CHAPTER 12

12.1 Reorder so that equations are diagonally dominant:

$$6x_1 - x_2 = 5$$
$$3x_1 + 8x_2 = 11$$

First iteration:

$$x_1 = \frac{5 + x_2}{6} = \frac{5+0}{6} = 0.8333$$
$$x_{1,r} = \lambda x_1 + (1-\lambda)x_1^{old} = 1.25(0.8333) - 0.25(0) = 1.041667$$
$$x_2 = \frac{11 - 3x_1}{8} = \frac{11 - 3(1.041667)}{8} = 0.984375$$
$$x_{2,r} = \lambda x_2 + (1-\lambda)x_2^{old} = 1.25(0.984375) - 0.25(0) = 1.2305$$

$$\varepsilon_{a,1} = \left|\frac{1.041667 - 0}{1.041667}\right| \times 100\% = 100\% \qquad \varepsilon_{a,2} = \left|\frac{1.2305 - 0}{1.2305}\right| \times 100\% = 100\%$$

Second iteration:

$$x_1 = \frac{5 + 1.2305}{6} = 1.0384$$
$$x_{1,r} = 1.25(1.0384) - 0.25(1.041667) = 1.0376$$
$$x_2 = \frac{11 - 3(1.2305)}{8} = 0.9859$$
$$x_{2,r} = 1.25(0.9859) - 0.25(1.2305) = 0.92476$$
$$\varepsilon_{a,1} = \left|\frac{1.0376 - 1.041667}{1.0376}\right| \times 100\% = 0.39\% \qquad \varepsilon_{a,2} = \left|\frac{0.92476 - 1.2305}{0.92476}\right| \times 100\% = 33.06\%$$

Third iteration:

$$x_1 = \frac{5 + 0.924759}{6} = 0.98746$$
$$x_{1,r} = 1.25(0.98746) - 0.25(1.037598) = 0.974925$$
$$x_2 = \frac{11 - 3(0.974925)}{8} = 1.009403$$
$$x_{2,r} = 1.25(1.009403) - 0.25(0.924759) = 1.030564$$

$$\varepsilon_{a,1} = \left|\frac{0.974925 - 1.037598}{0.974925}\right| \times 100\% = 6.43\% \qquad \varepsilon_{a,2} = \left|\frac{1.030564 - 0.924759}{1.030564}\right| \times 100\% = 10.27\%$$

The exact solution is $x_1 = 1$ and $x_2 = 1$ and the true errors are

$$\varepsilon_{t,1} = \left|\frac{1 - 0.974925}{1}\right| \times 100\% = 2.51\% \qquad \varepsilon_{t,2} = \left|\frac{1 - 1.030564}{1}\right| \times 100\% = 3.06\%$$

12.2 The first iteration can be implemented as

$$x_1 = \frac{41 + 0.4x_2}{0.8} = \frac{41 + 0.4(0)}{0.8} = 51.25$$

$$x_2 = \frac{25 + 0.4x_1 + 0.4x_3}{0.8} = \frac{25 + 0.4(51.25) + 0.4(0)}{0.8} = 56.875$$

$$x_3 = \frac{105 + 0.4x_2}{0.8} = \frac{105 + 0.4(56.875)}{0.8} = 159.6875$$

Second iteration:

$$x_1 = \frac{41 + 0.4(56.875)}{0.8} = 79.6875$$

$$x_2 = \frac{25 + 0.4(79.6875) + 0.4(159.6875)}{0.8} = 150.9375$$

$$x_3 = \frac{105 + 0.4(150.9375)}{0.8} = 206.7188$$

The error estimates can be computed as

$$\varepsilon_{a,1} = \left| \frac{79.6875 - 51.25}{79.6875} \right| \times 100\% = 35.69\%$$

$$\varepsilon_{a,2} = \left| \frac{150.9375 - 56.875}{150.9375} \right| \times 100\% = 62.32\%$$

$$\varepsilon_{a,3} = \left| \frac{206.7188 - 159.6875}{206.7188} \right| \times 100\% = 22.75\%$$

The remainder of the calculation proceeds until all the errors fall below the stopping criterion of 5%. The entire computation can be summarized as

iteration	unknown	value	ε_a	maximum ε_a
1	x_1	51.25	100.00%	
	x_2	56.875	100.00%	
	x_3	159.6875	100.00%	100.00%
2	x_1	79.6875	35.69%	
	x_2	150.9375	62.32%	
	x_3	206.7188	22.75%	62.32%
3	x_1	126.7188	37.11%	
	x_2	197.9688	23.76%	
	x_3	230.2344	10.21%	37.11%
4	x_1	150.2344	15.65%	
	x_2	221.4844	10.62%	
	x_3	241.9922	4.86%	15.65%
5	x_1	161.9922	7.26%	
	x_2	233.2422	5.04%	
	x_3	247.8711	2.37%	7.26%
6	x_1	167.8711	3.50%	
	x_2	239.1211	2.46%	
	x_3	250.8105	1.17%	3.50%

Thus, after 6 iterations, the maximum error is 3.5% and we arrive at the result: x_1 = 167.8711, x_2 = 239.1211 and x_3 = 250.8105.

(b) The same computation can be developed with relaxation where λ = 1.2.

