. The resulting tridiagonal set of equation can be solved just as Equation (4.11). (The steady, axisymmetric version of Equation (4.31) would be used.)

PROBLEM 4.48

Determine the appropriate difference equations for an axisymmetric, steady, spherical geometry with volumetric heat generation. Explain how to solve the equations.

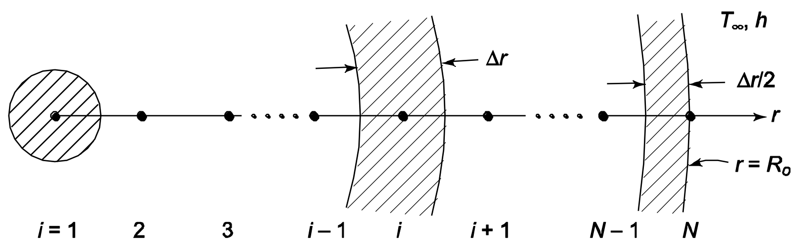
GIVEN

- Axisymmetric, steady, spherical geometry with heat generation

FIND

(a) Difference equations

SKETCH



SOLUTION

We need to perform an energy balance on the three shaded control volumes shown in the text. For the node at the sphere center, $i = 1$

$$\text{Volume} = \frac{4}{3}\pi\left(\frac{\Delta r}{2}\right)^3 = \frac{\pi}{6}\Delta r^3$$

$$\text{Surface} = 4\pi\left(\frac{\Delta r}{2}\right)^2 = \pi\Delta r^2$$

The energy balance is

$$k\,\frac{T_2 - T_1}{\Delta r}\,\pi\Delta r^2 + \dot{q}_G\,\frac{\pi}{6}\,\Delta r^3 = 0$$

In the tridiagonal form

$$T_1\,k\Delta r = T_2\,k\Delta r + \dot{q}_G\,\frac{\Delta r^3}{6}$$

For interior control volumes, $1 < i < N$

$$\text{Volume} = \frac{4}{3}\pi\left[\left(i\Delta r + \frac{\Delta r}{2}\right)^3 - \left(i\Delta r - \frac{\Delta r}{2}\right)^3\right] = \frac{4}{3}\pi\Delta r^3\left(3i^2 + \frac{1}{4}\right) \equiv V_i$$

$$\text{Inner surface area} = 4\pi\left(i\Delta r - \frac{\Delta r}{2}\right)^2 = 4\pi\Delta r^2\left(i - \frac{1}{2}\right)^2 \equiv A_{ii}$$

$$\text{Outer surface area} = 4\pi\left(i\Delta r + \frac{\Delta r}{2}\right)^2 = 4\pi\Delta r^2\left(i + \frac{1}{2}\right)^2 \equiv A_{io}$$

The energy balance is

$$k\,\frac{T_{i-1}-T_i}{\Delta r}\,A_{ii} + k\,\frac{T_{i+1}-T_i}{\Delta r}\,A_{io} + \dot{q}_G\,V_i = 0$$

In the tridiagonal form this becomes

$$T_i\left[\frac{k}{\Delta r}\;A_{ii} + A_{io}\;\right] = T_{i-1}\left[\frac{k}{\Delta r}A_{ii}\right] + T_{i+1}\left[\frac{k}{\Delta r}A_{io}\right] + \dot{q}_G\,V_i$$

For the control volume at the surface of the sphere

$$\text{Volume} = \frac{4}{3}\pi\left[R_o^{\,3} - \left(R_o - \frac{\Delta r}{2}\right)^3\right] \equiv V_o$$

$$\text{Inner surface area} = 4\pi\left(R_o - \frac{\Delta r}{2}\right)^2 \equiv A_{Ni}$$

$$\text{Outer surface area} = 4\pi R_o^2 \equiv A_{No}$$

The energy balance for the surface control volume is

$$k\,\frac{T_{N-1}-T_N}{\Delta r}\,A_{Ni} + A_{No}\,h\,(T_\infty - T_N) + \dot{q}_G\,V_o = 0$$

In the tridiagonal form

$$T_N\left[\frac{k}{\Delta r}A_{Ni} + hA_{No}\right] = T_{N-1}\left[\frac{k}{\Delta r}A_{Ni}\right] + A_{No}\,h + \dot{q}_G\,V_o$$

From the three control volume difference equations given above in the tridiagonal form, we can determine the matrix coefficients

$$a_1 = k\Delta r \quad b_1 = k\Delta r \quad c_1 = 0 \quad d_1 = \dot{q}_G\,\frac{\Delta r^3}{6}$$

$$a_i = \frac{k}{\Delta r}\,(A_{ii} + A_{io}) \quad b_i = \frac{k}{\Delta r}A_{io} \quad c_i = \frac{k}{\Delta r}A_{ii} \quad d_i = \dot{q}_G\,V_i \quad 1 < i < N$$

$$a_N = \frac{k}{\Delta r}A_{Ni} + hA_{No} \quad b_N = 0 \quad c_N = \frac{k}{\Delta r}A_{Ni} \quad d_N = A_{No}\,h + \dot{q}_G\,V_o$$

To solve this set of equations, we insert these coefficients into the matrix in Equation (3.11) and solve the tridiagonal matrix as was done for other one-dimensional problems.

PROBLEM 4.49

How would the results of Problem 4.47 be modified if the problem were not axisymmetric?

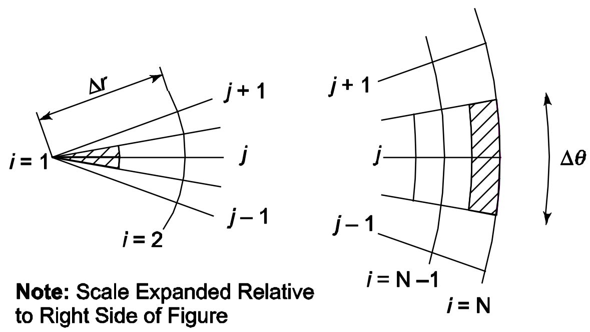
GIVEN

- Non-axisymmetric, steady, cylindrical geometry with heat generation and convective boundary condition.

FIND

(a) Difference equations for the centerline and surface control volumes

SKETCH



Note: Scale Expanded Relative to Right Side of Figure

SOLUTION

For the control volume at the centerline we have

$$\text{Volume } = \pi \left(\frac{\Delta r}{2}\right)^2 \frac{\Delta\theta}{2\pi} = \frac{\Delta r^2 \Delta\theta}{8}$$

$$\text{Surface area } = 2\pi \,\frac{\Delta r}{2}\,\frac{\Delta\theta}{2\pi} = \frac{\Delta r \Delta\theta}{2}$$

and the energy balance gives

$$k\, \frac{T_{2,j} - T_{i=1}}{\Delta r}\,\frac{\Delta r \Delta\theta}{2} + \dot{q}_G\, \frac{\Delta r^2 \Delta\theta}{8} = 0$$

For the control volume at the surface, we have for the volume per unit length

$$\text{Volume } = \pi\,\Delta r \left(R_o - \frac{\Delta r}{4}\right)\frac{\Delta\theta}{2\pi} = \Delta r\left(R_o - \frac{\Delta r}{4}\right)\frac{\Delta\theta}{2}$$

and for the surface area (per unit length) of the circumferential face inside R_o

$$\text{Inner circumferential surface area} = 2\pi\left(R_o - \frac{\Delta r}{2}\right)\frac{\Delta\theta}{2\pi} = \left(R_o - \frac{\Delta r}{2}\right)\Delta\theta$$

For the surface area of the circumferential face at R_o we have

$$\text{Outer circumferential surface area} = 2\pi R_o\,\frac{\Delta\theta}{2\pi} = R_o\,\Delta\theta$$

The surface area per unit length of the radial faces of the control volume are

$$\text{radial surface area} = \frac{\Delta r}{2}$$

The energy balance is

$$k\,\frac{T_{N-1,j}-T_{N,j}}{\Delta r}\left(R_o-\frac{\Delta r}{2}\right)\Delta\theta+k\,\frac{T_{N,j-1}-T_{N,j}}{R_o\Delta\theta}\frac{\Delta r}{2}+k\,\frac{T_{N,j+1}-T_{N,j}}{R_o\Delta\theta}\frac{\Delta r}{2}+h\,R_o\,\Delta\theta\,(T_\infty-T_{N,j})$$

$$+\ \dot{q}_G\,\Delta r\left(R_o-\frac{\Delta r}{4}\right)\frac{\Delta\theta}{2}=0$$

The solution of the above set of equations would be carried out in parallel with the method explained in Section 4.4.3.1, for two-dimensional steady problems. The difference equation for the interior nodes given by Equation (4.31) (steady state terms only) would be added to the above difference equations and an iterative solution procedure would be employed to find the solution.

