Problem 6.1

[Difficulty: 2]

6.1 An incompressible frictionless flow field is given by $\vec{V} = (Ax + By)\hat{i} + (Bx - Ay)\hat{j}$, where $A = 2\ \mathrm{s}^{-1}$ and $B = 2\ \mathrm{s}^{-1}$, and the coordinates are measured in meters. Find the magnitude and direction of the acceleration of a fluid particle at point $(x, y) = (2, 2)$. Find the pressure gradient at the same point, if $\vec{g} = -g\hat{j}$ and the fluid is water.

Given: Velocity field

Find: Acceleration of particle and pressure gradient at (2,2)

Solution:

Basic equations

$$\vec{a}_p = \underset{\substack{\text{total}\\\text{acceleration}\\\text{of a particle}}}{\frac{D\vec{V}}{Dt}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\\text{acceleration}}} + \underset{\substack{\text{local}\\\text{acceleration}}}{\frac{\partial \vec{V}}{\partial t}} \qquad \rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

Given data

$$A = 1\cdot\frac{1}{s} \qquad B = 3\cdot\frac{1}{s} \qquad x = 2\cdot m \qquad y = 2\cdot m \qquad \rho = 999\cdot\frac{kg}{m^3}$$

For this flow

$$u(x,y) = A\cdot x + B\cdot y \qquad v(x,y) = B\cdot x - A\cdot y$$

$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u = (A\cdot x + B\cdot y)\cdot\frac{\partial}{\partial x}(A\cdot x + B\cdot y) + (B\cdot x - A\cdot y)\cdot\frac{\partial}{\partial y}(A\cdot x + B\cdot y) \qquad a_x = \left(A^2 + B^2\right)\cdot x$$

$$a_y = u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v = (A\cdot x + B\cdot y)\cdot\frac{\partial}{\partial x}(B\cdot x - A\cdot y) + (B\cdot x - A\cdot y)\cdot\frac{\partial}{\partial y}(B\cdot x - A\cdot y) \qquad a_y = \left(A^2 + B^2\right)\cdot y$$

Hence at (2,2)

$$a_x = (1 + 9)\frac{1}{s} \times 2\cdot m \qquad a_x = 20\frac{m}{s} \qquad a_y = (1 + 9)\frac{1}{s} \times 2\cdot m \qquad a_y = 20\frac{m}{s}$$

$$a = \sqrt{a_x^2 + a_y^2} \qquad \theta = \operatorname{atan}\left(\frac{a_y}{a_x}\right) \qquad a = 28.28\frac{m}{s} \qquad \theta = 45\cdot deg$$

For the pressure gradient

$$\frac{\partial}{\partial x}p = \rho\cdot g_x - \rho\cdot a_x = -999\cdot\frac{kg}{m^3} \times 20\cdot\frac{m}{s^2} \times \frac{N\cdot s^2}{kg\cdot m} \qquad \frac{\partial}{\partial x}p = -20000\frac{Pa}{m} = -20.0\cdot\frac{kPa}{m}$$

$$\frac{\partial}{\partial y}p = -\rho\cdot g_y - \rho\cdot a_y = 999\cdot\frac{kg}{m^3} \times (-9.81 - 20)\cdot\frac{m}{s^2} \times \frac{N\cdot s^2}{kg\cdot m} \qquad \frac{\partial}{\partial y}p = -29800\frac{Pa}{m} = -29.8\cdot\frac{kPa}{m}$$

Problem 6.2 [Difficulty: 2]

6.2 A velocity field in a fluid with density of 1000 kg/m^3 is given by $\vec{V} = (-Ax + By)t\hat{i} + (Ay + Bx)t\hat{j}$, where $A = 2\ \mathrm{s}^{-2}$ and $B = 1\,\mathrm{s}^{-2}$, x and y are in meters, and t is in seconds. Body forces are negligible. Evaluate ∇p at point $(x, y) = (1, 1)$ at $t = 1$ s.

Given: Velocity field

Find: Pressure gradient at (1,1) at 1 s

Solution:

Basic equations

$$\vec{a}_p = \underset{\substack{\text{total}\\\text{acceleration}\\\text{of a particle}}}{\frac{D\vec{V}}{Dt}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\\text{acceleration}}} + \underset{\substack{\text{local}\\\text{acceleration}}}{\frac{\partial \vec{V}}{\partial t}} \qquad \rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

Given data $\quad A = 2\cdot\frac{1}{s^2} \quad B = 1\cdot\frac{1}{s^2} \quad x = 1\cdot m \quad y = 1\cdot m \quad t = 1\cdot s \quad \rho = 1000\cdot\frac{kg}{m^3}$

$$u(x,y,t) = (-A\cdot x + B\cdot y)\cdot t \qquad v(x,y,t) = (A\cdot y + B\cdot x)\cdot t$$

The acceleration components and values are

$$a_{xt}(x,y,t) = \frac{\partial}{\partial t}u(x,y,t) = B\cdot y - A\cdot x \quad a_{xt}(x,y,t) = B\cdot y - A\cdot x \quad a_{xt}(x,y,t) = -1\frac{m}{s^2}$$

$$a_{xc}(x,y,t) = u(x,y,t)\cdot\frac{\partial}{\partial x}u(x,y,t) + v(x,y,t)\cdot\frac{\partial}{\partial y}u(x,y,t) \quad a_{xc}(x,y,t) = t^2\cdot x\cdot\left(A^2 + B^2\right) \quad a_{xc}(x,y,t) = 5\frac{m}{s^2}$$

$$a_{yt}(x,y,t) = \frac{\partial}{\partial t}v(x,y,t) \quad a_{yt}(x,y,t) = A\cdot y + B\cdot x \quad a_{yt}(x,y,t) = 3\frac{m}{s^2}$$

$$a_{yc}(x,y,t) = u(x,y,t)\cdot\frac{\partial}{\partial x}v(x,y,t) + v(x,y,t)\cdot\frac{\partial}{\partial y}v(x,y,t) \quad a_{yc}(x,y,t) = t^2\cdot y\cdot\left(A^2 + B^2\right) \quad a_{yc}(x,y,t) = 5\frac{m}{s^2}$$

$$a_x(x,y,t) = a_{xt}(x,y,t) + a_{xc}(x,y,t) \quad a_x(x,y,t) = x\cdot A^2\cdot t^2 - x\cdot A + x\cdot B^2\cdot t^2 + y\cdot B \quad a_x(x,y,t) = 4\frac{m}{s^2}$$

$$a_y(x,y,t) = a_{yt}(x,y,t) + a_{yc}(x,y,t) \quad a_y(x,y,t) = y\cdot A^2\cdot t^2 + y\cdot A + y\cdot B^2\cdot t^2 + x\cdot B \quad a_y(x,y,t) = 8\frac{m}{s^2}$$

Hence for the pressure gradient

$$\frac{\partial}{\partial x}p = -\rho\cdot a_x = -1000\cdot\frac{kg}{m^3}\times 4\cdot\frac{m}{s^2}\times\frac{N\cdot s^2}{kg\cdot m} \qquad \frac{\partial}{\partial x}p = -4000\cdot\frac{Pa}{m} = -4\cdot\frac{kPa}{m}$$

$$\frac{\partial}{\partial y}p = -\rho\cdot a_y = -1000\cdot\frac{kg}{m^3}\times 8\cdot\frac{m}{s^2}\times\frac{N\cdot s^2}{kg\cdot m} \qquad \frac{\partial}{\partial y}p = -8000\cdot\frac{Pa}{m} = -8\cdot\frac{kPa}{m}$$

Problem 6.3

[Difficulty: 2]

6.3 The x component of velocity in an incompressible flow field is given by $u = Ax$, where $A = 2\ s^{-1}$ and the coordinates are measured in meters. The pressure at point $(x, y) = (0, 0)$ is $p_0 = 190$ kPa (gage). The density is $\rho = 1.50\ kg/m^3$ and the z axis is vertical. Evaluate the simplest possible y component of velocity. Calculate the fluid acceleration and determine the pressure gradient at point $(x, y) = (2, 1)$. Find the pressure distribution along the positive x axis.

Given: Velocity field

Find: Simplest y component of velocity; Acceleration of particle and pressure gradient at (2,1); pressure on x axis

Solution:

Basic equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\text{total acceleration of a particle}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\text{convective acceleration}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{local acceleration}} \qquad \rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

For this flow $\quad u(x,y) = A\cdot x \qquad \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0 \quad$ so $\quad v(x,y) = -\int \frac{\partial}{\partial x}u\,dy = -\int A\,dy = -A\cdot y + c$

Hence $\quad v(x,y) = -A\cdot y \quad$ is the simplest y component of velocity

For acceleration

$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u = A\cdot x\cdot\frac{\partial}{\partial x}(A\cdot x) + (-A\cdot y)\cdot\frac{\partial}{\partial y}(A\cdot x) = A^2\cdot x \qquad a_x = A^2\cdot x$$

$$a_y = u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v = A\cdot x\cdot\frac{\partial}{\partial x}(-A\cdot y) + (-A\cdot y)\cdot\frac{\partial}{\partial y}(-A\cdot y) \qquad a_y = A^2\cdot y$$

Hence at (2,1)

$$a_x = \left(\frac{2}{s}\right)^2 \times 2\cdot m \qquad a_y = \left(\frac{2}{s}\right)^2 \times 1\cdot m \qquad a_x = 8\frac{m}{s^2} \qquad a_y = 4\frac{m}{s^2}$$

$$a = \sqrt{a_x^2 + a_y^2} \qquad \theta = \operatorname{atan}\left(\frac{a_y}{a_x}\right) \qquad a = 8.94\frac{m}{s^2} \qquad \theta = 26.6\cdot deg$$

For the pressure gradient

$$\frac{\partial}{\partial x}p = \rho\cdot g_x - \rho\cdot a_x = -1.50\cdot\frac{kg}{m^3} \times 8\cdot\frac{m}{s^2} \times \frac{N\cdot s^2}{kg\cdot m} \qquad \frac{\partial}{\partial x}p = -12\cdot\frac{Pa}{m}$$

$$\frac{\partial}{\partial y}p = \rho\cdot g_y - \rho\cdot a_y = -1.50\cdot\frac{kg}{m^3} \times 4\cdot\frac{m}{s^2} \times \frac{N\cdot s^2}{kg\cdot m} \qquad \frac{\partial}{\partial y}p = -6\cdot\frac{Pa}{m}$$

$$\frac{\partial}{\partial z}p = \rho\cdot g_z - \rho\cdot a_z = 1.50 \times \frac{kg}{m^3} \times (-9.81)\cdot\frac{m}{s^2} \times \frac{N\cdot s^2}{kg\cdot m} \qquad \frac{\partial}{\partial y}p = -14.7\cdot\frac{Pa}{m}$$

For the pressure on the x axis $\quad dp = \frac{\partial}{\partial x}p \qquad p - p_0 = \int_0^x (\rho\cdot g_x - \rho\cdot a_x)\,dx = \int_0^x (-\rho\cdot A^2\cdot x)\,dx = -\frac{1}{2}\cdot\rho\cdot A^2\cdot x^2$

$$p(x) = p_0 - \frac{1}{2}\cdot\rho\cdot A^2\cdot x^2 \qquad p(x) = 190\cdot kPa - \frac{1}{2}\cdot 1.5\cdot\frac{kg}{m^3} \times \left(\frac{2}{s}\right)^2 \times \frac{N\cdot s^2}{kg\cdot m} \times x^2 \qquad p(x) = 190 - \frac{3}{1000}\cdot x^2$$

(p in kPa, x in m)

Problem 6.4

(Difficulty 2)

6.4 Consider the flow field with the velocity given by $\vec{V} = 3\hat{\imath} + 5t\hat{\jmath} + 8t^2\hat{k}$, where the velocity is in $\frac{m}{s}$ and t is in seconds. The fluid density is 800 $\frac{kg}{m^3}$ and gravity acts in the negative z direction. Determine the velocity, acceleration, and pressure gradient of the fluid at one second time increments from time= 0,1 to time= 5 seconds.

Find: The velocity, acceleration and pressure gradient at different time.

Assumption: Flow is frictionless and incompressible

Solution: Use Euler's equation to find the pressure gradient

$$\rho \frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

For the velocity field we have the expression as:

$$\vec{V} = 3\hat{\imath} + 5t\hat{\jmath} + 8t^2\hat{k}$$

The velocity is found at the different times. For $t = 1\ s$:

$$\vec{V} = 3\hat{\imath} + 5\hat{\jmath} + 8\hat{k}\ \frac{m}{s}$$

At $t = 2\ s$:

$$\vec{V} = 3\hat{\imath} + 10\hat{\jmath} + 32\hat{k}\ \frac{m}{s}$$

At $t = 3\ s$:

$$\vec{V} = 3\hat{\imath} + 15\hat{\jmath} + 72\hat{k}\ \frac{m}{s}$$

At $t = 4\ s$:

$$\vec{V} = 3\hat{\imath} + 20\hat{\jmath} + 128\hat{k}$$

At $t = 5\ s$:

$$\vec{V} = 3\hat{\imath} + 25\hat{\jmath} + 200\hat{k}$$

For the acceleration, we have the following definition:

$$\vec{a}_p = \frac{D\vec{V}}{Dt}$$

Where the acceleration scalar values are then

$$a_{x_p} = \frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

$$a_{y_p} = \frac{Dv}{Dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_{z_p} = \frac{Dw}{Dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

Thus

$$a_{x_p} = 0\ \frac{m}{s^2}$$

$$a_{y_p} = 5\ \frac{m}{s^2}$$

$$a_{z_p} = 16t\ \frac{m}{s^2}$$

$$\vec{a}_p = 5\hat{\jmath} + 16t\hat{k}$$

At $t = 1\ s$:

$$\vec{a}_p = 5\hat{\jmath} + 16\ \hat{k}\ \frac{m}{s^2}$$

At $t = 2\ s$:

$$\vec{a}_p = 5\hat{\jmath} + 32\ \hat{k}\ \frac{m}{s^2}$$

At $t = 3\ s$:

$$\vec{a}_p = 5\hat{\jmath} + 48\ \hat{k}\ \frac{m}{s^2}$$

At $t = 4\ s$:

$$\vec{a}_p = 5\hat{\jmath} + 64\ \hat{k}\ \frac{m}{s^2}$$

At $t = 5\ s$:

$$\vec{a}_p = 5\hat{\jmath} + 80\ \hat{k}\ \frac{m}{s^2}$$

For the frictionless flow we have Euler's equation:

$$\rho \frac{D\vec{V}}{Dt} = \rho \vec{g} - \nabla p$$

Thus pressure gradient is

$$\nabla p = \rho \vec{g} - \rho \frac{D\vec{V}}{Dt}$$

The density is

$$\rho = 800 \ \frac{kg}{m^3}$$

As the gravity acts in the negative z direction, we have:

$$\nabla p = -\rho g \hat{k} - \rho\left(5\hat{j} + 16t\hat{k}\right)$$

$$\nabla p = -5\rho\hat{j} - \rho(g + 16t)\hat{k}$$

$$g = 9.81 \ \frac{m}{s^2}$$

At $t = 1\ s$:

$$\nabla p = -4000\hat{j} - 20648\hat{k} \ \frac{Pa}{m}$$

At $t = 2\ s$:

$$\nabla p = -4000\hat{j} - 33448\hat{k} \ \frac{Pa}{m}$$

At $t = 3\ s$:

$$\nabla p = -4000\hat{j} - 46248\hat{k} \ \frac{Pa}{m}$$

At $t = 4\ s$:

$$\nabla p = -4000\hat{j} - 59048\hat{k} \ \frac{Pa}{m}$$

At $t = 5\ s$:

$$\nabla p = -4000\hat{j} - 71848\hat{k} \ \frac{Pa}{m}$$

Problem 6.5

(Difficulty 2)

6.5 Consider the flow field with the velocity given by $\vec{V} = 4y\hat{\imath} + 3x\hat{\jmath}$, where the velocity is in $\frac{ft}{s}$ and the coordinate are in feet. The fluid density is $\rho = 1.5\ \frac{slug}{ft^3}$ and gravity acts in the negative y direction. Determine the general expressions for the acceleration and pressure gradient. Plot the acceleration and pressure gradient in the y direction for x = 0 and x = 2 ft.

Find: The acceleration and pressure gradient.

Assumption: Flow is frictionless and incompressible

Solution: Use Euler's equation to find the pressure gradient

$$\rho \frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

For the velocity field we have the expression as:

$$\vec{V} = 4y\hat{\imath} + 3x\hat{\jmath}$$

For the acceleration, we have the following definition of acceleration:

$$\vec{a}_p = \frac{D\vec{V}}{Dt}$$

Or in terms of the scalar accelerations

$$a_{x_p} = \frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

$$a_{y_p} = \frac{Dv}{Dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_{z_p} = \frac{Dw}{Dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

Thus

$$a_{x_p} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = 4y \times 0 + 3x \times 4 = 12x\ \frac{ft}{s^2}$$

$$a_{y_p} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = 4y \times 3 + 3x \times 0 = 12y\ \frac{ft}{s^2}$$

$$a_{z_p} = 0\ \frac{ft}{s^2}$$

So the general expression for acceleration is:

$$\vec{a}_p = 12x\hat{\imath} + 12y\hat{\jmath}$$

For the pressure gradient we use Euler's equation:

$$\rho \frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

Thus the pressure gradient is

$$\nabla p = \rho\vec{g} - \rho \frac{D\vec{V}}{Dt}$$

Where the density is

$$\rho = 1.5\ \frac{slug}{ft^3} = 1.5\ \frac{lbf \cdot s^2}{ft^4}$$

As the gravity acts in the negative z direction, we have:

$$\nabla p = -\rho g\hat{\jmath} - \rho(12x\hat{\imath} + 12y\hat{\jmath})$$

Thus the general expression for pressure gradient is:

$$\nabla p = -12\rho x\hat{\imath} - \rho(g + 12y)\hat{\jmath}$$

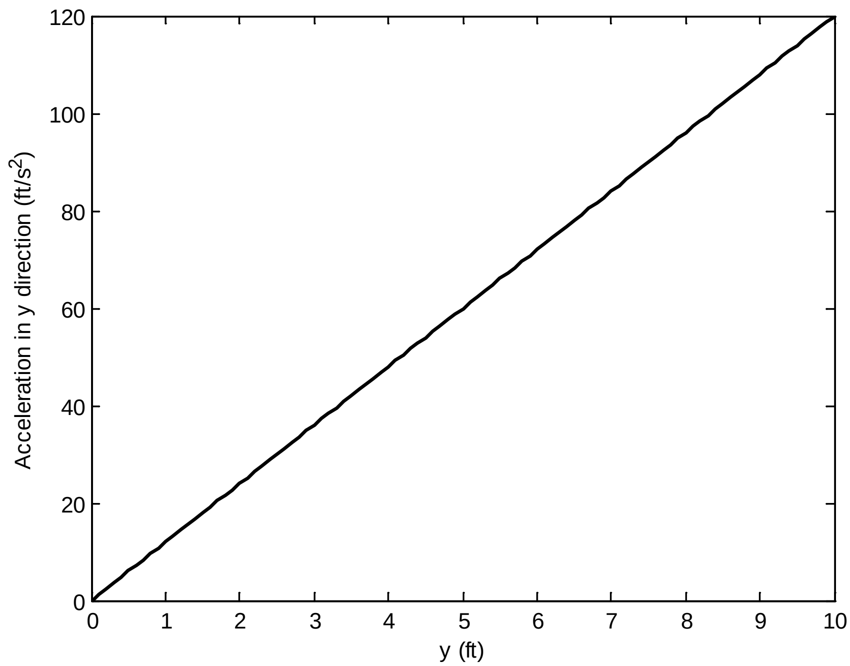
For the acceleration and pressure gradient in the y direction we have:

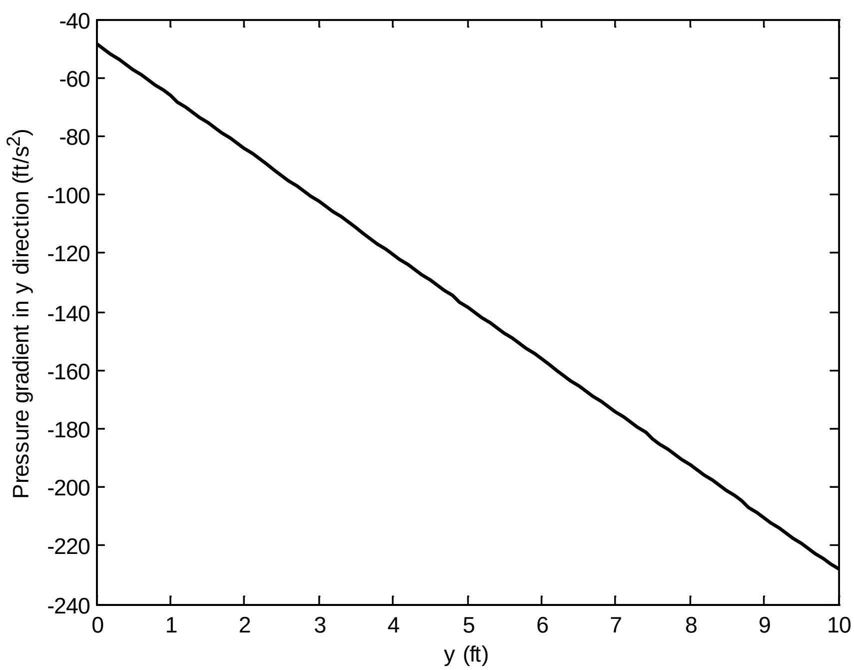
$$a_{y_p} = 12y\ \frac{ft}{s^2}$$

$$\frac{\partial p}{\partial y} = -\rho(g + 12y) = -1.5\ \frac{lbf \cdot s^2}{ft^4} \times \left(32.2\ \frac{ft}{s^2} + 12y\ \frac{ft}{s^2}\right)$$

$$\frac{\partial p}{\partial y} = -(48.3 + 18y)\frac{lbf}{ft^3}$$

The plots are shown in the figures.


Acceleration in y direction (ft/s^2)
y (ft)
0
20
40
60
80
100
120
0
1
2
3
4
5
6
7
8
9
10


Pressure gradient in y direction (ft/s^2)
y (ft)
-40
-60
-80
-100
-120
-140
-160
-180
-200
-220
-240
0
1
2
3
4
5
6
7
8
9
10

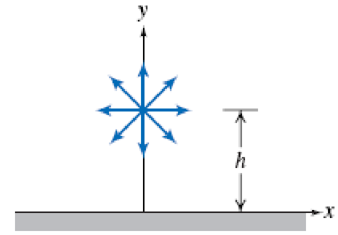
Problem 6.6

[Difficulty: 3]

6.6 The velocity field for a plane source located distance $h = 1$ m above an infinite wall aligned along the x axis is given by

$$\vec{V} = \frac{q}{2\pi[x^2 + (y-h)^2]}[x\hat{i} + (y-h)\hat{j}] + \frac{q}{2\pi[x^2 + (y+h)^2]}[x\hat{i} + (y+h)\hat{j}]$$

where $q = 2\ \mathrm{m^3/s/m}$. The fluid density is 1000 kg/m^3 and body forces are negligible. Derive expressions for the velocity and acceleration of a fluid particle that moves along the wall, and plot from $x = 0$ to $x = +10h$. Verify that the velocity and acceleration normal to the wall are zero. Plot the pressure gradient $\partial p/\partial x$ along the wall. Is the pressure gradient along the wall adverse (does it oppose fluid motion) or not?



Given: Velocity field

Find: Expressions for velocity and acceleration along wall; plot; verify vertical components are zero; plot pressure gradient

Solution:

The given data is $\quad q = 2\cdot\frac{\frac{m^3}{s}}{m} \qquad h = 1\cdot m \qquad \rho = 1000\cdot\frac{kg}{m^3}$

$$u = \frac{q\cdot x}{2\cdot\pi\left[x^2 + (y-h)^2\right]} + \frac{q\cdot x}{2\cdot\pi\left[x^2 + (y+h)^2\right]} \qquad v = \frac{q\cdot(y-h)}{2\cdot\pi\left[x^2 + (y-h)^2\right]} + \frac{q\cdot(y+h)}{2\cdot\pi\left[x^2 + (y+h)^2\right]}$$

The governing equation for acceleration is

$$\underset{\substack{\text{total}\\\text{acceleration}\\\text{of a particle}}}{\vec{a}_p = \frac{D\vec{V}}{Dt}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\\text{acceleration}}} + \underset{\substack{\text{local}\\\text{acceleration}}}{\frac{\partial \vec{V}}{\partial t}}$$

For steady, 2D flow this reduces to (after considerable math!)

x - component
$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u = -\frac{q^2\cdot x\cdot\left[\left(x^2+y^2\right)^2 - h^2\cdot\left(h^2 - 4\cdot y^2\right)\right]}{\left[x^2 + (y+h)^2\right]^2\cdot\left[x^2 + (y-h)^2\right]^2\cdot\pi^2}$$

y - component
$$a_y = u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v = -\frac{q^2\cdot y\cdot\left[\left(x^2+y^2\right)^2 - h^2\cdot\left(h^2 + 4\cdot x^2\right)\right]}{\pi^2\cdot\left[x^2 + (y+h)^2\right]^2\cdot\left[x^2 + (y-h)^2\right]^2}$$

For motion along the wall $\qquad y = 0\cdot m$

$$u = \frac{q\cdot x}{\pi\cdot\left(x^2+h^2\right)} \qquad v = 0 \quad \text{(No normal velocity)} \qquad a_x = -\frac{q^2\cdot x\cdot\left(x^2-h^2\right)}{\pi^2\cdot\left(x^2+h^2\right)^3} \qquad a_y = 0 \quad \text{(No normal acceleration)}$$

The governing equation (assuming inviscid flow) for computing the pressure gradient is

$$\rho \frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

Hence, the component of pressure gradient (neglecting gravity) along the wall is

$$\frac{\partial}{\partial x}p = -\rho\cdot\frac{Du}{Dt} \qquad \frac{\partial}{\partial x}p = \frac{\rho\cdot q^2\cdot x\cdot\left(x^2 - h^2\right)}{\pi^2\cdot\left(x^2 + h^2\right)^3}$$

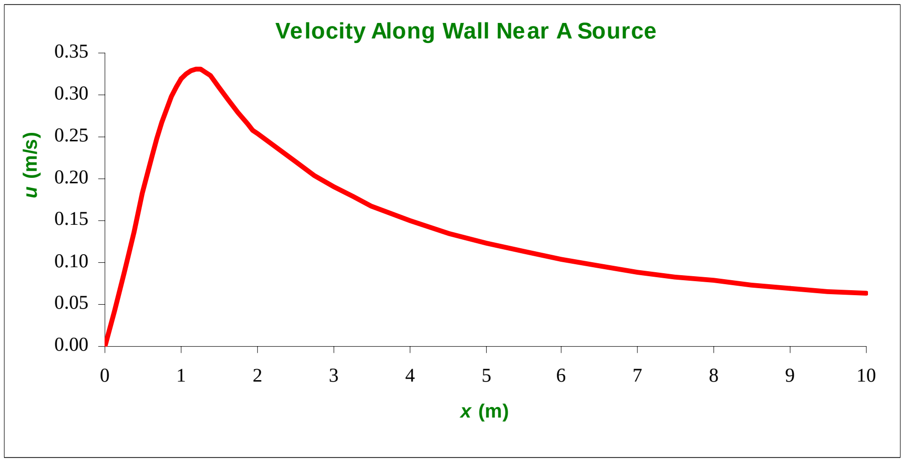
The plots of velocity, acceleration, and pressure gradient are shown below, done in *Excel*. From the plots it is clear that the fluid experiences an adverse pressure gradient from the origin to x = 1 m, then a negative one promoting fluid acceleration. If flow separates, it will likely be in the region x = 0 to x = h.

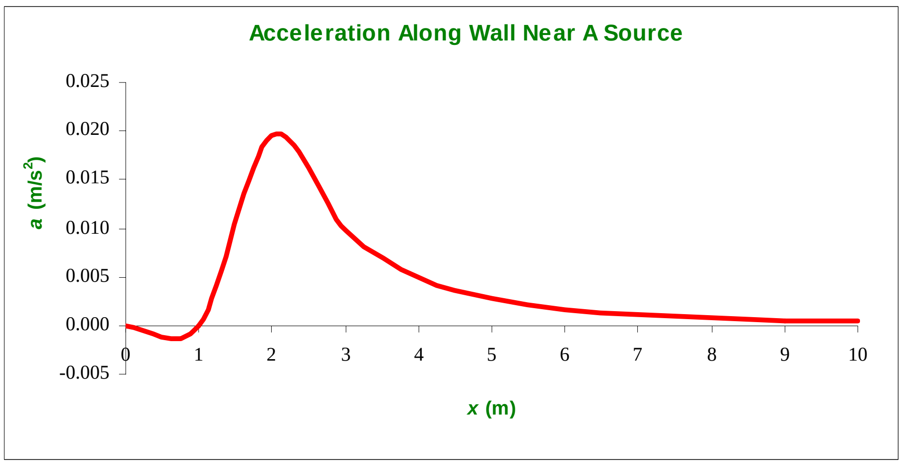
q = 2 $m^3/s/m$

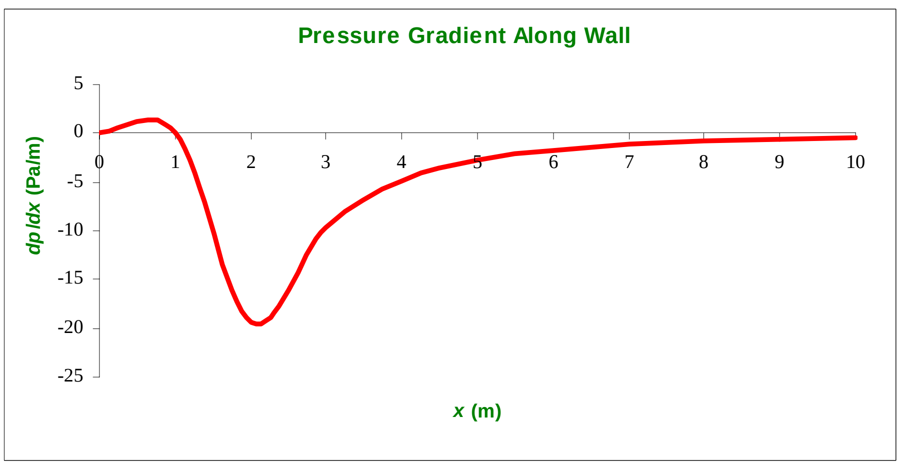
h = 1 m

∠= 1000 kg/m^3

x (m)	u (m/s)	a (m/s^2)	dp/dx (Pa/m)
0.0	0.00	0.00000	0.00
1.0	0.32	0.00000	0.00
2.0	0.25	0.01945	-19.45
3.0	0.19	0.00973	-9.73
4.0	0.15	0.00495	-4.95
5.0	0.12	0.00277	-2.77
6.0	0.10	0.00168	-1.68
7.0	0.09	0.00109	-1.09
8.0	0.08	0.00074	-0.74
9.0	0.07	0.00053	-0.53
10.0	0.06	0.00039	-0.39







Problem 6.7

[Difficulty: 2]

6.7 In a two-dimensional frictionless, incompressible ($\rho = 1500$ kg/m^3) flow, the velocity field in meters per second is given by $\vec{V} = (Ax + By)\hat{i} + (Bx - Ay)\hat{j}$; the coordinates are measured in meters, and $A = 4\ \mathrm{s}^{-1}$ and $B = 2\ \mathrm{s}^{-1}$. The pressure is $p_0 = 200$ kPa at point $(x, y) = (0, 0)$. Obtain an expression for the pressure field, $p(x, y)$ in terms of p_0, A, and B, and evaluate at point $(x, y) = (2, 2)$.

Given: Velocity field

Find: Expression for pressure field; evaluate at (2,2)

Solution:

Basic equations

$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\substack{\text{total}\\\text{acceleration}\\\text{of a particle}}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\\text{acceleration}}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\substack{\text{local}\\\text{acceleration}}} \qquad \rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

Given data $\quad A = 4\cdot\frac{1}{\mathrm{s}} \quad B = 2\cdot\frac{1}{\mathrm{s}} \quad x = 2\cdot\mathrm{m} \quad y = 2\cdot\mathrm{m} \quad \rho = 1500\cdot\frac{\mathrm{kg}}{\mathrm{m}^3} \quad p_0 = 200\cdot\mathrm{kPa}$

For this flow $\quad u(x,y) = A\cdot x + B\cdot y \qquad v(x,y) = B\cdot x - A\cdot y$

Note that $\quad \frac{\partial}{\partial x}u(x,y) + \frac{\partial}{\partial y}v(x,y) = 0 \qquad \frac{\partial}{\partial x}v(x,y) - \frac{\partial}{\partial y}u(x,y) = 0$

Then $\quad a_x(x,y) = u(x,y)\cdot\frac{\partial}{\partial x}u(x,y) + v(x,y)\cdot\frac{\partial}{\partial y}u(x,y) \qquad a_x(x,y) = x\cdot\left(A^2 + B^2\right) \qquad a_x(x,y) = 40\,\frac{\mathrm{m}}{\mathrm{s}^2}$

$a_y(x,y) = u(x,y)\cdot\frac{\partial}{\partial x}v(x,y) + v(x,y)\cdot\frac{\partial}{\partial y}v(x,y) \qquad a_y(x,y) = y\cdot\left(A^2 + B^2\right) \qquad a_y(x,y) = 40\,\frac{\mathrm{m}}{\mathrm{s}^2}$

The momentum equation becomes $\quad \frac{\partial}{\partial x}p = -\rho\cdot a_x \qquad \frac{\partial}{\partial y}p = -\rho\cdot a_y \quad$ and $\quad p = dx\cdot\frac{\partial}{\partial x}p + dy\cdot\frac{\partial}{\partial y}p$

Integrating

$$p(x,y) = p_0 - \rho\cdot\int_0^x a_x(x,y)\,dx - \rho\cdot\int_0^y a_y(x,y)\,dy$$

$$p(x,y) = p_0 - \frac{\rho\cdot\left(A^2 + B^2\right)\cdot y^2}{2} - \frac{\rho\cdot\left(A^2 + B^2\right)\cdot x^2}{2}$$

$$p(x,y) = 80\cdot\mathrm{kPa}$$

Problem 6.8

(Difficulty 2)

6.8 Consider a two-dimensional incompressible flow flowing downward against a plate. The velocity is given by $\vec{V} = Ax\hat{\imath} - Ay\hat{\jmath}$, where $A = 2\ s^{-1}$ and x and y are in meters. Determine general expressions for the acceleration and pressure gradient in the $x-$ and $y-$directions. Plot the pressure gradient along the plate ($y = 0$) from $x = 0$ to $x = 3$ m, and the pressure gradient along the centerline ($x = 0$) from $y = 0$ to $y = 3\ m$.

Find: The acceleration and pressure gradient.

Assumption: Flow is frictionless and incompressible

Solution: Use Euler's equation to find the pressure gradient

$$\rho \frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

For the velocity field we have the expression as:

$$\vec{V} = Ax\hat{\imath} - Ay\hat{\jmath}$$

$$A = 2\ s^{-1}$$

$$\vec{V} = 2x\hat{\imath} - 2y\hat{\jmath}$$

For the acceleration, we have the following definition of acceleration:

$$\vec{a}_p = \frac{D\vec{V}}{Dt}$$

Or in terms of the scalar accelerations

$$a_{x_p} = \frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

$$a_{y_p} = \frac{Dv}{Dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_{z_p} = \frac{Dw}{Dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

Thus

$$a_{x_p} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = 2x \times 2 + (-2y) \times 0 = 4x\ \frac{m}{s^2}$$

$$a_{y_p} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = 2x \times 0 + (-2y) \times (-2) = 4y\ \frac{m}{s^2}$$

$$a_{z_p} = 0\ \frac{m}{s^2}$$

So the general expression for acceleration is:

$$\vec{a}_p = 4x\hat{\imath} + 4y\hat{\jmath}$$

For the pressure gradient we use Euler's equation:

$$\rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

Thus the pressure gradient is

$$\nabla p = \rho\vec{g} - \rho\frac{D\vec{V}}{Dt}$$

$$\rho = 998\ \frac{kg}{m^3}$$

As the gravity acts in the negative z direction, we have:

$$\nabla p = -\rho g\hat{\jmath} - \rho(4x\hat{\imath} + 4y\hat{\jmath})$$

Thus the general expression for pressure gradient is:

$$\nabla p = -4\rho x\hat{\imath} - \rho(g + 4y)\hat{\jmath}$$

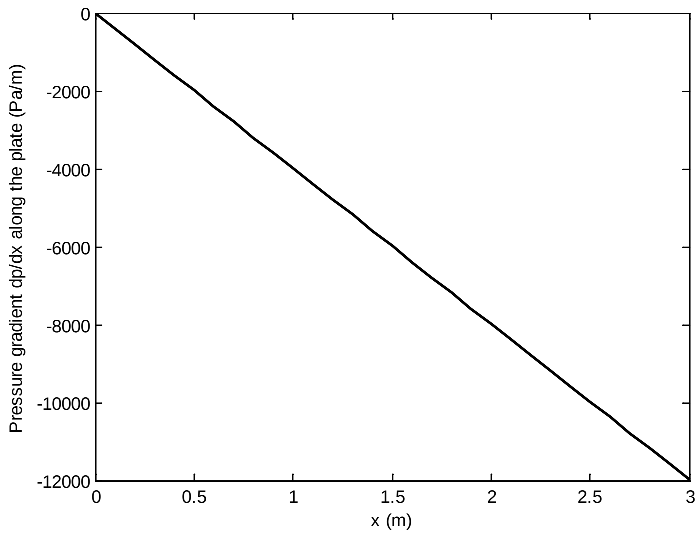
$$\nabla p = -4\rho x\hat{\imath} - \rho(g + 4y)\hat{\jmath}$$

$$\nabla p = -3992x\hat{\imath} - (9810 + 3992y)\hat{\jmath}\ \frac{Pa}{m}$$

The pressure gradient along the plate $(y = 0)$ is:

$$\frac{\partial p}{\partial x} = -3992x\ \frac{Pa}{m}$$

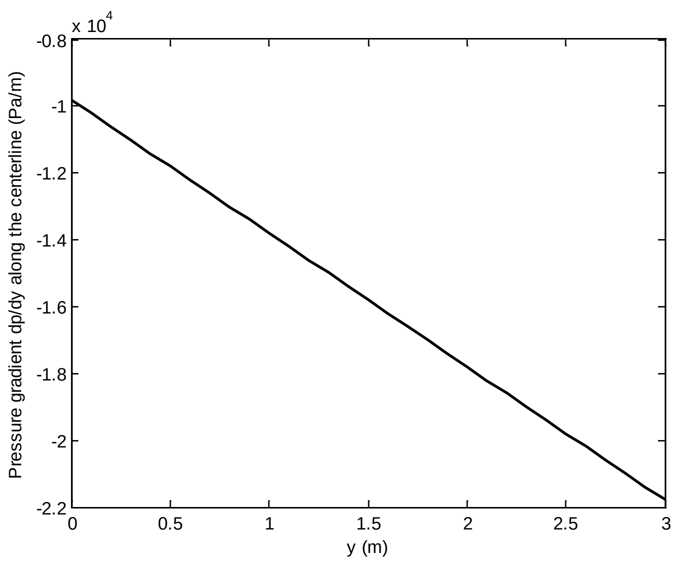
The plot of pressure gradient along the plate from $x = 0$ to $x = 3$ m is shown as:



The pressure gradient along the centerline $(x = 0)$ is:

$$\frac{\partial p}{\partial y} = -(9810 + 3992y)\ \frac{Pa}{m}$$

The plot of pressure gradient along the centerline from $y = 0$ to $y = 3$ m is shown as:



Problem 6.9

[Difficulty: 2]

6.9 An incompressible liquid with a density of 900 kg/m^3 and negligible viscosity flows steadily through a horizontal pipe of constant diameter. In a porous section of length $L = 2$ m, liquid is removed at a variable rate along the length so that the uniform axial velocity in the pipe is $u(x) = Ue^{-x/L}$, where $U = 20$ m/s. Develop expressions for and plot the acceleration of a fluid particle along the centerline of the porous section and the pressure gradient along the centerline. Evaluate the outlet pressure if the pressure at the inlet to the porous section is 50 kPa (gage).