<u>First iteration:</u>

$$x_1 = \frac{41 + 0.4x_2}{0.8} = \frac{41 + 0.4(0)}{0.8} = 51.25$$

Relaxation yields: $x_1 = 1.2(51.25) - 0.2(0) = 61.5$

$$x_2 = \frac{25 + 0.4x_1 + 0.4x_3}{0.8} = \frac{25 + 0.4(61.5) + 0.4(0)}{0.8} = 62$$

Relaxation yields: $x_2 = 1.2(62) - 0.2(0) = 74.4$

$$x_3 = \frac{105 + 0.4x_2}{0.8} = \frac{105 + 0.4(62)}{0.8} = 168.45$$

Relaxation yields: $x_3 = 1.2(168.45) - 0.2(0) = 202.14$

<u>Second iteration:</u>

$$x_1 = \frac{41 + 0.4(62)}{0.8} = 88.45$$

Relaxation yields: $x_1 = 1.2(88.45) - 0.2(61.5) = 93.84$

$$x_2 = \frac{25 + 0.4(93.84) + 0.4(202.14)}{0.8} = 179.24$$

Relaxation yields: $x_2 = 1.2(179.24) - 0.2(74.4) = 200.208$

$$x_3 = \frac{105 + 0.4(200.208)}{0.8} = 231.354$$

Relaxation yields: $x_3 = 1.2(231.354) - 0.2(202.14) = 237.1968$

The error estimates can be computed as

$$\varepsilon_{a,1} = \left| \frac{93.84 - 61.5}{93.84} \right| \times 100\% = 34.46\%$$
$$\varepsilon_{a,2} = \left| \frac{200.208 - 74.4}{200.208} \right| \times 100\% = 62.84\%$$
$$\varepsilon_{a,3} = \left| \frac{237.1968 - 202.14}{237.1968} \right| \times 100\% = 14.78\%$$

The remainder of the calculation proceeds until all the errors fall below the stopping criterion of 5%. The entire computation can be summarized as

iteration	unknown	value	relaxation	ε_a	maximum ε_a
1	x_1	51.25	61.5	100.00%	
	x_2	62	74.4	100.00%	
	x_3	168.45	202.14	100.00%	100.000%
2	x_1	88.45	93.84	34.46%	
	x_2	179.24	200.208	62.84%	
	x_3	231.354	237.1968	14.78%	62.839%
3	x_1	151.354	162.8568	42.38%	
	x_2	231.2768	237.49056	15.70%	
	x_3	249.99528	252.55498	6.08%	42.379%
4	x_1	169.99528	171.42298	5.00%	
	x_2	243.23898	244.38866	2.82%	
	x_3	253.44433	253.6222	0.42%	4.997%

Thus, relaxation speeds up convergence. After 4 iterations, the maximum error is 4.997% and we arrive at the result: x_1 = 171.423, x_2 = 244.389 and x_3 = 253.622.

12.3 The first iteration can be implemented as

$$x_1 = \frac{27 - 2x_2 + x_3}{10} = \frac{27 - 2(0) + 0}{10} = 2.7$$

$$x_2 = \frac{-61.5 + 3x_1 - 2x_3}{-6} = \frac{-61.5 + 3(2.7) - 2(0)}{-6} = 8.9$$

$$x_3 = \frac{-21.5 - x_1 - x_2}{5} = \frac{-21.5 - (2.7) - 8.9}{5} = -6.62$$

Second iteration:

$$x_1 = \frac{27 - 2(8.9) - 6.62}{10} = 0.258$$

$$x_2 = \frac{-61.5 + 3(0.258) - 2(-6.62)}{-6} = 7.914333$$

$$x_3 = \frac{-21.5 - (0.258) - 7.914333}{5} = -5.934467$$

The error estimates can be computed as

$$\varepsilon_{a,1} = \left|\frac{0.258 - 2.7}{0.258}\right| \times 100\% = 947\%$$

$$\varepsilon_{a,2} = \left|\frac{7.914333 - 8.9}{7.914333}\right| \times 100\% = 12.45\%$$

$$\varepsilon_{a,3} = \left|\frac{-5.934467 - (-6.62)}{-5.934467}\right| \times 100\% = 11.55\%$$

Note that because $\varepsilon_{a,1}$ is greater than the stopping criterion, the computation of $\varepsilon_{a,2}$ and $\varepsilon_{a,3}$ is unnecessary for this iteration. The remainder of the calculation proceeds until all the errors fall below the stopping criterion of 5%. The entire computation can be summarized as

iteration	unknown	value	ε_a	maximum ε_a
1	x_1	2.7	100.00%	
	x_2	8.9	100.00%	
	x_3	-6.62	100.00%	100%
2	x_1	0.258	946.51%	
	x_2	7.914333	12.45%	
	x_3	-5.93447	11.55%	946%
3	x_1	0.523687	50.73%	
	x_2	8.010001	1.19%	
	x_3	-6.00674	1.20%	50.73%
4	x_1	0.497326	5.30%	
	x_2	7.999091	0.14%	
	x_3	-5.99928	0.12%	5.30%
5	x_1	0.500253	0.59%	
	x_2	8.000112	0.01%	
	x_3	-6.00007	0.01%	0.59%

Thus, after 5 iterations, the maximum error is 0.59% and we arrive at the result: x_1 = 0.500253, x_2 = 8.000112 and x_3 = –6.00007.