PROBLEM 4.50

For the geometry shown in the sketch below, determine the layout of nodes and control volumes. Provide a scale drawing showing the problem geometry overlaid with the nodes and control volumes. Explain how to derive the energy balance equation for all the boundary control volumes.

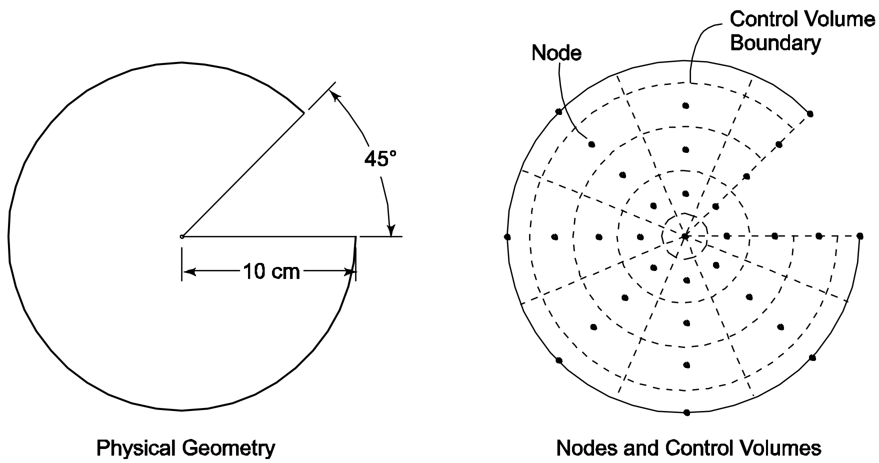
GIVEN

- Cylindrical geometry shown in the figure.

FIND

(a) A reasonable layout of nodes and control volumes

SKETCH



SOLUTION

We do not know what the boundary conditions are so we cannot make any judgment about symmetry. Therefore, we must assume that the problem is axisymmetric. Since R_o = 10 cm, let's use Δr = 2.5 cm giving 5 radial nodes. To accommodate the 45° cutout, let's use a circumferential node spacing, $\Delta\theta$ = 45°. The right side of the sketch shows the resulting layout of nodes and control volumes. Energy balance equations for the control volumes at the circumferential boundary would be derived as described in Section 4.5. For those control volumes, we have conduction from three surrounding control volumes and either convection, specified flux, or a specified temperature at the fourth surface, depending on the boundary condition. For the control volumes along the two radial boundaries, we have conduction from two surrounding control volumes. Treatment of the third surface would depend on the boundary condition.

PROBLEM 4.51

Hot flue gases from a combustion furnace flow through a chimney, which is 7-m-tall and has a hollow cylindrical cross section with inner diameter d_i = 30 cm and outer diameter d_o = 50 cm. The flue gases flow with an average temperature of T_g = 300°C and convective heat transfer coefficient of h_g = 75 W/(m^2 K). The chimney is made of concrete, which has a thermal conductivity of k = 1.4 W/(m K). It is exposed to outside air that has an average temperature of T_a = 25°C and convective heat transfer coefficient of 15 W/(m^2 K). For steady-state conditions, (a) determine the inner and outer wall temperatures, (b) plot the temperature distribution along the thickness of the chimney wall, and (c) determine the rate of heat loss to outside air from the chimney. Solve the problem by numerical analysis using a nodal network with Δr = 2 cm and $\Delta\theta$ = 10°.

GIVEN

- Hot flue gas flow in a hollow cylindrical 7-m-tall chimney with inner and outer diameters of d_i = 0.3 m and d_o = 0.5 m, and thermal conductivity k = 1.4 W/(m K).
- Average hot gas temperature T_g = 300°C and heat transfer coefficient $\bar{h}_{c,g} = 75$ W/(m^2 K).
- Outside air average temperature T_a = 25°C and heat transfer coefficient $\bar{h}_{c,a} = 15$ W/(m^2 K).

FIND

- The temperature distribution in chimney wall and the inside and outside wall temperatures.
- Rate of heat loss from gas to outside air through the chimney.

ASSUMPTIONS

- Steady state conditions, and there is no heat generation in the chimney wall and the conduction along the height of the chimney is negligible.

SOLUTION

(a) For steady-state conditions, the heat conduction equation is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0$$

which is subject to the following boundary conditions

$$q'' = -k\left.\frac{dT}{dr}\right|_{r=r_i} = \bar{h}_{c,g}\ \ T_{w,i} - T_g \quad \text{and } q'' = -k\left.\frac{dT}{dr}\right|_{r=r_o} = \bar{h}_{c,a}\ \ T_{w,o} - T_a$$

Construct a nodal network with 2 cm spacing in the radial direction in the thickness of the chimney's hollow cylinder (6 nodal points total, including the nodal points on the inner and outer wall) and 10-degree spacing in the angular direction (36 nodal points). This would result in the following form of discretized equation

$$\frac{\Delta r}{r\Delta\theta}\ \ T_{i,j+1} - 2T_{i,j} + T_{i,j+1}\ \ + \frac{r\Delta\theta}{\Delta r}\ \ T_{i+1,j} - 2T_{i,j} + T_{i-1,j}\ \ + \frac{\Delta\theta}{2}\ \ T_{i+1,j} - T_{i-1,j}\ \ = \ 0$$

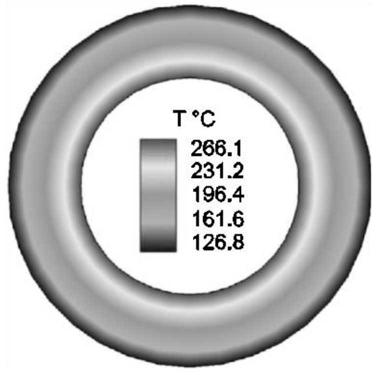
Though, it may be noted that because of the circular symmetry, this can be solved as a one-dimensional problem without using the angular nodes.

The numerical solution (carried out on MATLAB) yields the following temperature values at the six (6) radial nodes along the wall thickness

Inner Wall					Outer Wall
	Node 2	Node 3	Node 4	Node 5	
Node 1					Node 6

266.1°C 235.7°C 208.5°C 183.7°C 160.9°C 126.8°

(b) This temperature distribution is depicted as an isotherm contour plot in the figure below.



(c) Also, the rate of heat loss from the outer wall of the chimney to air can be calculated as

$$q = \bar{h}_{c,a} \left(\pi d_o L\right)\left(T_{w,o} - T_a\right) = 15\left(\pi \times 0.5 \times 7\right)\left(126.8 - 25\right) = 16{,}790 \text{ W}$$

PROBLEM 4.52

It has been proposed that highly concentrating solar power such as the one below can be used to process materials economically when it is desirable to heat the material surface rapidly without significantly heating the bulk. In one such process for case hardening low-cost carbon steel, the surface of a thin disk is to be exposed to concentrated solar flux. The distribution of absorbed solar flux on the disk is given by

$$q''(r) = q''_{max} \left(1 - 0.09\left(\frac{r}{R_o}\right)^2\right)$$

where r is the distance from the disk axis and q''_{max} and R_o are parameters that describe the flux distribution. The disk diameter is $2R_s$, its thickness is Z_s, its thermal conductivity is k and its thermal diffusivity is α. The disk is initially at temperature T_{init} and at time t = 0 it is suddenly exposed to the concentrated flux. Derive the set of explicit difference equations needed to predict how the disk temperature distribution evolves with time. The edge and bottom surface of the disk are insulated and reradiation from the disk is neglected.