Given: Velocity field

Find: Expression for acceleration and pressure gradient; plot; evaluate pressure at outlet

Solution:

Basic equations

$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\text{total acceleration of a particle}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\text{convective acceleration}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{local acceleration}}$$

$$\rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

Given data $\quad U = 20\cdot\frac{m}{s} \quad L = 2\cdot m \quad p_{in} = 50\cdot kPa \quad \rho = 900\cdot\frac{kg}{m^3}$

Here $\quad u(x) = U\cdot e^{-\frac{x}{L}} \quad u(0) = 20\frac{m}{s} \quad u(L) = 7.36\frac{m}{s}$

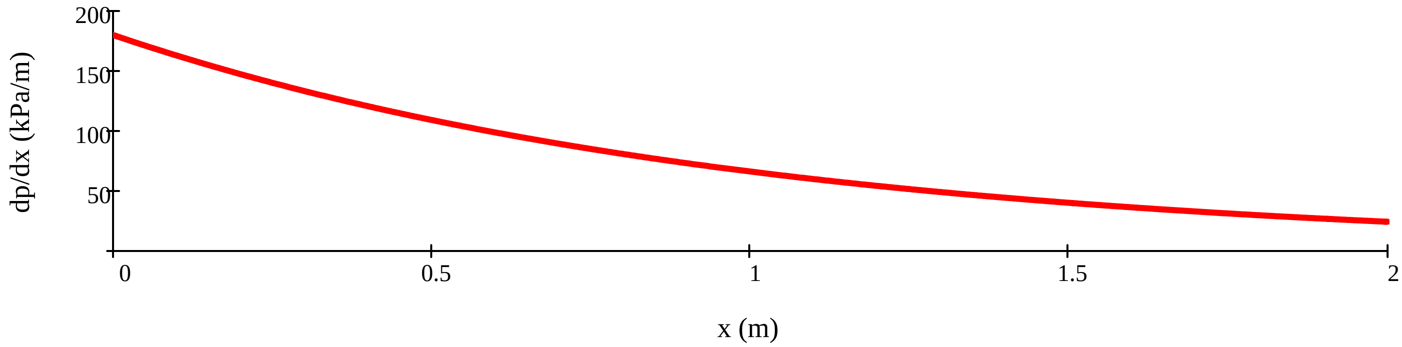
The x component of acceleration is then $\quad a_x(x) = u(x)\cdot\frac{\partial}{\partial x}u(x) \quad a_x(x) = -\frac{U^2\cdot e^{-\frac{2\cdot x}{L}}}{L}$

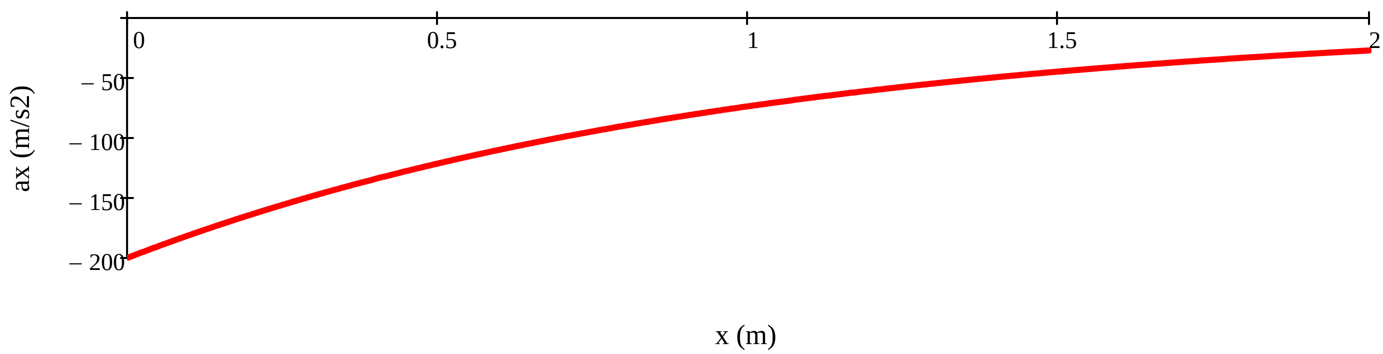
The x momentum becomes $\quad \rho\cdot u\cdot\frac{d}{dx}u = \rho\cdot a_a = -\frac{d}{dx}p$

The pressure gradient is then $\quad \frac{dp}{dx} = \frac{\rho}{L}\cdot U^2\cdot e^{-\frac{2\cdot x}{L}}$

Integrating momentum $\quad p(x) = p_{in} - \rho\cdot\int_0^x a_x(x)\,dx \quad p(x) = p_{in} - \frac{U^2\cdot\rho\cdot\left(e^{-\frac{2\cdot x}{L}} - 1\right)}{2}$

Hence $\quad p(L) = p_{in} - \frac{U^2\cdot\rho\cdot\left(e^{-2} - 1\right)}{2} \quad p(L) = 206\cdot kPa$


dp/dx (kPa/m)
200
150
100
50
0
0.5
1
1.5
2
x (m)


ax (m/s2)
0
0.5
1
1.5
2
− 50
− 100
− 150
− 200
x (m)

Problem 6.10

(Difficulty 2)

6.10 Consider a flow of water in pipe. What is the pressure gradient required to accelerate the water at $20\ \frac{ft}{s^2}$ if the pipe is (a) horizontal, (b) vertical with the water flowing upward, and (c) vertical with the water flowing downward. Explain why the pressure gradient depends on orientation and why the pressure gradient differs in sign between case (b) and (c).

Assumption: Frictionless, incompressible, and unidirectional flow in the pipe.

Solution: Use Euler's equation to find the pressure gradient:

$$\rho \frac{D\vec{V}}{Dt} = \rho \vec{g} - \nabla p$$

The pressure gradient is then

$$\nabla p = \rho \vec{g} - \rho \frac{D\vec{V}}{Dt}$$

Or in terms of acceleration

$$\nabla p = \rho \vec{g} - \rho \vec{a}_p$$

The density is:

$$\rho = 1.9\ \frac{slug}{ft^3} = 1.9\ \frac{lbf \cdot s^2}{ft^4}$$

(a) For the horizontal pipe there is no effect of gravity, and Euler's equation becomes

$$\nabla p = -\rho \vec{a}_p$$

The acceleration is

$$\vec{a}_p = 20\hat{\imath}\ \frac{ft}{s^2}$$

The pressure gradient we need is:

$$\frac{\partial p}{\partial x} = -\rho \vec{a}_p = -1.9\ \frac{lbf \cdot s^2}{ft^4} \times 20\ \frac{ft}{s^2} = -38\ \frac{lbf}{ft^3}$$

(b) For the vertical pipe with water flowing upward, the upward direction is positive y coordinate and gravity acts downward. Euler's equation is

$$\nabla p = \rho \vec{g} - \rho \vec{a}_p$$

Where the acceleration is

$$\vec{a}_p = 20\hat{j}\ \frac{ft}{s^2}$$

The pressure gradient is then

$$\frac{\partial p}{\partial y} = \rho g - \rho \vec{a}_p = 1.9\ \frac{lbf \cdot s^2}{ft^4} \times \left(-32.2 \frac{ft}{s^2}\right) - 1.9\ \frac{lbf \cdot s^2}{ft^4} \times 20\ \frac{ft}{s^2} = -99.2\ \ \frac{lbf}{ft^3}$$

(c) For the vertical pipe with water flowing downward:

$$\vec{a}_p = -20\hat{j}\ \frac{ft}{s^2}$$

$$\frac{\partial p}{\partial y} = \rho g - \rho \vec{a}_p = 1.9\ \frac{lbf \cdot s^2}{ft^4} \times \left(-32.2 \frac{ft}{s^2}\right) - 1.9\ \frac{lbf \cdot s^2}{ft^4} \times \left(-20\ \frac{ft}{s^2}\right) = -23\ \ \frac{lbf}{ft^3}$$

The pressure gradient is the driven force in this fluid flow, when we have different acceleration with different orientation, the pressure gradient will be different.

For the case (b) and case (c), the fluid velocity is opposite, the acceleration is opposite, then the pressure gradient will be different.

Problem 6.11 [Difficulty: 3]

6.11 The velocity field for a plane vortex sink is given by $\vec{V} = (-q/2\pi r)\hat{e}_r + (K/2\pi r)\hat{e}_\theta$, where $q = 2$ m³/s/m and $K = 1$ m³/s/m. The fluid density is 1000 kg/m³. Find the acceleration at (1, 0), (1, π/2), and (2, 0). Evaluate ∇p under the same conditions.

Given: Velocity field

Find: The acceleration at several points; evaluate pressure gradient

Solution:

The given data is $q = 2\cdot\dfrac{\frac{m^3}{s}}{m}$ $\quad K = 1\cdot\dfrac{\frac{m^3}{s}}{m}$ $\quad \rho = 1000\cdot\dfrac{kg}{m^3}$ $\quad V_r = -\dfrac{q}{2\cdot\pi\cdot r}$ $\quad V_\theta = \dfrac{K}{2\cdot\pi\cdot r}$

The governing equations for this 2D flow are

$$\rho a_r = \rho\left(\frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z\frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r}\right) = \rho g_r - \frac{\partial p}{\partial r} \qquad (6.3a)$$

$$\rho a_\theta = \rho\left(\frac{\partial V_\theta}{\partial t} + V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_z\frac{\partial V_\theta}{\partial z} + \frac{V_r V_\theta}{r}\right) = \rho g_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} \qquad (6.3b)$$

The total acceleration for this steady flow is then

r - component $\quad a_r = V_r\cdot\dfrac{\partial}{\partial r}V_r + \dfrac{V_\theta}{r}\cdot\dfrac{\partial}{\partial \theta}V_r - \dfrac{V_\theta^2}{r}$ $\quad a_r = -\dfrac{q^2 + K^2}{4\cdot\pi^2\cdot r^3}$

θ - component $\quad a_\theta = V_r\cdot\dfrac{\partial}{\partial r}V_\theta + \dfrac{V_\theta}{r}\cdot\dfrac{\partial}{\partial \theta}V_\theta + \dfrac{V_r\cdot V_\theta}{r}$ $\quad a_\theta = 0$

Evaluating at point (1,0) $\quad a_r = -0.127\,\dfrac{m}{s^2}$ $\quad a_\theta = 0$

Evaluating at point (1,π/2) $\quad a_r = -0.127\,\dfrac{m}{s^2}$ $\quad a_\theta = 0$

Evaluating at point (2,0) $\quad a_r = -0.0158\,\dfrac{m}{s^2}$ $\quad a_\theta = 0$

From Eq. 6.3, pressure gradient is $\quad \dfrac{\partial}{\partial r}p = -\rho\cdot a_r$ $\quad \dfrac{\partial}{\partial r}p = \dfrac{\rho\cdot\left(q^2 + K^2\right)}{4\cdot\pi^2\cdot r^3}$

$\dfrac{1}{r}\cdot\dfrac{\partial}{\partial \theta}p = -\rho\cdot a_\theta$ $\quad \dfrac{1}{r}\cdot\dfrac{\partial}{\partial \theta}p = 0$

Evaluating at point (1,0) $\quad \dfrac{\partial}{\partial r}p = 127\cdot\dfrac{Pa}{m}$ $\quad \dfrac{1}{r}\cdot\dfrac{\partial}{\partial \theta}p = 0$

Evaluating at point (1,π/2) $\quad \dfrac{\partial}{\partial r}p = 127\cdot\dfrac{Pa}{m}$ $\quad \dfrac{1}{r}\cdot\dfrac{\partial}{\partial \theta}p = 0$

Evaluating at point (2,0) $\quad \dfrac{\partial}{\partial r}p = 15.8\cdot\dfrac{Pa}{m}$ $\quad \dfrac{1}{r}\cdot\dfrac{\partial}{\partial \theta}p = 0$

Problem 6.12

[Difficulty: 3]

6.12 An incompressible liquid with negligible viscosity and density $\rho = 1.75$ slug/ft^3 flows steadily through a horizontal pipe. The pipe cross-section area linearly varies from 15 in^2 to 2.5 in^2 over a length of 10 feet. Develop an expression for and plot the pressure gradient and pressure versus position along the pipe, if the inlet centerline velocity is 5 ft/s and inlet pressure is 35 psi. What is the exit pressure? *Hint:* Use relation

$$u\frac{\partial u}{\partial x} = \frac{1}{2}\frac{\partial}{\partial x}(u^2)$$

Given: Flow in a pipe with variable area

Find: Expression for pressure gradient and pressure; Plot them; exit pressure

Solution:

Assumptions: 1) Incompressible flow 2) Flow profile remains unchanged so centerline velocity can represent average velocity

Basic equations

$$Q = V \cdot A \qquad \vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\substack{\text{total}\\ \text{acceleration}\\ \text{of a particle}}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\ \text{acceleration}}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\substack{\text{local}\\ \text{acceleration}}} \qquad \rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

Given data

$$\rho = 1.75\cdot\frac{\text{slug}}{\text{ft}^3} \qquad p_i = 35\cdot\text{psi} \qquad A_i = 15\cdot\text{in}^2 \qquad A_e = 2.5\cdot\text{in}^2 \qquad L = 10\cdot\text{ft} \qquad u_i = 5\cdot\frac{\text{ft}}{\text{s}}$$

For this 1D flow

$$Q = u_i\cdot A_i = u\cdot A \qquad A = A_i - \frac{(A_i - A_e)}{L}\cdot x \qquad \text{so} \qquad u(x) = u_i\cdot\frac{A_i}{A} = u_i\cdot\frac{A_i}{A_i - \left[\frac{(A_i - A_e)}{L}\cdot x\right]}$$

$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u = u_i\cdot\frac{A_i}{A_i - \left[\frac{(A_i - A_e)}{L}\cdot x\right]}\cdot\frac{\partial}{\partial x}\left[u_i\cdot\frac{A_i}{A_i - \left[\frac{(A_i - A_e)}{L}\cdot x\right]}\right] = \frac{A_i^2\cdot L^2\cdot u_i^2\cdot(A_e - A_i)}{(A_i\cdot L + A_e\cdot x - A_i\cdot x)^3}$$

For the pressure

$$\frac{\partial}{\partial x}p = -\rho\cdot a_x - \rho\cdot g_x = -\frac{\rho\cdot A_i^2\cdot L^2\cdot u_i^2\cdot(A_e - A_i)}{(A_i\cdot L + A_e\cdot x - A_i\cdot x)^3}$$

and

$$dp = \frac{\partial}{\partial x}p\cdot dx \qquad p - p_i = \int_0^x \frac{\partial}{\partial x}p\,dx = \int_0^x -\frac{\rho\cdot A_i^2\cdot L^2\cdot u_i^2\cdot(A_e - A_i)}{(A_i\cdot L + A_e\cdot x - A_i\cdot x)^3}\,dx$$

This is a tricky integral, so instead consider the following:

$$\frac{\partial}{\partial x}p = -\rho\cdot a_x = -\rho\cdot u\cdot\frac{\partial}{\partial x}u = -\frac{1}{2}\cdot\rho\cdot\frac{\partial}{\partial x}\left(u^2\right)$$

Hence

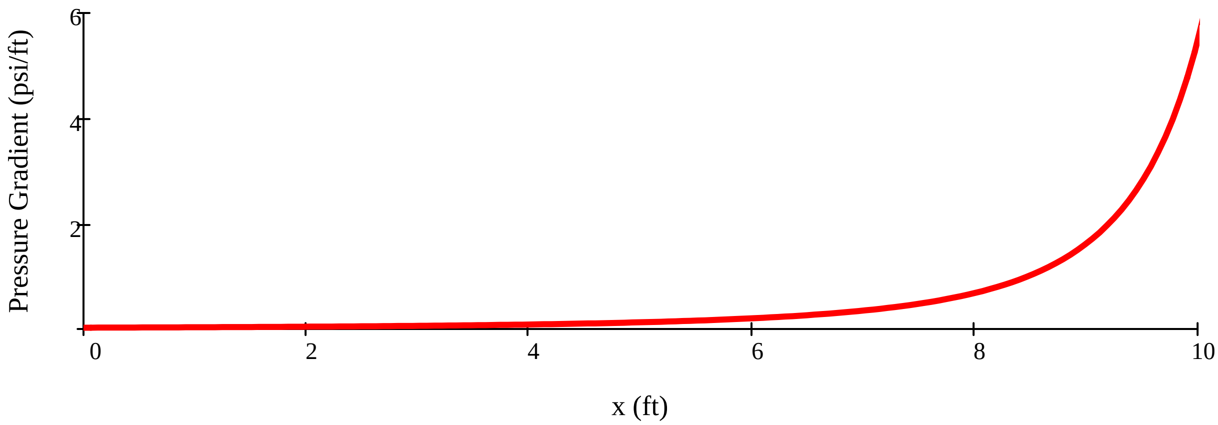
$$p - p_i = \int_0^x \frac{\partial}{\partial x}p\,dx = -\frac{\rho}{2}\cdot\int_0^x \frac{\partial}{\partial x}\left(u^2\right)dx = \frac{\rho}{2}\cdot\left(u(x=0)^2 - u(x)^2\right)$$

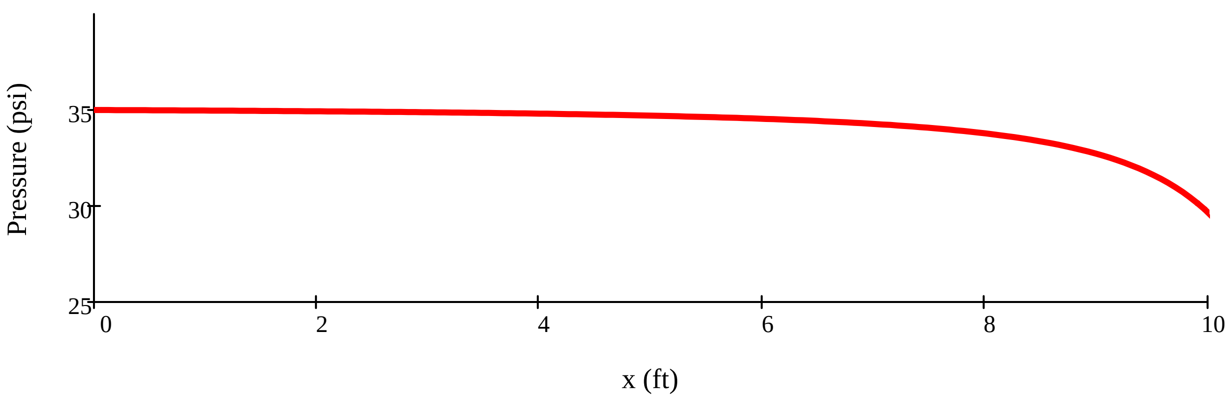
$$p(x) = p_i + \frac{\rho}{2}\cdot\left(u_i^{\,2} - u(x)^2\right)$$ which we recognise as the Bernoulli equation!

Hence
$$p(x) = p_i + \frac{\rho\cdot u_i^{\,2}}{2}\cdot\left[1 - \left[\frac{A_i}{A_i - \left[\frac{\left(A_i - A_e\right)}{L}\cdot x\right]}\right]^2\right]$$

At the exit $p(L) = 29.7\,\text{psi}$

The following plots can be done in *Excel*

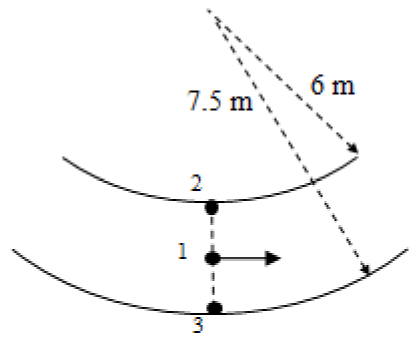




Problem 6.13

(Difficulty: 2)

6.13 Consider water flowing in a circular section of a two-dimensional channel. Assume the velocity is uniform across the channel at $12\ \frac{m}{s}$. The pressure is $120\ kPa$ at centerline (point 1). Determine the pressure at point 2 and 3 for the case of (a) flow in the horizontal plane (b) gravity acting in the direction of 2 to 3



Find: The pressures of the fluid

Assumption: The flow is frictionless and steady

Solution: Apply Euler's equation

$$\rho \frac{D\vec{V}}{Dt} = \rho \vec{g} - \nabla p$$

In cylindrical coordinates, for this steady two-dimensional flow we have:

$$\rho a_r = \rho \left(V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r} \frac{\partial V_r}{\partial \theta} - \frac{V_\theta^2}{r} \right) = \rho g_r - \frac{\partial p}{\partial r}$$

$$\rho a_\theta = \rho \left(V_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} \frac{\partial V_\theta}{\partial \theta} - \frac{V_\theta V_r}{r} \right) = \rho g_\theta - \frac{1}{r} \frac{\partial p}{\partial \theta}$$

For this flow we have:

$$V_r = 0$$

$$\frac{\partial V_\theta}{\partial r} = 0$$

$$V_\theta = 12\ \frac{m}{s}$$

The governing equation can be simplified to:

$$\rho\left(-\frac{V_\theta^2}{r}\right) = \rho g_r - \frac{\partial p}{\partial r}$$

$$\frac{\partial p}{\partial r} = \rho g_r + \rho\left(\frac{V_\theta^2}{r}\right)$$

(a) For the case flow in the horizontal plane, we have:

$$\frac{\partial p}{\partial r} = \rho\left(\frac{V_\theta^2}{r}\right)$$

$$p_1 - p_2 = \int_2^1 \frac{\partial p}{\partial r} dr = \int_2^1 \left[\rho\left(\frac{V_\theta^2}{r}\right)\right] dr$$

$$p_1 - p_2 = \rho V_\theta^2 \ln\left(\frac{r_1}{r_2}\right)$$

$$p_2 = p_1 - \rho V_\theta^2 \ln\left(\frac{r_1}{r_2}\right)$$

$$p_2 = 120\ kPa - 998\ \frac{kg}{m^3} \times \left(12\ \frac{m}{s}\right)^2 \times \ln\left(\frac{6.75}{6}\right) = 103.1\ kPa$$

$$p_3 - p_1 = \int_1^3 \frac{\partial p}{\partial r} dr = \int_1^3 \left[\rho\left(\frac{V_\theta^2}{r}\right)\right] dr$$

$$p_3 - p_1 = \rho V_\theta^2 \ln\left(\frac{r_3}{r_1}\right)$$

$$p_3 = p_1 + \rho V_\theta^2 \ln\left(\frac{r_3}{r_1}\right)$$

$$p_3 = 120\ kPa + 998\ \frac{kg}{m^3} \times \left(12\ \frac{m}{s}\right)^2 \times \ln\left(\frac{7.5}{6.75}\right) = 135.1\ kPa$$

(b) Gravity normal to the flow, we have:

$$\frac{\partial p}{\partial r} = -\rho g + \rho\left(\frac{V_\theta^2}{r}\right)$$

$$p_1 - p_2 = \int_2^1 \frac{\partial p}{\partial r} dr = \int_2^1 \left[-\rho g + \rho\left(\frac{V_\theta^2}{r}\right)\right] dr$$

$$p_1 - p_2 = -\rho g(r_1 - r_2) + \rho V_\theta^2 \ln\left(\frac{r_1}{r_2}\right)$$

$$p_2 = p_1 + \rho g(r_1 - r_2) - \rho V_\theta^2 \ln\left(\frac{r_1}{r_2}\right)$$

$$p_2 = 120\ kPa + 9.81\ \frac{kPa}{m} \times 0.75\ m - 998\ \frac{kg}{m^3} \times \left(12\ \frac{m}{s}\right)^2 \times \ln\left(\frac{6.75}{6}\right)$$

$$p_2 = 110.4\ kPa$$

Also we have:

$$p_3 - p_1 = \int_1^3 \frac{\partial p}{\partial r} dr = \int_1^3 \left[-\rho g + \rho\left(\frac{V_\theta^2}{r}\right)\right] dr$$

$$p_3 - p_1 = -\rho g(r_3 - r_1) + \rho V_\theta^2 \ln\left(\frac{r_3}{r_1}\right)$$

$$p_3 = p_1 - \rho g(r_3 - r_1) + \rho V_\theta^2 \ln\left(\frac{r_3}{r_1}\right)$$

$$p_3 = 120\ kPa - 9.81\ \frac{kPa}{m} \times 0.75\ m + 998\ \frac{kg}{m^3} \times \left(12\ \frac{m}{s}\right)^2 \times \ln\left(\frac{7.5}{6.75}\right)$$

$$p_3 = 127.8\ kPa$$

Problem 6.14

(Difficulty 2)

6.14 Consider a tornado as air moving in a circular pattern in the horizontal plane. If the wind speed is 200 mph and the diameter of the tornado is 200 ft, determine the radial pressure gradient. If it is desired to model the tornado using water in a 6 in diameter tube, what speed is needed to give the same radial pressure gradient?

Find: The pressure gradient for prototype and model case.

Assumption: Flow is frictionless and incompressible

Solution: Use Euler's equation to find the pressure gradient

$$\rho \frac{D\vec{V}}{Dt} = \rho \vec{g} - \nabla p$$

The pressure gradient is

$$\nabla p = \rho \vec{g} - \rho \frac{D\vec{V}}{Dt}$$

For this two-dimensional flow we have in cylindrical coordinates:

$$\rho a_r = \rho \left(\frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r} \frac{\partial V_r}{\partial \theta} - \frac{V_\theta^2}{r} \right) = \rho g_r - \frac{\partial p}{\partial r}$$

$$\rho a_\theta = \rho \left(\frac{\partial V_\theta}{\partial t} + V_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} \frac{\partial V_\theta}{\partial \theta} - \frac{V_\theta V_r}{r} \right) = \rho g_\theta - \frac{1}{r} \frac{\partial p}{\partial \theta}$$

For this specific flow we have:

$$V_r = 0$$

$$\frac{\partial V_\theta}{\partial r} = 0$$

The tangential velocity is

$$V_\theta = 200\ mph = 293\ \frac{ft}{s}$$

$$r = \frac{1}{2} D = 100\ ft$$

There is no gravity force in the horizontal plane.

The governing equation are then simplified as:

$$\rho\left(-\frac{V_\theta^2}{r}\right) = -\frac{\partial p}{\partial r}$$

$$\frac{\partial p}{\partial r} = \rho\left(\frac{V_\theta^2}{r}\right)$$

The density of the air is:

$$\rho = 0.0024\ \frac{slug}{ft^3} = 0.0024\ \frac{lbf \cdot s^2}{ft^4}$$

Thus the radial pressure gradient is

$$\frac{\partial p}{\partial r} = 0.0024\ \frac{lbf \cdot s^2}{ft^4} \times \frac{\left(293\ \frac{ft}{s}\right)^2}{100\ ft} = 2.06\ \frac{lbf}{ft^3}$$

For the model, we have:

$$\rho_m = 1.94\ \frac{slug}{ft^3} = 1.94\ \frac{lbf \cdot s^2}{ft^4}$$

$$r_m = 3\ in = 0.25\ ft$$

The model radial pressure gradient is

$$\frac{\partial p}{\partial r_m} = \rho_m\left(\frac{V_{\theta m}^2}{r_m}\right)$$

Or the velocity is

$$V_{\theta m}^2 = \frac{r_m}{\rho_m}\frac{\partial p}{\partial r_m}$$

Where the radial gradient is the same as for the prototype

$$\frac{\partial p}{\partial r_m} = 2.06\ \frac{lbf}{ft^3}$$

The tangential velocity is

$$V_{\theta m} = \sqrt{\frac{r_m}{\rho_m}\frac{\partial p}{\partial r_m}} = \sqrt{\frac{0.25\ ft}{1.94\ \frac{lbf \cdot s^2}{ft^4}} \times 2.06\ \frac{lbf}{ft^3}} = 0.515\ \frac{ft}{s}$$

This velocity is easily attainable in water.

Problem 6.15

[Difficulty: 3]

6.15 A nozzle for an incompressible, inviscid fluid of density $\rho = 1000\ \text{kg/m}^3$ consists of a converging section of pipe. At the inlet the diameter is $D_i = 100$ mm, and at the outlet the diameter is $D_o = 20$ mm. The nozzle length is $L = 500$ mm, and the diameter decreases linearly with distance x along the nozzle. Derive and plot the acceleration of a fluid particle, assuming uniform flow at each section, if the speed at the inlet is $V_i = 1$ m/s. Plot the pressure gradient through the nozzle, and find its maximum absolute value. If the pressure gradient must be no greater than 5 MPa/m in absolute value, how long would the nozzle have to be?

Given: Nozzle geometry

Find: Acceleration of fluid particle; Plot; Plot pressure gradient; find L such that pressure gradient < 5 MPa/m in absolute value

Solution:

The given data is $D_i = 0.1 \cdot m$ $\quad D_o = 0.02 \cdot m$ $\quad L = 0.5 \cdot m$ $\quad V_i = 1 \cdot \frac{m}{s}$ $\quad \rho = 1000 \cdot \frac{kg}{m^3}$

For a linear decrease in diameter $\quad D(x) = D_i + \frac{D_o - D_i}{L} \cdot x$

From continuity $\quad Q = V \cdot A = V \cdot \frac{\pi}{4} \cdot D^2 = V_i \cdot \frac{\pi}{4} \cdot D_i^2 \qquad Q = 0.00785 \frac{m^3}{s}$

Hence $\quad V(x) \cdot \frac{\pi}{4} \cdot D(x)^2 = Q \qquad V(x) = \dfrac{4 \cdot Q}{\pi \cdot \left(D_i + \frac{D_o - D_i}{L} \cdot x\right)^2}$

or $\quad V(x) = \dfrac{V_i}{\left(1 + \frac{D_o - D_i}{L \cdot D_i} \cdot x\right)^2}$

The governing equation for this flow is

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} \qquad (6.2a)$$

or, for steady 1D flow, in the notation of the problem

$$a_x = V \cdot \frac{d}{dx} V = \frac{V_i}{\left(1 + \frac{D_o - D_i}{L \cdot D_i} \cdot x\right)^2} \cdot \frac{d}{dx} \frac{V_i}{\left(1 + \frac{D_o - D_i}{L \cdot D_i} \cdot x\right)^2} \qquad a_x(x) = -\frac{2 \cdot V_i^2 \cdot (D_o - D_i)}{D_i \cdot L \cdot \left[1 + \frac{(D_o - D_i)}{D_i \cdot L} \cdot x\right]^5}$$

This is plotted in the associated *Excel* workbook

From Eq. 6.2a, pressure gradient is

$$\frac{\partial}{\partial x} p = -\rho \cdot a_x \qquad \frac{\partial}{\partial x} p = \frac{2 \cdot \rho \cdot V_i^2 \cdot (D_o - D_i)}{D_i \cdot L \cdot \left[1 + \frac{(D_o - D_i)}{D_i \cdot L} \cdot x\right]^5}$$

This is also plotted in the associated *Excel* workbook. Note that the pressure gradient is always negative: separation is unlikely to occur in the nozzle

At the inlet $\frac{\partial}{\partial x}p = -3.2 \cdot \frac{kPa}{m}$ At the exit $\frac{\partial}{\partial x}p = -10 \cdot \frac{MPa}{m}$

To find the length *L* for which the absolute pressure gradient is no more than 5 MPa/m, we need to solve

$$\left|\frac{\partial}{\partial x}p\right| \le 5 \cdot \frac{MPa}{m} = \frac{2 \cdot \rho \cdot V_i^{\,2} \cdot \left(D_o - D_i\right)}{D_i \cdot L \cdot \left[1 + \frac{\left(D_o - D_i\right)}{D_i \cdot L} \cdot x\right]^5}$$

with $x = L$ m (the largest pressure gradient is at the outlet)

Hence $$L \ge \frac{2 \cdot \rho \cdot V_i^{\,2} \cdot \left(D_o - D_i\right)}{D_i \cdot \left(\frac{D_o}{D_i}\right)^5 \cdot \left|\frac{\partial}{\partial x}p\right|}$$ $L \ge 1 \cdot m$

This result is also obtained using *Goal Seek* in the *Excel* workbook

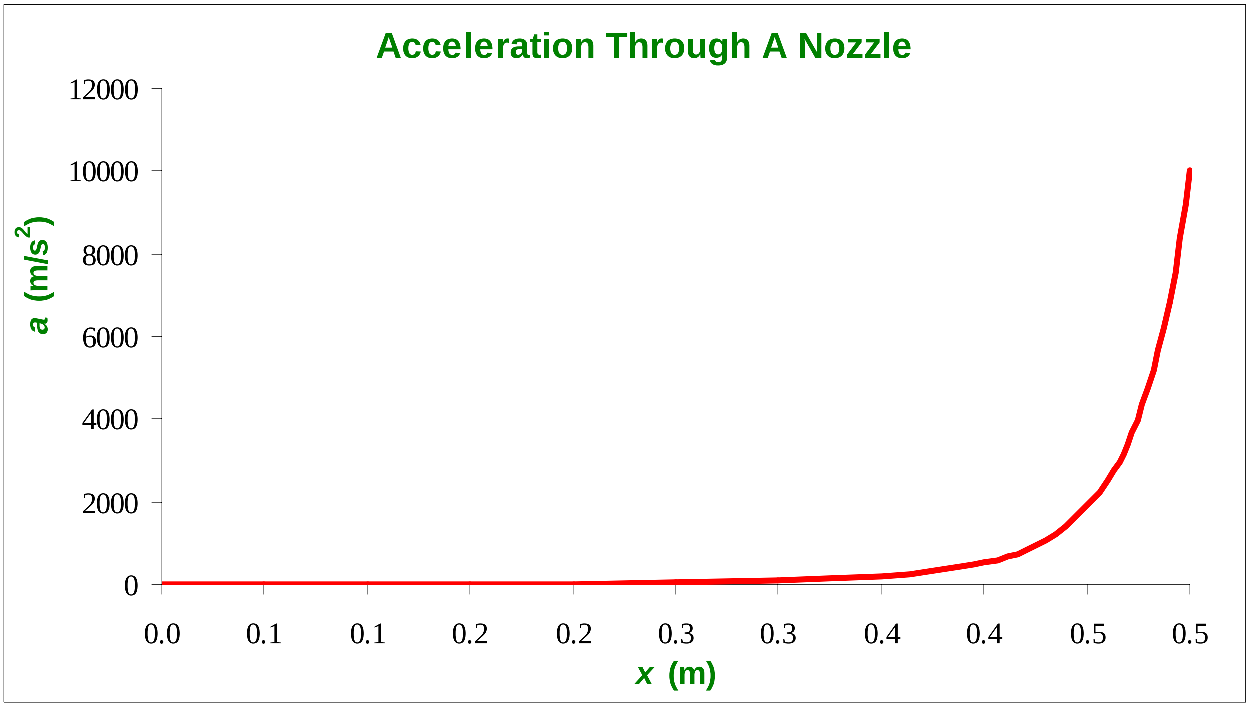
From Excel

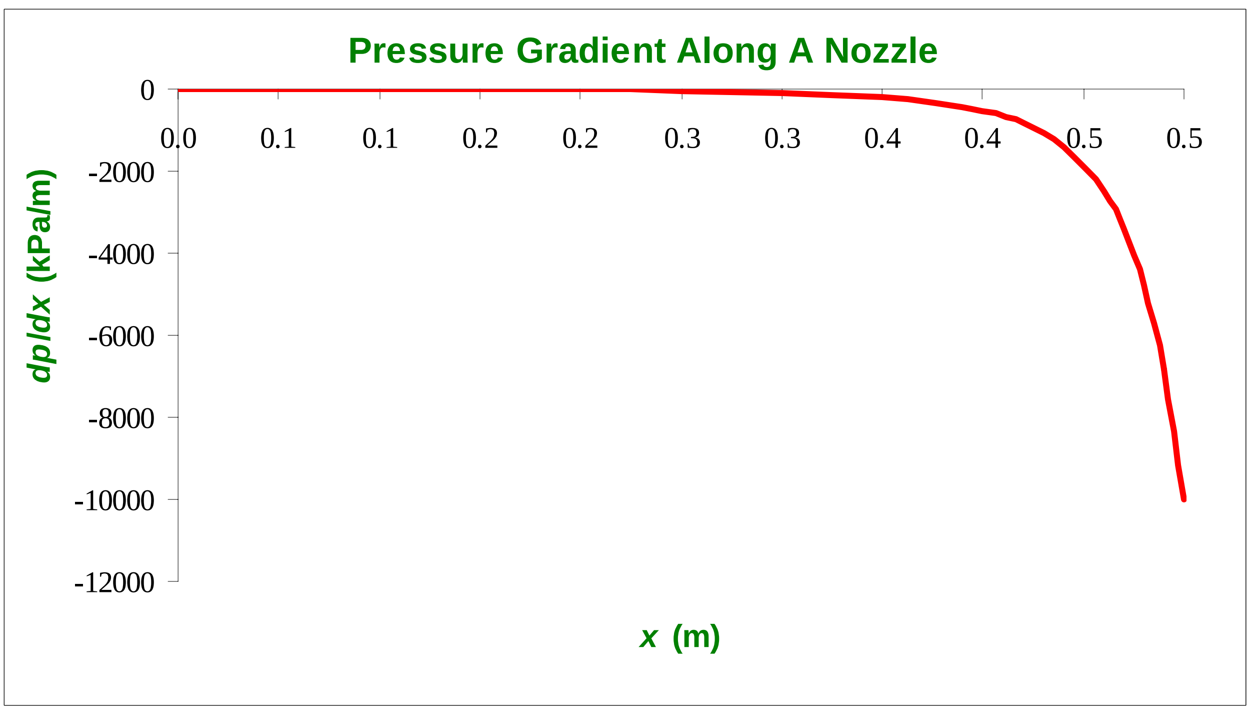
x **(m)**	***a*** **(m/s²)**	***dp*** **/*****dx*** **(kPa/m)**
0.000	3.20	-3.20
0.050	4.86	-4.86
0.100	7.65	-7.65
0.150	12.6	-12.6
0.200	22.0	-22.0
0.250	41.2	-41.2
0.300	84.2	-84.2
0.350	194	-194
0.400	529	-529
0.420	843	-843
0.440	1408	-1408
0.460	2495	-2495
0.470	3411	-3411
0.480	4761	-4761
0.490	6806	-6806
0.500	10000	-10000

For the length *L* required
for the pressure gradient
to be less than 5 MPa/m (abs)
use *Goal Seek*

L = 1.00 m

x **(m)**	***dp*** **/*****dx*** **(kPa/m)**
1.00	-5000


Acceleration Through A Nozzle
12000
10000
8000
6000
4000
2000
0
a (m/s2)
0.0
0.1
0.1
0.2
0.2
0.3
0.3
0.4
0.4
0.5
0.5
x (m)


Pressure Gradient Along A Nozzle
0
-2000
-4000
-6000
-8000
-10000
-12000
dp/dx (kPa/m)
0.0
0.1
0.1
0.2
0.2
0.3
0.3
0.4
0.4
0.5
0.5
x (m)

Problem 6.16

[Difficulty: 3]

6.16 A diffuser for an incompressible, inviscid fluid of density $\rho = 1000$ kg/m^3 consists of a diverging section of pipe. At the inlet the diameter is $D_i = 0.25$ m, and at the outlet the diameter is $D_o = 0.75$ m. The diffuser length is $L = 1$ m, and the diameter increases linearly with distance x along the diffuser. Derive and plot the acceleration of a fluid particle, assuming uniform flow at each section, if the speed at the inlet is $V_i = 5$ m/s. Plot the pressure gradient through the diffuser, and find its maximum value. If the pressure gradient must be no greater than 25 kPa/m, how long would the diffuser have to be?

Given: Diffuser geometry

Find: Acceleration of a fluid particle; plot it; plot pressure gradient; find *L* such that pressure gradient is less than 25 kPa/m

Solution:

The given data is $D_i = 0.25 \cdot m$ $\quad D_o = 0.75 \cdot m$ $\quad L = 1 \cdot m$ $\quad V_i = 5 \cdot \frac{m}{s}$ $\quad \rho = 1000 \cdot \frac{kg}{m^3}$

For a linear increase in diameter $\quad D(x) = D_i + \frac{D_o - D_i}{L} \cdot x$

From continuity $\quad Q = V \cdot A = V \cdot \frac{\pi}{4} \cdot D^2 = V_i \cdot \frac{\pi}{4} \cdot D_i^2 \qquad Q = 0.245 \frac{m^3}{s}$

Hence $\quad V(x) \cdot \frac{\pi}{4} \cdot D(x)^2 = Q \qquad V(x) = \dfrac{4 \cdot Q}{\pi \cdot \left(D_i + \frac{D_o - D_i}{L} \cdot x\right)^2}$ $\quad$ or $\quad V(x) = \dfrac{V_i}{\left(1 + \frac{D_o - D_i}{L \cdot D_i} \cdot x\right)^2}$

The governing equation for this flow is

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} \qquad (6.2a)$$

or, for steady 1D flow, in the notation of the problem

$$a_x = V \cdot \frac{d}{dx} V = \frac{V_i}{\left(1 + \frac{D_o - D_i}{L \cdot D_i} \cdot x\right)^2} \cdot \frac{d}{dx} \frac{V_i}{\left(1 + \frac{D_o - D_i}{L \cdot D_i} \cdot x\right)^2}$$

Hence

$$a_x(x) = -\frac{2 \cdot V_i^2 \cdot \left(D_o - D_i\right)}{D_i \cdot L \cdot \left[1 + \frac{\left(D_o - D_i\right)}{D_i \cdot L} \cdot x\right]^5}$$

This can be plotted in *Excel* (see below)

From Eq. 6.2a, pressure gradient is $\quad \frac{\partial}{\partial x} p = -\rho \cdot a_x \qquad \frac{\partial}{\partial x} p = \dfrac{2 \cdot \rho \cdot V_i^2 \cdot \left(D_o - D_i\right)}{D_i \cdot L \cdot \left[1 + \frac{\left(D_o - D_i\right)}{D_i \cdot L} \cdot x\right]^5}$

This can also plotted in *Excel*. Note that the pressure gradient is adverse: separation is likely to occur in the diffuser, and occur near the entrance

At the inlet $\frac{\partial}{\partial x}p = 100 \cdot \frac{kPa}{m}$ At the exit $\frac{\partial}{\partial x}p = 412 \cdot \frac{Pa}{m}$

To find the length *L* for which the pressure gradient is no more than 25 kPa/m, we need to solve

$$\frac{\partial}{\partial x}p \le 25 \cdot \frac{kPa}{m} = \frac{2 \cdot \rho \cdot V_i^{\,2} \cdot \left(D_o - D_i\right)}{D_i \cdot L \cdot \left[1 + \frac{\left(D_o - D_i\right)}{D_i \cdot L} \cdot x\right]^5}$$

with *x* = 0 m (the largest pressure gradient is at the inlet)

Hence $$L \ge \frac{2 \cdot \rho \cdot V_i^{\,2} \cdot \left(D_o - D_i\right)}{D_i \cdot \frac{\partial}{\partial x}p}$$ $L \ge 4 \cdot m$

This result is also obtained using *Goal Seek* in *Excel*.

