Problem 2.1 [Difficulty: 1]

2.1 For the velocity fields given below, determine:
a. whether the flow field is one-, two-, or three-dimensional, and why.
b. whether the flow is steady or unsteady, and why.
(The quantities a and b are constants.)
(1) $\vec{V} = [(ax+t)e^{by}]\hat{i}$ (2) $\vec{V} = (ax - by)\hat{i}$
(3) $\vec{V} = ax\hat{i} + [e^{bx}]\hat{j}$ (4) $\vec{V} = ax\hat{i} + bx^2\hat{j} + ax\hat{k}$
(5) $\vec{V} = ax\hat{i} + [e^{bt}]\hat{j}$ (6) $\vec{V} = ax\hat{i} + bx^2\hat{j} + ay\hat{k}$
(7) $\vec{V} = ax\hat{i} + [e^{bt}]\hat{j} + ay\hat{k}$ (8) $\vec{V} = ax\hat{i} + [e^{by}]\hat{j} + az\hat{k}$

Given: Velocity fields

Find: Whether flows are 1, 2 or 3D, steady or unsteady.

Solution:

(1)	$\vec{V} = \vec{V}(x,y)$	2D	$\vec{V} = \vec{V}(t)$	Unsteady
(2)	$\vec{V} = \vec{V}(x,y)$	2D	$\vec{V} \neq \vec{V}(t)$	Steady
(3)	$\vec{V} = \vec{V}(x)$	1D	$\vec{V} \neq \vec{V}(t)$	Steady
(4)	$\vec{V} = \vec{V}(x)$	1D	$\vec{V} \neq \vec{V}(t)$	Steady
(5)	$\vec{V} = \vec{V}(x)$	1D	$\vec{V} = \vec{V}(t)$	Unsteady
(6)	$\vec{V} = \vec{V}(x,y)$	2D	$\vec{V} \neq \vec{V}(t)$	Steady
(7)	$\vec{V} = \vec{V}(x,y)$	2D	$\vec{V} = \vec{V}(t)$	Unsteady
(8)	$\vec{V} = \vec{V}(x,y,z)$	3D	$\vec{V} \neq \vec{V}(t)$	Steady

Problem 2.2 [Difficulty: 1]

2.2 For the velocity fields given below, determine:
a. whether the flow field is one-, two-, or three-dimensional, and why.
b. whether the flow is steady or unsteady, and why.
(The quantities a and b are constants.)
(1) $\vec{V} = [ay^2e^{-bt}]\hat{i}$ (2) $\vec{V} = ax^2\hat{i} + bx\hat{j} + c\hat{k}$
(3) $\vec{V} = axy\hat{i} - byt\hat{j}$ (4) $\vec{V} = ax\hat{i} - by\hat{j} + ct\hat{k}$
(5) $\vec{V} = [ae^{-bx}]\hat{i} + bt^2\hat{j}$ (6) $\vec{V} = a(x^2 + y^2)^{1/2}(1/z^3)\hat{k}$
(7) $\vec{V} = (ax + t)\hat{i} - by^2\hat{j}$ (8) $\vec{V} = ax^2\hat{i} + bxz\hat{j} + cy\hat{k}$

Given: Velocity fields

Find: Whether flows are 1, 2 or 3D, steady or unsteady.

Solution:

(1)	$\vec{V} = \vec{V}(y)$	1D	$\vec{V} = \vec{V}(t)$	Unsteady
(2)	$\vec{V} = \vec{V}(x)$	1D	$\vec{V} \neq \vec{V}(t)$	Steady
(3)	$\vec{V} = \vec{V}(x, y)$	2D	$\vec{V} = \vec{V}(t)$	Unsteady
(4)	$\vec{V} = \vec{V}(x, y)$	2D	$\vec{V} = \vec{V}(t)$	Unsteady
(5)	$\vec{V} = \vec{V}(x)$	1D	$\vec{V} = \vec{V}(t)$	Unsteady
(6)	$\vec{V} = \vec{V}(x, y, z)$	3D	$\vec{V} \neq \vec{V}(t)$	Steady
(7)	$\vec{V} = \vec{V}(x, y)$	2D	$\vec{V} = \vec{V}(t)$	Unsteady
(8)	$\vec{V} = \vec{V}(x, y, z)$	3D	$\vec{V} \neq \vec{V}(t)$	Steady

Problem 2.3

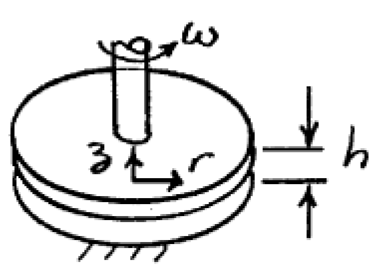
[Difficulty: 2]

2.3 A viscous liquid is sheared between two parallel disks; the upper disk rotates and the lower one is fixed. The velocity field between the disks is given by $\vec{V} = \hat{e}_\theta r\omega z/h$. (The origin of coordinates is located at the center of the lower disk; the upper disk is located at $z = h$.) What are the dimensions of this velocity field? Does this velocity field satisfy appropriate physical boundary conditions? What are they?

Given: Viscous liquid sheared between parallel disks.

Upper disk rotates, lower fixed.

Velocity field is: $\vec{V} = \hat{e}_\theta \dfrac{r\omega z}{h}$



Find:

a. Dimensions of velocity field.

b. Satisfy physical boundary conditions.

Solution: To find dimensions, compare to $\vec{V} = \vec{V}(x, y, z)$ form.

The given field is $\vec{V} = \vec{V}(r, z)$. Two space coordinates are included, so the field is 2-D.

Flow must satisfy the no-slip condition:

1. At lower disk, $\vec{V} = 0$ since stationary.

   $z = 0$, so $\vec{V} = \hat{e}_\theta \dfrac{r\omega 0}{h} = 0$, so satisfied.

2. At upper disk, $\vec{V} = \hat{e}_\theta r\omega$ since it rotates as a solid body.

   $z = h$, so $\vec{V} = \hat{e}_\theta \dfrac{r\omega h}{h} = \hat{e}_\theta r\omega$, so satisfied.

Problem 2.4

[Difficulty: 1]

2.4 For the velocity field $\vec{V} = Ax^2y\hat{i} + Bxy^2\hat{j}$, where $A = 2\ \text{m}^{-2}\text{s}^{-1}$ and $B = 1\ \text{m}^{-2}\text{s}^{-1}$, and the coordinates are measured in meters, obtain an equation for the flow streamlines. Plot several streamlines in the first quadrant.

Given: Velocity field

Find: Equation for streamlines

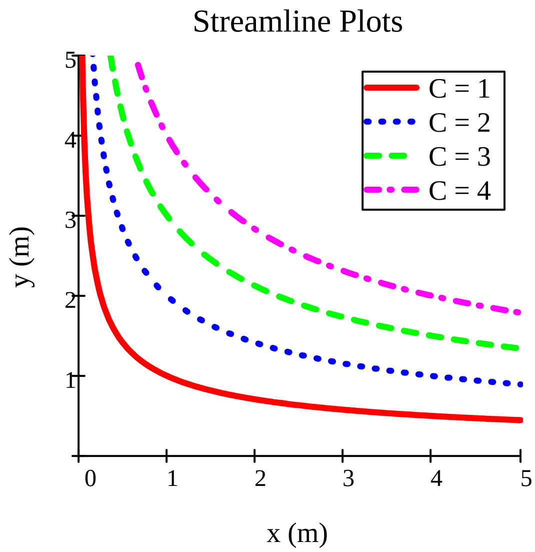
Solution:

For streamlines
$$\frac{v}{u} = \frac{dy}{dx} = \frac{B\cdot x\cdot y^2}{A\cdot x^2\cdot y} = \frac{B\cdot y}{A\cdot x}$$

So, separating variables
$$\frac{dy}{y} = \frac{B}{A}\cdot\frac{dx}{x}$$

Integrating
$$\ln(y) = \frac{B}{A}\cdot\ln(x) + c = -\frac{1}{2}\cdot\ln(x) + c$$

The solution is
$$y = \frac{C}{\sqrt{x}}$$



The plot can be easily done in *Excel*.

Problem 2.5

(Difficulty: 2)

2.5 A fluid flow has the following velocity components: $u = 1\ m/s$ and $v = 2x\ m/s$. Find an equation for and sketch the streamlines of this flow.

Given: The velocity components: $u = 1\ m/s$ and $v = 2x\ m/s$.

Find: The equation for streamlines and sketch the streamlines.

Assumption: The flow is steady and incompressible

Solution: Use the definition of streamlines in terms of velocity to determine the equation for the streamlines. By definition, we have:

$$\frac{dx}{u} = \frac{dy}{v}$$

Or

$$dx = \frac{u}{v} dy$$

Substituting in the velocity components in we obtain:

$$dx = \frac{1}{2x} dy$$

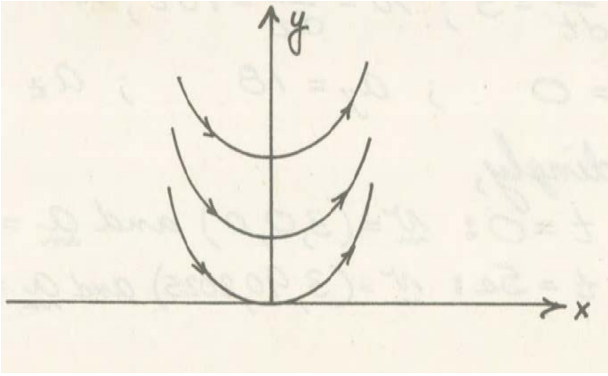
$$2x dx = dy$$

Integrating both sides, we get:

$$x^2 = y + c$$

Where c is a constant that can be found for each specific problem.

To plot the streamlines, we write: $y = x^2 - c$. The plot is shown in the figure.



Problem 2.6

(Difficulty: 2)

2.6 When an incompressible, non-viscous fluid flows against a plate (two-dimensional) flow, an exact solution for the equations of motion for this flow is $u = Ax$, $v = -Ay$, with $A > 0$. The coordinate origin is located at the stagnation point 0, where the flow divides and the local velocity is zero. Find the streamlines.

Given: The velocity components: $u = Ax, v = -Ay$

Find: The equation for streamlines and sketch the streamlines.

Assumption: The flow is steady and incompressible

Solution: Use the definition of streamlines in terms of velocity to determine the equation for the streamlines. By definition, we have:

$$\frac{dx}{u} = \frac{dy}{v}$$

Substituting in for the velocity components we obtain:

$$\frac{dx}{Ax} = \frac{dy}{-Ay}$$

$$\frac{dx}{x} = \frac{dy}{-y}$$

Integrating both sides, we get:

$$\int \frac{dx}{x} = -\int \frac{dy}{y}$$

$$\ln x = -\ln y + c$$

where c is a constant that can be found for each problem

Using the relation for logarithms, the streamline equation is:

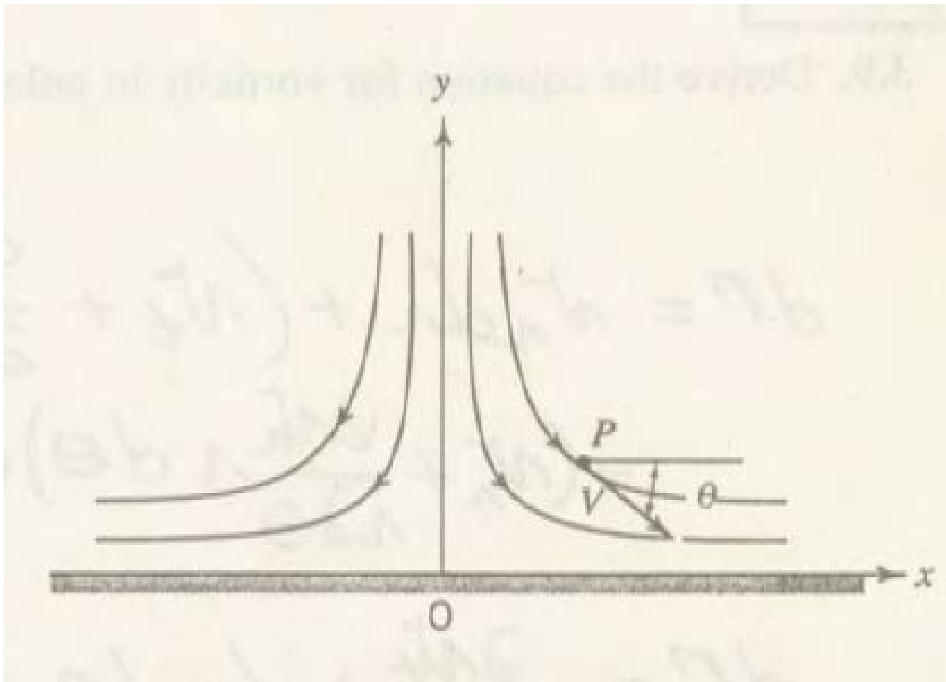
$$\ln x\, y = +c$$

Or we can rewrite as:

$$xy = c_1$$

Where c_1 is a constant.

The plot of the streamlines is shown in the figure as an example:



Problem 2.7

(Difficulty: 2)

2.7 For the free vortex flow the velocities are $V_t = \frac{5}{r}$ and $V_r = 0$. Assume that lengths are in feet or meters and times are in seconds. Plot the streamlines of this flow. How does the velocity vary with distance from the origin? What is the velocity at the origin $(0,0)$?

Given: The velocity components: $V_t = \frac{5}{r}$, $V_r = 0$

Find: The streamline, how the velocity varies with distance from the origin, and the velocity at the origin $(0,0)$.

Assumption: The flow is steady and incompressible

Solution: Use the definition of streamlines in terms of velocity to determine the equation for the streamlines. By definition, in radial coordinates we have:

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{V_r}{V_t}$$

Substituting the velocity components we obtain:

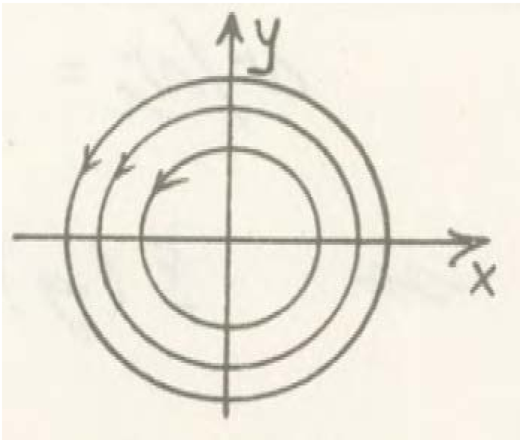
$$\frac{dr}{d\theta} = 0$$

Integrating both sides, we get:

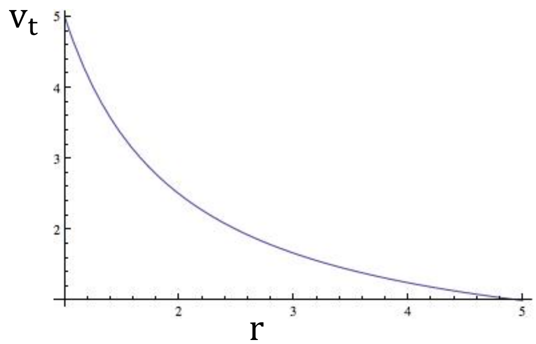
$$r = c$$

c is a constant

So the streamline can be plotted as:



The velocity will decrease as the distance to the origin r increases as shown in the figure.



The velocity at the origin $(0,0)$ is

$$V_r = 0$$

$$V_t = \infty$$

Problem 2.8

(Difficulty: 2)

2.8 For the forced vortex flow the velocities are $V_t = \omega r$ and $V_r = 0$. Plot the streamlines of this flow. How does the velocity vary with distance from the origin? What is the velocity at the origin$(0,0)$?

Given: The velocity components: $V_t = \omega r, V_r = 0$

Assumption: The flow is steady and incompressible

Find: The equation for the streamlines, how the velocity varies with distance from origin, the velocity at origin $(0,0)$.

Solution: Use the definition of streamlines in terms of velocity to determine the equation for the streamlines. By definition, in radial coordinates we have:

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{V_r}{V_t}$$

Substituting the velocity components in we obtain:

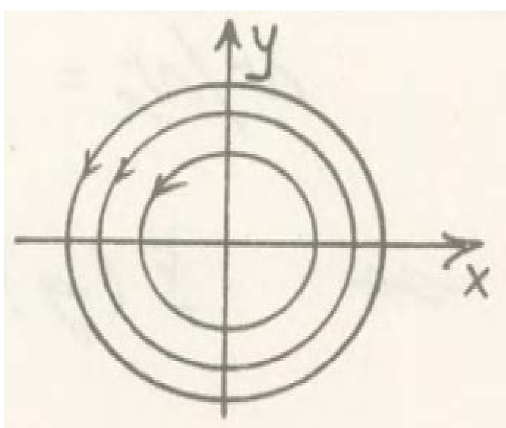
$$\frac{dr}{d\theta} = 0$$

Integrating both sides, we get:

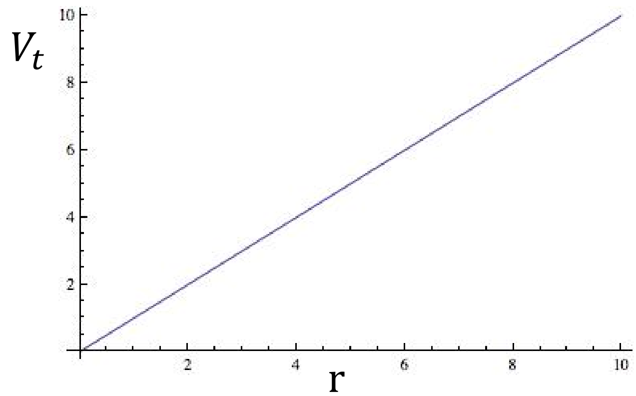
$$r = c$$

c is a constant

So the streamline can be plotted as:



The velocity will increase as the distance to the origin r increases. For example $\omega = 1$.



The velocity at the origin $(0,0)$ is

$$V_r = 0$$

$$V_t = 0$$

Problem 2.9

[Difficulty: 1]

2.9 A velocity field is specified as $\vec{V} = axy\hat{i} + by^2\hat{j}$, where $a = 2\ \mathrm{m^{-1}s^{-1}}$, $b = -6\ \mathrm{m^{-1}s^{-1}}$, and the coordinates are measured in meters. Is the flow field one-, two-, or three-dimensional? Why? Calculate the velocity components at the point (2, ½). Develop an equation for the streamline passing through this point. Plot several streamlines in the first quadrant including the one that passes through the point (2, ½).

Given: Velocity field

Find: Whether field is 1D, 2D or 3D; Velocity components at (2,1/2); Equation for streamlines; Plot

Solution:

The velocity field is a function of *x* and *y*. It is therefore 2D.

At point (2,1/2), the velocity components are

$$u = a\cdot x\cdot y = 2\cdot\frac{1}{\mathrm{m\cdot s}} \times 2\cdot\mathrm{m} \times \frac{1}{2}\cdot\mathrm{m} \qquad u = 2\cdot\frac{\mathrm{m}}{\mathrm{s}}$$

$$v = b\cdot y^2 = -6\cdot\frac{1}{\mathrm{m\cdot s}} \times \left(\frac{1}{2}\cdot\mathrm{m}\right)^2 \qquad v = -\frac{3}{2}\cdot\frac{\mathrm{m}}{\mathrm{s}}$$

For streamlines

$$\frac{v}{u} = \frac{dy}{dx} = \frac{b\cdot y^2}{a\cdot x\cdot y} = \frac{b\cdot y}{a\cdot x}$$

So, separating variables

$$\frac{dy}{y} = \frac{b}{a}\cdot\frac{dx}{x}$$

Integrating

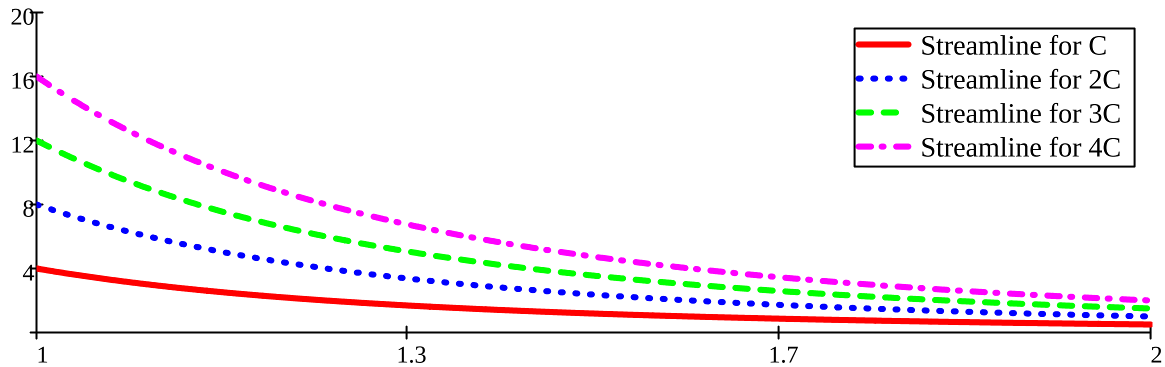
$$\ln(y) = \frac{b}{a}\cdot\ln(x) + c \qquad y = C\cdot x^{\frac{b}{a}}$$

The solution is

$$y = C\cdot x^{-3}$$

The streamline passing through point (2,1/2) is given by

$$\frac{1}{2} = C\cdot 2^{-3} \qquad C = \frac{1}{2}\cdot 2^3 \qquad C = 4 \qquad y = \frac{4}{x^3}$$



This can be plotted in *Excel*.

Problem 2.10 [Difficulty: 2]

2.10 A velocity field is given by $\vec{V} = ax^3\hat{i} + bxy^3\hat{j}$, where $a = 1\ \text{m}^{-2}\text{s}^{-1}$ and $b = 1\ \text{m}^{-3}\text{s}^{-1}$. Find the equation of the streamlines. Plot several streamlines in the first quadrant.

Given: Velocity field

Find: Equation for streamlines; Plot streamlines

Solution:

Streamlines are given by $\quad \dfrac{v}{u} = \dfrac{dy}{dx} = \dfrac{b \cdot x \cdot y^3}{a \cdot x^3}$

So, separating variables $\quad \dfrac{dy}{y^3} = \dfrac{b \cdot dx}{a \cdot x^2}$

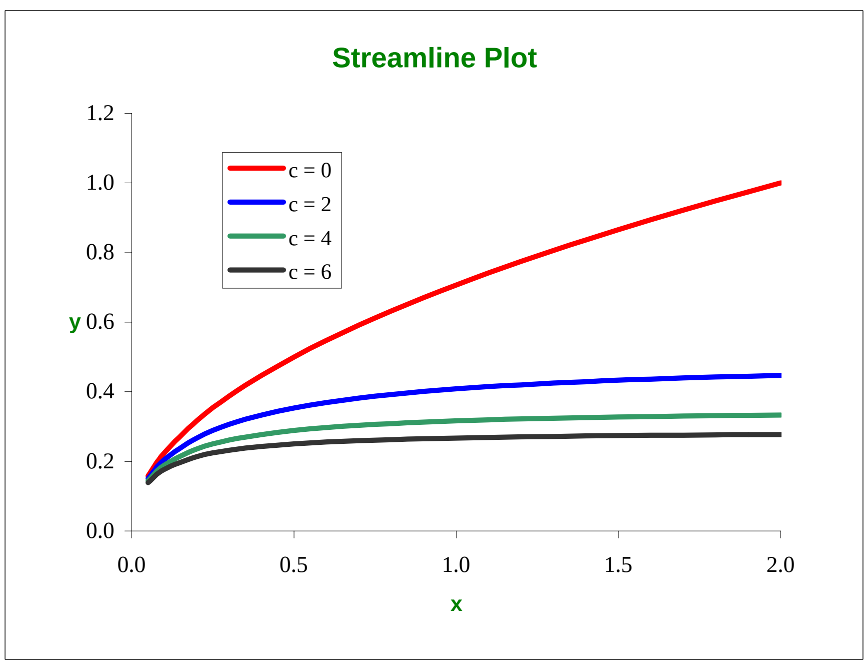
Integrating $\quad -\dfrac{1}{2 \cdot y^2} = \dfrac{b}{a} \cdot \left(-\dfrac{1}{x}\right) + C$

The solution is $\quad y = \dfrac{1}{\sqrt{2 \cdot \left(\dfrac{b}{a \cdot x} + C\right)}}$

Note: For convenience the sign of C is changed.

a = **1**
b = **1**

C =	**0**	**2**	**4**	**6**
x	**y**	**y**	**y**	**y**
0.05	0.16	0.15	0.14	0.14
0.10	0.22	0.20	0.19	0.18
0.20	0.32	0.27	0.24	0.21
0.30	0.39	0.31	0.26	0.23
0.40	0.45	0.33	0.28	0.24
0.50	0.50	0.35	0.29	0.25
0.60	0.55	0.37	0.30	0.26
0.70	0.59	0.38	0.30	0.26
0.80	0.63	0.39	0.31	0.26
0.90	0.67	0.40	0.31	0.27
1.00	0.71	0.41	0.32	0.27
1.10	0.74	0.41	0.32	0.27
1.20	0.77	0.42	0.32	0.27
1.30	0.81	0.42	0.32	0.27
1.40	0.84	0.43	0.33	0.27
1.50	0.87	0.43	0.33	0.27
1.60	0.89	0.44	0.33	0.27
1.70	0.92	0.44	0.33	0.28
1.80	0.95	0.44	0.33	0.28
1.90	0.97	0.44	0.33	0.28
2.00	1.00	0.45	0.33	0.28



Problem 2.11

[Difficulty: 2]

2.11 The velocity for a steady, incompressible flow in the xy plane is given by $\vec{V} = \hat{i}A/x + \hat{j}Ay/x^2$, where $A = 2\ \text{m}^2/\text{s}$, and the coordinates are measured in meters. Obtain an equation for the streamline that passes through the point $(x, y) = (1, 3)$. Calculate the time required for a fluid particle to move from $x = 1$ m to $x = 2$ m in this flow field.

Given: Velocity field

Find: Equation for streamline through (1,3)

Solution:

For streamlines $$\frac{v}{u} = \frac{dy}{dx} = \frac{A\cdot\frac{y}{x^2}}{\frac{A}{x}} = \frac{y}{x}$$

So, separating variables $$\frac{dy}{y} = \frac{dx}{x}$$

Integrating $$\ln(y) = \ln(x) + c$$

The solution is $$y = C\cdot x$$ which is the equation of a straight line.

For the streamline through point (1,3) $\quad 3 = C\cdot 1 \quad C = 3 \quad$ and $\quad y = 3\cdot x$

For a particle $$u_p = \frac{dx}{dt} = \frac{A}{x} \quad \text{or} \quad x\cdot dx = A\cdot dt \quad x = \sqrt{2\cdot A\cdot t + c} \quad t = \frac{x^2}{2\cdot A} - \frac{c}{2\cdot A}$$

Hence the time for a particle to go from x = 1 to x = 2 m is

$$\Delta t = t(x = 2) - t(x = 1) \qquad \Delta t = \frac{(2\cdot m)^2 - c}{2\cdot A} - \frac{(1\cdot m)^2 - c}{2\cdot A} = \frac{4\cdot m^2 - 1\cdot m^2}{2\times 2\cdot\frac{m^2}{s}} \qquad \Delta t = 0.75\cdot s$$

Problem 2.12

[Difficulty: 3]

2.12 The flow field for an atmospheric flow is given by

$$\vec{V} = -\frac{Ky}{2\pi(x^2+y^2)}\hat{i} + \frac{Kx}{2\pi(x^2+y^2)}\hat{j}$$

where $K = 10^5\ \text{m}^2/\text{s}$, and the x and y coordinates are parallel to the local latitude and longitude. Plot the velocity magnitude along the x axis, along the y axis, and along the line $y = x$, and discuss the velocity direction with respect to these three axes. For each plot use a range x or $y = -1$ km to 1 km, excluding $|x|$ or $|y| < 100$ m. Find the equation for the streamlines and sketch several of them. What does this flow field model?

Given: Flow field

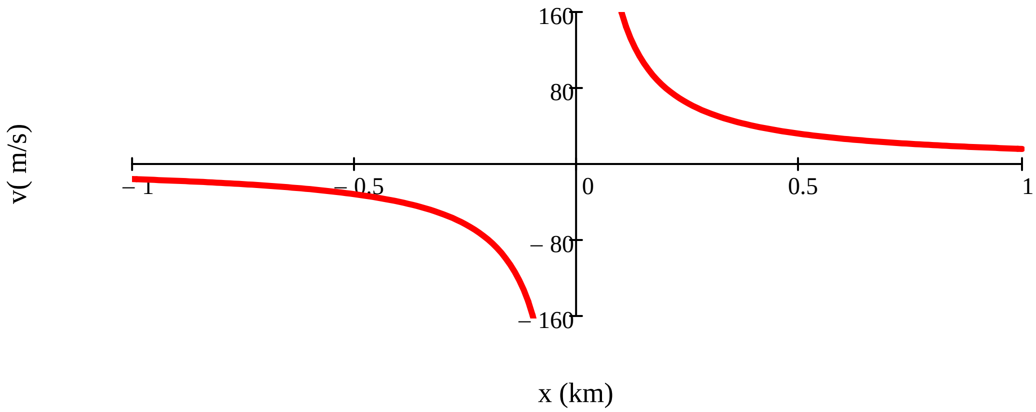
Find: Plot of velocity magnitude along axes, and y = x; Equation of streamlines

Solution:

On the x axis, y = 0, so

$$u = -\frac{K\cdot y}{2\cdot\pi\cdot\left(x^2+y^2\right)} = 0 \qquad v = \frac{K\cdot x}{2\cdot\pi\cdot\left(x^2+y^2\right)} = \frac{K}{2\cdot\pi\cdot x}$$

Plotting



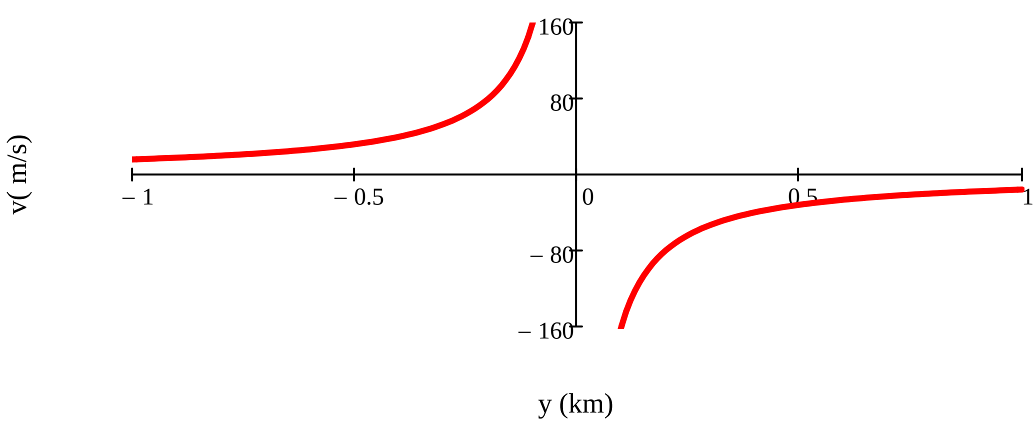
The velocity is perpendicular to the axis, is very high close to the origin, and falls off to zero.

This can also be plotted in Excel.

On the y axis, x = 0, so

$$u = -\frac{K\cdot y}{2\cdot\pi\cdot\left(x^2+y^2\right)} = -\frac{K}{2\cdot\pi\cdot y} \qquad v = \frac{K\cdot x}{2\cdot\pi\cdot\left(x^2+y^2\right)} = 0$$

Plotting



The velocity is perpendicular to the axis, is very high close to the origin, and falls off to zero.
This can also be plotted in Excel.

On the y = x axis $$u = -\frac{K\cdot x}{2\cdot\pi\cdot\left(x^2 + x^2\right)} = -\frac{K}{4\cdot\pi\cdot x} \qquad v = \frac{K\cdot x}{2\cdot\pi\cdot\left(x^2 + x^2\right)} = \frac{K}{4\cdot\pi\cdot x}$$

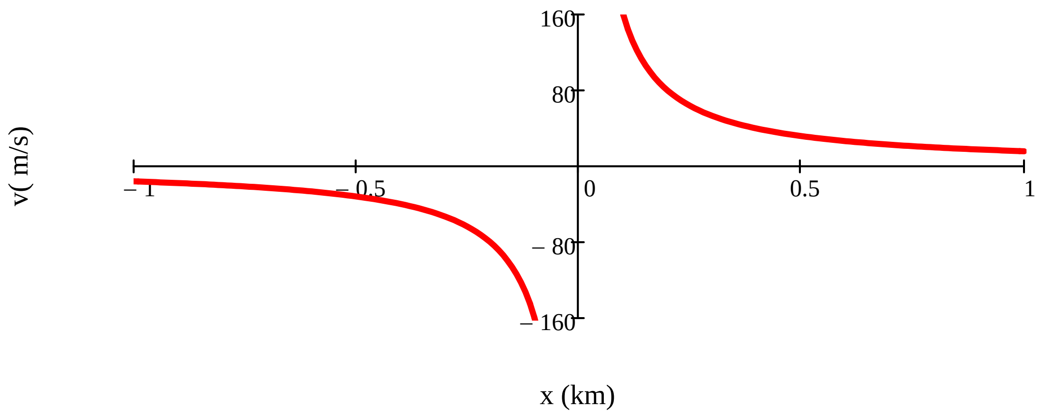
The flow is perpendicular to line y = x: Slope of line y = x: 1

Slope of trajectory of motion: $\frac{u}{v} = -1$

If we define the radial position: $r = \sqrt{x^2 + y^2}$ then along y = x $r = \sqrt{x^2 + x^2} = \sqrt{2}\cdot x$

Then the magnitude of the velocity along y = x is $$V = \sqrt{u^2 + v^2} = \frac{K}{4\cdot\pi}\cdot\sqrt{\frac{1}{x^2} + \frac{1}{x^2}} = \frac{K}{2\cdot\pi\cdot\sqrt{2}\cdot x} = \frac{K}{2\cdot\pi\cdot r}$$

Plotting



This can also be plotted in Excel.

For streamlines $$\frac{v}{u} = \frac{dy}{dx} = \frac{\dfrac{K\cdot x}{2\cdot\pi\cdot\left(x^2+y^2\right)}}{-\dfrac{K\cdot y}{2\cdot\pi\cdot\left(x^2 + y^2\right)}} = -\frac{x}{y}$$

So, separating variables $y\cdot dy = -x\cdot dx$

Integrating $$\frac{y^2}{2} = -\frac{x^2}{2} + c$$

The solution is $x^2 + y^2 = C$ which is the equation of a circle.

Streamlines form a set of concentric circles.

This flow models a vortex flow.

Problem 2.13

(Difficulty: 2)

2.13 For the velocity field $\vec{V} = Ax\vec{i} - Ay\vec{j}$, where $A = 2\ s^{-1}$, which can be interpreted to represent flow in a corner, show that the parametric equations for particle motion are given by $x_p = c_1 e^{At}$ and $y_p = c_2 e^{-At}$. Obtain the equation for the pathline of the particle located at the point $(x, y) = (2,2)$ at the instant $t = 0$. Compare this pathline with streamline through the same point.

Find: The pathlines and streamlines .

Assumption: The flow is steady and incompressible

Solution: Use the definitions of pathlines and streamlines in terms of velocity. We relate the velocities to the change in position with time. For the particle motion we have:

$$\frac{dx}{dt} = u = Ax$$

$$\frac{dy}{dt} = v = -Ay$$

Or

$$\frac{dx}{x} = A\,dt$$

$$\frac{dy}{y} = -A\,dt$$

Integrating both sides of the equation, we get:

$$\ln x = At + c$$

$$\ln y = -At + c$$

So the parametric equations for particle motion are given by:

$$x_p = e^{(At+c)} = c_1 e^{At}$$

$$y_p = e^{(-At+c)} = c_2 e^{-At}$$

With $A = 2\ s^{-1}$:

$$x_p = e^{(At+c)} = c_1 e^{2t}$$

$$y_p = e^{(-At+c)} = c_2 e^{-2t}$$

For the pathline:

At $t = 0$, $x_p = x_0 = 2$, $y_p = y_0 = 2$. So the equation for the pathline is

$$x_p y_p = x_0 y_0 = 4$$

For the streamline:

$$\frac{dy}{dx} = \frac{v}{u} = \frac{-Ay}{Ax} = \frac{-y}{x}$$

$$\frac{dy}{y} = \frac{-dx}{x}$$

Integrating both sides of the equation we get:

$$\ln y = -\ln x + c$$

$$xy = c$$

For points $(x, y) = (2,2)$, the constant c = 4 and the equation for the streamline is:

$$xy = 4$$

Comparing the pathline and streamline, it is seen that for steady flow the pathline and streamline coincide as expected.