GIVEN

- Steel disk exposed to concentrated solar flux

FIND

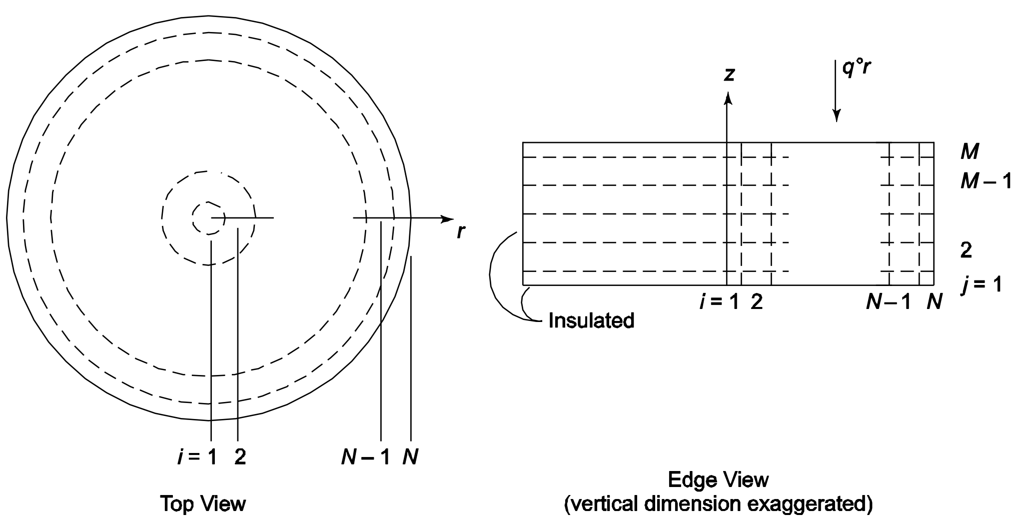
(a) Explicit difference equations that describe evolution of disk temperature

SOLUTION

The problem is a two-dimensional cylindrical geometry in the coordinates r and z. There are no gradients in the circumferential direction. Let there be N radial nodes and M axial nodes as shown in the sketch below. Then the size of the control volumes and the node locations are given by

$$\Delta r = \frac{R_s}{N-1} \qquad r_i = (i-1)\,\Delta r \qquad i = 1, 2, \ldots \mathrm{N}$$

$$\Delta z = \frac{Z_s}{M-1} \qquad z_j = (j-1)\,\Delta z \qquad j = 1, 2, \ldots M$$



There are a total of $N \times M$ control volumes and each has the shape of a ring with rectangular cross-section. We need to develop an energy balance equation for each control volume. First, let us determine the volume and surface area of each control volume since these will be needed in the energy balance equations.

The top or bottom face surface area of each control volume is A_{fi}

$$A_{fi} = \pi \left(\frac{\Delta r}{2}\right)^2 \quad i = 1$$

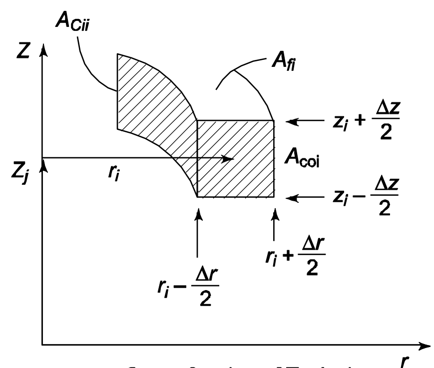
$$A_{fi} = \pi \left(\left(r_i + \frac{\Delta r}{2}\right)^2 - \left(r_i - \frac{\Delta r}{2}\right)^2 \right) = 2\pi \Delta r^2 \,(i-1) \qquad i = 2, 3, \ldots N-1$$

$$A_{fi} = \pi \left(R_s^2 - \left(R_s - \frac{\Delta r}{2}\right)^2 \right) = \pi \,\Delta r \left(R_s - \frac{\Delta r}{4}\right) \qquad i = N$$

Now, the volume of each control volume is just

$$V_i = A_{fi} \,\Delta z \qquad i = 1, 2, \ldots N$$

Except for the control volume at node $i = 1$, each control volume has two curved surfaces, an outer surface and an inner surface, see sketch below.



Cross Section of Typical
Control Volume

The surface area of the outer curved surface is

$$A_{\infty i} = 2\pi \left(r_i - \frac{\Delta r}{2}\right) \Delta z = 2\pi \Delta rz \left(i - \frac{1}{2}\right) \qquad i = 1, 2, \ldots N-1$$

$$A_{\infty i} = 2\pi R_s \,\Delta z \qquad i = N$$

The surface area of the inner surface is

$$A_{cii} = 2\pi \left(r_i - \frac{\Delta r}{2}\right) \Delta z = 2\pi \Delta r \Delta z \left(i - \frac{3}{2}\right) \qquad i = 2, 3, \ldots N$$

By definition

$$A_{cii} = 0 \qquad i = 1$$

(In the above notation for A_{cii}, the first i in the subscript refers to the inner curved surface and the second i is the node index.)

The control volumes along the exposed surface absorb solar flux given by the equation in the problem statement. We need to integrate this flux equation over each control volume to determine the solar energy absorbed for each control volume. The following equation expresses this

$$\overline{q_i} \equiv \int_o^{r_i + \frac{\Delta r}{2}} q''(r)\, 2\pi r dr \qquad i = 1$$

$$\overline{q_i} \equiv \int_{r_i+\frac{\Delta r}{2}}^{r_i+\frac{\Delta r}{2}} q''(r)\, 2\pi r dr \qquad i = 2, 3, \dots N-1$$

$$\overline{q_i} \equiv \int_{r_i-\frac{\Delta r}{2}}^{r_i} q''(r)\, 2\pi r dr \qquad i = N$$

Carrying out the integration and simplifying we find

$$\overline{q_i} = \frac{\pi}{4}\, q''_{\max}\, \Delta r^2 \left(1 - \frac{0.09}{8}\left(\frac{\Delta r}{R_o}\right)^2\right) \qquad i = 1$$

$$\overline{q_i} = \pi\, q''_{\max}\, \Delta r^2 \left[2(i-1) - \frac{0.09}{8}\left(\frac{\Delta r}{R_o}\right)^2 (4i^3 - 12i^2 + 13i - 5)\right] \qquad i = 2, 3, \dots N-1$$

$$\overline{q_i} = \pi\, q''_{\max}\, \Delta r^2 \left[N - \frac{5}{4} - \frac{0.09}{8}\left(\frac{\Delta r}{R_o}\right)^2 \left(2N^3 - \frac{15}{2}N^2 + \frac{19}{2}N - \frac{65}{16}\right)\right] i = N$$

We are now in a position to evaluate the energy balance for each control volume. We actually only need to develop 9 unique difference equations. These are for the interior nodes, the nodes at the four corners, and the nodes on the axis, and on the three outer surfaces.

The explicit energy balance equation for all interior nodes is

$$k\left[\frac{T_{i+1,j,k} - T_{i,j,k}}{\Delta r} A_{coi} + \frac{T_{i-1,j,k} - T_{i,j,k}}{\Delta r} A_{cii} + \frac{T_{i,j+1,k} - T_{i,j,k}}{\Delta z} A_{fi} + \frac{T_{i,j-1,k} - T_{i,j,k}}{\Delta z} A_{fi}\right]$$

$$= \rho c V_i \frac{T_{i,j,k+1} - T_{i,j,k}}{\Delta t}$$

Solving for the node temperatures

$$T_{i,j,k+1} = T_{i,j,k} + \frac{\alpha \Delta t}{V_i}\left[\frac{T_{i+1,j,k} - T_{i,j,k}}{\Delta r} A_{coi} + \frac{T_{i-1,j,k} - T_{i,j,k}}{\Delta r} A_{cii} + \right.$$

$$\left. \frac{T_{i,j+1,k} - T_{i,j,k}}{\Delta z} A_{fi} + \frac{T_{i,j-1,k} - T_{i,j,k}}{\Delta z} A_{fi}\right]$$

$$i = 2, 3, \dots N-1 \qquad j = 2, 3, \dots M-1$$

For the interior nodes along the axis we have

$$k\left[\frac{T_{i+1,j,k} - T_{i,j,k}}{\Delta r} A_{coi} + \frac{T_{i,j+1,k} - T_{i,j,k}}{\Delta z} A_{fi} + \frac{T_{i,j-1,k} - T_{i,j,k}}{\Delta z} A_{fi}\right] = \rho c V_i \frac{T_{i,j,k+1} - T_{i,j,k}}{\Delta t}$$