In *Excel*:

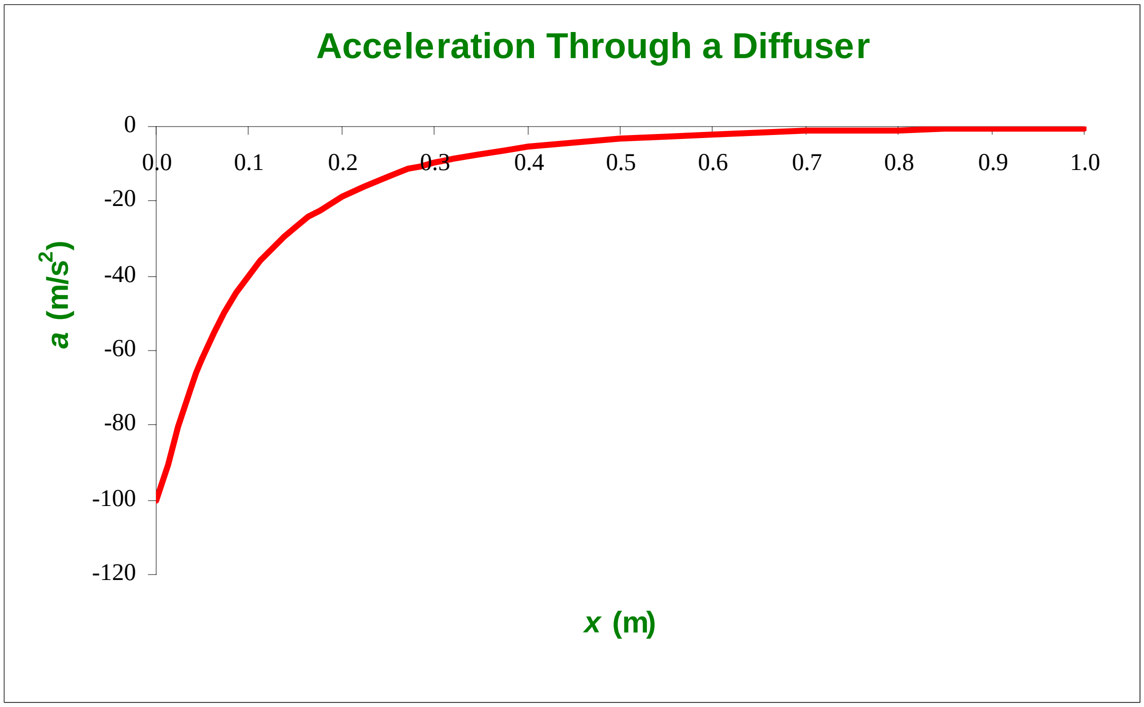
D_i =	0.25	m
D_o =	0.75	m
L =	1	m
V_i =	5	m/s
(=	1000	kg/m^3

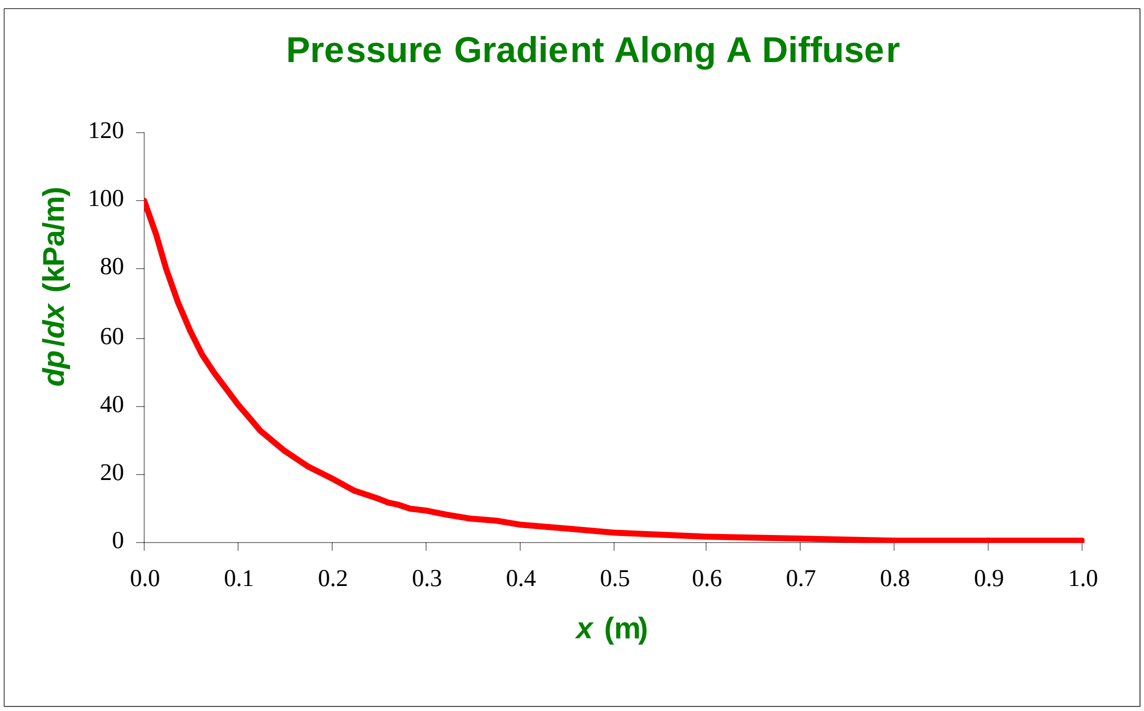
x **(m)**	***a*** **(m/s^2)**	***dp*** **/*****dx*** **(kPa/m)**
0.00	-100	100
0.05	-62.1	62.1
0.10	-40.2	40.2
0.15	-26.9	26.93
0.20	-18.59	18.59
0.25	-13.17	13.17
0.30	-9.54	9.54
0.40	-5.29	5.29
0.50	-3.125	3.125
0.60	-1.940	1.940
0.70	-1.256	1.256
0.80	-0.842	0.842
0.90	-0.581	0.581
1.00	-0.412	0.412

For the length *L* required for the pressure gradient to be less than 25 kPa/m use *Goal Seek*

L = 4.00 m

x **(m)**	***dp*** **/*****dx*** **(kPa/m)**
0.0	25.0

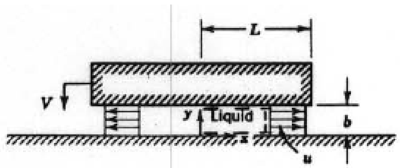

Acceleration Through a Diffuser
0
-20
-40
-60
-80
-100
-120
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
a (m/s2)
x (m)


Pressure Gradient Along A Diffuser
120
100
80
60
40
20
0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
dp/dx (kPa/m)
x (m)

Problem 6.17

(Difficulty: 3)

6.17 A liquid layer separates two plane surfaces as shown. The lower surface is stationary; the upper surface moves downward at constant speed V. The moving surface has width w, perpendicular to the plane of the diagram, and $w \gg L$. The incompressible liquid layer, of density ρ, is squeezed from between the surfaces. Assume the flow is uniform at any cross section and neglect viscosity as a first approximation. Use a suitable chosen control volume to show that $u = Vx/b$ within the gap, where $b = b_0 - Vt$. Obtain an algebraic expression for the acceleration of a fluid particle located at x. Determine the pressure gradient, $\partial p/\partial x$, in the liquid layer. Find the pressure distribution, $p(x)$. Obtain an expression for the net pressure force that acts on the upper (moving) flat surface.



Solution: Use the continuity and Euler (momentum) equations

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho d\forall + \int_{CS} \rho \bar{V} \cdot d\bar{A}$$

$$-\nabla p + \rho \vec{g} = \rho \frac{D\bar{V}}{Dt}$$

$$\vec{F} = -\int p d\vec{A}$$

(a) For the deformable CV shown,

$$0 = \frac{\partial}{\partial t}\int_0^y \rho wxdy + \rho uwy = \rho wx \frac{dy}{dt} + \rho uwy$$

But we have:

$$\frac{dy}{dt} = -V$$

$$u = \frac{Vx}{y}$$

If $y = b_0$ at $t = 0$, then $y = b = b_0 - Vt$ at anytime t.

$$u = \frac{Vx}{b}$$

(b)

$$a_x = \frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \omega\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

With the assumptions, $u \neq u(y), \omega = 0$ so that:

$$a_x = \frac{Vx}{b}\left(\frac{V}{b}\right) + \frac{\partial u}{\partial b}\frac{\partial b}{\partial t} = \frac{V^2 x}{b^2} + \left(-\frac{Vx}{b^2}\right)(-V) = \frac{2V^2 x}{b^2}$$

(c) From Euler's equation in the x direction with $g_x = 0$,

$$\frac{\partial p}{\partial x} = -\rho a_x = -\frac{2\rho V^2 x}{b^2}$$

(d)

$$p - p_{atm} = \int_L^x \frac{\partial p}{\partial x} dx = \int_L^x -\frac{2\rho V^2}{b^2} x dx = -\frac{\rho V^2 x^2}{b^2}\Bigg|_L^x = -\frac{\rho V^2 x^2}{b^2} + \frac{\rho V^2 L^2}{b^2}$$

$$p - p_{atm} = \frac{\rho V^2}{b^2}(L^2 - x^2) = \frac{\rho V^2 L^2}{b^2}\left[1 - \left(\frac{x}{L}\right)^2\right]$$

(e)

$$F_y = \int_A (p - p_{atm})\, dA = 2\int_0^L \frac{\rho V^2 L^2}{b^2}\left[1 - \left(\frac{x}{L}\right)^2\right] w dx$$

$$F_y = \frac{2\rho V^2 L^2 w}{b^2}\left[x - \frac{x^3}{3L^2}\right]_0^L = \frac{2\rho V^2 L^2 w}{b^2}\frac{2L}{3}$$

$$F_y = \frac{4\rho V^2 L^3 w}{3b^2}$$

The direction is upward.

Problem 6.18

(Difficulty: 3)

6.18 Consider problem 6.15 with the nozzle directed upward. Assuming that the flow is uniform at each section, derive and plot the acceleration of a fluid particle for an inlet speed of $V_i = 2\ \frac{m}{s}$. Plot the pressure gradient through the nozzle, and its maximum absolute value. If the pressure gradient must be no greater than 7 $\frac{MPa}{m}$ in absolute value, how long would the nozzle have to be?

Find: the flow properties of the nozzle

Assumption: The flow is ideal

Solution: Apply the continuity and Euler's equation

From the continuity equation we have:

$$Q = VA = \frac{\pi}{4} D^2 v = \frac{\pi}{4} D_i^2 V_i = \frac{\pi}{4} \times (0.1\ m)^2 \times 2\ \frac{m}{s} = 0.0157\ \frac{m^3}{s}$$

The velocity v in t he y-direction is

$$v = \frac{Q}{\frac{\pi}{4} D^2} = \frac{4Q}{\pi \left(D_i + \frac{D_o - D_i}{L} y\right)^2} = \frac{V_i}{\left(1 + \frac{D_o - D_i}{D_i L} y\right)^2}$$

For this 1D flow in the positive y direction, we have the acceleration as:

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = v\frac{dv}{dy}$$

For a linear decrease in diameter, the diameter of the nozzle at any location is

$$D(x) = D_i + \frac{D_o - D_i}{L} y$$

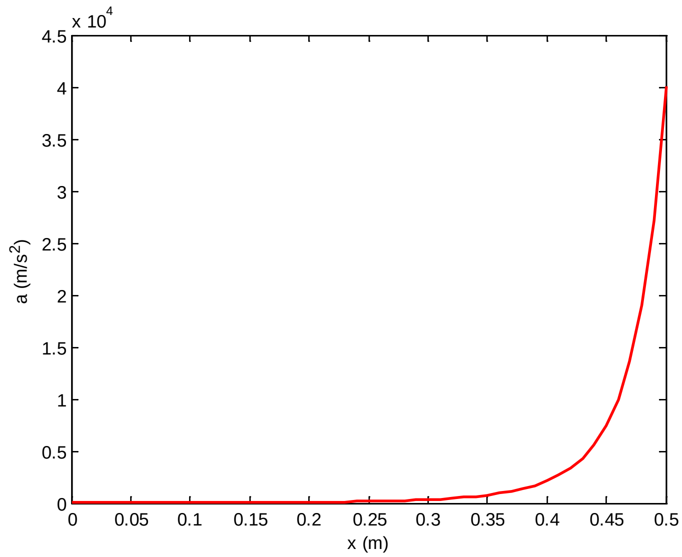
We have the following parameters for this nozzle

$$D_i = 0.1\ m,\ D_o = 0.02\ m, L = 0.5\ m\ ,V_i = 2\ \frac{m}{s}, \text{ and } \rho = 1000\ \frac{kg}{m^3}$$

The acceleration at any location is given by

$$a_y = \frac{V_i}{\left(1 + \frac{D_o - D_i}{D_i L} y\right)^2} \frac{d\dfrac{V_i}{\left(1 + \frac{D_o - D_i}{D_i L} y\right)^2}}{dy} = -\frac{2V_i^2 (D_o - D_i)}{D_i L \left(1 + \frac{D_o - D_i}{D_i L} y\right)^5} = \frac{12.8}{(1 - 1.6y)^5}\ \frac{m}{s^2}$$

The plot is shown as:



To find the pressure gradient, we use the momentum equation:

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial p}{\partial y}$$

For steady flow in the vertical direction, the equation reduces to

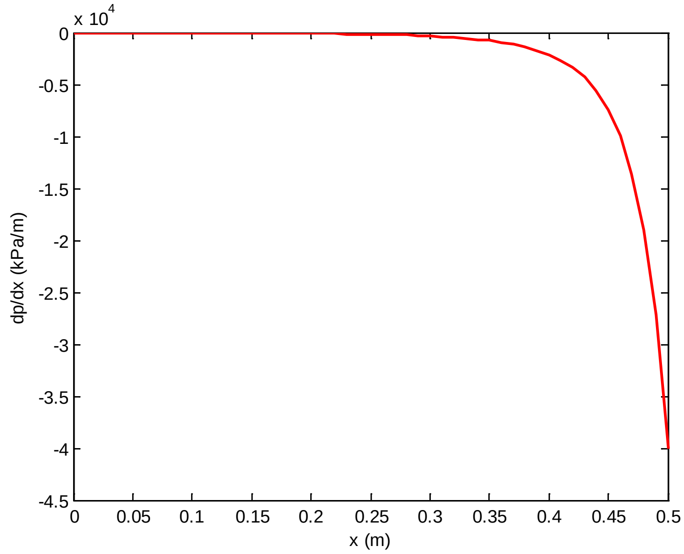
$$\rho v\frac{\partial v}{\partial y}=-\rho g-\frac{\partial p}{\partial y}$$

The pressure gradient is

$$\frac{\partial p}{\partial y}=-\rho g-\rho v\frac{\partial v}{\partial y}=-\rho g-\rho v\frac{dv}{dy}=-\rho g+\rho\frac{2V_i^2(D_o-D_i)}{D_iL\left(1+\frac{D_o-D_i}{D_iL}y\right)^5}=-\rho g-\rho a_y$$

$$\frac{\partial p}{\partial y}=-9.8-\frac{12.8}{(1-1.6y)^5}\,\frac{kPa}{m}$$

The plot for the pressure gradient is:



At the inlet:

$$\frac{\partial p}{\partial y} = -22.6\ kPa$$

At the outlet:

$$\frac{\partial p}{\partial y} = -40.1\ MPa$$

The maximum absolute value of pressure gradient is 40.1 MPa.

If the pressure gradient must be no great than 7 $\frac{MPa}{m}$, we have:

$$\frac{\partial p}{\partial y} = \left| -\rho g + \rho \frac{2V_i^2(D_o - D_i)}{D_i L \left(1 + \frac{D_o - D_i}{D_i L} y\right)^5} \right| \leq 7 \frac{MPa}{m}$$

At the outlet we have the maximum pressure gradient so:

$$\left| -9800 - \frac{6400}{L(1-0.8)^5} \right| \leq 7000000$$

Or

$$9800 + \frac{6400}{L(1-0.8)^5} \leq 7000000$$

Or

$$\frac{6400}{L(1-0.8)^5} \leq 6990200$$

So the length must be

$$L \geq \frac{6400}{6990200 \times (1-0.8)^5}\ m = 2.86\ m$$

Problem 6.19

(Difficulty: 3)

6.19 Consider problem 6.16 with the diffuser directed upward. Assuming that the flow is uniform at each section, derive and plot the acceleration of a fluid particle for an inlet speed of $V_i = 12\ \frac{m}{s}$. Plot the pressure gradient through the diffuser, and its maximum absolute value. If the pressure gradient must be no greater than 20 $\frac{kPa}{m}$, how long would the diffuser have to be?

Find: the flow properties of the diffuser

Assumption: The flow is ideal

Solution: Apply the continuity and Euler's equation

From the continuity equation we have:

$$Q = VA = \frac{\pi}{4}D^2 v = \frac{\pi}{4}D_i^2 V_i = \frac{\pi}{4} \times (0.25\ m)^2 \times 12\ \frac{m}{s} = 0.589\ \frac{m^3}{s}$$

The velocity v in the y-direction is

$$v = \frac{Q}{\frac{\pi}{4}D^2} = \frac{4Q}{\pi\left(D_i + \frac{D_o - D_i}{L}y\right)^2} = \frac{V_i}{\left(1 + \frac{D_o - D_i}{D_i L}y\right)^2}$$

For this 1D flow in the positive y direction, we have the acceleration as:

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = v\frac{dv}{dy}$$

For the linear decrease in diameter, the diameter at any location is:

$$D(x) = D_i + \frac{D_o - D_i}{L}y$$

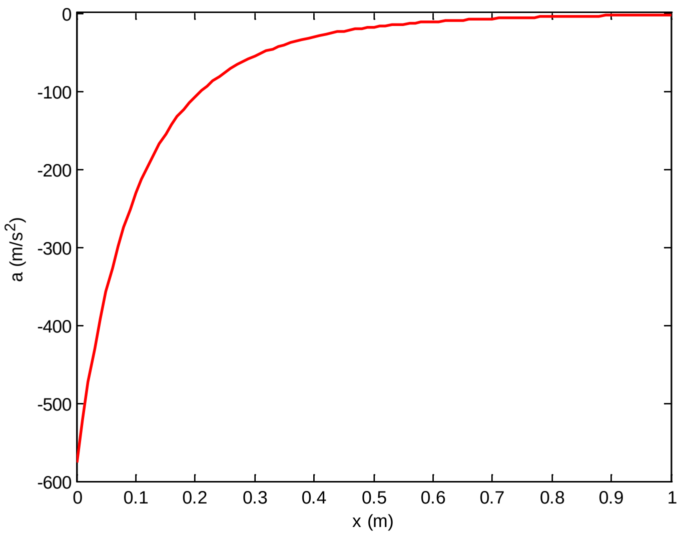
The acceleration is then

$$a_y = \frac{V_i}{\left(1 + \frac{D_o - D_i}{D_i L}y\right)^2} \frac{d\frac{V_i}{\left(1 + \frac{D_o - D_i}{D_i L}y\right)^2}}{dy} = -\frac{2V_i^2(D_o - D_i)}{D_i L\left(1 + \frac{D_o - D_i}{D_i L}y\right)^5} = -\frac{576}{(1+2y)^5}\ \frac{m}{s^2}$$

For this problem we have the following parameters:

$$D_i = 0.25\ m,\ D_o = 0.75\ m, L = 1\ m, V_i = 5\ \frac{m}{s}, and\ \rho = 1000\ \frac{kg}{m^3}$$

The plot for acceleration is:



To find the pressure gradient, we use the momentum equation:

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial p}{\partial y}$$

This reduces to

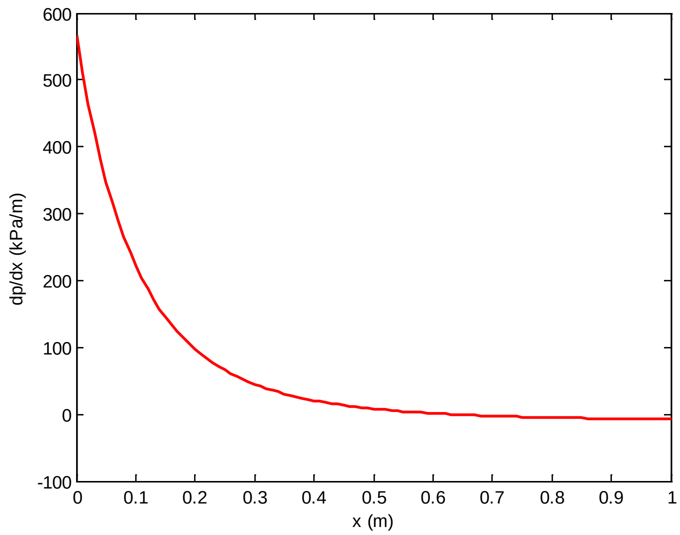
$$\rho v\frac{\partial v}{\partial y}=-\rho g-\frac{\partial p}{\partial y}$$

The pressure gradient is then

$$\frac{\partial p}{\partial y}=-\rho g-\rho v\frac{\partial v}{\partial y}=-\rho g-\rho v\frac{dv}{dy}=-\rho g+\rho\frac{2V_i^2(D_o-D_i)}{D_iL\left(1+\frac{D_o-D_i}{D_iL}y\right)^5}=-\rho g-\rho a_y$$

$$\frac{\partial p}{\partial y}=-9.8+\frac{576}{(1+2y)^5}\,\frac{kPa}{m}$$

The plot is shown as:



At the inlet:

$$\frac{\partial p}{\partial y} = 566.2\ kPa$$

At the outlet:

$$\frac{\partial p}{\partial y} = -7.43\ kPa$$

The maximum absolute value of pressure gradient is $566.2\ \ kPa$.

If the pressure gradient must be no great than $20\ \frac{kPa}{m}$, we have:

$$\frac{\partial p}{\partial y} = \left| -\rho g + \rho \frac{2V_i^2(D_o - D_i)}{D_i L \left(1 + \frac{D_o - D_i}{D_i L} y\right)^5} \right| \leq 20\ \frac{kPa}{m}$$

At the inlet we have the maximum pressure gradient so:

$$\left| -9800 + \frac{576000}{L(1)^5} \right| \leq 20000$$

Or

$$-9800 + \frac{576000}{L} \leq 20000$$

Or

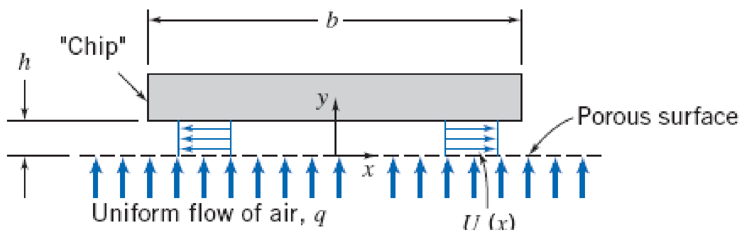
$$\frac{576000}{L} \leq 29800$$

So the length must be

$$L \geq \frac{576000}{29800}\ m = 19.33\ m$$

Problem 6.20 [Difficulty: 4]

6.20 A rectangular computer chip floats on a thin layer of air, $h = 0.5$ mm thick, above a porous surface. The chip width is $b = 40$ mm, as shown. Its length, L, is very long in the direction perpendicular to the diagram. There is no flow in the z direction. Assume flow in the x direction in the gap under the chip is uniform. Flow is incompressible, and frictional effects may be neglected. Use a suitably chosen control volume to show that $U(x) = qx/h$ in the gap. Find a general expression for the (2D) acceleration of a fluid particle in the gap in terms of q, h, x, and y. Obtain an expression for the pressure gradient $\partial p/\partial x$. Assuming atmospheric pressure on the chip upper surface, find an expression for the net pressure force on the chip; is it directed upward or downward? Explain. Find the required flow rate q ($m^3/s/m^2$) and the maximum velocity, if the mass per unit length of the chip is 0.005 kg/m. Plot the pressure distribution as part of your explanation of the direction of the net force.



Given: Rectangular chip flow

Find: Velocity field; acceleration; pressure gradient; net force; required flow rate; plot pressure

Solution:

Basic equations
$$\sum_{CS} \left(\vec{V}\cdot\vec{A}\right) = 0 \qquad \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$$

$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\text{total acceleration of a particle}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\text{convective acceleration}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{local acceleration}} \qquad \rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

The given data is $\rho = 1.23\cdot\frac{kg}{m^3}$ $\quad p_{atm} = 101\cdot kPa$ $\quad h = 0.5\cdot mm$ $\quad b = 40\cdot mm$ $\quad M_{length} = 0.005\cdot\frac{kg}{m}$

Assuming a CV that is from the centerline to any point x, and noting that q is inflow per unit area, continuity leads to

$$q\cdot x\cdot L = U\cdot h\cdot L \qquad \text{or} \qquad u(x) = U(x) = q\cdot\frac{x}{h}$$

For acceleration we will need the vertical velocity v; we can use

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0 \qquad \text{or} \qquad \frac{\partial}{\partial y}v = -\frac{\partial}{\partial x}u = -\frac{du}{dx} = -\frac{d}{dx}\left(q\cdot\frac{x}{h}\right) = -\frac{q}{h}$$

Hence $\quad v(y = y) - v(y = 0) = -\int_0^y \frac{q}{h}\,dy = -q\cdot\frac{y}{h}$

But $\quad v(y = 0) = q \qquad$ so $\qquad v(y) = q\cdot\left(1 - \frac{y}{h}\right)$

For the x acceleration $\quad a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u \qquad a_x = q\cdot\frac{x}{h}\cdot\left(\frac{q}{h}\right) + q\cdot\left(1 - \frac{y}{h}\right)\cdot(0) \qquad a_x = \frac{q^2}{h^2}\cdot x$

For the y acceleration $a_y = u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v$ $\quad a_y = q\cdot\frac{x}{h}\cdot(0) + q\cdot\left(1 - \frac{y}{h}\right)\cdot\left(-\frac{q}{h}\right)$ $\quad a_x = \frac{q^2}{h}\cdot\left(\frac{y}{h} - 1\right)$

For the pressure gradient we use x and y momentum (Euler equation) $\quad \rho\cdot\frac{Du}{Dx} = \rho\cdot\left(u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u\right) = \rho\cdot a_x = \frac{\partial}{\partial x}p$

Hence $\quad \frac{\partial}{\partial x}p = -\rho\cdot\frac{q^2}{h^2}\cdot x$

Also $\quad \rho\cdot\frac{Dv}{Dx} = \rho\cdot\left(u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v\right) = \rho\cdot a_y = -\frac{\partial}{\partial y}p$ $\quad \frac{\partial}{\partial y}p = \rho\cdot\frac{q^2}{h}\cdot\left(1 - \frac{y}{h}\right)$

For the pressure distribution, integrating from the outside edge (x = b/2) to any point x

$$p(x = x) - p\left(x = \frac{b}{2}\right) = p(x) - p_{atm} = \int_{\frac{b}{2}}^{x}\frac{\partial}{\partial x}p\,dx = \int_{\frac{b}{2}}^{x}-\rho\cdot\frac{q^2}{h^2}\cdot x\,dx = -\rho\cdot\frac{q^2}{2\cdot h^2}\cdot x^2 + \rho\cdot\frac{q^2}{8\cdot h^2}\cdot b^2$$

$$p(x) = p_{atm} + \rho\cdot\frac{q^2\cdot b^2}{8\cdot h^2}\cdot\left[1 - 4\cdot\left(\frac{x}{b}\right)^2\right]$$

For the net force we need to integrate this ... actually the gage pressure, as this pressure is opposed on the outer surface by p_{atm}

$$p_g(x) = \frac{\rho\cdot q^2\cdot b^2}{8\cdot h^2}\cdot\left[1 - 4\cdot\left(\frac{x}{b}\right)^2\right]$$

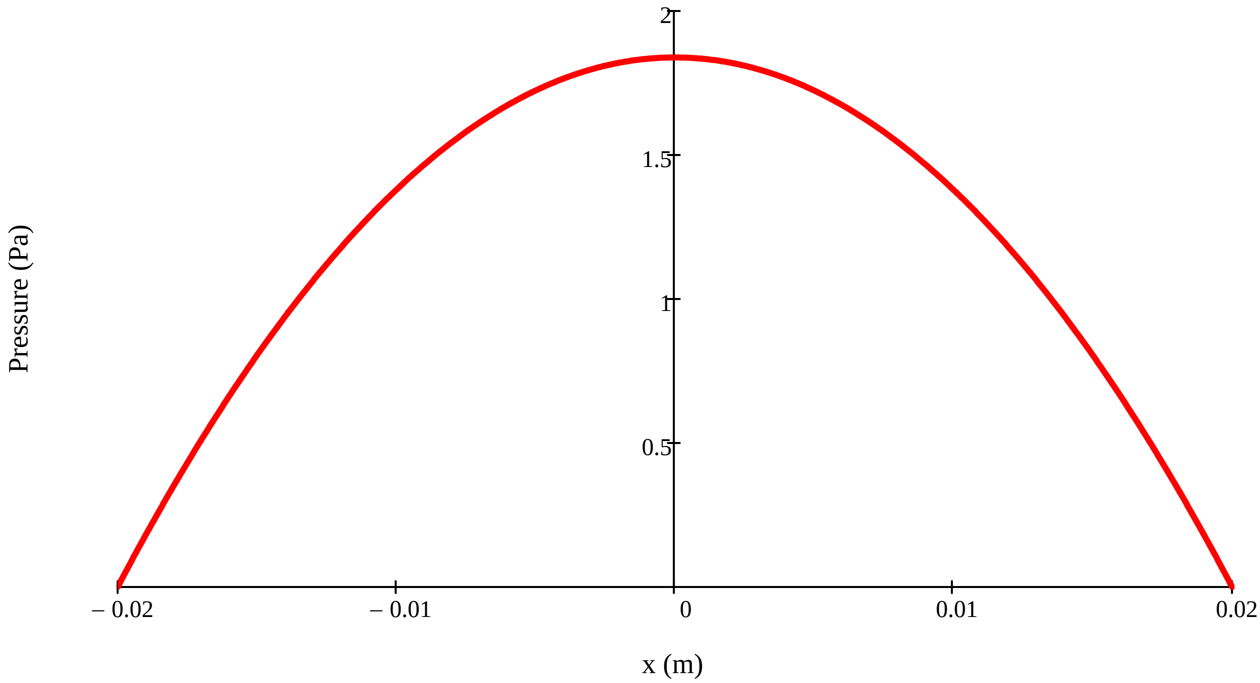
$$F_{net} = 2\cdot L\cdot\int_0^{\frac{b}{2}}p_g(x)\,dx = 2\cdot L\cdot\int_0^{\frac{b}{2}}\frac{\rho\cdot q^2\cdot b^2}{8\cdot h^2}\cdot\left[1 - 4\cdot\left(\frac{x}{b}\right)^2\right]dx = \frac{\rho\cdot q^2\cdot b^2\cdot L}{4\cdot h^2}\cdot\left(\frac{b}{2} - \frac{1}{3}\cdot\frac{b}{2}\right) \qquad F_{net} = \frac{\rho\cdot q^2\cdot b^3\cdot L}{12\cdot h^2}$$

The weight of the chip must balance this force $\quad M\cdot g = M_{length}\cdot L\cdot g = F_{net} = \frac{\rho\cdot q^2\cdot b^3\cdot L}{12\cdot h^2}$ $\quad$ or $\quad M_{length}\cdot g = \frac{\rho\cdot q^2\cdot b^3}{12\cdot h^2}$

Solving for q for the given mass/length $\quad q = \sqrt{\frac{12\cdot h^2\cdot g\cdot M_{length}}{\rho\cdot b^3}}$ $\quad q = 0.0432\cdot\frac{\frac{m^3}{s}}{m^2}$

The maximum speed $\quad U_{max} = u\left(x = \frac{b}{2}\right) = q\cdot\frac{\frac{b}{2}}{h}$ $\quad U_{max} = \frac{b\cdot q}{2\cdot h}$ $\quad U_{max} = 1.73\frac{m}{s}$

The following plot can be done in *Excel* $\quad p_g(x) = \frac{\rho\cdot q^2\cdot b^2}{8\cdot h^2}\cdot\left[1 - 4\cdot\left(\frac{x}{b}\right)^2\right]$

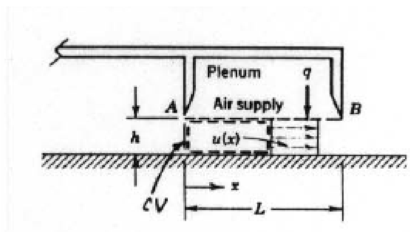


The net force is such that the chip is floating on air due to a Bernoulli effect: the speed is maximum at the edges and zero at the center; pressure has the opposite trend - pressure is minimum (p_{atm}) at the edges and maximum at the center.

Problem 6.21

(Difficulty: 3)

6.21 Heavy weights can be moved with relative ease on air cushions by using a load pallet as shown. Air is supplied from the plenum through porous surface AB. It enters the gap vertically at uniform speed, q. Once in the gap, all air flows in the positive x direction (there is no flow across the plane at $x = 0$). Assume air flow in the gap is incompressible and uniform at each cross section, with speed $u(x)$, as shown in the enlarged view. Although the gap is narrow ($h \ll L$), neglect frictional effects as a first approximation. Use a suitably chosen control volume to show that $u(x) = qx/h$ in the gap. Calculate the acceleration of a fluid particle in the gap. Evaluate the pressure gradient, $\partial p/\partial x$, and sketch the pressure distribution within the gap. Be sure to indicate the pressure at $x = L$.



Assumptions: (1) steady flow (2) incompressible flow (3) uniform flow at each section (4) no variation with z (5) horizontal, so $g_x = 0$

Solution: Use the continuity and Euler equations

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho d\forall + \int_{CS} \rho \bar{V} \cdot d\bar{A}$$

$$a_{px} = \frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \omega\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

$$-\frac{\partial p}{\partial x} + \rho g_x = \rho a_x$$

Choose a CV that extends upward from the surface in the gap, from 0 to h, as shown.

From the continuity equation,

$$0 = \{-|\rho q w x|\} + \{|\rho u(x) w h|\}$$

$$u(x) = \frac{qx}{h}$$

The acceleration is:

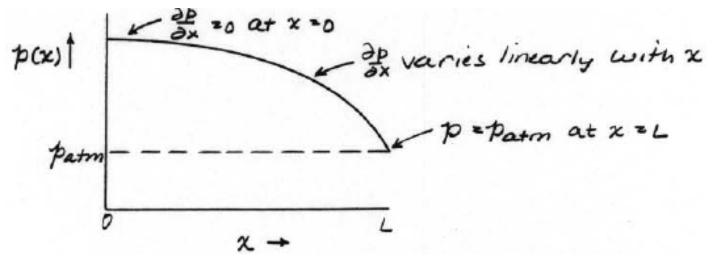
$$a_{px} = \frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \omega\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x}$$

$$a_{px} = \frac{qx}{h}\frac{q}{h} = \frac{q^2x}{h^2}$$

The pressure gradient is:

$$\frac{\partial p}{\partial x} = -\rho a_x = -\frac{\rho q^2 x}{h^2}$$

Then we have for pressure as a function of x:



Problem 6.22

(Difficulty: 4)

6.22 The y component of velocity in a two-dimensional incompressible flow field is given by $v = -Axy$, where v is in m/s, the coordinates are measured in meters, and $A = 1\ m^{-1} \cdot s^{-1}$. There is no velocity component or variation in the z direction. Calculate the acceleration of a fluid particle at point $(x, y) = (1,2)$. Estimate the radius of curvature of the streamline passing through this point. Plot the streamline and show both the velocity vector and the acceleration vector on the plot. (Assume the simplest form of the x component of velocity.)

Solution: Use the continuity equation and the Euler relations for ideal 2-dimensional incompressible flow. The continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Or

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$$

Evaluating the velocity

$$u = \int \frac{\partial u}{\partial x} dx + f(y) = \int -\frac{\partial v}{\partial y} dx + f(y) = -\int (-Ax) dx + f(y) = \frac{Ax^2}{2} + f(y)$$

Choose the simplest solution for f(y)

$$f(y) = 0$$

Then the velocity is

$$u = \frac{Ax^2}{2}$$

Hence

$$\bar{V} = \frac{Ax^2}{2}\hat{\imath} - Axy\hat{\jmath} = A\left(\frac{x^2}{2}\hat{\imath} - xy\hat{\jmath}\right)$$

The acceleration of a fluid particle is given by:

$$\bar{a}_p = u\frac{\partial \bar{V}}{\partial x} + v\frac{\partial \bar{V}}{\partial y} = \frac{Ax^2}{2}(Ax\hat{\imath} - Ay\hat{\jmath}) - Axy(-Ax\hat{\jmath})$$

$$\bar{a}_p = \frac{A^2x^3}{2}\hat{\imath} + \frac{A^2x^2y}{2}\hat{\jmath} = \frac{A^2}{2}(x^3\hat{\imath} + x^2y\hat{\jmath})$$

At the point (1,2)

$$\bar{a}_p = \frac{1}{2} \times \frac{1}{m^2s^2} \times [(1\ m)^3\hat{\imath} + (1\ m)^2(2\ m)\hat{\jmath}] = 0.5\hat{\imath} + \hat{\jmath}\ \frac{m}{s^2}$$

$$\bar{V} = \frac{1}{m \cdot s}\left[\frac{1}{2}(1\ m)^2\hat{\imath} - (1\ m)(2\ m)\hat{\jmath}\right] = 0.5\hat{\imath} - 2\hat{\jmath}\ \frac{m}{s}$$

The unit vector tangent to the streamline is:

$$\hat{e}_t = \frac{\bar{V}}{|\bar{V}|} = \frac{0.5\hat{\imath} - 2\hat{\jmath}}{\sqrt{(0.5)^2 + (-2)^2}} = 0.243\hat{\imath} - 0.970\hat{\jmath}$$

The unit vector normal to the streamline is:

$$\hat{e}_n = \hat{e}_t \times \hat{k} = (0.243\hat{\imath} - 0.970\hat{\jmath}) \times \hat{k} = -0.970\hat{\imath} - 0.243\hat{\jmath}$$

The normal component of the acceleration is:

$$a_n = -\frac{V^2}{R} = \bar{a} \cdot \hat{e}_n = (0.5\hat{\imath} + \hat{\jmath}\ ) \cdot (-0.970\hat{\imath} - 0.243\hat{\jmath})$$

$$-\frac{V^2}{R} = -0.728\ \frac{m}{s^2}$$

$$R = \frac{V^2}{0.728\ \frac{m}{s^2}} = \frac{4.25\ \frac{m^2}{s^2}}{0.728\ \frac{m}{s^2}} = 5.84\ m$$

The slope of the streamlines is given by:

$$\frac{dy}{dx} = \frac{v}{u} = \frac{-Axy}{\frac{Ax^2}{2}} = -\frac{2y}{x}$$

Thus

$$\frac{dy}{y} + 2\frac{dx}{x} = 0$$

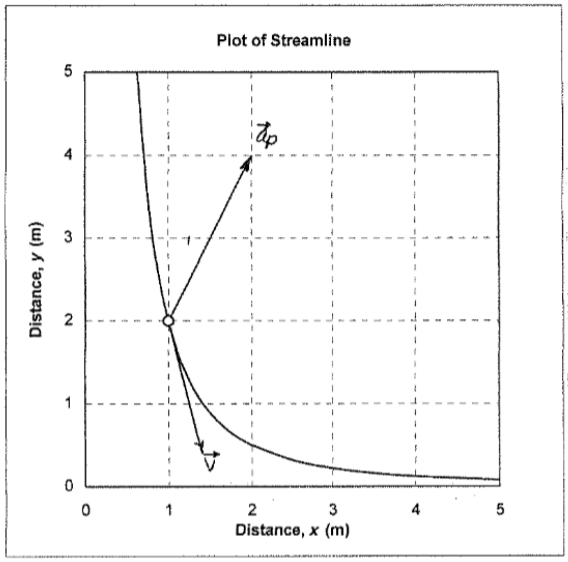
and

$$\ln y + \ln x^2 = \ln c$$

$$x^2y = c$$

The equation of the streamline through (1,2) is $x^2y = 2$.