Problem 2.14

(Difficulty: 2)

2.14 A velocity field in polar coordinates is given with the radial velocity as $V_r = -\frac{A}{r}$ and the tangential velocity as $V_\theta = \frac{A}{r}$, where r is in m and $A = 10\ m^2$. Plot the streamlines passing through the $\theta = 0$ and $r = 1\ m, 2m, and\ 3m$. What does the flow field model?

Assumption: The flow is steady and incompressible

Solution: Use the definition of streamlines in terms of velocity. The definition of a streamline in radial coordinates is:

$$\frac{1}{r}\frac{dr}{V_r} = \frac{d\theta}{V_\theta}$$

With the velocity components

$$\frac{1}{r}\left(\frac{dr}{-\frac{A}{r}}\right) = \frac{d\theta}{\frac{A}{r}}$$

Or

$$-\frac{dr}{r} = d\theta$$

Integrating both sides:

$$-\ln r = \theta + c$$

For the case of $\theta = 0$ and $r = 1\ m$, we have:

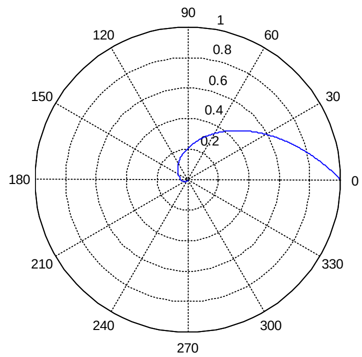
$$c = 0$$

So the streamline is:

$$\ln r = -\theta$$

$$r = exp(-\theta)$$

The plot of the streamline is



For the case $\theta = 0$ and $r = 2\ m$, we have:

$$c = -\ln 2$$

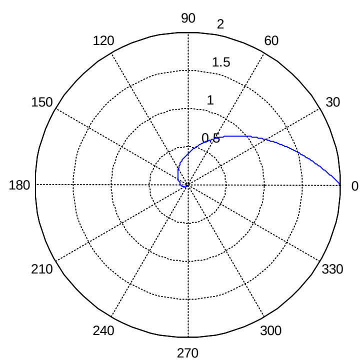
So the streamline is:

$$\ln r = -\theta - c = -\theta + \ln 2$$

$$\ln r - \ln 2 = -\theta$$

$$\ln \frac{r}{2} = -\theta$$

$$r = 2exp(-\theta)$$



For the case $\theta = 0$ and $r = 3\ m$, we have:

$$c = -\ln 3$$

So the streamline is:

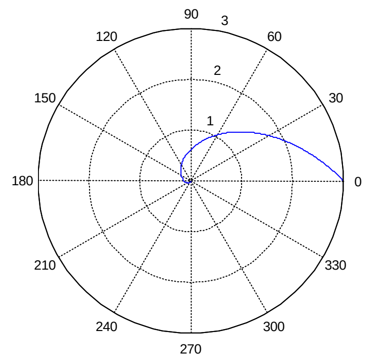
$$\ln r = -\theta - c = -\theta + \ln 3$$

$$\ln r - \ln 3 = -\theta$$

$$\ln \frac{r}{3} = -\theta$$

$$r = 3exp(-\theta)$$

The flow field models the circular flow from the center at the origin.



Problem 2.15

(Difficulty: 2)

2.15 The flow of air near the earth's surface is affected both by the wind and thermal currents. In certain circumstances the velocity field can be represented by $\vec{V} = a\vec{\imath} + b\left(1 - \frac{y}{h}\right)\vec{\jmath}$ for $y < h$ and by $\vec{V} = a\vec{\imath}$ for $y > h$. Plot the streamlines for the flow for $\frac{b}{a} = 0.01, 0.1\ and\ 1$.

Assumption: The flow is steady and incompressible

Solution: Use the definition of streamlines in terms of velocity to determine the equation for the streamlines. By definition, we have:

$$\frac{dy}{dx} = \frac{v}{u}$$

$$\frac{dx}{u} = \frac{dy}{v}$$

Or, substituting the equations for the velocities:

$$\frac{dx}{a} = \frac{dy}{b\left(1 - \frac{y}{h}\right)}$$

And when y < h

$$x = \frac{dy}{\frac{b}{a}\left(1 - \frac{y}{h}\right)}$$

And when $y > h$

$$\frac{dy}{dx} = 0$$

Integrating both sides of the equation:

when $y < h$.

$$x = -\frac{ah}{b}\ln(bh - by) + c_1$$

when $y > h$.

$$y = c_2$$

For the critical point $y = h$, we have $c_2 = h$

For example, the streamline passing through $(0,0)$:

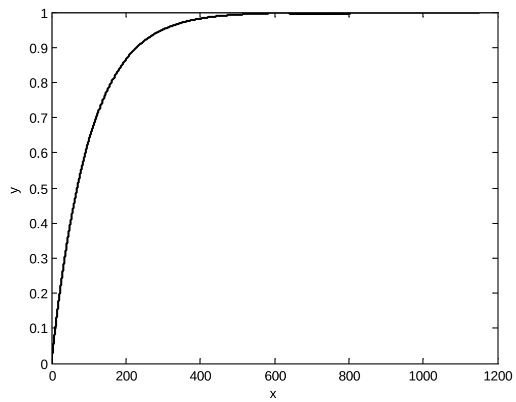
$$c_1 = \frac{ah}{b}\ln(bh)$$

$$x = -\frac{ah}{b}\ln(bh - by) + \frac{ah}{b}\ln(bh)$$

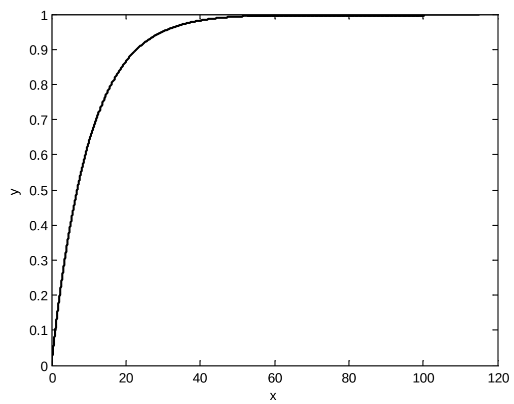
when $y < h$.

Assume $h = b = 1$,. For the first case $a = 100$, the streamline is shown as:

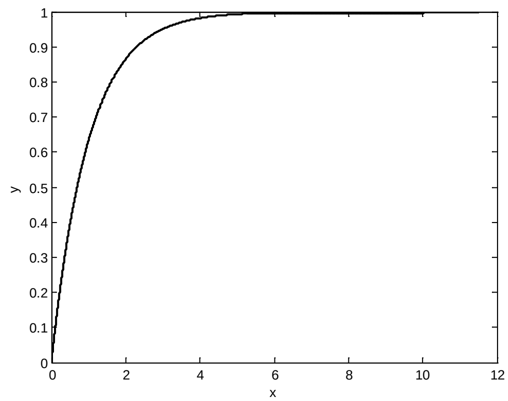
The streamline is shown:



For the second case $a = 10$, the streamline is shown as:



For the third case $a = 10$, the streamline is shown as:


1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
y
0
2
4
6
8
10
12
x

Problem 2.16 [Difficulty: 2]

2.16 A velocity field is given by $\vec{V} = ayt\hat{i} - bx\hat{j}$, where $a = 1\ \mathrm{s}^{-2}$ and $b = 4\ \mathrm{s}^{-1}$. Find the equation of the streamlines at any time t. Plot several streamlines at $t = 0$ s, $t = 1$ s, and $t = 20$ s.

Given: Velocity field

Find: Equation of streamlines; Plot streamlines

Solution:

Streamlines are given by $$\frac{v}{u} = \frac{dy}{dx} = \frac{-b \cdot x}{a \cdot y \cdot t}$$

So, separating variables $$a \cdot t \cdot y \cdot dy = -b \cdot x \cdot dx$$

Integrating $$\frac{1}{2} \cdot a \cdot t \cdot y^2 = -\frac{1}{2} \cdot b \cdot x^2 + C$$

The solution is $$y = \sqrt{C - \frac{b \cdot x^2}{a \cdot t}}$$

For $t = 0$ s $x = c$

For $t = 1$ s $y = \sqrt{C - 4 \cdot x^2}$

For $t = 20$ s $y = \sqrt{C - \frac{x^2}{5}}$

t = 0

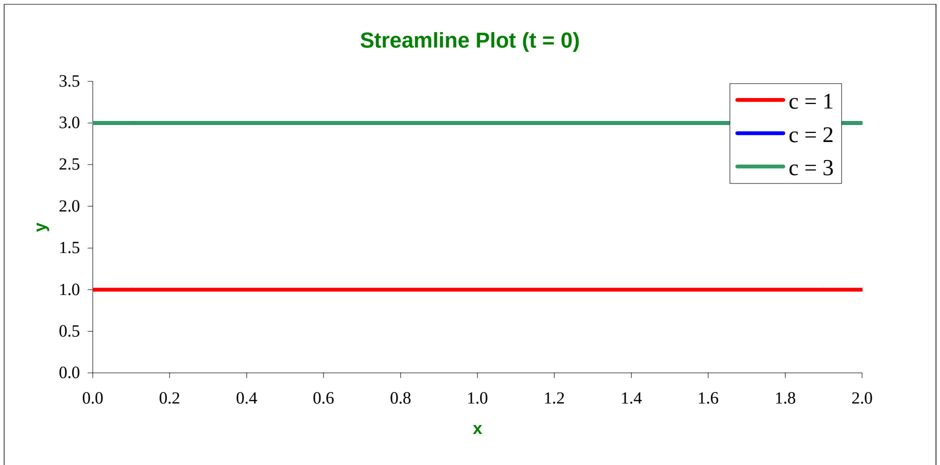
x	y (C = 1)	y (C = 2)	y (C = 3)
0.00	1.00	2.00	3.00
0.10	1.00	2.00	3.00
0.20	1.00	2.00	3.00
0.30	1.00	2.00	3.00
0.40	1.00	2.00	3.00
0.50	1.00	2.00	3.00
0.60	1.00	2.00	3.00
0.70	1.00	2.00	3.00
0.80	1.00	2.00	3.00
0.90	1.00	2.00	3.00
1.00	1.00	2.00	3.00
1.10	1.00	2.00	3.00
1.20	1.00	2.00	3.00
1.30	1.00	2.00	3.00
1.40	1.00	2.00	3.00
1.50	1.00	2.00	3.00
1.60	1.00	2.00	3.00
1.70	1.00	2.00	3.00
1.80	1.00	2.00	3.00
1.90	1.00	2.00	3.00
2.00	1.00	2.00	3.00

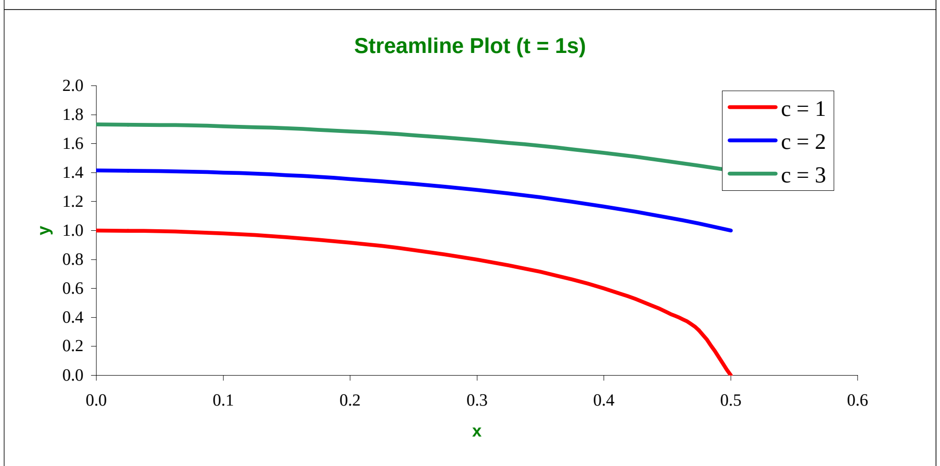
t = 1 s

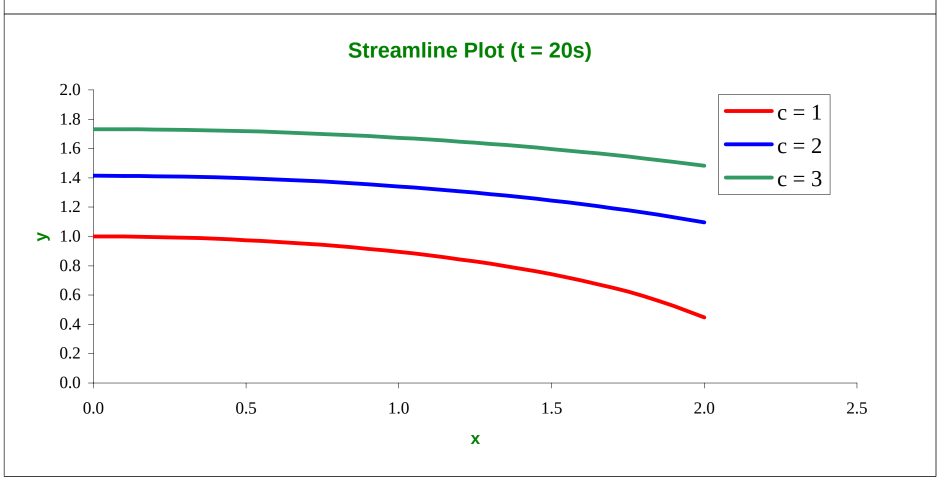
x	y (C = 1)	y (C = 2)	y (C = 3)
0.000	1.00	1.41	1.73
0.025	1.00	1.41	1.73
0.050	0.99	1.41	1.73
0.075	0.99	1.41	1.73
0.100	0.98	1.40	1.72
0.125	0.97	1.39	1.71
0.150	0.95	1.38	1.71
0.175	0.94	1.37	1.70
0.200	0.92	1.36	1.69
0.225	0.89	1.34	1.67
0.250	0.87	1.32	1.66
0.275	0.84	1.30	1.64
0.300	0.80	1.28	1.62
0.325	0.76	1.26	1.61
0.350	0.71	1.23	1.58
0.375	0.66	1.20	1.56
0.400	0.60	1.17	1.54
0.425	0.53	1.13	1.51
0.450	0.44	1.09	1.48
0.475	0.31	1.05	1.45
0.500	0.00	1.00	1.41

t = 20 s

x	y (C = 1)	y (C = 2)	y (C = 3)
0.00	1.00	1.41	1.73
0.10	1.00	1.41	1.73
0.20	1.00	1.41	1.73
0.30	0.99	1.41	1.73
0.40	0.98	1.40	1.72
0.50	0.97	1.40	1.72
0.60	0.96	1.39	1.71
0.70	0.95	1.38	1.70
0.80	0.93	1.37	1.69
0.90	0.92	1.36	1.68
1.00	0.89	1.34	1.67
1.10	0.87	1.33	1.66
1.20	0.84	1.31	1.65
1.30	0.81	1.29	1.63
1.40	0.78	1.27	1.61
1.50	0.74	1.24	1.60
1.60	0.70	1.22	1.58
1.70	0.65	1.19	1.56
1.80	0.59	1.16	1.53
1.90	0.53	1.13	1.51
2.00	0.45	1.10	1.48


Streamline Plot (t = 0)
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
y
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
x
c = 1
c = 2
c = 3


Streamline Plot (t = 1s)
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
y
0.0
0.1
0.2
0.3
0.4
0.5
0.6
x
c = 1
c = 2
c = 3


Streamline Plot (t = 20s)
2.0
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
y
0.0
0.5
1.0
1.5
2.0
2.5
x
c = 1
c = 2
c = 3

Problem 2.17

[Difficulty: 2]

2.17 Air flows downward toward an infinitely wide horizontal flat plate. The velocity field is given by $\vec{V} = (ax\hat{i} - ay\hat{j})(2 + \cos\omega t)$, where $a = 5\ s^{-1}$, $\omega = 2\pi\ s^{-1}$, x and y (measured in meters) are horizontal and vertically upward, respectively, and t is in s. Obtain an algebraic equation for a streamline at $t = 0$. Plot the streamline that passes through point $(x, y) = (3, 3)$ at this instant. Will the streamline change with time? Explain briefly. Show the velocity vector on your plot at the same point and time. Is the velocity vector tangent to the streamline? Explain.

Given: Time-varying velocity field

Find: Streamlines at t = 0 s; Streamline through (3,3); velocity vector; will streamlines change with time

Solution:

For streamlines $$\frac{v}{u} = \frac{dy}{dx} = -\frac{a\cdot y\cdot(2+\cos(\omega\cdot t))}{a\cdot x\cdot(2+\cos(\omega\cdot t))} = -\frac{y}{x}$$

At t = 0 (actually all times!) $$\frac{dy}{dx} = -\frac{y}{x}$$

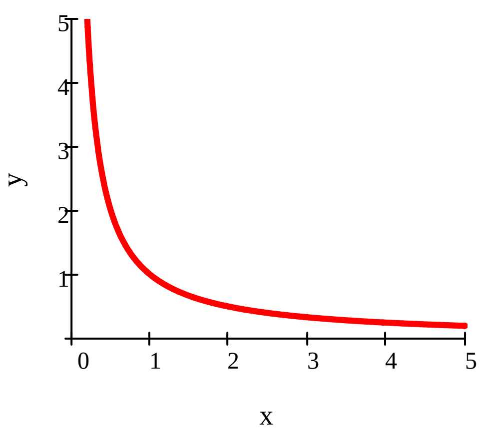
So, separating variables $$\frac{dy}{y} = -\frac{dx}{x}$$

Integrating $$\ln(y) = -\ln(x) + c$$

The solution is $$y = \frac{C}{x}$$ which is the equation of a hyperbola.

For the streamline through point (3,3) $$C = \frac{3}{3} \qquad C = 1 \qquad \text{and} \qquad y = \frac{1}{x}$$

The streamlines will not change with time since dy/dx does not change with time.



This curve can be plotted in Excel.

At t = 0 $$u = a\cdot x\cdot(2+\cos(\omega\cdot t)) = 5\cdot\frac{1}{s}\times 3\cdot m\times 3$$

$$u = 45\cdot\frac{m}{s}$$

$$v = -a\cdot y\cdot(2+\cos(\omega\cdot t)) = 5\cdot\frac{1}{s}\times 3\cdot m\times 3$$

$$v = -45\cdot\frac{m}{s}$$

The velocity vector is tangent to the curve;

Tangent of curve at (3,3) is $$\frac{dy}{dx} = -\frac{y}{x} = -1$$

Direction of velocity at (3,3) is $$\frac{v}{u} = -1$$

Problem 2.18 [Difficulty: 3]

2.18 Consider the flow described by the velocity field $\vec{V} = Bx(1 + At)\hat{i} + Cy\hat{j}$, with $A = 0.5\ \text{s}^{-1}$ and $B = C = 1\ \text{s}^{-1}$. Coordinates are measured in meters. Plot the pathline traced out by the particle that passes through the point (1, 1) at time $t = 0$. Compare with the streamlines plotted through the same point at the instants $t = 0$, 1, and 2 s.

Given: Velocity field

Find: Plot of pathline traced out by particle that passes through point (1,1) at t = 0; compare to streamlines through same point at the instants t = 0, 1 and 2s

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $v_p = \frac{dy}{dt}$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Assumption: 2D flow

Hence for pathlines $u_p = \frac{dx}{dt} = B\cdot x\cdot(1 + A\cdot t)$ $A = 0.5\cdot\frac{1}{s}$ $B = 1\cdot\frac{1}{s}$ $v_p = \frac{dy}{dt} = C\cdot y$ $C = 1\cdot\frac{1}{s}$

So, separating variables $\frac{dx}{x} = B\cdot(1 + A\cdot t)\cdot dt$ $\frac{dy}{y} = C\cdot dt$

Integrating $\ln(x) = B\cdot\left(t + A\cdot\frac{t^2}{2}\right) + C_1$ $\ln(y) = C\cdot t + C_2$

$$x = e^{B\cdot\left(t+A\cdot\frac{t^2}{2}\right)+C_1} = e^{C_1}\cdot e^{B\cdot\left(t+A\cdot\frac{t^2}{2}\right)} = c_1\cdot e^{B\cdot\left(t+A\cdot\frac{t^2}{2}\right)} \qquad y = e^{C\cdot t+C_2} = e^{C_2}\cdot e^{C\cdot t} = c_2\cdot e^{C\cdot t}$$

The pathlines are $x = c_1\cdot e^{B\cdot\left(t+A\cdot\frac{t^2}{2}\right)}$ $y = c_2\cdot e^{C\cdot t}$

Using given data $x = e^{B\cdot\left(t+A\cdot\frac{t^2}{2}\right)}$ $y = e^{C\cdot t}$

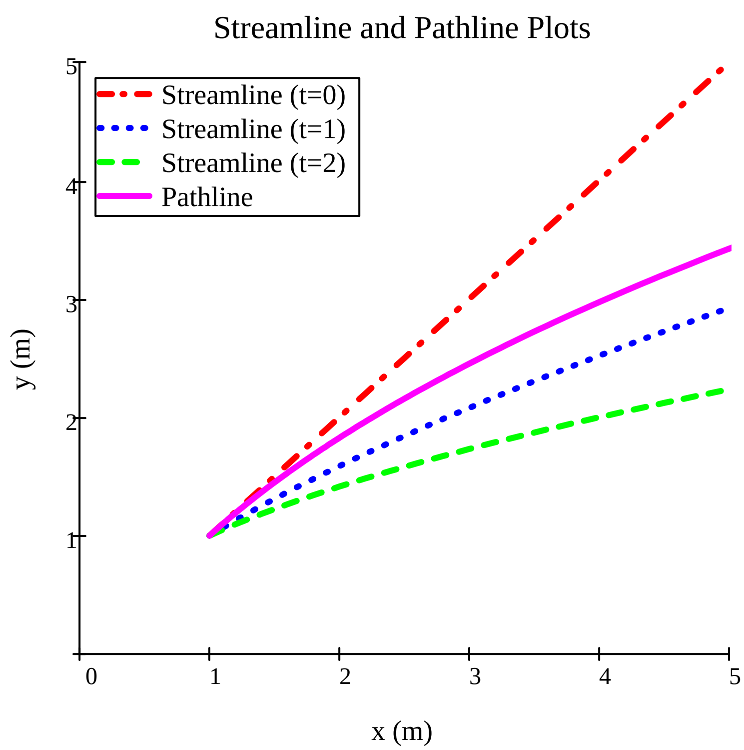
For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{C\cdot y}{B\cdot x\cdot(1 + A\cdot t)}$

So, separating variables $(1 + A\cdot t)\cdot\frac{dy}{y} = \frac{C}{B}\cdot\frac{dx}{x}$ which we can integrate for any given t (t is treated as a constant)

Integrating $(1 + A\cdot t)\cdot\ln(y) = \frac{C}{B}\cdot\ln(x) + c$

The solution is $\quad y^{1+A\cdot t} = \text{const}\cdot x^{\frac{C}{B}}$ $\quad$ or $\quad y = \text{const}\cdot x$

For particles at (1,1) at t = 0, 1, and 2s $\quad y = x^{\frac{C}{B}}$ $\quad y = x^{\frac{C}{(1+A)B}}$ $\quad y = x^{\frac{C}{(1+2\cdot A)B}}$



Problem 2.19 [Difficulty: 3]

2.19 Consider the velocity field $V = ax\hat{i} + by(1 + ct)\hat{j}$, where $a = b = 2\ \mathrm{s}^{-1}$ and $c = 0.4\ \mathrm{s}^{-1}$. Coordinates are measured in meters. For the particle that passes through the point $(x, y) = (1, 1)$ at the instant $t = 0$, plot the pathline during the interval from $t = 0$ to 1.5 s. Compare this pathline with the streamlines plotted through the same point at the instants $t = 0$, 1, and 1.5 s.

Given: Velocity field

Find: Plot of pathline of particle for t = 0 to 1.5 s that was at point (1,1) at t = 0; compare to streamlines through same point at the instants t = 0, 1 and 1.5 s

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $v_p = \frac{dy}{dt}$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Assumption: 2D flow

Hence for pathlines $u_p = \frac{dx}{dt} = ax$ $a = 2\ \frac{1}{s}$ $v_p = \frac{dy}{dt} = b\cdot y\cdot(1 + c\cdot t)$ $b = 2\ \frac{1}{s^2}$ $c = 0.4\ \frac{1}{s}$

So, separating variables $\frac{dx}{x} = a\cdot dt$ $dy = b\cdot y\cdot(1 + c\cdot t)\cdot dt$ $\frac{dy}{y} = b\cdot(1 + c\cdot t)\cdot dt$

Integrating $\ln\left(\frac{x}{x_0}\right) = a\cdot t$ $x_0 = 1\ \mathrm{m}$ $\ln\left(\frac{y}{y_0}\right) = b\cdot\left(t + \frac{1}{2}\cdot c\cdot t^2\right)$ $y_0 = 1\ \mathrm{m}$

Hence $x(t) = x_0\cdot e^{a\cdot t}$ $y(t) = e^{b\cdot\left(t+\frac{1}{2}\cdot c\cdot t^2\right)}$

Using given data $x(t) = e^{2\cdot t}$ $y(t) = e^{2\cdot t+0.4\cdot t^2}$

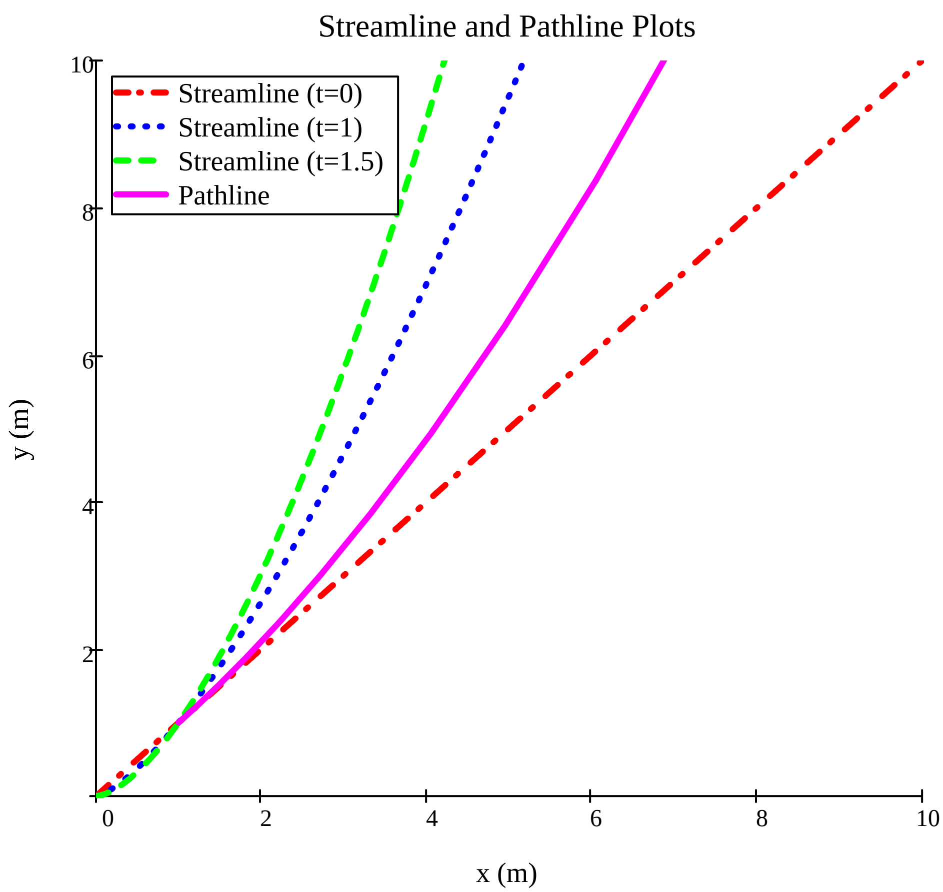
For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{b\cdot y\cdot(1 + c\cdot t)}{a\cdot x}$

So, separating variables $\frac{dy}{y} = \frac{b\cdot(1 + c\cdot t)}{a\cdot x}\cdot dx$ which we can integrate for any given t (t is treated as a constant)

Hence $\ln\left(\frac{y}{y_0}\right) = \frac{b}{a}\cdot(1 + c\cdot t)\cdot\ln\left(\frac{x}{x_0}\right)$

The solution is $y = y_0\cdot\left(\frac{x}{x_0}\right)^{\frac{b}{a}\cdot(1+c\cdot t)}$

For $t = 0$ $\quad y = y_0 \cdot \left(\frac{x}{x_0}\right)^{\frac{b}{a}\cdot(1+c\cdot t)} = x$ $\quad t = 1$ $\quad y = y_0 \cdot \left(\frac{x}{x_0}\right)^{\frac{b}{a}\cdot(1+c\cdot t)} = x^{1.4}$ $\quad t = 1.5$ $\quad y = y_0 \cdot \left(\frac{x}{x_0}\right)^{\frac{b}{a}\cdot(1+c\cdot t)} = x^{1.6}$



Problem 2.20 [Difficulty: 3]

2.20 Consider the flow field given in Eulerian description by the expression $\vec{V} = ax\hat{i} + byt\hat{j}$, where $a = 0.2\ \text{s}^{-1}$, $b = 0.04\ \text{s}^{-2}$, and the coordinates are measured in meters. Derive the Lagrangian position functions for the fluid particle that was located at the point $(x, y) = (1, 1)$ at the instant $t = 0$. Obtain an algebraic expression for the pathline followed by this particle. Plot the pathline and compare with the streamlines plotted through the same point at the instants $t = 0$, 10, and 20 s.

Given: Velocity field

Find: Plot of pathline of particle for t = 0 to 1.5 s that was at point (1,1) at t = 0; compare to streamlines through same point at the instants t = 0, 1 and 1.5 s

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $v_p = \frac{dy}{dt}$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Assumption: 2D flow

Hence for pathlines $u_p = \frac{dx}{dt} = a \cdot x$ $a = \frac{1}{5} \frac{1}{s}$ $v_p = \frac{dy}{dt} = b \cdot y \cdot t$ $b = \frac{1}{25} \frac{1}{s^2}$

So, separating variables $\frac{dx}{x} = a \cdot dt$ $dy = b \cdot y \cdot t \cdot dt$ $\frac{dy}{y} = b \cdot t \cdot dt$

Integrating $\ln\left(\frac{x}{x_0}\right) = a \cdot t$ $x_0 = 1\ \ m$ $\ln\left(\frac{y}{y_0}\right) = b \cdot \frac{1}{2} \cdot t^2$ $y_0 = 1\ \ m$

Hence $x(t) = x_0 \cdot e^{a \cdot t}$ $y(t) = y_0 \cdot e^{\frac{1}{2} \cdot b \cdot t^2}$

Using given data $x(t) = e^{\frac{t}{5}}$ $y(t) = e^{\frac{t^2}{50}}$

For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{b \cdot y \cdot t}{a \cdot x}$

So, separating variables $\frac{dy}{y} = \frac{b \cdot t}{a \cdot x} \cdot dx$ which we can integrate for any given t (t is treated as a constant)

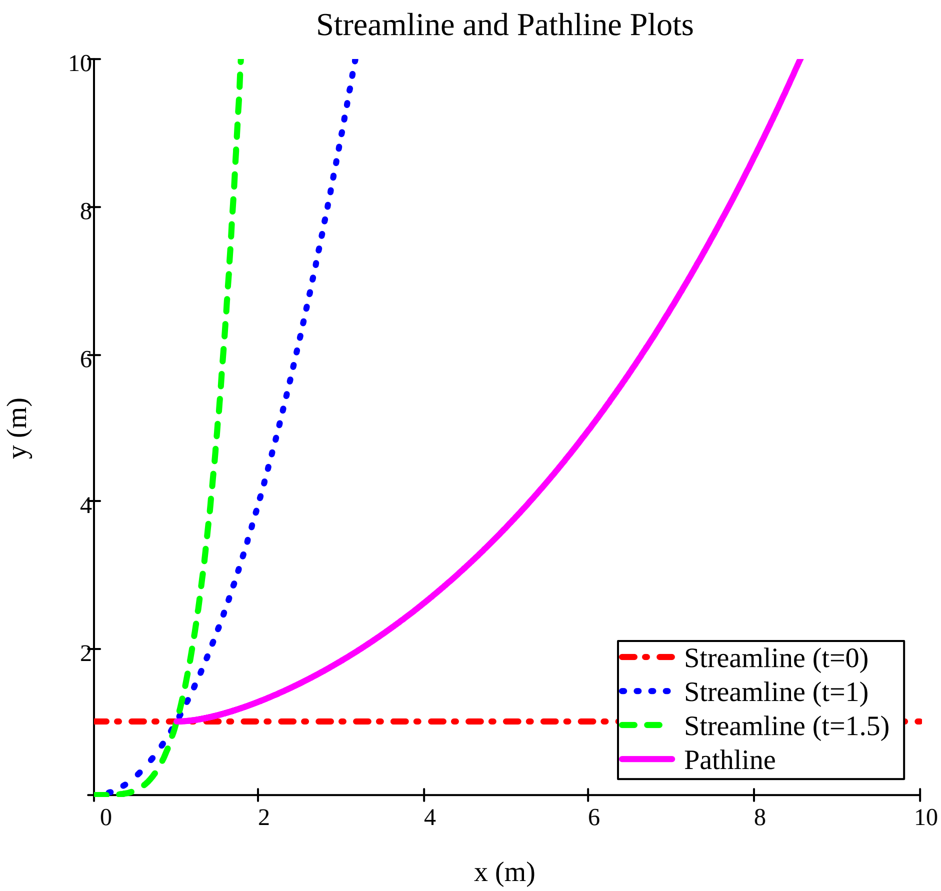
Hence $\ln\left(\frac{y}{y_0}\right) = \frac{b}{a} \cdot t \cdot \ln\left(\frac{x}{x_0}\right)$

The solution is $y = y_0 \cdot \left(\frac{x}{x_0}\right)^{\frac{b}{a} \cdot t}$ $\frac{b}{a} = 0.2$ $x_0 = 1$ $y_0 = 1$

For $\quad t = 0 \qquad y = y_0 \cdot \left(\frac{x}{x_0}\right)^{\frac{b}{a} \cdot t} = 1$

$t = 5 \qquad y = y_0 \cdot \left(\frac{x}{x_0}\right)^{\frac{b}{a} \cdot t} = x \qquad \frac{b}{a} \cdot t = 1$

$t = 10 \qquad y = y_0 \cdot \left(\frac{x}{x_0}\right)^{\frac{b}{a} \cdot t} = x^2 \qquad \frac{b}{a} \cdot t = 2$



Problem 2.21

[Difficulty: 3]

2.21 A velocity field is given by $\vec{V} = axt\hat{i} - by\hat{j}$, where $a = 0.1\ \mathrm{s}^{-2}$ and $b = 1\ \mathrm{s}^{-1}$. For the particle that passes through the point $(x, y) = (1, 1)$ at instant $t = 0$ s, plot the pathline during the interval from $t = 0$ to $t = 3$ s. Compare with the streamlines plotted through the same point at the instants $t = 0$, 1, and 2 s.