Solving for the node temperatures

$$T_{i,j,k+1} = T_{i,j,k} + \frac{\alpha \Delta t}{V_i}\left[\frac{T_{i+1,j,k} - T_{i,j,k}}{\Delta r} A_{coi} + \frac{T_{i,j+1,k} - T_{i,j,k}}{\Delta z} A_{fi} + \frac{T_{i,j-1,k} - T_{i,j,k}}{\Delta z} A_{fi}\right]$$

$$i = 1 \qquad j = 2, 3, \dots M-1$$

For the node on the top of the axis

$$k\left[\frac{T_{i+1,j,k}-T_{i,j,k}}{\Delta r}A_{coi}+\frac{T_{i,j-1,k}-T_{i,j,k}}{\Delta z}A_{fi}\right]+\overline{q_i} = \rho\, c\, V_{\mathrm{i}}\,\frac{T_{i,j,k+1}-T_{i,j,k}}{\Delta t}$$

Solving for the node temperature

$$T_{i,j,k+1} = T_{i,j,k}+\frac{\alpha\Delta t}{V_i}\left[\frac{T_{i+1,j,k}-T_{i,j,k}}{\Delta r}A_{coi}+\frac{T_{i,j-1,k}-T_{i,j,k}}{\Delta z}A_{fi}\right]\frac{\overline{q_i}\,\Delta t}{\rho c V_i}$$

$$i = 1 \qquad j = M$$

For the node on the bottom of the axis

$$k\left[\frac{T_{i+1,j,k}-T_{i,j,k}}{\Delta r}A_{coi}+\frac{T_{i,j+1,k}-T_{i,j,k}}{\Delta z}A_{fi}\right] = \rho\, c\, V_i\,\frac{T_{i,j,k+1}-T_{i,j,k}}{\Delta t}$$

$$i = 1 \qquad j = 1$$

For the interior nodes along the outer curved surface

$$k\left[\frac{T_{i-1,j,k}-T_{i,j,k}}{\Delta r}A_{cii}+\frac{T_{i,j+1,k}-T_{i,j,k}}{\Delta z}A_{fi}+\frac{T_{i,j-1,k}-T_{i,j,k}}{\Delta z}A_{fi}\right] = \rho\, c\, V_i\,\frac{T_{i,j,k+1}-T_{i,j,k}}{\Delta t}$$

Solving for the node temperatures

$$T_{i,j,k+1} = T_{i,j,k}+\frac{\alpha\Delta t}{V_i}\left[\frac{T_{i-1,j,k}-T_{i,j,k}}{\Delta r}A_{cii}+\frac{T_{i,j+1,k}-T_{i,j,k}}{\Delta z}A_{fi}+\frac{T_{i,j-1,k}-T_{i,j,k}}{\Delta z}A_{fi}\right]$$

$$i = N \qquad j = 2, 3, \ldots M-1$$

For the node on the top of the outer curved surface

$$k\left[\frac{T_{i-1,j,k}-T_{i,j,k}}{\Delta r}A_{cii}+\frac{T_{i,j-1,k}-T_{i,j,k}}{\Delta z}A_{fi}\right]\overline{q_i} = \rho\, c\, V_{\mathrm{i}}\,\frac{T_{i,j,k+1}-T_{i,j,k}}{\Delta t}$$

Solving for the node temperature

$$T_{i,j,k+1} = T_{i,j,k}+\frac{\alpha\Delta t}{V_i}\left[\frac{T_{i-1,j,k}-T_{i,j,k}}{\Delta r}A_{cii}+\frac{T_{i,j-1,k}-T_{i,j,k}}{\Delta z}A_{fi}\right]+\frac{\overline{q_i}\,\Delta t}{\rho c V_i}$$

$$i = N \qquad j = M$$

For the node on the bottom of the outer curved surface

$$k\left[\frac{T_{i-1,j,k}-T_{i,j,k}}{\Delta r}A_{cii}+\frac{T_{i,j+1,k}-T_{i,j,k}}{\Delta z}A_{fi}\right] = \rho\, c\, V_i\,\frac{T_{i,j,k+1}-T_{i,j,k}}{\Delta t}$$

Solving for the node temperature

$$T_{i,j,k+1} = T_{i,j,k}+\frac{\alpha\Delta t}{V_i}\left[\frac{T_{i-1,j,k}-T_{i,j,k}}{\Delta r}A_{cii}+\frac{T_{i,j+1,k}-T_{i,j,k}}{\Delta z}A_{fi}\right]$$

$$i = N \qquad j = 1$$

For the interior nodes on the bottom surface

$$k\left[\frac{T_{i-1,j,k}-T_{i,j,k}}{\Delta r}A_{cii}+\frac{T_{i+1,j,k}-T_{i,j,k}}{\Delta r}A_{\infty i}+\frac{T_{i,j+1,k}-T_{i,j,k}}{\Delta z}A_{fi}\right] = \rho\, c\, V_i\,\frac{T_{i,j,k+1}-T_{i,j,k}}{\Delta t}$$

Solving for the node temperature

$$T_{i,j,k+1} = T_{i,j,k} + \frac{\alpha \Delta t}{V_i}\left[\frac{T_{i-1,j,k} - T_{i,j,k}}{\Delta r} A_{cii} + \frac{T_{i+1,j,k} - T_{i,j,k}}{\Delta r} A_{\infty i} + \frac{T_{i,j+1,k} - T_{i,j,k}}{\Delta z} A_{fi}\right]$$

$$i = 2, 3, \ldots N-1 \qquad j = 1$$

Finally, for the interior nodes on the top surface

$$k\left[\frac{T_{i-1,j,k} - T_{i,j,k}}{\Delta r} A_{cii} + \frac{T_{i+1,j,k} - T_{i,j,k}}{\Delta r} A_{\infty i} + \frac{T_{i,j-1,k} - T_{i,j,k}}{\Delta z} A_{fi}\right] + \overline{q_i} = \rho\, c\, V_i \frac{T_{i,j,k+1} - T_{i,j,k}}{\Delta t}$$

Solving for the node temperature

$$T_{i,j,k+1} = T_{i,j,k} + \frac{\alpha \Delta t}{V_i}\left[\frac{T_{i-1,j,k} - T_{i,j,k}}{\Delta r} A_{cii} + \frac{T_{i+1,j,k} - T_{i,j,k}}{\Delta r} A_{\infty i} + \frac{T_{i,j-1,k} - T_{i,j,k}}{\Delta z} A_{fi}\right] + \frac{\overline{q_i} \Delta t}{\rho c V_i}$$

$$i = 2, 3, \ldots N-1 \qquad j = M$$

PROBLEM 4.53

Solve the set of difference equations derived in Problem 4.52 given the following values of the problem parameters

k = 40.0 W/(mK), disk thermal conductivity
α = 1 × 10^{-5} m^2/s, disk thermal diffusivity
R_s = 25 mm, disk radius
Z_s = 5 mm, disk thickness
q''_{max} = 3 × 10^6 W/m^2, peak absorbed flux
R_o = 50 mm, parameter in flux distribution
T_{init} = 20°C, disk initial temperature

Determine the temperature distribution in the disk when the maximum temperature is 300°C.

GIVEN

- Difference equations developed in Problem 4.52 given the following values of the problem parameters

FIND

(a) Disk temperature distribution when the maximum temperature is 300°C

SOLUTION

All 9 difference equations can be written in the form

$$T_{i,j,k+1} = T_{i,j,k}\left(1 - \frac{\alpha\Delta t}{V_i}\left(R_{i,j} + L_{i,j} + U_{i,j} + D_{i,j}\right)\right)$$
$$+ \frac{\alpha\Delta t}{V_i}\left(R_{i,j}\,T_{i+1,j,k} + L_{i,j}\,T_{i-1,j,k} + U_{i,j}\,T_{i,j+1,k} + D_{i,j}\,T_{i,j-1,k}\right) + C_{i,j}$$

Where the coefficients $C_{i,j}$, $R_{i,j}$, $L_{i,j}$, $U_{i,j}$, $D_{i,j}$ are defined in the table below. Note that to use the above general equation for all nodes, we must allow the matrix of node temperatures to extend to indices $i = 0$, $i = N + 1$, $j = 0$, and $j = M + 1$. Temperatures at these nodes do not have meaning, but to apply the general difference equation along the axis or on the boundaries, they must be defined. Their values do not matter because the coefficients are set up to account for the special equations on the axis or on the boundaries.