A plot is given below



Problem 6.23 [Difficulty: 2]

6.23 The velocity field for a plane doublet is given in Table 6.2. Find an expression for the pressure gradient at any point (r, θ).

Given: Velocity field for doublet

Find: Expression for pressure gradient

Solution:

Basic equations

$$\rho a_r = \rho\left(\frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z\frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r}\right) = \rho g_r - \frac{\partial p}{\partial r}$$

$$\rho a_\theta = \rho\left(\frac{\partial V_\theta}{\partial t} + V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_z\frac{\partial V_\theta}{\partial z} + \frac{V_r V_\theta}{r}\right) = \rho g_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta}$$

For this flow $\quad V_r(r,\theta) = -\frac{\Lambda}{r^2}\cdot\cos(\theta) \qquad V_\theta(r,\theta) = -\frac{\Lambda}{r^2}\cdot\sin(\theta) \qquad V_z = 0$

Hence for r momentum $\quad \rho\cdot g_r - \frac{\partial}{\partial r}p = \rho\cdot\left(V_r\cdot\frac{\partial}{\partial r}V_r + \frac{V_\theta}{r}\cdot\frac{\partial}{\partial \theta}V_r - \frac{V_\theta^2}{r}\right)$

Ignoring gravity

$$\frac{\partial}{\partial r}p = -\rho\cdot\left[\left(-\frac{\Lambda}{r^2}\cdot\cos(\theta)\right)\cdot\frac{\partial}{\partial r}\left(-\frac{\Lambda}{r^2}\cdot\cos(\theta)\right) + \frac{\left(-\frac{\Lambda}{r^2}\cdot\sin(\theta)\right)}{r}\cdot\frac{\partial}{\partial \theta}\left(-\frac{\Lambda}{r^2}\cdot\cos(\theta)\right) - \frac{\left(-\frac{\Lambda}{r^2}\cdot\sin(\theta)\right)^2}{r}\right] \qquad \frac{\partial}{\partial r}p = \frac{2\cdot\Lambda^2\cdot\rho}{r^5}$$

For θ momentum $\quad \rho\cdot g_\theta - \frac{1}{r}\cdot\frac{\partial}{\partial \theta}p = \rho\cdot\left(V_r\cdot\frac{\partial}{\partial r}V_\theta + \frac{V_\theta}{r}\cdot\frac{\partial}{\partial \theta}V_\theta + \frac{V_r\cdot V_\theta}{r}\right)$

Ignoring gravity

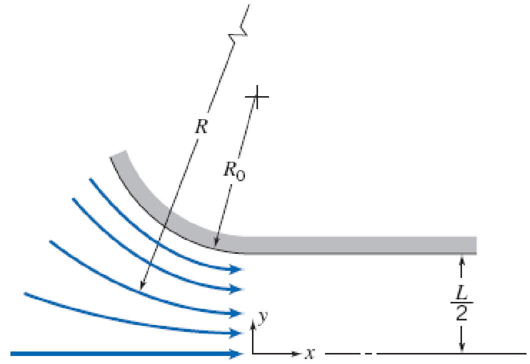
$$\frac{\partial}{\partial \theta}p = -r\cdot\rho\cdot\left[\left(-\frac{\Lambda}{r^2}\cdot\cos(\theta)\right)\cdot\frac{\partial}{\partial r}\left(-\frac{\Lambda}{r^2}\cdot\sin(\theta)\right) + \frac{\left(-\frac{\Lambda}{r^2}\cdot\sin(\theta)\right)}{r}\cdot\frac{\partial}{\partial \theta}\left(-\frac{\Lambda}{r^2}\cdot\sin(\theta)\right) + \frac{\left(-\frac{\Lambda}{r^2}\cdot\sin(\theta)\right)\cdot\left(-\frac{\Lambda}{r^2}\cdot\cos(\theta)\right)}{r}\right] \qquad \frac{\partial}{\partial \theta}p = 0$$

The pressure gradient is purely radial

Problem 6.24

[Difficulty: 2]

6.24 To model the velocity distribution in the curved inlet section of a water channel, the radius of curvature of the streamlines is expressed as $R = LR_0/2y$. As an approximation, assume the water speed along each streamline is $V = 10$ m/s. Find an expression for and plot the pressure distribution from $y = 0$ to the tunnel wall at $y = L/2$, if the centerline pressure (gage) is 50 kPa, $L = 75$ mm, and $R_0 = 0.2$ m. Find the value of V for which the wall static pressure becomes 35 kPa.



Given: Flow in a curved section

Find: Expression for pressure distribution; plot; V for wall static pressure of 35 kPa

Solution:

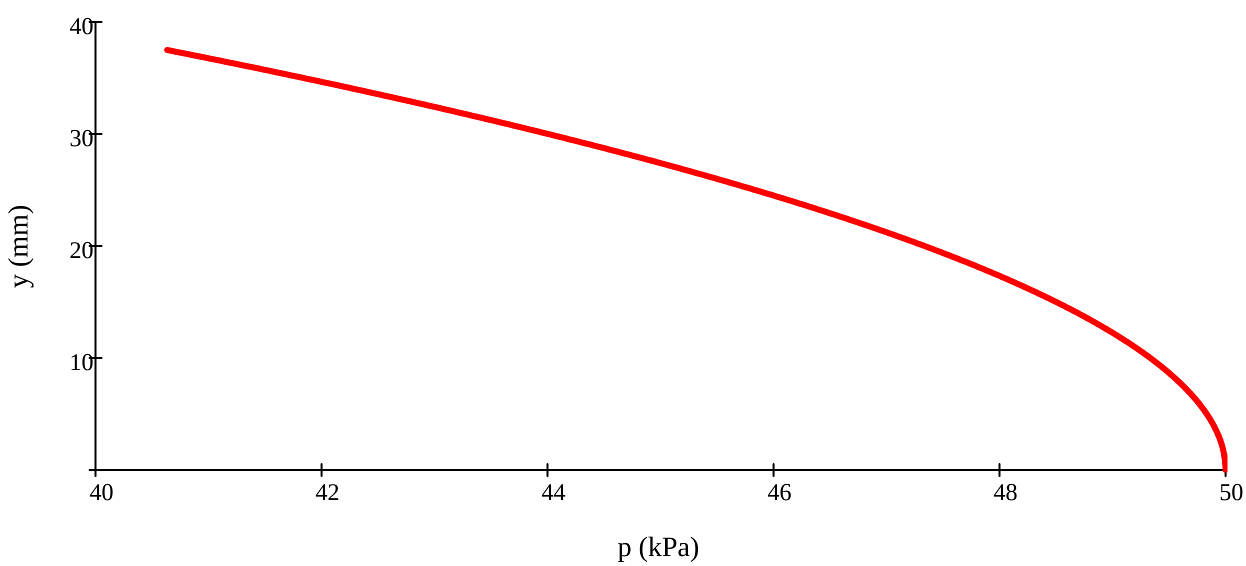
Basic equation $\quad \frac{\partial}{\partial n} p = \rho \cdot \frac{V^2}{R}$

Assumptions: Steady; frictionless; no body force; constant speed along streamline

Given data $\quad \rho = 999 \cdot \frac{kg}{m^3} \quad V = 10 \cdot \frac{m}{s} \quad L = 75 \cdot mm \quad R_0 = 0.2 \cdot m \quad p_c = 50 \cdot kPa$

At the inlet section $\quad p = p(y) \quad$ hence $\quad \frac{\partial}{\partial n} p = -\frac{dp}{dy} = \rho \cdot \frac{V^2}{R} = \rho \cdot V^2 \cdot \frac{2 \cdot y}{L \cdot R_0} \quad dp = -\rho \cdot V^2 \cdot \frac{2 \cdot y}{L \cdot R_0} \cdot dy$

Integrating from y = 0 to y = y $\quad p(y) = p_c - \frac{\rho \cdot V^2}{R_0 \cdot L} \cdot y^2 \quad (1) \quad p(0) = 50 \cdot kPa \quad p\left(\frac{L}{2}\right) = 40.6 \cdot kPa$



For a new wall pressure $\quad p_{wall} = 35 \cdot kPa \quad$ solving Eq 1 for V gives $\quad V = \sqrt{\frac{4 \cdot R_0 \cdot (p_c - p_{wall})}{\rho \cdot L}} \quad V = 12.7 \frac{m}{s}$

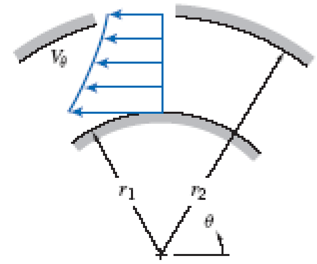
Problem 6.25

[Difficulty: 3]

6.25 Repeat Example 6.1, but with the somewhat more realistic assumption that the flow is similar to a free vortex (irrotational) profile, $V_\theta = c/r$ (where c is a constant), as shown in Fig. P6.25 In doing so, prove that the flow rate is given by $Q = k\sqrt{\Delta p}$, where k is

$$k = w \ln\left(\frac{r_2}{r_1}\right)\sqrt{\frac{2r_2^2 r_1^2}{\rho(r_2^2 - r_1^2)}}$$

and w is the depth of the bend.



Given: Velocity field for free vortex flow in elbow

Find: Similar solution to Example 6.1; find k (above)

Solution:

Basic equation $\quad \frac{\partial}{\partial r}p = \frac{\rho \cdot V^2}{r} \quad$ with $\quad V = V_\theta = \frac{c}{r}$

Assumptions: 1) Frictionless 2) Incompressible 3) free vortex

For this flow $\quad p \neq p(\theta) \quad$ so $\quad \frac{\partial}{\partial r}p = \frac{d}{dr}p = \frac{\rho \cdot V^2}{r} = \frac{\rho \cdot c^2}{r^3}$

Hence
$$\Delta p = p_2 - p_1 = \int_{r_1}^{r_2} \frac{\rho \cdot c^2}{r^3}\, dr = \frac{\rho \cdot c^2}{2} \cdot \left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) = \frac{\rho \cdot c^2 \cdot \left(r_2^2 - r_1^2\right)}{2 \cdot r_1^2 \cdot r_2^2} \qquad (1)$$

Next we obtain c in terms of Q

$$Q = \int \vec{V}\, d\vec{A} = \int_{r_1}^{r_2} V \cdot w\, dr = \int_{r_1}^{r_2} \frac{w \cdot c}{r}\, dr = w \cdot c \cdot \ln\left(\frac{r_2}{r_1}\right)$$

Hence
$$c = \frac{Q}{w \cdot \ln\left(\frac{r_2}{r_1}\right)}$$

Using this in Eq 1
$$\Delta p = p_2 - p_1 = \frac{\rho \cdot c^2 \cdot \left(r_2^2 - r_1^2\right)}{2 \cdot r_1^2 \cdot r_2^2} = \frac{\rho \cdot Q^2 \cdot \left(r_2^2 - r_1^2\right)}{2 \cdot w^2 \cdot \ln\left(\frac{r_2}{r_1}\right)^2 \cdot r_1^2 \cdot r_2^2}$$

Solving for Q
$$Q = w \cdot \ln\left(\frac{r_2}{r_1}\right) \cdot \sqrt{\frac{2 \cdot r_1^2 \cdot r_2^2}{\rho \cdot \left(r_2^2 - r_1^2\right)}} \cdot \sqrt{\Delta p} \qquad k = w \cdot \ln\left(\frac{r_2}{r_1}\right) \cdot \sqrt{\frac{2 \cdot r_1^2 \cdot r_2^2}{\rho \cdot \left(r_2^2 - r_1^2\right)}}$$

Problem 6.26 [Difficulty: 2]

6.26 Using the analyses of Example 6.1 and Problem 6.25, plot the discrepancy (percent) between the flow rates obtained from assuming uniform flow and the free vortex (irrotational) profile as a function of inner radius r2/r1

Given: Flow rates in elbow for uniform flow and free vortes

Find: Plot discrepancy

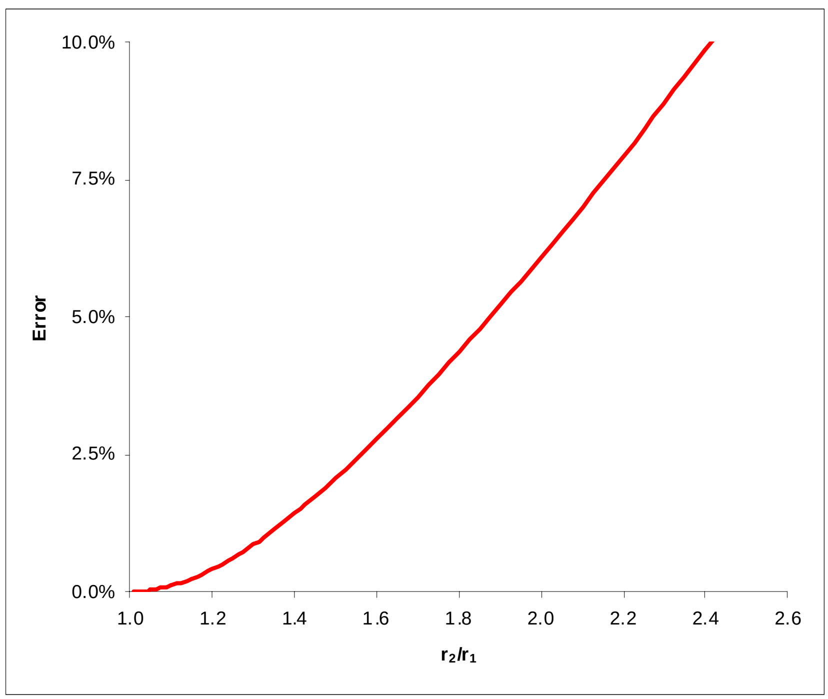
Solution:

For Example 6.1 $$Q_{Uniform} = V\cdot A = w\cdot(r_2 - r_1)\cdot\sqrt{\frac{1}{\rho\cdot\ln\left(\frac{r_2}{r_1}\right)}}\cdot\sqrt{\Delta p} \quad \text{or} \quad \frac{Q_{Uniform}\cdot\sqrt{\rho}}{w\cdot r_1\cdot\sqrt{\Delta p}} = \frac{\left(\frac{r_2}{r_1} - 1\right)}{\sqrt{\ln\left(\frac{r_2}{r_1}\right)}} \qquad (1)$$

For Problem 6.25 $$Q = w\cdot\ln\left(\frac{r_2}{r_1}\right)\cdot\sqrt{\frac{2\cdot r_1^{\,2}\cdot r_2^{\,2}}{\rho\cdot\left(r_2^{\,2} - r_1^{\,2}\right)}}\cdot\sqrt{\Delta p} \quad \text{or} \quad \frac{Q\cdot\sqrt{\rho}}{w\cdot r_1\cdot\sqrt{\Delta p}} = \left(\frac{r_2}{r_1}\right)\cdot\ln\left(\frac{r_2}{r_1}\right)\cdot\sqrt{\frac{2}{\left[\left(\frac{r_2}{r_1}\right)^2 - 1\right]}} \qquad (2)$$

It is convenient to plot these as functions of r_2/r_1

r_2/r_1	Eq. 1	Eq. 2	Error
1.01	0.100	0.100	0.0%
1.05	0.226	0.226	0.0%
1.10	0.324	0.324	0.1%
1.15	0.401	0.400	0.2%
1.20	0.468	0.466	0.4%
1.25	0.529	0.526	0.6%
1.30	0.586	0.581	0.9%
1.35	0.639	0.632	1.1%
1.40	0.690	0.680	1.4%
1.45	0.738	0.726	1.7%
1.50	0.785	0.769	2.1%
1.55	0.831	0.811	2.4%
1.60	0.875	0.851	2.8%
1.65	0.919	0.890	3.2%
1.70	0.961	0.928	3.6%
1.75	1.003	0.964	4.0%
1.80	1.043	1.000	4.4%
1.85	1.084	1.034	4.8%
1.90	1.123	1.068	5.2%
1.95	1.162	1.100	5.7%
2.00	1.201	1.132	6.1%
2.05	1.239	1.163	6.6%
2.10	1.277	1.193	7.0%
2.15	1.314	1.223	7.5%
2.20	1.351	1.252	8.0%
2.25	1.388	1.280	8.4%
2.30	1.424	1.308	8.9%
2.35	1.460	1.335	9.4%
2.40	1.496	1.362	9.9%
2.45	1.532	1.388	10.3%
2.50	1.567	1.414	10.8%



Problem 6.27 [Difficulty: 4]

6.27 The x component of velocity in a two-dimensional incompressible flow field is given by $u = -\Lambda(x^2 - y^2)/(x^2 + y^2)^2$, where u is in m/s, the coordinates are measured in meters, and $\Lambda = 2\ \mathrm{m^3 \cdot s^{-1}}$. Show that the simplest form of the y component of velocity is given by $v = -2\Lambda xy/(x^2 + y^2)^2$. There is no velocity component or variation in the z direction. Calculate the acceleration of fluid particles at points $(x, y) = (0, 1)$, $(0, 2)$, and $(0, 3)$. Estimate the radius of curvature of the streamlines passing through these points. What does the relation among the three points and their radii of curvature suggest to you about the flow field? Verify this by plotting these streamlines. [*Hint*: You will need to use an integrating factor.]

Given: x component of velocity field

Find: y component of velocity field; acceleration at several points; estimate radius of curvature; plot streamlines

Solution:

The given data is $\Lambda = 2 \cdot \frac{m^3}{s}$ $\quad u = -\frac{\Lambda \cdot (x^2 - y^2)}{(x^2 + y^2)^2}$

The basic equation (continuity) is $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$

The basic equation for acceleration is

$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\substack{\text{total}\\ \text{acceleration}\\ \text{of a particle}}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\ \text{acceleration}}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\substack{\text{local}\\ \text{acceleration}}}$$

Hence $$v = -\int \frac{du}{dx}\,dy = -\int \frac{2 \cdot \Lambda \cdot x \cdot (x^2 - 3 \cdot y^2)}{(x^2 + y^2)^3}\,dy$$

Integrating (using an integrating factor) $$v = -\frac{2 \cdot \Lambda \cdot x \cdot y}{(x^2 + y^2)^2}$$

Alternatively, we could check that the given velocities u and v satisfy continuity

$$u = -\frac{\Lambda \cdot (x^2 - y^2)}{(x^2 + y^2)^2} \qquad \frac{\partial}{\partial x}u = \frac{2 \cdot \Lambda \cdot x \cdot (x^2 - 3 \cdot y^2)}{(x^2 + y^2)^3} \qquad v = -\frac{2 \cdot \Lambda \cdot x \cdot y}{(x^2 + y^2)^2} \qquad \frac{\partial}{\partial y}v = -\frac{2 \cdot \Lambda \cdot x \cdot (x^2 - 3 \cdot y^2)}{(x^2 + y^2)^3}$$

so $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$

For steady, 2D flow the acceleration components reduce to (after considerable math!):

x - component

$$a_x = u\cdot\frac{\partial}{\partial x}u + v\cdot\frac{\partial}{\partial y}u$$

$$a_x = \left[-\frac{\Lambda\cdot\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}\right]\cdot\left[\frac{2\cdot\Lambda\cdot x\cdot\left(x^2-3\cdot y^2\right)}{\left(x^2+y^2\right)^3}\right] + \left[-\frac{2\cdot\Lambda\cdot x\cdot y}{\left(x^2+y^2\right)^2}\right]\cdot\left[\frac{2\cdot\Lambda\cdot y\cdot\left(3\cdot x^2-y^2\right)}{\left(x^2+y^2\right)^3}\right] \quad a_x = -\frac{2\cdot\Lambda^2\cdot x}{\left(x^2+y^2\right)^3}$$

y - component

$$a_y = u\cdot\frac{\partial}{\partial x}v + v\cdot\frac{\partial}{\partial y}v$$

$$a_y = \left[-\frac{\Lambda\cdot\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}\right]\cdot\left[\frac{2\cdot\Lambda\cdot y\cdot\left(3\cdot x^2-y^2\right)}{\left(x^2+y^2\right)^3}\right] + \left[-\frac{2\cdot\Lambda\cdot x\cdot y}{\left(x^2+y^2\right)^2}\right]\cdot\left[\frac{2\cdot\Lambda\cdot y\cdot\left(3\cdot y^2-x^2\right)}{\left(x^2+y^2\right)^3}\right] \quad a_y = -\frac{2\cdot\Lambda^2\cdot y}{\left(x^2+y^2\right)^3}$$

Evaluating at point (0,1) $\quad u = 2\cdot\frac{m}{s} \quad v = 0\cdot\frac{m}{s} \quad a_x = 0\cdot\frac{m}{s^2} \quad a_y = -8\cdot\frac{m}{s^2}$

Evaluating at point (0,2) $\quad u = 0.5\cdot\frac{m}{s} \quad v = 0\cdot\frac{m}{s} \quad a_x = 0\cdot\frac{m}{s^2} \quad a_y = -0.25\cdot\frac{m}{s^2}$

Evaluating at point (0,3) $\quad u = 0.222\cdot\frac{m}{s} \quad v = 0\cdot\frac{m}{s} \quad a_x = 0\cdot\frac{m}{s^2} \quad a_y = -0.0333\cdot\frac{m}{s^2}$

The instantaneous radius of curvature is obtained from $\quad a_{radial} = -a_y = -\frac{u^2}{r} \quad$ or $\quad r = -\frac{u^2}{a_y}$

For the three points

$$y = 1\,m \qquad r = \frac{\left(2\cdot\frac{m}{s}\right)^2}{8\cdot\frac{m}{s^2}} \qquad r = 0.5\,m$$

$$y = 2\,m \qquad r = \frac{\left(0.5\cdot\frac{m}{s}\right)^2}{0.25\cdot\frac{m}{s^2}} \qquad r = 1\,m$$

$$y = 3\,m \qquad r = \frac{\left(0.2222\cdot\frac{m}{s}\right)^2}{0.03333\cdot\frac{m}{s^2}} \qquad r = 1.5\cdot m$$

The radius of curvature in each case is 1/2 of the vertical distance from the origin. The streamlines form circles tangent to the *x* axis

The streamlines are given by

$$\frac{dy}{dx} = \frac{v}{u} = \frac{-\dfrac{2\cdot\Lambda\cdot x\cdot y}{\left(x^2+y^2\right)^2}}{-\dfrac{\Lambda\cdot\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}} = \frac{2\cdot x\cdot y}{\left(x^2-y^2\right)}$$

so

$$-2\cdot x\cdot y\cdot dx + \left(x^2-y^2\right)\cdot dy = 0$$

This is an inexact integral, so an integrating factor is needed

First we try

$$R = \frac{1}{-2\cdot x\cdot y}\cdot\left[\frac{d}{dx}\left(x^2-y^2\right) - \frac{d}{dy}(-2\cdot x\cdot y)\right] = -\frac{2}{y}$$

Then the integrating factor is

$$F = e^{\int -\frac{2}{y}\,dy} = \frac{1}{y^2}$$

The equation becomes an exact integral

$$-2\cdot\frac{x}{y}\cdot dx + \frac{\left(x^2-y^2\right)}{y^2}\cdot dy = 0$$

So

$$u = \int -2\cdot\frac{x}{y}\,dx = -\frac{x^2}{y} + f(y) \quad \text{and} \quad u = \int \frac{\left(x^2-y^2\right)}{y^2}\,dy = -\frac{x^2}{y} - y + g(x)$$

Comparing solutions

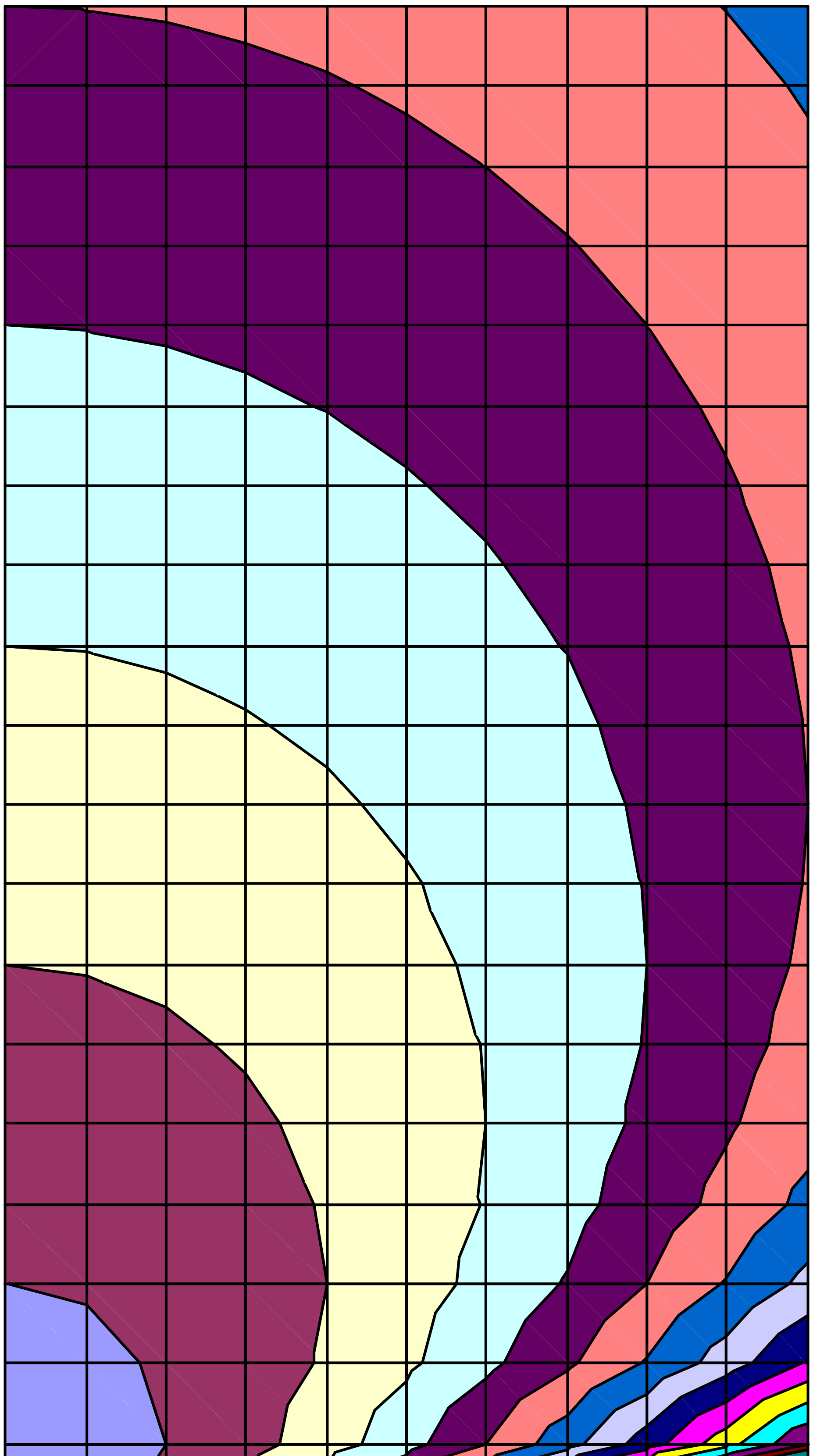
$$\psi = \frac{x^2}{y} + y \qquad (1) \qquad \text{or} \qquad x^2 + y^2 = \psi\cdot y = \text{const}\cdot y$$

These form circles that are tangential to the *x* axis, as can be shown in *Excel*:

The stream function can be evaluated using Eq 1

	y values																				
x values	**0.10**	**0.25**	**0.50**	**0.75**	**1.00**	**1.25**	**1.50**	**1.75**	**2.00**	**2.25**	**2.50**	**2.75**	**3.00**	**3.25**	**3.50**	**3.75**	**4.00**	**4.25**	**4.50**	**4.75**	**5.00**
2.50	62.6	25.3	13.0	9.08	7.25	6.25	5.67	5.32	5.13	5.03	5.00	5.02	5.08	5.17	5.29	5.42	5.56	5.72	5.89	6.07	6.25
2.25	50.7	20.5	10.6	7.50	6.06	5.30	4.88	4.64	4.53	4.50	4.53	4.59	4.69	4.81	4.95	5.10	5.27	5.44	5.63	5.82	6.01
2.00	40.1	16.3	8.50	6.08	5.00	4.45	4.17	4.04	4.00	4.03	4.10	4.20	4.33	4.48	4.64	4.82	5.00	5.19	5.39	5.59	5.80
1.75	30.7	12.5	6.63	4.83	4.06	3.70	3.54	3.50	3.53	3.61	3.73	3.86	4.02	4.19	4.38	4.57	4.77	4.97	5.18	5.39	5.61
1.50	22.6	9.25	5.00	3.75	3.25	3.05	3.00	3.04	3.13	3.25	3.40	3.57	3.75	3.94	4.14	4.35	4.56	4.78	5.00	5.22	5.45
1.25	15.7	6.50	3.63	2.83	2.56	2.50	2.54	2.64	2.78	2.94	3.13	3.32	3.52	3.73	3.95	4.17	4.39	4.62	4.85	5.08	5.31
1.00	10.1	4.25	2.50	2.08	2.00	2.05	2.17	2.32	2.50	2.69	2.90	3.11	3.33	3.56	3.79	4.02	4.25	4.49	4.72	4.96	5.20
0.75	5.73	2.50	1.63	1.50	1.56	1.70	1.88	2.07	2.28	2.50	2.73	2.95	3.19	3.42	3.66	3.90	4.14	4.38	4.63	4.87	5.11
0.50	2.60	1.25	1.00	1.08	1.25	1.45	1.67	1.89	2.13	2.36	2.60	2.84	3.08	3.33	3.57	3.82	4.06	4.31	4.56	4.80	5.05
0.25	0.73	0.50	0.63	0.83	1.06	1.30	1.54	1.79	2.03	2.28	2.53	2.77	3.02	3.27	3.52	3.77	4.02	4.26	4.51	4.76	5.01
0.00	0.10	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00	3.25	3.50	3.75	4.00	4.25	4.50	4.75	5.00

See next page for plot:



Problem 6.28

[Difficulty: 1]

6.28 Water flows at a speed of 25 ft/s. Calculate the dynamic pressure of this flow. Express your answer in inches of mercury.

Given: Water at speed 25 ft/s

Find: Dynamic pressure in in. Hg

Solution:

Basic equations $$p_{dynamic} = \frac{1}{2}\cdot\rho\cdot V^2 \qquad p = \rho_{Hg}\cdot g\cdot\Delta h = SG_{Hg}\cdot\rho\cdot g\cdot\Delta h$$

Hence $$\Delta h = \frac{\rho\cdot V^2}{2\cdot SG_{Hg}\cdot\rho\cdot g} = \frac{V^2}{2\cdot SG_{Hg}\cdot g}$$

$$\Delta h = \frac{1}{2}\times\left(25\cdot\frac{ft}{s}\right)^2\times\frac{1}{13.6}\times\frac{s^2}{32.2\cdot ft}\times\frac{12\cdot in}{1\cdot ft} \qquad \Delta h = 8.56\,in$$

Problem 6.29

[Difficulty: 2]

6.29 Plot the speed of air versus the dynamic pressure (in millimeters of mercury), up to a dynamic pressure of 250 mm Hg.

Given: Air speed

Find: Plot dynamic pressure in mm Hg

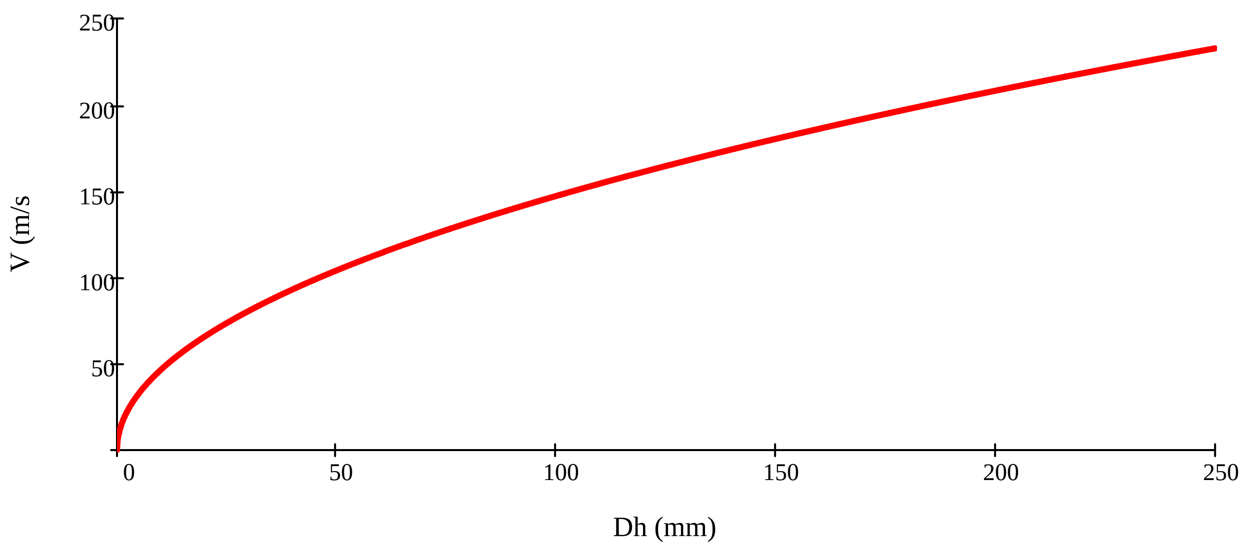
Solution:

Basic equations $p_{dynamic} = \frac{1}{2}\cdot\rho_{air}\cdot V^2$ $\quad p = \rho_{Hg}\cdot g\cdot\Delta h = SG_{Hg}\cdot\rho_w\cdot g\cdot\Delta h$

Available data $\rho_w = 999\cdot\frac{kg}{m^3}$ $\quad \rho_{air} = 1.23\cdot\frac{kg}{m^3}$ $\quad SG_{Hg} = 13.6$

Hence $\frac{1}{2}\cdot\rho_{air}\cdot V^2 = SG_{Hg}\cdot\rho_w\cdot g\cdot\Delta h$

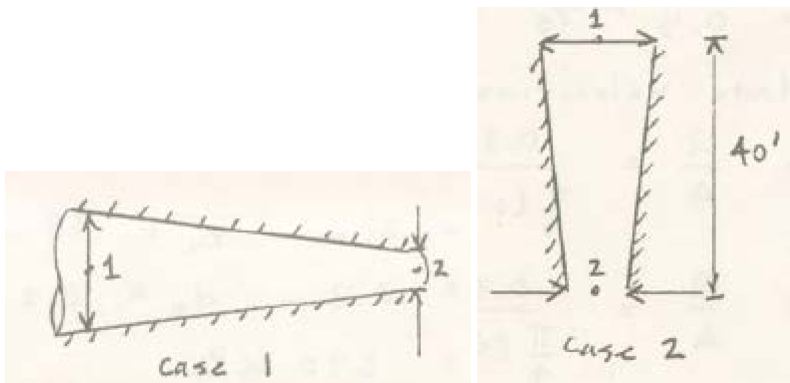
Solving for V $V(\Delta h) = \sqrt{\frac{2\cdot SG_{Hg}\cdot\rho_w\cdot g\cdot\Delta h}{\rho_{air}}}$



Problem 6.30

(Difficulty 2)

6.30 Water flows in a pipeline. At a point in the line where the diameter is $7\ in$, the velocity is $12\ \frac{ft}{s}$ and the pressure is $50\ psi$. At a point $40\ ft$ away the diameter reduces to $3\ in$. Calculate the pressure here when the pipe is (a) horizontal, and (b) vertical with flow downward, and (c) vertical with the flow upward. Explain why there is a difference in the pressure for the different situations.



Find: Calculate the pressure p_2.

Assumption: The flow is steady, incompressible, uniform, and frictionless.

Solution: Apply the continuity and Bernoulli equations to find the pressure

The continuity equation is:

$$Q = V_1 A_1 = V_2 A_2$$

The Bernoulli equation along a streamline is

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = constant$$

The velocity is the same for each orientation and is calculated using the continuity equation. The flow areas are:

$$A_1 = \frac{\pi D_1^2}{4}$$

$$A_2 = \frac{\pi D_2^2}{4}$$

The velocity V_2 is

$$V_2 = \frac{V_1 A_1}{A_2} = \frac{V_1 D_1^2}{D_2^2} = \frac{12\ \frac{ft}{s} \times \left(\frac{7}{12}\ ft\right)^2}{\left(\frac{3}{12}\ ft\right)^2} = 65.3\ \frac{ft}{s}$$

(a) For the horizontal pipe, the height cancels out. The Bernoulli equation along the center streamline is:

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} = \frac{p_2}{\rho} + \frac{V_2^2}{2}$$

The inlet pressure is

$$p_1 = 50\,psi = 7200\ \frac{lbf}{ft^2}$$

$$\rho = 1.938\ \frac{slug}{ft^3} = 1.938\ \frac{lbf \cdot s^2}{ft^4}$$

So the pressure is computed as:

$$p_2 = p_1 + \frac{\rho}{2}(V_1^2 - V_2^2)$$

$$p_2 = 7200\ \frac{lbf}{ft^2} + \frac{1.938\ \frac{lbf \cdot s^2}{ft^4}}{2} \times \left(\left(12\ \frac{ft}{s}\right)^2 - \left(65.3\ \frac{ft}{s}\right)^2\right) = 3208\ \frac{lbf}{ft^2} = 22.3\,psi$$

(b) For the vertical pipe with the flow downward, we have for the Bernoulli equation:

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

$$p_2 = p_1 + \frac{\rho}{2}(V_1^2 - V_2^2) + \rho g(z_1 - z_2)$$

Using the same values for velocity and pressure p_1 as for the horizontal situation, we have the additional pressure due to the height difference

$$p_2 = 3208\ \frac{lbf}{ft^2} + \rho g(z_1 - z_2) = 3208\ \frac{lbf}{ft^2} + 1.938\ \frac{lbf \cdot s^2}{ft^4} \times 32.2\ \frac{ft}{s^2} \times (40 - 0)ft$$

$$p_2 = 5704\ \frac{lbf}{ft^2} = 39.6\,psi$$

(c) For the vertical pipe with the flow upward, we have for the Bernoulli equation, where the flow is now from 2 to 1:

$$\frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 = \frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1$$

The pressure at the small end (2) is then

$$p_2 = p_1 + \frac{\rho}{2}(V_1^2 - V_2^2) + \rho g(z_1 - z_2)$$

Using the same values for velocity and pressure p_1 as for the horizontal situation, we have

$$p_2 = 3208\ \frac{lbf}{ft^2} + \rho g(z_1 - z_2) = 3208\ \frac{lbf}{ft^2} + 1.938\ \frac{lbf \cdot s^2}{ft^4} \times 32.2\ \frac{ft}{s^2} \times (40 - 0) ft$$

$$p_2 = 5704\ \frac{lbf}{ft^2} = 39.6\, psi$$

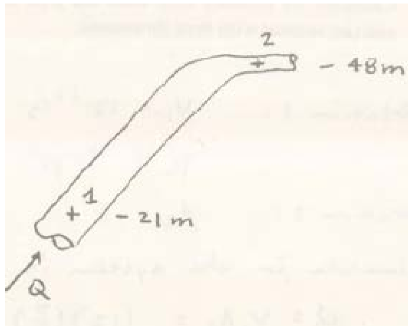
which is the same as for the flow downward.

The pressures for the vertical orientation are greater than for the horizontal orientation due to the hydrostatic pressure. The pressures for the vertical directions are the same since the hydrostatic pressure difference is the same regardless of flow direction.

Problem 6.31

(Difficulty 2)

6.31 In a pipe $0.3\ m$ in diameter, $0.3\ \frac{m^3}{s}$ of water are pumped up a hill. On the hilltop (elevation 48), the line reduces to $0.2\ m$ diameter. If the pump maintains a pressure of $690\ kPa$ at elevation 21, calculate the pressure in the pipe on the hilltop.



Find: Calculate the pressure p_2.

Assumption: The flow is steady, incompressible, uniform, and frictionless.