Given: Velocity field

Find: Plot pathlines and streamlines

Solution:

Pathlines are given by $\dfrac{dx}{dt} = u = a \cdot x \cdot t$ $\qquad$ $\dfrac{dy}{dt} = v = -b \cdot y$

So, separating variables $\dfrac{dx}{x} = a \cdot t \cdot dt$ $\qquad$ $\dfrac{dy}{y} = -b \cdot dt$

Integrating $\ln(x) = \dfrac{1}{2} \cdot a \cdot t^2 + c_1$ $\qquad$ $\ln(y) = -b \cdot t + c_2$

For initial position (x_0, y_0) $\qquad$ $x = x_0 \cdot e^{\frac{a}{2} \cdot t^2}$ $\qquad$ $y = y_0 \cdot e^{-b \cdot t}$

Using the given data, and IC $(x_0, y_0) = (1,1)$ at $t = 0$

$x = e^{0.05 \cdot t^2}$ $\qquad$ $y = e^{-t}$

Streamlines are given by $\dfrac{v}{u} = \dfrac{dy}{dx} = \dfrac{-b \cdot y}{a \cdot x \cdot t}$

So, separating variables $\dfrac{dy}{y} = -\dfrac{b}{a \cdot t} \cdot \dfrac{dx}{x}$ $\qquad$ Integrating $\qquad$ $\ln(y) = -\dfrac{b}{a \cdot t} \cdot \ln(x) + C$

The solution is $y = C \cdot x^{-\frac{b}{a \cdot t}}$

For streamline at (1,1) at $t = 0$ s $\qquad$ $x = c$

For streamline at (1,1) at $t = 1$ s $\qquad$ $y = x^{-10}$

For streamline at (1,1) at $t = 2$ s $\qquad$ $y = x^{-5}$

Pathline

t	x	y
0.00	1.00	1.00
0.25	1.00	0.78
0.50	1.01	0.61
0.75	1.03	0.47
1.00	1.05	0.37
1.25	1.08	0.29
1.50	1.12	0.22
1.75	1.17	0.17
2.00	1.22	0.14
2.25	1.29	0.11
2.50	1.37	0.08
2.75	1.46	0.06
3.00	1.57	0.05
3.25	1.70	0.04
3.50	1.85	0.03
3.75	2.02	0.02
4.00	2.23	0.02
4.25	2.47	0.01
4.50	2.75	0.01
4.75	3.09	0.01
5.00	3.49	0.01

Streamlines

t = 0

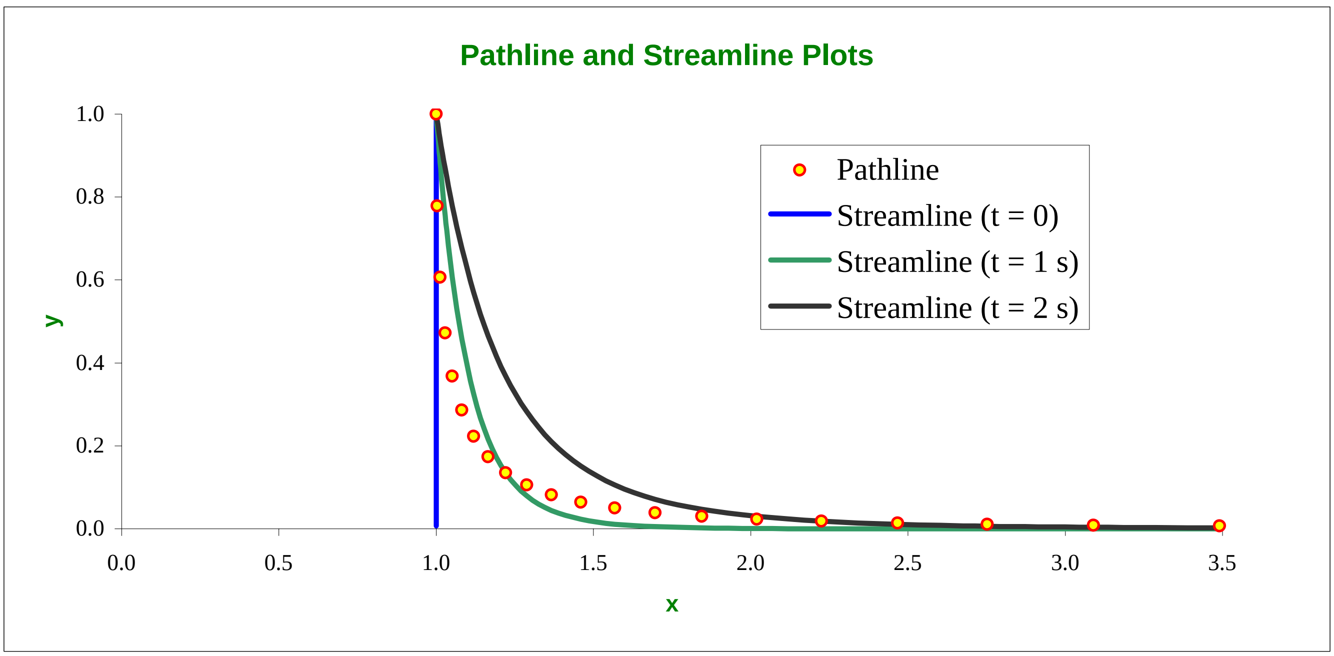
x	y
1.00	1.00
1.00	0.78
1.00	0.61
1.00	0.47
1.00	0.37
1.00	0.29
1.00	0.22
1.00	0.17
1.00	0.14
1.00	0.11
1.00	0.08
1.00	0.06
1.00	0.05
1.00	0.04
1.00	0.03
1.00	0.02
1.00	0.02
1.00	0.01
1.00	0.01
1.00	0.01
1.00	0.01

t = 1 s

x	y
1.00	1.00
1.00	0.97
1.01	0.88
1.03	0.75
1.05	0.61
1.08	0.46
1.12	0.32
1.17	0.22
1.22	0.14
1.29	0.08
1.37	0.04
1.46	0.02
1.57	0.01
1.70	0.01
1.85	0.00
2.02	0.00
2.23	0.00
2.47	0.00
2.75	0.00
3.09	0.00
3.49	0.00

t = 2 s

x	y
1.00	1.00
1.00	0.98
1.01	0.94
1.03	0.87
1.05	0.78
1.08	0.68
1.12	0.57
1.17	0.47
1.22	0.37
1.29	0.28
1.37	0.21
1.46	0.15
1.57	0.11
1.70	0.07
1.85	0.05
2.02	0.03
2.23	0.02
2.47	0.01
2.75	0.01
3.09	0.00
3.49	0.00



Problem 2.22

[Difficulty: 4]

2.22 Consider the garden hose of Fig. 2.5. Suppose the velocity field is given by $\vec{V} = u_0\hat{i} + v_0 \sin[\omega(t - x/u_0)]\hat{j}$, where the x direction is horizontal and the origin is at the mean position of the hose, $u_0 = 10$ m/s, $v_0 = 2$ m/s, and $\omega = 5$ cycle/s. Find and plot on one graph the instantaneous streamlines that pass through the origin at $t = 0$ s, 0.05 s, 0.1 s, and 0.15 s. Also find and plot on one graph the pathlines of particles that left the origin at the same four times.

Given: Velocity field

Find: Plot streamlines that are at origin at various times and pathlines that left origin at these times

Solution:

For streamlines
$$\frac{v}{u} = \frac{dy}{dx} = \frac{v_0 \cdot \sin\left[\omega \cdot \left(t - \frac{x}{u_0}\right)\right]}{u_0}$$

So, separating variables (t=const)
$$dy = \frac{v_0 \cdot \sin\left[\omega \cdot \left(t - \frac{x}{u_0}\right)\right]}{u_0} \cdot dx$$

Integrating
$$y = \frac{v_0 \cdot \cos\left[\omega \cdot \left(t - \frac{x}{u_0}\right)\right]}{\omega} + c$$

Using condition y = 0 when x = 0
$$y = \frac{v_0 \cdot \left[\cos\left[\omega \cdot \left(t - \frac{x}{u_0}\right)\right] - \cos(\omega \cdot t)\right]}{\omega}$$
This gives streamlines y(x) at each time t

For particle paths, first find x(t)
$$\frac{dx}{dt} = u = u_0$$

Separating variables and integrating $dx = u_0 \cdot dt$ or $x = u_0 \cdot t + c_1$

Using initial condition x = 0 at t = τ $c_1 = -u_0 \cdot \tau$ $x = u_0 \cdot (t - \tau)$

For y(t) we have
$$\frac{dy}{dt} = v = v_0 \cdot \sin\left[\omega \cdot \left(t - \frac{x}{u_0}\right)\right] \quad \text{so} \quad \frac{dy}{dt} = v = v_0 \cdot \sin\left[\omega \cdot \left[t - \frac{u_0 \cdot (t - \tau)}{u_0}\right]\right]$$

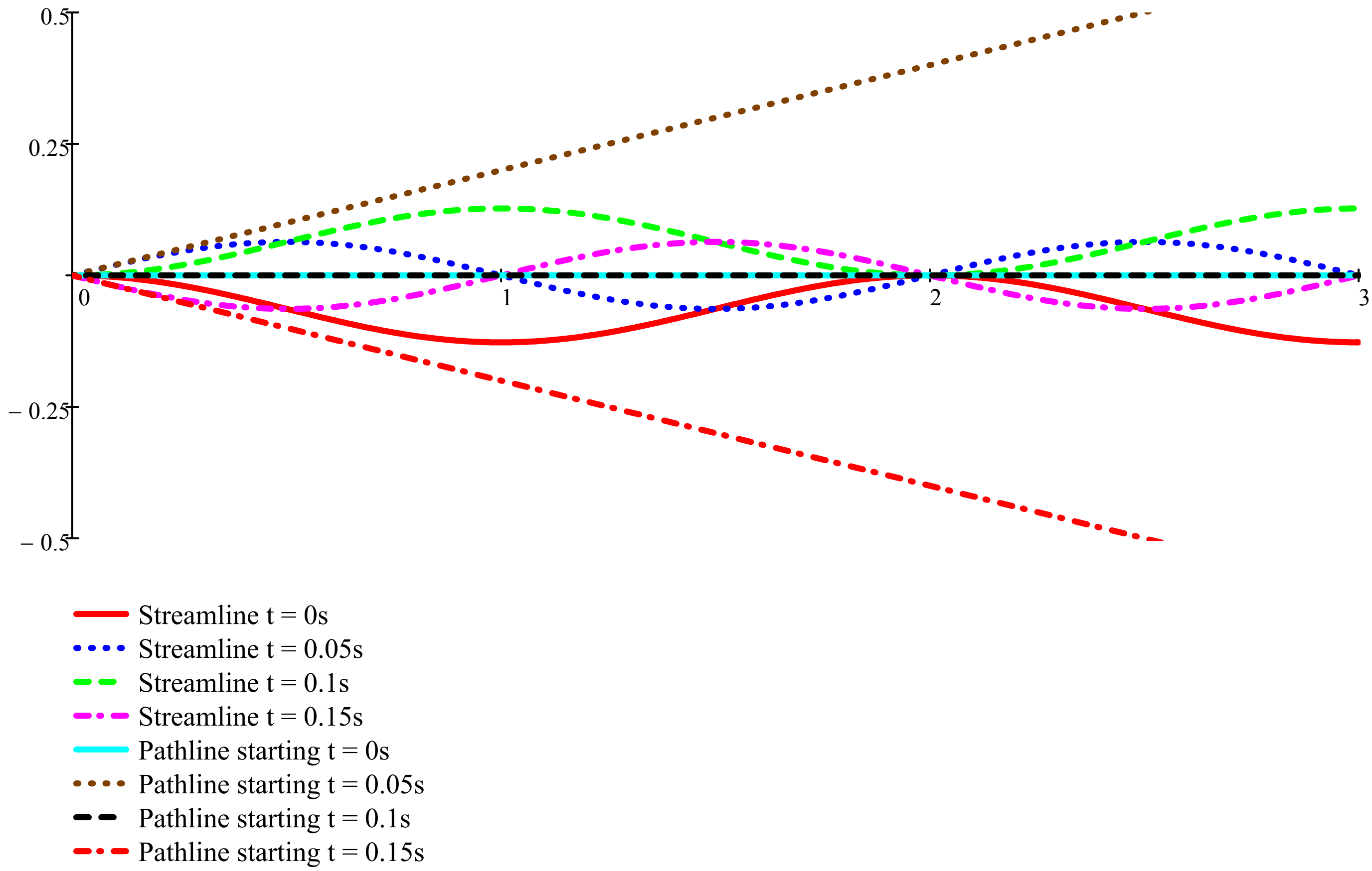
and
$$\frac{dy}{dt} = v = v_0 \cdot \sin(\omega \cdot \tau)$$

Separating variables and integrating $dy = v_0 \cdot \sin(\omega \cdot \tau) \cdot dt$ $y = v_0 \cdot \sin(\omega \cdot \tau) \cdot t + c_2$

Using initial condition y = 0 at t = τ $c_2 = -v_0 \cdot \sin(\omega \cdot \tau) \cdot \tau$ $y = v_0 \cdot \sin(\omega \cdot \tau) \cdot (t - \tau)$

The pathline is then

$x(t, \tau) = u_0 \cdot (t - \tau)$ $y(t, \tau) = v_0 \cdot \sin(\omega \cdot \tau) \cdot (t - \tau)$ These terms give the path of a particle (x(t),y(t)) that started at t = τ.



The streamlines are sinusoids; the pathlines are straight (once a water particle is fired it travels in a straight line).

These curves can be plotted in *Excel*.

Problem 2.23 [Difficulty: 4]

2.23 Consider the velocity field of Problem 2.18 Plot the streakline formed by particles that passed through the point (1, 1) during the interval from $t = 0$ to $t = 3$ s. Compare with the streamlines plotted through the same point at the instants $t = 0$, 1, and 2 s.

Given: Velocity field

Find: Plot of streakline for t = 0 to 3 s at point (1,1); compare to streamlines through same point at the instants t = 0, 1 and 2 s

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $\quad v_p = \frac{dy}{dt}$ $\quad$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Following the discussion leading up to Eq. we first find equations for the pathlines in form

$$x_p(t) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_p(t) = y(t, x_0, y_0, t_0)$$

$$x_{st}(t_0) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_{st}(t_0) = y(t, x_0, y_0, t_0)$$

which gives the streakline at t, where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

Assumption: 2D flow

For pathlines $u_p = \frac{dx}{dt} = B \cdot x \cdot (1 + A \cdot t)$ $\quad A = 0.5 \cdot \frac{1}{s}$ $\quad B = 1 \cdot \frac{1}{s}$ $\quad v_p = \frac{dy}{dt} = C \cdot y$ $\quad C = 1 \cdot \frac{1}{s}$

So, separating variables $\frac{dx}{x} = B \cdot (1 + A \cdot t) \cdot dt$ $\qquad \frac{dy}{y} = C \cdot dt$

Integrating $\ln\left(\frac{x}{x_0}\right) = B \cdot \left(t - t_0 + A \cdot \frac{t^2 - t_0^{\,2}}{2}\right)$ $\qquad \ln\left(\frac{y}{y_0}\right) = C \cdot (t - t_0)$

$$x = x_0 \cdot e^{B \cdot \left(t - t_0 + A \cdot \frac{t^2 - t_0^2}{2}\right)} \qquad y = y_0 \cdot e^{C \cdot (t - t_0)}$$

The pathlines are $x_p(t) = x_0 \cdot e^{B \cdot \left(t - t_0 + A \cdot \frac{t^2 - t_0^2}{2}\right)}$ $\qquad y_p(t) = y_0 \cdot e^{C \cdot (t - t_0)}$

where x_0, y_0 is the position of the particle at $t = t_0$. Re-interpreting the results as streaklines:

The streaklines are then $x_{st}(t_0) = x_0 \cdot e^{B \cdot \left(t - t_0 + A \cdot \frac{t^2 - t_0^2}{2}\right)}$ $\qquad y_{st}(t_0) = y_0 \cdot e^{C \cdot (t - t_0)}$

where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

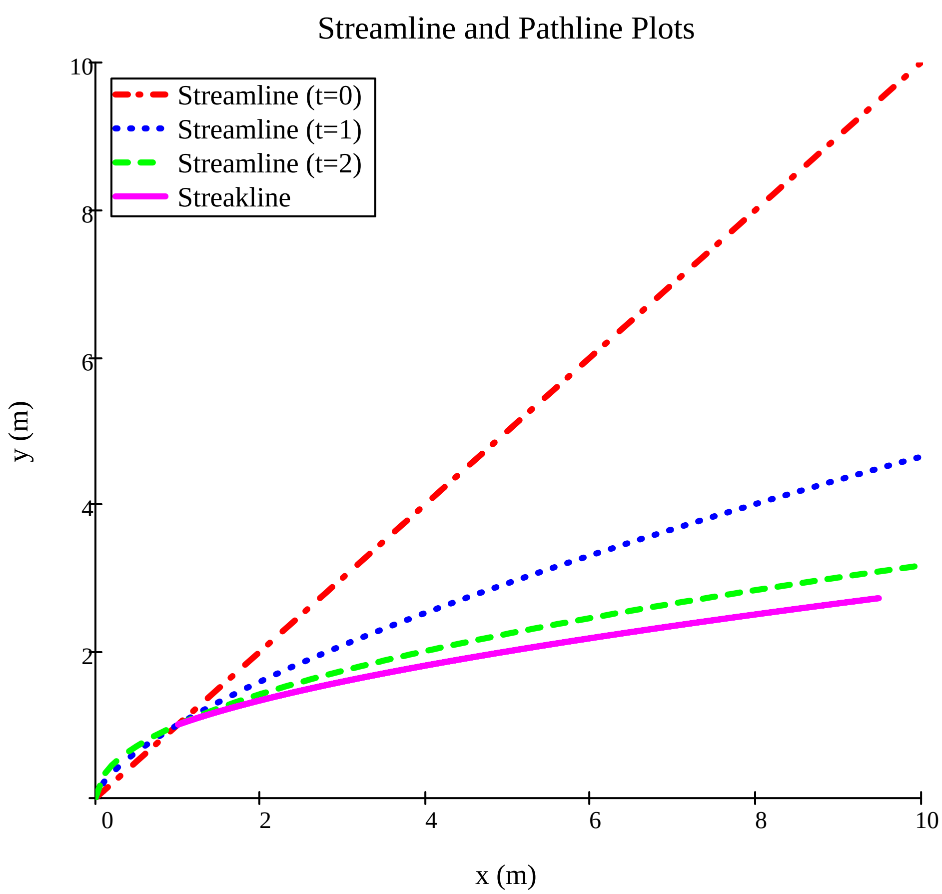
For streamlines $\quad \frac{v}{u} = \frac{dy}{dx} = \frac{C \cdot y}{B \cdot x \cdot (1 + A \cdot t)}$

So, separating variables $\quad (1 + A \cdot t) \cdot \frac{dy}{y} = \frac{C}{B} \cdot \frac{dx}{x}$ $\quad$ which we can integrate for any given t (t is treated as a constant)

Integrating $\quad (1 + A \cdot t) \cdot \ln(y) = \frac{C}{B} \cdot \ln(x) + \text{const}$

The solution is $\quad y^{1+A \cdot t} = \text{const} \cdot x^{\frac{C}{B}}$

For particles at (1,1) at t = 0, 1, and 2s $\quad y = x \qquad y = x^{\frac{2}{3}} \qquad y = x^{\frac{1}{2}}$



Problem 2.24 [Difficulty: 4]

2.24 Streaklines are traced out by neutrally buoyant marker fluid injected into a flow field from a fixed point in space. A particle of the marker fluid that is at point (x, y) at time t must have passed through the injection point (x_0, y_0) at some earlier instant $t = \tau$. The time history of a marker particle may be found by solving the pathline equations for the initial conditions that $x = x_0$, $y = y_0$ when $t = \tau$. The present locations of particles on the streakline are obtained by setting τ equal to values in the range $0 \le \tau \le t$. Consider the flow field $\vec{V} = ax(1 + bt)\hat{i} + cy\hat{j}$, where $a = c = 1\ s^{-1}$ and $b = 0.2\ s^{-1}$. Coordinates are measured in meters. Plot the streakline that passes through the initial point $(x_0, y_0) = (1, 1)$, during the interval from $t = 0$ to $t = 3$ s. Compare with the streamline plotted through the same point at the instants $t = 0$, 1, and 2 s.

Given: Velocity field

Find: Plot of streakline for t = 0 to 3 s at point (1,1); compare to streamlines through same point at the instants t = 0, 1 and 2 s

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $\quad v_p = \frac{dy}{dt}$ $\quad$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Following the discussion leading up to Eq. 2.10, we first find equations for the pathlines in form

$$x_p(t) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_p(t) = y(t, x_0, y_0, t_0)$$

$$x_{st}(t_0) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_{st}(t_0) = y(t, x_0, y_0, t_0)$$

which gives the streakline at t, where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

Assumption: 2D flow

For pathlines $u_p = \frac{dx}{dt} = a \cdot x \cdot (1 + b \cdot t)$ $\quad a = 1\ \frac{1}{s}$ $\quad b = \frac{1}{5}\ \frac{1}{s}$ $\quad v_p = \frac{dy}{dt} = c \cdot y$ $\quad c = 1\ \frac{1}{s}$

So, separating variables $\frac{dx}{x} = a \cdot (1 + b \cdot t) \cdot dt$ $\qquad \frac{dy}{y} = c \cdot dt$

Integrating $\ln\left(\frac{x}{x_0}\right) = a \cdot \left(t - t_0 + b \cdot \frac{t^2 - t_0^2}{2}\right)$ $\qquad \ln\left(\frac{y}{y_0}\right) = c \cdot (t - t_0)$

$$x = x_0 \cdot e^{a \cdot \left(t - t_0 + b \cdot \frac{t^2 - t_0^2}{2}\right)} \qquad y = y_0 \cdot e^{c \cdot (t - t_0)}$$

The pathlines are $\quad x_p(t) = x_0 \cdot e^{a\cdot\left(t-t_0+b\cdot\frac{t^2-t_0{}^2}{2}\right)} \qquad y_p(t) = y_0 \cdot e^{c\cdot(t-t_0)}$

where x_0, y_0 is the position of the particle at $t = t_0$. Re-interpreting the results as streaklines:

The streaklines are then $\quad x_{st}(t_0) = x_0 \cdot e^{a\cdot\left(t-t_0+b\cdot\frac{t^2-t_0{}^2}{2}\right)} \qquad y_{st}(t_0) = y_0 \cdot e^{c\cdot(t-t_0)}$

where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

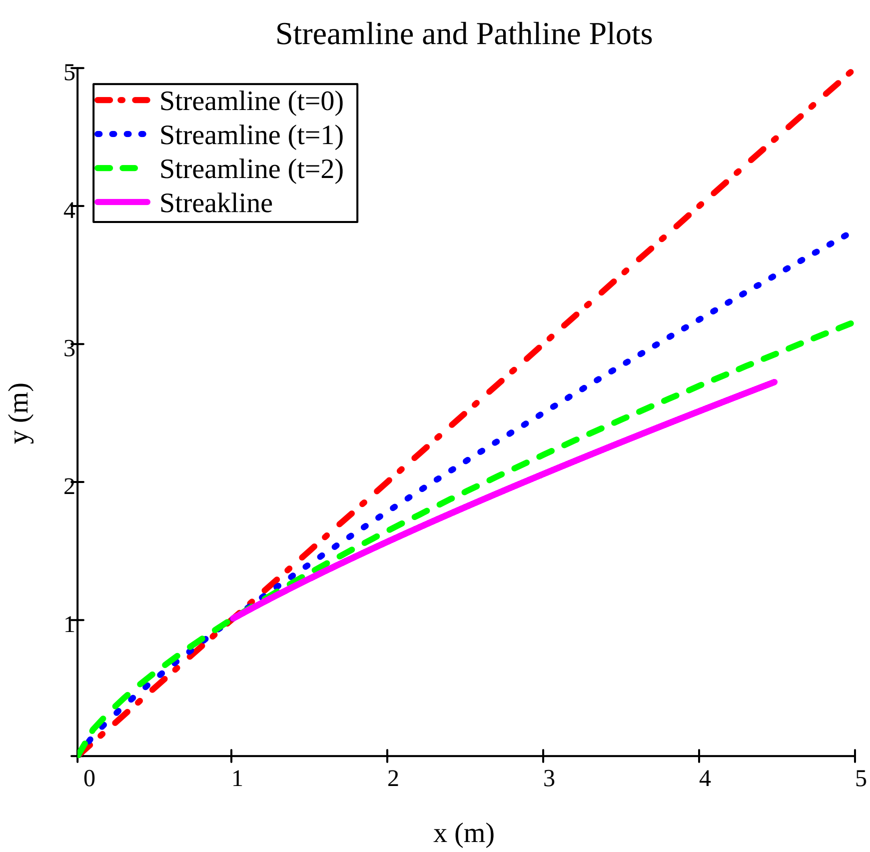
For streamlines $\quad \frac{v}{u} = \frac{dy}{dx} = \frac{c\cdot y}{a\cdot x\cdot(1+b\cdot t)}$

So, separating variables $\quad (1+b\cdot t)\cdot\frac{dy}{y} = \frac{c}{a}\cdot\frac{dx}{x}$ $\qquad$ which we can integrate for any given t (t is treated as a constant)

Integrating $\quad (1+b\cdot t)\cdot\ln(y) = \frac{c}{a}\cdot\ln(x) + \text{const}$

The solution is $\quad y^{1+b\cdot t} = \text{const}\cdot x^{\frac{c}{a}}$

For particles at (1,1) at t = 0, 1, and 2s $\qquad y = x \qquad y = x^{\frac{2}{3}} \qquad y = x^{\frac{1}{2}}$



2.25 Consider the flow field $\vec{V} = axt\hat{i} + b\hat{j}$, where $a = 1/4\ s^{-2}$ and $b = 1/3$ m/s. Coordinates are measured in meters. For the particle that passes through the point $(x, y) = (1, 2)$ at the instant $t = 0$, plot the pathline during the time interval from $t = 0$ to 3 s. Compare this pathline with the streakline through the same point at the instant $t = 3$ s.

Given: Velocity field

Find: Plot of pathline for t = 0 to 3 s for particle that started at point (1,2) at t = 0; compare to streakline through same point at the instant t = 3

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $\quad v_p = \frac{dy}{dt}$

Following the discussion leading up to Eq. we first find equations for the pathlines in form

$$x_p(t) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_p(t) = y(t, x_0, y_0, t_0)$$

$$x_{st}(t_0) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_{st}(t_0) = y(t, x_0, y_0, t_0)$$

which gives the streakline at t, where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

Assumption: 2D flow

For pathlines $u_p = \frac{dx}{dt} = a \cdot x \cdot t$ $\quad a = \frac{1}{4} \cdot \frac{1}{s^2}$ $\quad b = \frac{1}{3} \cdot \frac{m}{s}$ $\quad v_p = \frac{dy}{dt} = b$

So, separating variables $\frac{dx}{x} = a \cdot t \cdot dt$ $\quad dy = b \cdot dt$

Integrating $\ln\left(\frac{x}{x_0}\right) = \frac{a}{2} \cdot \left(t^2 - t_0^{\,2}\right)$ $\quad y - y_0 = b \cdot (t - t_0)$

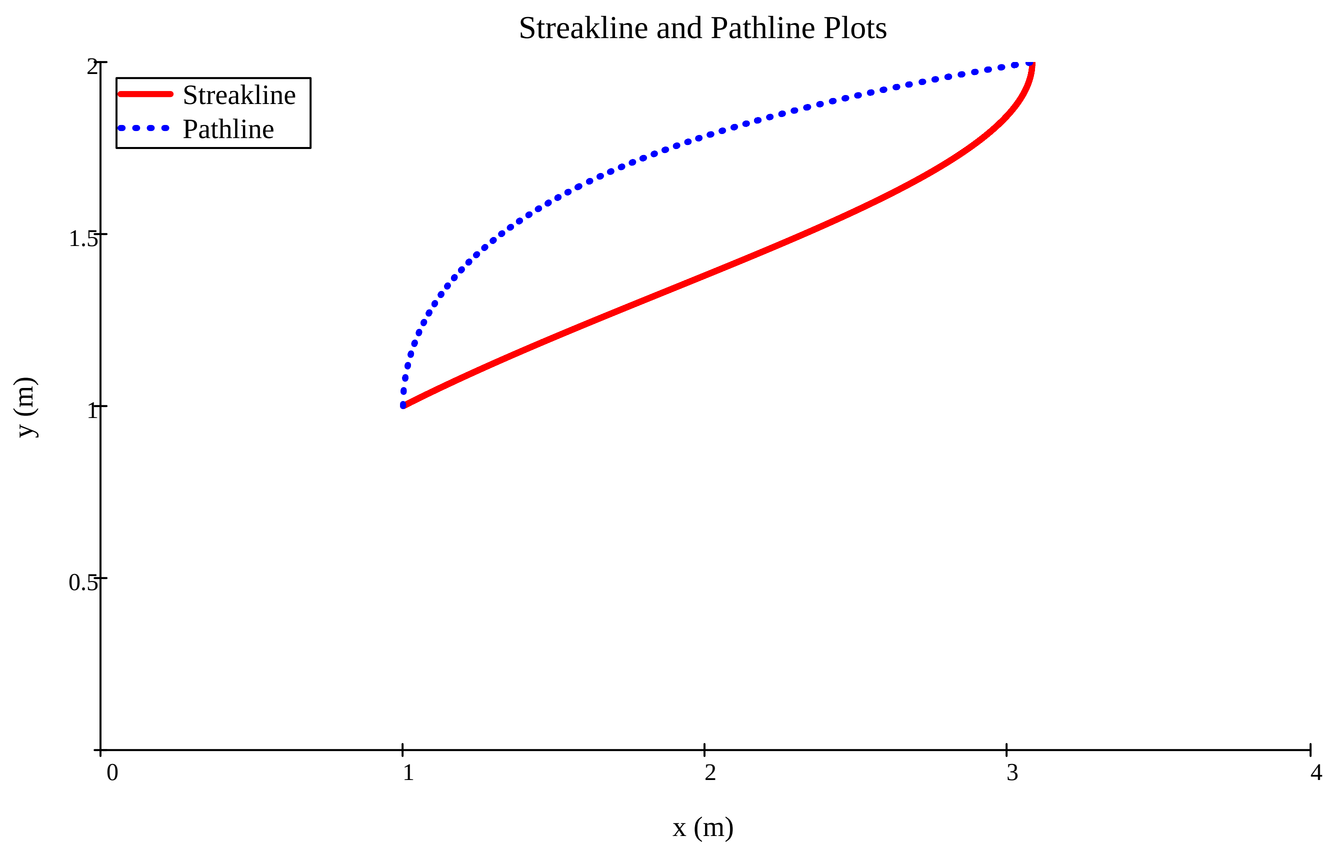
$$x = x_0 \cdot e^{\frac{a}{2} \cdot \left(t^2 - t_0^{\,2}\right)} \qquad y = y_0 + b \cdot (t - t_0)$$

The pathlines are $x_p(t) = x_0 \cdot e^{\frac{a}{2} \cdot \left(t^2 - t_0^{\,2}\right)}$ $\quad y_p(t) = y_0 + b \cdot (t - t_0)$

where x_0, y_0 is the position of the particle at $t = t_0$. Re-interpreting the results as streaklines:

The pathlines are then $x_{st}(t_0) = x_0 \cdot e^{\frac{a}{2} \cdot \left(t^2 - t_0^{\,2}\right)}$ $\quad y_{st}(t_0) = y_0 + b \cdot (t - t_0)$

where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)


Streakline and Pathline Plots
Streakline
Pathline
y (m)
x (m)
0
0.5
1
1.5
2
0
1
2
3
4

Problem 2.26

[Difficulty: 4]

2.26 A flow is described by velocity field $\vec{V} = ay^2\hat{i} + b\hat{j}$, where $a = 1\ \mathrm{m^{-1}s^{-1}}$ and $b = 2$ m/s. Coordinates are measured in meters. Obtain the equation for the streamline passing through point (6, 6). At $t = 1$ s, what are the coordinates of the particle that passed through point (1, 4) at $t = 0$? At $t = 3$ s, what are the coordinates of the particle that passed through point $(-3, 0)$ 2 s earlier? Show that pathlines, streamlines, and streaklines for this flow coincide.

Given: 2D velocity field

Find: Streamlines passing through (6,6); Coordinates of particle starting at (1,4); that pathlines, streamlines and streaklines coincide

Solution:

For streamlines $\quad \frac{v}{u} = \frac{dy}{dx} = \frac{b}{a\cdot y^2} \quad$ or $\quad \int a\cdot y^2\, dy = \int b\, dx$

Integrating $\quad \frac{a\cdot y^3}{3} = b\cdot x + c$

For the streamline through point (6,6) $\quad c = 60$ and $\quad y^3 = 6\cdot x + 180$

For particle that passed through (1,4) at t = 0 $\quad u = \frac{dx}{dt} = a\cdot y^2 \quad \int 1\, dx = x - x_0 = \int a\cdot y^2\, dt$ We need y(t)

$v = \frac{dy}{dt} = b \quad \int 1\, dy = \int b\, dt \quad y = y_0 + b\cdot t = y_0 + 2\cdot t$

Then $\quad x - x_0 = \int_0^t a\cdot\left(y_0 + b\cdot t\right)^2 dt \quad x = x_0 + a\cdot\left(y_0^{\,2}\cdot t + b\cdot y_0\cdot t^2 + \frac{b^2\cdot t^3}{3}\right)$

Hence, with $x_0 = 1 \quad y_0 = 4 \quad x = 1 + 16\cdot t + 8\cdot t^2 + \frac{4}{3}\cdot t^3$ At t = 1 s $\quad x = 26.3\cdot m$

$y = 4 + 2\cdot t \quad y = 6\cdot m$

For particle that passed through (-3,0) at t = 1 $\quad \int 1\, dy = \int b\, dt \quad y = y_0 + b\cdot\left(t - t_0\right)$

$$x - x_0 = \int_{t_0}^{t} a\cdot\left(y_0 + b\cdot t\right)^2 dt \qquad x = x_0 + a\cdot\left[y_0^{\,2}\cdot\left(t - t_0\right) + b\cdot y_0\cdot\left(t^2 - t_0^{\,2}\right) + \frac{b^2}{3}\cdot\left(t^3 - t_0^{\,3}\right)\right]$$

Hence, with $x_0 = -3$, $y_0 = 0$ at $t_0 = 1$ $\quad x = -3 + \frac{4}{3}\cdot\left(t^3 - 1\right) = \frac{1}{3}\cdot\left(4\cdot t^3 - 13\right) \quad y = 2\cdot(t - 1)$

Evaluating at t = 3 $\quad x = 31.7\cdot m \quad y = 4\cdot m$

This is a steady flow, so pathlines, streamlines and streaklines always coincide

Problem 2.27

[Difficulty: 3]

2.27 Tiny hydrogen bubbles are being used as tracers to visualize a flow. All the bubbles are generated at the origin ($x = 0$, $y = 0$). The velocity field is unsteady and obeys the equations:

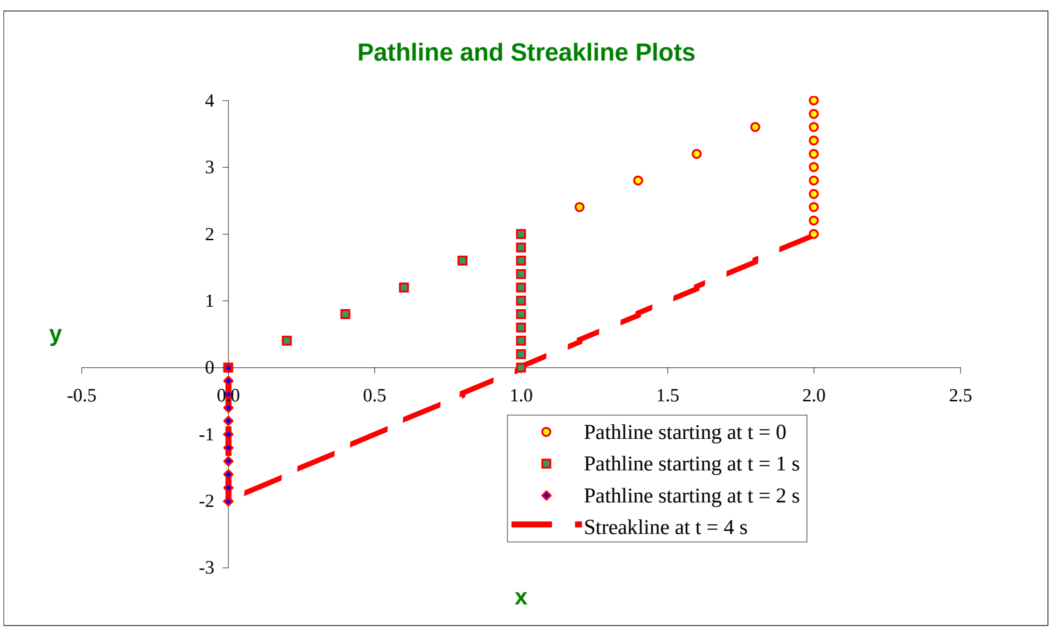
$$u = 1\ \text{m/s} \qquad v = 2\ \text{m/s} \qquad 0 \le t < 2\ \text{s}$$
$$u = 0 \qquad v = -1\ \text{m/s} \qquad 0 \le t \le 4\ \text{s}$$

Plot the pathlines of bubbles that leave the origin at $t = 0$, 1, 2, 3, and 4 s. Mark the locations of these five bubbles at $t = 4$ s. Use a dashed line to indicate the position of a streakline at $t = 4$ s.