Table of Coefficients for the Difference Equation

$C_{i,j}$	$R_{i,j}$	$L_{i,j}$	$U_{i,j}$	$D_{i,j}$	Applicable Range of i	Applicable Range of j
0	$A_{coi}/\Delta r$	$A_{cii}/\Delta r$	$A_{fi}/\Delta z$	$A_{fi}/\Delta z$	2, 3 … $N - 1$	2, 3 … $M - 1$
0	$A_{coi}/\Delta r$	0	$A_{fi}/\Delta z$	$A_{fi}/\Delta z$	1	2, 3 … $M - 1$
$\frac{\Delta t\bar{q}_i}{\rho c V_i}$	$A_{coi}/\Delta r$	0	0	$A_{fi}/\Delta z$	1	M
0	$A_{coi}/\Delta r$	0	$A_{fi}/\Delta z$	0	1	1
0	0	$A_{cii}/\Delta r$	$A_{fi}/\Delta z$	$A_{fi}/\Delta z$	N	2, 3 … $M - 1$
$\frac{\Delta t\bar{q}_i}{\rho c V_i}$	0	$A_{cii}/\Delta r$	0	$A_{fi}/\Delta z$	N	M
0	0	$A_{cii}/\Delta r$	$A_{fi}/\Delta z$	0	N	1
$\frac{\Delta t\bar{q}_i}{\rho c V_i}$	$A_{coi}/\Delta r$	$A_{cii}/\Delta r$	0	$A_{fi}/\Delta z$	2, 3 … $N - 1$	M

0	$A_{coi}/\Delta r$	$A_{cii}/\Delta r$	$A_{fi}/\Delta z$	0	2, 3 … $N-1$	1

To maintain a positive coefficient on $T_{i,j,k}$ for stability we must have

$$\frac{\alpha\Delta t}{V_i}(R_{i,j} + L_{i,j} + U_{i,j} + D_{i,j}) < 1 \qquad i = 1, 2, \dots N \quad j = 1, 2 \dots M$$

The largest permissible time step Δt is therefore

$$\Delta t_{\max} = \left\{ \frac{V_i}{\alpha \; R_{i,j} + L_{i,j} + U_{i,j} + D_{i,j}} \right\}\Bigg|_{\max}$$

We will use some fraction of this time step in the program execution.

Note that we must first calculate the coefficients R, L, U, and D before determining $\Delta t_{\max}$. Then we can fill in the coefficients C.

The Pascal program listed below solves the difference equations as described above.

```
Program Prob3_M;        {solution to Problem 3_M}
uses crt, printer;
const   N = 11;         {radial nodes}
        M = 21;         {axial nodes}
        k = 40.0;       {thermal conductivity (W/mK)}
        alpha = 1e-5;   {thermal diffusivity (m^2/s)}
        Rs = 0.025;     {disk radius (m)}
        Zs = 0.005;     {disk thickness (m)}
        qmax = 3.0e6;   {peak flux (W/m^2)}
        Ro = 0.05;      {parameter in flux equation (m)}
        Tinit = 20.0;   {initial temperature (C)}
        Tmax = 300.0;   {maximum desired temperature at top of axis (C)}
var     dr, dz, rhoC, dtMax, dt, time:real;
        i, j : integer;
        C,R,L,U,D : array [1..N,1..M] of real;
        q,V,Aco,Aci,Af : array [1..N] of real;
        Toid,Tnew : array [1..N+1,1..M+1] of real;
begin
        {calculate size of the control volumes}
        dr : = Rs/(N - 1);
        dz : = Zs/(M - 1);
        rhoC: = k/alpha;

        {calculate control volume surface areas and volumes}
        Af[1]  : = pi*dr*dr/4.0;
        Af[N]  : = pi*dr*(Rs - dr/4.0);
        for i  : = 2 to N - 1 do Af[i]  : = 2.0*pi*dr*dr*(i - 1);
        for i  : = 1 to N do V[i] : = Af[i]*dz;

        Aco[N] : = 2.0*pi*Rs*dz;
        for i : = 1 to N - 1 do Aco[i] : = 2.0*pi*dr*dz*(i - 0.5);

        Aci[1] : = 0.0;
        for i : = 2 to N do Aci[i] : = 2.0*pi*dr*dz*(i - 1.5);

        {calculate absorbed flux as function of i}
        q [1] :  = pi/4.0*qmax*dr*dr*(1.0 - 0.9/8.0*sqr(dr/Ro));
        q [N]  : = pi*qmax*dr*dr*(N - 1.25-0.9/8.0*sqr(dr/Ro)*(2.0*N*N*N -7.5*N*N
        + 9.5*N - 65.0/16.0));
        for i :  = 2 to N - 1 do
        q [i]  : = pi*qmax*dr*dr*(2.0*(1 - 1.0) - 0.9/4.0*sqr(dr/Ro)*(4.0*i*i*i - 12.0*i*
        i + 13.0*i - 5.0));

        {fill in the coefficient matrices}

        for i : = 1 to N do    {first, zero all of them out}
        for j : = 1 to M do
```

```
begin
       R[i, j]  :  = 0.0;
       L[i, j]  :  = 0.0;
       U[i, j]  :  = 0.0;
       D[i, j]  :  = 0.0;
       C[i, j]  :  = 0.0;
end

for i : = 2 to N - 1 do
for j : = 2 to M - 1 do
begin
       R[i, j]  :  = Aco[i]/dr;
       L[i, j]  :  = Aci[i]/dr;
       U[i, j]  :  = Af[i]/dz;
       D[i, j]  :  = Af[i]/dz;
end;

i : = 1;
for j : = 2 to M - 1 do
begin
       R[i, j] : = Aco[i]/dr;
       U[i, j] : = Af[i]/dz;
       D[i, j] : = Af[i]/dz;
end;

i : = 1;
j : = M;
R[i, j]  : = Aco[i]/dr;
D[i, j]  : = Af[i]/dz;
i : = 1;
j : = 1;
R[i, j]  : = Aco[i]/dr;
U[i, j]  : = Af[i]/dz;

i : = N;
for j : = 2 to M - 1 do
begin
       L[i, j]  : = Aci[i]/dr;
       U[i, j]  : = Af[i]/dz;
       D[i, j]  : = Af[i]/dz;
end;

i : = N;
j : = M;
L[i, j]  : = Aci[i]/dr;
D[i, j]  : = Af[i]/dz;

i : = N;
j : = 1;
L[i, j]  : = Aci[i]/dr;
U[i, j]  : = Af[i]/dz;

j : = M;
for I : = 2 to N - 1 do
begin
       R[i, j]  :  = Aco[i]/dr;
       L[i, j]  :  = Aci[i]/dr;
       D [i, j]  :  = Af[i]/dz;
end;

j : = 1;
for I := 2 to N - 1 do
begin
       R [i, j]  : = Aco[i]/dr;
       L [i, j]  : = Aci[i]/dr;
```