Solution: Apply the continuity and Bernoulli equations to find the pressure

The continuity equation is:

$$Q = V_1 A_1 = V_2 A_2$$

The Bernoulli equation along a streamline is

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = constant$$

The velocity can be calculated using the continuity equation:

$$A_1 = \frac{\pi D_1^2}{4}$$

$$A_2 = \frac{\pi D_2^2}{4}$$

The velocity at location 1 is

$$V_1 = \frac{Q}{A_1} = \frac{0.3\ \frac{m^3}{s}}{\frac{\pi \times (0.3\ m)^2}{4}} = 4.24\ \frac{m}{s}$$

And at location 2

$$V_2 = \frac{Q}{A_2} = \frac{0.3\ \frac{m^3}{s}}{\frac{\pi \times (0.2\ m)^2}{4}} = 9.55\ \frac{m}{s}$$

Applying the Bernoulli equation from inlet and outlet, we have:

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

Or the pressure is

$$p_2 = p_1 + \frac{\rho}{2}(V_1^2 - V_2^2) + \rho g(z_1 - z_2)$$

$$p_2 = 690\ kPa + \frac{998\ \frac{kg}{m^3}}{2} \times \left(\left(4.24\ \frac{m}{s}\right)^2 - \left(9.55\ \frac{m}{s}\right)^2\right) + 998\ \frac{kg}{m^3} \times 9.81\ \frac{m}{s^2} \times (-27\ m)$$

$$p_2 = 389\ kPa$$

Problem 6.32

[Difficulty: 2]

6.32 A jet of air from a nozzle is blown at right angles against a wall in which two pressure taps are located. A manometer connected to the tap directly in front of the jet shows a head of 25 mm of mercury above atmospheric. Determine the approximate speed of the air leaving the nozzle if it is at −10°C and 200 kPa. At the second tap a manometer indicates a head of 5 mm of mercury above atmospheric; what is the approximate speed of the air there?

Given: Air jet hitting wall generating pressures

Find: Speed of air at two locations

Solution:

Basic equations $\frac{p}{\rho_{air}} + \frac{V^2}{2} + g \cdot z = const$ $\rho_{air} = \frac{p}{R_{air} \cdot T}$ $\Delta p = \rho_{Hg} \cdot g \cdot \Delta h = SG_{Hg} \cdot \rho \cdot g \cdot \Delta h$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Available data $R = 287 \cdot \frac{J}{kg \cdot K}$ $T = -10\,°C$ $\rho = 999 \cdot \frac{kg}{m^3}$ $p = 200 \cdot kPa$ $SG_{Hg} = 13.6$

For the air $\rho_{air} = \frac{p}{R \cdot T}$ $\rho_{air} = 2.65 \frac{kg}{m^3}$

Hence, applying Bernoulli between the jet and where it hits the wall directly

$$\frac{p_{atm}}{\rho_{air}} + \frac{V_j^2}{2} = \frac{p_{wall}}{\rho_{air}} \qquad p_{wall} = \frac{\rho_{air} \cdot V_j^2}{2} \qquad \text{(working in gage pressures)}$$

Hence $p_{wall} = SG_{Hg} \cdot \rho \cdot g \cdot \Delta h = \frac{\rho_{air} \cdot V_j^2}{2}$ so $V_j = \sqrt{\frac{2 \cdot SG_{Hg} \cdot \rho \cdot g \cdot \Delta h}{\rho_{air}}}$

$\Delta h = 25 \cdot mm$ hence $V_j = \sqrt{2 \times 13.6 \times 999 \cdot \frac{kg}{m^3} \times \frac{1}{2.65} \cdot \frac{m^3}{kg} \times 9.81 \cdot \frac{m}{s^2} \times 25 \cdot mm \times \frac{1 \cdot m}{1000 \cdot mm}}$ $V_j = 50.1 \frac{m}{s}$

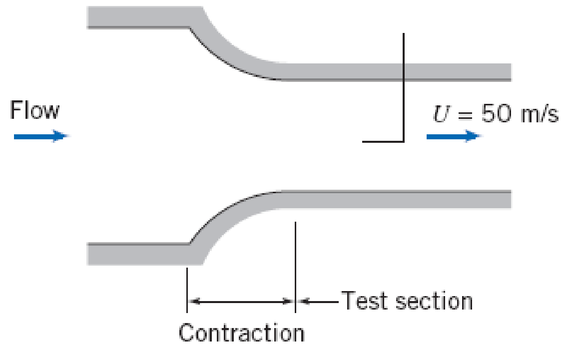
Repeating the analysis for the second point

$\Delta h = 5 \cdot mm$ $\frac{p_{atm}}{\rho_{air}} + \frac{V_j^2}{2} = \frac{p_{wall}}{\rho_{air}} + \frac{V^2}{2}$ $V = \sqrt{V_j^2 - \frac{2 \cdot p_{wall}}{\rho_{air}}} = \sqrt{V_j^2 - \frac{2 \cdot SG_{Hg} \cdot \rho \cdot g \cdot \Delta h}{\rho_{air}}}$

Hence $V = \sqrt{\left(50.1 \cdot \frac{m}{s}\right)^2 - 2 \times 13.6 \times 999 \cdot \frac{kg}{m^3} \times \frac{1}{2.65} \cdot \frac{m^3}{kg} \times 9.81 \cdot \frac{m}{s^2} \times 5 \cdot mm \times \frac{1 \cdot m}{1000 \cdot mm}}$ $V = 44.8 \frac{m}{s}$

Problem 6.33 [Difficulty: 2]

6.33 The inlet contraction and test section of a laboratory wind tunnel are shown. The air speed in the test section is $U = 50$ m/s. A total-head tube pointed upstream indicates that the stagnation pressure on the test section centerline is 10 mm of water below atmospheric. The laboratory is maintained at atmospheric pressure and a temperature of $-5°$C. Evaluate the dynamic pressure on the centerline of the wind tunnel test section. Compute the static pressure at the same point. Qualitatively compare the static pressure at the tunnel wall with that at the centerline. Explain why the two may not be identical.



Given: Wind tunnel with inlet section

Find: Dynamic and static pressures on centerline; compare with Speed of air at two locations

Solution:

Basic equations $p_{dyn} = \frac{1}{2}\cdot\rho_{air}\cdot U^2$ $\quad p_0 = p_s + p_{dyn}$ $\quad \rho_{air} = \frac{p}{R_{air}\cdot T}$ $\quad \Delta p = \rho_w\cdot g\cdot \Delta h$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Available data $T = -5\,°C$ $\quad U = 5($ $R = 287\cdot\frac{J}{kg\cdot K}$ $\quad p_{atm} = 101\cdot kPa$ $\quad h_0 = -10\cdot mm$ $\quad \rho_w = 999\cdot\frac{kg}{m^3}$

For air $\rho_{air} = \frac{p_{atm}}{R\cdot T}$ $\quad \rho_{air} = 1.31\frac{kg}{m^3}$

$p_{dyn} = \frac{1}{2}\cdot\rho_{air}\cdot U^2$ $\quad p_{dyn} = 1.64\cdot kPa$

Also $p_0 = \rho_w\cdot g\cdot h_0$ $\quad p_0 = -98.0\,Pa$ (gage)

and $p_0 = p_s + p_{dyn}$ so $p_s = p_0 - p_{dyn}$ $\quad p_s = -1.738\,kPa$ (gage) $\quad h_s = \frac{p_s}{\rho_w\cdot g}$ $\quad h_s = -177\,mm$

Streamlines in the test section are straight so $\frac{\partial}{\partial n}p = 0$ and $p_w = p_{centerline}$

In the curved section $\frac{\partial}{\partial n}p = \rho_{air}\cdot\frac{V^2}{R}$ so $p_w < p_{centerline}$

Problem 6.34

(Difficulty: 2)

6.34 Maintenance work on high-pressure hydraulic systems requires special precautions. A small leak can result in a high speed jet of hydraulic fluid that can penetrate the skin and cause serious injury (therefore troubleshooters are cautioned to use a piece of paper or cardboard, not a finger, to search for leaks). Calculate and plot the jet speed of a leak versus system pressure, for pressures up to 40 MPa $(gage)$. Explain how a high-speed jet of hydraulic fluid can cause injury.

Assumptions: (1) steady flow (2) incompressible flow (3) frictionless flow (4) flow along a streamline

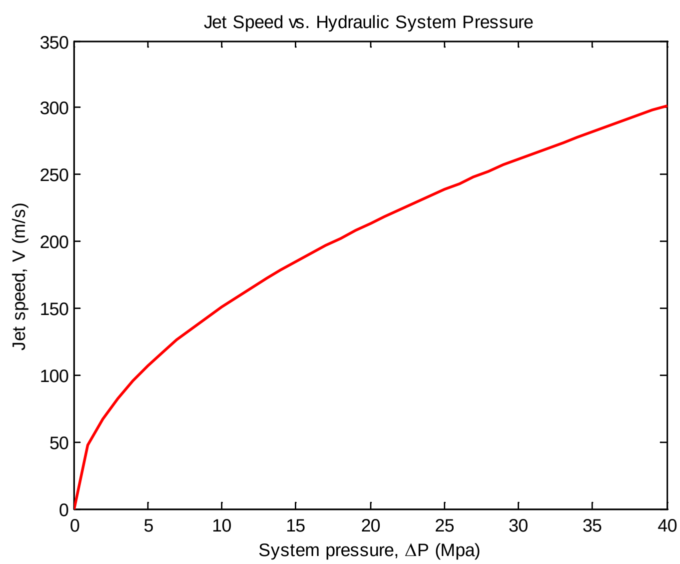
Solution: Use the Bernoulli equation:

$$\frac{p}{\rho}+\frac{V^2}{2}+gz=constant$$

The Bernoulli equation gives:

$$V=\left[\frac{2(p_0-p_{atm})}{\rho}\right]^{\frac{1}{2}}$$

From Table in Appendix A for lubricating oil $SG=0.88$.

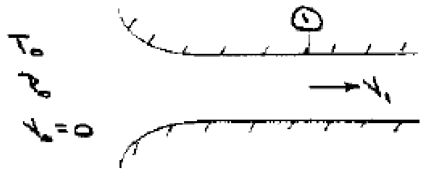


The high stagnation pressure ruptures the skin causing the jet to penetrate the tissue.

Problem 6.35

(Difficulty: 2)

6.35 An open-circuit wind tunnel draws in air from the atmosphere through a well-contoured nozzle. In the test section, where the flow is straight and nearly uniform, a static pressure tap is drilled into the tunnel wall. A manometer connected to the tap shows that static pressure within the tunnel is 45 mm of water below atmospheric. Assume that the air is incompressible, and at 25 ℃, 100 kPa (abs). Calculate the air speed in the wind-tunnel test section.



Assumptions: (1) steady flow (2) incompressible flow (3) frictionless flow (4) flow along a streamline (5) air behaves as an ideal gas (6) stagnation pressure=p_{atm}

Solution: Use the Bernoulli equation:

$$\frac{p}{\rho}+\frac{V^2}{2}+gz=constant$$

From the Bernoulli equation thwn,

$$\frac{p_0}{\rho}=\frac{p_1}{\rho}+\frac{V_1^2}{2}$$

$$p_0-p_1=p_{atm}-p_1=\frac{1}{2}\rho V_1^2$$

$$V_1=\left[\frac{2(p_{atm}-p_1)}{\rho}\right]^{\frac{1}{2}}$$

From the manometer reading,

$$p_{atm}-p_1=\rho_{H_2o}gh$$

$$V_1=\left[\frac{2\rho_{H_2o}gh}{\rho}\right]^{\frac{1}{2}}$$

From the ideal gas equation of state:

$$\rho = \frac{p}{RT} = \frac{100 \times 10^3\ \dfrac{N}{m^2}}{287\ \dfrac{N \cdot m}{kg \cdot K} \times 298\ K} = 1.17\ \frac{kg}{m^3}$$

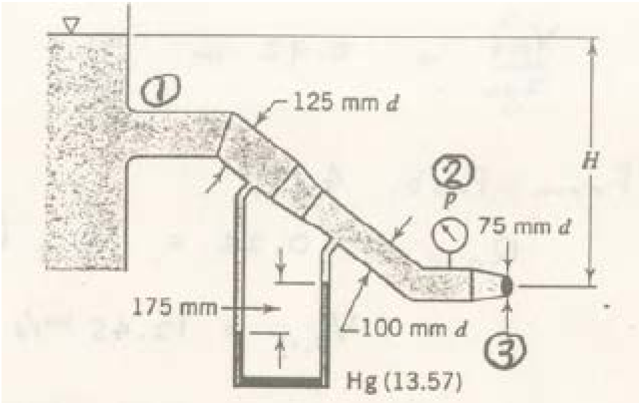
Thus the velocity is

$$V_1 = \left[\frac{2 \times 999\ \dfrac{kg}{m^3} \times 9.81\ \dfrac{m}{s^2} \times 0.046\ m}{1.17\ \dfrac{kg}{m^3}}\right]^{\frac{1}{2}} = 27.7\ \frac{m}{s}$$

Problem 6.36

(Difficulty 3)

6.36 Water is flowing. Calculate $H\ (m)$ and $p\ (kPa)$.



Find: The manometer reading and the pressure

Assumption: The flow is steady, incompressible, uniform, and frictionless.

Solution: Apply the continuity and Bernoulli equations to find the pressure

The continuity equation is:

$$Q = V_1 A_1 = V_2 A_2$$

The Bernoulli equation along a streamline is

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = constant$$

From the continuity equation we have:

$$Q = V_1 A_1 = V_2 A_2 = V_3 A_3$$

$$\frac{V_1^2}{V_3^2} = \frac{A_3^2}{A_1^2} = \frac{D_3^4}{D_1^4}$$

$$\frac{V_2^2}{V_3^2} = \frac{A_3^2}{A_2^2} = \frac{D_3^4}{D_2^4}$$

The hydrostatic pressure is determined from the manometer reading:

$$p_1 + \gamma_{H_2o} z_1 = p_2 + \gamma_{H_2o}(z_2 - 0.175) + \gamma_{Hg}(0.175)$$

$$\left(\frac{p_1}{\gamma_{H_2o}} + z_1\right) - \left(\frac{p_2}{\gamma_{H_2o}} + z_2\right) = -0.175 + \frac{\gamma_{Hg}}{\gamma_{H_2o}}(0.175)$$

Appling the Bernoulli equation between points 1 and 2 as:

$$\frac{p_1}{\rho_{H_2o}} + gz_1 + \frac{1}{2}V_1^2 = \frac{p_2}{\rho_{H_2o}} + gz_2 + \frac{1}{2}V_2^2$$

$$\frac{p_1}{\gamma_{H_2o}} + z_1 + \frac{1}{2g}V_1^2 = \frac{p_2}{\gamma_{H_2o}} + z_2 + \frac{1}{2g}V_2^2$$

Or

$$\left(\frac{p_1}{\gamma_{H_2o}} + z_1\right) - \left(\frac{p_2}{\gamma_{H_2o}} + z_2\right) = \frac{V_2^2}{2g} - \frac{V_1^2}{2g} = \left(\frac{D_3^4}{D_2^4} - \frac{D_3^4}{D_1^4}\right)\frac{V_3^2}{2g}$$

Combining the equations we have:

$$\left(\frac{D_3^4}{D_2^4} - \frac{D_3^4}{D_1^4}\right)\frac{V_3^2}{2g} = -0.175 + \frac{\gamma_{Hg}}{\gamma_{H_2o}}(0.175)$$

$$\frac{\gamma_{Hg}}{\gamma_{H_2o}} = 13.57$$

$$\frac{V_3^2}{2g} = \frac{-0.175\ m + 13.57 \times (0.175\ m)}{\left(\frac{(0.075\ m)^4}{(0.1\ m)^4} - \frac{(0.075\ m)^4}{(0.125\ m)^4}\right)} = 11.78\ m$$

Appling the Bernoulli equation from the water surface to the out let we have:

$$H = \frac{V_3^2}{2g} = 11.78\ m$$

Appling the Bernoulli equation from section 2 to section 3

$$\frac{p_2}{\gamma_{H_2o}} + \frac{V_2^2}{2g} = \frac{V_3^2}{2g}$$

$$\frac{p_2}{\gamma_{H_2o}} = \frac{V_3^2}{2g}\left(1 - \frac{D_3^4}{D_2^4}\right)$$

Thus the pressure at location 2 is

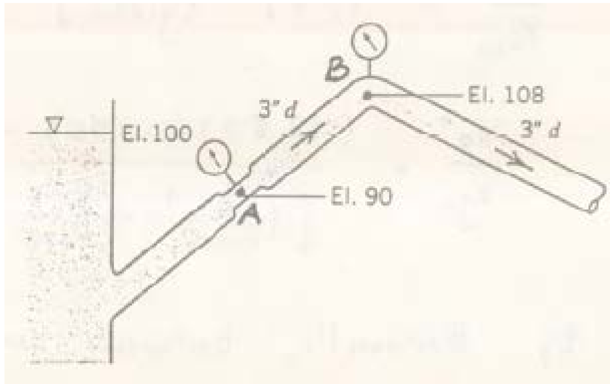
$$p_2 = \frac{\gamma_{H_2o}V_3^2}{2g}\left(1 - \frac{D_3^4}{D_2^4}\right)$$

$$p_2 = 9810\ \frac{N}{m^3} \times 11.78\ m \times \left(1 - \frac{(0.075\ m)^4}{(0.1\ m)^4}\right) = 78.9\ kPa$$

Problem 6.37

(Difficulty 2)

6.37 If each gage shows the same reading for a flow rate of 1.00 $\frac{ft^3}{s}$. What is the diameter of the constriction?



Find: Calculate the diameter of the constriction D_A.

Assumption: The flow is steady, incompressible, uniform, and frictionless.

Solution: Apply the continuity and Bernoulli equations to find the pressure

The continuity equation is:

$$Q = V_1 A_1 = V_2 A_2$$

The Bernoulli equation along a streamline is

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = constant$$

From the continuity equation:

$$Q = V_A A_A = V_B A_B$$

The velocity at B can be calculated by:

$$A_A = \frac{\pi D_A^2}{4}$$

$$A_B = \frac{\pi D_B^2}{4}$$

$$V_B = \frac{Q}{A_B} = \frac{1.00\ \frac{ft^3}{s}}{\frac{\pi \times \left(\frac{3}{12}\ ft\right)^2}{4}} = 20.4\ \frac{ft}{s}$$

Apply the Bernoulli equation for the streamline from A to B we have:

$$\frac{p_A}{\rho} + \frac{V_A^2}{2} + gz_A = \frac{p_B}{\rho} + \frac{V_B^2}{2} + gz_B$$

As each gage shows the same reading:

$$p_A = p_B$$

The velocity at A is then

$$V_A = \sqrt{V_B^2 + 2g(z_B - z_A)} = \sqrt{\left(20.4\ \frac{ft}{s}\right)^2 + 2 \times 32.2\ \frac{ft}{s^2} \times (108\ ft - 90\ ft)} = 39.7\ \frac{ft}{s}$$

So the constriction diameter is from the continuity equation:

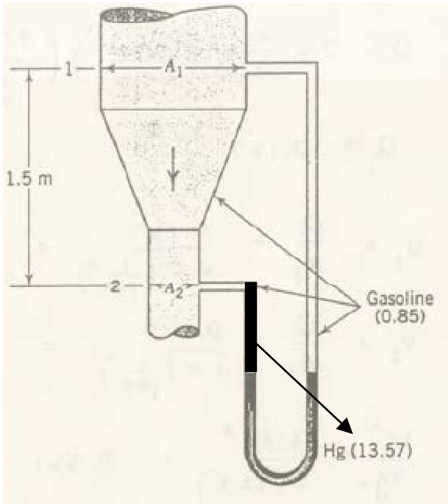
$$A_A = \frac{\pi D_A^2}{4} = \frac{Q}{V_A}$$

$$D_A = \sqrt{\frac{4Q}{\pi V_A}} = \sqrt{\frac{4 \times 1.00\ \frac{ft^3}{s}}{\pi \times 39.7\ \frac{ft}{s}}} = 0.179\ ft = 2.15\ in$$

Problem 6.38

(Difficulty 2)

6.38 Derive a relation between A_1 and A_2 so that for a flow rate of $0.28\ \frac{m^3}{s}$ the static pressure will be the same at sections 1 and 2. Also calculate the manometer reading for this condition and state which leg has the higher mercury column.



Assumption: The flow is steady and incompressible

Solution: Use the continuity and Bernoulli equations together with manometer relations to find the height of the mercury column.

From the continuity equation:

$$Q = V_1 A_1 = V_2 A_2$$

Appling the Bernoulli equation from section 1 to section 2 we have:

$$\frac{p_1}{\rho_{gas}} + g z_1 + \frac{1}{2} V_1^2 = \frac{p_2}{\rho_{gas}} + g z_2 + \frac{1}{2} V_2^2$$

We also have the pressures at location 1 and 2 as specified as equal:

$$p_1 = p_2$$

From the Bernoulli equation

$$\frac{1}{2} V_1^2 - \frac{1}{2} V_2^2 = g(z_2 - z_1)$$

Using the continuity equation to replace the velocities with the volume flow rate and areas

$$V_1^2 - V_2^2 = \frac{Q^2}{A_1^2} - \frac{Q^2}{A_2^2} = 2g(z_2 - z_1)$$

The relation between the areas is then

$$\frac{1}{A_1^2} - \frac{1}{A_2^2} = \frac{2g(z_2 - z_1)}{Q^2} = \frac{2 \times 9.81\ \frac{m}{s^2} \times (-1.5\ m)}{\left(0.28\ \frac{m^3}{s}\right)^2} = -375\ \frac{1}{m^4}$$

For the static pressure equation of the manometer we have:

$$p_1 + \gamma_{gas}(1.5 + h) = p_2 + \gamma_{Hg} h$$

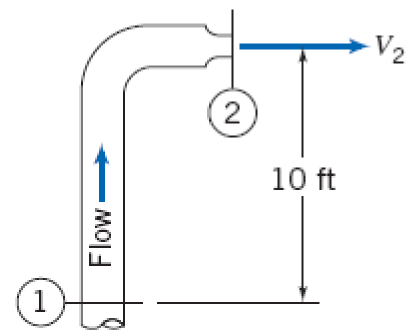
$$\gamma_{gas}(1.5 + h) = \gamma_{Hg} h$$

$$h = \frac{\gamma_{gas} \times 1.5\ m}{\gamma_{Hg} - \gamma_{gas}} = \frac{0.85 \times 1.5\ m}{13.57 - 0.85} = 0.1002\ m = 100.2\ mm$$

The mercury column is higher in the left hand leg of the manometer.

Problem 6.39 [Difficulty: 2]

6.39 Water flows steadily up the vertical 1-in.-diameter pipe and out the nozzle, which is 0.5 in. in diameter, discharging to atmospheric pressure. The stream velocity at the nozzle exit must be 30 ft/s. Calculate the minimum gage pressure required at section (1). If the device were inverted, what would be the required minimum pressure at section (1) to maintain the nozzle exit velocity at 30 ft/s?



Given: Flow in pipe/nozzle device

Find: Gage pressure needed for flow rate; repeat for inverted

Solution:

Basic equations $Q = V \cdot A$ $\qquad \frac{p}{\rho} + \frac{V^2}{2} + g \cdot z = \text{const}$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Available data $D_1 = 1 \cdot in$ $\quad D_2 = 0.5 \cdot in$ $\quad V_2 = 30 \cdot \frac{ft}{s}$ $\quad z_2 = 10 \cdot ft$ $\quad \rho = 1.94 \cdot \frac{slug}{ft^3}$

From continuity $Q = V_1 \cdot A_1 = V_2 \cdot A_2$ $\quad V_1 = V_2 \cdot \frac{A_2}{A_1}$ or $\quad V_1 = V_2 \cdot \left(\frac{D_2}{D_1}\right)^2$ $\quad V_1 = 7.50 \frac{ft}{s}$

Hence, applying Bernoulli between locations 1 and 2

$$\frac{p_1}{\rho} + \frac{V_1^{\,2}}{2} + 0 = \frac{p_2}{\rho} + \frac{V_2^{\,2}}{2} + g \cdot z_2 = 0 + \frac{V_2^{\,2}}{2} + g \cdot z_2 \text{working in gage pressures}$$

Solving for p_1 (gage) $\quad p_1 = \rho \cdot \left(\frac{V_2^{\,2} - V_1^{\,2}}{2} + g \cdot z_2\right)$ $\qquad p_1 = 10.0 \cdot psi$

When it is inverted $\quad z_2 = -10 \cdot ft$

$$p_2 = \rho \cdot \left(\frac{V_2^{\,2} - V_1^{\,2}}{2} + g \cdot z_2\right) \qquad p_2 = 1.35 \cdot psi$$

Problem 6.40

[Difficulty: 2]

6.40 You are on a date. Your date runs out of gas unexpectedly. You come to your own rescue by siphoning gas from another car. The height difference for the siphon is about 1 ft. The hose diameter is 0.5 in. What is your gasoline flow rate?

Given: Siphoning of gasoline

Find: Flow rate

Solution:

Basic equation

$$\frac{p}{\rho_{gas}} + \frac{V^2}{2} + g \cdot z = \text{const}$$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Hence, applying Bernoulli between the gas tank free surface and the siphon exit

$$\frac{p_{atm}}{\rho_{gas}} = \frac{p_{atm}}{\rho_{gas}} + \frac{V^2}{2} - g \cdot h$$

where we assume the tank free surface is slowly changing so V_{tank} <<, and h is the difference in levels

Hence

$$V = \sqrt{2 \cdot g \cdot h}$$

The flow rate is then

$$Q = V \cdot A = \frac{\pi \cdot D^2}{4} \cdot \sqrt{2 \cdot g \cdot h}$$

$$Q = \frac{\pi}{4} \times (.5 \cdot in)^2 \times \frac{1 \cdot ft^2}{144 \cdot in^2} \times \sqrt{2 \times 32.2 \frac{ft}{s^2} \times 1 \cdot ft} \qquad Q = 0.0109 \cdot \frac{ft^3}{s} \qquad Q = 4.91 \cdot \frac{gal}{min}$$

Problem 6.41

[Difficulty: 2]

6.41 A tank at a pressure of 50 kPa (gage) gets a pinhole rupture and benzene shoots into the air. Ignoring losses, to what height will the benzene rise?

Given: Ruptured pipe

Find: Height benzene rises from tank

Solution:

Basic equation $$\frac{p}{\rho_{ben}} + \frac{V^2}{2} + g \cdot z = \text{const}$$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Available data $\rho = 999 \cdot \frac{kg}{m^3}$ $\quad p_{ben} = 50 \cdot kPa$ (gage) $\quad$ From Table A.2 $\quad SG_{ben} = 0.879$

Hence, applying Bernoulli between the pipe and the rise height of the benzene

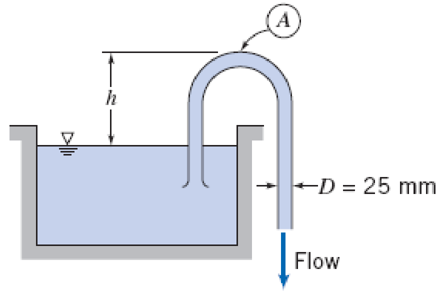
$$\frac{p_{ben}}{\rho_{ben}} = \frac{p_{atm}}{\rho_{ben}} + g \cdot h$$ where we assume V_{pipe} <<, and h is the rise height

Hence $$h = \frac{p_{ben}}{SG_{ben} \cdot \rho \cdot g}$$ where p_{ben} is now the gage pressure

$$h = 5.81\,m$$

Problem 6.42 [Difficulty: 3]

6.42 The water flow rate through the siphon is 5 L/s, its temperature is 20°C, and the pipe diameter is 25 mm. Compute the maximum allowable height, h, so that the pressure at point A is above the vapor pressure of the water. (Assume the flow is frictionless.)



Given: Flow rate through siphon

Find: Maximum height h to avoid cavitation

Solution:

Basic equation $\frac{p}{\rho} + \frac{V^2}{2} + g \cdot z = \text{const}$ $\qquad Q = V \cdot A$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Available data $Q = 5 \cdot \frac{L}{s}$ $\qquad Q = 5 \times 10^{-3} \frac{m^3}{s}$ $\qquad D = 25 \cdot mm$ $\qquad \rho = 999 \cdot \frac{kg}{m^3}$ $\qquad p_{atm} = 101 \cdot kPa$

From continuity $V = \frac{Q}{A} = \frac{4 \cdot Q}{\pi \cdot D^2}$ $\qquad V = \frac{4}{\pi} \times 0.005 \frac{m^3}{s} \times \left(\frac{1}{.025 m}\right)^2$ $\qquad V = 10.2 \frac{m}{s}$

Hence, applying Bernoulli between the free surface and point A

$$\frac{p_{atm}}{\rho} = \frac{p_A}{\rho} + g \cdot h + \frac{V^2}{2}$$

where we assume $V_{Surface}$ <<

Hence $p_A = p_{atm} - \rho \cdot g \cdot h - \rho \cdot \frac{V^2}{2}$

From the steam tables, at 20ºC the vapor pressure is $\qquad p_v = 2.358 \cdot kPa$

This is the lowest permissible value of p_A

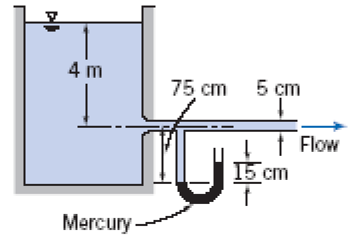
Hence $p_A = p_v = p_{atm} - \rho \cdot g \cdot h - \rho \cdot \frac{V^2}{2}$ or $\qquad h = \frac{p_{atm} - p_v}{\rho \cdot g} - \frac{V^2}{2 \cdot g}$

Hence $h = (101 - 2.358) \times 10^3 \cdot \frac{N}{m^2} \times \frac{1}{999} \cdot \frac{m^3}{kg} \times \frac{s^2}{9.81 \cdot m} \times \frac{kg \cdot m}{N \cdot s^2} - \frac{1}{2} \times \left(10.2 \frac{m}{s}\right)^2 \times \frac{s^2}{9.81 \cdot m}$ $\qquad h = 4.76 m$

Problem 6.43

[Difficulty: 2]

6.43 Water flows from a very large tank through a 5-cm-diameter tube. The dark liquid in the manometer is mercury. Estimate the velocity in the pipe and the rate of discharge from the tank. (Assume the flow is frictionless.)



Given: Flow through tank-pipe system

Find: Velocity in pipe; Rate of discharge

Solution:

Basic equations $\quad \frac{p}{\rho} + \frac{V^2}{2} + g\cdot z = \text{const} \qquad \Delta p = \rho\cdot g\cdot \Delta h \qquad Q = V\cdot A$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Hence, applying Bernoulli between the free surface and the manometer location

$$\frac{p_{atm}}{\rho} = \frac{p}{\rho} - g\cdot H + \frac{V^2}{2}$$

where we assume $V_{Surface}$ <<, and H = 4 m

Hence $\quad p = p_{atm} + \rho\cdot g\cdot H - \rho\cdot\frac{V^2}{2}$

For the manometer $\quad p - p_{atm} = SG_{Hg}\cdot\rho\cdot g\cdot h_2 - \rho\cdot g\cdot h_1$

Note that we have water on one side and mercury on the other of the manometer

Combining equations $\quad \rho\cdot g\cdot H - \rho\cdot\frac{V^2}{2} = SG_{Hg}\cdot\rho\cdot g\cdot h_2 - \rho\cdot g\cdot h_1 \quad$ or $\quad V = \sqrt{2\cdot g\cdot\left(H - SG_{Hg}\cdot h_2 + h_2\right)}$

Hence $\quad V = \sqrt{2\times 9.81\cdot\frac{m}{s^2}\times(4 - 13.6\times 0.15 + 0.75)\cdot m} \qquad V = 7.29\frac{m}{s}$

The flow rate is $\quad Q = V\cdot\frac{\pi\cdot D^2}{4} \qquad Q = \frac{\pi}{4}\times 7.29\cdot\frac{m}{s}\times(0.05\cdot m)^2 \qquad Q = 0.0143\frac{m^3}{s}$

Problem 6.44 [Difficulty: 3]

6.44 Consider frictionless, incompressible flow of air over the wing of an airplane flying at 200 km/hr. The air approaching the wing is at 65 kPa and −10°C. At a certain point in the flow, the pressure is 60 kPa. Calculate the speed of the air relative to the wing at this point and the absolute air speed.

Given: Air flow over a wing

Find: Air speed relative to wing at a point; absolute air speed

Solution:

Basic equation $\quad \frac{p}{\rho} + \frac{V^2}{2} + g\cdot z = \text{const} \qquad p = \rho\cdot R\cdot T$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Available data $\quad T = -10\,°C \qquad p_1 = 65\cdot kPa \qquad V_1 = 200\cdot\frac{km}{hr} \qquad p_2 = 60\cdot kPa \qquad R = 286.9\cdot\frac{N\cdot m}{kg\cdot K}$

For air $\quad \rho = \frac{p_1}{R\cdot T} \qquad \rho = (65)\times 10^3\cdot\frac{N}{m^2}\times\frac{kg\cdot K}{286.9\cdot N\cdot m}\times\frac{1}{(-10+273)\cdot K} \qquad \rho = 0.861\,\frac{kg}{m^3}$

Hence, applying Bernoulli between the upstream point (1) and the point on the wing (2)

$$\frac{p_1}{\rho} + \frac{V_1^{\,2}}{2} = \frac{p_2}{\rho} + \frac{V_2^{\,2}}{2} \qquad \text{where we ignore gravity effects}$$

Hence $\quad V_2 = \sqrt{V_1^{\,2} + 2\cdot\frac{(p_1 - p_2)}{\rho}}$

Then $\quad V_2 = \sqrt{\left(200\cdot\frac{km}{hr}\right)^2\times\left(\frac{1000\cdot m}{1\cdot km}\right)^2\times\left(\frac{1\cdot hr}{3600\cdot s}\right)^2 + 2\times\frac{m^3}{0.861\cdot kg}\times(65-60)\times 10^3\cdot\frac{N}{m^2}\times\frac{kg\cdot m}{N\cdot s^2}} \qquad V_2 = 121\,\frac{m}{s}$

NOTE: At this speed, significant density changes will occur, so this result is not very realistic

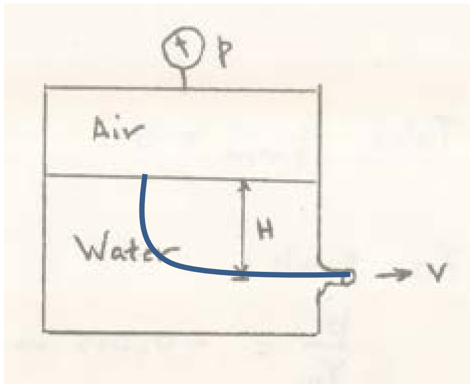
The absolute velocity is

$$V_{2abs} = V_2 - V_1 \qquad V_{2abs} = 65.7\,\frac{m}{s}$$

Problem 6.45

(Difficulty 1)

6.45 A closed tank contains water with air above it. The air is maintained at a gage pressure of 150 kPa and 3 m below the water surface a nozzle discharges into the atmosphere. At what velocity will water emerge from the nozzle?



Find: Calculate the manometer reading for this condition.

Assumption: The flow is steady, incompressible, uniform, and frictionless.

Solution: Apply the Bernoulli equation to find the pressure

The Bernoulli equation along a streamline is

$$\frac{p}{\rho}+\frac{V^2}{2}+gz=constant$$

Appling the Bernoulli equation for the streamline from the interface to outlet we have:

$$\frac{p_1}{\rho}+\frac{V_1^2}{2}+gz_1=\frac{p_2}{\rho}+\frac{V_2^2}{2}+gz_2$$

The pressure at the nozzle is atmosphere pressure:

$$p_2=0$$

And we also have:

$$z_2=0$$

$$V_1=0$$

So we obtain:

$$\frac{p_1}{\rho} + gz_1 = \frac{V_2^2}{2}$$

The exit velocity is

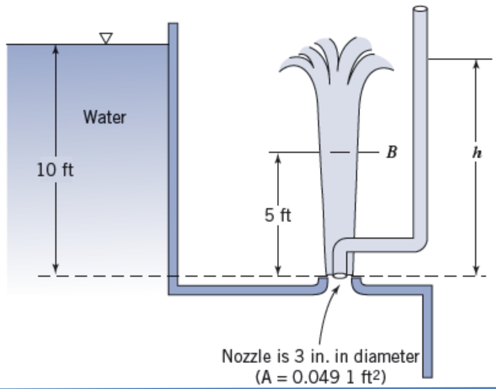
$$V_2 = \sqrt{\frac{2p_1}{\rho} + 2gz_1}$$

$$V_2 = \sqrt{\frac{2 \times 150 \times 10^3\ Pa}{998\ \frac{kg}{m^3}} + 2 \times 9.81\ \frac{m}{s^2} \times 3m} = 19.0\ \frac{m}{s}$$

Problem 6.46

(Difficulty 2)

6.46 Water jets upward through a 3 in diameter nozzle under a head of 10 ft. At what height h will be the liquid stand in the pitot tube? What is the cross-sectional area of jet at section B?



Find: The height h and cross-sectional area A_B.

Assumption: The flow is steady, incompressible, uniform, and frictionless.

Solution: Apply the continuity and Bernoulli equations to find the pressure

The continuity equation is:

$$Q = V_1 A_1 = V_2 A_2$$

The Bernoulli equation along a streamline is

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = constant$$

The pitot tube reading will be identical to total head at the nozzle outlet:

$$h = 10\ ft$$

Appling the Bernoulli equation of the streamline from top to the bottom across B we have:

$$\frac{p_B}{\rho} + \frac{V_B^2}{2} + gz_B = \frac{p_o}{\rho} + \frac{V_o^2}{2} + gz_o = \frac{p_t}{\rho} + \frac{V_t^2}{2} + gz_t$$

Where we have for this situation

$$p_t = 0, V_t = 0, z_t = h, p_o = 0, z_o = 0, p_B = 0$$

The Bernoulli equation reduces to

$$0 + \frac{V_B^2}{2} + gz_B = 0 + \frac{V_o^2}{2} + 0 = 0 + 0 + gh$$

The velocity for the outlet can be found as:

$$V_o = \sqrt{2gh} = \sqrt{2 \times 32.2\ \frac{ft}{s^2} \times 10\ ft} = 25.4\ \frac{ft}{s}$$

The cross-sectional area at outlet is:

$$A_o = \frac{\pi D_o^2}{4} = \frac{\pi \times \left(\frac{3}{12}\ ft\right)^2}{4} = 0.0491\ ft^2$$

The volumetric flow rate is:

$$Q = V_o A_0 = 25.4\ \frac{ft}{s} \times 0.0491\ ft^2 = 1.247\ \frac{ft^3}{s}$$

For the jet cross-section B, from the Bernoulli equation:

$$V_B = \sqrt{2g(z_t - z_B)} = \sqrt{2 \times 32.2\ \frac{ft}{s^2} \times 5\ ft} = 17.94\ \frac{ft}{s}$$

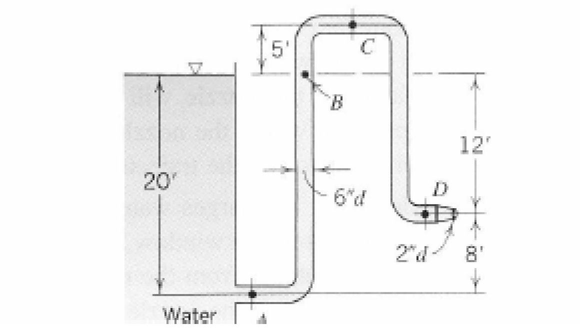
The outlet area is then

$$A_B = \frac{Q}{V_B} = \frac{1.247\ \frac{ft^3}{s}}{17.94\ \frac{ft}{s}} = 0.0695\ ft^2$$

Problem 6.47

(Difficulty 2)

6.47 Calculate the rate of flow through this pipeline and the pressures at A, B, C and D. Sketch the EL and HGL showing vertical distances.



Find: The flow rate and the pressures

Assumption: The flow is steady, incompressible, uniform, and frictionless.