12.4 The first iteration can be implemented as

$$x_1 = \frac{27 - 2x_2 + x_3}{10} = \frac{27 - 2(0) + 0}{10} = 2.7$$

$$x_2 = \frac{-61.5 + 3x_1 - 2x_3}{-6} = \frac{-61.5 + 3(0) - 2(0)}{-6} = 10.25$$

$$x_3 = \frac{-21.5 - x_1 - x_2}{5} = \frac{-21.5 - 0 - 0}{5} = -4.3$$

Second iteration:

$$x_1 = \frac{27 - 2(10.25) - 4.3}{10} = 0.22$$

$$x_2 = \frac{-61.5 + 3(2.7) - 2(-4.3)}{-6} = 7.466667$$

$$x_3 = \frac{-21.5 - (2.7) - 10.25}{5} = -6.89$$

The error estimates can be computed as

$$\varepsilon_{a,1} = \left|\frac{0.22 - 2.7}{0.258}\right| \times 100\% = 1127\%$$

$$\varepsilon_{a,2} = \left|\frac{7.466667 - 10.25}{7.466667}\right| \times 100\% = 37.28\%$$

$$\varepsilon_{a,3} = \left|\frac{-6.89 - (-4.3)}{-6.89}\right| \times 100\% = 37.59\%$$

Note that because $\varepsilon_{a,1}$ is greater than the stopping criterion, the computation of $\varepsilon_{a,2}$ and $\varepsilon_{a,3}$ is unnecessary for this iteration. The remainder of the calculation proceeds until all the errors fall below the stopping criterion of 5%. The entire computation can be summarized as

iteration	unknown	value	ε_a	maximum ε_a
1	x_1	2.7		
	x_2	10.25		
	x_3	-4.3		
2	x_1	0.22	1127.27%	
	x_2	7.466667	37.28%	
	x_3	-6.89	37.59%	1127.27%
3	x_1	0.517667	57.50%	
	x_2	7.843333	4.80%	
	x_3	-5.83733	18.03%	57.50%
4	x_1	0.5476	5.47%	
	x_2	8.045389	2.51%	
	x_3	-5.9722	2.26%	5.47%
5	x_1	0.493702	10.92%	
	x_2	7.985467	0.75%	
	x_3	-6.0186	0.77%	10.92%
6	x_1	0.501047	1.47%	
	x_2	7.99695	0.14%	
	x_3	-5.99583	0.38%	1.47%

Thus, after 6 iterations, the maximum error is 1.47% and we arrive at the result: x_1 = 0.501047, x_2 = 7.99695 and x_3 = –5.99583.

12.5 The first iteration can be implemented as

$$c_1 = \frac{3800 + 3c_2 + c_3}{15} = \frac{3800 + 3(0) + 0}{15} = 253.3333$$

$$c_2 = \frac{1200 + 3c_1 + 6c_3}{18} = \frac{1200 + 3(253.3333) + 6(0)}{18} = 108.8889$$

$$c_3 = \frac{2350 + 4c_1 + c_2}{12} = \frac{2350 + 4(253.3333) + 108.8889}{12} = 289.3519$$

Second iteration:

$$c_1 = \frac{3800 + 3(108.889) + 289.3519}{15} = 294.4012$$

$$c_2 = \frac{1200 + 3(294.4012) + 6(289.3519)}{18} = 212.1842$$

$$c_3 = \frac{2350 + 4(294.4012) + 212.1842}{12} = 311.6491$$

The error estimates can be computed as

$$\varepsilon_{a,1} = \left|\frac{294.4012 - 253.3333}{294.4012}\right| \times 100\% = 13.95\%$$

$$\varepsilon_{a,2} = \left|\frac{212.1842 - 108.8889}{212.1842}\right| \times 100\% = 48.68\%$$

$$\varepsilon_{a,3} = \left|\frac{311.6491 - 289.3519}{311.6491}\right| \times 100\% = 7.15\%$$

Note that because $\varepsilon_{a,1}$ is greater than the stopping criterion, the computation of $\varepsilon_{a,2}$ and $\varepsilon_{a,3}$ is unnecessary for this iteration. The remainder of the calculation can be summarized as

iteration	unknown	value	ε_a	maximum ε_a
1	x_1	253.3333		
	x_2	108.8889		
	x_3	289.3519		
2	x_1	294.4012	13.95%	
	x_2	212.1842	48.68%	
	x_3	311.6491	7.15%	48.68%
3	x_1	316.5468	7.00%	
	x_2	223.3075	4.98%	
	x_3	319.9579	2.60%	7.00%
4	x_1	319.3254	0.87%	
	x_2	226.5402	1.43%	
	x_3	321.1535	0.37%	1.43%

We can stop at this point as all the approximate errors are less than the 5% stopping criterion. Note that after several more iterations, we arrive at the result: x_1 = 320.2073, x_2 = 227.2021 and x_3 = 321.5026.

12.6 The equations must first be rearranged so that they are diagonally dominant

$$-8x_1 + x_2 - 2x_3 = -20$$
$$2x_1 - 6x_2 - x_3 = -38$$
$$-3x_1 - x_2 + 7x_3 = -34$$

(a) The first iteration can be implemented as

$$x_1 = \frac{-20 - x_2 + 2x_3}{-8} = \frac{-20 - 0 + 2(0)}{-8} = 2.5$$
$$x_2 = \frac{-38 - 2x_1 + x_3}{-6} = \frac{-38 - 2(2.5) + 0}{-6} = 7.166667$$
$$x_3 = \frac{-34 + 3x_1 + x_2}{7} = \frac{-34 + 3(2.5) + 7.166667}{7} = -2.761905$$