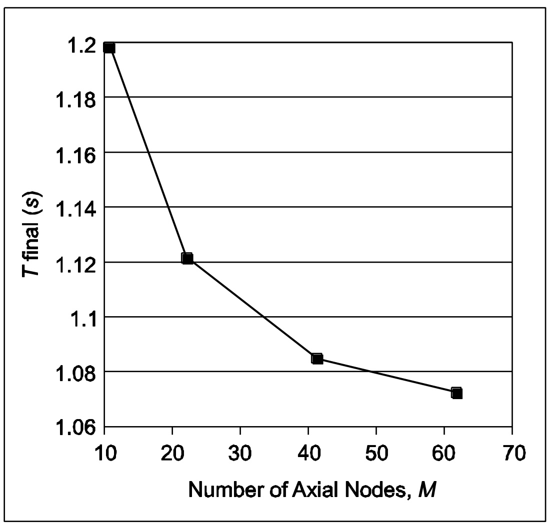
```
        U [i, j]  : = Af[i]/dz;
end;

{find maximum permissible dt}
dtMax : = 0.0;
for i : = 1 to N do
for j : = 1 to M do
begin
        dt : = V[i]/alpha/(R[i, j] + L[i, j] + U[i, j] + D[i, j]);
        if = dt > dtMax then dtMax : = dt;
end;
dt : = 0.5*dtMax;      {actual value to be used in solution}

{fill in the cij matrix}
for i : = 1 to N do C[i, M] : = dt*q [i]/rhoC/V[i];

{establish the initial conditions}
for i : = 1 to N do
for j : = 1 to M do
Told [i, j] : = Tinit;

{carry out the solution}
time : = 0.0;
repeat
        time: = time + dt;
        writeln (time: 10:5);
        for i: = 1 to N do
        for j: = 1 to M do
        Tnew [i,j] : = Told [i,j]*(1.0 - alpha*dt/V[i]*(R[i,j]+ L[i,j]+ U[i,j]+
        D[i,j]))
+ alpha*dt/V[i]*(R[i, j]*Told [i + 1, j] + L[i, j]*Told [i - 1, j] + U[i, j]*Told
[i, j + 1] + D[i, j]*Told [i, j - 1]) + C[i, j];

if Tnew[1, m] > Tmax then     {print out distribution and quit}
begin
        writeln(1st, time : 8 : 4, 'sec dt = ', dt : 15 : 10);
        write(1st,'   ');
        for i : = 1 to N do write (1st, 1 : 10);
        writeln(1st);
        for j : = M downto 1 do
        begin
                write(1st, j : 4);
                for i : = 1 to N do write(1st, Tnew[i, j]; 10 : 5);
                writeln(1st);
        end;
        writeln(1st);
        halt;
end

(otherwise, keep going}
for i : = 1 to N do
for j : = 1 to M do
Told[i, j] ; = Tnew[i, j];

until time < - 1.0;

end.
```

Now, we need to determine the node spacing and Δt required for an accurate solution. As suggested in the text, trial and error is the best method.

First, let's determine the required spatial resolution, that is, the values of N and M. Since the gradients in the radial direction are small compared to those in the axial direction, we don't expect much influence on the results by varying N so we will use N = 11. Now, pick values of M = 11, 21, 41, and 61. A time step of Δt = 0.0003 s will give stable results for all these values of M. The table below gives the results for these 4 runs.

M	Time (s) for T_{max} = 300°C	T [1, 1]	T [N, 1]	T [N, M]
11	1.1979	116.29	109.86	281.97
21	1.1223	115.75	109.32	282.05
41	1.0845	115.49	109.06	282.08
61	1.0719	115.40	108.97	282.09

Notice that the temperatures in the table are not significantly affected by M but the time required to reach 300°C is somewhat sensitive. This is displayed in the graph shown to the right. It appears that the time gradually decreases but between M = 41 and M = 61, the graph levels off significantly. At M = 81, the time would probably be somewhat less than that at M = 61 but clearly we have reached the point of diminishing return. We will choose M = 41 as begin a reasonable compromise.



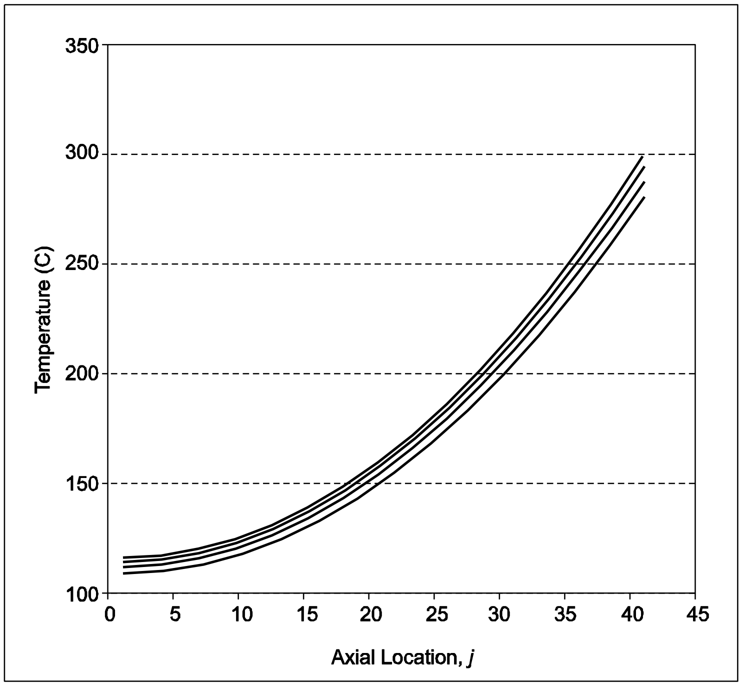
Time Required to Reach a Maximum Temperature of 300°c as a Function of Number of Axial Nodes

Next, we need to determine the appropriate time step, Δt. For N = 11, M = 41, the maximum permissible time step according to the equation given previously is 0.00155 s. In practice, values larger than 1/2 of this maximum result in instability. Running with 0.00015, 0.0003, and 0.0006 seconds, we find

Δt (s)	Time (s) for T_{max} = 300°C	T [1, 1]	T [N, 1]	T [N, M]
0.00015	1.0845	115.488	109.06	282.08
0.0003	1.0845	115.48659	109.06	282.08
0.0006	1.0848	115.52582	109.09	282.13

From the results in the table, it is clear that there is little benefit from a time step of less than 0.0003 s. For a reasonable compromise, choose Δt = 0.0006 s.

Using these choices, M = 41, N = 11, and Δt = 0.0006 s, the results are plotted below



Temperature Distribution along Six Axial Lines

The solution shows that 1.0848 seconds is required for the maximum temperature in the disk to reach 300°C. Furthermore, the graph demonstrates that the temperature gradient axially through the disk is not especially large as was required for the case hardening application. This indicates that the incident flux needs to be increased.

PROBLEM 4.54

A circumferential fin with a rectangular cross section is made of mild steel has an outer diameter of 3.7 cm OD, and a thickness of 0.3 cm. An array of these circular fins 5 are attached to a circular tube with an outer diameter of 2.5 cm as shown. Cooling air blowing over the fin produces a heat transfer coefficient of 28.4 W/(m^2 K). If the temperatures of the base of the fin and of the flowing air are 260°C and 38°C, respectively, numerically determine the temperature distribution in the fin and the heat transfer rate from the fin. Compare your results with those obtained by fin analysis of Chapter 2. Also, how different is the performance if the fins are made of aluminum instead?

GIVEN

- Circumferential fin of mild steel with rectangular cross section.
- Fin outer diameter (d_o)= 3.7 cm=0.037 m
- Fin thickness(t)= 0.3 cm =0.003 m
- Tube outer diameter (d_i)=2.5 cm=0.025 m
- Convection heat transfer coefficient($\bar{h}_c$)= 28.4 W/(m^2 K)
- Temperature of fin base (T_b)= 260^{0}C
- Ambient temperatue (T_∞)= 38^{0}C

FIND

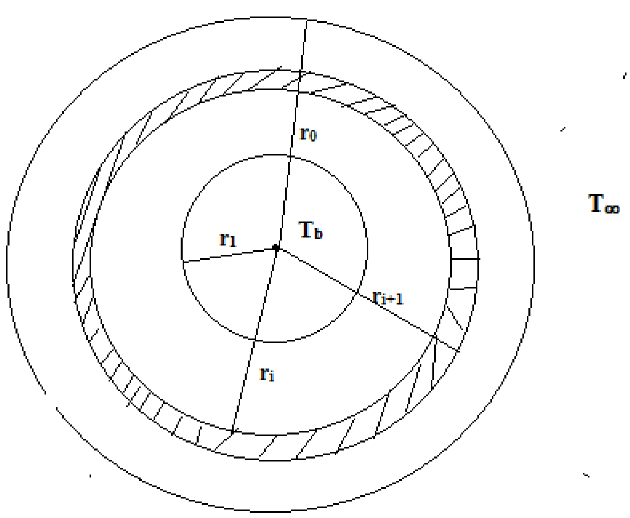
(a) Temperature distribution in fin
(b) Rate of heat transfer in fin
(c) Difference in performance when fins are made of aluminum instead.