Solution: Apply the continuity and Bernoulli equations to find the pressure

The continuity equation is:

$$Q = V_1 A_1 = V_2 A_2$$

The Bernoulli equation along a streamline is

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = constant$$

Applying the Bernoulli equation from the water surface 1 to the outlet 2, we have:

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

Where

$$p_1 = 0, V_1 = 0, z_1 = H, p_2 = 0, z_2 = 0,$$

The Bernoulli equation then becomes

$$0 + 0 + gH = 0 + \frac{V_2^2}{2} + 0$$

Or, solving for the outlet velocity

$$\frac{V_2^2}{2g} = H$$

$$V_2 = \sqrt{2gH} = \sqrt{2 \times 32.2\ \frac{ft}{s^2} \times 12\ ft} = 27.8\ \frac{ft}{s}$$

The volumetric flow rate is the

$$Q = VA_2 = 27.8\ \frac{ft}{s} \times \frac{\pi}{4} \times \left(\frac{2}{12}\ ft\right)^2 = 0.606\ \frac{ft^3}{s}$$

The velocities at $A, B, C\ and\ D$ are the same and using the continuity equation are:

$$V_A = V_B = V_C = V_D = \frac{Q}{A} = \frac{0.606\ \frac{ft^3}{s}}{\frac{\pi}{4} \times \left(\frac{6}{12}\ ft\right)^2} = 3.09\ \frac{ft}{s}$$

The velocity head in pipe is the same at these locations:

$$\frac{V^2}{2g} = \frac{\left(3.09\ \frac{ft}{s}\right)^2}{2 \times 32.2\ \frac{ft}{s^2}} = 0.148\ ft$$

Apply Bernoulli equation from water surface to A we have:

$$H = \frac{p_A}{\gamma} + \frac{V_A^2}{2g} + z_A$$

$$p_A = \gamma H - \gamma \frac{V_A^2}{2g} = 62.4\ \frac{lbf}{ft^3} \times 20\ ft - 62.4\ \frac{lbf}{ft^3} \times 0.148\ ft = 1238\ \frac{lbf}{ft^2} = 8.59\ psi$$

From the surface to B

$$H = \frac{p_B}{\gamma} + \frac{V_B^2}{2g} + H$$

$$p_B = -\gamma \frac{V_B^2}{2g} = -62.4\ \frac{lbf}{ft^3} \times 0.148\ ft = -9.2352\ \frac{lbf}{ft^2} = -0.0641\ psi$$

From the surface to C

$$H = \frac{p_c}{\gamma} + \frac{V_c^2}{2g} + H + 5$$

$$\frac{p_c}{\gamma} = -5\ ft - 0.148\ ft = -5.148\ ft$$

$$p_c = -62.4\ \frac{lbf}{ft^3} \times 5.148\ ft = -321\frac{lbf}{ft^2} = -2.23\ psi$$

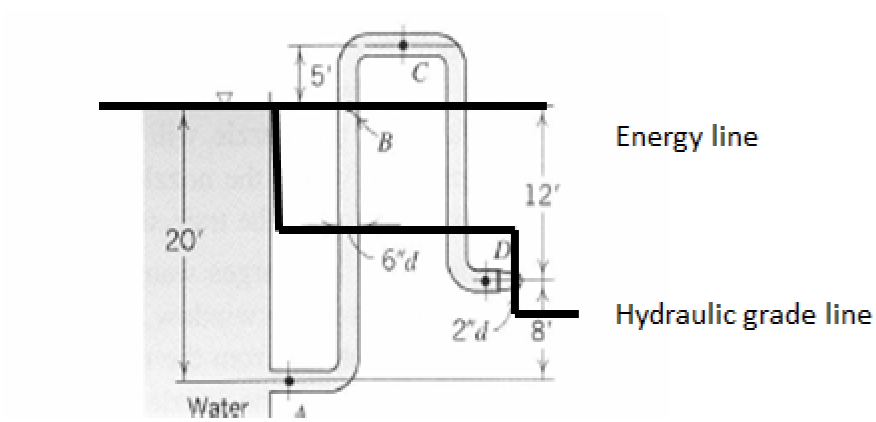
From the surface to D

$$H = \frac{p_D}{\gamma} + \frac{V_D^2}{2g} + z_D$$

$$\frac{p_D}{\gamma} = 20\ ft - 8ft - 0.148\ ft = 11.852\ ft$$

$$p_D = 62.4\ \frac{lbf}{ft^3} \times 11.852\ ft = 740\ \frac{lbf}{ft^2} = 5.14\ psi$$

The energy and hydraulic grade lines are sketched below



Problem 6.48

(Difficulty: 3)

6.48 A mercury barometer is carried in a car on a day when there is no wind. The temperature is $20\ ℃$ and the corrected barometer height is $761\ mm$ of mercury. One window is open slightly as the car travels at $105\ \ km/hr$. The barometer reading in the moving car is $5\ mm$ lower than when the car is stationary. Explain what is happening. Calculate the local speed of the air flowing past the window, relative to the automobile.

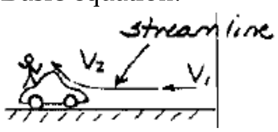
Assumptions: (1) steady flow (2) incompressible flow (3) neglect the friction (4) flow along a streamline (5) neglect Δz

Solution:

Physically, the air speed relative to car is higher than in the freestream, thus lowering the pressure at window.

Apply the Bernoulli equation as seen by an observer on the car:

Basic equation:



$$\frac{p_1}{\rho}+\frac{V_1^2}{2}+gz_1=\frac{p_2}{\rho}+\frac{V_2^2}{2}+gz_2$$

Then

$$V_2^2=V_1^2+2\frac{p_1-p_2}{\rho}$$

$$V_2=\left[V_1^2+2\frac{p_1-p_2}{\rho}\right]^{\frac{1}{2}}$$

From fluid statics,

$$p_1-p_2=\rho g(h_1-h_2)=SG\rho_{H_2o}g\Delta h$$

$$p_1-p_2=13.6\times 1000\ \frac{kg}{m^3}\times 9.81\ \frac{m}{s^2}\times 0.005\ m=667\ \frac{N}{m^2}$$

From ideal gas:

$$\rho = \frac{p}{RT} = \frac{13.6 \times 1000\ \frac{kg}{m^3} \times 9.81\ \frac{m}{s^2} \times 0.761\ m}{287\ \frac{N \cdot m}{kg \cdot K} \times (273 + 20)\ K} = 1.21\ \frac{kg}{m^3}$$

Substituting, we get the velocity relative to the car

$$V_2 = \left[V_1^2 + 2\frac{p_1 - p_2}{\rho}\right]^{\frac{1}{2}} = \left[\left(105\ \frac{km}{hr} \times 1000\ \frac{m}{km} \times \frac{1}{3600}\ \frac{hr}{s}\right)^2 + 2 \times \frac{667\ \frac{N}{m^2}}{1.21\ \frac{kg}{m^3}}\right]^{\frac{1}{2}}$$

$$V_2 = 44.2\ \frac{m}{s} = 159\ \frac{km}{hr}$$

Problem 6.49

[Difficulty: 2]

6.49 A racing car travels at 235 mph along a straightaway. The team engineer wishes to locate an air inlet on the body of the car to obtain cooling air for the driver's suit. The plan is to place the inlet at a location where the air speed is 60 mph along the surface of the car. Calculate the static pressure at the proposed inlet location. Express the pressure rise above ambient as a fraction of the freestream dynamic pressure.

Given: Race car on straightaway

Find: Air inlet where speed is 60 mph; static pressure; pressure rise

Solution:

Basic equation $\frac{p}{\rho} + \frac{V^2}{2} + g \cdot z = \text{const}$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline 5) Standard atmosphere

Available data $p_{atm} = 101 \cdot kPa$ $\rho = 0.002377 \frac{slug}{ft^3}$ $V_1 = 235 \cdot mph$ $V_2 = 60 \cdot mph$

Between location 1 (the upstream flow at 235 mph with respect to the car), and point 2 (on the car where V = 60 mph), Bernoulli becomes

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} = \frac{p_{atm}}{\rho} + \frac{V_1^2}{2} = \frac{p_2}{\rho} + \frac{V_2^2}{2}$$

Hence $p_2 = p_{atm} + \frac{1}{2} \cdot \rho \cdot V_1^2 \cdot \left[1 - \left(\frac{V_2}{V_1}\right)^2\right]$ $p_2 = 15.6\,psi$

Note that the pressure rise is $\Delta p = \frac{1}{2} \cdot \rho \cdot V_1^2 \cdot \left[1 - \left(\frac{V_2}{V_1}\right)^2\right]$ $\Delta p = 0.917 \cdot psi$

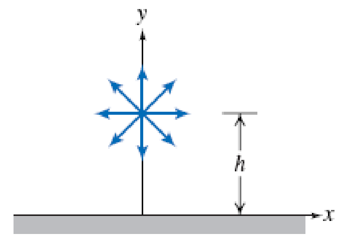
The freestream dynamic pressure is $q = \frac{1}{2} \cdot \rho \cdot V_1^2$ $q = 0.980 \cdot psi$

Then $\frac{\Delta p}{q} = 93.5 \cdot \%$

Note that at this speed the flow is borderline compressible!

Problem 6.50 [Difficulty: 3]

6.50 The velocity field for a plane source at a distance h above an infinite wall aligned along the x axis was given in Problem 6.6. Using the data from that problem, plot the pressure distribution along the wall from $x = -10h$ to $x = +10h$ (assume the pressure at infinity is atmospheric). Find the net force on the wall if the pressure on the lower surface is atmospheric. Does the force tend to pull the wall towards the source, or push it away?



Given: Velocity field

Find: Pressure distribution along wall; plot distribution; net force on wall

Solution:

The given data is $q = 2 \cdot \dfrac{\frac{m^3}{s}}{m}$ $\quad h = 1 \cdot m \quad$ $\rho = 1000 \cdot \dfrac{kg}{m^3}$

$$u = \frac{q \cdot x}{2 \cdot \pi \left[x^2 + (y-h)^2\right]} + \frac{q \cdot x}{2 \cdot \pi \left[x^2 + (y+h)^2\right]} \qquad v = \frac{q \cdot (y-h)}{2 \cdot \pi \left[x^2 + (y-h)^2\right]} + \frac{q \cdot (y+h)}{2 \cdot \pi \left[x^2 + (y+h)^2\right]}$$

The governing equation is the Bernoulli equation

$$\frac{p}{\rho} + \frac{1}{2} \cdot V^2 + g \cdot z = \text{const} \qquad \text{where} \qquad V = \sqrt{u^2 + v^2}$$

Apply this to point arbitrary point (x,0) on the wall and at infinity (neglecting gravity)

At $\quad |x| \rightarrow 0 \qquad u \rightarrow 0 \qquad v \rightarrow 0 \qquad V \rightarrow 0$

At point (x,0) $\quad u = \dfrac{q \cdot x}{\pi \cdot \left(x^2 + h^2\right)} \qquad v = 0 \qquad V = \dfrac{q \cdot x}{\pi \cdot \left(x^2 + h^2\right)}$

Hence the Bernoulli equation becomes $\quad \dfrac{p_{atm}}{\rho} = \dfrac{p}{\rho} + \dfrac{1}{2} \cdot \left[\dfrac{q \cdot x}{\pi \cdot \left(x^2 + h^2\right)}\right]^2$

or (with pressure expressed as gage pressure) $\quad p(x) = -\dfrac{\rho}{2} \cdot \left[\dfrac{q \cdot x}{\pi \cdot \left(x^2 + h^2\right)}\right]^2$

(Alternatively, the pressure distribution could have been obtained from Problem 6.8, where the momentum equation was used to find the pressure gradient $\dfrac{\partial}{\partial x} p = \dfrac{\rho \cdot q^2 \cdot x \cdot \left(x^2 - h^2\right)}{\pi^2 \cdot \left(x^2 + h^2\right)^3}$ along the wall. Integration of this with respect to x leads to the same result for $p(x)$)

The plot of pressure can be done in *Excel* (see below). From the plot it is clear that the wall experiences a negative gage pressure on the upper surface (and zero gage pressure on the lower), so the net force on the wall is upwards, towards the source

The force per width on the wall is given by $\quad F = \int_{-10\cdot h}^{10\cdot h} (p_{upper} - p_{lower})\,dx \qquad F = -\frac{\rho\cdot q^2}{2\cdot\pi^2}\cdot\int_{-10\cdot h}^{10\cdot h} \frac{x^2}{(x^2+h^2)^2}\,dx$

The integral is

$$\int \frac{x^2}{(x^2+h^2)^2}\,dx = \frac{\operatorname{atan}\left(\frac{x}{h}\right)}{2\cdot h} - \frac{x}{2\cdot h^2 + 2\cdot x^2}$$

so

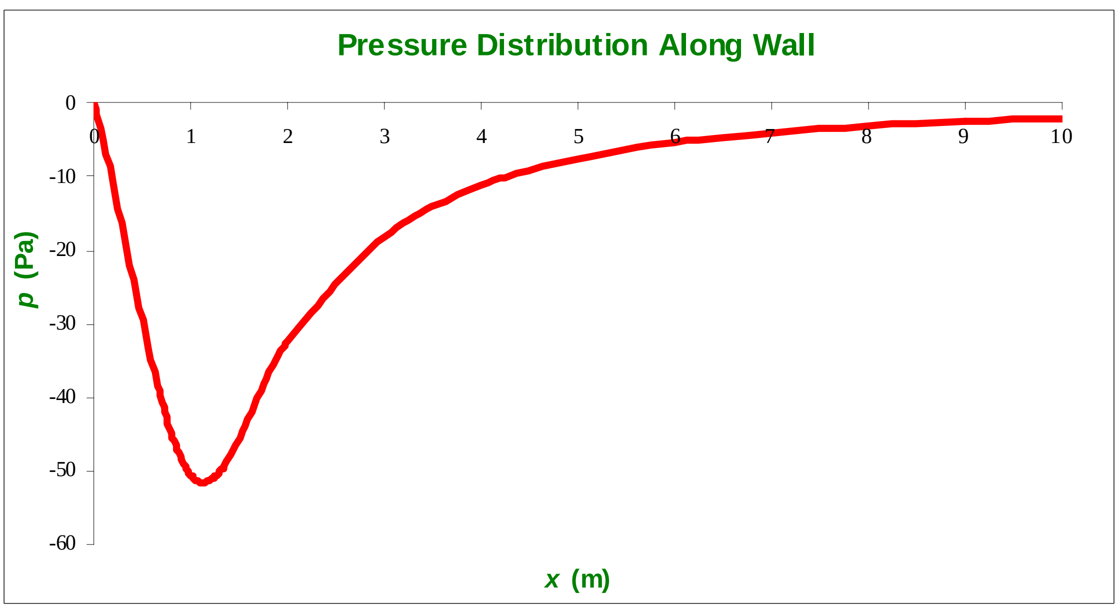
$$F = -\frac{\rho\cdot q^2}{2\cdot\pi^2\cdot h}\cdot\left(-\frac{10}{101} + \operatorname{atan}(10)\right)$$

$$F = -\frac{1}{2\cdot\pi^2}\times 1000\cdot\frac{kg}{m^3}\times\left(2\cdot\frac{m^2}{s}\right)^2\times\frac{1}{1\cdot m}\times\left(-\frac{10}{101}+\operatorname{atan}(10)\right)\times\frac{N\cdot s^2}{kg\cdot m} \qquad F = -278\cdot\frac{N}{m}$$

In *Excel*:

q = 2 $m^3/s/m$
h = 1 m
ℵ = 1000 kg/m^3

x (m)	*p* (Pa)
0.0	0.00
1.0	-50.66
2.0	-32.42
3.0	-18.24
4.0	-11.22
5.0	-7.49
6.0	-5.33
7.0	-3.97
8.0	-3.07
9.0	-2.44
10.0	-1.99

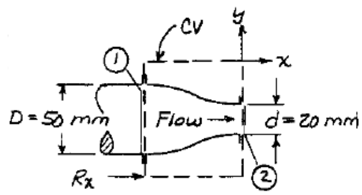


Problem 6.51

(Difficulty: 3)

6.51 A smoothly contoured nozzle, with outlet diameter $d = 20\ mm$, is coupled to a straight pipe by means of flanges. Water flows in the pipe, of diameter $D = 50\ mm$, and the nozzle discharges to the atmosphere. For steady flow and neglecting the effects of viscosity, find the volume flow rate in the pipe corresponding to a calculated axial force of $45.5\ N$ needed to keep the nozzle attached to the pipe.

Assumptions: (1) steady flow (2) uniform flow at each section (3) flow along a streamline (4) incompressible flow (5) no friction (6) horizontal, $F_{Bx} = 0, z_1 = z_2$ (7) use gage pressures



Solution: Use the continuity, Bernoulli, and momentum equations

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

$$F_{sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho d\forall + \int_{CS} u\rho \vec{V} \cdot d\vec{A}$$

Apply continuity, Bernoulli and x momentum equations we have:

$$0 = -V_1 A_1 + V_2 A_2$$

$$V_2 = V_1 \frac{A_1}{A_2} = V_1 \left(\frac{D}{d}\right)^2$$

$$Q = V_1 A_1 = V_2 A_2$$

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} = \frac{V_2^2}{2}$$

$$p_1 = \frac{\rho}{2}(V_2^2 - V_1^2) = \frac{\rho V_1^2}{2}\left[\left(\frac{V_2}{V_1}\right)^2 - 1\right] = \frac{\rho V_1^2}{2}\left[\left(\frac{D}{d}\right)^4 - 1\right]$$

$$R_x + p_1 A_1 = u_1\{-|\rho V_1 A_1|\} + u_2\{-|\rho V_2 A_2|\} = \rho V_1 A_1 (V_2 - V_1)$$

$$R_x + \frac{\rho V_1^2}{2}\left[\left(\frac{D}{d}\right)^4 - 1\right]A_1 = \rho V_1^2 A_1 \left(\frac{V_2}{V_1} - 1\right) = \rho V_1^2 A_1 \left[\left(\frac{D}{d}\right)^2 - 1\right]$$

Using the equations above:

$$V_1^2 = -\frac{2R_x}{\rho A_1}\frac{1}{\left(\frac{D}{d}\right)^4 - 1 - 2\left[\left(\frac{D}{d}\right)^2 - 1\right]} = -\frac{2R_x}{\rho A_1}\frac{1}{\left(\frac{D}{d}\right)^4 - 2\left(\frac{D}{d}\right)^2 + 1}$$

$$V_1 = \sqrt{-\frac{2R_x}{\rho A_1}}\frac{1}{\left(\frac{D}{d}\right)^2 - 1}$$

Thus

$$V_1 = \sqrt{-\frac{2 \times (-45.5\ N)}{999\ \frac{kg}{m^3} \times \frac{\pi}{4} \times (0.05\ m)^2}} \times \frac{1}{\left(\frac{50}{20}\right)^2 - 1} = 1.30\ \frac{m}{s}$$

Finally we have:

$$Q = V_1 A_1 = 1.30\ \frac{m}{s} \times \frac{\pi}{4} \times (0.05\ m)^2 = 2.55 \times 10^{-3}\ \frac{m^3}{s}$$

{Note: It is necessary to recognize that $R_x < 0$ for a nozzle}

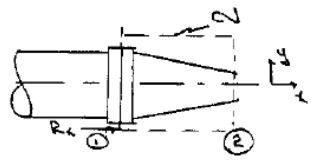
Problem 6.52

(Difficulty: 3)

6.52 Water flows steadily through a 3.25-in-diameter pipe and discharges through a 1.25-in-diameter nozzle to atmospheric pressure. The flow rate is 24.5 gpm. Calculate the minimum static pressure required in the pipe to produce this flow rate. Evaluate the axial force of the nozzle assembly on the pipe flange.

Assumptions: (1) steady flow (2) incompressible flow (3) frictionless flow (4) flow a long a streamline (5) $\Delta z = 0$ (6) uniform flow at each section

Solution:



Apply the Bernoulli equation along the central streamline between sections (1) and (2):

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

Then

$$p_1 = p_2 + \frac{\rho}{2}(V_2^2 - V_1^2) = p_2 + \frac{\rho V_2^2}{2}\left[1 - \left(\frac{V_1}{V_2}\right)^2\right]$$

$$p_2 = p_{atm}$$

From continuity equation,

$$A_2 V_2 = A_1 V_1$$

Then

$$p_{1g} = \frac{\rho V_2^2}{2}\left[1 - \left(\frac{A_2}{A_1}\right)^2\right] = \frac{\rho V_2^2}{2}\left[1 - \left(\frac{D_2}{D_1}\right)^4\right]$$

$$Q = 24.5\ gpm = 0.0546\ \frac{ft^3}{s}$$

$$V_2 = \frac{Q}{A} = \frac{0.0546\ \frac{ft^3}{s}}{\frac{\pi}{4} \times \left(\frac{1.25}{12}\ ft\right)^2} = 6.41\ \frac{ft}{s}$$

So we have:

$$p_{1g} = \frac{1.94\ \frac{lbf\cdot s^2}{ft^4} \times \left(6.41\ \frac{ft}{s}\right)^2}{2} \times \left[1 - \left(\frac{1.25}{3.25}\right)^4\right] = 39.0\ \frac{lbf}{ft^2}$$

Apply the x-momentum equation to the CV,

$$F_{sx} + F_{Bx} = \frac{\partial}{\partial t}\int_{CV} u\rho d\forall + \int_{CS} u\rho \vec{V}\cdot d\vec{A}$$

$$R_x + p_{1g}A_1 = u_1\{-\dot{m}\} + u_2\{\dot{m}\} = -V_1\dot{m} + V_2\dot{m}$$

$$R_x = -p_{1g}A_1 + \dot{m}(V_2 - V_1) = -p_{1g}A_1 + \rho Q V_2\left(1 - \frac{V_1}{V_2}\right) == -p_{1g}A_1 + \rho Q V_2\left(1 - \left(\frac{D_2}{D_1}\right)^2\right)$$

$$R_x = -39.0\ \frac{lbf}{ft^2} \times \frac{\pi}{4} \times \left(\frac{3.25}{12}\ ft\right)^2 + 1.94\ \frac{lbf\cdot s^2}{ft^4} \times 0.0546\ \frac{ft^3}{s} \times 6.41\ \frac{ft}{s} \times \left(1 - \left(\frac{1.25}{3.25}\right)^2\right)$$

$$R_x = -1.67\ lbf$$

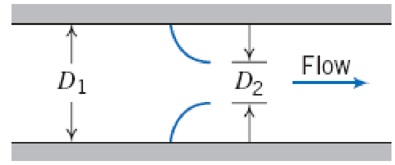
Force of nozzle on flange:

$$K_x = -R_x = 1.67\ lbf$$

Problem 6.53

[Difficulty: 3]

6.53 A flow nozzle is a device for measuring the flow rate in a pipe. This particular nozzle is to be used to measure low-speed air flow for which compressibility may be neglected. During operation, the pressures p_1 and p_2 are recorded, as well as upstream temperature, T_1. Find the mass flow rate in terms of $\Delta p = p_2 - p_1$ and T_1, the gas constant for air, and device diameters D_1 and D_2. Assume the flow is frictionless. Will the actual flow be more or less than this predicted flow? Why?



Given: Flow nozzle

Find: Mass flow rate in terms of Δp, T_1 and D_1 and D_2

Solution:

Basic equation
$$\frac{p}{\rho} + \frac{V^2}{2} + g \cdot z = \text{const} \qquad Q = V \cdot A$$

Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Hence, applying Bernoulli between the inlet (1) and exit (2)

$$\frac{p_1}{\rho} + \frac{V_1^{\,2}}{2} = \frac{p_2}{\rho} + \frac{V_2^{\,2}}{2} \qquad \text{where we ignore gravity effects}$$

But we have
$$Q = V_1 \cdot A_1 = V_1 \cdot \frac{\pi \cdot D_1^{\,2}}{4} = V_2 \cdot \frac{\pi \cdot D_2^{\,2}}{4} \qquad \text{so} \qquad V_1 = V_2 \cdot \left(\frac{D_2}{D_1}\right)^2$$

Note that we assume the flow at D_2 is at the same pressure as the entire section 2; this will be true if there is turbulent mixing

Hence
$$V_2^{\,2} - V_2^{\,2} \cdot \left(\frac{D_2}{D_1}\right)^4 = \frac{2 \cdot (p_2 - p_1)}{\rho} \qquad \text{or} \qquad V_2 = \sqrt{\frac{2 \cdot (p_1 - p_2)}{\rho \cdot \left[1 - \left(\frac{D_2}{D_1}\right)^4\right]}}$$

Then the mass flow rate is
$$m_{flow} = \rho \cdot V_2 \cdot A_2 = \rho \cdot \frac{\pi \cdot D_2^{\,2}}{4} \cdot \sqrt{\frac{2 \cdot (p_1 - p_2)}{\rho \cdot \left[1 - \left(\frac{D_2}{D_1}\right)^4\right]}} = \frac{\pi \cdot D_2^{\,2}}{2 \cdot \sqrt{2}} \cdot \sqrt{\frac{\Delta p \cdot \rho}{\left[1 - \left(\frac{D_2}{D_1}\right)^4\right]}}$$

Using
$$p = \rho \cdot R \cdot T \qquad m_{flow} = \frac{\pi \cdot D_2^{\,2}}{2 \cdot \sqrt{2}} \cdot \sqrt{\frac{\Delta p \cdot p_1}{R \cdot T_1 \cdot \left[1 - \left(\frac{D_2}{D_1}\right)^4\right]}}$$

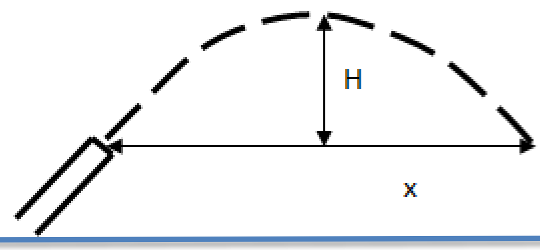
For a flow nozzle
$$m_{flow} = k \cdot \sqrt{\Delta p} \quad \text{where} \qquad k = \frac{\pi \cdot D_2^{\,2}}{2 \cdot \sqrt{2}} \cdot \sqrt{\frac{p_1}{R \cdot T_1 \cdot \left[1 - \left(\frac{D_2}{D_1}\right)^4\right]}}$$

We can expect the actual flow will be less because there is actually significant loss in the device. Also the flow will experience a vena contracta so that the minimum diameter is actually smaller than D_2. We will discuss this device in Chapter 8.

Problem 6.54

(Difficulty 3)

6.54 The head of water on a 50 mm diameter smooth nozzle is 3 m. If the nozzle is directed upward at angles of (a) 30°, (b) 45°, (c) 60°, and (d) 90°, how high above the nozzle will the jet rise, and how far from the nozzle will the jet pass through the horizontal plane in which the nozzle lies? What is the diameter of the jet at the top of the trajectory?



Find: The height h, the distance x and the diameter of jet on the top.

Assumption: The flow is steady, incompressible, uniform, and frictionless.

Solution: Apply the continuity and Bernoulli equations to find the pressure

The continuity equation is:

$$Q = V_1 A_1 = V_2 A_2$$

The Bernoulli equation along a streamline is

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = constant$$

The total head is the 3 m. The velocity leaving the nozzle is then:

$$V = \sqrt{2gH} = \sqrt{2 \times 9.81\ \frac{m}{s^2} \times 3m} = 7.67\ \frac{m}{s}$$

The horizontal velocity at the outlet is:

$$V_x = V \cos\theta$$

The vertical velocity at the outlet is:

$$V_y = V \sin\theta$$

The maximum height can be calculated by:

$$h = \frac{V_y^2}{2g}$$

or

$$h = \frac{V_y^2}{2g} = \frac{\left(7.67\ \frac{m}{s} \times \sin 90°\right)^2}{2 \times 9.81\ \frac{m}{s^2}} = 3.00\ m = H$$

The maximum height is the total head.

To calculate the jet trajectory, we follow a particle of water from the time it leaves the nozzle. Applying Newton's second law

$$F = ma = m\frac{dV_y}{dt}$$

The force on the particle is its weight W, and acts downward. The mass is the weight divided by g. Newton's law is then

$$-W = \frac{W}{g}\frac{dV_y}{dt}$$

Or, separating variables

$$dV_y = -g\ dt$$

Integrating from the initial velocity Vy at time equal 0, the y-component of velocity at any time is

$$V_y(t) = V_y - gt$$

The time for jet to return to the outlet elevation is:

$$t = 2\frac{V_y}{g}$$

The distance x can be calculated by:

$$x = V_x t = V_x 2\frac{V_y}{g}$$

At the top of the jet,

$$V_y = 0$$

The volumetric flow rate is calculated from the velocity leaving the nozzle, and equals the volume flow rate at the top of the trajectory:

$$Q = VA = V\frac{\pi d_o^2}{4} = V_x\frac{\pi d_t^2}{4}$$

The diameter of the jet at the top of the trajectory is related to the nozzle diameter as

$$d_t = d_o \sqrt{\frac{V}{V_x}}$$

$$d_o = 0.05\ m$$

(a) For $\theta = 30°$, the height is

$$h = \frac{V_y^2}{2g} = \frac{\left(7.67\ \frac{m}{s} \times \sin 30°\right)^2}{2 \times 9.81\ \frac{m}{s^2}} = 0.75\ m$$

and the distance is

$$x = 2V_x \frac{V_y}{g} = \frac{2 \times \left(7.67\ \frac{m}{s} \times \sin 30°\right) \times \left(7.67\ \frac{m}{s} \times \cos 30°\right)}{9.81\ \frac{m}{s^2}} = 5.19\ m$$

The jet diameter is

$$d_t = d_o \sqrt{\frac{V}{V_x}} = 0.05\ m \times \sqrt{\frac{7.67\ \frac{m}{s}}{7.67\ \frac{m}{s} \times \cos 30°}} = 53.7\ mm$$

(b) For $\theta = 45°$, the height is

$$h = \frac{V_y^2}{2g} = \frac{\left(7.67\ \frac{m}{s} \times \sin 45°\right)^2}{2 \times 9.81\ \frac{m}{s^2}} = 1.50\ m$$

and the distance is

$$x = 2V_x \frac{V_y}{g} = \frac{2 \times \left(7.67\ \frac{m}{s} \times \sin 45°\right) \times \left(7.67\ \frac{m}{s} \times \cos 45°\right)}{9.81\ \frac{m}{s^2}} = 6\ m$$

The jet diameter is

$$d_t = d_o \sqrt{\frac{V}{V_x}} = 0.05\ m \times \sqrt{\frac{7.67\ \frac{m}{s}}{7.67\ \frac{m}{s} \times \cos 45°}} = 59.5\ mm$$

(c) For $\theta = 60°$, the height is

$$h = \frac{V_y^2}{2g} = \frac{\left(7.67\ \frac{m}{s} \times \sin 60°\right)^2}{2 \times 9.81\ \frac{m}{s^2}} = 2.25\ m$$

and the distance is

$$x = 2V_x \frac{V_y}{g} = \frac{2 \times \left(7.67\ \frac{m}{s} \times \sin 60°\right) \times \left(7.67\ \frac{m}{s} \times \cos 60°\right)}{9.81\ \frac{m}{s^2}} = 5.19\ m$$

The jet diameter is

$$d_t = d_o\sqrt{\frac{V}{V_x}} = 0.05\ m \times \sqrt{\frac{7.67\ \frac{m}{s}}{7.67\ \frac{m}{s} \times \cos 60°}} = 70.7\ mm$$

(d) For $\theta = 90°$ the distance is

$$x = 2V_x\frac{V_y}{g} = \frac{2 \times \left(7.67\ \frac{m}{s} \times \sin 90°\right) \times \left(7.67\ \frac{m}{s} \times \cos 90°\right)}{9.81\ \frac{m}{s^2}} = 0\ m$$

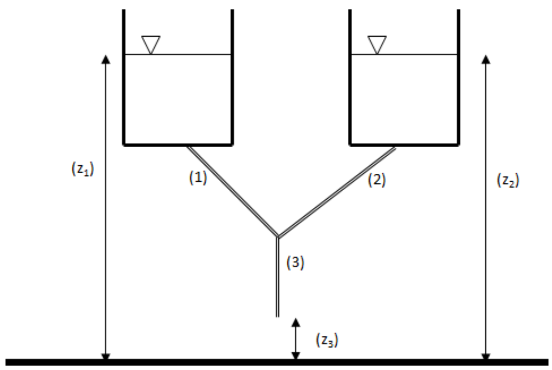
and the diameter is

$$d_t = d_o\sqrt{\frac{V}{V_x}} = 0.05\ m \times \sqrt{\frac{7.67\ \frac{m}{s}}{7.67\ \frac{m}{s} \times \cos 90°}} = infinity$$

Problem 6.55

(Difficulty 3)

6.55 Water flows from one reservoir in a 200 mm pipe, while water flows from a second reservoir in a 150 mm pipe. The two pipes meet in a "tee" junction with a 300 mm pipe that discharges to the atmosphere at an elevation of 20 m. If the water surface in the reservoirs is at 30 m elevation, what is the total flow rate?



Find: The total flow rate.

Assumption: The flow is steady, incompressible, uniform, and frictionless.

Solution: Apply the continuity and Bernoulli equations to find the pressure

The continuity equation is:

$$Q = V\ A$$

The Bernoulli equation along a streamline is

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = constant$$

We will apply the Bernoulli equation along a streamline from water surface of the left hand reservoir to the discharge of the pipe at z_4. This assumes that the flow is frictionless without any irreversibilities such as mixing of the fluid at the junction of pipe 1 and pipe 2. We will discuss this assumption at the end of the problem.

The Bernoulli equation becomes

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_4}{\rho} + \frac{V_4^2}{2} + gz_4$$

For this situation, we have $p_1 = 0$, $V_1 = 0$, and $p_4 = 0$. The Bernoulli equation becomes

$$gz_1 = \frac{V_4^2}{2} + gz_4$$

Or, V_4 is given by

$$V_4 = \sqrt{2g(z_1 - z_4)} = \sqrt{2 \times 9.81\ \frac{m}{s^2} \times (30\ m - 20\ m)} = 14\ \frac{m}{s}$$

The area at 4 is

$$A_4 = \frac{1}{4}\pi D_4^2 = \frac{1}{4} \times \pi \times (0.3\ m)^2 = 0.0707\ m^2$$

And the total flow rate is

$$Q = V_4 A_4 = 14\ \frac{m}{s} \times 0.0707\ m^2 = 0.99\ \frac{m^3}{s}$$

With the assumption that the flow is frictionless from the surface of either reservoir to the outlet, the diameters of pipe 1 and pipe 2 do not matter. In reality they would matter. The combined flow area of these two pipes is 0.049 m2, which is about 70 % of that of pipe 3. Therefore the velocities in pipes 1 and 2 would be 45% greater than that in pipe 3. The deceleration of the flows would create mixing and the frictionless flow assumption would not strictly valid.

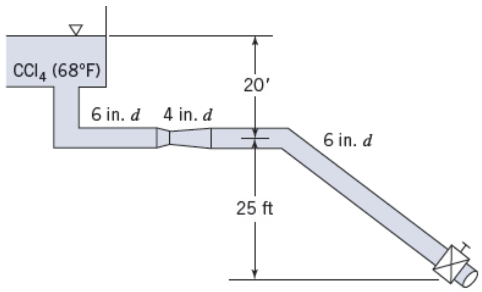
The velocities of the flows in pipes 1 and 2 are equal since the heads are equal. There would then be no mixing between these two streams.

Another factor in the flow might be whether the pressure at the junction was low enough for cavitation to occur.

Problem 6.56

(Difficulty 2)

6.56 Barometric pressure is 14.0 psi. What is the maximum flow rate that can be obtained by opening the valve if (a) cavitation is not a consideration and (b) cavitation needs to be prevented?



Assumption: The flow is steady, incompressible, uniform, and frictionless.

Solution: Apply the Bernoulli equations to find the minimum pressure in the system and check for cavitation.

The Bernoulli equation along a streamline is

$$\frac{p}{\rho}+\frac{V^2}{2}+gz=constant$$

For CCl_4 we have:

$$p_v = 1.9\, psi\ and\ SG = 1.59$$

$$p_{atm} = 14\, psi$$

Assume that the lowest pressure will occur at the 4 in constriction where the velocity is highest. We apply the Bernoulli equation from the water surface where the velocity is zero to the 4 in constriction. We take the height datum as the constriction.

(a) If cavitation is not a problem, we apply the Bernoulli equation from the water surface to the valve:

$$\frac{V_4^2}{2g}=h$$

$$V_4=\sqrt{2gh}=\sqrt{2\times 32.2\,\frac{ft}{s^2}\times 45\,ft}=53.8\,\frac{ft}{s}$$

$$Q=V_4A_4=53.8\,\frac{ft}{s}\times\frac{\pi}{4}\times\left(\frac{6}{12}\,ft\right)^2=10.56\,\frac{ft^3}{s}$$

(b) If we wish to prevent cavitation, the minimum pressure at the constriction will be the vapor pressure p_v:

$$h + \frac{p_{atm}}{\gamma} = \frac{p_v}{\gamma} + \frac{V_4^2}{2g}$$

$$V_4^2 = 2g\left(h + \frac{p_{atm}}{\gamma} - \frac{p_v}{\gamma}\right)$$

The velocity at the constriction is

$$V_4 = \sqrt{2g\left(h + \frac{p_{atm}}{\gamma} - \frac{p_v}{\gamma}\right)} = \sqrt{2 \times 32.2\ \frac{ft}{s^2} \times \left(20\ ft + \frac{(14 - 1.9) \times 144\ \frac{lbf}{ft^2}}{1.59 \times 62.4\ \frac{lbf}{ft^3}}\right)}$$

$$V_4 = 49.2\ \frac{ft}{s}$$

The volumetric flow rate is:

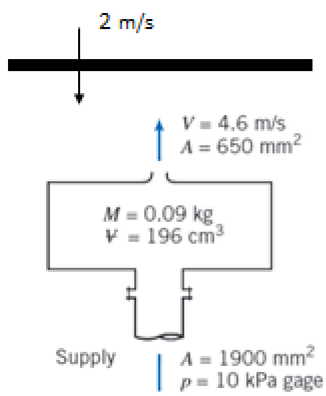
$$Q = V_4 A_4 = 49.2\ \frac{ft}{s} \times \frac{\pi}{4} \times \left(\frac{4}{12}\ ft\right)^2 = 4.29\ \frac{ft^3}{s}$$

This is the maximum flow rate which can occur without cavitation.

Problem 6.57

(Difficulty: 3)

6.57 A spray system is shown in the diagram. Water is supplied at $p = 10\ kPa$ gage, through the flanged opening of area $A = 1900\ mm^2$. The water leaves in a steady free jet at atmospheric pressure. The jet area and speed are $A = 650\ mm^2$ and $V = 4.6\ \frac{m}{s}$. The mass of the spray system is $0.09\ kg$ and it contains $V = 196\ cm^3$ of water. An object, with a flat horizontal lower surface, moves downward into the jet of the spray system with speed $U = 2\ \frac{m}{s}$. Determine the minimum supply pressure needed to produce the jet leaving the spray system. Calculate the maximum pressure exerted by the liquid jet on the flat object at the instant when the object is $h = 0.5\ m$ above the jet exit. Estimate the force of the water jet on the flat object.



P6.57

Given: As shown in the figure.

Assumptions: (1) steady flow

(2) incompressible flow

(3) no friction

(4) flow along a streamline

(5) uniform flow at inlet and outlet

(6) $p_2 = p_{atm}$

(7) neglect z_1 and z_2

(8) neglect body force

Solution:

The minimum pressure occurs when friction is neglected, so we apply the Bernoulli equation from inlet (1) to outlet (2):

$$\frac{p_1}{\rho}+\frac{V_1^2}{2}+gz_1=\frac{p_2}{\rho}+\frac{V_2^2}{2}+gz_2$$

Then

$$p_1=\frac{\rho}{2}(V_2^2-V_1^2)=\frac{\rho}{2}V_2^2\left(1-\left(\frac{V_1}{V_2}\right)^2\right)$$

From the continuity equation we have:

$$V_1A_1=V_2A_2$$

$$\frac{V_1}{V_2}=\frac{A_2}{A_1}$$

$$V_2=4.6\ \frac{m}{s}$$

The minimum pressure is:

$$p_1=\frac{1000\ \frac{kg}{m^3}}{2}\times\left(4.6\ \frac{m}{s}\right)^2\times\left(1-\left(\frac{650\ mm^2}{1900\ mm^2}\right)^2\right)=9.34\ kPa$$

Friction effects would cause this value to be higher.

The maximum pressure of the jet on the object is the stagnation pressure on the object so we have:

$$p_s=\frac{1}{2}\rho V_r^2$$

V_r is the the velocity of the impinging jet relative to the object. At $h=0.5\ m$, the jet velocity can be calculated as:

$$\frac{p_2}{\rho}+\frac{V_2^2}{2}+gz_2=\frac{p_4}{\rho}+\frac{V_4^2}{2}+gz_4$$

$$V_4=\sqrt{V_2^2-2g(z_4-z_2)}=\sqrt{\left(4.6\ \frac{m}{s}\right)^2-2\times9.81\ \frac{m}{s^2}\times(0.5\ m)}=3.37\frac{m}{s}$$

$$V_r=V_4+U=3.37\ \frac{m}{s}+2.0\ \frac{m}{s}=5.37\ \ 5\ \frac{m}{s}$$

So the maximum pressure is:

$$p_s = \frac{1}{2} \times 1000 \ \frac{kg}{m^3} \times \left(5.37 \ \frac{m}{s}\right)^2 = 14.4 \ kPa$$

To determine the force of the water on the object, we apply the z component of the momentum equation to the control vlume of the object:

$$F_{sz} + F_{Bz} = \frac{\partial}{\partial t} \int_{CV} w\rho d\forall + \int_{CS} w\rho \vec{V} \cdot d\vec{A}$$

Then we have:

$$-F_1 = -w|\rho V_r A_4|$$

Where F_1 is applied force necessary to maintain motion or plate at constant speed U.