Solution

The particle starting at t = 3 s follows the particle starting at t = 2 s;
The particle starting at t = 4 s doesn't move!

Pathlines:	**Starting at t = 0**		**Starting at t = 1 s**		**Starting at t = 2 s**		**Streakline at t = 4 s**	
t	**x**	**y**	**x**	**y**	**x**	**y**	**x**	**y**
0.00	0.00	0.00					2.00	2.00
0.20	0.20	0.40					1.80	1.60
0.40	0.40	0.80					1.60	1.20
0.60	0.60	1.20					1.40	0.80
0.80	0.80	1.60					1.20	0.40
1.00	1.00	2.00	0.00	0.00			1.00	0.00
1.20	1.20	2.40	0.20	0.40			0.80	-0.40
1.40	1.40	2.80	0.40	0.80			0.60	-0.80
1.60	1.60	3.20	0.60	1.20			0.40	-1.20
1.80	1.80	3.60	0.80	1.60			0.20	-1.60
2.00	2.00	4.00	1.00	2.00	0.00	0.00	0.00	-2.00
2.20	2.00	3.80	1.00	1.80	0.00	-0.20	0.00	-1.80
2.40	2.00	3.60	1.00	1.60	0.00	-0.40	0.00	-1.60
2.60	2.00	3.40	1.00	1.40	0.00	-0.60	0.00	-1.40
2.80	2.00	3.20	1.00	1.20	0.00	-0.80	0.00	-1.20
3.00	2.00	3.00	1.00	1.00	0.00	-1.00	0.00	-1.00
3.20	2.00	2.80	1.00	0.80	0.00	-1.20	0.00	-0.80
3.40	2.00	2.60	1.00	0.60	0.00	-1.40	0.00	-0.60
3.60	2.00	2.40	1.00	0.40	0.00	-1.60	0.00	-0.40
3.80	2.00	2.20	1.00	0.20	0.00	-1.80	0.00	-0.20
4.00	2.00	2.00	1.00	0.00	0.00	-2.00	0.00	0.00



Problem 2.28

[Difficulty: 3]

2.28 A flow is described by velocity field $\vec{V} = a\hat{i} + bx\hat{j}$, where $a = 2$ m/s and $b = 1\ s^{-1}$. Coordinates are measured in meters. Obtain the equation for the streamline passing through point (2, 5). At $t = 2$ s, what are the coordinates of the particle that passed through point (0, 4) at $t = 0$? At $t = 3$ s, what are the coordinates of the particle that passed through point (1, 4.25) 2 s earlier? What conclusions can you draw about the pathline, streamline, and streakline for this flow?

Given: Velocity field

Find: Equation for streamline through point (2.5); coordinates of particle at t = 2 s that was at (0,4) at t = 0; coordinates of particle at t = 3 s that was at (1,4.25) at t = 1 s; compare pathline, streamline, streakline

Solution:

Governing equations: For streamlines $\frac{v}{u} = \frac{dy}{dx}$ For pathlines $u_p = \frac{dx}{dt}$ $v_p = \frac{dy}{dt}$

Assumption: 2D flow

Given data $a = 2\,\frac{m}{s}$ $b = 1\,\frac{1}{s}$ $x_0 = 2$ $y_0 = 5$ $x = 1$ $x = x$

For streamlines $\frac{v}{u} = \frac{dy}{dx} = \frac{b \cdot x}{a}$

So, separating variables $\frac{a}{b} \cdot dy = x \cdot dx$

Integrating $\frac{a}{b} \cdot \left(y - y_0\right) = \frac{1}{2} \cdot \left(x^2 - x_0^{\,2}\right)$

The solution is then $y = y_0 + \frac{b}{2 \cdot a} \cdot \left(x^2 - x_0^{\,2}\right) = \frac{x^2}{4} + 4$

Hence for pathlines $u_p = \frac{dx}{dt} = a$ $v_p = \frac{dy}{dt} = b \cdot x$

Hence $dx = a \cdot dt$ $dy = b \cdot x \cdot dt$

Integrating $x - x_0 = a \cdot \left(t - t_0\right)$ $dy = b \cdot \left[x_0 + a \cdot \left(t - t_0\right)\right] \cdot dt$

$$y - y_0 = b \cdot \left[x_0 \cdot \left(t - t_0\right) + \frac{a}{2} \cdot \left(\left(t^2 - t_0^{\,2}\right)\right) - a \cdot t_0 \cdot \left(t - t_0\right)\right]$$

The pathlines are $x = x_0 + a \cdot \left(t - t_0\right)$ $y = y_0 + b \cdot \left[x_0 \cdot \left(t - t_0\right) + \frac{a}{2} \cdot \left(\left(t^2 - t_0^{\,2}\right)\right) - a \cdot t_0 \cdot \left(t - t_0\right)\right]$

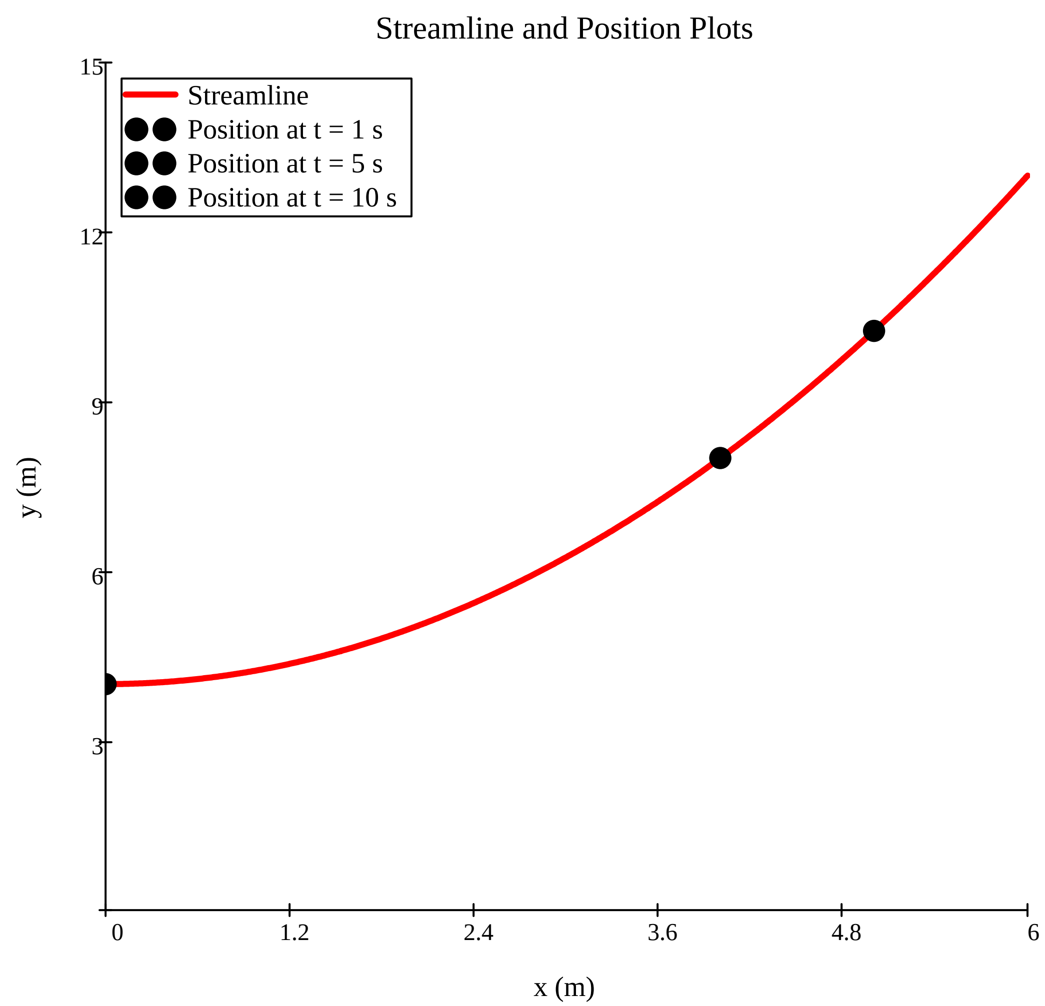
For a particle that was at $x_0 = 0\,m$, $y_0 = 4\,m$ at $t_0 = 0s$, at time $t = 2\,s$ we find the position is

$$x = x_0 + a\cdot(t - t_0) = 4\,m \qquad y = y_0 + b\cdot\left[x_0\cdot(t - t_0) + \frac{a}{2}\cdot\left(\left(t^2 - t_0^{\,2}\right)\right) - a\cdot t_0\cdot(t - t_0)\right] = \quad m$$

For a particle that was at $x_0 = 1\,m$, $y_0 = 4.25\,m$ at $t_0 = 1\,s$, at time $t = 3\,s$ we find the position is

$$x = x_0 + a\cdot(t - t_0) = 5\,m \qquad y = y_0 + b\cdot\left[x_0\cdot(t - t_0) + \frac{a}{2}\cdot\left(\left(t^2 - t_0^{\,2}\right)\right) - a\cdot t_0\cdot(t - t_0)\right] = 10.\,m$$

For this steady flow streamlines, streaklines and pathlines coincide; the particles refered to are the same particle!



Problem 2.29 [Difficulty: 4]

2.29 A flow is described by velocity field $\vec{V} = ay\hat{i} + bt\hat{j}$, where $a = 0.2\ \text{s}^{-1}$ and $b = 0.4\ \text{m/s}^2$. At $t = 2$ s, what are the coordinates of the particle that passed through point (1, 2) at $t = 0$? At $t = 3$ s, what are the coordinates of the particle that passed through point (1, 2) at $t = 2$ s? Plot the pathline and streakline through point (1, 2), and plot the streamlines through the same point at the instants $t = 0$, 1, 2, and 3 s.

Given: Velocity field

Find: Coordinates of particle at t = 2 s that was at (1,2) at t = 0; coordinates of particle at t = 3 s that was at (1,2) at t = 2 s; plot pathline and streakline through point (1,2) and compare with streamlines through same point at t = 0, 1 and 2 s

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $v_p = \frac{dy}{dt}$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Following the discussion leading up to Eq. 2.10, we first find equations for the pathlines in form

$$x_p(t) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_p(t) = y(t, x_0, y_0, t_0)$$

$$x_{st}(t_0) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_{st}(t_0) = y(t, x_0, y_0, t_0)$$

which gives the streakline at t, where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

Assumption: 2D flow

Given data $a = 0.2 \frac{1}{s}$ $b = 0.4 \frac{m}{s^2}$

Hence for pathlines $u_p = \frac{dx}{dt} = a \cdot y$ $v_p = \frac{dy}{dt} = b \cdot t$

Hence $dx = a \cdot y \cdot dt$ $dy = b \cdot t \cdot dt$ $y - y_0 = \frac{b}{2} \cdot \left(t^2 - t_0^{\,2}\right)$

For x $dx = \left[a \cdot y_0 + a \cdot \frac{b}{2} \cdot \left(t^2 - t_0^{\,2}\right)\right] \cdot dt$

Integrating $x - x_0 = a \cdot y_0 \cdot (t - t_0) + a \cdot \frac{b}{2} \cdot \left[\frac{t^3}{3} - \frac{t_0^{\,3}}{3} - t_0^{\,2} \cdot (t - t_0)\right]$

The pathlines are $x(t) = x_0 + a \cdot y_0 \cdot (t - t_0) + a \cdot \frac{b}{2} \cdot \left[\frac{t^3}{3} - \frac{t_0^{\,3}}{3} - t_0^{\,2} \cdot (t - t_0)\right]$ $y(t) = y_0 + \frac{b}{2} \cdot \left(t^2 - t_0^{\,2}\right)$

These give the position (x,y) at any time t of a particle that was at (x_0,y_0) at time t_0

Note that streaklines are obtained using the logic of the Governing equations, above

The streaklines are $x(t_0) = x_0 + a\cdot y_0\cdot(t - t_0) + a\cdot\frac{b}{2}\cdot\left[\frac{t^3}{3} - \frac{t_0^{\,3}}{3} - t_0^{\,2}\cdot(t - t_0)\right]$ $\qquad y(t_0) = y_0 + \frac{b}{2}\cdot\left(t^2 - t_0^{\,2}\right)$

These gives the streakline at t, where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

For a particle that was at $x_0 = 1\,m$, $y_0 = 2\,m$ at $t_0 = 0s$, at time $t = 2\,s$ we find the position is (from pathline equations)

$$x = x_0 + a\cdot y_0\cdot(t - t_0) + a\cdot\frac{b}{2}\cdot\left[\frac{t^3}{3} - \frac{t_0^{\,3}}{3} - t_0^{\,2}\cdot(t - t_0)\right] = 1.9\,m \qquad y = y_0 + \frac{b}{2}\cdot\left(t^2 - t_0^{\,2}\right) = 2.8\,m$$

For a particle that was at $x_0 = 1\,m$, $y_0 = 2\,m$ at $t_0 = 2\,s$, at time $t = 3\,s$ we find the position is

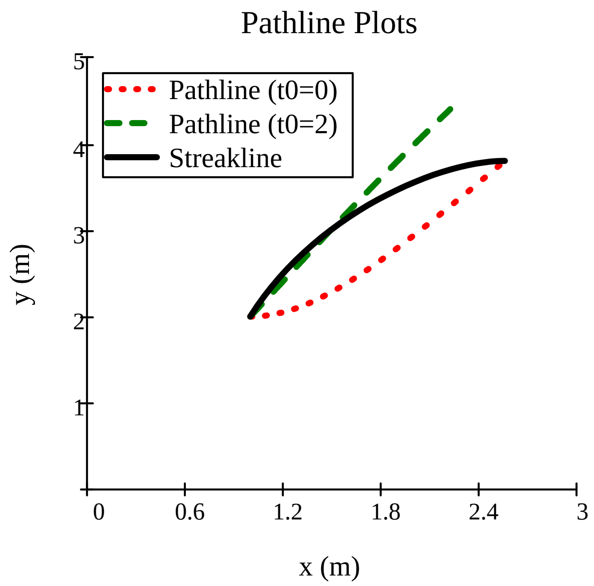
$$x = x_0 + a\cdot y_0\cdot(t - t_0) + a\cdot\frac{b}{2}\cdot\left[\frac{t^3}{3} - \frac{t_0^{\,3}}{3} - t_0^{\,2}\cdot(t - t_0)\right] = 1.4\,m \qquad y = y_0 + \frac{b}{2}\cdot\left(t^2 - t_0^{\,2}\right) = 3.0\quad m$$

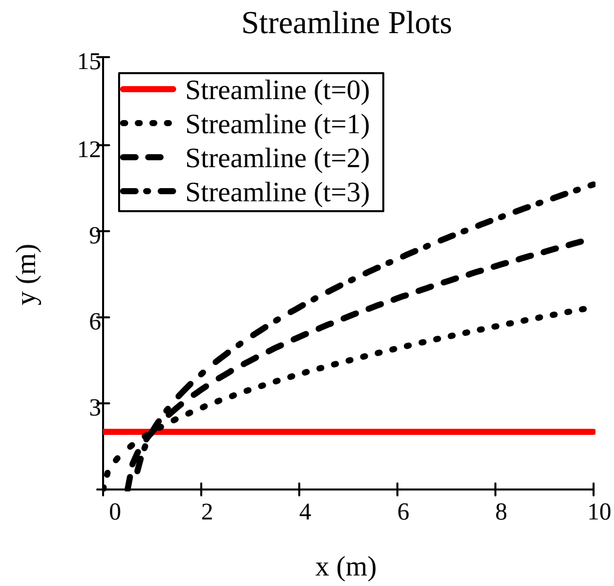
For streamlines $\quad \frac{v}{u} = \frac{dy}{dx} = \frac{b\cdot t}{a\cdot y}$

So, separating variables $\quad y\cdot dy = \frac{b}{a}\cdot t\cdot dx \qquad$ where we treat t as a constant

Integrating $\quad \frac{y^2 - y_0^{\,2}}{2} = \frac{b\cdot t}{a}\cdot(x - x_0) \qquad$ and we have $\quad x_0 = 1\quad m \quad y_0 = 2\quad m$

The streamlines are then $\quad y = \sqrt{y_0^{\,2} + \frac{2\cdot b\cdot t}{a}\cdot(x - x_0)} = \sqrt{4\cdot t\cdot(x - 1) + 4}$


Pathline Plots
Pathline (t0=0)
Pathline (t0=2)
Streakline
y (m)
x (m)
0
0.6
1.2
1.8
2.4
3
1
2
3
4
5


Streamline Plots
Streamline (t=0)
Streamline (t=1)
Streamline (t=2)
Streamline (t=3)
y (m)
x (m)
0
2
4
6
8
10
3
6
9
12
15

Problem 2.30

[Difficulty: 4]

2.30 A flow is described by velocity field $\vec{V} = at\hat{i} + b\hat{j}$, where $a = 0.4$ m/s^2 and $b = 2$ m/s. At $t = 2$ s, what are the coordinates of the particle that passed through point (2, 1) at $t = 0$? At $t = 3$ s, what are the coordinates of the particle that passed through point (2, 1) at $t = 2$ s? Plot the pathline and streakline through point (2, 1) and compare with the streamlines through the same point at the instants $t = 0$, 1, and 2 s.

Given: Velocity field

Find: Coordinates of particle at t = 2 s that was at (2,1) at t = 0; coordinates of particle at t = 3 s that was at (2,1) at t = 2 s; plot pathline and streakline through point (2,1) and compare with streamlines through same point at t = 0, 1 and 2 s

Solution:

Governing equations: For pathlines $u_p = \frac{dx}{dt}$ $\quad v_p = \frac{dy}{dt}$ $\quad$ For streamlines $\frac{v}{u} = \frac{dy}{dx}$

Following the discussion leading up to Eq. we first find equations for the pathlines in form

$$x_p(t) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_p(t) = y(t, x_0, y_0, t_0)$$

$$x_{st}(t_0) = x(t, x_0, y_0, t_0) \quad \text{and} \quad y_{st}(t_0) = y(t, x_0, y_0, t_0)$$

which gives the streakline at t, where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

Assumption: 2D flow

Given data $\quad a = 0.4\,\frac{m}{s^2} \quad b = 2\,\frac{m}{s^2}$

Hence for pathlines $\quad u_p = \frac{dx}{dt} = a \cdot t \qquad v_p = \frac{dy}{dt} = b$

Hence $\quad dx = a \cdot t \cdot dt \qquad dy = b \cdot dt$

Integrating $\quad x - x_0 = \frac{a}{2} \cdot \left(t^2 - t_0^{\,2}\right) \qquad y - y_0 = b \cdot (t - t_0)$

The pathlines are $\quad x(t) = x_0 + \frac{a}{2} \cdot \left(t^2 - t_0^{\,2}\right) \qquad y(t) = y_0 + b \cdot (t - t_0)$

These give the position (x,y) at any time t of a particle that was at (x_0,y_0) at time t_0

Note that streaklines are obtained using the logic of the Governing equations, above

The streaklines are $\quad x(t_0) = x_0 + \frac{a}{2} \cdot \left(t^2 - t_0^{\,2}\right) \qquad y(t_0) = y_0 + b \cdot (t - t_0)$

These gives the streakline at t, where x_0, y_0 is the point at which dye is released (t_0 is varied from 0 to t)

For a particle that was at $x_0 = 2\,m$, $y_0 = 1\,m$ at $t_0 = 0s$, at time $t = 2\,s$ we find the position is (from pathline equations)

$$x = x_0 + \frac{a}{2}\cdot\left(t^2 - t_0^{\,2}\right) = 2.8\,m \qquad\qquad y = y_0 + b\cdot\left(t - t_0\right) = 5\ \ m$$

For a particle that was at $x_0 = 2\,m$, $y_0 = 1\,m$ at $t_0 = 2\,s$, at time $t = 3\,s$ we find the position is

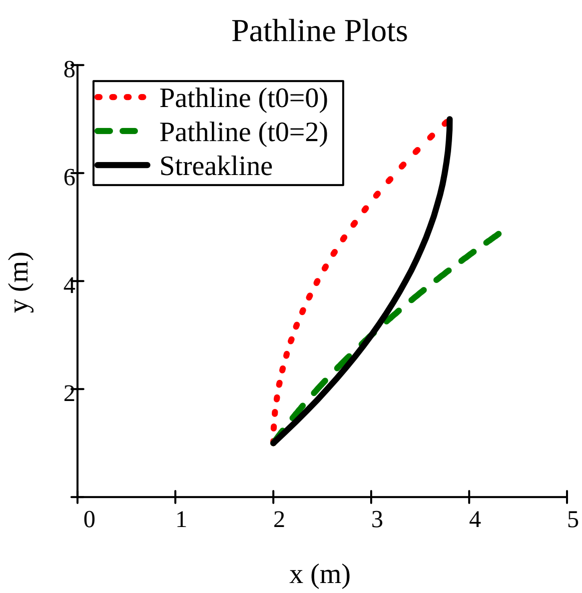
$$x = x_0 + \frac{a}{2}\cdot\left(t^2 - t_0^{\,2}\right) = 3\ \ m \qquad\qquad y = y_0 + b\cdot\left(t - t_0\right) = 3\ \ m$$

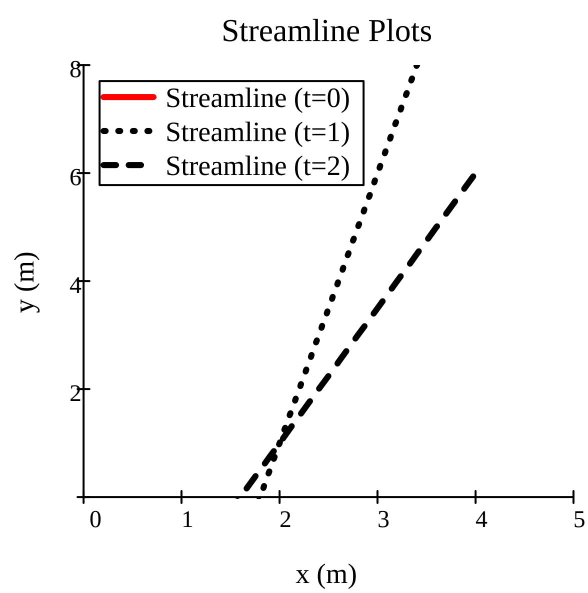
For streamlines $\quad \frac{v}{u} = \frac{dy}{dx} = \frac{b}{a\cdot t}$

So, separating variables $\quad dy = \frac{b}{a\cdot t}\cdot dx \qquad$ where we treat t as a constant

Integrating $\quad y - y_0 = \frac{b}{a\cdot t}\cdot\left(x - x_0\right) \qquad$ and we have $\quad x_0 = 2\ \ m \quad y_0 = 1\ \ m$

The streamlines are then $\quad y = y_0 + \frac{b}{a\cdot t}\cdot\left(x - x_0\right) = \frac{5\cdot(x-2)}{t} + 1$





Problem 2.31

[Difficulty: 2]

2.31 The variation with temperature of the viscosity of air is represented well by the empirical Sutherland correlation

$$\mu = \frac{bT^{1/2}}{1+S/T}$$

Best-fit values of b and S are given in Appendix A. Develop an equation in SI units for kinematic viscosity versus temperature for air at atmospheric pressure. Assume ideal gas behavior. Check by using the equation to compute the kinematic viscosity of air at 0°C and at 100°C and comparing to the data in Appendix A (Table A.10); plot the kinematic viscosity for a temperature range of 0°C to 100°C, using the equation and the data in Table A.10.

Given: Sutherland equation

Find: Corresponding equation for kinematic viscosity

Solution:

Governing equation: $\mu = \dfrac{b\cdot T^{\frac{1}{2}}}{1+\frac{S}{T}}$ Sutherland equation $\quad p = \rho\cdot R\cdot T$ Ideal gas equation

Assumptions: Sutherland equation is valid; air is an ideal gas

The given data is $\quad b = 1.458\times 10^{-6}\cdot\dfrac{kg}{m\cdot s\cdot K^{\frac{1}{2}}} \quad S = 110.4\cdot K \quad R = 286.9\cdot\dfrac{J}{kg\cdot K} \quad p = 101.3\cdot kPa$

The kinematic viscosity is $\quad \nu = \dfrac{\mu}{\rho} = \dfrac{\mu\cdot R\cdot T}{p} = \dfrac{R\cdot T}{p}\cdot\dfrac{b\cdot T^{\frac{1}{2}}}{1+\frac{S}{T}} = \dfrac{R\cdot b}{p}\cdot\dfrac{T^{\frac{3}{2}}}{1+\frac{S}{T}} = \dfrac{b'\cdot T^{\frac{3}{2}}}{1+\frac{S}{T}}$

where $\quad b' = \dfrac{R\cdot b}{p} \quad b' = 4.129\times 10^{-9}\dfrac{m^2}{K^{1.5}\cdot s}$

$$b' = 286.9\cdot\frac{N\cdot m}{kg\cdot K}\times 1.458\times 10^{-6}\cdot\frac{kg}{m\cdot s\cdot K^{\frac{1}{2}}}\times\frac{m^2}{101.3\times 10^3\cdot N} = 4.129\times 10^{-9}\cdot\frac{m^2}{s\cdot K^{\frac{3}{2}}}$$

Hence $\quad \nu = \dfrac{b'\cdot T^{\frac{3}{2}}}{1+\frac{S}{T}} \quad$ with $\quad b' = 4.129\times 10^{-9}\cdot\dfrac{m^2}{s\cdot K^{\frac{3}{2}}} \quad S = 110.4\,K$

Check with Appendix A, Table A.10. At $T = 0\,°C$ we find $T = 273.1\,K$ $\nu = 1.33 \times 10^{-5} \cdot \frac{m^2}{s}$

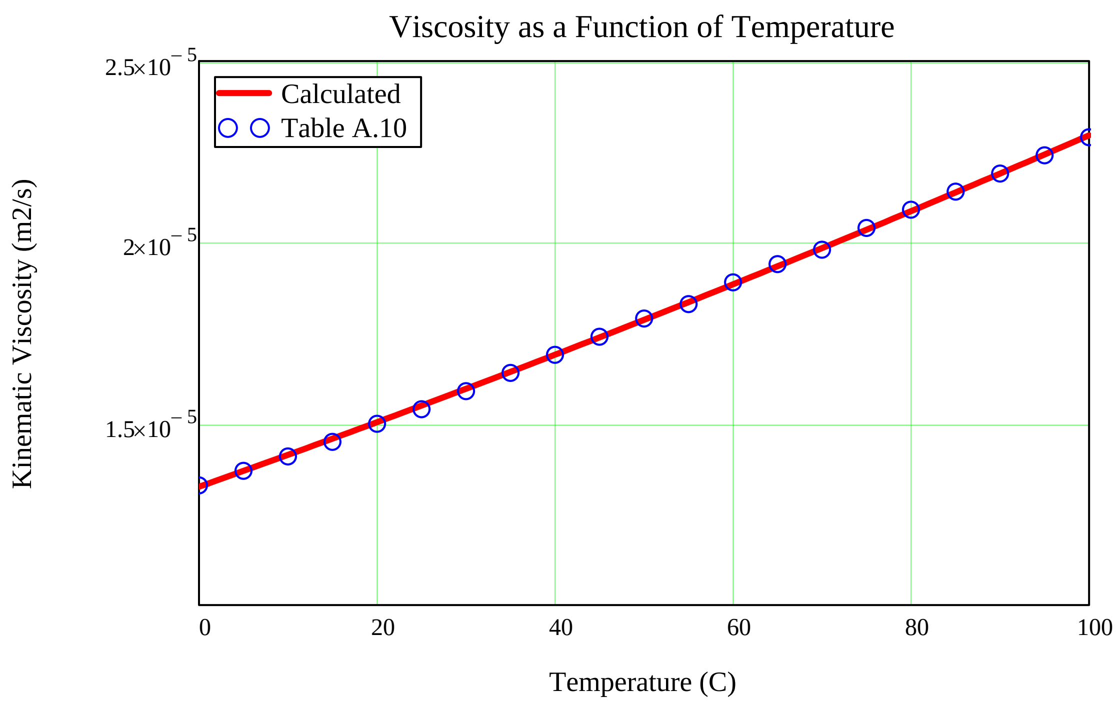
$$\nu = \frac{4.129 \times 10^{-9} \frac{m^2}{s \cdot K^{\frac{3}{2}}} \times (273.1 \cdot K)^{\frac{3}{2}}}{1 + \frac{110.4}{273.1}} \qquad \nu = 1.33 \times 10^{-5} \cdot \frac{m^2}{s}$$

Check!

At $T = 100\,°C$ we find $T = 373.1\,K$ $\nu = 2.29 \times 10^{-5} \cdot \frac{m^2}{s}$

$$\nu = \frac{4.129 \times 10^{-9} \frac{m^2}{s \cdot K^{\frac{3}{2}}} \times (373.1 \cdot K)^{\frac{3}{2}}}{1 + \frac{110.4}{373.1}} \qquad \nu = 2.30 \times 10^{-5} \cdot \frac{m^2}{s}$$

Check!



Problem 2.32

[Difficulty: 2]

2.32 The variation with temperature of the viscosity of air is correlated well by the empirical Sutherland equation

$$\mu = \frac{bT^{1/2}}{1 + S/T}$$

Best-fit values of b and S are given in Appendix A for use with SI units. Use these values to develop an equation for calculating air viscosity in British Gravitational units as a function of absolute temperature in degrees Rankine. Check your result using data from Appendix A.

Given: Sutherland equation with SI units

Find: Corresponding equation in BG units

Solution:

Governing equation: $\mu = \dfrac{b \cdot T^{\frac{1}{2}}}{1 + \dfrac{S}{T}}$ Sutherland equation

Assumption: Sutherland equation is valid

The given data is $b = 1.458 \times 10^{-6} \cdot \dfrac{kg}{m \cdot s \cdot K^{\frac{1}{2}}}$ $S = 110.4 \cdot K$

Converting constants $b = 1.458 \times 10^{-6} \cdot \dfrac{kg}{m \cdot s \cdot K^{\frac{1}{2}}} \times \dfrac{lbm}{0.454 \cdot kg} \times \dfrac{slug}{32.2 \cdot lbm} \times \dfrac{0.3048 \cdot m}{ft} \times \left(\dfrac{5 \cdot K}{9 \cdot R}\right)^{\frac{1}{2}}$ $b = 2.27 \times 10^{-8} \cdot \dfrac{slug}{ft \cdot s \cdot R^{\frac{1}{2}}}$

Alternatively $b = 2.27 \times 10^{-8} \dfrac{slug}{ft \cdot s \cdot R^{\frac{1}{2}}} \times \dfrac{lbf \cdot s^2}{slug \cdot ft}$ $b = 2.27 \times 10^{-8} \cdot \dfrac{lbf \cdot s}{ft^2 \cdot R^{\frac{1}{2}}}$

Also $S = 110.4 \cdot K \times \dfrac{9 \cdot R}{5 \cdot K}$ $S = 198.7 \cdot R$

and $\mu = \dfrac{b \cdot T^{\frac{1}{2}}}{1 + \dfrac{S}{T}}$ with T in Rankine, μ in $\dfrac{lbf \cdot s}{ft^2}$

Check with Appendix A, Table A.9. At $T = 68\,°F$ we find $\quad T = 527.7 \cdot R \quad \mu = 3.79 \times 10^{-7} \cdot \frac{lbf \cdot s}{ft^2}$

$$\mu = \frac{2.27 \times 10^{-8} \frac{lbf \cdot s}{ft^2 \cdot R^{\frac{1}{2}}} \times (527.7 \cdot R)^{\frac{1}{2}}}{1 + \frac{198.7}{527.7}} \qquad \mu = 3.79 \times 10^{-7} \cdot \frac{lbf \cdot s}{ft^2}$$

Check!

At $T = 200\,°F$ we find $\quad T = 659.7 \cdot R \quad \mu = 4.48 \times 10^{-7} \cdot \frac{lbf \cdot s}{ft^2}$

$$\mu = \frac{2.27 \times 10^{-8} \frac{lbf \cdot s}{ft^2 \cdot R^{\frac{1}{2}}} \times (659.7 \cdot R)^{\frac{1}{2}}}{1 + \frac{198.7}{659.7}} \qquad \mu = 4.48 \times 10^{-7} \cdot \frac{lbf \cdot s}{ft^2}$$

Check!

Problem 2.33

[Difficulty: 3]

2.33 Some experimental data for the viscosity of helium at 1 atm are

T, °C	0	100	200	300	400
μ, N·s/m^2(× 10^5)	1.86	2.31	2.72	3.11	3.46

Using the approach described in Appendix A.3, correlate these data to the empirical Sutherland equation

$$\mu = \frac{bT^{1/2}}{1 + S/T}$$

(where T is in kelvin) and obtain values for constants b and S.

Given: Viscosity data

Find: Obtain values for coefficients in Sutherland equation

Solution:

Data:

T (°C)	T (K)	μ(x10^5)
0	273	1.86E-05
100	373	2.31E-05
200	473	2.72E-05
300	573	3.11E-05
400	673	3.46E-05

Using procedure of Appendix A.3:

T (K)	$T^{3/2}/\mu$
273	2.43E+08
373	3.12E+08
473	3.78E+08
573	4.41E+08
673	5.05E+08

The equation to solve for coefficients S and b is

$$\frac{T^{3/2}}{\mu} = \left(\frac{1}{b}\right)T + \frac{S}{b}$$

From the built-in *Excel Linear Regression* functions:

Slope = 6.534E+05

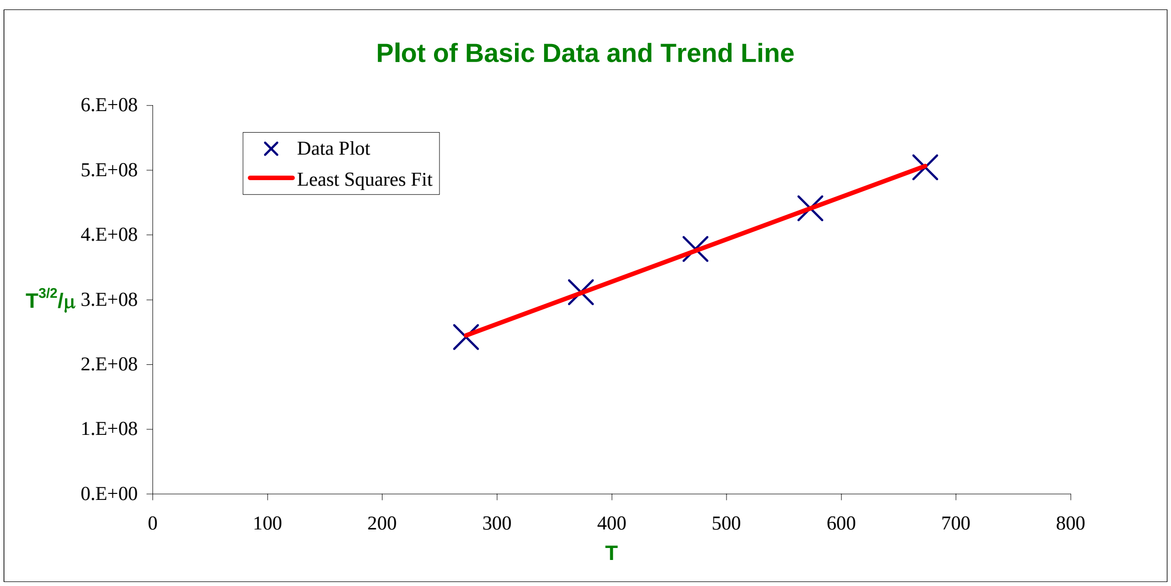
Intercept = 6.660E+07

R^2 = 0.9996

Hence:

b = 1.531E-06 kg/m·s·K$^{1/2}$

S = 101.9 K



Problem 2.34

[Difficulty: 2]

2.34 The velocity distribution for laminar flow between parallel plates is given by

$$\frac{u}{u_{max}} = 1 - \left(\frac{2y}{h}\right)^2$$

where h is the distance separating the plates and the origin is placed midway between the plates. Consider a flow of water at 15°C, with $u_{max} = 0.10$ m/s and $h = 0.1$ mm. Calculate the shear stress on the upper plate and give its direction. Sketch the variation of shear stress across the channel.