PROPERTIES

From Appendix 2 Table

- Thermal conductivity of mild steel(k)= 43 W/(m K)
- Thermal conductivity of Aluminum (k)=238 W/(m K)

SKETCH



SOLUTION

For the node and control volume arrangement shown in the sketch, we have

$$x_i = \Delta x(i-1) \quad i = 1, 2, \ldots, N = 9 \quad \Delta x = \frac{r_0 - r_1}{N-1}$$

For the control volume at i = 1, we have a specified temperature, therefore

$$T_1 = T_{\text{root}}$$

For the interior control volumes, i = 2, 3, 4, 5,… , 8 an energy balance gives

$$\left\{ kA_i \frac{T_{i+1} - T_i}{\Delta r} + kA_{i-1} \frac{T_{i-1} - T_i}{\Delta r} \right\} + 2\pi(r_i^2 - r_{i-1}^2)\, h\,(T_\infty - T_i) = 0$$

Where $A_i = 2\pi r_i t$

Writing this in the tridiagonal form

$$T_i \left[2\pi k r_{i-1} t / dr + 2\pi k r_i t / dr + 2\pi h * (r_i^2 - r_{i-1}^2) \right] =$$

$$\left[2\pi k r_{i-1} t / dr * T_{i-1} + 2\pi k r_i t / dr * T_{i+1} + 2\pi h * (r_i^2 - r_{i-1}^2) * T_\infty \right]$$

=> $$T_i = \frac{\left[k r_{i-1} t / dr * T_{i-1} + k r_i t / dr * T_{i+1} + h * (r_i^2 - r_{i-1}^2) * T_\infty \right]}{\left[k r_{i-1} t / dr + k r_i t / dr + h * (r_i^2 - r_{i-1}^2) \right]}$$

Neglecting heat loss from tip of the fin as fin thickness is very small we have

$T_N = T_{N-1}$

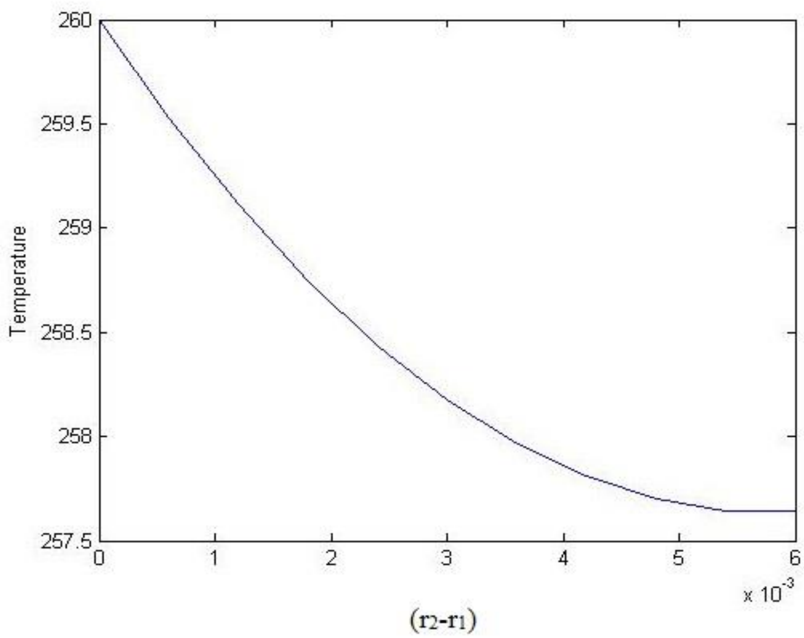
and $T_1 = T_b$ as boundary conditions

The above problem is solved by discretization method using matlab. The matlab code is given below.

```
% Matlab code for problem_4_54

L=6*10^-3;  % in m
N=11;
delr=L/(N-1);     % in m
t=0.003;           % in m
h=28.4;           % W/(m^2 K)
k=43;           % W/(m K)
T(1)=260;         % Celsius
T(2:N)=0;          % Celsius
Tinf=38;           % Celsius
for i=1:N
r(i)=(0.0125+(i-1)*delr);
end

for j=1:1:300
for i=N-1:-1:2
    Tf=T;
T(i)=(k*r(i-1)*t/delr*T(i-1)+k*r(i)*t/delr*T(i+1)+h*(r(i)^2-r(i-
1)^2)*Tinf)/(k*r(i-1)/delr*t+k*r(i)/delr*t+h*(r(i)^2-r(i-1)^2));
T(N)=T(N-1);
```

```
end;
count=0;
for i=1:N
    if abs(Tf(i)-T(i))<10^-5
        count=count+1;
    end

end
    if count==N
        break
    end
end

for i=1:1:N
   x(i)=(i-1)*delr;
end
plot(x,T)
q=k*2*pi*r(1)*t*(T(1)-T(2))/delr
```

The temperature distribution found through above discretization in matlab is given below



The heat flow through the fin is given by conduction through first control volume which is calculated as

$Q= k*2\pi*r_1*t*(T(1)-T(2))/\Delta r$

which is obtained as Q= 8.104 Watt

Now, from chapter 2.

$$\left(r_0+\frac{t}{2}-r_i\right)^{3/2} = 6.5*10^{-4}\ m^{3/2}$$

$\left[2\bar{h}/kt(r_0-r_1)\right]^{1/2}$ =270.9

$\left(r_0+\frac{t}{2}\right)\Big/ r_i$ =2.02

$\left[2\bar{h}/kt(r_0-r_1)\right]^{1/2} * \left(r_0+\frac{t}{2}-r_i\right)^{3/2}$ =0.176

From figure 2.22 for efficiency of circumferential fins

η_f=98%

Thus the heat transfer through the fin is given by

$$Q_f= \eta_f \bar{h} * \pi * \left(\left(r_0+\frac{t}{2}\right)^2 - r_1^2\right) * (T_b - T_\infty)$$

$$= 0.97 * 28.4 * 2\pi * \left(0.02^2 - 0.0125^2\right) * (260-38) \text{ W}$$

=9.36 Watts

which is near to the discretization solution we obtained through matlab. If Aluminum is used instead of mild steel. Substituting the value of thermal conductivity in above matlab code we get

$\left[2\bar{h}/kt(r_0-r_1)\right]^{1/2}$ =115

$\left(r_0+\frac{t}{2}\right)\Big/ r_i$ =2.02

$\left(r_0+\frac{t}{2}-r_i\right)^{3/2}$ =6.5*10^{-4} m$^{3/2}$

$\left[2\bar{h}/kt(r_0-r_1)\right]^{1/2} * \left(r_0+\frac{t}{2}-r_i\right)^{3/2}$ =0.075

η_f=99.5%

$$Q_f= \eta_f \bar{h} * \pi * \left(\left(r_0+\frac{t}{2}\right)^2 - r_1^2\right) * (T_b - T_\infty)$$

$$= 0.995 * 28.4 * 2\pi * \left(0.02^2 - 0.0125^2\right) * (260-38) \text{ W}$$

=9.6 Watts

Q= 9.6 Watt. Thus heat dissipation increases if Aluminum fin is used.

PROBLEM 4.55

Consider two-dimensional steady conduction near a curved boundary. Determine the difference equation for an appropriate control volume near the node (*i*, *j*). The boundary experiences convective heat transfer through a coefficient *h* to ambient temperature T_a. The surface of the boundary is given by $y_s = f(x)$.