From the continuity equation we have:

$$A_2 V_2 = A_4 V_4$$

$$A_4 = \frac{V_2}{V_4} A_2 = \frac{4.6 \ \frac{m}{s}}{3.37 \ \frac{m}{s}} \times 650 \ mm^2 = 887 \ mm^2 = 8.87 \times 10^{-4} \ m^2$$

$$F_1 = w|\rho V_r A_4| = \rho V_r^2 A_4$$

$$F_1 = 1000 \ \frac{kg}{m^3} \times \left(5.37 \ \frac{m}{s}\right)^2 \times 8.87 \times 10^{-4} \ m^2 = 25.6 \ N$$

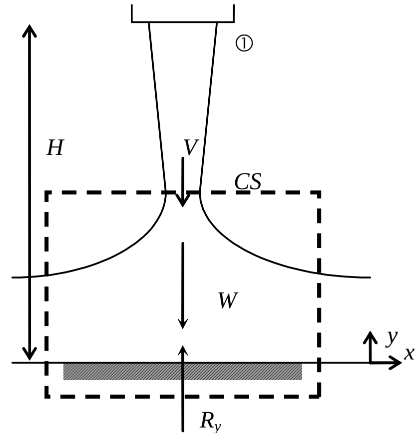
Neglecting the weight of the plate then we have:

$$F_{H_2o} = 25.6 \ N$$

Problem 6.58

[Difficulty: 3]

6.58 Water flows out of a kitchen faucet of 1.25 cm diameter at the rate of 0.1 L/s. The bottom of the sink is 45 cm below the faucet outlet. Will the cross-sectional area of the fluid stream increase, decrease, or remain constant between the faucet outlet and the bottom of the sink? Explain briefly. Obtain an expression for the stream cross section as a function of distance *y* above the sink bottom. If a plate is held directly under the faucet, how will the force required to hold the plate in a horizontal position vary with height above the sink? Explain briefly.



Given: Flow through kitchen faucet

Find: Area variation with height; force to hold plate as function of height

Solution:

Basic equation
$$\frac{p}{\rho} + \frac{V^2}{2} + g \cdot z = \text{const} \qquad Q = V \cdot A \qquad F_y = F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV} v\,\rho\, dV + \int_{CS} v\,\rho\,\vec{V}\cdot d\vec{A}$$

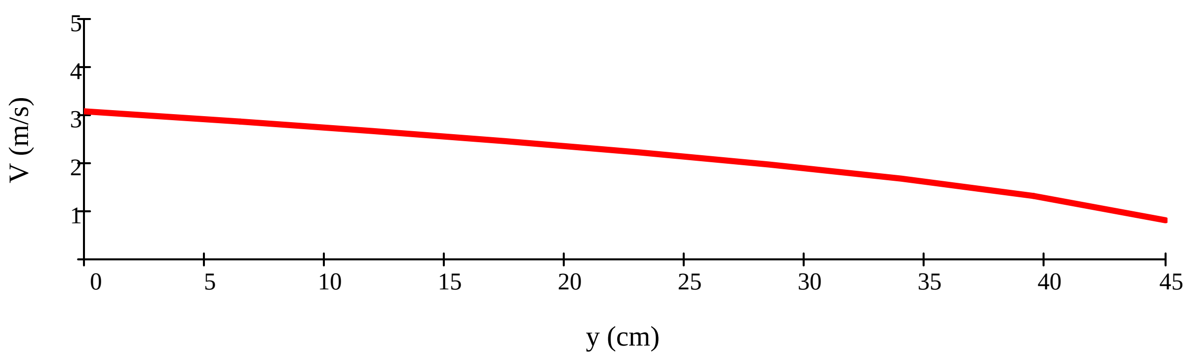
Assumptions: 1) Incompressible flow 2) Inviscid 3) Steady 4) Along a streamline

Hence, applying Bernoulli between the faucet (1) and any height y

$$\frac{V_1^{\ 2}}{2} + g \cdot H = \frac{V^2}{2} + g \cdot y$$ where we assume the water is at p_{atm}

Hence $$V(y) = \sqrt{V_1^{\ 2} + 2 \cdot g \cdot (H - y)}$$

The problem doesn't require a plot, but it looks like $V_1 = 0.815 \frac{m}{s}$ $V(0 \cdot m) = 3.08 \frac{m}{s}$



The speed increases as y decreases because the fluid particles "trade" potential energy for kinetic, just as a falling solid particle does!

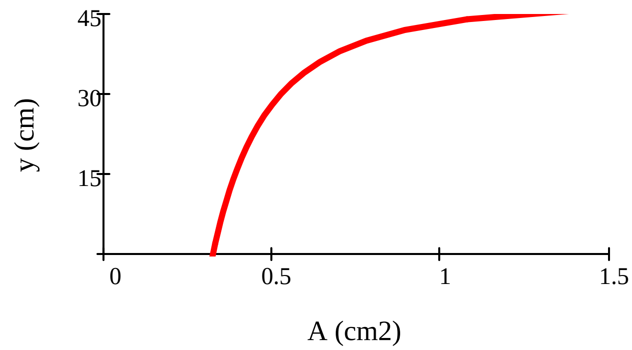
But we have $$Q = V_1 \cdot A_1 = V_1 \cdot \frac{\pi \cdot D^2}{4} = V \cdot A$$

Hence $$A = \frac{V_1 \cdot A_1}{V} \qquad A(y) = \frac{\pi \cdot D_1^{\ 2} \cdot V_1}{4 \cdot \sqrt{V_1^{\ 2} + 2 \cdot g \cdot (H - y)}}$$

The problem doesn't require a plot, but it looks like

$$A(H) = 1.23 \cdot cm^2$$

$$A(0) = 0.325 \cdot cm^2$$



The area decreases as the speed increases. If the stream falls far enough the flow will change to turbulent.

For the CV above

$$R_y - W = u_{in} \cdot \left(-\rho \cdot V_{in} \cdot A_{in}\right) = -V \cdot (-\rho \cdot Q)$$

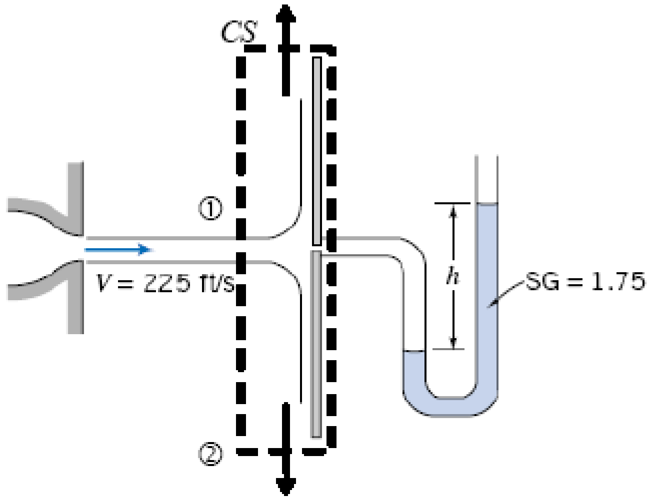
$$R_y = W + \rho \cdot V^2 \cdot A = W + \rho \cdot Q \cdot \sqrt{{V_1}^2 + 2 \cdot g \cdot (H - y)}$$

Hence R_y increases in the same way as V as the height y varies; the maximum force is when y = H

$$R_{ymax} = W + \rho \cdot Q \cdot \sqrt{{V_1}^2 + 2 \cdot g \cdot H}$$

Problem 6.59 [Difficulty: 4]

6.59 A horizontal axisymmetric jet of air with 0.4 in. diameter strikes a stationary vertical disk of 7.5 in. diameter. The jet speed is 225 ft/s at the nozzle exit. A manometer is connected to the center of the disk. Calculate (a) the deflection, if the manometer liquid has SG = 1.75, (b) the force exerted by the jet on the disk, and (c) the force exerted on the disk if it is assumed that the stagnation pressure acts on the entire forward surface of the disk. Sketch the streamline pattern and plot the distribution of pressure on the face of the disk.



Given: Air jet striking disk

Find: Manometer deflection; Force to hold disk; Force assuming p_0 on entire disk; plot pressure distribution

Solution:

Basic equations: Hydrostatic pressure, Bernoulli, and momentum flux in x direction

$$\Delta p = SG\cdot\rho\cdot g\cdot\Delta h \qquad \frac{p}{\rho} + \frac{V^2}{2} + g\cdot z = \text{constant} \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\, d\mathcal{V} + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible 3) No friction 4) Flow along streamline 5) Uniform flow 6) Horizontal flow ($g_x = 0$)

Applying Bernoulli between jet exit and stagnation point

$$\frac{p_{atm}}{\rho_{air}} + \frac{V^2}{2} = \frac{p_0}{\rho_{air}} + 0 \qquad p_0 - p_{atm} = \frac{1}{2}\cdot\rho_{air}\cdot V^2$$

But from hydrostatics $\quad p_0 - p_{atm} = SG\cdot\rho\cdot g\cdot\Delta h \quad$ so $\quad \Delta h = \dfrac{\frac{1}{2}\cdot\rho_{air}\cdot V^2}{SG\cdot\rho\cdot g} = \dfrac{\rho_{air}\cdot V^2}{2\cdot SG\cdot\rho\cdot g}$

$$\Delta h = 0.002377\cdot\frac{slug}{ft^3}\times\left(225\cdot\frac{ft}{s}\right)^2\times\frac{1}{2\cdot 1.75}\times\frac{ft^3}{1.94\cdot slug}\times\frac{s^2}{32.2\cdot ft} \qquad \Delta h = 0.55\cdot ft \qquad \Delta h = 6.60\cdot in$$

For x momentum $\quad R_x = V\cdot\left(-\rho_{air}\cdot A\cdot V\right) = -\rho_{air}\cdot V^2\cdot\dfrac{\pi\cdot d^2}{4}$

$$R_x = -0.002377\cdot\frac{slug}{ft^3}\times\left(225\cdot\frac{ft}{s}\right)^2\times\frac{\pi\cdot\left(\frac{0.4}{12}\cdot ft\right)^2}{4}\times\frac{lbf\cdot s^2}{slug\cdot ft} \qquad R_x = -0.105\cdot lbf$$

The force of the jet on the plate is then $\quad F = -R_x \qquad F = 0.105\cdot lbf$

The stagnation pressure is $\quad p_0 = p_{atm} + \dfrac{1}{2}\cdot\rho_{air}\cdot V^2$

The force on the plate, assuming stagnation pressure on the front face, is

$$F = \left(p_0 - p\right)\cdot A = \frac{1}{2}\cdot\rho_{air}\cdot V^2\cdot\frac{\pi\cdot D^2}{4}$$

$$F = \frac{\pi}{8} \times 0.002377\cdot\frac{\text{slug}}{\text{ft}^3} \times \left(225\cdot\frac{\text{ft}}{\text{s}}\right)^2 \times \left(\frac{7.5}{12}\cdot\text{ft}\right)^2 \times \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} \qquad F = 18.5\cdot\text{lbf}$$

Obviously this is a huge overestimate!

For the pressure distribution on the disk, we use Bernoulli between the disk outside edge any radius r for radial flow

$$\frac{p_{atm}}{\rho_{air}} + \frac{1}{2}\cdot v_{edge}^2 = \frac{p}{\rho_{air}} + \frac{1}{2}\cdot v^2$$

We need to obtain the speed v as a function of radius. If we assume the flow remains constant thickness h, then

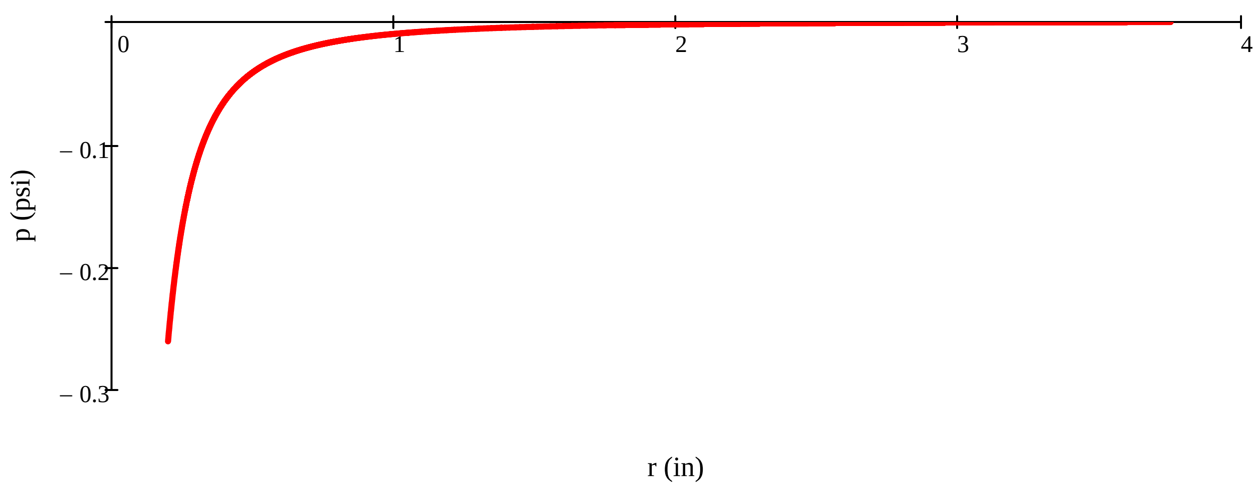
$$Q = v\cdot 2\cdot\pi\cdot r\cdot h = V\cdot\frac{\pi\cdot d^2}{4} \qquad\qquad v(r) = V\cdot\frac{d^2}{8\cdot h\cdot r}$$

We need an estimate for h. As an approximation, we assume that h = d (this assumption will change the scale of p(r) but not the basic shape)

Hence $\qquad v(r) = V\cdot\frac{d}{8\cdot r}$

Using this in Bernoulli $\qquad p(r) = p_{atm} + \frac{1}{2}\cdot\rho_{air}\cdot\left(v_{edge}^2 - v(r)^2\right) = p_{atm} + \frac{\rho_{air}\cdot V^2\cdot d^2}{128}\cdot\left(\frac{4}{D^2} - \frac{1}{r^2}\right)$

Expressed as a gage pressure $\qquad p(r) = \frac{\rho_{air}\cdot V^2\cdot d^2}{128}\cdot\left(\frac{4}{D^2} - \frac{1}{r^2}\right)$

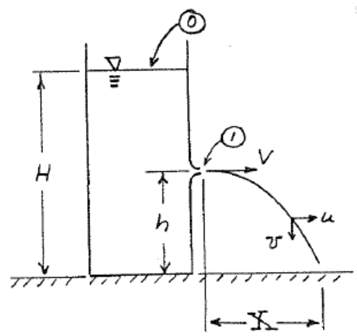


Problem 6.60

(Difficulty: 4)

6.60 The water level in a large tank is maintained at height H above the surrounding level terrain. A rounded nozzle placed in the side of the tank discharges a horizontal jet. Neglecting friction, determine the height h at which the orifice should be placed so the water strikes the ground at the maximum horizontal distance X from the tank. Plot jet speed V and distance X as functions of h $(0 < h < H)$.

Assumptions: (1) steady flow (2) incompressible flow (3) flow along streamline (4) no friction



Solution: Apply Bernoulli equation between tank surface and jet:

$$\frac{p_0}{\rho} + \frac{V_0^2}{2} + gz_0 = \frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1$$

Then

$$gH = \frac{V_1^2}{2} + gh$$

$$V_1 = \sqrt{2g(H-h)}$$

Assume no air resistance in the stream. Then $u = constant$, and $X = ut = \sqrt{2g(H-h)}t$.

The only force acting on the stream is gravity,

$$\sum F_y = -Mg = Ma_y = M\frac{dv}{dt}$$

$$\frac{dv}{dt} = -g$$

Integrating we obtain,

$$v = -gt$$

$$y = y_0 - \frac{1}{2}gt^2$$

$$t = \left[\frac{2(y_0 - y)}{g}\right]^{\frac{1}{2}}$$

The time of flight is then,

$$t = \sqrt{\frac{2y_0}{g}} = \sqrt{\frac{2h}{g}}$$

Substituting into the equation:

$$X = \sqrt{2g(H-h)}\sqrt{\frac{2h}{g}} = 2\sqrt{h(H-h)}$$

X will be maximized when $h(H-h)$ is maximum, or when:

$$\frac{d}{dh}[h(H-h)] = 0$$

$$H - 2h = 0$$

$$h = \frac{H}{2}$$

The corresponding range is:

$$X = 2\sqrt{\frac{H}{2}\left(\frac{H}{2}\right)} = H$$

We have the equations as:

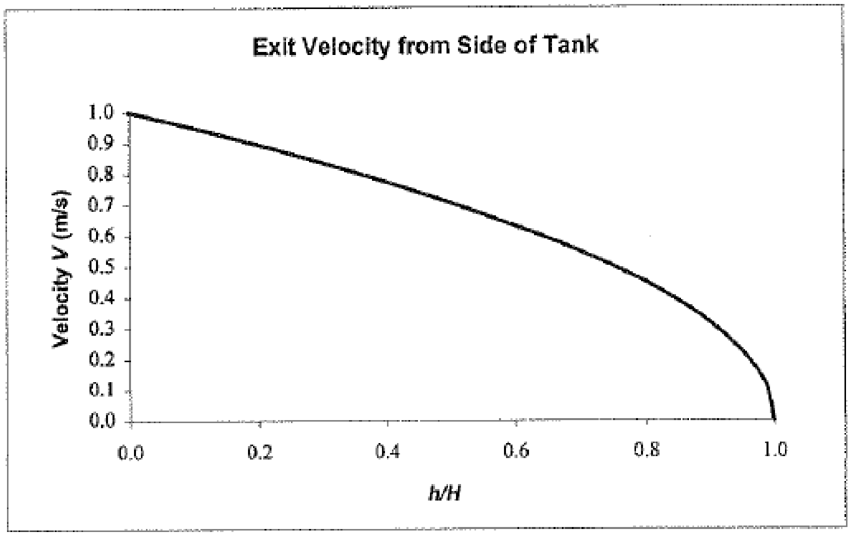
$$\frac{V_1}{\sqrt{2gH}} = \sqrt{1 - \frac{h}{H}}$$

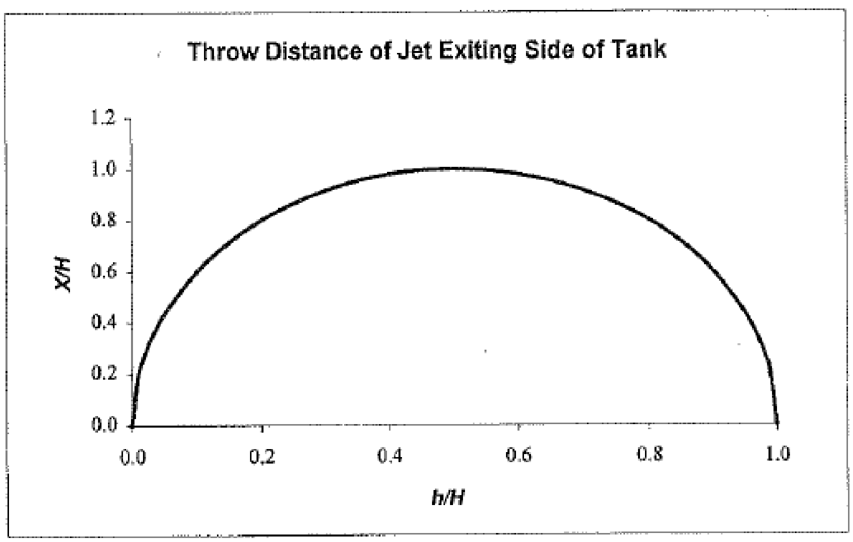
$$\frac{X}{H} = 2\sqrt{\frac{h}{H}\left(1 - \frac{h}{H}\right)}$$

The plots are shown as:

Exit velocity and throw distance from orifice in side of tank, versus height *h/H*

h/H	$V/(2gH)^{1/2}$	*X/H*
0.00	1.00	0.000
0.01	0.995	0.199
0.02	0.990	0.280
0.03	0.985	0.341
0.04	0.980	0.392
0.05	0.975	0.436
0.10	0.949	0.600
0.15	0.922	0.714
0.20	0.894	0.800
0.25	0.866	0.866
0.30	0.837	0.917
0.35	0.806	0.954
0.40	0.775	0.980
0.45	0.742	0.995
0.50	0.707	1.000
0.55	0.671	0.995
0.60	0.632	0.980
0.65	0.592	0.954
0.70	0.548	0.917
0.75	0.500	0.866
0.80	0.447	0.800
0.85	0.387	0.714
0.90	0.316	0.600
0.95	0.224	0.436
0.96	0.200	0.392
0.97	0.173	0.341
0.98	0.141	0.280
0.99	0.100	0.199
1.00	0.00	0.00





Problem 6.61

[Difficulty: 3]

6.61 Many recreation facilities use inflatable "bubble" structures. A tennis bubble to enclose four courts is shaped roughly as a circular semicylinder with a diameter of 50 ft and a length of 50 ft. The blowers used to inflate the structure can maintain the air pressure inside the bubble at 0.75 in. of water above ambient pressure. The bubble is subjected to a wind that blows at 35 mph in a direction perpendicular to the axis of the semicylindrical shape. Using polar coordinates, with angle θ measured from the ground on the upwind side of the structure, the resulting pressure distribution may be expressed as

$$\frac{p - p_\infty}{\frac{1}{2}\rho V_\infty^2} = 1 - 4\sin^2\theta$$

where p is the pressure at the surface, p_∞ the atmospheric pressure, and V_w the wind speed. Determine the net vertical force exerted on the structure.

Given: Air flow over "bubble" structure

Find: Net vertical force

Solution: The net force is given by $\vec{F} = \int p\, d\vec{A}$ also $\Delta p = \rho \cdot g \cdot \Delta h$

Available data $L = 50\cdot ft$ $R = 25ft$ $V = 35\cdot mph$ $\Delta h = 0.75\cdot in$ $\rho = 1.94\cdot\frac{slug}{ft^3}$ $\rho_{air} = 0.00238\cdot\frac{slug}{ft^3}$

The internal pressure is $\Delta p = \rho \cdot g \cdot \Delta h$ $\Delta p = 187\ Pa$

Through symmetry only the vertical component of force is no-zero $F_V = \int_0^\pi (p_i - p)\cdot \sin(\theta)\cdot R\cdot L\, d\theta$

where pi is the internal pressure and p the external $p_i = p_{atm} + \Delta p$ $p = p_{atm} - \frac{1}{2}\cdot\rho_{air}\cdot V^2\cdot\left(1 - 4\cdot\sin(\theta)^2\right)$

Hence

$$F_V = \int_0^\pi \left[\Delta p - \frac{1}{2}\cdot\rho_{air}\cdot V^2\cdot\left(1 - 4\cdot\sin(\theta)^2\right)\right]\cdot\sin(\theta)\cdot R\cdot L\, d\theta$$

$$F_V = R\cdot L\cdot\Delta p\cdot\int_0^\pi \sin(\theta)\, d\theta - R\cdot L\cdot\frac{1}{2}\cdot\rho_{air}\cdot V^2\cdot\int_0^\pi \left(1 - 4\cdot\sin(\theta)^2\right)\cdot\sin(\theta)\, d\theta$$

But $\int \left(\sin(\theta) - 4\cdot\sin(\theta)^3\right) d\theta = -\cos(\theta) + 4\cdot\left(\cos(\theta) - \frac{1}{3}\cdot\cos(\theta)^3\right)$ so $\int_0^\pi \left(\sin(\theta) - 4\cdot\sin(\theta)^3\right) d\theta = -\frac{10}{3}$

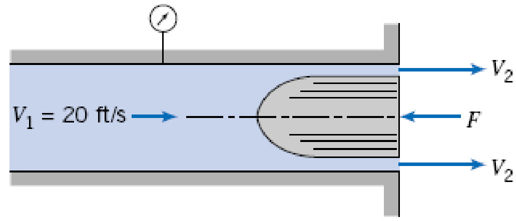
$\int \sin(\theta)\, d\theta = -\cos(\theta)$ so $\int_0^\pi \sin(\theta)\, d\theta = 2$

Combining results $F_V = R\cdot L\cdot\left(2\cdot\Delta p + \frac{5}{3}\cdot\rho_{air}\cdot V^2\right)$ $F_V = 2.28\times 10^4\cdot lbf$ $F_V = 22.8\cdot kip$

Problem 6.62

[Difficulty: 4]

6.62 Water flows at low speed through a circular tube with inside diameter of 2 in. A smoothly contoured body of 1.5 in. diameter is held in the end of the tube where the water discharges to atmosphere. Neglect frictional effects and assume uniform velocity profiles at each section. Determine the pressure measured by the gage and the force required to hold the body.



Given: Water flow out of tube

Find: Pressure indicated by gage; force to hold body in place

Solution:

Basic equations: Bernoulli, and momentum flux in x direction

$$\frac{p}{\rho} + \frac{V^2}{2} + g \cdot z = \text{constant} \qquad Q = V \cdot A \qquad F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\,\rho\,dV\!\!\!\!- + \int_{CS} u\,\rho\,\vec{V}\cdot d\vec{A}$$

Assumptions: 1) Steady flow 2) Incompressible 3) No friction 4) Flow along streamline 5) Uniform flow 6) Horizontal flow ($g_x = 0$)

Applying Bernoulli between jet exit and stagnation point

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} = \frac{p_2}{\rho} + \frac{V_2^2}{2} = \frac{V_2^2}{2} \qquad \text{where we work in gage pressure}$$

$$p_1 = \frac{\rho}{2}\cdot\left(V_2^2 - V_1^2\right)$$

But from continuity $Q = V_1 \cdot A_1 = V_2 \cdot A_2$ $\qquad V_2 = V_1 \cdot \frac{A_1}{A_2} = V_1 \cdot \frac{D^2}{D^2 - d^2}$ $\qquad$ where D = 2 in and d = 1.5 in

$$V_2 = 20 \cdot \frac{ft}{s} \cdot \left(\frac{2^2}{2^2 - 1.5^2}\right) \qquad V_2 = 45.7 \cdot \frac{ft}{s}$$

Hence

$$p_1 = \frac{1}{2} \times 1.94 \cdot \frac{slug}{ft^3} \times \left(45.7^2 - 20^2\right)\cdot\left(\frac{ft}{s}\right)^2 \times \frac{lbf \cdot s^2}{slug \cdot ft} \qquad p_1 = 1638 \cdot \frac{lbf}{ft^2} \qquad p_1 = 11.4 \cdot psi \qquad \text{(gage)}$$

The x mometum is $\quad -F + p_1 \cdot A_1 - p_2 \cdot A_2 = u_1 \cdot \left(-\rho \cdot V_1 \cdot A_1\right) + u_2 \cdot \left(\rho \cdot V_2 \cdot A_2\right)$

$$F = p_1 \cdot A_1 + \rho \cdot \left(V_1^2 \cdot A_1 - V_2^2 \cdot A_2\right) \qquad \text{using gage pressures}$$

$$F = 11.4 \cdot \frac{lbf}{in^2} \times \frac{\pi \cdot (2 \cdot in)^2}{4} + 1.94 \cdot \frac{slug}{ft^3} \times \left[\left(20 \cdot \frac{ft}{s}\right)^2 \times \frac{\pi \cdot (2 \cdot in)^2}{4} - \left(45.7 \cdot \frac{ft}{s}\right)^2 \times \frac{\pi \cdot \left[(2 \cdot in)^2 - (1.5 \cdot in)^2\right]}{4}\right] \times \left(\frac{1 \cdot ft}{12 \cdot in}\right)^2 \times \frac{lbf \cdot s^2}{slug \cdot ft}$$

$$F = 14.1 \cdot lbf \qquad \text{in the direction shown}$$

Problem 6.63

[Difficulty: 5]

6.63 Describe the pressure distribution on the exterior of a multistory building in a steady wind. Identify the locations of the maximum and minimum pressures on the outside of the building. Discuss the effect of these pressures on infiltration of outside air into the building.

Open-Ended Problem Statement: Describe the pressure distribution on the exterior of a multistory building in a steady wind. Identify the locations of the maximum and minimum pressures on the outside of the building. Discuss the effect of these pressures on infiltration of outside air into the building.

Discussion: A multi-story building acts as a bluff-body obstruction in a thick atmospheric boundary layer. The boundary-layer velocity profile causes the air speed near the top of the building to be highest and that toward the ground to be lower.

Obstruction of air flow by the building causes regions of stagnation pressure on upwind surfaces. The stagnation pressure is highest where the air speed is highest. Therefore the maximum surface pressure occurs near the roof on the upwind side of the building. Minimum pressure on the upwind surface of the building occurs near the ground where the air speed is lowest.

The minimum pressure on the entire building will likely be in the low-speed, low-pressure wake region on the downwind side of the building.

Static pressure inside the building will tend to be an average of all the surface pressures that act on the outside of the building. It is never possible to seal all openings completely. Therefore air will tend to infiltrate into the building in regions where the outside surface pressure is above the interior pressure, and will tend to pass out of the building in regions where the outside surface pressure is below the interior pressure. Thus generally air will tend to move through the building from the upper floors toward the lower floors, and from the upwind side to the downwind side.

Problem 6.64

[Difficulty: 5]

6.64 An aspirator provides suction by using a stream of water flowing through a venturi. Analyze the shape and dimensions of such a device. Comment on any limitations on its use.

Open-Ended Problem Statement: An aspirator provides suction by using a stream of water flowing through a venturi. Analyze the shape and dimensions of such a device. Comment on any limitations on its use.

Discussion: The basic shape of the aspirator channel should be a converging nozzle section to reduce pressure followed by a diverging diffuser section to promote pressure recovery. The basic shape is that of a venturi flow meter.

If the diffuser exhausts to atmosphere, the exit pressure will be atmospheric. The pressure rise in the diffuser will cause the pressure at the diffuser inlet (venturi throat) to be below atmospheric.

A small tube can be brought in from the side of the throat to aspirate another liquid or gas into the throat as a result of the reduced pressure there.

The following comments can be made about limitations on the aspirator:

1. It is desirable to minimize the area of the aspirator tube compared to the flow area of the venturi throat. This minimizes the disturbance of the main flow through the venturi and promotes the best possible pressure recovery in the diffuser.
2. It is desirable to avoid cavitation in the throat of the venturi. Cavitation alters the effective shape of the flow channel and destroys the pressure recovery in the diffuser. To avoid cavitation, the reduced pressure must always be above the vapor pressure of the driver liquid.
3. It is desirable to limit the flow rate of gas into the venturi throat. A large amount of gas can alter the flow pattern and adversely affect pressure recovery in the diffuser.

The best combination of specific dimensions could be determined experimentally by a systematic study of aspirator performance. A good starting point probably would be to use dimensions similar to those of a commercially available venturi flow meter.

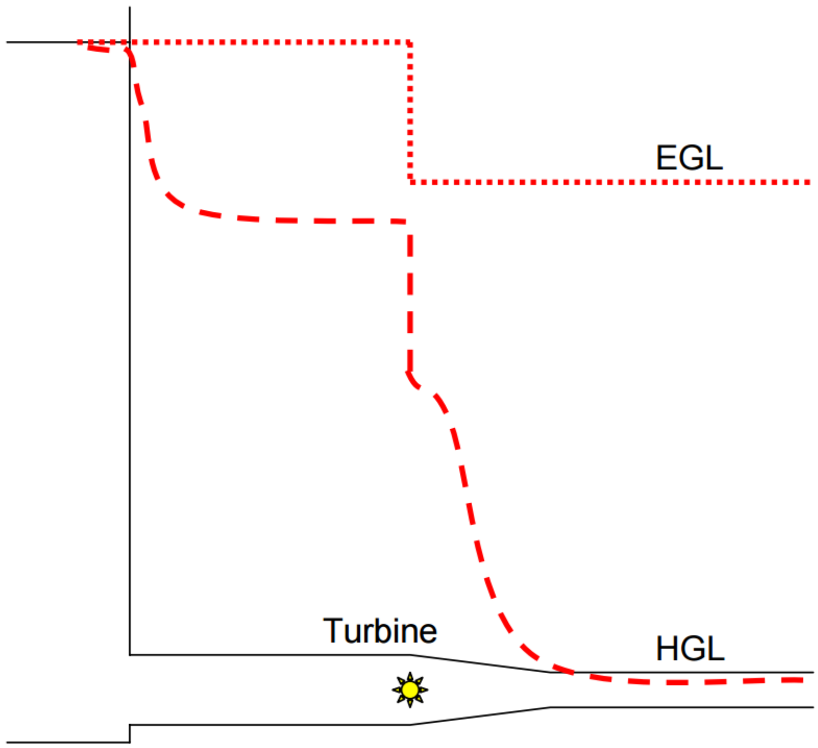
Problem 6.65

(Difficulty: 2)

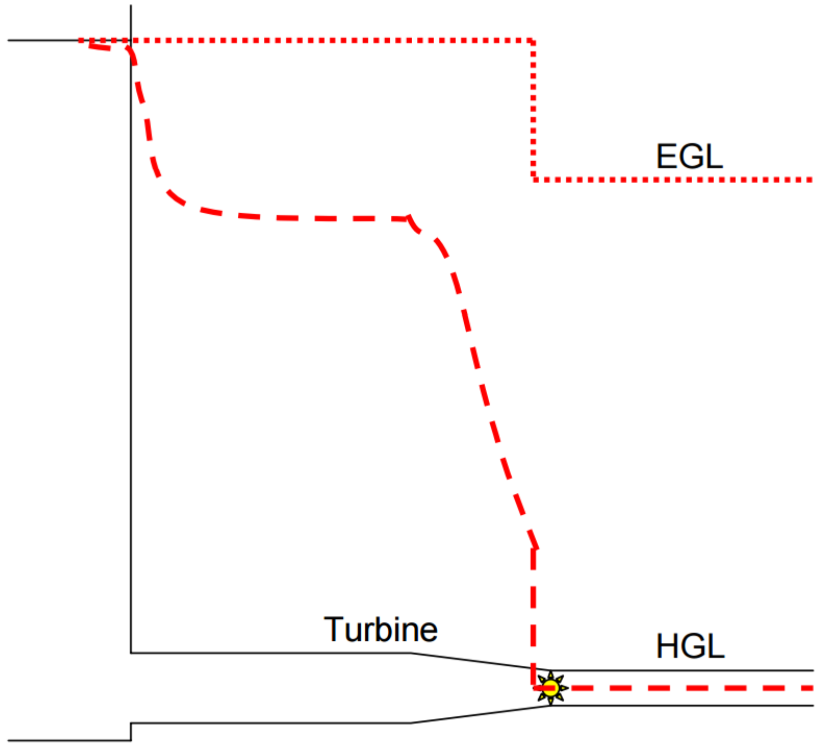
6.65 Carefully sketch the energy grade lines (EGL) and hydraulic grade lines (HGL) for the system shown in Fig. 6.6 if the pipe is horizontal (i.e., the outlet is at the base of the reservoir), and a water turbine (extracting energy) is located at point 2, or at point 3. In chapter 8 we will investigate the effects of friction on internal flows. Can you anticipate and sketch the effect of the friction on the EGL and HGL for the two cases?

Solution:

(a) For the turbine located at point 2, the EGL and HGL would be:



(b) For the turbine located at point 3, the RGL and HGL would be



The effect of friction would be that the EGL would tend to drop: suddenly at the contraction, gradually in the large pipe, more steeply in the small pipe. The HGL would then “hang” below the HGL in a manner similar to that shown.

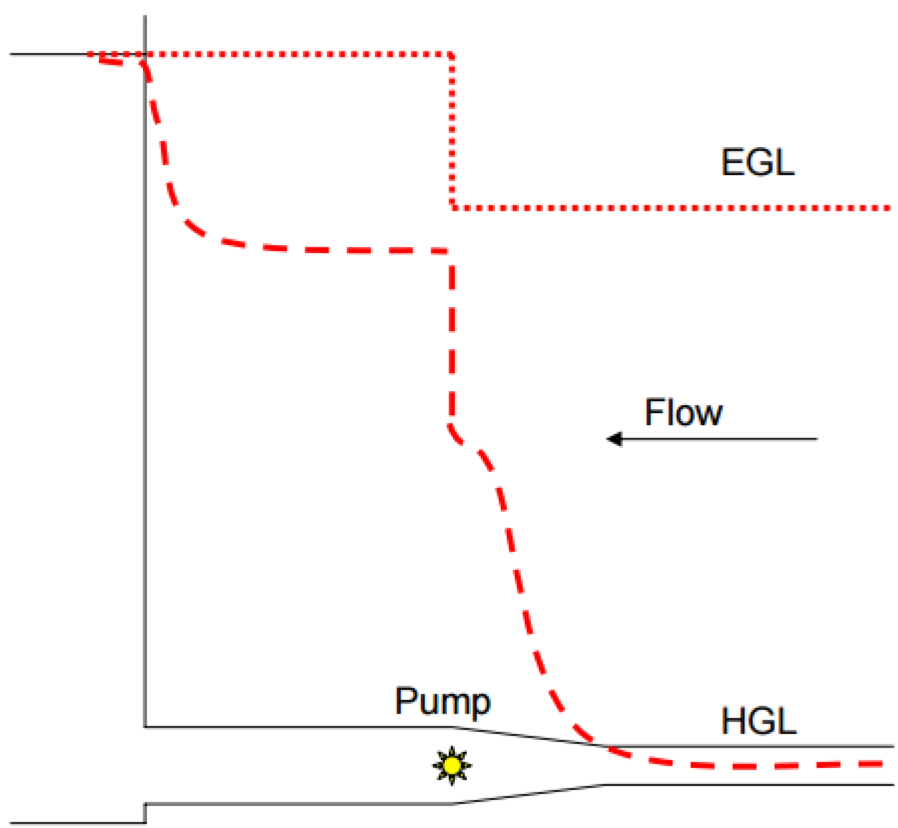
Problem 6.66

(Difficulty: 2)

6.66 Carefully sketch the energy grade lines (EGL) and hydraulic grade lines (HGL) for the system shown in Fig. 6.6 if a pump adding energy to the fluid is located at point (2), such that flow is into the reservoir. In Chapter 8 we will investigate the effects of friction on internal flows. Can you anticipate and sketch the effect of friction on the EGL and HGL for the two cases?

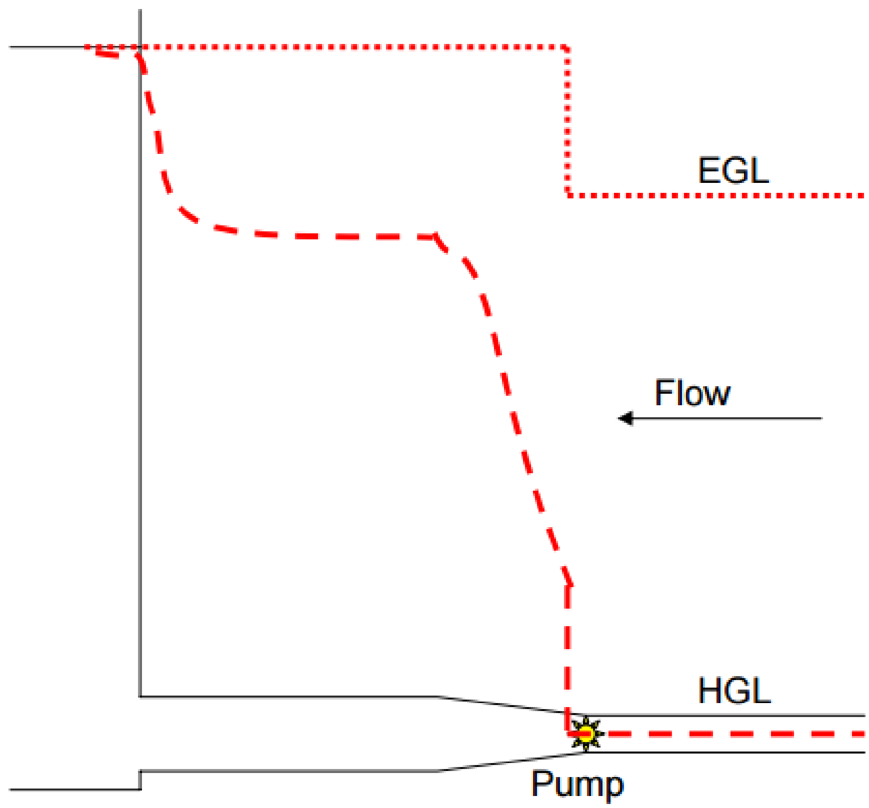
Solution:

(a) The EGL and HGL for the a pump at point 2 is:



Note that the effect of friction would be that the EGL would tend to drop from right to left: steeply in the small pipe, gradually in the large pipe, and suddenly at the expansion. The HGL would then "hang" below the EGL in a manner similar to that shown.

(b) Note that the effect of friction would be that the EGL would tend to drop from right to left: steeply in the small pipe, gradually in the large pipe, and suddenly at the expansion. The HGL would then "hang" below the HGL in a manner similar to that shown.