Given: Velocity distribution between flat plates

Find: Shear stress on upper plate; Sketch stress distribution

Solution:

Basic equation $\tau_{yx} = \mu\cdot\frac{du}{dy}$ $\qquad \frac{du}{dy} = \frac{d}{dy}u_{max}\cdot\left[1 - \left(\frac{2\cdot y}{h}\right)^2\right] = u_{max}\cdot\left(-\frac{4}{h^2}\right)\cdot 2\cdot y = -\frac{8\cdot u_{max}\cdot y}{h^2}$

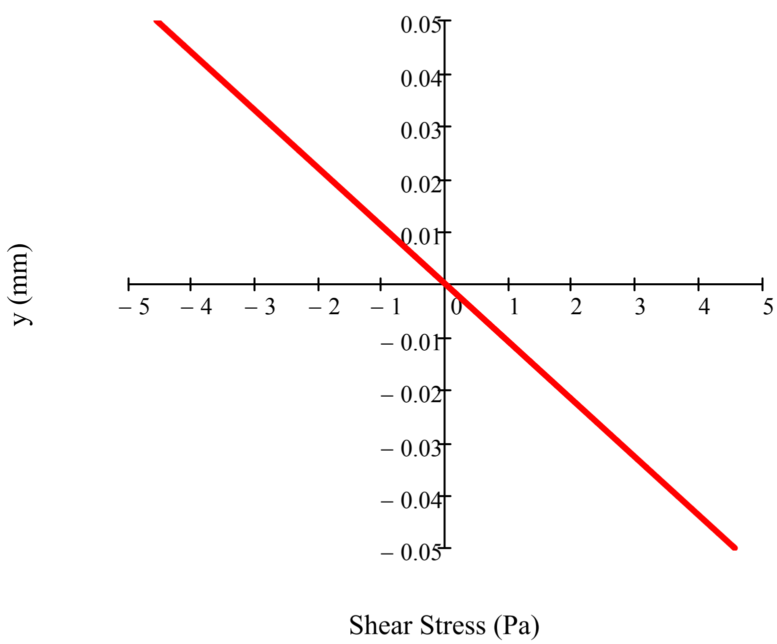
$$\tau_{yx} = -\frac{8\cdot\mu\cdot u_{max}\cdot y}{h^2}$$

At the upper surface $y = \frac{h}{2}$ and $h = 0.1\cdot mm$ $\qquad u_{max} = 0.1\cdot\frac{m}{s}$ $\qquad \mu = 1.14\times 10^{-3}\cdot\frac{N\cdot s}{m^2}$

Hence $\tau_{yx} = -8\times 1.14\times 10^{-3}\cdot\frac{N\cdot s}{m^2}\times 0.1\cdot\frac{m}{s}\times\frac{0.1}{2}\cdot mm\times\frac{1\cdot m}{1000\cdot mm}\times\left(\frac{1}{0.1\cdot mm}\times\frac{1000\cdot mm}{1\cdot m}\right)^2$ $\qquad \tau_{yx} = -4.56\cdot\frac{N}{m^2}$

The upper plate is a minus y surface. Since $\tau_{yx} < 0$, the shear stress on the upper plate must act in the plus x direction.

The shear stress varies linearly with y $\qquad \tau_{yx}(y) = -\left(\frac{8\cdot\mu\cdot u_{max}}{h^2}\right)\cdot y$



Problem 2.35

(Difficulty: 1)

2.35 What is the ratio between the viscosities of air and water at 10°C ? What is the ratio between their kinematic viscosities at this temperature and standard barometric pressure ?

Given: The temperature 10°C .

Find: Ratio between the viscosities of air and water at 10°C . Ratio of kinematic viscosities at this temperature and pressure.

Assumption: The standard barometric pressure is sea level pressure. Air can be treated as an ideal gas

Solution:

At 10°C, for the viscosities:

$$\mu_{air} = 0.018 \times 10^{-3}\ Pa \quad and \quad \mu_{H_2o} = 1.4 \times 10^{-3}\ Pa \cdot s$$

$$\frac{\mu_{air}}{\mu_{H_2o}} = \frac{0.018 \times 10^{-3}\ Pa}{1.4 \times 10^{-3}\ \ Pa} = 0.013$$

For the densities at STP:

$$\rho_{H_2o} = 1000\ \frac{kg}{m^3}$$

$$\rho_{air} = 1.225\ \frac{kg}{m^3}$$

at 15°C, using the ideal gas relation where $\rho \propto \frac{1}{T}$ at constant pressure

$$\rho_{air} = 1.225 \times \frac{(15 + 273)}{(10 + 273)}\ \frac{kg}{m^3} = 1.247\ \frac{kg}{m^3}$$

The ration of kinematic viscosities at 10°C.

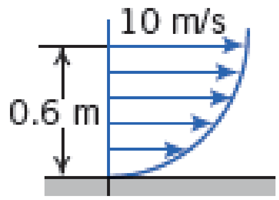
$$\frac{v_{air}}{v_{H_2o}} = \frac{\dfrac{0.018 \times 10^{-3}\ Pa}{1.247\ \frac{kg}{m^3}}}{\dfrac{1.4 \times 10^{-3}\ Pa}{1000\ \frac{kg}{m^3}}} = 10.3$$

The dynamic viscosity of air is much less than that of water but the kinematic viscosity is greater.

Problem 2.36

(Difficulty: 2)

2.36 Calculate the velocity gradients and shear stress for $y = 0, 0.2, 0.4$ and $0.6\ m$, if the velocity profile is a quarter-circle center having its center $0.6\ m$ from the boundary. The fluid viscosity is $7.5 \times 10^{-4}\ \frac{Ns}{m^2}$.



Given: The fluid viscosity $\mu = 7.5 \times 10^{-4}\ \frac{Ns}{m^2}$.

Find: The velocity gradient $\frac{du}{dy}$

Solution:

The equation for a quarter-circle with y measured up from the surface of the plate is:

$$\left(\frac{u}{10}\right)^2 + \left(\frac{y}{0.6} - 1\right)^2 = 1$$

Or, expanding the expression:

$$u^2 = 278(1.2y - y^2)$$

The velocity gradient is:

$$2\text{u}\frac{du}{dy} = 278(1.2 - 2y)$$

$$\frac{du}{dy} = 139\left(\frac{1.2 - 2y}{u}\right)$$

And the shear stress is

$$\tau = \mu\frac{du}{dy} = 7.5 \times 10^{-4}\ \frac{Ns}{m^2} \cdot 139\left(\frac{1.2 - 2y}{u}\right) = 0.104\left(\frac{1.2 - 2y}{u}\right)$$

When $y = 0$, from the equation for the velocity we have

$$u = 0\ \frac{m}{s}$$

And for the gradient we have

$$\frac{du}{dy} = \infty \ \frac{1}{s}$$

And

$$\tau = \infty \ \frac{N}{m^2}$$

When $y = 0.2$

$$u = 7.46 \ \frac{m}{s}$$

$$\frac{du}{dy} = 14.9 \ \frac{1}{s}$$

$$\tau = 0.0111 \ \frac{N}{m^2}$$

When $y = 0.4$

$$u = 9.43 \ \frac{m}{s}$$

$$\frac{du}{dy} = 5.90 \ \frac{1}{s}$$

$$\tau = 0.0044 \frac{N}{m^2}$$

When $y = 0.6$

$$u = 10 \ \frac{m}{s}$$

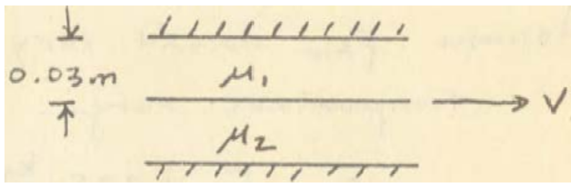
$$\frac{du}{dy} = 0 \ \frac{1}{s}$$

$$\tau = 0 \ \frac{N}{m^2}$$

Problem 2.37

(Difficulty: 2)

2.37 A very large thin plate is centered in a gap of width $0.06\ m$ with different oils of unknown viscosities above and below; one viscosity is twice the other. When the plate is pulled at a velocity of $0.3\ \frac{m}{s}$, the resulting force on one square meter of plate due to the viscous shear on both sides is $29\ N$.. Assuming viscous flow and neglecting all end effects, calculate the viscosities of the oils.



Given: Viscosity: $\mu_2 = 2\mu_1$. Width of gap: $h = 0.06\ m$. Velocity:$V = 0.3\ \frac{m}{s}$. Force per square meter: $F = 29\ \frac{N}{m^2}$

Find: μ_1 and μ_2

Assumption: Viscous flow with linear velocity profiles, negligible end effects.

Solution: Use Newton's law relating shear stress to viscosity and velocity gradient. The relation between the two viscosities is

$$\mu_2 = 2\mu_1$$

Because the gaps are equal and the plate velocity is the same for both fluids, the velocity gradient is the same for both sides of the plate:

$$\frac{dV}{dy} = \frac{V}{0.5h} = \frac{0.3\ \frac{m}{s}.}{0.5 \times 0.06\ m} = 10\ \frac{1}{s}$$

For a Newtonian fluid with a linear velocity profile, we have

$$\tau = \mu \frac{dV}{dy} = \mu \frac{\Delta V}{\Delta y}$$

The force on the plate due to the top layer of fluid is

$$\tau_1 = \mu_1 \frac{\Delta V}{\Delta y} = \mu_1 \frac{0.3 \frac{m}{s}}{0.03\ m} = \mu_1 101$$

Similarly, the force on bottom of the plate is

$$\tau_2 = \mu_2 \frac{\Delta V}{\Delta y} = \mu_2 \frac{0.3 \frac{m}{s}}{0.03\ m} = \mu_2 10 \frac{1}{s}$$

The total force per unit area equals the sum of the two shear stresses, where for the 1 m^2 plate the shear stress is equal to 29 N/m^2.

$$\frac{F}{A} = \tau_1 + \tau_2 = 29\ \frac{N}{m^2}$$

Or, since μ_2 = times μ_1

$$\mu_1 101 + \mu_2 10\frac{1}{s} = 3\mu_1 10\frac{1}{s} = 29\ \frac{N}{m^2}$$

$$\mu_1 = \frac{1}{3} \times 29\ \frac{N}{m^2} \times \frac{s}{10} = 0.967\frac{N \cdot s}{m^2} = 0.967\ Pa \cdot s$$

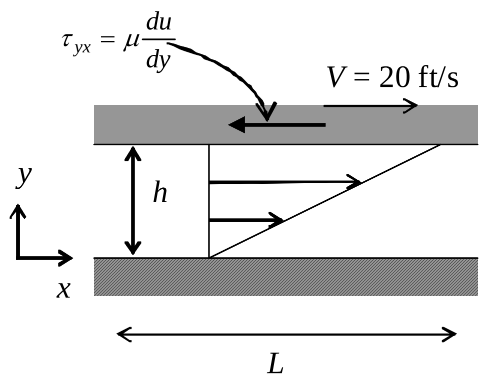
$$\mu_2 = 2\mu_1 = 1.934\ Pa \cdot s$$

Problem 2.38 [Difficulty: 2]

2.38 A female freestyle ice skater, weighing 100 lbf, glides on one skate at speed $V = 20$ ft/s. Her weight is supported by a thin film of liquid water melted from the ice by the pressure of the skate blade. Assume the blade is $L = 11.5$ in. long and $w = 0.125$ in. wide, and that the water film is $h = 0.0000575$ in. thick. Estimate the deceleration of the skater that results from viscous shear in the water film, if end effects are neglected.

Given: Ice skater and skate geometry

Find: Deceleration of skater



Solution:

Governing equation: $\tau_{yx} = \mu \cdot \frac{du}{dy}$ $\Sigma F_x = M \cdot a_x$

Assumptions: Laminar flow

The given data is $W = 100 \cdot lbf$ $V = 20 \cdot \frac{ft}{s}$ $L = 11.5 \cdot in$ $w = 0.125 \cdot in$ $h = 0.0000575 \cdot in$

$\mu = 3.68 \times 10^{-5} \cdot \frac{lbf \cdot s}{ft^2}$ Table A.7 @32ºF

Then $\tau_{yx} = \mu \cdot \frac{du}{dy} = \mu \cdot \frac{V}{h} = 3.68 \times 10^{-5} \cdot \frac{lbf \cdot s}{ft^2} \times 20 \cdot \frac{ft}{s} \times \frac{1}{0.0000575 \cdot in} \times \frac{12 \cdot in}{ft}$

$\tau_{yx} = 154 \cdot \frac{lbf}{ft^2}$

Equation of motion $\Sigma F_x = M \cdot a_x$ or $\tau_{yx} \cdot A = \frac{-W}{g} \cdot a_x$

$$a_x = -\frac{\tau_{yx} \cdot A \cdot g}{W} = -\frac{\tau_{yx} \cdot L \cdot w \cdot g}{W}$$

$$a_x = -154 \frac{lbf}{ft^2} \times 11.5 \cdot in \times 0.125 \cdot in \times 32.2 \cdot \frac{ft}{s^2} \times \frac{1}{100 \cdot lbf} \times \frac{ft^2}{(12 \cdot in)^2}$$

$$a_x = -0.495 \cdot \frac{ft}{s^2}$$

Problem 2.39

[Difficulty: 2]

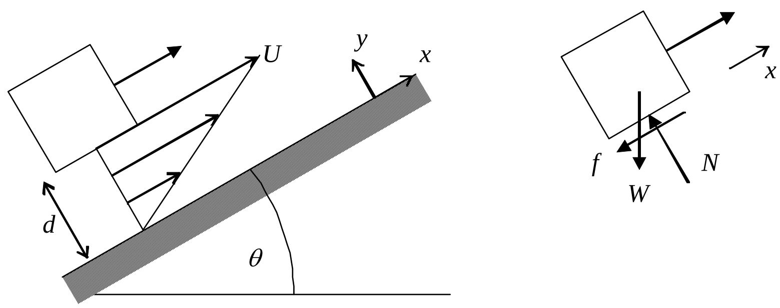
2.39 A block of mass 10 kg and measuring 250 mm on each edge is pulled up an inclined surface on which there is a film of SAE 10W-30 oil at 30°F (the oil film is 0.025 mm thick). Find the steady speed of the block if it is released. If a force of 75 N is applied to pull the block up the incline, find the steady speed of the block. If the force is now applied to push the block down the incline, find the steady speed of the block. Assume the velocity distribution in the oil film is linear. The surface is inclined at an angle of 30° from the horizontal.

Given: Block moving on incline on oil layer

Find: Speed of block when free, pulled, and pushed

Solution:

Governing equations: $\tau_{yx} = \mu \cdot \frac{du}{dy}$ $\qquad \Sigma F_x = M \cdot a_x$



Assumptions: Laminar flow

The given data is $M = 10 \cdot kg$ $\quad W = M \cdot g$ $\quad W = 98.066\,N$ $\quad w = 250 \cdot mm$

$d = 0.025 \cdot mm$ $\quad \theta = 30 \cdot deg$ $\quad F = 75 \cdot N$

$\mu = 10^{-1} \cdot \frac{N \cdot s}{m^2}$ $\qquad$ Fig. A.2 SAE 10-39 @30ºC

Equation of motion $\quad \Sigma F_x = M \cdot a_x = 0 \quad$ so $\quad F - f - W \cdot \sin(\theta) = 0$

The friction force is $\quad f = \tau_{yx} \cdot A = \mu \cdot \frac{du}{dy} \cdot A = \mu \cdot \frac{U}{d} \cdot w^2$

Hence for uphill motion $\quad F = f + W \cdot \sin(\theta) = \mu \cdot \frac{U}{d} \cdot w^2 + W \cdot \sin(\theta) \qquad U = \frac{d \cdot (F - W \cdot \sin(\theta))}{\mu \cdot w^2}$ $\quad$ (For downpush change sign of W)

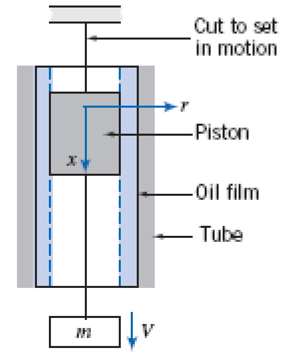
For no force: $\quad U = \frac{d \cdot W \cdot \sin(\theta)}{\mu \cdot w^2} \qquad U = 0.196 \frac{m}{s}$

Pushing up: $\quad U = \frac{d \cdot (F - W \cdot \sin(\theta))}{\mu \cdot w^2} \qquad U = 0.104 \frac{m}{s}$ $\qquad$ Pushing down: $\quad U = \frac{d \cdot (F + W \cdot \sin(\theta))}{\mu \cdot w^2} \qquad U = 0.496 \frac{m}{s}$

Problem 2.40

[Difficulty: 2]

2.40 A 73-mm-diameter aluminum (SG = 2.64) piston of 100-mm length resides in a stationary 75-mm-inner-diameter steel tube lined with SAE 10W-30 oil at 25°C. A mass $m = 2$ kg is suspended from the free end of the piston. The piston is set into motion by cutting a support cord. What is the terminal velocity of mass m? Assume a linear velocity profile within the oil.



Given: Flow data on apparatus

Find: The terminal velocity of mass *m*

Solution:

Given data: $D_{piston} = 73 \cdot mm$ $\quad D_{tube} = 75 \cdot mm$ $\quad Mass = 2 \cdot kg$ $\quad L = 100 \cdot mm$ $\quad SG_{Al} = 2.64$

Reference data: $\rho_{water} = 1000 \cdot \frac{kg}{m^3}$ (maximum density of water)

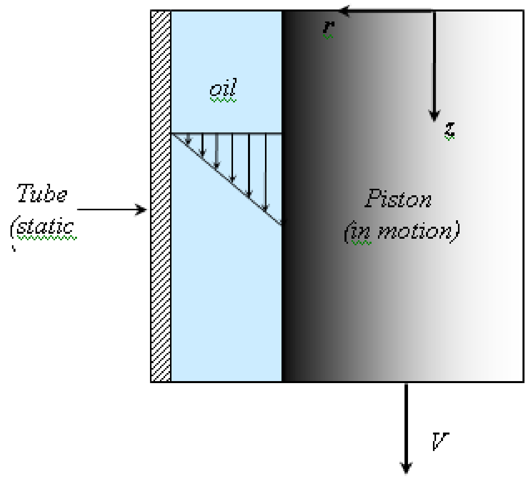
From Fig. A.2:, the dynamic viscosity of SAE 10W-30 oil at 25°C is: $\mu = 0.13 \cdot \frac{N \cdot s}{m^2}$

The terminal velocity of the mass *m* is equivalent to the terminal velocity of the piston. At that terminal speed, the acceleration of the piston is zero. Therefore, all forces acting on the piston must be balanced. This means that the force driving the motion (i.e. the weight of mass *m* and the piston) balances the viscous forces acting on the surface of the piston. Thus, at r = R_{piston}:

$$\left[Mass + SG_{Al} \cdot \rho_{water} \cdot \left(\frac{\pi \cdot D_{piston}^2 \cdot L}{4}\right)\right] \cdot g = \tau_{rz} \cdot A = \left(\mu \cdot \frac{d}{dr} V_z\right) \cdot \left(\pi \cdot D_{piston} \cdot L\right)$$

The velocity profile within the oil film is linear ...

Therefore $$\frac{d}{dr} V_z = \frac{V}{\left(\frac{D_{tube} - D_{piston}}{2}\right)}$$



Thus, the terminal velocity of the piston, *V*, is:

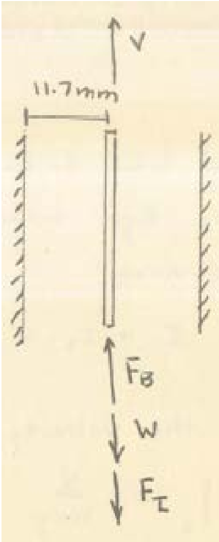
$$V = \frac{g \cdot \left(SG_{Al} \cdot \rho_{water} \cdot \pi \cdot D_{piston}^2 \cdot L + 4 \cdot Mass\right) \cdot \left(D_{tube} - D_{piston}\right)}{8 \cdot \mu \cdot \pi \cdot D_{piston} \cdot L}$$

or $V = 10.2 \frac{m}{s}$

Problem 2.41

(Difficulty: 2)

2.41 A vertical gap 25 mm wide of infinite extent contains oil of specific gravity 0.95 and viscosity 2.4 Pa · s. A metal plate 1.5 m × 1.5 m × 1.6 mm weighting 45 N is to be lifted through the gap at a constant speed of 0.06 m/s.Estimate the force required.



Given: Plate size:1.5 m × 1.5 m × 1.6 mm .Width of gap: 25 mm. Specific gravity:0.95.

Viscosity: 2.4 Pa · s. Weight: 45 N. Speed: 0.06 m/s.

Find: The force required F_T.

Assumption: Viscous flow. Neglecting all end effects. Linear velocity profile in the gap.

Solution: Make a force balance on the plate. Use Newton's law of viscosity to relate the viscous force on the plate to the viscosity and velocity.

We need to calculate all the individual forces. There are the force due to gravity (weight), the buoyancy force, and the drag force.

Buoyancy force:

$$F_B = \rho g V = SG \cdot \rho_{H_2O} g V = 0.95 \times 998.2 \times 9.81 \times 1.5 \times 1.5 \times 0.0016 = 33.5\ N$$

Drag force: The viscous shear stress is given by

$$\tau = \mu \frac{du}{dx} = \mu \frac{\Delta u}{\Delta x}$$

The force on both sides of the plate is

$$F_\tau = 2\,\mu \frac{\Delta u}{\Delta x} A = 2 \times 2.4\ Pa\ \ s \times \frac{0.06\,\frac{m}{s}}{0.0117\ m} \times (1.5\ m)^2 = 55.4\ N$$

The force required to maintain motion is F_T, by the force balance equation we have:

$$F_T + F_B = W + F_\tau$$

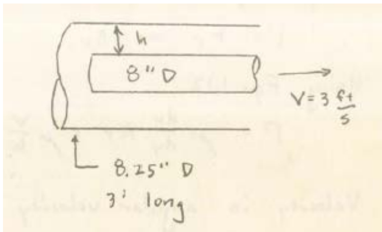
The total force is then

$$F_T = W + F_\tau - F_B = 45\ N + 55.4\ N - 33.5\ N = 66.9\ N$$

Problem 2.42

(Difficulty: 2)

2.42 A cylinder 8 in in diameter and 3 ft long is concentric with a pipe of 8.25 in. Between cylinder and pipe there is an oil film. What force is required to move the cylinder along the pipe at a constant velocity of 3 fps? The kinematic viscosity of the oil is 0.006 $\frac{ft^2}{s}$. The specific gravity is 0.92.



Given: Cylinder diameter: $D_c = 8\ in$. Cylinder length: $\mathrm{L} = 3\ ft$. Pipe diameter: $D_p = 8.25\ in$. Cylinder velocity:$\mathrm{V} = 3\ \frac{ft}{s}$. Oil viscosity: $v = 0.006\ \frac{ft^2}{s}$. Specific gravity:$SG = 0.92$.

Find: The force required F_T.

Assumption: Viscous flow with linear velocity profile in oil, negligible end effects.

Solution: Use Newton's law of viscosity to evaluate the viscous force.

The gap h between the cylinder and pipe is:

$$h = \frac{8.25 - 8}{2}\ in = 0.125\ in = 0.0104\ ft$$

The contact area A between cylinder and oil is:

$$1\ ft = 12\ in$$

$$A = \pi D_c L = \pi \times \frac{8}{12} \times 3\ ft^2 = 6.28\ ft^2$$

The dynamic viscosity:

$$\mu = v\rho = v \cdot SG \cdot \rho_{H_2o} = 0.006\ \frac{ft^2}{s} \times 0.92 \times 62.4 \frac{lbm}{ft^3} = 0.344\ \frac{lbm}{ft \cdot s}$$

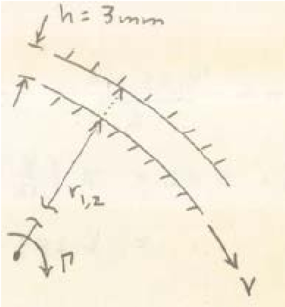
The drag force is, assuming a linear velocity profile in the fluid

$$\mathrm{F}_D = \mu A \frac{du}{dy} = \mu A \frac{V}{h} = 0.344\ \frac{lbm}{ft \cdot s} \times 6.28\ ft^2 \times \frac{3\ \frac{ft}{s}}{0.0104\ ft} = 605\ \frac{lbm \cdot ft}{s^2} = 19\ lbf$$

Problem 2.43

(Difficulty: 2)

2.43 Crude oil at 20°C fills the space between two concentric cylinders 250 mm high and with diameters of 150 mm and 156 mm. What torque is required to rotate the inner cylinder at 12 rpm, the outer cylinder remaining stationary?



Given: Temperature: T = 20 °C. Cylinder height: H = 250 mm. Outer cylinder diameter: $D_o = 156\ mm$. Inner cylinder diameter: $D_I = 150\ mm$. Rotating speed: 12 rpm.

Find: The required torque Γ.

Assumption: Linear velocity profile in fluid, viscous flow, neglect all end effects.

Solution: Use Newton's law of viscosity to find the force on the surfaces

The torque equals force times radius:

$$T = F_D R_I$$

The velocity of inner cylinder is:

$$V = \omega R_I = 12\ \frac{r}{min} \times \frac{1\ min}{60\ s} \times \left(2\pi\frac{1}{r}\right) \times \frac{150}{2 \times 1000}\ m = 0.0942\ \frac{m}{s}$$

The dynamic viscosity of crude oil at Temperature = 20 ℃.:

$$\mu = 0.00718\ Pa \cdot s$$

Newton's law of viscosity with a linear velocity profile is

$$\tau = \mu\frac{du}{dx} = \mu\frac{\Delta u}{\Delta r}$$

The drag force is:

$$\mathrm{F}_D = \mu\frac{V}{0.5 \times (D_o - D_I)}A$$

$$\mathrm{F}_D = 0.00718\ Pa \cdot s \times \frac{0.0942\ \frac{m}{s}}{0.5 \times (0.006\ m)} \times (\pi \times 0.15\ m \times 0.25\ m) = 0.0266\ N$$

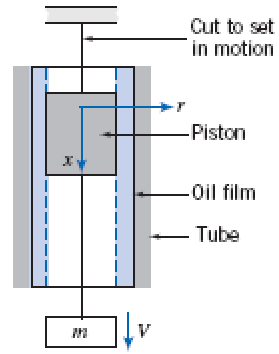
The torque is then

$$T = F_D R_I = 0.0266\ N \times 0.075\ m = 0.002\ N \cdot m$$

Problem 2.44

[Difficulty: 3]

2.44 The piston in Problem 2.40 is traveling at terminal speed. The mass m now disconnects from the piston. Plot the piston speed vs. time. How long does it take the piston to come within 1 percent of its new terminal speed?



Given: Flow data on apparatus

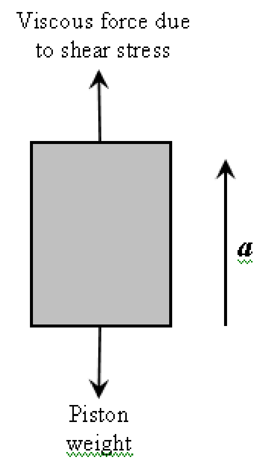
Find: Sketch of piston speed vs time; the time needed for the piston to reach 99% of its new terminal speed.

Solution:

Given data: $D_{piston} = 73\cdot mm$ $\quad D_{tube} = 75\cdot mm$ $\quad L = 100\cdot mm$ $\quad SG_{Al} = 2.64$ $\quad V_0 = 10.2\cdot\frac{m}{s}$ (From Problem 2.40)

Reference data: $\rho_{water} = 1000\cdot\frac{kg}{m^3}$ (maximum density of water)

the dynamic viscosity of SAE 10W-30 oil at 25ºC is: $\mu = 0.13\cdot\frac{N\cdot s}{m^2}$



The free body diagram of the piston after the cord is cut is:

Piston weight: $$W_{piston} = SG_{Al}\cdot\rho_{water}\cdot g\cdot\left(\frac{\pi\cdot D_{piston}^{\ 2}}{4}\right)\cdot L$$

Viscous force: $$F_{viscous}(V) = \tau_{rz}\cdot A$$

or $$F_{viscous}(V) = \mu\cdot\left[\frac{V}{\frac{1}{2}\cdot\left(D_{tube} - D_{piston}\right)}\right]\cdot\left(\pi\cdot D_{piston}\cdot L\right)$$

Applying Newton's second law: $$m_{piston}\cdot\frac{dV}{dt} = W_{piston} - F_{viscous}(V)$$

Therefore $\frac{dV}{dt} = g - a\cdot V$ where $$a = \frac{8\cdot\mu}{SG_{Al}\cdot\rho_{water}\cdot D_{piston}\cdot\left(D_{tube} - D_{piston}\right)}$$

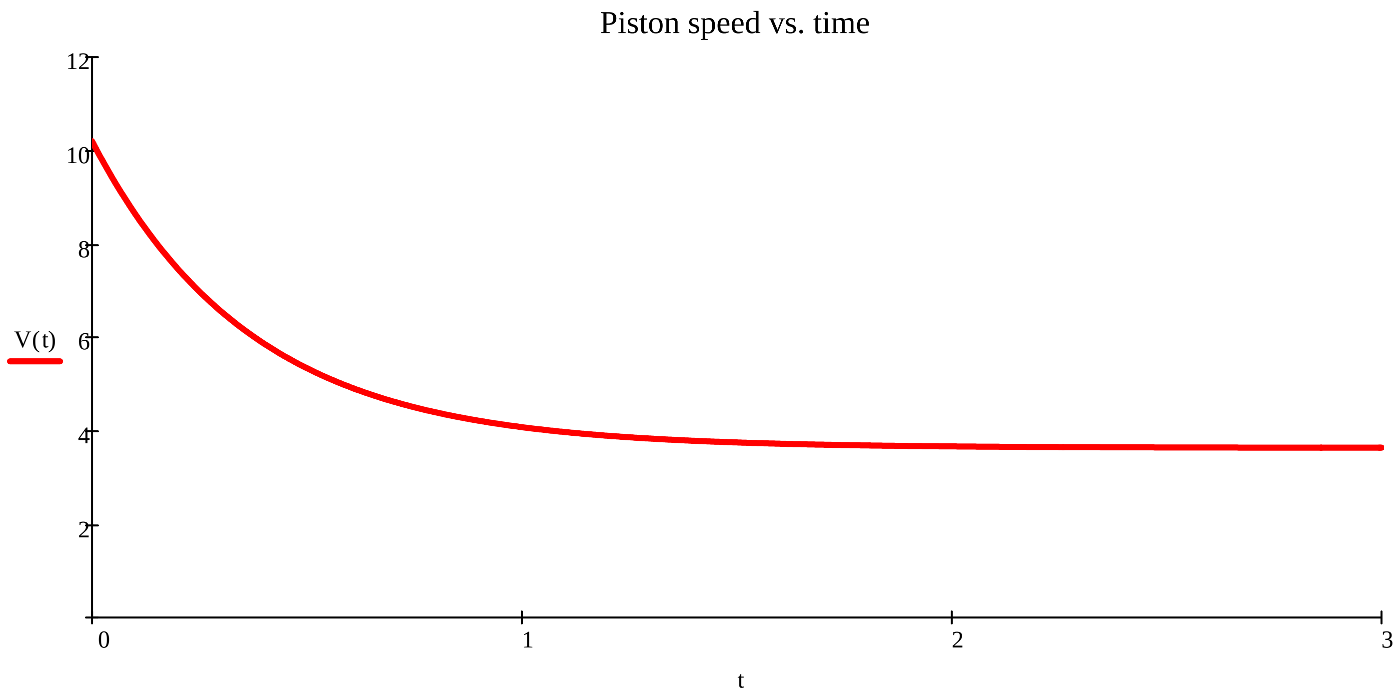
If $V = g - a\cdot V$ then $\frac{dX}{dt} = -a\cdot\frac{dV}{dt}$

The differential equation becomes $\frac{dX}{dt} = -a\cdot X$ where $X(0) = g - a\cdot V_0$

The solution to this differential equation is: $X(t) = X_0\cdot e^{-a\cdot t}$ or $g - a\cdot V(t) = \left(g - a\cdot V_0\right)\cdot e^{-a\cdot t}$

Therefore $$V(t) = \left(V_0 - \frac{g}{a}\right)\cdot e^{(-a\cdot t)} + \frac{g}{a}$$

Plotting piston speed vs. time (which can be done in Excel)



The terminal speed of the piston, V_t, is evaluated as t approaches infinity

$$V_t = \frac{g}{a} \qquad \text{or} \qquad V_t = 3.63\,\frac{m}{s}$$

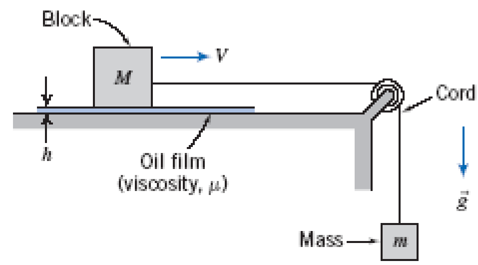
The time needed for the piston to slow down to within 1% of its terminal velocity is:

$$t = \frac{1}{a}\cdot \ln\left(\frac{V_0 - \frac{g}{a}}{1.01\cdot V_t - \frac{g}{a}}\right) \qquad \text{or} \qquad t = 1.93\,s$$

Problem 2.45

[Difficulty: 3]

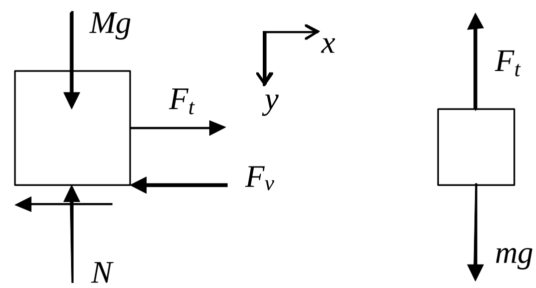
2.45 A block of mass M slides on a thin film of oil. The film thickness is h and the area of the block is A. When released, mass m exerts tension on the cord, causing the block to accelerate. Neglect friction in the pulley and air resistance. Develop an algebraic expression for the viscous force that acts on the block when it moves at speed V. Derive a differential equation for the block speed as a function of time. Obtain an expression for the block speed as a function of time. The mass $M = 5$ kg, $m = 1$ kg, $A = 25\ \text{cm}^2$, and $h = 0.5$ mm. If it takes 1 s for the speed to reach 1 m/s, find the oil viscosity μ. Plot the curve for $V(t)$.