Second iteration:

$$x_1 = \frac{-20 - 7.166667 + 2(-2.761905)}{-8} = 4.08631$$
$$x_2 = \frac{-38 - 2x_1 + x_3}{-6} = \frac{-38 - 2(4.08631) + (-2.761905)}{-6} = 8.155754$$
$$x_3 = \frac{-34 + 3x_1 + x_2}{7} = \frac{-34 + 3(4.08631) + 8.155754}{7} = -1.94076$$

The error estimates can be computed as

$$\varepsilon_{a,1} = \left|\frac{4.08631 - 2.5}{4.08631}\right| \times 100\% = 38.82\%$$
$$\varepsilon_{a,2} = \left|\frac{8.155754 - 7.166667}{8.155754}\right| \times 100\% = 12.13\%$$

$$\varepsilon_{a,3} = \left| \frac{-1.94076 - (-2.761905)}{-1.94076} \right| \times 100\% = 42.31\%$$

The remainder of the calculation proceeds until all the errors fall below the stopping criterion of 5%. The entire computation can be summarized as

iteration	unknown	value	ε_a	maximum ε_a
0	x_1	0		
	x_2	0		
	x_3	0		
1	x_1	2.5	100.00%	
	x_2	7.166667	100.00%	
	x_3	-2.7619	100.00%	100.00%
2	x_1	4.08631	38.82%	
	x_2	8.155754	12.13%	
	x_3	-1.94076	42.31%	42.31%
3	x_1	4.004659	2.04%	
	x_2	7.99168	2.05%	
	x_3	-1.99919	2.92%	2.92%

Thus, after 3 iterations, the maximum error is 2.92% and we arrive at the result: x_1 = 4.004659, x_2 = 7.99168 and x_3 = −1.99919.

(b) The same computation can be developed with relaxation where λ = 1.2.

First iteration:

$$x_1 = \frac{-20 - x_2 + 2x_3}{-8} = \frac{-20 - 0 + 2(0)}{-8} = 2.5$$

Relaxation yields: $x_1 = 1.2(2.5) - 0.2(0) = 3$

$$x_2 = \frac{-38 - 2x_1 + x_3}{-6} = \frac{-38 - 2(3) + 0}{-6} = 7.333333$$

Relaxation yields: $x_2 = 1.2(7.333333) - 0.2(0) = 8.8$

$$x_3 = \frac{-34 + 3x_1 + x_2}{7} = \frac{-34 + 3(3) + 8.8}{7} = -2.3142857$$

Relaxation yields: $x_3 = 1.2(-2.3142857) - 0.2(0) = -2.7771429$

Second iteration:

$$x_1 = \frac{-20 - x_2 + 2x_3}{-8} = \frac{-20 - 8.8 + 2(-2.7771429)}{-8} = 4.2942857$$

Relaxation yields: $x_1 = 1.2(4.2942857) - 0.2(3) = 4.5531429$

$$x_2 = \frac{-38 - 2x_1 + x_3}{-6} = \frac{-38 - 2(4.5531429) - 2.7771429}{-6} = 8.3139048$$

Relaxation yields: $x_2 = 1.2(8.3139048) - 0.2(8.8) = 8.2166857$

$$x_3 = \frac{-34 + 3x_1 + x_2}{7} = \frac{-34 + 3(4.5531429) + 8.2166857}{7} = -1.7319837$$

Relaxation yields: $x_3 = 1.2(-1.7319837) - 0.2(-2.7771429) = -1.5229518$

The error estimates can be computed as

$$\varepsilon_{a,1} = \left|\frac{4.5531429 - 3}{4.5531429}\right| \times 100\% = 34.11\%$$
$$\varepsilon_{a,2} = \left|\frac{8.2166857 - 8.8}{8.2166857}\right| \times 100\% = 7.1\%$$
$$\varepsilon_{a,3} = \left|\frac{-1.5229518 - (-2.7771429)}{-1.5229518}\right| \times 100\% = 82.35\%$$

The remainder of the calculation proceeds until all the errors fall below the stopping criterion of 5%. The entire computation can be summarized as

iteration	unknown	value	relaxation	ε_a	maximum ε_a
1	x_1	2.5	3	100.00%	
	x_2	7.3333333	8.8	100.00%	
	x_3	-2.314286	-2.777143	100.00%	100.000%
2	x_1	4.2942857	4.5531429	34.11%	
	x_2	8.3139048	8.2166857	7.10%	
	x_3	-1.731984	-1.522952	82.35%	82.353%
3	x_1	3.9078237	3.7787598	20.49%	
	x_2	7.8467453	7.7727572	5.71%	
	x_3	-2.12728	-2.248146	32.26%	32.257%
4	x_1	4.0336312	4.0846055	7.49%	
	x_2	8.0695595	8.12892	4.38%	
	x_3	-1.945323	-1.884759	19.28%	19.280%
5	x_1	3.9873047	3.9678445	2.94%	
	x_2	7.9700747	7.9383056	2.40%	
	x_3	-2.022594	-2.050162	8.07%	8.068%
6	x_1	4.0048286	4.0122254	1.11%	
	x_2	8.0124354	8.0272613	1.11%	
	x_3	-1.990866	-1.979007	3.60%	3.595%

Thus, relaxation actually seems to retard convergence. After 6 iterations, the maximum error is 3.595% and we arrive at the result: $x_1 = 4.0122254$, $x_2 = 8.0272613$ and $x_3 = -1.979007$.

12.7 As ordered, none of the sets are guaranteed to converge. However, if Set 1 and 2 are reordered so that they are diagonally dominant, they will converge on the solution of (1, 1, 1).