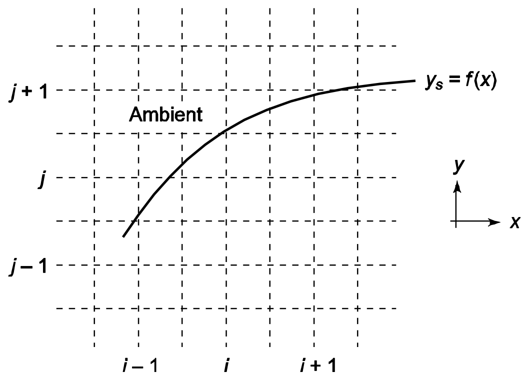
GIVEN

- Two-dimensional steady conduction near a curved surface
- Convective boundary condition
- Curved surface given by $y_s = f(x)$

FIND

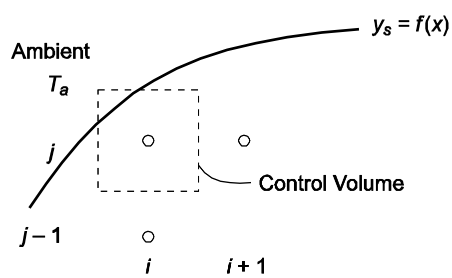
(a) Difference equation for a node *i*, *j* near the surface

SKETCH



SOLUTION

Consider a control volume for the node *i*, *j* as shown below



Heat can flow into or out of the control volume at four surfaces. An energy balance on the control volume is given by

$$k\left\{\frac{T_{i,j-1}-T_{i,j}}{\Delta y}\Delta x+\frac{T_{i+1,j}-T_{i,j}}{\Delta x}\Delta y\right\} + h_c\,\{(T_a - T_{i,j})\,\Delta x + (T_a - T_{i,j})\,\Delta y\} = 0$$

PROBLEM 4.56

Derive the control volume energy balance equation for three-dimensional transient conduction with heat generation in a rectangular coordinate system.

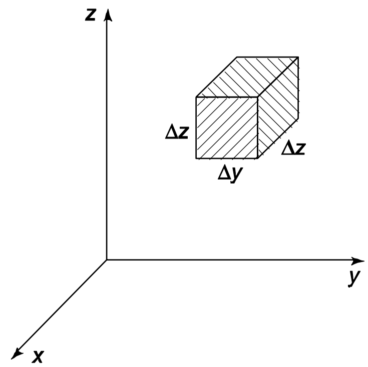
GIVEN

- Three-dimensional transient conduction with heat generation in a rectangular coordinate system

FIND

(a) The control volume energy balance equation

SKETCH



SOLUTION

The control volume, in an x, y, z coordinate system, is shown in the sketch above. Define the nodal indices as follows

$$x = (i - 1)\,\Delta x$$

$$y = (j - 1)\,\Delta y$$

$$z = (l - 1)\,\Delta z$$

and for simplicity define

$$T \equiv T_{i,\,j,\,1,\,m}$$

Now, heat conducted into the control volume is

$$k\Big(\left(T_{i+1,j,l,m} - T + T_{i-1,j,l,m} - T\right) \frac{\Delta y \Delta z}{\Delta x} + \left(T_{i,j+1,l,m} - T + T_{i,j-1,l,m} - T\right) \frac{\Delta x \Delta z}{\Delta y}$$

$$+ \left(T_{i,j,l+1,m} - T + T_{i,j,l-1,m} - T\right) \frac{\Delta x \Delta y}{\Delta z} \Big)$$

The heat generated in the control volume is

$$\dot{q}_{G,i,j,1,m}\ \Delta x\ \Delta y\ \Delta z$$

and the rate at which thermal energy is stored in the control volume is

$$\rho\, c\ \Delta x\ \Delta y\ \frac{T_{i,j,l,m+1} - T}{\Delta t}$$

Since the heat conducted into the control volume plus the rate at which heat is generated in the control volume must equal the rate at which energy is stored in the control volume, the difference equation is

$$k\left(\left(T_{i+1,j,l,m} - 2T + T_{i-1,j,l,m}\right) \frac{\Delta y \Delta z}{\Delta x} + \left(T_{i,j+1,l,m} - 2T + T_{i,j-1,l,m}\right) \frac{\Delta x \Delta z}{\Delta y} \right.$$

$$\left. + \left(T_{i,j,l+1,m} - 2T + T_{i,j,l-1,m}\right) \frac{\Delta x \Delta y}{\Delta z} \right)$$

$$+ \dot{q}_{G,i,j,1,m}\, \Delta x\, \Delta y\, \Delta z = \rho\, c\, \Delta x\, \Delta y\, \Delta z\, \frac{T_{i,j,l,m+1} - T}{\Delta t}$$

Dividing by $k\, \Delta x\, \Delta y\, \Delta z$ we have

$$\frac{T_{i+1,j,l,m} - 2T + T_{i-1,j,l,m}}{\Delta x^2} + \frac{T_{i,j+1,l,m} - 2T + T_{i,j-1,l,m}}{\Delta y^2} + \frac{T_{i,j,l+1,m} - 2T + T_{i,j,l-1,m}}{\Delta z^2}$$

$$+ \frac{\dot{q}_{G,i,j,l,m}}{k} = \frac{1}{\alpha} \frac{T_{i,j,l,m+1} - T}{\Delta t}$$

PROBLEM 4.57

Derive the energy balance equation for a corner control volume in a three-dimensional steady conduction problem with heat generation in a rectangular coordinate system. Assume an adiabatic boundary condition and equal node spacing in all three dimensions.

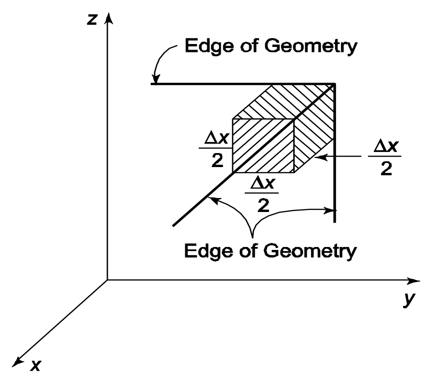
GIVEN

- Three-dimensional steady conduction in a rectangular coordinate system, corner boundary control volume with specified temperature boundary condition

FIND

(a) Energy balance equation for the control volume

SKETCH



SOLUTION

First, define the nodal indices as follows

$$x = (i-1)\,\Delta x \qquad y = (j-1)\,\Delta y \qquad z = (l-1)\,\Delta z$$

and for simplicity, let

$$T \equiv T_{i,j,l,m}$$

where, as usual, m is the time index. Note that the volume of the control volume is

$$\frac{\Delta x \Delta y \Delta z}{8}$$

Referring to the sketch above, we see that there are three surfaces across which heat is transferred by conduction. For these surfaces, the heat transferred into the control volume is

$$k\left\{\frac{T_{i+1,j,l,m}-T}{\Delta x}\frac{\Delta y \Delta z}{4}+\frac{T_{i,j-1,l,m}-T}{\Delta y}\frac{\Delta x \Delta z}{4}+\frac{T_{i,j,l-1,m}-T}{\Delta z}\frac{\Delta y \Delta x}{4}\right\}$$

Heat generation in the control volume is $\quad \dot{q}_{G,i,j,l,m}\dfrac{\Delta x \Delta y \Delta z}{8}$

and the rate at which energy is stored in the control volume is $\quad \rho\, c\, \dfrac{T_{i,j,l,m+1}-T}{\Delta t}\dfrac{\Delta x \Delta y \Delta z}{8}$

The resulting energy balance equation for the control volume is

$$\frac{T_{i+1,j,l,m}+T_{i,j-1,l,m}+T_{i,j,l-1,m}-3T}{4\Delta x^2}+\frac{\dot{q}_{G,i,j,l,m}}{8k}=\frac{1}{8\alpha}\frac{T_{i,j,l,m+1}-T}{\Delta t}$$

PROBLEM 4.58

Determine the stability criterion for an explicit solution of three-dimensional transient conduction in a rectangular geometry.

GIVEN

- Three-dimensional transient conduction in a rectangular geometry

FIND

(a) The stability criterion for an explicit situation

SOLUTION

From the solution of Problem 3.49, the control volume energy balance equation is

$$\frac{T_{i+1,j,l,m} - 2T + T_{i-1,j,l,m}}{\Delta x^2} + \frac{T_{i,j+1,l,m} - 2T + T_{i,j-1,l,m}}{\Delta y^2} + \frac{T_{i,j,l+1,m} - 2T + T_{i,j,l-1,m}}{\Delta z^2}$$

$$+ \frac{\dot{q}_{G,i,j,l,m}}{k} = \frac{1}{\alpha}\frac{T_{i,j,l,m+1} - T}{\Delta t}$$

so the coefficient on T is

$$-2\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2}\right) + \frac{1}{\alpha \Delta t}$$

Since this coefficient must be greater than zero to ensure stability, we have

$$\Delta t < \frac{1}{2\alpha}\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2}\right)^{-1}$$

Note that this expression is consistent with the extension from one-dimensional to two-dimensional stability criteria.