Given: Block on oil layer pulled by hanging weight

Find: Expression for viscous force at speed V; differential equation for motion; block speed as function of time; oil viscosity

Solution:



Governing equations: $\tau_{yx} = \mu\cdot\frac{du}{dy}$ $\qquad \Sigma F_x = M\cdot a_x$

Assumptions: Laminar flow; linear velocity profile in oil layer

The given data is $\quad M = 5\cdot kg \quad W = m\cdot g = 9.81\cdot N \quad A = 25\cdot cm^2 \quad h = 0.05\cdot mm$

Equation of motion (block) $\quad \Sigma F_x = M\cdot a_x \quad$ so $\quad F_t - F_v = M\cdot\frac{dV}{dt} \qquad (1)$

Equation of motion (block) $\quad \Sigma F_y = m\cdot a_y \quad$ so $\quad m\cdot g - F_t = m\cdot\frac{dV}{dt} \qquad (2)$

Adding Eqs. (1) and (2) $\quad m\cdot g - F_v = (M + m)\cdot\frac{dV}{dt}$

The friction force is $\quad F_v = \tau_{yx}\cdot A = \mu\cdot\frac{du}{dy}\cdot A = \mu\cdot\frac{V}{h}\cdot A$

Hence $\quad m\cdot g - \frac{\mu\cdot A}{h}\cdot V = (M + m)\cdot\frac{dV}{dt}$

To solve separate variables $\quad dt = \dfrac{M + m}{m\cdot g - \frac{\mu\cdot A}{h}\cdot V}\cdot dV$

$$t = -\frac{(M + m)\cdot h}{\mu\cdot A}\cdot\left(\ln\left(m\cdot g - \frac{\mu\cdot A}{h}\cdot V\right) - \ln(m\cdot g)\right) = -\frac{(M + m)\cdot h}{\mu\cdot A}\cdot\ln\left(1 - \frac{\mu\cdot A}{m\cdot g\cdot h}\cdot V\right)$$

Hence taking antilogarithms $\quad 1 - \frac{\mu\cdot A}{m\cdot g\cdot h}\cdot V = e^{-\frac{\mu\cdot A}{(M+m)\cdot h}\cdot t}$

Finally $$V = \frac{m \cdot g \cdot h}{\mu \cdot A} \cdot \left[1 - e^{-\frac{\mu \cdot A}{(M+m) \cdot h} \cdot t} \right]$$ The maximum velocity is $$V = \frac{m \cdot g \cdot h}{\mu \cdot A}$$

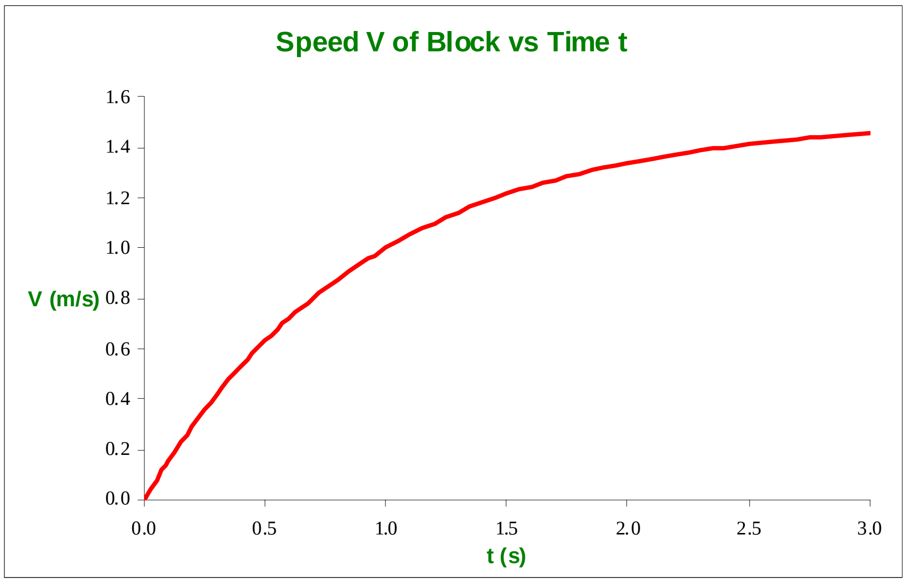
In Excel:

The data is

M =	5.00	kg
m =	1.00	kg
g =	9.81	m/s^2
0 =	1.30	$N.s/m^2$
A =	25	cm^2
h =	0.5	mm

To find the viscosity for which the speed is 1 m/s after 1 s
use *Goal Seek* with the velocity targeted to be 1 m/s by varying
the viscosity in the set of cell below:

t (s)	V (m/s)
1.00	1.000

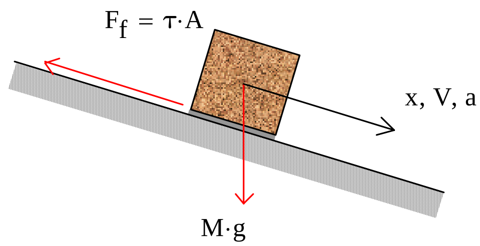


t (s)	V (m/s)
0.00	0.000
0.10	0.155
0.20	0.294
0.30	0.419
0.40	0.531
0.50	0.632
0.60	0.722
0.70	0.803
0.80	0.876
0.90	0.941
1.00	1.00
1.10	1.05
1.20	1.10
1.30	1.14
1.40	1.18
1.50	1.21
1.60	1.25
1.70	1.27
1.80	1.30
1.90	1.32
2.00	1.34
2.10	1.36
2.20	1.37
2.30	1.39
2.40	1.40
2.50	1.41
2.60	1.42
2.70	1.43
2.80	1.44
2.90	1.45
3.00	1.46

Problem 2.46

[Difficulty: 4]

2.46 A block 0.1 m square, with 5 kg mass, slides down a smooth incline, 30° below the horizontal, on a film of SAE 30 oil at 20°C that is 0.20 mm thick. If the block is released from rest at $t = 0$, what is its initial acceleration? Derive an expression for the speed of the block as a function of time. Plot the curve for $V(t)$. Find the speed after 0.1 s. If we want the mass to instead reach a speed of 0.3 m/s at this time, find the viscosity μ of the oil we would have to use.



Given: Data on the block and incline

Find: Initial acceleration; formula for speed of block; plot; find speed after 0.1 s. Find oil viscosity if speed is 0.3 m/s after 0.1 s

Solution:

Given data $\quad M = 5 \cdot kg \quad A = (0.1 \cdot m)^2 \quad d = 0.2 \cdot mm \quad \theta = 30 \cdot deg$

From Fig. $\quad \mu = 0.4 \cdot \frac{N \cdot s}{m^2}$

Applying Newton's 2nd law to initial instant (no friction) $\quad M \cdot a = M \cdot g \cdot \sin(\theta) - F_f = M \cdot g \cdot \sin(\theta)$

so $\quad a_{init} = g \cdot \sin(\theta) = 9.81 \cdot \frac{m}{s^2} \times \sin(30 \cdot deg) \qquad a_{init} = 4.9 \frac{m}{s^2}$

Applying Newton's 2nd law at any instant $\quad M \cdot a = M \cdot g \cdot \sin(\theta) - F_f \quad$ and $\quad F_f = \tau \cdot A = \mu \cdot \frac{du}{dy} \cdot A = \mu \cdot \frac{V}{d} \cdot A$

so $\quad M \cdot a = M \cdot \frac{dV}{dt} = M \cdot g \cdot \sin(\theta) - \frac{\mu \cdot A}{d} \cdot V$

Separating variables $\quad \dfrac{dV}{g \cdot \sin(\theta) - \frac{\mu \cdot A}{M \cdot d} \cdot V} = dt$

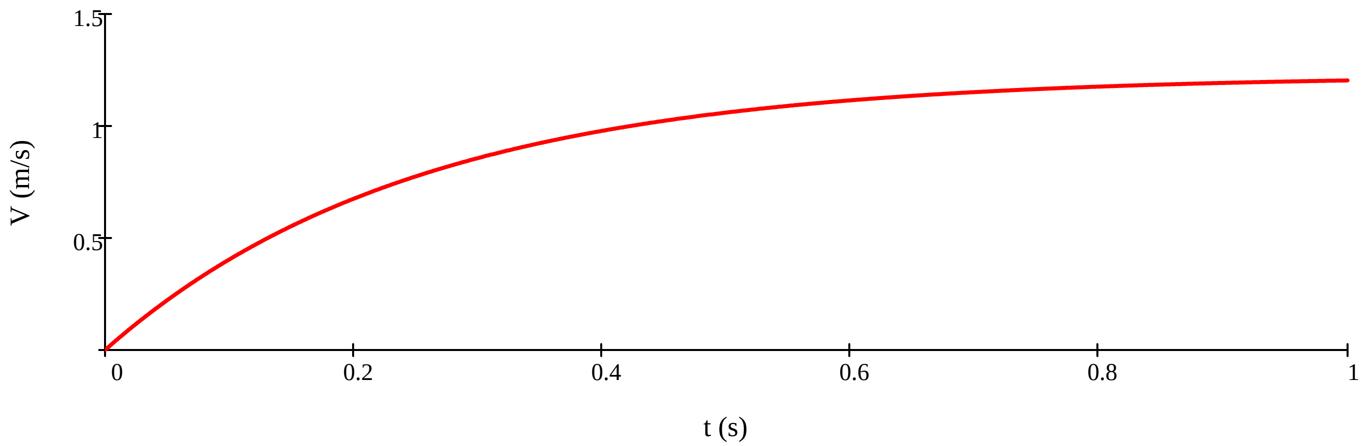
Integrating and using limits $\quad -\frac{M \cdot d}{\mu \cdot A} \cdot \ln\left(1 - \frac{\mu \cdot A}{M \cdot g \cdot d \cdot \sin(\theta)} \cdot V\right) = t$

or $\quad V(t) = \frac{M \cdot g \cdot d \cdot \sin(\theta)}{\mu \cdot A} \cdot \left(1 - e^{\frac{-\mu \cdot A}{M \cdot d} \cdot t}\right)$

At t = 0.1 s $\quad V = 5 \cdot kg \times 9.81 \cdot \frac{m}{s^2} \times 0.0002 \cdot m \cdot \sin(30 \cdot deg) \times \frac{m^2}{0.4 \cdot N \cdot s \cdot (0.1 \cdot m)^2} \times \frac{N \cdot s^2}{kg \cdot m} \times \left[1 - e^{-\left(\frac{0.4 \cdot 0.01}{5 \cdot 0.0002} \cdot 0.1\right)}\right]$

$V(0.1 \cdot s) = 0.404 \cdot \frac{m}{s}$

The plot looks like



To find the viscosity for which V(0.1 s) = 0.3 m/s, we must solve

$$V(t = 0.1\cdot s) = \frac{M\cdot g\cdot d\cdot \sin(\theta)}{\mu\cdot A}\cdot\left[1 - e^{\frac{-\mu\cdot A}{M\cdot d}\cdot(t=0.1\cdot s)}\right]$$

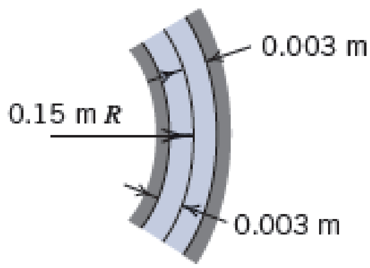
The viscosity μ is implicit in this equation, so solution must be found by manual iteration, or by any of a number of classic root-finding numerical methods, or by using *Excel*'s *Goal Seek*

Using *Excel*: $\mu = 1.08\cdot\frac{N\cdot s}{m^2}$

Problem 2.47

(Difficulty: 1)

2.47 A torque of $4\ N \cdot m$ is required to rotate the intermediate cylinder at $30\ \frac{r}{min}$. Calculate the viscosity of the oil. All the cylinders are $450\ mm$ long. Neglect end effects.



Given: Cylinder height: $\mathrm{H} = 450\ mm$. Rotation speed: $30\ \frac{r}{min}$. Gap between cylinder: $h = 0.003\ m$.

Radius of intermediate cylinder: $R = 0.15\ m$. Torque: $\mathrm{T} = 4\ N \cdot m$.

Find: The oil viscosity μ.

Assumption: Linear velocity profile in fluid, viscous flow, negligible end effects.

Solution: Use Newtons's viscosity law to evaluate the force on the cylinder

Newton's law of viscosity for a linear velocity profile is

$$\tau = \mu \frac{du}{dr} = \mu \frac{\Delta u}{\Delta r}$$

The velocity of intermediate cylinder:

$$V = \omega R = 30\ \frac{r}{min} \times \frac{1\ min}{60\ s} \times 2\pi\ \frac{1}{r} \times R\ m = 0.471\ \frac{m}{s}$$

The drag force on both sides of the cylinder is:

$$F_D = 2\mu A \frac{V}{h}$$

The torque is given by:

$$T = F_D \cdot R = 2\mu A \frac{V}{h} \cdot R$$

The area is:

$$\mathrm{A} = 2\pi \mathrm{RH}$$

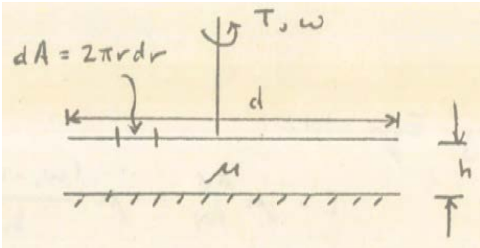
The viscosity is then:

$$\mu = \frac{T\ h}{2AVR} = \frac{4\ N \cdot m \times 0.003\ m}{2 \times 2\pi \times 0.15\ m \times 0.45\ m \times 0.471\ \frac{m}{s} \times 0.15\ m} = 0.2\ \frac{N \cdot S}{m^2} = 0.2\ Pa \cdot s$$

Problem 2.48

(Difficulty: 2)

2.48 A circular disk of diameter d is slowly rotated in a liquid of large viscosity μ at a small distance h from a fixed surface. Derive an expression for the torque T necessary to maintain an angular velocity ω. Neglect centrifugal effects.



Given: Disk diameter: d. Distance to fixed surface: h. Viscosity: μ. Angular velocity: ω. Radius of

Find: Torque: T.

Assumption: Linear velocity profile in gap between the two disks, viscous flow, negligible end effects, negligible centrifugal force effects.

Solution: Use Newton's law of viscosity with a linear velocity profile to find the forces

$$\tau = \mu \frac{du}{dy} = \mu \frac{\Delta u}{\Delta y}$$

The velocity at the interface of the fluid and disk varies with radius

$$V = \omega\, r$$

The expression for shear stress is then

$$\tau = \mu \frac{\omega\, r}{h}$$

The incremental torque the product of the radius and the force per unit area and the area from the center to the outer radius. The total torque is the integral from the centerline to the outer radius.

$$T = \int_0^{d/2} r \left(\mu \frac{\omega\, r}{h}\right) 2\pi r\, dr$$

Or

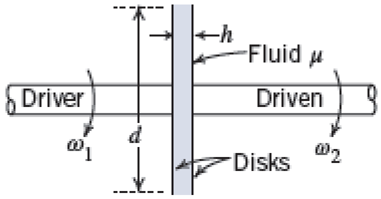
$$T = \frac{\mu\,\omega}{h} 2\pi \int_0^{d/2} r^3 \; dr$$

$$T = \frac{\pi}{32}\,\frac{\mu\,\omega\,d^4}{h}$$

Problem 2.49

(Difficulty: 2)

2.49 The fluid drive shown transmits a torque T for steady-state conditions (ω_1 and ω_2 constant). Derive an expression for the slip $(\omega_1 - \omega_2)$ in terms of $T, \mu, d\ and\ h$. For values $d = 6\ in$, $h = 0.2\ in.$, SAE 30 oil at 75 F, a shaft rotation of 120 rpm, and a torque of 0.003 ft-lbf, determine the slip.



Given: $d = 6\ in$, $h = 0.2\ in$, rotation: 120 rpm. SAE 30 oil at 75 F

Find: The slip $(\omega_1 - \omega_2)$.

Assume: Linear velocity profile in the viscous fluid

Solution: Use Newton's law of viscosity to relate the viscous forces to the torque and slip

From the force balance equation we have:

$$T = \int_0^{\frac{d}{2}} \tau \cdot dA \cdot r = \int_0^{\frac{d}{2}} \tau \cdot 2\pi r dr \cdot r$$

Assuming a linear velocity profile in the space between the two disks, Newton's law of viscosity is

$$\tau = \mu \frac{du}{dx} = \mu \frac{\Delta u}{\Delta x}$$

The velocity difference varies with radius

$$\Delta u = (\omega_1 - \omega_2) r$$

So the shear stress is

$$\tau = \mu \frac{(\omega_1 - \omega_2) r}{h}$$

The torque is then:

$$T = \int_0^{\frac{d}{2}} 2\pi\mu \frac{(\omega_1 - \omega_2)}{h} r^3 dr = \pi\mu \frac{(\omega_1 - \omega_2)}{32h} d^4$$

Solving for the slip:

$$(\omega_1 - \omega_2) = \frac{32\,Th}{\pi\,\mu\,d^4}$$

The viscosity for SAE 30 oil at 75 F is:

$$\mu = 0.008\ \frac{slug}{ft \cdot s} = 0.008\ \frac{lbf \cdot s^2}{ft}\frac{1}{ft \cdot s} = 0.008\ \frac{lbf \cdot s}{ft^2}$$

The slip is then:

$$(\omega_1 - \omega_2) = \frac{32\ \times 0.003\,ft - lbf \times 0.0167\,ft}{\pi \times\ 0.008\ \frac{lbf \cdot s}{ft^2}\ \times (0.5\,ft)^4} = 1.019\ \frac{rad}{s} = 9.73\ \ rpm$$

Problem 2.50

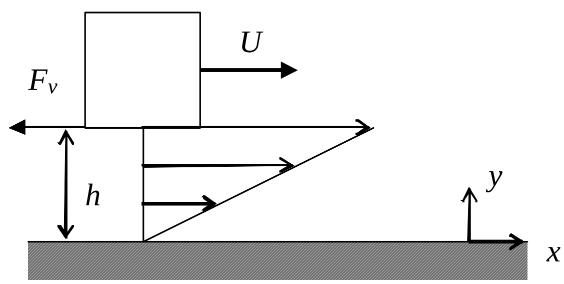
[Difficulty: 3]

2.50 A block that is a mm square slides across a flat plate on a thin film of oil. The oil has viscosity μ and the film is h mm thick. The block of mass M moves at steady speed U under the influence of constant force F. Indicate the magnitude and direction of the shear stresses on the bottom of the block and the plate. If the force is removed suddenly and the block begins to slow, sketch the resulting speed versus time curve for the block. Obtain an expression for the time required for the block to lose 95 percent of its initial speed.

Given: Block sliding on oil layer

Find: Direction of friction on bottom of block and on plate; expression for speed U versus time; time required to lose 95% of initial speed

Solution:



Governing equations: $\tau_{yx} = \mu \cdot \frac{du}{dy}$ $\qquad \Sigma F_x = M \cdot a_x$

Assumptions: Laminar flow; linear velocity profile in oil layer

The bottom of the block is a -y surface, so τ_{yx} acts to the left; The plate is a +y surface, so τ_{yx} acts to the right

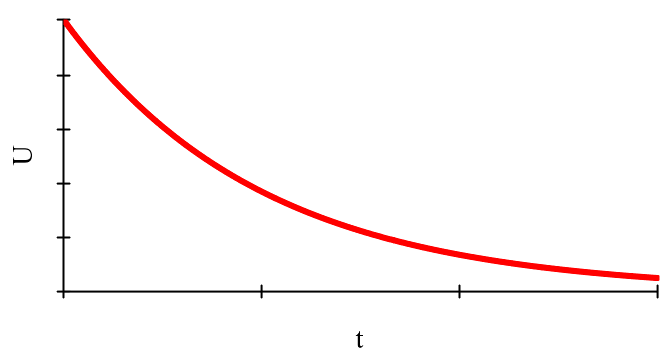
Equation of motion $\quad \Sigma F_x = M \cdot a_x \quad$ so $\quad F_v = M \cdot \frac{dU}{dt}$

The friction force is $\quad F_v = \tau_{yx} \cdot A = \mu \cdot \frac{du}{dy} \cdot A = \mu \cdot \frac{U}{h} \cdot a^2$

Hence $\quad -\frac{\mu \cdot a^2}{h} \cdot U = M \cdot \frac{dU}{dt}$

To solve separate variables $\quad \frac{1}{U} \cdot dU = -\frac{\mu \cdot a^2}{M \cdot h} \cdot dt$

$$\ln\left(\frac{U}{U_0}\right) = -\frac{\mu \cdot a^2}{M \cdot h} \cdot t$$



Hence taking antilogarithms $\quad U = U_0 \cdot e^{-\frac{\mu \cdot a^2}{M \cdot h} \cdot t}$

Solving for t $\quad t = -\frac{M \cdot h}{\mu \cdot a^2} \cdot \ln\left(\frac{U}{U_0}\right)$

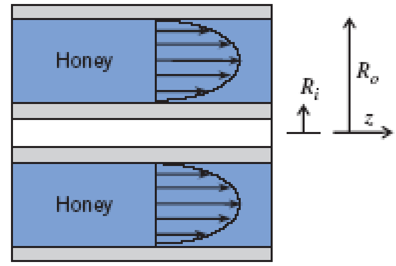
Hence for $\frac{U}{U_0} = 0.05 \quad t = 3.0 \cdot \frac{M \cdot h}{\mu \cdot a^2}$

Problem 2.51 [Difficulty: 3]

2.51 In a food-processing plant, honey is pumped through an annular tube. The tube is $L = 2$ m long, with inner and outer radii of $R_i = 5$ mm and $R_o = 25$ mm, respectively. The applied pressure difference is $\Delta p = 125$ kPa, and the honey viscosity is $\mu = 5$ N·s/m^2. The theoretical velocity profile for laminar flow through an annulus is:

$$u_z(r) = \frac{1}{4\mu}\left(\frac{\Delta p}{L}\right)\left[R_i^2 - r^2 - \frac{R_o^2 - R_i^2}{\ln\left(\frac{R_i}{R_o}\right)} \cdot \ln\left(\frac{r}{R_i}\right)\right]$$

Show that the no-slip condition is satisfied by this expression. Find the location at which the shear stress is zero. Find the viscous forces acting on the inner and outer surfaces, and compare these to the force $\Delta p \pi (R_o^2 - R_i^2)$. Explain.



Given: Data on annular tube

Find: Whether no-slip is satisfied; location of zeroshear stress; viscous forces

Solution:

The velocity profile is
$$u_z(r) = \frac{1}{4\cdot\mu}\cdot\frac{\Delta p}{L}\cdot\left(R_i^2 - r^2 - \frac{R_o^2 - R_i^2}{\ln\left(\frac{R_i}{R_o}\right)}\cdot\ln\left(\frac{r}{Ri}\right)\right)$$

Check the no-slip condition. When $r = R_o$
$$u_z(R_o) = \frac{1}{4\cdot\mu}\cdot\frac{\Delta p}{L}\cdot\left(R_i^2 - R_o^2 - \frac{R_o^2 - R_i^2}{\ln\left(\frac{R_i}{R_o}\right)}\cdot\ln\left(\frac{R_o}{Ri}\right)\right)$$

$$u_z(R_o) = \frac{1}{4\cdot\mu}\cdot\frac{\Delta p}{L}\cdot\left[R_i^2 - R_o^2 + \left(R_o^2 - R_i^2\right)\right] = 0$$

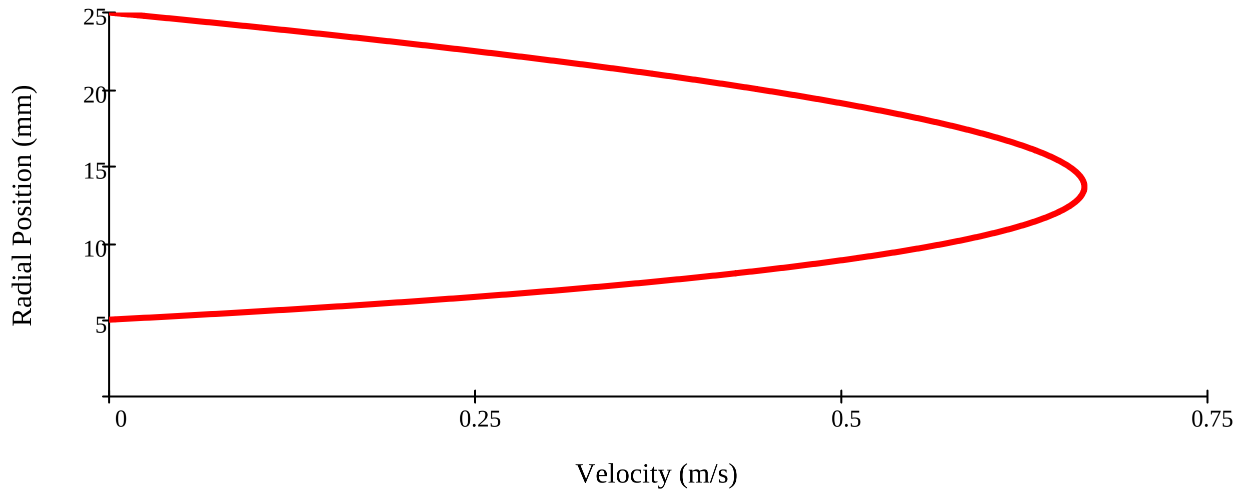
When $r = R_i$
$$u_z(R_i) = \frac{1}{4\cdot\mu}\cdot\frac{\Delta p}{L}\cdot\left(R_i^2 - R_i^2 - \frac{R_o^2 - R_i^2}{\ln\left(\frac{R_i}{R_o}\right)}\cdot\ln\left(\frac{R_i}{R_i}\right)\right) = 0$$

The no-slip condition is satisfied.

The given data is $R_i = 5\cdot mm$ $\quad R_o = 25\cdot mm$ $\quad \Delta p = 125\cdot kPa$ $\quad L = 2\cdot m$

The viscosity of the honey is $\mu = 5\cdot\frac{N\cdot s}{m^2}$

The plot looks like



For each, shear stress is given by $\tau_{rx} = \mu \cdot \frac{du}{dr}$

$$\tau_{rx} = \mu \cdot \frac{du_z(r)}{dr} = \mu \cdot \frac{d}{dr}\left[\frac{1}{4 \cdot \mu} \cdot \frac{\Delta p}{L} \cdot \left(R_i^{\,2} - r^2 - \frac{R_o^{\,2} - R_i^{\,2}}{\ln\left(\frac{R_i}{R_o}\right)} \cdot \ln\left(\frac{r}{Ri}\right)\right)\right]$$

Hence $$\tau_{rx} = \frac{1}{4} \cdot \frac{\Delta p}{L} \cdot \left(-2 \cdot r - \frac{R_o^{\,2} - R_i^{\,2}}{\ln\left(\frac{R_i}{R_o}\right) \cdot r}\right)$$

For zero stress $$-2 \cdot r - \frac{R_o^{\,2} - R_i^{\,2}}{\ln\left(\frac{R_i}{R_o}\right) \cdot r} = 0 \qquad \text{or} \qquad r = \sqrt{\frac{R_i^{\,2} - R_o^{\,2}}{2 \cdot \ln\left(\frac{R_i}{R_o}\right)}} \qquad r = 13.7 \cdot mm$$

On the outer surface $$F_o = \tau_{rx} \cdot A = \frac{1}{4} \cdot \frac{\Delta p}{L} \cdot \left(-2 \cdot R_o - \frac{R_o^{\,2} - R_i^{\,2}}{\ln\left(\frac{R_i}{R_o}\right) \cdot R_o}\right) \cdot 2 \cdot \pi \cdot R_o \cdot L$$

$$F_o = \Delta p \cdot \pi \cdot \left(-R_o^{\,2} - \frac{R_o^{\,2} - R_i^{\,2}}{2 \cdot \ln\left(\frac{R_i}{R_o}\right)}\right)$$

$$F_o = 125 \times 10^3 \cdot \frac{N}{m^2} \times \pi \times \Bigg[-\left(25 \cdot mm \times \frac{1 \cdot m}{1000 \cdot mm}\right)^2 - \frac{\left[(25 \cdot mm)^2 - (5 \cdot mm)^2\right] \times \left(\frac{1 \cdot m}{1000 \cdot mm}\right.}{2 \cdot \ln\left(\frac{5}{25}\right)}$$

$$F_o = -172\ N$$

On the inner surface

$$F_i = \tau_{rx} \cdot A = \frac{1}{4} \cdot \frac{\Delta p}{L} \cdot \left(-2 \cdot R_i - \frac{R_o^{\ 2} - R_i^{\ 2}}{\ln\left(\frac{R_i}{R_o}\right) \cdot R_i} \right) \cdot 2 \cdot \pi \cdot R_i \cdot L$$

$$F_i = \Delta p \cdot \pi \cdot \left(-R_i^{\ 2} - \frac{R_o^{\ 2} - R_i^{\ 2}}{2 \cdot \ln\left(\frac{R_i}{R_o}\right)} \right)$$

Hence

$$F_i = 125 \times 10^3 \cdot \frac{N}{m^2} \times \pi \times \Bigg[-\left(5 \cdot mm \times \frac{1 \cdot m}{1000 \cdot mm}\right)^2 - \frac{\left[(25 \cdot mm)^2 - (5 \cdot mm)^2\right] \times \left(\frac{1 \cdot m}{1000 \cdot mm}\right)^2}{2 \cdot \ln\left(\frac{5}{25}\right)}$$

$$F_i = 63.4\ N$$

Note that

$$F_o - F_i = -236\ N$$ and $$\Delta p \cdot \pi \cdot \left(R_o^{\ 2} - R_i^{\ 2}\right) = 236\ N$$

The net pressure force just balances the net viscous force!

Problem 2.52

[Difficulty: 3]

2.52 SAE 10W-30 oil at 100°C is pumped through a tube L = 10 m long, diameter D = 20 mm. The applied pressure difference is Δp = 5 kPa. On the centerline of the tube is a metal filament of diameter d = 1 μm. The theoretical velocity profile for laminar flow through the tube is:

$$V(r) = \frac{1}{16\mu}\left(\frac{\Delta p}{L}\right)\left[d^2 - 4r^2 - \frac{D^2 - d^2}{\ln\left(\frac{d}{D}\right)} \cdot \ln\left(\frac{2r}{d}\right)\right]$$

Show that the no-slip condition is satisfied by this expression. Find the location at which the shear stress is zero, and the stress on the tube and on the filament. Plot the velocity distribution and the stress distribution. (For the stress curve, set an upper limit on stress of 5 Pa.) Discuss the results.

Given: Data on flow through a tube with a filament

Find: Whether no-slip is satisfied; location of zero stress;stress on tube and filament

Solution:

The velocity profile is
$$V(r) = \frac{1}{16\cdot\mu}\cdot\frac{\Delta p}{L}\cdot\left(d^2 - 4\cdot r^2 - \frac{D^2 - d^2}{\ln\left(\frac{d}{D}\right)}\cdot\ln\left(\frac{2\cdot r}{d}\right)\right)$$

Check the no-slip condition. When $r = \frac{D}{2}$
$$V\left(\frac{D}{2}\right) = \frac{1}{16\cdot\mu}\cdot\frac{\Delta p}{L}\cdot\left(d^2 - D^2 - \frac{D^2 - d^2}{\ln\left(\frac{d}{D}\right)}\cdot\ln\left(\frac{D}{d}\right)\right)$$

$$V(D) = \frac{1}{16\cdot\mu}\cdot\frac{\Delta p}{L}\cdot\left[d^2 - D^2 + \left(D^2 - d^2\right)\right] = 0$$

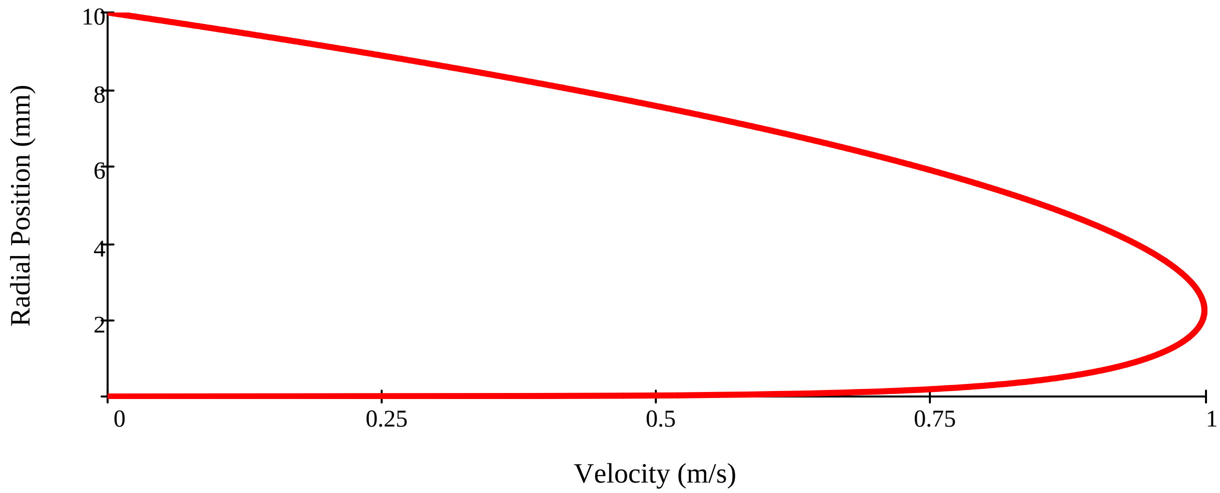
When $r = \frac{d}{2}$
$$V(d) = \frac{1}{16\cdot\mu}\cdot\frac{\Delta p}{L}\cdot\left(d^2 - d^2 - \frac{D^2 - d^2}{\ln\left(\frac{d}{D}\right)}\cdot\ln\left(\frac{d}{d}\right)\right) = 0$$

The no-slip condition is satisfied.

The given data is $d = 1\cdot\mu m$ $\quad D = 20\cdot mm$ $\quad \Delta p = 5\cdot kPa$ $\quad L = 10\cdot m$

The viscosity of SAE 10-30 oil at 100°C is $\mu = 1\times10^{-2}\cdot\frac{N\cdot s}{m^2}$

The plot looks like

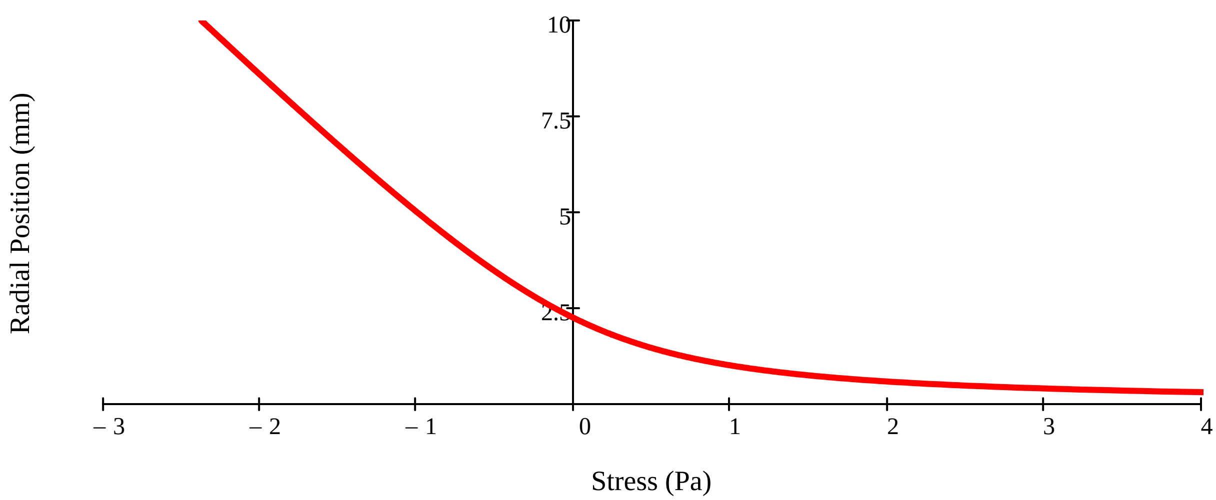


For each, shear stress is given by $\tau_{rx} = \mu \cdot \frac{du}{dr}$

$$\tau_{rx} = \mu \cdot \frac{dV(r)}{dr} = \mu \cdot \frac{d}{dr}\left[\frac{1}{16 \cdot \mu} \cdot \frac{\Delta p}{L} \cdot \left(d^2 - 4 \cdot r^2 - \frac{D^2 - d^2}{\ln\left(\frac{d}{D}\right)} \cdot \ln\left(\frac{2 \cdot r}{D_i}\right)\right)\right]$$

$$\tau_{rx}(r) = \frac{1}{16} \cdot \frac{\Delta p}{L} \cdot \left(-8 \cdot r - \frac{D^2 - d^2}{\ln\left(\frac{d}{D}\right) \cdot r}\right)$$

For the zero-stress point $-8 \cdot r - \frac{D^2 - d^2}{\ln\left(\frac{d}{D}\right) \cdot r} = 0$ or $r = \sqrt{\frac{d^2 - D^2}{8 \cdot \ln\left(\frac{d}{D}\right)}}$ $r = 2.25 \cdot mm$

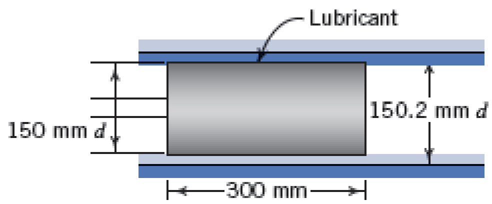


Using the stress formula $\tau_{rx}\left(\frac{D}{2}\right) = -2.374 \, Pa$ $\tau_{rx}\left(\frac{d}{2}\right) = 2.524 \cdot kPa$

Problem 2.53

(Difficulty: 2)

2.53 The lubricant has a kinematic viscosity of $2.8 \times 10^{-5}\ \frac{m^2}{s}$ and SG of 0.92. If the mean velocity of the piston is $6\ \frac{m}{s}$, approximately what is the power dissipated in the friction?