Set 1:
$$\begin{aligned} 8x + 3y + z &= 12 \\ 2x + 4y - z &= 5 \\ -6x \quad\quad + 7z &= 1 \end{aligned}$$

Set 2:
$$\begin{aligned} 3x + y - z &= 3 \\ x + 4y - z &= 4 \\ x + y + 5z &= 7 \end{aligned}$$

Because it is not diagonally dominant, Set 3 will not converge on the correct solution of (1, 1, 1). For example, if they are ordered as

$$\begin{aligned} -2x + 4y - 5z &= -3 \\ 2y - z &= 1 \\ -x + 3y + 5z &= 7 \end{aligned}$$

For this case, Gauss-Seidel iterations yields

iteration	unknown	value	ε_a	maximum ε_a
1	x_1	1.5	100.00%	
	x_2	0.5	100.00%	
	x_3	1.4	100.00%	100.00%
2	x_1	-1	250.00%	
	x_2	1.2	58.33%	
	x_3	0.48	191.67%	250.00%
3	x_1	2.7	137.04%	
	x_2	0.74	62.16%	
	x_3	1.496	67.91%	137.04%
4	x_1	-0.76	455.26%	
	x_2	1.248	40.71%	
	x_3	0.4992	199.68%	455.26%
5	x_1	2.748	127.66%	
	x_2	0.7496	66.49%	
	x_3	1.49984	66.72%	127.66%
6	x_1	-0.7504	466.20%	
	x_2	1.24992	40.03%	
	x_3	0.499968	199.99%	466.20%
7	x_1	2.74992	127.29%	
	x_2	0.749984	66.66%	
	x_3	1.499994	66.67%	127.29%
8	x_1	-0.75002	466.65%	
	x_2	1.249997	40.00%	
	x_3	0.499999	200.00%	466.65%

Alternatively, they can be ordered as

$$\begin{aligned} -x + 3y + 5z &= 7 \\ 2y - z &= 1 \\ -2x + 4y - 5z &= -3 \end{aligned}$$

For this case, Gauss-Seidel iterations yields

iteration	unknown	value	ε_a	maximum ε_a
1	x_1	-7	100.00%	
	x_2	0.5	100.00%	
	x_3	3.8	100.00%	100.00%
2	x_1	13.5	151.85%	
	x_2	2.4	79.17%	
	x_3	-2.88	231.94%	231.94%
3	x_1	-14.2	195.07%	
	x_2	-0.94	355.32%	
	x_3	5.528	152.10%	355.32%
4	x_1	17.82	179.69%	
	x_2	3.264	128.80%	
	x_3	-3.9168	241.14%	241.14%

5	x_1	-16.792	206.12%	
	x_2	-1.4584	323.81%	
	x_3	6.15008	163.69%	323.81%
6	x_1	19.3752	186.67%	
	x_2	3.57504	140.79%	
	x_3	-4.29005	243.36%	243.36%
7	x_1	-17.7251	209.31%	
	x_2	-1.64502	317.32%	
	x_3	6.374029	167.31%	317.32%
8	x_1	19.93507	188.91%	
	x_2	3.687014	144.62%	
	x_3	-4.42442	244.06%	244.06%

12.8 The equations to be solved are

$$f_1(x,y) = -x^2 + x + 0.5 - y$$
$$f_2(x,y) = x^2 - y - 5xy$$

The partial derivatives can be computed and evaluated at the initial guesses

$$\frac{\partial f_{1,0}}{\partial x} = -2x + 1 \;= -2(1.2) + 1 = -1.4 \qquad \frac{\partial f_{1,0}}{\partial y} = -1$$
$$\frac{\partial f_{2,0}}{\partial x} = 2x - 5y \;= 2(1.2) - 5(1.2) = -3.6 \qquad \frac{\partial f_{2,0}}{\partial y} = -1 - 5x = -1 - 5(1.2) = -7$$

They can then be used to compute the determinant of the Jacobian for the first iteration is

$$-1.4(-7) - (-1)(-3.6) = 6.2$$

The values of the functions can be evaluated at the initial guesses as

$$f_{1,0} = -1.2^2 + 1.2 + 0.5 - 1.2 = -0.94$$
$$f_{2,0} = 1.2^2 - 5(1.2)(1.2) - 1.2 = -6.96$$

These values can be substituted into Eq. (12.12) to give

$$x_1 = 1.2 - \frac{-0.94(-3.6) - (-6.96)(-1)}{6.2} = 1.26129$$
$$x_2 = 1.2 - \frac{-6.96(-1.4) - (-0.94)(-3.6)}{6.2} = 0.174194$$

The computation can be repeated until an acceptable accuracy is obtained. The results are summarized as

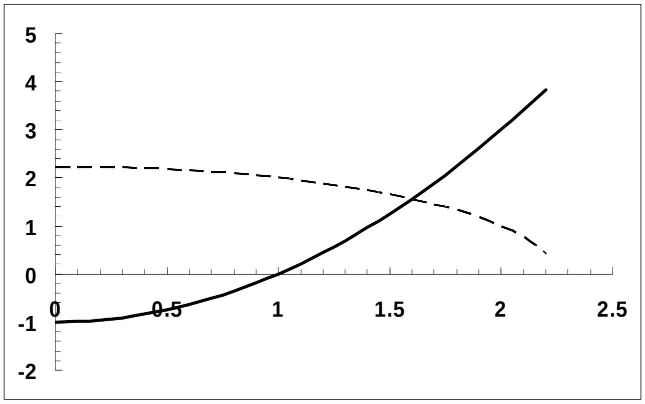
iteration	x	y	ε_{a1}	ε_{a2}
0	1.2	1.2		
1	1.26129	0.174194	4.859%	588.889%
2	1.234243	0.211619	2.191%	17.685%
3	1.233319	0.212245	0.075%	0.295%
4	1.233318	0.212245	0.000%	0.000%

12.9 **(a)** The equations can be set up in a form amenable to plotting as

$y = x^2 - 1$

$y = \sqrt{5 - x^2}$

These can be plotted as



Thus, a solution seems to lie at about $x = y = 1.6$.