Given: The kinematic viscosity: $\nu = 2.8 \times 10^{-5}\ \frac{m^2}{s}$.Specific gravity: $SG = 0.92$. Mean velocity: $V = 6\ \frac{m}{s}$. The configuration is shown in the figure.

Assumption: Linear velocity profile in the lubricant, negligible end effects.

Find: Power P_f dissipated in the friction.

Solution: Use Newton's law of viscosity to relate the viscous shear stress to the velocities

The shear stress is given by Newton's law of viscosity

$$\tau = \mu \frac{du}{dr} = \mu \frac{\Delta u}{\Delta r}$$

The density of the lubricant is:

$$\rho = SG \cdot \rho_{H_20} = 0.92 \times 998\ \frac{kg}{m^3} = 918\ \frac{kg}{m^3}$$

The dynamic viscosity of the lubricant is:

$$\mu = \nu\rho = 2.8 \times 10^{-5}\ \frac{m^2}{s} \times 918\ \frac{kg}{m^3} = 2.57 \times 10^{-2}\ \frac{kg}{m \cdot s} = 2.57 \times 10^{-2}\ Pa \cdot s$$

The drag force:

$$F_D = \mu A \frac{\Delta u}{\Delta r}$$

$$F_D = 2.57 \times 10^{-2}\ Pa \cdot s \times (\pi \times 0.15\ m \times 0.3\ m) \times \frac{6\ \frac{m}{s}}{0.0001\ m} = 218\ N$$

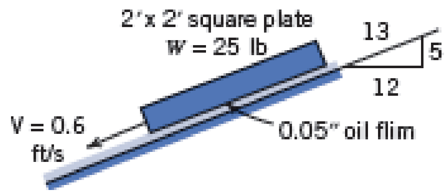
The power P_f dissipated in the friction is the product of the force and velocity:

$$P_f = F_D V = 218\ N \times 6\ \frac{m}{s} = 1308\ W$$

Problem 2.54

(Difficulty: 1)

2.54 Calculate the approximate viscosity of the oil.



Given: Velocity: $V = 0.6\ \frac{ft}{s}$. Gravity: $W = 25\ lbf$. Area: $2\ ft \times 2ft$. Gap: $h = 0.05\ in$.

Slope: $sin\theta = \frac{5}{13}$.

Assumption: Linear velocity profile in oil, negligible end effects.

Find: Viscosity of the oil.

Solution: Use Newton's law of viscosity to find the relation between shear stress and velocity.

The force balance equation is that the drag force equals the component of the weight along the surface:

$$F_D = W \cdot sin\theta = \frac{5}{13}W$$

The drag force is found using Newton's law of viscosity

$$\tau = \mu\frac{du}{dy} = \mu A\frac{\Delta u}{\Delta y}$$

The drag force is then, where the velocity profile is assumed linear:

$$F_D = \mu A\frac{\Delta u}{\Delta y} = \mu A\frac{V}{h}$$

From the force balance

$$\mu A\frac{V}{h} = \frac{5}{13}W$$

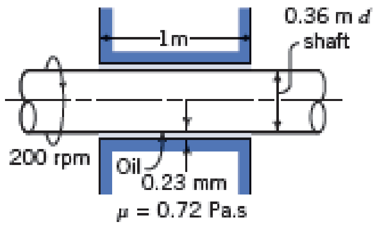
The viscosity of the oil is:

$$\mu = \frac{5}{13}W \cdot \frac{h}{AV} = \frac{5}{13} \times 25\ lbf \times \frac{\frac{0.05}{12}\ ft}{4\ ft^2 \times 0.6\ \frac{ft}{s}} = 0.0167\ \frac{lbf \cdot s}{ft^2}$$

Problem 2.55

(Difficulty: 2)

2.55 Calculate the approximate power lost in friction in this ship propeller shaft bearing.



Given: Rotation speed: 200 $\frac{r}{min}$. Gap: $h = 0.23\ mm$. Length: $L = 1\ m$. Shaft diameter: $D = 0.36\ m$. Viscosity: $\mu = 0.72\ Pa \cdot s$.

Assumption: Linear velocity profile in fluid, negligible end effects.

Find: Power P_d lost in friction.

Solution: Use Newton's law of viscosity to relate the viscous force to the velocity

$$\tau = \mu \frac{du}{dr} = \mu \frac{\Delta u}{\Delta r}$$

The drag force is given by the product of the shear stress and area. For the linear velocity in the fluid:

$$F_D = A\,\mu \frac{\Delta u}{\Delta r} = A\,\mu \frac{V}{h}$$

The velocity is given by:

$$V = \omega R = 200\ \frac{r}{min} \times \frac{1\ min}{60\ s} \times \left(2\pi\ \frac{rad}{r}\right) \times \frac{0.36}{2}\ m = 3.77\ \frac{m}{s}$$

So we have:

$$F_D = 0.72\ Pa \cdot s \times (\pi \times 0.36\ m \times 1\ m) \times \frac{3.77\ \frac{m}{s}}{0.00023\ m} = 13340\ N$$

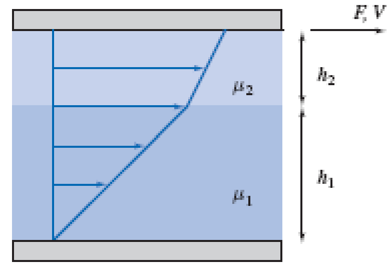
The power P_d lost in friction is the product of force and velocity:

$$P_d = F.V = 13340\ N \times 3.77\ \frac{m}{s} = 50.3\ kW$$

Problem 2.56

[Difficulty: 2]

2.56 Fluids of viscosities $\mu_1 = 0.1\ \mathrm{N\cdot s/m^2}$ and $\mu_2 = 0.15\ \mathrm{N\cdot s/m^2}$ are contained between two plates (each plate is $1\ \mathrm{m^2}$ in area). The thicknesses are $h_1 = 0.5$ mm and $h_2 = 0.3$ mm, respectively. Find the force F to make the upper plate move at a speed of 1 m/s. What is the fluid velocity at the interface between the two fluids?



Given: Flow between two plates

Find: Force to move upper plate; Interface velocity

Solution:

The shear stress is the same throughout (the velocity gradients are linear, and the stresses in the fluid at the interface must be equal and opposite).

Hence $\tau = \mu_1 \cdot \frac{du_1}{dy} = \mu_2 \cdot \frac{du_2}{dy}$ or $\mu_1 \cdot \frac{V_i}{h_1} = \mu_2 \cdot \frac{(V - V_i)}{h_2}$ where V_i is the interface velocity

Solving for the interface velocity V_i

$$V_i = \frac{V}{1 + \frac{\mu_1}{\mu_2} \cdot \frac{h_2}{h_1}} = \frac{1 \cdot \frac{m}{s}}{1 + \frac{0.1}{0.15} \cdot \frac{0.3}{0.5}} \qquad V_i = 0.714 \frac{m}{s}$$

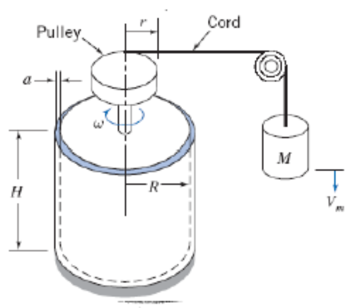
Then the force required is

$$F = \tau \cdot A = \mu_1 \cdot \frac{V_i}{h_1} \cdot A = 0.1 \cdot \frac{N \cdot s}{m^2} \times 0.714 \cdot \frac{m}{s} \times \frac{1}{0.5 \cdot mm} \times \frac{1000 \cdot mm}{1 \cdot m} \times 1 \cdot m^2 \qquad F = 143\ N$$

Problem 2.57

(Difficulty: 2)

2.57 A concentric cylinder viscometer may be formed by rotating the inner member of a pair of closely fitting cylinders. The annular gap is small so that a linear velocity profile will exist in the liquid sample. Consider a viscometer with an inner cylinder of 4 $in.$ diameter and 8 $in.$ height, and a clearance gap width of 0.001 $in,$ filled with castor oil at 90 °F. Determine the torque required to turn the inner cylinder at 400 $rpm.$



Assumptions: (1) Newtonian liquid (2) Narrow gap, so linear velocity profile (3)steady angular speed

Solution: For Newtonian fluid we have Newton's law of viscosity:

$$\tau = \mu \frac{du}{dy}$$

The required torque must balance the resisting torque of the shear force. The shear force is given by:

$$F = \tau A$$

$$A = 2\pi R h$$

For small gap (linear profile) we have:

$$\tau = \mu \frac{V}{d}$$

Where V is the tangential velocity of inner cylinder given by:

$$V = R\omega$$

Hence

$$F = \tau A = \mu \frac{R\omega}{d} 2\pi R h = \frac{2\pi \mu R^2 \omega h}{d}$$

And the torque is:

$$T = RF = \frac{2\pi\mu R^3 \omega h}{d}$$

From Fig. A.2, for castor oil at 90 °F,

$$\mu = 0.38\ \frac{N \cdot s}{m^2} = 0.0079\ \frac{lbf \cdot s}{ft^2}$$

$$\omega = 400\ rpm = 41.9\ \frac{rad}{s}$$

Substituting into the expression for torque, we find:

$$T = \frac{2\pi\mu R^3 \omega h}{d} = \frac{2\pi \times 0.0079\ \frac{lbf \cdot s}{ft^2} \times \left(\frac{2}{12}\ ft\right)^3 \times 41.9\ \frac{rad}{s} \times \frac{8}{12}\ ft}{\frac{0.001}{12}\ ft}$$

$$T = 77.4\ lbf \cdot ft$$

Problem 2.58

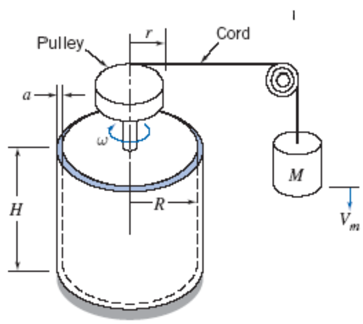
(Difficulty: 2)

2.58 A concentric cylinder viscometer is driven by a falling mass M connected by a cord and pulley to the inner cylinder, as shown. The liquid to be tested fills the annular gap of width a and height H. After a brief starting transient, the mass falls at a constant speed V_m. Develop an algebraic expression for the viscosity of the liquid in the device in terms of $M, g, V_m, r, R, a, and\ H$. Evaluate the viscosity of the liquid using:

$$M = 0.10\ kg \qquad r = 25\ mm$$

$$R = 50\ mm \qquad a = 0.20\ mm$$

$$H = 80\ mm \qquad V_m = 30\ \ mm/s$$



Assumptions: (1) Newtonian liquid (2) Narrow gap, so linear velocity profile (3)steady angular speed

Solution: Apply Newton's law of viscosity. The basic equations are:

$$\tau = \mu \frac{du}{dy}$$

$$\sum M_t = 0$$

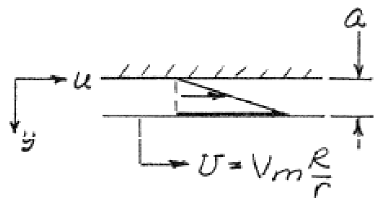
$$T = \tau A R$$

Summing torques on the rotor:

$$\sum M_t = Mgr - \tau A R = 0$$

$$A = 2\pi R H$$

Because $a \ll R$, treat the gap as plane, then



$$\tau = \mu\frac{du}{dy} = \mu\frac{\Delta u}{\Delta y} = \mu\frac{U}{a} = \frac{\mu V_m R}{ar}$$

Substituting into the expression for moments,

$$Mgr - \frac{\mu V_m R}{ar} 2\pi RHR = Mgr - \frac{2\pi\mu\, V_m R^3 H}{ar} = 0$$

So we have for the viscosity:

$$\mu = \frac{Mgr^2 a}{2\pi\, V_m R^3 H}$$

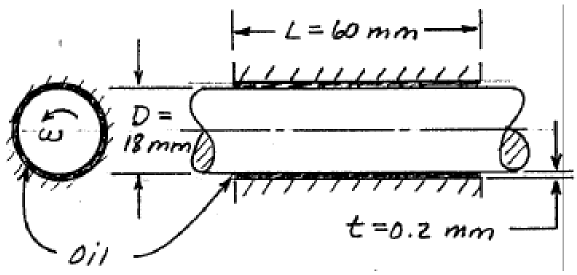
For the given data, the viscosity is

$$\mu = \frac{0.10\ kg \times 9.81\ \frac{m}{s^2} \times (0.025\ m)^2 \times 0.0002\ m}{2\pi \times 0.03\ \frac{m}{s} \times (0.05\ m)^3 \times 0.08\ m} = 0.0651\ \frac{N \cdot s}{m^2}$$

Problem 2.59

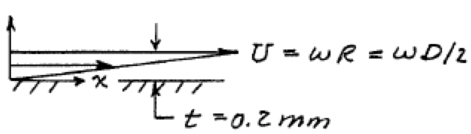
(Difficulty: 2)

2.59 A shaft with outside diameter of 18 mm turns at 20 revolutions per second inside a stationary journal bearing 60 mm long. A thin film of oil 0.2 mm thick fills the concentric annulus between the shaft and journal. The torque needed to turn the shaft is 0.0036 $N \cdot m$. Estimate the viscosity of the oil that fills the gap.



Assumptions: (1) Newtonian fluid (2) Gap is narrow, so velocity profile is linear, $\frac{du}{dy} \approx \frac{\Delta u}{\Delta y}$

We have the following geometry:



Solution: Use Newton's law of viscosity:

$$\tau_{yx} \approx \mu \frac{\Delta u}{\Delta y} = \mu \frac{U}{t} = \frac{\mu \omega D}{2t}$$

Neglecting end effects, the torque is:

$$T = FR = \tau_{yx} AR = \tau_{yx} (\pi D L) \frac{D}{2} = \frac{\mu \pi \omega D^3 L}{4t}$$

Solving for viscosity,

$$\mu = \frac{4tT}{\pi \omega D^3 L}$$

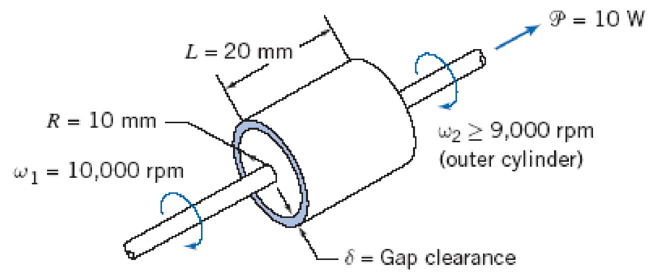
The value of viscosity is

$$\mu = \frac{4 \times 0.0002\ m \times 0.0036\ N \cdot m}{\pi \times \left(20 \times 2\pi\ \frac{rad}{s}\right) \times (0.018\ m)^3 \times 0.06\ m} = 0.0208\ \frac{N \cdot s}{m^2}$$

Problem 2.60

[Difficulty: 3]

2.60 A shock-free coupling for a low-power mechanical drive is to be made from a pair of concentric cylinders. The annular space between the cylinders is to be filled with oil. The drive must transmit power, $\mathscr{P} = 10$ W. Other dimensions and properties are as shown. Neglect any bearing friction and end effects. Assume the minimum practical gap clearance δ for the device is $\delta = 0.25$ mm. Dow manufactures silicone fluids with viscosities as high as 10^6 centipoise. Determine the viscosity that should be specified to satisfy the requirement for this device.



Given: Shock-free coupling assembly

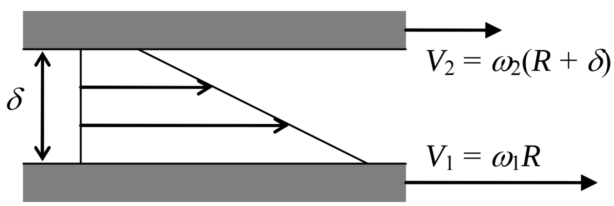
Find: Required viscosity

Solution:

Basic equation $\qquad \tau_{r\theta} = \mu \cdot \frac{du}{dr} \qquad$ Shear force $\quad F = \tau \cdot A \qquad$ Torque $\quad T = F \cdot R \qquad$ Power $\quad P = T \cdot \omega$

Assumptions: Newtonian fluid, linear velocity profile

$$\tau_{r\theta} = \mu \cdot \frac{du}{dr} = \mu \cdot \frac{\Delta V}{\Delta r} = \mu \cdot \frac{\left[\omega_1 \cdot R - \omega_2 \cdot (R + \delta)\right]}{\delta}$$



$$\tau_{r\theta} = \mu \cdot \frac{(\omega_1 - \omega_2) \cdot R}{\delta} \qquad \text{Because } \delta << R$$

Then $\qquad P = T \cdot \omega_2 = F \cdot R \cdot \omega_2 = \tau \cdot A_2 \cdot R \cdot \omega_2 = \frac{\mu \cdot (\omega_1 - \omega_2) \cdot R}{\delta} \cdot 2 \cdot \pi \cdot R \cdot L \cdot R \cdot \omega_2$

$$P = \frac{2 \cdot \pi \cdot \mu \cdot \omega_2 \cdot (\omega_1 - \omega_2) \cdot R^3 \cdot L}{\delta}$$

Hence $\qquad \mu = \frac{P \cdot \delta}{2 \cdot \pi \cdot \omega_2 \cdot (\omega_1 - \omega_2) \cdot R^3 \cdot L}$

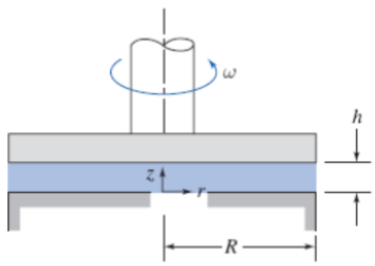
$$\mu = \frac{10 \cdot W \times 2.5 \times 10^{-4} \cdot m}{2 \cdot \pi} \times \frac{1}{9000} \cdot \frac{min}{rev} \times \frac{1}{1000} \cdot \frac{min}{rev} \times \frac{1}{(.01 \cdot m)^3} \times \frac{1}{0.02 \cdot m} \times \frac{N \cdot m}{s \cdot W} \times \left(\frac{rev}{2 \cdot \pi \cdot rad}\right)^2 \times \left(\frac{60 \cdot s}{min}\right)^2$$

$\mu = 0.202 \cdot \frac{N \cdot s}{m^2} \qquad \mu = 2.02 \cdot poise \qquad$ which corresponds to SAE 30 oil at 30ºC.

Problem 2.61

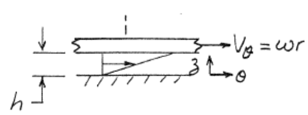
(Difficulty: 4)

2.61 A proposal has been made to use a pair of parallel disks to measure the viscosity of a liquid sample. The upper disk rotates at height h above the lower disk. The viscosity of the liquid in the gap is to be calculated from measurements of the torque needed turn the upper disk steadily. Obtain an algebraic expression for the torque needed to turn the disk. Could we use this device to measure the viscosity of a non-Newtonian fluid? Explain.



Assumptions: (1) Newtonian fluid (2) No-slip condition (3) Linear velocity profile (in a narrow gap)

Use the r, θ, z coordinates as shown as:



Solution: Use Newton's law of viscosity:

$$\tau_{z\theta} = \mu \frac{dV_\theta}{dz}$$

$$dT = rdF = r\tau_{z\theta}dA$$

The velocity at any radial location on the rotating disk is:

$$V_\theta = \omega r$$

Since the velocity profile is linear then:

$$\tau_{z\theta} = \mu \frac{dV_\theta}{dz} = \mu \frac{\omega r}{h}$$

$$dT = r\mu \frac{\omega r}{h} 2\pi r dr = \frac{2\pi\mu\omega r^3}{h} dr$$

By integrating, the torque is,

$$T = \int_0^A dT = \int_0^R \frac{2\pi\mu\omega r^3}{h} dr = \frac{\pi\mu\omega r^4}{2h}\Bigg|_0^R = \frac{\pi\mu\omega R^4}{2h}$$

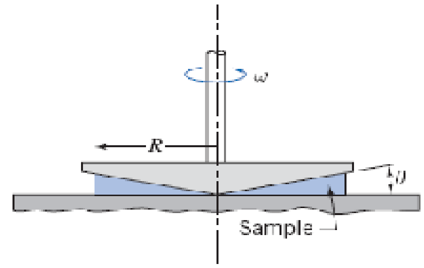
The device could not be used to measure the viscosity of a non-Newtonian fluid because the applied shear stress is not uniform. It varies from zero at the center of the disks to $\mu\omega R/h$ at the edge.

Problem 2.62

[Difficulty: 4]

2.62 The cone and plate viscometer shown is an instrument used frequently to characterize non-Newtonian fluids. It consists of a flat plate and a rotating cone with a very obtuse angle (typically θ is less than 0.5 degrees). The apex of the cone just touches the plate surface and the liquid to be tested fills the narrow gap formed by the cone and plate. The viscometer is used to measure the apparent viscosity of a fluid. The data below are obtained. What kind of non-Newtonian fluid is this? Find the values of k and n used in Eqs. 2.16 and 2.17 in defining the apparent viscosity of a fluid. (Assume θ is 0.5 degrees.) Predict the viscosity at 90 and 100 rpm.

Speed (rpm)	10	20	30	40	50	60	70	80
μ(N·s/m^2)	0.121	0.139	0.153	0.159	0.172	0.172	0.183	0.185



Given: Data on the viscometer

Find: The values of coefficients k and n; determine the kind of non-Newtonial fluid it is; estimate viscosity at 90 and 100 rpm

Solution:

The velocity gradient at any radius r is $$\frac{du}{dy} = \frac{r\cdot\omega}{r\cdot\tan(\theta)}$$

where ω (rad/s) is the angular velocity $$\omega = \frac{2\cdot\pi\cdot N}{60}$$ where N is the speed in rpm

For small θ, tan(θ) can be replace with θ, so $$\frac{du}{dy} = \frac{\omega}{\theta}$$

From Eq $$k\cdot\left(\left|\frac{du}{dy}\right|\right)^{n-1}\frac{du}{dy} = \eta\cdot\frac{du}{dy}$$

where η is the apparent viscosity. Hence $$\eta = k\cdot\left(\frac{du}{dy}\right)^{n-1} = k\cdot\left(\frac{\omega}{\theta}\right)^{n-1}$$

The data is

N (rpm)	μ (N·s/m^2)
10	0.121
20	0.139
30	0.153
40	0.159
50	0.172
60	0.172
70	0.183
80	0.185

The computed data is

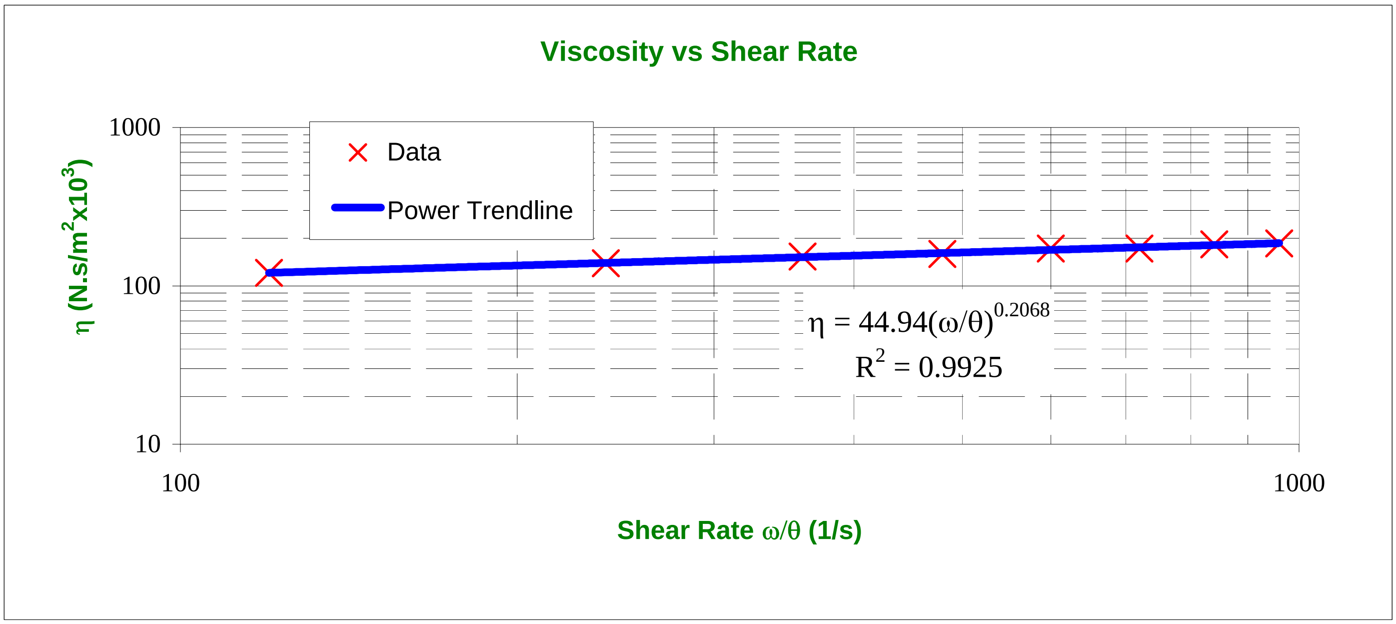
ω (rad/s)	ω/θ (1/s)	η (N·s/m²x10³)
1.047	120	121
2.094	240	139
3.142	360	153
4.189	480	159
5.236	600	172
6.283	720	172
7.330	840	183
8.378	960	185

From the *Trendline* analysis

k = 0.0449
n - 1 = 0.2068
n = 1.21 The fluid is dilatant

The apparent viscosities at 90 and 100 rpm can now be computed

N (rpm)	ω (rad/s)	ω/θ (1/s)	η (N·s/m²x10³)
90	9.42	1080	191
100	10.47	1200	195

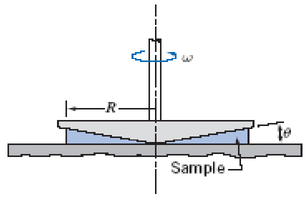


Problem 2.63

[Difficulty: 3]

2.63 A viscometer is used to measure the viscosity of a patient's blood. The deformation rate (shear rate)–shear stress data is shown below. Plot the apparent viscosity versus deformation rate. Find the value of k and n in Eq. 2.17, and from this examine the aphorism "Blood is thicker than water."

du/dy (s^{-1})	5	10	25	50	100	200	300	400
τ (Pa)	0.0457	0.119	0.241	0.375	0.634	1.06	1.46	1.78



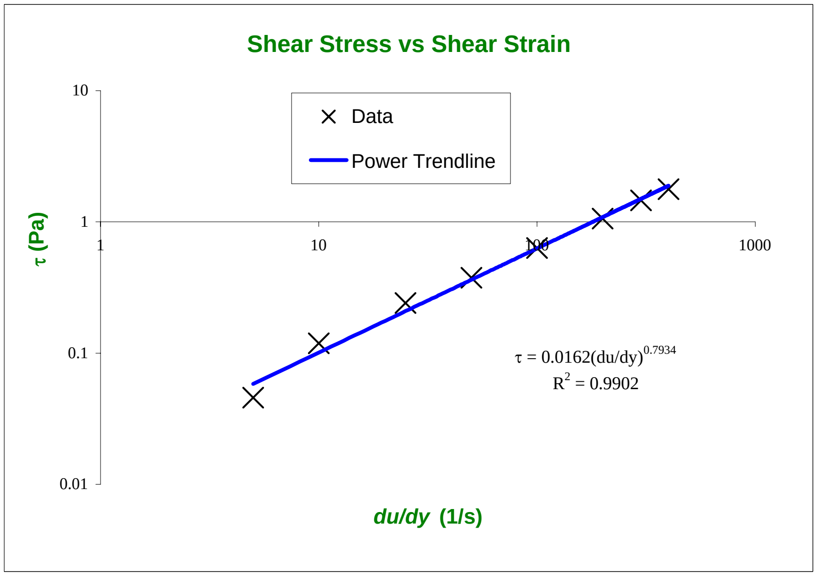
Given: Viscometer data

Find: Value of k and n in Eq. 2.17

Solution:

The data is

τ (Pa)	du/dy (s^{-1})
0.0457	5
0.119	10
0.241	25
0.375	50
0.634	100
1.06	200
1.46	300
1.78	400



Hence we have $k = 0.0162$

$n = 0.7934$ Blood is pseudoplastic (shear thinning)

The apparent viscosity from $\eta = k\,(du/dy)^{n-1}$

du/dy (s^{-1})	η ($N \cdot s/m^2$)
5	0.0116
10	0.0101
25	0.0083
50	0.0072
100	0.0063
200	0.0054
300	0.0050
400	0.0047

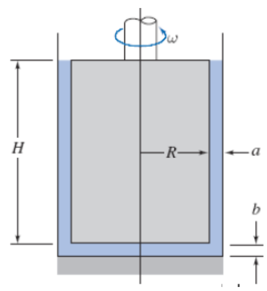
$\mu_{water} = 0.001\ N \cdot s/m^2$ at $20^{\circ}C$

Hence, blood is "thicker" than water!

Problem 2.64

(Difficulty: 5)

2.64 A concentric-cylinder viscometer is shown. Viscous torque is produced by the annular gap around the inner cylinder. Additional viscous torque is produced by the flat bottom of the stationary outer cylinder. Obtain an algebraic expression for the viscous torque due to flow in the annular gap of width a. Obtain an algebraic expression for the viscous torque due to flow in the bottom clearance gap of height b. Prepare a plot showing the ratio, b/a, required to hold the bottom torque to 1 percent or less of the annulus torque, versus the other geometric variables. What are the design implications? What modifications to the design can you recommend?

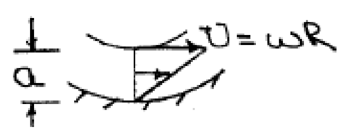


Assumptions: (1) Newtonian fluid (2) Linear velocity profile (in a narrow gap)

Solution: Use Newton's law of viscosity:

$$\tau_{yx} = \mu \frac{du}{dy}$$

(a) In annular gap, the velocity profile is:
(b)



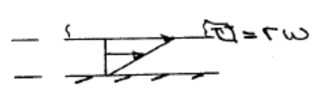
The shear stress is then

$$\tau = \mu \frac{du}{dr} = \mu \frac{\Delta u}{\Delta r} = \mu \frac{U}{a} = \mu \frac{\omega R}{a}$$

The torque is calculated as:

$$T_{an} = F_f R = \tau A R = \mu \frac{\omega R}{a}(2\pi R H) R = \frac{2\pi\mu\omega R^3 H}{a}$$

(c) In bottom gap:



$$\tau = \mu \frac{du}{dz} = \mu \frac{\Delta u}{\Delta z} = \mu \frac{U}{b} = \frac{\mu \omega r}{b}$$

So we have for the torque here:

$$T_{bot} = \int_0^A dT = \int_0^A r dF = \int_0^A r \tau dA = \int_0^R r \frac{\mu \omega r}{b} (2\pi r)\, dr$$

$$T_{bot} = \int_0^R \frac{2\pi \mu \omega r^3}{b} dr = \frac{\pi \mu \omega r^4}{2b}\Bigg|_0^R = \frac{\pi \mu \omega R^4}{2b}$$

(d) For the condition that:

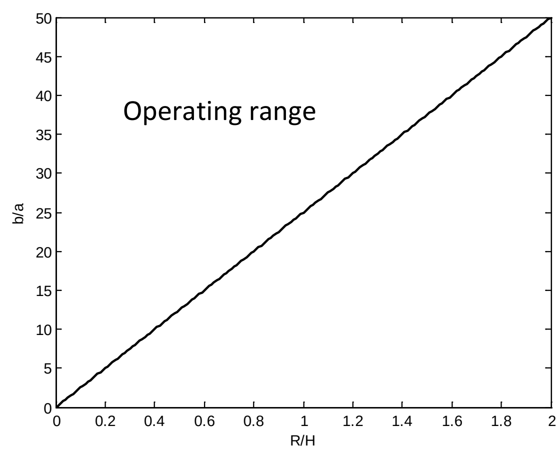
$$\frac{T_{bot}}{T_{an}} \leq \frac{1}{100}$$

Then

$$\frac{T_{bot}}{T_{an}} = \frac{\frac{\pi \mu \omega R^4}{2b}}{\frac{2\pi \mu \omega R^3 H}{a}} = \frac{aR}{4bH} \leq \frac{1}{100}$$

$$\frac{b}{a} \geq 25 \frac{R}{H}$$

So we have the plot:



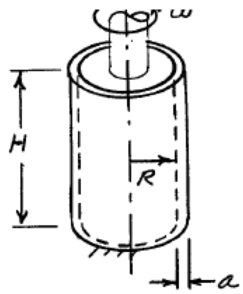
(e) The plot shows the operating range specific design would depend on other constraints, for $a = 1\ mm$ with $R/H = 0.5$ gives $b = 12.5\ mm$.

(f) For a given value of R/H, the dimension b could be efficiently increased by “hollowing out” the inner cylinders as shown by the data lines in the diagram above.

Problem 2.65

(Difficulty: 4)

2.65 Design a concentric-cylinder viscometer to measure the viscosity of a liquid similar to water. The goal is to achieve a measurement accuracy of ± 1 percent. Specify the configuration and dimensions of the viscometer. Indicate what measured parameter will be used to infer the viscosity of the liquid sample.

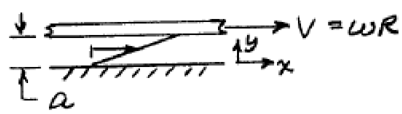


Assumptions: (1) Steady (2) Newtonian liquid (3) Narrow gap, so "unroll" it (4) Linear velocity profile in gap (5) Neglect end effects

Solution: Use Newton's law of viscosity.

$$\tau = \mu \frac{du}{dy}$$

For the model below we have:



$$u = V\frac{y}{a} = \frac{\omega R y}{a}$$

$$\frac{du}{dy} = \frac{\omega R}{a}$$

Thus

$$\tau = \mu \frac{du}{dy} = \mu \frac{\omega R}{a}$$

The torque is

$$T = R\tau A = R\mu \frac{\omega R}{a}(2\pi R H) = \frac{2\pi\mu\omega R^3 H}{a}$$

So we have for the viscosity:

$$\mu = \frac{Ta}{2\pi\omega R^3 H}$$

From this equation the uncertainty in μ is:

$$u_\mu = \pm[u_T^2 + u_a^2 + u_w^2 + (3u_R)^2 + u_H^2]^{\frac{1}{2}} = \pm[13u^2]^{\frac{1}{2}} = \pm 3.61u$$

If the uncertainty of each parameter equals u, then

$$u = \pm\frac{u_\mu}{3.61} = \pm\frac{1\ percent}{3.61} = \pm 0.277\ percent$$

Typical dimensions for a bench-top unit might be:

$$H = 200\ mm, R = 75\ mm, a = 0.02\ mm, and\ \omega = 10.5\ \ rad/s$$

From Appendix A, Table A.8, water has a viscosity of:

$$\mu = 1.0 \times 10^{-3}\ \frac{N \cdot s}{m^2}$$

The corresponding torque would be:

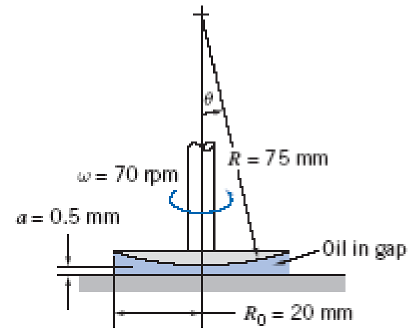
$$T = \frac{2\pi \times 1.0 \times 10^{-3}\ \frac{N \cdot s}{m^2} \times 10.5\ \frac{rad}{s} \times (0.075\ m)^3 \times 0.2\ m}{0.00002\ m} = 0.278\ N \cdot m$$

It should be possible to measure this torque quite accurately.