(b) The equations can be solved in a number of different ways. For example, the first equation can be solved for x and the second solved for y. For this case, successive substitution does not work

First iteration:

$x = \sqrt{5 - y^2} = \sqrt{5 - (1.5)^2} = 1.658312$

$y = (1.658312)^2 - 1 = 1.75$

Second iteration:

$x = \sqrt{5 - (1.75)^2} = 1.391941$

$y = (1.391941)^2 - 1 = 0.9375$

Third iteration:

$x = \sqrt{5 - (0.9375)^2} = 2.030048$

$y = (2.030048)^2 - 1 = 3.12094$

Thus, the solution is moving away from the solution that lies at approximately $x = y = 1.6$.

An alternative solution involves solving the second equation for x and the first for y. For this case, successive substitution does work

First iteration:

$x = \sqrt{y + 1} = \sqrt{1.5 + 1} = 1.581139$

$y = \sqrt{5 - x^2} = \sqrt{5 - (1.581139)^2} = 1.581139$

Second iteration:

$$x = \sqrt{1.581139} = 1.606592$$
$$y = \sqrt{5-(1.606592)^2} = 1.555269$$

Third iteration:

$$x = \sqrt{5-(1.555269)^2} = 1.598521$$
$$y = (1.598521)^2 - 1 = 1.563564$$

After several more iterations, the calculation converges on the solution of x = 1.600485 and y = 1.561553.

(c) The equations to be solved are

$$f_1(x, y) = x^2 - y - 1$$
$$f_2(x, y) = 5 - y^2 - x^2$$

The partial derivatives can be computed and evaluated at the initial guesses

$$\frac{\partial f_{1,0}}{\partial x} = 2x \qquad \frac{\partial f_{1,0}}{\partial y} = -1$$
$$\frac{\partial f_{2,0}}{\partial x} = -2x \qquad \frac{\partial f_{2,0}}{\partial y} = -2y$$

They can then be used to compute the determinant of the Jacobian for the first iteration is

$$-1.4(-7) - (-1)(-3.6) = 6.2$$

The values of the functions can be evaluated at the initial guesses as

$$f_{1,0} = -1.2^2 + 1.2 + 0.5 - 1.2 = -0.94$$
$$f_{2,0} = 1.2^2 - 5(1.2)(1.2) - 1.2 = -6.96$$

These values can be substituted into Eq. (12.12) to give

$$x_1 = 1.2 - \frac{-0.94(-3.6) - (-6.96)(-1)}{6.2} = 1.26129$$
$$x_2 = 1.2 - \frac{-6.96(-1.4) - (-0.94)(-3.6)}{6.2} = 0.174194$$

The computation can be repeated until an acceptable accuracy is obtained. The results are summarized as

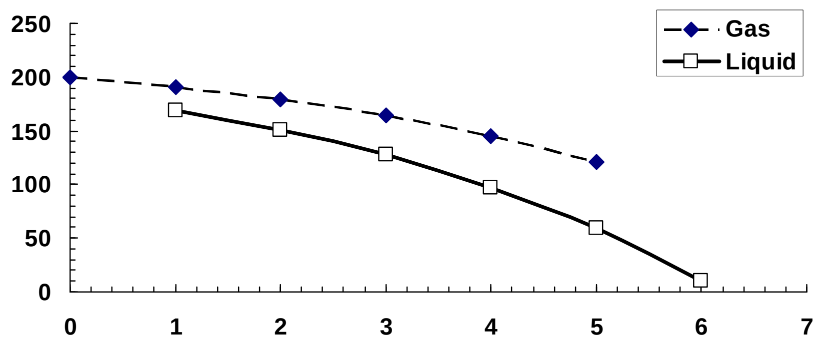
iteration	x_1	x_2	ε_{a1}	ε_{a2}
0	1.5	1.5		
1	1.604167	1.5625	6.494%	4.000%
2	1.600489	1.561553	0.230%	0.061%
3	1.600485	1.561553	0.000%	0.000%

12.10 The mass balances can be expressed in matrix form as

$$\begin{bmatrix} 2.8 & 0 & 0 & 0 & 0 & -0.8 & 0 & 0 & 0 & 0 \\ -2 & 2.8 & 0 & 0 & 0 & 0 & -0.8 & 0 & 0 & 0 \\ 0 & -2 & 2.8 & 0 & 0 & 0 & 0 & -0.8 & 0 & 0 \\ 0 & 0 & -2 & 2.8 & 0 & 0 & 0 & 0 & -0.8 & 0 \\ 0 & 0 & 0 & -2 & 2.8 & 0 & 0 & 0 & 0 & -0.8 \\ -0.8 & 0 & 0 & 0 & 0 & 1.8 & -1 & 0 & 0 & 0 \\ 0 & -0.8 & 0 & 0 & 0 & 0 & 1.8 & -1 & 0 & 0 \\ 0 & 0 & -0.8 & 0 & 0 & 0 & 0 & 1.8 & -1 & 0 \\ 0 & 0 & 0 & -0.8 & 0 & 0 & 0 & 0 & 1.8 & -1 \\ 0 & 0 & 0 & 0 & -0.8 & 0 & 0 & 0 & 0 & 1.8 \end{bmatrix} \begin{Bmatrix} c_{G1} \\ c_{G2} \\ c_{G3} \\ c_{G4} \\ c_{G5} \\ c_{L1} \\ c_{L2} \\ c_{L3} \\ c_{L4} \\ c_{L5} \end{Bmatrix} = \begin{Bmatrix} 200 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 10 \end{Bmatrix}$$

These equations can then be solved. The results are tabulated and plotted below:

Reactor	Gas	Liquid
0	200	
1	190.9925	168.4737
2	179.4114	150.4587
3	164.5214	127.2965
4	145.3772	97.51656
5	120.7631	59.22807
6		10



12.11 Substituting centered difference finite differences, the Laplace equation can be written for the node (1, 1) as

$$0 = \frac{T_{21} - 2T_{11} + T_{01}}{\Delta x^2} + \frac{T_{12} - 2T_{11} + T_{10}}{\Delta y^2}$$

Because the grid is square ($\Delta x = \Delta y$), this equation can be expressed as

$$0 = T_{21} - 4T_{11} + T_{01} + T_{12} + T_{10}$$

The boundary node values (T_{01} = 100 and T_{10} = 75) can be substituted to give

$$4T_{11} - T_{12} - T_{21} = 175$$

The same approach can be written for the other interior nodes. When this is done, the following system of equations results

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \begin{Bmatrix} T_{11} \\ T_{12} \\ T_{21} \\ T_{22} \end{Bmatrix} = \begin{Bmatrix} 175 \\ 125 \\ 75 \\ 25 \end{Bmatrix}$$

These equations can be solved using the Gauss-Seidel method. For example, the first iteration would be

$$T_{11} = \frac{175 + T_{12} + T_{21}}{4} = \frac{175 + 0 + 0}{4} = 43.75$$
$$T_{12} = \frac{125 + T_{11} + T_{22}}{4} = \frac{125 + 43.75 + 0}{4} = 42.1875$$
$$T_{21} = \frac{75 + T_{11} + T_{22}}{4} = \frac{75 + 43.75 + 0}{4} = 29.6875$$
$$T_{22} = \frac{25 + T_{12} + T_{21}}{4} = \frac{25 + 42.1875 + 29.6875}{4} = 24.21875$$

The computation can be continued as follows:

iteration	unknown	value	ε_a	maximum ε_a
0	x_1	0		
	x_2	0		
	x_3	0		
	x_4	0		
1	x_1	43.75	100.00%	
	x_2	42.1875	100.00%	
	x_3	29.6875	100.00%	
	x_4	24.21875	100.00%	100.00%
2	x_1	61.71875	29.11%	
	x_2	52.73438	20.00%	
	x_3	40.23438	26.21%	
	x_4	29.49219	17.88%	29.11%
3	x_1	66.99219	7.87%	
	x_2	55.37109	4.76%	
	x_3	42.87109	6.15%	
	x_4	30.81055	4.28%	7.87%
4	x_1	68.31055	1.93%	
	x_2	56.03027	1.18%	
	x_3	43.53027	1.51%	
	x_4	31.14014	1.06%	1.93%
5	x_1	68.64014	0.48%	
	x_2	56.19507	0.29%	
	x_3	43.69507	0.38%	
	x_4	31.22253	0.26%	0.48%

Thus, after 5 iterations, the maximum error is 0.48% and we are converging on the final result: T_{11} = 68.75, T_{12} = 56.25, T_{21} = 43.75, and T_{22} = 31.25.

12.12

```
function [x,ea,iter] = GaussSeidel(A,b,es,maxit)
% GaussSeidel: Gauss Seidel method
%   x = GaussSeidel(A,b): Gauss Seidel without relaxation
% input:
%   A = coefficient matrix
%   b = right hand side vector
%   es = stop criterion (default = 0.00001%)
```

```
%   maxit = max iterations (default = 50)
% output:
%   x = solution vector
%   ea = maximum relative error (%)
%   iter = number of iterations

if nargin<2,error('at least 2 input arguments required'),end
if nargin<3|isempty(es),es=0.00001;end
if nargin<4|isempty(maxit),maxit=50;end
[m,n] = size(A);
if m~=n, error('Matrix A must be square'); end
C = A;
for i = 1:n
  C(i,i) = 0;
  x(i) = 0;
end
x = x';
for i = 1:n
  C(i,1:n) = C(i,1:n)/A(i,i);
end
for i = 1:n
  d(i) = b(i)/A(i,i);
end
iter = 0;
while (1)
  xold = x;
  for i = 1:n
    x(i) = d(i)-C(i,:)*x;
    if x(i) ~= 0
      ea(i) = abs((x(i) - xold(i))/x(i)) * 100;
    end
  end
  iter = iter+1;
  if max(ea)<=es | iter >= maxit, break, end
end
ea=max(ea);
```

<u>Example 12.1:</u>

```
>> clear,clc
>> A=[3 -.1 -.2;.1 7 -.3;.3 -.2 10];
>> b=[7.85;-19.3;71.4];
>> [x,ea,iter] = GaussSeidel(A,b)

x =
    3.0000
   -2.5000
    7.0000
ea =
  6.4417e-008
iter =
     6
```