{Many details would need to be considered. (e.g. bearings, temperature rise, etc.) to produce a workable device.}

Problem 2.66 [Difficulty: 5]

2.66 A cross section of a rotating bearing is shown. The spherical member rotates with angular speed ω, a small distance, a, above the plane surface. The narrow gap is filled with viscous oil, having $\mu = 1250$ cp. Obtain an algebraic expression for the shear stress acting on the spherical member. Evaluate the maximum shear stress that acts on the spherical member for the conditions shown. (Is the maximum necessarily located at the maximum radius?) Develop an algebraic expression (in the form of an integral) for the total viscous shear torque that acts on the spherical member. Calculate the torque using the dimensions shown.



Given: Geometry of rotating bearing

Find: Expression for shear stress; Maximum shear stress; Expression for total torque; Total torque

Solution:

Basic equation $\quad \tau = \mu \cdot \frac{du}{dy} \qquad dT = r \cdot \tau \cdot dA$

Assumptions: Newtonian fluid, narrow clearance gap, laminar motion

From the figure $\quad r = R \cdot \sin(\theta) \qquad u = \omega \cdot r = \omega \cdot R \cdot \sin(\theta) \qquad \frac{du}{dy} = \frac{u - 0}{h} = \frac{u}{h}$

$$h = a + R \cdot (1 - \cos(\theta)) \qquad dA = 2 \cdot \pi \cdot r \cdot dr = 2 \cdot \pi\, R \cdot \sin(\theta) \cdot R \cdot \cos(\theta) \cdot d\theta$$

Then $\quad \tau = \mu \cdot \frac{du}{dy} = \frac{\mu \cdot \omega \cdot R \cdot \sin(\theta)}{a + R \cdot (1 - \cos(\theta))}$

To find the maximum τ set $\quad \frac{d}{d\theta}\left[\frac{\mu \cdot \omega \cdot R \cdot \sin(\theta)}{a + R \cdot (1 - \cos(\theta))}\right] = 0 \qquad$ so $\qquad \frac{R \cdot \mu \cdot \omega \cdot (R \cdot \cos(\theta) - R + a \cdot \cos(\theta))}{(R + a - R \cdot \cos(\theta))^2} = 0$

$$R \cdot \cos(\theta) - R + a \cdot \cos(\theta) = 0 \qquad \theta = \operatorname{acos}\left(\frac{R}{R + a}\right) = \operatorname{acos}\left(\frac{75}{75 + 0.5}\right) \qquad \theta = 6.6 \cdot \text{deg}$$

$$\tau = 12.5 \cdot \text{poise} \times 0.1 \cdot \frac{\frac{\text{kg}}{\text{m} \cdot \text{s}}}{\text{poise}} \times 2 \cdot \pi \cdot \frac{70}{60} \cdot \frac{\text{rad}}{\text{s}} \times 0.075 \cdot \text{m} \times \sin(6.6 \cdot \text{deg}) \times \frac{1}{[0.0005 + 0.075 \cdot (1 - \cos(6.6 \cdot \text{deg}))] \cdot \text{m}} \times \frac{\text{N} \cdot \text{s}^2}{\text{m} \cdot \text{kg}}$$

$$\tau = 79.2 \cdot \frac{\text{N}}{\text{m}^2}$$

The torque is $\quad T = \int r \cdot \tau \cdot A\, d\theta = \int_0^{\theta_{max}} \frac{\mu \cdot \omega \cdot R^4 \cdot \sin(\theta)^2 \cdot \cos(\theta)}{a + R \cdot (1 - \cos(\theta))}\, d\theta \qquad$ wher e $\quad \theta_{max} = \operatorname{asin}\left(\frac{R_0}{R}\right) \qquad \theta_{max} = 15.5 \cdot \text{deg}$

This integral is best evaluated numerically using Excel, Mathcad, or a good calculator $\quad T = 1.02 \times 10^{-3} \cdot \text{N} \cdot \text{m}$

Problem 2.67

(Difficulty: 2)

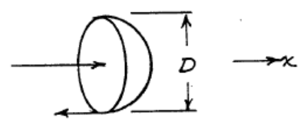
2.67 Small gas bubbles form in soda when a bottle or can is opened. The average bubble diameter is about 0.1 mm. Estimate the pressure difference between the inside and outside of such a bubble.

Assumptions: The only forces are due to surface tension and pressure

Solution:

Consider a free-body diagram of half a bubble:

Two forces act:



Pressure:

$$F_p = \Delta p \frac{\pi D^2}{4}$$

Surface tension:

$$F_\sigma = \sigma \pi D$$

Summing forces for equilibrium:

$$\sum F_x = F_p - F_\sigma = \Delta p \frac{\pi D^2}{4} - \sigma \pi D = 0$$

So we have:

$$\frac{\Delta p D}{4} - \sigma = 0$$

$$\Delta p = \frac{4\sigma}{D}$$

Assuming soda-gas interface is similar to water-air, then $\sigma = 72.8\ \frac{mN}{m}$, and

$$\Delta p = \frac{4 \times 0.0728\ \frac{N}{m}}{0.0001\ m} = 2.91\ kPa$$

Problem 2.68

[Difficulty: 2]

2.68 You intend to gently place several steel needles on the free surface of the water in a large tank. The needles come in two lengths: Some are 5 cm long, and some are 10 cm long. Needles of each length are available with diameters of 1 mm, 2.5 mm, and 5 mm. Make a prediction as to which needles, if any, will float.

Given: Data on size of various needles

Find: Which needles, if any, will float

Solution:

For a steel needle of length *L*, diameter *D*, density ρ_s, to float in water with surface tension σ and contact angle θ, the vertical force due to surface tension must equal or exceed the weight

$$2 \cdot L \cdot \sigma \cdot \cos(\theta) \ge W = m \cdot g = \frac{\pi \cdot D^2}{4} \cdot \rho_s \cdot L \cdot g \qquad \text{or} \qquad D \le \sqrt{\frac{8 \cdot \sigma \cdot \cos(\theta)}{\pi \cdot \rho_s \cdot g}}$$

From Table $\quad \sigma = 72.8 \times 10^{-3} \cdot \frac{N}{m} \quad \theta = 0 \cdot deg \quad$ and for water $\quad \rho = 1000 \cdot \frac{kg}{m^3}$

From Table for steel $\quad SG = 7.83$

Hence

$$\sqrt{\frac{8 \cdot \sigma \cdot \cos(\theta)}{\pi \cdot SG \cdot \rho \cdot g}} = \sqrt{\frac{8}{\pi \cdot 7.83} \times 72.8 \times 10^{-3} \cdot \frac{N}{m} \times \frac{m^3}{999 \cdot kg} \times \frac{s^2}{9.81 \cdot m} \times \frac{kg \cdot m}{N \cdot s^2}} = 1.55 \times 10^{-3} \cdot m = 1.55 \cdot mm$$

Hence *D* < 1.55 mm. Only the 1 mm needles float (needle length is irrelevant)

Problem 2.69

[Difficulty: 3]

2.69 According to Folsom [6], the capillary rise Δh (in.) of a water-air interface in a tube is correlated by the following empirical expression:

$$\Delta h = Ae^{-b \cdot D}$$

where D (in.) is the tube diameter, $A = 0.400$, and $b = 4.37$. You do an experiment to measure Δh versus D and obtain:

D (in.)	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1
Δh (in.)	0.232	0.183	0.09	0.059	0.052	0.033	0.017	0.01	0.006	0.004	0.003

What are the values of A and b that best fit this data using *Excel's Trendline* feature? Do they agree with Folsom's values? How good is the data?

Given: Caplillary rise data

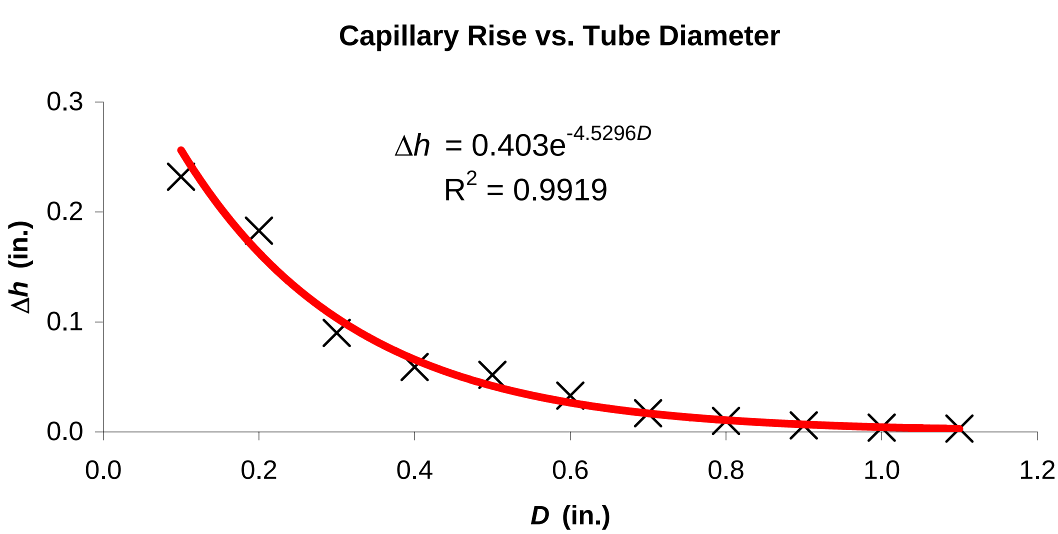
Find: Values of A and b

Solution:

D (in.)	Δh (in.)
0.1	0.232
0.2	0.183
0.3	0.090
0.4	0.059
0.5	0.052
0.6	0.033
0.7	0.017
0.8	0.010
0.9	0.006
1.0	0.004
1.1	0.003

$$\boxed{\begin{aligned} A &= 0.403 \\ b &= 4.530 \end{aligned}}$$

The fit is a good one (R^2 = 0.9919)



Problem 2.70

(Difficulty: 3)

2.70 Calculate and plot the maximum capillary rise of water at 20 C to be expected in a vertical glass tube as a function of tube diameters from 0.5 to 2.5 mm.

Given: Temperature: $T = 20\ ℃$. Diameter: $D\ from\ 0.5\ to\ 2.5\ mm$.

Find: Maximum capillary rise Δh.

Solution: Use the relation for capillary force to find the rise.

For the force balance on the water we have the capillary force and the weight of the volume of water:

$$\sum F_z = \sigma_{H_2o} \pi D \cos\theta - \rho_{H_2o} g \Delta V = 0$$

$$\Delta V = \frac{1}{4}\pi D^2 \Delta h$$

So we have:

$$\Delta h = \frac{4\sigma_{H_2o} \pi D \cos\theta}{\rho_{H_2o} g \pi D^2} = \frac{4\sigma_{H_2o} \cos\theta}{\rho_{H_2o} g D}$$

When $\theta = 0$, we have the maximum Δh for specific tube diameter . The surface tension for water is given by

$$\sigma_{H_2o} = 0.0728\ \frac{N}{m}$$

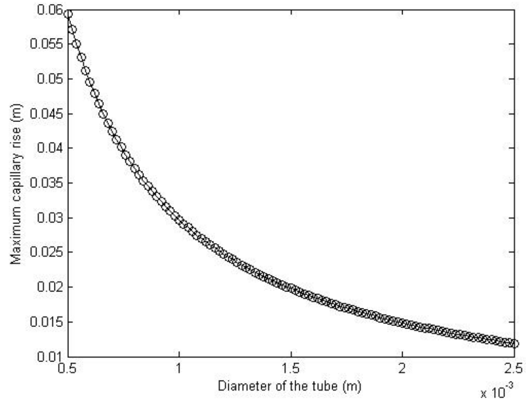
The rise is then

$$\Delta h = \frac{4\sigma_{H_2o} \cos\theta}{\rho_{H_2o} g D} = \frac{4 \times 0.0728 \frac{N}{m} \times 1}{998 \frac{kg}{m^3} \times 9.81 \frac{m}{s^2} \times D} = \frac{2.97 \times 10^{-5}}{D}\quad m$$

For the range

$$0.0005\ m \le D \le 0.0025\ m$$

The plot of rise versus diameter is



Problem 2.71

(Difficulty: 2)

2.71 Calculate the maximum capillary rise of water at $20\ ℃$ to be expected between two vertical, clean glass plates spaced $1mm$ apart.

Given: Temperature: $T = 20\ ℃$. Distance between two plate: $D = 1\ mm$.

Find: Maximum capillary rise Δh.

Solution: Use the relation for capillary force to find the rise

The force balance equation equates the capillary force to the weight of the water:

$$\sum F_z = \sigma_{H_2o} \cdot 2L\cos\theta - \rho_{H_2o} gDL\Delta h = 0$$

where L is the width of the plate. Solving for Δh:

$$\Delta h = \frac{\sigma_{H_2o} \cdot 2L\cos\theta}{\rho_{H_2o} gDL} = \frac{2\sigma_{H_2o}\cos\theta}{\rho_{H_2o} gD}$$

For the maximum capillary rise:

$$\theta = 0$$

The surface tension for water is given by

$$\sigma_{H_2o} = 0.0728\ \frac{N}{m}$$

$$\Delta h = \frac{2\sigma_{H_2o}\cos\theta}{\rho_{H_2o} gD} = \frac{2 \times 0.0728\ \frac{N}{m}}{998\ \frac{kg}{m^3} \times 9.8\ \frac{m}{s^2} \times 0.001\ m} = 0.0149\ m = 14.9\ mm$$

Problem 2.72

(Difficulty: 2)

2.72 Calculate the maximum capillary depression of mercury to be expected in the vertical glass tube $1\ mm$ in diameter at 15.5 ℃.

Given: Temperature: $T = 15.5\ ℃\ or\ 60°F$. Distance between two plate: $D = 1mm\ or\ 0.04\ in$.

Find: Maximum capillary depression Δh.

Solution: Use the relation for capillary force to find the rise

The force balance equation per width of the plate equates the capillary force to the weight of the water:

$$\sum F_z = \sigma \pi D \cos\theta - \rho g \Delta V = 0$$

Where the volume is

$$\Delta V = \frac{1}{4}\pi D^2 \Delta h$$

Solving for the depression:

$$\Delta h = \frac{4\sigma\pi D\cos\theta}{\rho g \pi D^2} = \frac{4\sigma\cos\theta}{\rho g D}$$

For mercury, the surface tension is

$$\sigma = 0.51\ \frac{N}{m}$$

And the density is

$$\gamma = \rho g = 133\ \frac{kN}{m^3}$$

For the maximum capillary depression:

$$\theta = 130\ °\ \text{for mercury.}$$

The depression is

$$\Delta h = \frac{4\sigma\cos\theta}{\gamma D} = \frac{4 \times 0.51 \times \cos(130\ °)}{133 \times 1000 \times 0.001}\ m = -9.86\ mm$$

Problem 2.73

(Difficulty: 2)

2.73 Water usually is assumed to be incompressible when evaluating static pressure variations. Actually it is 100 times more compressible than steel. Assuming the bulk modulus of water is constant, compute the percent change in density for water raised to a gage pressure of 100 atm. Plot the percentage change in water density as a function of p/p_{atm} up to a pressure of $50000\ psi$, which is the approximate pressure used for high-speed cutting jets of water to cut concrete and other composite materials. Would constant density be reasonable assumption for engineering calculations for cutting jets?

Solution: Use the definition of bulk modulus

$$E_v = \frac{dp}{\frac{d\rho}{\rho}}$$

Assume $E_v = constant$, then

$$\frac{d\rho}{\rho} = \frac{dp}{E_v}$$

Integrating from ρ_0 to ρ gives:

$$\ln \frac{\rho}{\rho_0} = \frac{p - p_0}{E_v} = \frac{\Delta p}{E_v}$$

$$\frac{\rho}{\rho_0} = e^{\frac{\Delta p}{E_v}}$$

The relative change in density is:

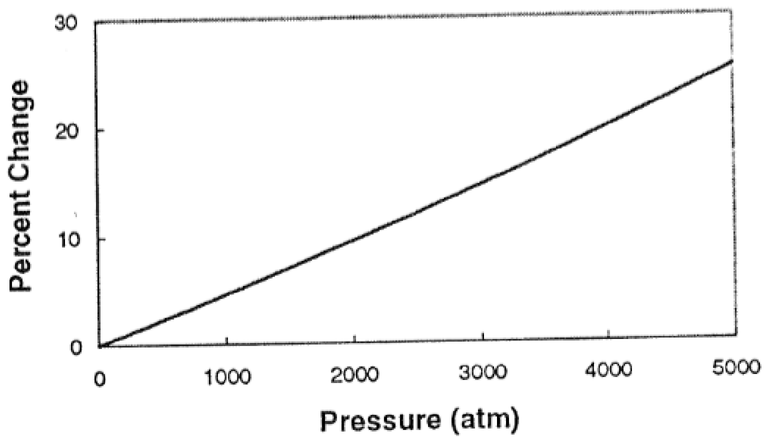
$$\frac{\Delta\rho}{\rho_0} = \frac{\rho - \rho_0}{\rho_0} = \frac{\rho}{\rho_0} - 1 = e^{\frac{\Delta p}{E_v}} - 1$$

From Table A.2, $E_v = 2.24\ GPa$ for water at 20℃.

For $p = 100\ atm\ (gage), \Delta p = 100\ am, so$

$$\frac{\Delta\rho}{\rho_0} = \exp\left(100atm \times \frac{1}{2.24 \times 10^9\ Pa} \times \frac{101.325 \times 10^3\ Pa}{atm}\right) - 1 = 0.166 = 16.6\%$$

Thus constant density is not a reasonable assumption for a cutting jet operating at 50,000 psi. Constant density (5% change) would be reasonable up to $\Delta p \approx 16{,}000\ psi$. The percent change as a function of pressure for a constant bulk modulus is



Problem 2.74

[Difficulty: 2]

2.74 The viscous boundary layer velocity profile shown in Fig. 2.15 can be approximated by a cubic equation,

$$u(y) = a + b\left(\frac{y}{\delta}\right) + c\left(\frac{y}{\delta}\right)^3$$

The boundary condition is $u = U$ (the free stream velocity) at the boundary edge δ (where the viscous friction becomes zero). Find the values of a, b, and c.

Given: Boundary layer velocity profile in terms of constants a, b and c

Find: Constants a, b and c

Solution:

Basic equation $$u = a + b\cdot\left(\frac{y}{\delta}\right) + c\cdot\left(\frac{y}{\delta}\right)^3$$

Assumptions: No slip, at outer edge $u = U$ and $\tau = 0$

At $y = 0$ $\quad 0 = a \quad\quad a = 0$

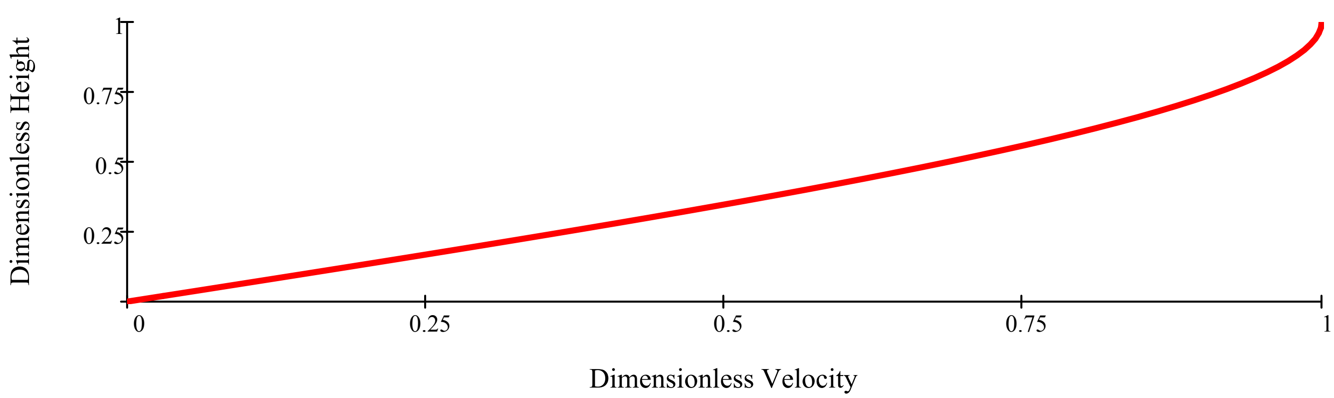
At $y = \delta$ $\quad U = a + b + c \quad\quad b + c = U \quad (1)$

At $y = \delta$ $\quad \tau = \mu\cdot\frac{du}{dy} = 0$

$$0 = \frac{d}{dy}\, a + b\cdot\left(\frac{y}{\delta}\right) + c\cdot\left(\frac{y}{\delta}\right)^3 = \frac{b}{\delta} + 3\cdot c\cdot\frac{y^2}{\delta^3} = \frac{b}{\delta} + 3\cdot\frac{c}{\delta} \quad\quad b + 3\cdot c = 0 \quad (2)$$

From 1 and 2 $\quad c = -\frac{U}{2} \quad\quad b = \frac{3}{2}\cdot U$

Hence $\quad u = \frac{3\cdot U}{2}\cdot\left(\frac{y}{\delta}\right) - \frac{U}{2}\cdot\left(\frac{y}{\delta}\right)^3 \quad\quad \frac{u}{U} = \frac{3}{2}\cdot\left(\frac{y}{\delta}\right) - \frac{1}{2}\cdot\left(\frac{y}{\delta}\right)^3$



Problem 2.75

[Difficulty: 3]

2.75 In a food industry process, carbon tetrachloride at 20°C flows through a tapered nozzle from an inlet diameter D_{in} = 50 mm to an outlet diameter of D_{out}. The area varies linearly with distance along the nozzle, and the exit area is one-fifth of the inlet area; the nozzle length is 250 mm. The flow rate is $Q = 2$ L/min. It is important for the process that the flow exits the nozzle as a turbulent flow. Does it? If so, at what point along the nozzle does the flow become turbulent?

Given: Geometry of and flow rate through tapered nozzle

Find: At which point becomes turbulent

Solution:

Basic equation For pipe flow (Section 2-6) $Re = \frac{\rho \cdot V \cdot D}{\mu} = 2300$ for transition to turbulence

Also flow rate Q is given by $Q = \frac{\pi \cdot D^2}{4} \cdot V$

We can combine these equations and eliminate V to obtain an expression for Re in terms of D and Q

$$Re = \frac{\rho \cdot V \cdot D}{\mu} = \frac{\rho \cdot D}{\mu} \cdot \frac{4 \cdot Q}{\pi \cdot D^2} = \frac{4 \cdot Q \cdot \rho}{\pi \cdot \mu \cdot D} \qquad Re = \frac{4 \cdot Q \cdot \rho}{\pi \cdot \mu \cdot D}$$

For a given flow rate Q, as the diameter is reduced the Reynolds number increases (due to the velocity increasing with A^{-1} or D^{-2}).

Hence for turbulence (Re = 2300), solving for D $D = \frac{4 \cdot Q \cdot \rho}{2300 \cdot \pi \cdot \mu}$

The nozzle is tapered: $D_{in} = 50 \cdot mm$ $\quad D_{out} = \frac{D_{in}}{\sqrt{5}}$ $\quad D_{out} = 22.4 \cdot mm$

Carbon tetrachloride: $\mu_{CT} = 10^{-3} \cdot \frac{N \cdot s}{m^2}$ (Fig A.2) For water $\rho = 1000 \cdot \frac{kg}{m^3}$

$SG = 1.595$ (Table A.2) $\rho_{CT} = SG \cdot \rho$ $\quad \rho_{CT} = 1595 \frac{kg}{m^3}$

For the given flow rate $Q = 2 \cdot \frac{L}{min}$ $\quad \frac{4 \cdot Q \cdot \rho_{CT}}{\pi \cdot \mu_{CT} \cdot D_{in}} = 1354$ LAMINAR $\quad \frac{4 \cdot Q \cdot \rho_{CT}}{\pi \cdot \mu_{CT} \cdot D_{out}} = 3027$ TURBULENT

For the diameter at which we reach turbulence $D = \frac{4 \cdot Q \cdot \rho_{CT}}{2300 \cdot \pi \cdot \mu_{CT}}$ $\quad D = 29.4 \cdot mm$

But $L = 250 \cdot mm$ and linear ratios leads to the distance from D_{in} at which $D = 29.4 \cdot mm$ $\quad \frac{L_{turb}}{L} = \frac{D - D_{in}}{D_{out} - D_{in}}$

$$L_{turb} = L \cdot \frac{D - D_{in}}{D_{out} - D_{in}} \qquad L_{turb} = 186 \cdot mm$$

Problem 2.76

[Difficulty: 2]

2.76 What is the Reynolds number of water at 20°C flowing at 0.25 m/s through a 5-mm-diameter tube? If the pipe is now heated, at what mean water temperature will the flow transition to turbulence? Assume the velocity of the flow remains constant.

Given: Data on water tube

Find: Reynolds number of flow; Temperature at which flow becomes turbulent

Solution:

Basic equation — For pipe flow (Section 2-6) $\quad Re = \frac{\rho \cdot V \cdot D}{\mu} = \frac{V \cdot D}{\nu}$

At 20°C, from Fig. A.3 $\quad \nu = 9 \times 10^{-7} \cdot \frac{m^2}{s}$ $\quad$ and so $\quad Re = 0.25 \cdot \frac{m}{s} \times 0.005 \cdot m \times \frac{1}{9 \times 10^{-7}} \cdot \frac{s}{m^2}$ $\quad Re = 1389$

For the heated pipe $\quad Re = \frac{V \cdot D}{\nu} = 2300$ $\quad$ for transition to turbulence

Hence $\quad \nu = \frac{V \cdot D}{2300} = \frac{1}{2300} \times 0.25 \cdot \frac{m}{s} \times 0.005 \cdot m$ $\quad \nu = 5.435 \times 10^{-7} \frac{m^2}{s}$

From Fig. A.3, the temperature of water at this viscosity is approximately $\quad T = 52 \cdot C$

2.77 A supersonic aircraft travels at 2700 km/hr at an altitude of 27 km. What is the Mach number of the aircraft? At what approximate distance measured from the leading edge of the aircraft's wing does the boundary layer change from laminar to turbulent?

Given: Data on supersonic aircraft

Find: Mach number; Point at which boundary layer becomes turbulent

Solution:

Basic equation $V = M \cdot c$ and $c = \sqrt{k \cdot R \cdot T}$ For air at STP, k = 1.40 and R = 286.9J/kg.K (53.33 ft.lbf/lbm°R).

Hence $M = \frac{V}{c} = \frac{V}{\sqrt{k \cdot R \cdot T}}$

At 27 km the temperature is approximately $T = 223.5 \cdot K$

$$M = \left(2700 \times 10^3 \cdot \frac{m}{hr} \times \frac{1 \cdot hr}{3600 \cdot s}\right) \cdot \left(\frac{1}{1.4} \times \frac{1}{286.9} \cdot \frac{kg \cdot K}{N \cdot m} \times \frac{1 \cdot N \cdot s^2}{kg \cdot m} \times \frac{1}{223.5} \cdot \frac{1}{K}\right)^{\frac{1}{2}} \qquad M = 2.5$$

For boundary layer transition, from Section 2-6 $Re_{trans} = 500000$

Then $Re_{trans} = \frac{\rho \cdot V \cdot x_{trans}}{\mu}$ so $x_{trans} = \frac{\mu \cdot Re_{trans}}{\rho \cdot V}$

We need to find the viscosity and density at this altitude and pressure. The viscosity depends on temperature only, but at 223.5 K = - 50°C, it is off scale of Fig. A.3. Instead we need to use formulas as in Appendix A

At this altitude the density is $\rho = 0.02422 \times 1.225 \frac{kg}{m^3}$ $\qquad \rho = 0.0297 \frac{kg}{m^3}$

For μ $\qquad \mu = \frac{b \cdot T^{\frac{1}{2}}}{1 + \frac{S}{T}}$ where $b = 1.458 \times 10^{-6} \cdot \frac{kg}{m \cdot s \cdot K^{\frac{1}{2}}}$ $\qquad S = 110.4 \cdot K$

$$\mu = 1.459 \times 10^{-5} \frac{kg}{m \cdot s} \qquad \mu = 1.459 \times 10^{-5} \cdot \frac{N \cdot s}{m^2}$$

Hence

$$x_{trans} = 1.459 \times 10^{-5} \cdot \frac{kg}{m \cdot s} \times 500000 \times \frac{1}{0.0297} \cdot \frac{m^3}{kg} \times \frac{1}{2700} \times \frac{1}{10^3} \cdot \frac{hr}{m} \times \frac{3600 \cdot s}{1 \cdot hr} \qquad x_{trans} = 0.327m$$

Problem 2.78

[Difficulty: 2]

2.78 SAE 30 oil at 100°C flows through a 12-mm-diameter stainless-steel tube. What is the specific gravity and specific weight of the oil? If the oil discharged from the tube fills a 100-mL graduated cylinder in 9 seconds, is the flow laminar or turbulent?

Given: Type of oil, flow rate, and tube geometry

Find: Whether flow is laminar or turbulent

Solution:

Data on SAE 30 oil SG or density is limited in the Appendix. We can Google it or use the following $\nu = \frac{\mu}{\rho}$ so $\rho = \frac{\mu}{\nu}$

At 100°C, from Figs. A.2 and A.3 $\mu = 9 \times 10^{-3} \cdot \frac{N \cdot s}{m^2}$ $\nu = 1 \times 10^{-5} \cdot \frac{m^2}{s}$

$$\rho = 9 \times 10^{-3} \cdot \frac{N \cdot s}{m^2} \times \frac{1}{1 \times 10^{-5}} \cdot \frac{s}{m^2} \times \frac{kg \cdot m}{s^2 \cdot N} \qquad \rho = 900 \frac{kg}{m^3}$$

Hence $SG = \frac{\rho}{\rho_{water}}$ $\rho_{water} = 1000 \cdot \frac{kg}{m^3}$ $SG = 0.9$

The specific weight is $\gamma = \rho \cdot g$

$$\gamma = 900 \cdot \frac{kg}{m^3} \times 9.81 \cdot \frac{m}{s^2} \times \frac{N \cdot s^2}{kg \cdot m} \qquad \gamma = 8.829 \times 10^3 \cdot \frac{N}{m^3}$$

For pipe flow $Q = \frac{\pi \cdot D^2}{4} \cdot V$ so $V = \frac{4 \cdot Q}{\pi \cdot D^2}$

$$Q = 100 \cdot mL \times \frac{10^{-6} \cdot m^3}{1 \cdot mL} \times \frac{1}{9} \cdot \frac{1}{s} \qquad Q = 1.111 \times 10^{-5} \frac{m^3}{s}$$

Then

$$V = \frac{4}{\pi} \times 1.11 \times 10^{-5} \cdot \frac{m^3}{s} \times \left(\frac{1}{12} \cdot \frac{1}{mm} \times \frac{1000 \cdot mm}{1 \cdot m} \right)^2 \qquad V = 0.0981 \frac{m}{s}$$

Hence $Re = \frac{\rho \cdot V \cdot D}{\mu}$

$$Re = 900 \cdot \frac{kg}{m^3} \times 0.0981 \cdot \frac{m}{s} \times 0.012 \cdot m \times \frac{1}{9 \times 10^{-3}} \cdot \frac{m^2}{N \cdot s} \times \frac{N \cdot s^2}{kg \cdot m} \qquad Re = 118$$

Flow is laminar

Problem 2.79

[Difficulty: 2]

2.79 A seaplane is flying at 100 mph through air at 45°F. At what distance from the leading edge of the underside of the fuselage does the boundary layer transition to turbulence? How does this boundary layer transition change as the underside of the fuselage touches the water during landing? Assume the water temperature is also 45°F.

Given: Data on seaplane

Find: Transition point of boundary layer

Solution:

For boundary layer transition, $Re_{trans} = 500000$

Then $Re_{trans} = \dfrac{\rho \cdot V \cdot x_{trans}}{\mu} = \dfrac{V \cdot x_{trans}}{\nu}$ so $x_{trans} = \dfrac{\nu \cdot Re_{trans}}{V}$

At 45°F = 7.2°C $\nu = 0.8 \times 10^{-5} \cdot \dfrac{m^2}{s} \times \dfrac{10.8 \cdot \frac{ft^2}{s}}{1 \cdot \frac{m^2}{s}}$ $\nu = 8.64 \times 10^{-5} \cdot \dfrac{ft^2}{s}$

$$x_{trans} = 8.64 \times 10^{-5} \cdot \frac{ft^2}{s} \cdot 500000 \times \frac{1}{100 \cdot mph} \times \frac{60 \cdot mph}{88 \cdot \frac{ft}{s}} \qquad x_{trans} = 0.295 \cdot ft$$

As the seaplane touches down:

At 45°F = 7.2°C $\nu = 1.5 \times 10^{-5} \cdot \dfrac{m^2}{s} \times \dfrac{10.8 \cdot \frac{ft^2}{s}}{1 \cdot \frac{m^2}{s}}$ $\nu = 1.62 \times 10^{-4} \cdot \dfrac{ft^2}{s}$

$$x_{trans} = 1.62 \times 10^{-4} \cdot \frac{ft^2}{s} \cdot 500000 \times \frac{1}{100 \cdot mph} \times \frac{60 \cdot mph}{88 \cdot \frac{ft}{s}} \qquad x_{trans} = 0.552 \cdot ft$$

Problem 2.80 [Difficulty: 3]

2.80 An airliner is cruising at an altitude of 5.5 km with a speed of 700 km/hr. As the airliner increases its altitude, it adjusts its speed so that the Mach number remains constant. Provide a sketch of speed vs. altitude. What is the speed of the airliner at an altitude of 8 km?

Given: Data on airliner

Find: Sketch of speed versus altitude (M = const)

Solution:

Data on temperature versus height can be obtained from appropriate Table

At 5.5 km the temperature is approximately 252 K

The speed of sound is obtained from $c = \sqrt{k \cdot R \cdot T}$

where $k = 1.4$

$R = 286.9$ J/kg·K

$c = 318$ m/s

We also have

$V = 700$ km/hr

or $V = 194$ m/s

Hence $M = V/c$ or

$M = 0.611$

To compute V for constant M, we use $V = M \cdot c = 0.611 \cdot c$

At a height of 8 km: $V = 677$ km/hr

NOTE: Realistically, the aiplane will fly to a maximum height of about 10 km!

z (km)	T (K)	c (m/s)	V (km/hr)
4	262	325	713
5	259	322	709
5	256	320	704
6	249	316	695
7	243	312	686
8	236	308	677
9	230	304	668
10	223	299	658
11	217	295	649
12	217	295	649
13	217	295	649
14	217	295	649
15	217	295	649
16	217	295	649
17	217	295	649
18	217	295	649
19	217	295	649
20	217	295	649
22	219	296	651
24	221	298	654
26	223	299	657
28	225	300	660
30	227	302	663
40	250	317	697
50	271	330	725
60	256	321	705
70	220	297	653
80	181	269	592
90	181	269	592