<u>Prob. 12.2*a*:</u>

```
>> clear,clc
>> A=[0.8 -0.4 0;-0.4 0.8 -0.4;0 -0.4 0.8];
>> b=[41;25;105];
>> [x,ea,iter] = GaussSeidel(A,b)

x =
```

```
  173.7500
  245.0000
  253.7500
ea =
  6.4536e-006
iter =
    25
```

12.13

```
function [x,ea,iter] = GaussSeidelRelax(A,b,lambda,es,maxit)
% GaussSeidel: Gauss Seidel method with relaxation
%   x = GaussSeidel(A,b): Gauss Seidel with relaxation
% input:
%   A = coefficient matrix
%   b = right hand side vector
%   lambda = relation factor (default = 1)
%   es = stop criterion (default = 0.00001%)
%   maxit = max iterations (default = 50)
% output:
%   x = solution vector
%   ea = maximum relative error (%)
%   iter = number of iterations

if nargin<2,error('at least 2 input arguments required'),end
if nargin<3|isempty(lambda),lambda=1;end
if nargin<4|isempty(es),es=0.00001;end
if nargin<5|isempty(maxit),maxit=50;end
[m,n] = size(A);
if m~=n, error('Matrix A must be square'); end
C = A;
for i = 1:n
  C(i,i) = 0;
  x(i) = 0;
end
x = x';
for i = 1:n
  C(i,1:n) = C(i,1:n)/A(i,i);
end
for i = 1:n
  d(i) = b(i)/A(i,i);
end
iter = 0;
while (1)
  xold = x;
  for i = 1:n
    x(i) = d(i)-C(i,:)*x;
    x(i) = lambda*x(i) + (1 - lambda)*xold(i);
    if x(i) ~= 0
      ea(i) = abs((x(i) - xold(i))/x(i)) * 100;
    end
  end
  iter = iter+1;
  if max(ea)<=es | iter >= maxit, break, end
end
ea=max(ea);
```

Example 12.2:

```
>> clear,clc
>> A=[10 -2;-3 12];
>> b=[8;9];
```

```
[x,ea,iter] = GaussSeidelRelax(A,b,1.2,[],2)

x =
    1.0531
    0.9783
ea =
   21.4307
iter =
     2
```

Prob. 12.2*b*:

```
>> clear,clc
>> A=[0.8 -0.4 0;-0.4 0.8 -0.4;0 -0.4 0.8];
>> b=[41;25;105];
>> [x,ea,iter] = GaussSeidelRelax(A,b,1.2)

x =
  173.7500
  245.0000
  253.7500
ea =
  6.1014e-006
iter =
    12
```

12.14

```
function [x,f,ea,iter]=newtmult(func,x0,es,maxit,varargin)
% newtmult: Newton-Raphson root zeroes nonlinear systems
% [x,f,ea,iter]=newtmult(func,x0,es,maxit,p1,p2,...):
%   uses the Newton-Raphson method to find the roots of
%   a system of nonlinear equations
% input:
%   func = name of function that returns f and J
%   x0 = initial guess
%   es = desired percent relative error (default = 0.0001%)
%   maxit = maximum allowable iterations (default = 50)
%   p1,p2,... = additional parameters used by function
% output:
%   x = vector of roots
%   f = vector of functions evaluated at roots
%   ea = approximate percent relative error (%)
%   iter = number of iterations

if nargin<2,error('at least 2 input arguments required'),end
if nargin<3|isempty(es),es=0.0001;end
if nargin<4|isempty(maxit),maxit=50;end
iter = 0;
x=x0;
while (1)
  [J,f]=func(x,varargin{:});
  dx=J\f;
  x=x-dx;
  iter = iter + 1;
  ea=100*max(abs(dx./x));
  if iter>=maxit|ea<=es, break, end
end

function [J,f]=jfreact(x,varargin)
del=0.000001;
df1dx1=(u(x(1)+del*x(1),x(2))-u(x(1),x(2)))/(del*x(1));
```

```
df1dx2=(u(x(1),x(2)+del*x(2))-u(x(1),x(2)))/(del*x(2));
df2dx1=(v(x(1)+del*x(1),x(2))-v(x(1),x(2)))/(del*x(1));
df2dx2=(v(x(1),x(2)+del*x(2))-v(x(1),x(2)))/(del*x(2));
J=[df1dx1 df1dx2;df2dx1 df2dx2];
f1=u(x(1),x(2));
f2=v(x(1),x(2));
f=[f1;f2];
end
```

Example 12.4: The functions are set up as

```
function f=u(x,y)
f=x^2+x*y-10;
end

function f=v(x,y)
f=y+3*x*y^2-57;
end

>> clear,clc
>> x0=[1.5;3.5];
>> [x,f,ea,iter]=newtmult(@jfreact,x0)

x =
    2.0000
    3.0000
f =
  1.0e-004 *
   -0.0129
   -0.2210
ea =
  1.9494e-005
iter =
     4
```

Prob. 12.8: The functions are set up as

```
function f=u(x,y)
f=-x^2+x+0.5-y;
end

function f=v(x,y)
f=x^2-y-5*x*y;
end

>> clear,clc
>> x0=[1.2;1.2];
>> [x,f,ea,iter]=newtmult(@jfreact,x0)

x =
    1.2333
    0.2122
f =
  1.0e-011 *
   -0.1711
    0.3206
ea =
  1.0523e-010
iter =
     5
```