Problem 6.67

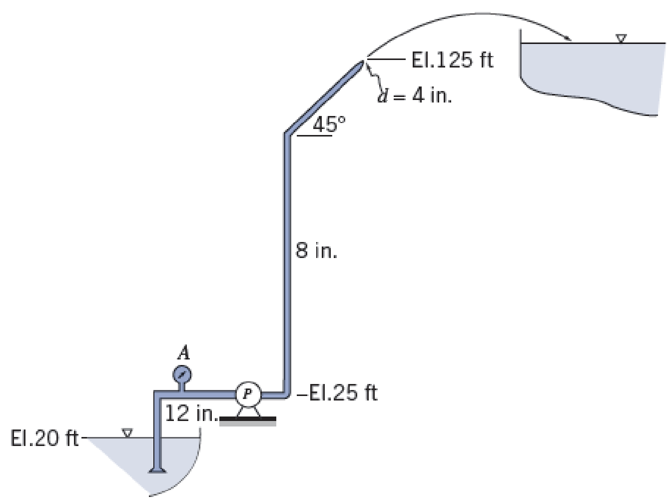
(Difficulty 2)

6.67 Water is being pumped from the lower reservoir through a nozzle into the upper reservoir. If the vacuum gage at A reads 2.4 psi vacuum,

(a) find the flow velocity through the nozzle.

(b) find the horsepower the pump must add to the water.

(c) draw the energy line and the hydraulic grade line.



Assumption: The flow is steady, incompressible, uniform, and frictionless.

Solution: Apply the continuity and Bernoulli equations to find the velocity and power.

The continuity equation is:

$$Q = V_1 A_1 = V_2 A_2$$

The Bernoulli equation along a streamline is

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = constant$$

The gage pressure at A is:

$$p_A = -2.4\ psi = -345.6\ \frac{lbf}{ft^2}$$

Applying the Bernoulli equation from station 1 to station A:

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_A}{\rho} + \frac{V_A^2}{2} + gz_A$$

Where $p_1 = 0$ and $V_1 = 0$

$$gz_1 = \frac{p_A}{\rho} + \frac{V_A^2}{2} + gz_A$$

The water density is

$$\rho = 1.938\ \frac{slug}{ft^3} = 1.938\ \frac{lbf \cdot s^2}{ft^4}$$

The velocity at A is then

$$V_A = \sqrt{2gz_1 - \frac{2p_A}{\rho} - 2gz_A} = \sqrt{2 \times 32.2\ \frac{ft}{s^2} \times (20\ ft - 25\ ft) - \frac{2 \times -345.6\ \frac{lbf}{ft^2}}{1.938\ \frac{lbf \cdot s^2}{ft^4}}} = 5.90\ \frac{ft}{s}$$

The volumetric flow rate is:

$$Q = V_A A_A = 5.90\ \frac{ft}{s} \times \frac{\pi}{4} \times \left(\frac{12}{12}\ ft\right)^2 = 4.63\ \frac{ft^3}{s}$$

Applying the continuity equation, the volumetric flow rate at location 2 is the same as at A:

$$V_2 = \frac{Q}{A_2} = \frac{4.63\ \frac{ft^3}{s}}{\frac{\pi}{4} \times \left(\frac{4}{12}\ ft\right)^2} = 53.1\ \frac{ft}{s}$$

Applying the energy equation from station 1 to station 2:

$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 + E_p = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

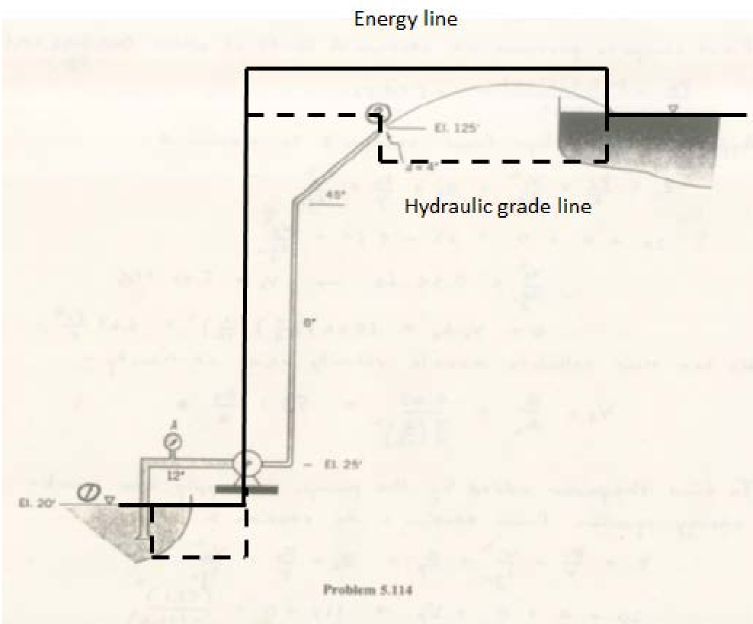
Where E_p is the head provided by the pump

$$E_p = (z_2 - z_1) + \frac{V_2^2}{2g} = 148.8\ ft$$

The pump power is then

$$\dot{W}_p = \frac{Q\rho g E_p}{550} = 70\ hp$$

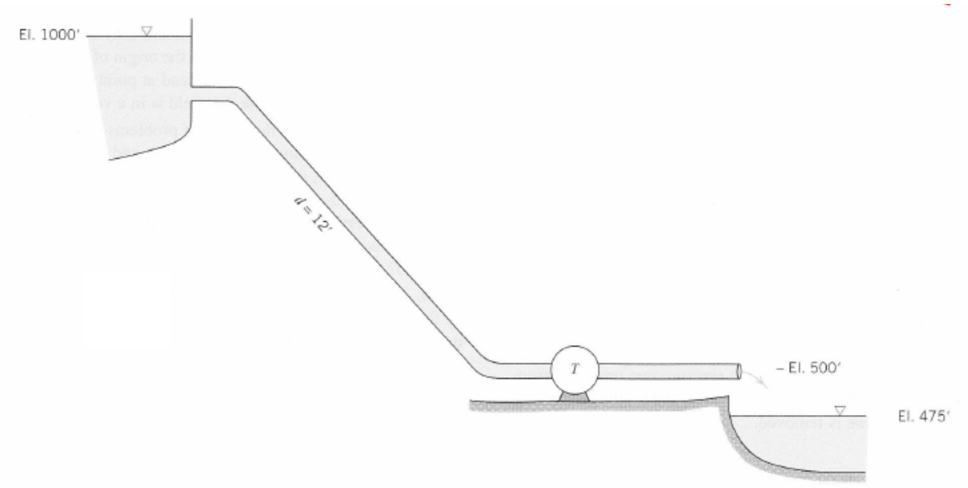
The energy line and hydraulic grade lines are



Problem 6.68

(Difficulty: 2)

6.68 The turbine extracts power from the water flowing from the reservoir. Find the horsepower extracted if the flow through the system is 1000 $\frac{ft^3}{s}$. Draw the energy line and the hydraulic grade line.



Find: The power produced by the turbine

Assumption: The flow is steady, incompressible, uniform, and frictionless.

Solution: Apply the continuity and Bernoulli equations to find the power.

The continuity equation is:

$$Q = V_1 A_1 = V_2 A_2$$

The Bernoulli equation along a streamline is

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = constant$$

The volumetric flow rate Q is:

$$Q = 1000\ \frac{ft^3}{s}$$

From the continuity equation, the velocity at station 2:

$$V_2 = \frac{Q}{A_2} = \frac{1000\ \frac{ft^3}{s}}{\frac{\pi}{4} \times (12\ ft)^2} = 8.84\ \frac{ft}{s}$$

Applying the energy equation from station 1 to station 2, where E_T is the turbine head:

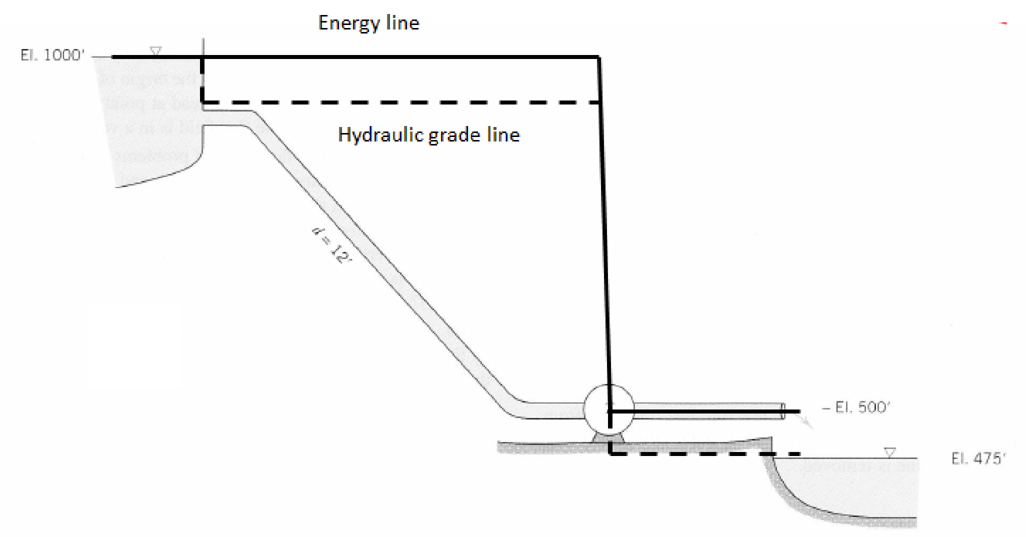
$$\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + z_1 = E_T + \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + z_2$$

$$E_T = 499\ ft$$

The power is be calculated by:

$$\dot{W}_T = \frac{Q\rho g E_T}{550} = 56600\ hp$$

The energy line and hydraulic grade line are shown in the figure.



Problem 6.69

(Difficulty: 2)

6.69 Consider a two-dimensional fluid flow: $u = ax + by$ and $v = cx + dy$, where a, b, c, and d are constant. If the flow is incompressible and irrotational, find the relationships among a, b, c, and d. Find the stream function and velocity potential function of this flow.

Given: 2D incompressible, inviscid flow field

Find: Relationships among constants; stream function; velocity potential.

Solution:

Basic equations

For incompressible flow the velocity components are

$$u = \frac{\partial \Psi}{\partial y}$$

$$v = -\frac{\partial \Psi}{\partial x}$$

For irrotational flow

$$u = -\frac{\partial \phi}{\partial x}$$

$$v = -\frac{\partial \phi}{\partial y}$$

Check to be certain the flow is incompressible:

$$\frac{\partial u(x,y)}{\partial x} + \frac{\partial v(x,y)}{\partial y} = a + d = 0$$

This requires that:

$$d = -a$$

Check to see if the flow is irrotational

$$\frac{\partial v(x,y)}{\partial x} - \frac{\partial u(x,y)}{\partial y} = c - b$$

This requires that:

$$c = b$$

Solving for the stream function:

$$\Psi(x,y) = \int u(x,y)dy = \frac{by^2}{2} + axy + f(x)$$

$$\frac{\partial \Psi}{\partial x} = ay + \frac{df}{dx} = -v = -cx - dy$$

$$\frac{df}{dx} = -ay - dy - cx = -cx$$

$$f(x) = -\frac{cx^2}{2}$$

So the stream function is:

$$\Psi(x,y) = \frac{by^2}{2} + axy - \frac{cx^2}{2} = axy + \frac{b}{2}(y^2 - x^2)$$

Solving for the velocity potential:

$$\phi(x,y) = -\int u(x,y)dx = -\frac{ax^2}{2} - bxy + g(y)$$

$$\frac{\partial \phi(x,y)}{\partial y} = -bx + \frac{dg}{dy} = -cx - dy$$

$$\frac{dg}{dy} = bx - cx - dy = -dy$$

$$g(y) = -\frac{dy^2}{2}$$

So the velocity potential is:

$$\phi(x,y) = -\frac{ax^2}{2} - bxy - \frac{dy^2}{2}$$

Problem 6.70

(Difficulty: 2)

6.70 The velocity field for a two-dimensional flow is $\vec{V} = (Ax - By)t\hat{\imath} - (Bx + Ay)t\hat{\jmath}$, where $A = 1\ s^{-2}$ $B = 2\ s^{-2}$, t is in seconds, and the coordinates are measured in meters. Is this a possible incompressible flow? Is the flow steady or unsteady? Show that the flow is irrotational and derive an expression for the velocity potential.

Assumption: The flow is ideal

Solution: Use the continuity expression and the relation for irrotational flow

For incompressible flow we have:

$$\nabla \cdot \vec{V} = 0$$

For given flow:

$$\nabla \cdot \vec{V} = \frac{\partial (Ax - By)t}{\partial x} - \frac{\partial (Bx + Ay)t}{\partial y} = At - At = 0$$

The velocity field represents a possible incompressible flow.

The flow is unsteady since we have:

$$\vec{V} = \vec{V}(x, y, t)$$

The rotation is given by:

$$\vec{\omega} = \frac{1}{2}\nabla \times \vec{V} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

$$\vec{\omega} = \frac{1}{2}\left[\frac{\partial - (Bx + Ay)t}{\partial x} - \frac{\partial (Ax - By)t}{\partial y}\right] = -Bt + Bt = 0$$

The flow is irrotational.

From the definition of the velocity potential,

$$\vec{V} = -\nabla \phi$$

$$u = -\frac{\partial \phi}{\partial x}$$

$$v = -\frac{\partial \phi}{\partial y}$$

$$\phi = \int -udx + f(y,t) = \int -(Ax - By)tdx + f(y,t)$$

$$\phi = \left(-A\frac{x^2}{2} + Bxy\right)t + f(y,t)$$

$$v = -\frac{\partial\phi}{\partial y} = -Bxt - \frac{\partial f(y,t)}{\partial y} = -(Bx + Ay)t$$

$$\frac{\partial f(y,t)}{\partial y} = Ayt$$

$$f(y,t) = \frac{1}{2}Ay^2t$$

Hence we have:

$$\phi = \left(-A\frac{x^2}{2} + Bxy\right)t + \frac{1}{2}Ay^2t$$

$$\phi = \left\{\frac{A}{2}(y^2 - x^2) + Bxy\right\}t$$

Problem 6.71

(Difficulty: 2)

6.71 A flow field is characterized by the stream function $\Psi = Axy$ where $A = 2\,s^{-1}$ and the coordinates are measured in feet. Verify that the flow is irrotational and determine the velocity potential ϕ. Plot the streamlines and potential lines and visually verify that they are orthogonal.

Find: The velocity potential ϕ and plot the streamline and potential lines.

Assumption: The flow is incompressible, steady, and frictionless

Solution: Apply the definitions of the stream function and the velocity potential

The flow is irrotational if the vorticity is zero. The vorticity is defined as:

$$\vec{\xi} = \nabla \times \vec{V}$$

As we have for the stream function:

$$\Psi = 2xy$$

The velocities u and v are

$$u = \frac{\partial \Psi}{\partial y} = 2x$$

$$v = -\frac{\partial \Psi}{\partial x} = -2y$$

Thus the vorticity is

$$\vec{\xi} = \nabla \times \vec{V} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k} = (0 - 0)\hat{k} = 0$$

This flow is irrotational.

For the velocity potential, u and v are given by:

$$u = -\frac{\partial \phi}{\partial x}$$

$$v = -\frac{\partial \phi}{\partial y}$$

So we have:

$$\frac{\partial \phi}{\partial x} = -u = -2x$$

$$\frac{\partial \phi}{\partial y} = -v = 2y$$

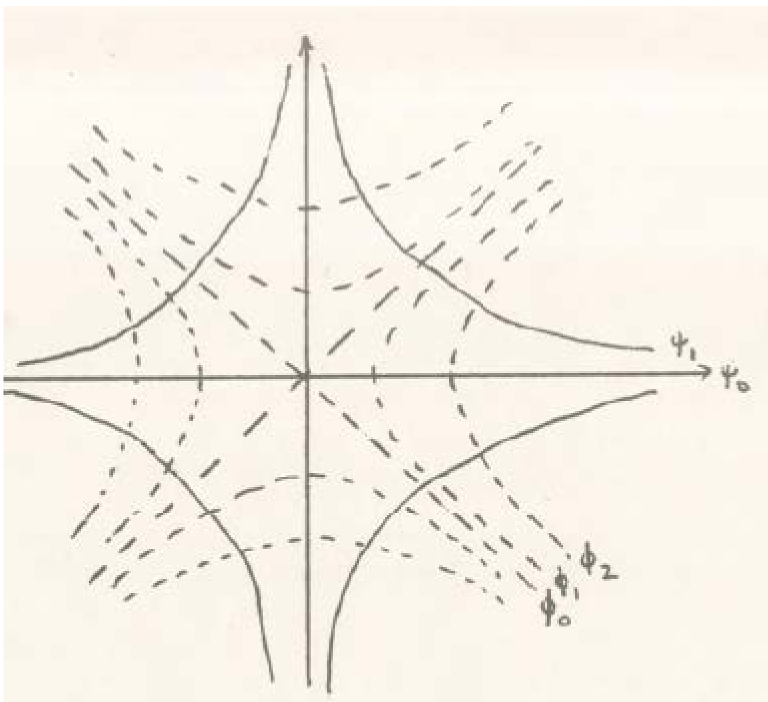
$$\phi(x, y) = -x^2 + f(y)$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial f}{\partial y} = 2y$$

$$f(y) = y^2 + c$$

$$\phi(x, y) = -x^2 + y^2 + c$$

Plotting the streamlines and potential lines:



Problem 6.72 [Difficulty: 3]

6.72 The flow field for a plane source at a distance h above an infinite wall aligned along the x axis is given by

$$\vec{V} = \frac{q}{2\pi[x^2 + (y-h)^2]}[x\hat{i} + (y-h)\hat{j}] + \frac{q}{2\pi[x^2 + (y+h)^2]}[x\hat{i} + (y+h)\hat{j}]$$

where q is the strength of the source. The flow is irrotational and incompressible. Derive the stream function and velocity potential. By choosing suitable values for q and h, plot the streamlines and lines of constant velocity potential. (*Hint*: Use the *Excel* workbook of Example 6.10.)

Given: Velocity field of irrotational and incompressible flow

Find: Stream function and velocity potential; plot

Solution:

The velocity field is

$$u = \frac{q\cdot x}{2\cdot\pi\left[x^2 + (y-h)^2\right]} + \frac{q\cdot x}{2\cdot\pi\left[x^2 + (y+h)^2\right]} \qquad v = \frac{q\cdot(y-h)}{2\cdot\pi\left[x^2 + (y-h)^2\right]} + \frac{q\cdot(y+h)}{2\cdot\pi\left[x^2 + (y+h)^2\right]}$$

The basic equations are

$$u = \frac{\partial}{\partial y}\psi \qquad v = -\frac{\partial}{\partial x}\psi \qquad u = -\frac{\partial}{\partial x}\phi \qquad v = -\frac{\partial}{\partial y}\phi$$

Hence for the stream function

$$\psi = \int u(x,y)\,dy = \frac{q}{2\cdot\pi}\cdot\left(\operatorname{atan}\left(\frac{y-h}{x}\right) + \operatorname{atan}\left(\frac{y+h}{x}\right)\right) + f(x)$$

$$\psi = -\int v(x,y)\,dx = \frac{q}{2\cdot\pi}\cdot\left(\operatorname{atan}\left(\frac{y-h}{x}\right) + \operatorname{atan}\left(\frac{y+h}{x}\right)\right) + g(y)$$

The simplest expression for ψ is

$$\psi(x,y) = \frac{q}{2\cdot\pi}\cdot\left(\operatorname{atan}\left(\frac{y-h}{x}\right) + \operatorname{atan}\left(\frac{y+h}{x}\right)\right)$$

For the stream function

$$\phi = -\int u(x,y)\,dx = -\frac{q}{4\cdot\pi}\cdot\ln\left[\left[x^2 + (y-h)^2\right]\cdot\left[x^2 + (y+h)^2\right]\right] + f(y)$$

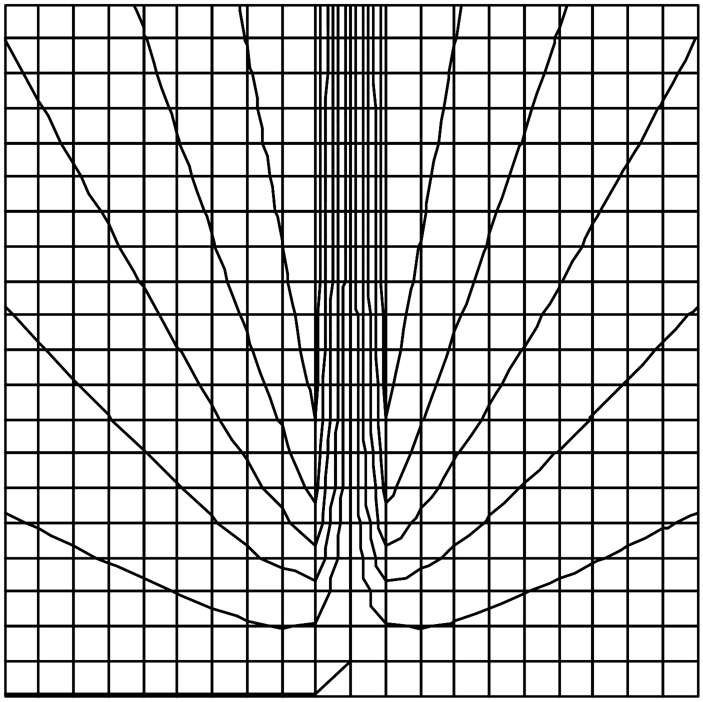
$$\phi = -\int v(x,y)\,dy = -\frac{q}{4\cdot\pi}\cdot\ln\left[\left[x^2 + (y-h)^2\right]\cdot\left[x^2 + (y+h)^2\right]\right] + g(x)$$

The simplest expression for φ is

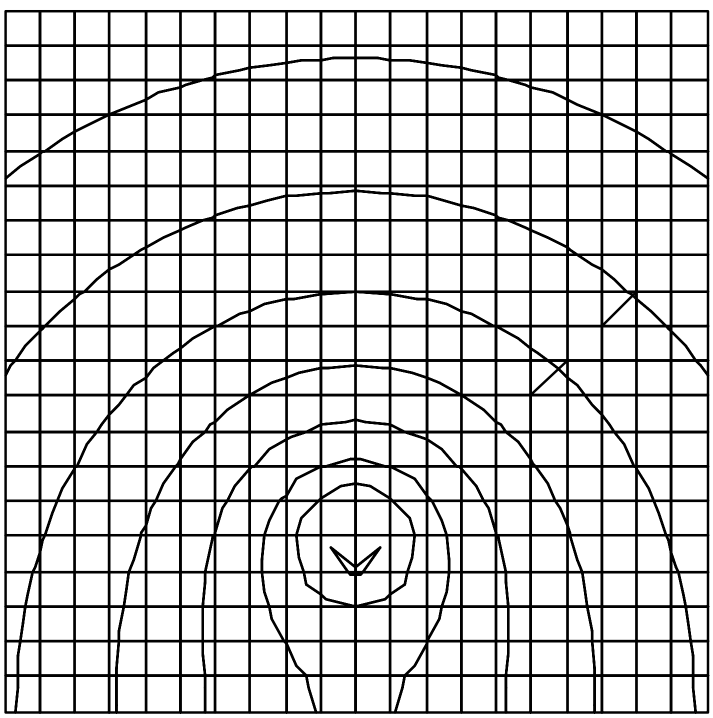
$$\phi(x,y) = -\frac{q}{4\cdot\pi}\cdot\ln\left[\left[x^2 + (y-h)^2\right]\cdot\left[x^2 + (y+h)^2\right]\right]$$

In *Excel*:

Stream Function



Velocity Potential



Problem 6.73 [Difficulty: 2]

6.73 The stream function of a flow field is $\psi = Ax^2y - By^3$, where $A = 1\ \text{m}^{-1}\cdot\text{s}^{-1}$, $B = \frac{1}{3}\ \text{m}^{-1}\cdot\text{s}^{-1}$, and the coordinates are measured in meters. Find an expression for the velocity potential.

Given: Stream function

Find: Velocity potential

Solution:

Basic equations: Incompressibility because ψ exists $\quad u = \frac{\partial}{\partial y}\psi \quad v = -\frac{\partial}{\partial x}\psi \quad u = -\frac{\partial}{\partial x}\varphi \quad v = -\frac{\partial}{\partial y}\varphi$

Irrotationality $\quad \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$

We have $\quad \psi(x,y) = A\cdot x^2\cdot y - B\cdot y^3$

Then $\quad u(x,y) = \frac{\partial}{\partial y}\psi(x,y) \qquad u(x,y) = A\cdot x^2 - 3\cdot B\cdot y^2$

$v(x,y) = -\frac{\partial}{\partial x}\psi(x,y) \qquad v(x,y) = -2\cdot A\cdot x\cdot y$

Then $\quad \frac{\partial}{\partial x}v(x,y) - \frac{\partial}{\partial y}u(x,y) = 6\cdot B\cdot y - 2\cdot A\cdot y \quad$ but $\quad 6\cdot B - 2\cdot A = 0\frac{1}{\text{m}\cdot\text{s}} \quad$ hence flow is IRROTATIONAL

Hence $\quad u = -\frac{\partial}{\partial x}\varphi \quad$ so $\quad \varphi(x,y) = -\int u(x,y)\,dx + f(y) \rightarrow \varphi(x,y) = f(y) - \frac{A\cdot x^3}{3} + 3\cdot B\cdot x\cdot y^2$

$v = -\frac{\partial}{\partial y}\varphi \quad$ so $\quad \varphi(x,y) = -\int v(x,y)\,dy + g(x) \rightarrow \varphi(x,y) = A\cdot x\cdot y^2 + g(x)$

Comparing, the simplest velocity potential is then $\quad \varphi(x,y) = A\cdot x\cdot y^2 - \frac{A\cdot x^3}{3}$

Problem 6.74

(Difficulty: 2)

6.74 A flow field is characterized by the stream function

$$\Psi = 2y + \frac{1}{2\pi}\left(\tan^{-1}\frac{y-a}{x} - \tan^{-1}\frac{y+a}{x}\right)$$

Derive an expression for the location of the stagnation points. Sketch the flow field.

Find: The stagnation points in the flow

Assumption: The flow is incompressible, steady, and frictionless

Solution: Apply the definitions of the stream function

The stream function is given by

$$\Psi = 2y + \frac{1}{2\pi}\left(\tan^{-1}\frac{y-a}{x} - \tan^{-1}\frac{y+a}{x}\right)$$

The velocity components u and v can be calculated as:

$$u = \frac{\partial \Psi}{\partial y} = 2 + \frac{1}{2\pi}\left[\frac{x}{x^2 + (y-a)^2} - \frac{x}{x^2 + (y+a)^2}\right]$$

$$v = -\frac{\partial \Psi}{\partial x} = -\frac{1}{2\pi}\left[\frac{y+a}{x^2 + (y+a)^2} - \frac{y-a}{x^2 + (y-a)^2}\right]$$

The stagnation points are where:

$$u = 0$$

$$v = 0$$

For the stagnation point we have for u = 0:

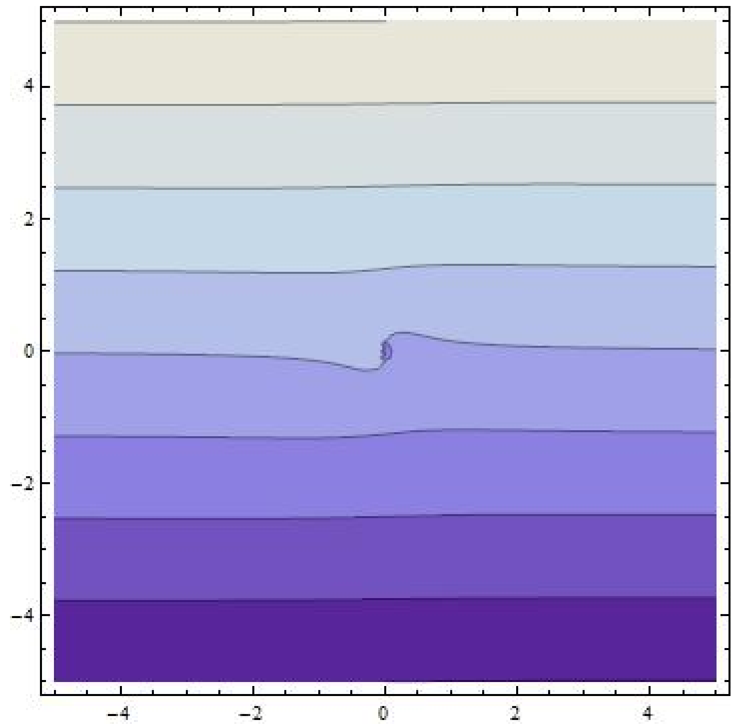
$$\frac{xy}{[x^2 + (y-a)^2][x^2 + (y+a)^2]} = -\frac{\pi}{a}$$

And for v = 0

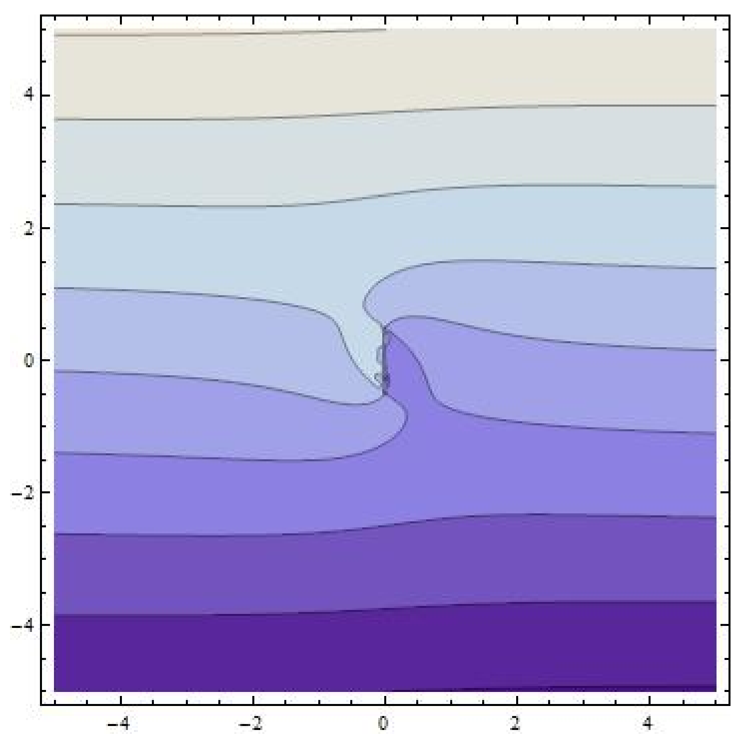
$$x^2 + a^2 = y^2$$

The above equations govern all of the stagnation points in the flow field. We plot the flow field as follows:

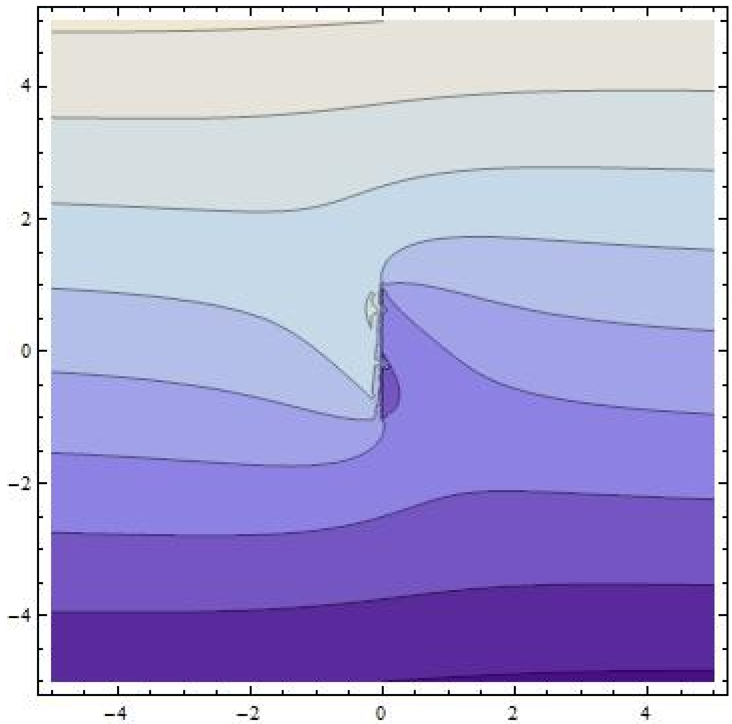
For $a = 0.1$,



For $a = 0.5$,



For $a = 1$,



Problem 6.75

(Difficulty: 2)

6.75 A flow field is characterized by the stream function

$$\Psi = xy^2 + Bx^3$$

What does the value of B need to be for the flow to be irrotational? For that value of B, determine the velocity potential ϕ. Sketch the streamlines and potential lines.

Find: The value of B for irrotational flow and the velocity potential

Assumption: The flow is incompressible, steady, and frictionless

Solution: Apply the definitions of the stream function

The flow is irrotational if the vorticity is zero. The vorticity is defined as:

$$\vec{\xi} = \nabla \times \vec{V} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

The velocity field is calculated in terms of the stream function as:

$$u = \frac{\partial \Psi}{\partial y} = 2xy$$

$$v = -\frac{\partial \Psi}{\partial x} = -y^2 - 3Bx^2$$

Thus

$$\nabla \times \boldsymbol{V} = -6Bx - 2x = 0$$

The vorticity will be zero if

$$B = -\frac{1}{3}$$

To find the velocity potential we use the definitions:

$$u = -\frac{\partial \phi}{\partial x}$$

$$v = -\frac{\partial \phi}{\partial y}$$

For the expression for u we have

$$\frac{\partial \phi}{\partial x} = -u = -2xy$$

Integrating

$$\phi = -x^2y + f(y)$$

From the expression for v we have

$$\frac{\partial \phi}{\partial y} = -v = y^2 - x^2 = -x^2 + \frac{\partial f}{\partial y}$$

The function f is then

$$\frac{\partial f}{\partial y} = y^2$$

$$f(y) = \frac{1}{3}y^3 + C$$

Where C is a constant.

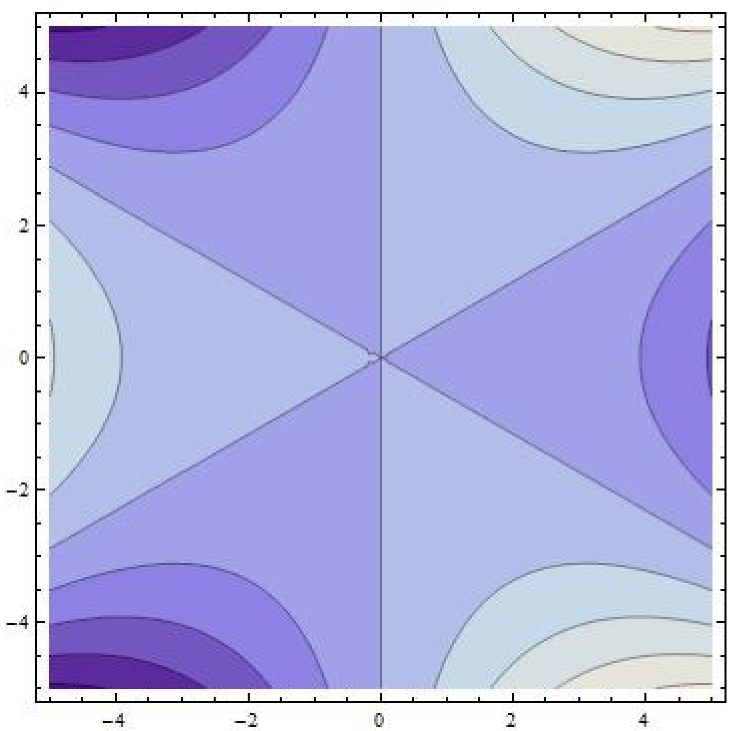
Thus the velocity potential is

$$\phi = -x^2y + \frac{1}{3}y^3 + C$$

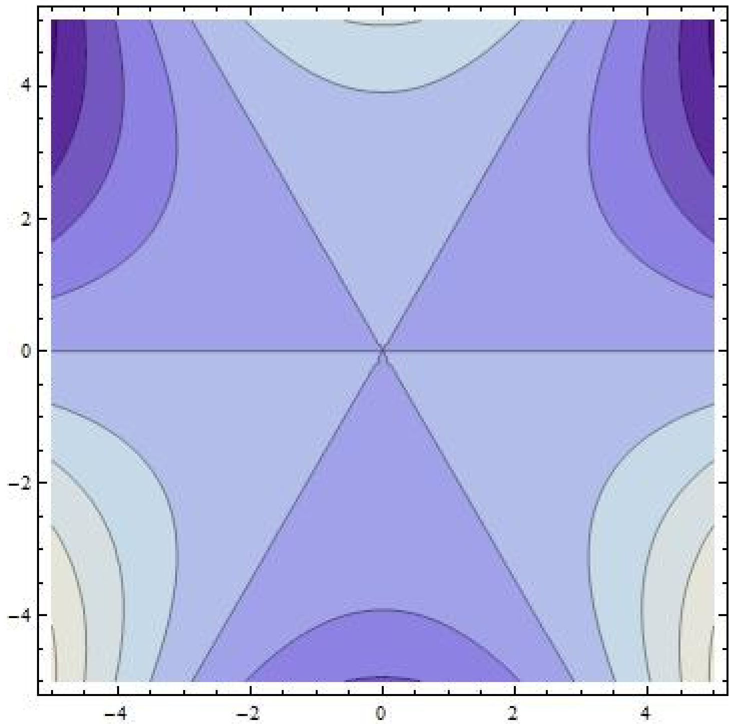
And the stream function is:

$$\Psi = xy^2 - \frac{1}{3}x^3$$

The streamlines are shown as:



The potential lines are shown as:



We can find that the streamlines and potential lines are orthogonal .

Problem 6.76

[Difficulty: 2]

6.76 The stream function of a flow field is $\psi = Ax^3 - Bxy^2$, where $A = 1\ \text{m}^{-1}\cdot\text{s}^{-1}$ and $B = 3\ \text{m}^{-1}\cdot\text{s}^{-1}$, and coordinates are measured in meters. Find an expression for the velocity potential.

Given: Stream function

Find: Velocity potential

Solution:

Basic equations: Incompressibility because ψ exists $\quad u = \frac{\partial}{\partial y}\psi \quad v = -\frac{\partial}{\partial x}\psi \quad u = -\frac{\partial}{\partial x}\varphi \quad v = -\frac{\partial}{\partial y}\varphi$

Irrotationality $\quad \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$

We have $\quad \psi(x,y) = A\cdot x^3 - B\cdot x\cdot y^2$

Then $\quad u(x,y) = \frac{\partial}{\partial y}\psi(x,y) \qquad u(x,y) = -2\cdot B\cdot x\cdot y$

$v(x,y) = -\frac{\partial}{\partial x}\psi(x,y) \qquad v(x,y) = B\cdot y^2 - 3\cdot A\cdot x^2$

Then $\quad \frac{\partial}{\partial x}v(x,y) - \frac{\partial}{\partial y}u(x,y) = 2\cdot B\cdot x - 6\cdot A\cdot x$ but $2\cdot B - 6\cdot A = 0\,\frac{1}{m\cdot s}$ hence flow is IRROTATIONAL

Hence $\quad u = -\frac{\partial}{\partial x}\varphi$ so $\quad \varphi(x,y) = -\int u(x,y)\,dx + f(y) \rightarrow \varphi(x,y) = B\cdot y\cdot x^2 + f(y)$

$v = -\frac{\partial}{\partial y}\varphi$ so $\quad \varphi(x,y) = -\int v(x,y)\,dy + g(x) \rightarrow \varphi(x,y) = g(x) - \frac{B\cdot y^3}{3} + 3\cdot A\cdot x^2\cdot y$

Comparing, the simplest velocity potential is then $\quad \varphi(x,y) = 3\cdot A\cdot x^2\cdot y - \frac{B\cdot y^3}{3}$

Problem 6.77

[Difficulty: 2]

6.77 A flow field is represented by the stream function $\psi = x^5 - 15x^4y^2 + 15x^2y^4 - y^6$. Find the corresponding velocity field. Show that this flow field is irrotational and obtain the potential function.

Given: Stream function

Find: Velocity field; Show flow is irrotational; Velocity potential

Solution:

Basic equations: Incompressibility because ψ exists $\quad u = \frac{\partial}{\partial y}\psi \quad v = -\frac{\partial}{\partial x}\psi \quad u = -\frac{\partial}{\partial x}\varphi \quad v = -\frac{\partial}{\partial y}\varphi$

Irrotationality $\quad \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$

We have $\quad \psi(x,y) = x^6 - 15\cdot x^4\cdot y^2 + 15\cdot x^2\cdot y^4 - y^6$

$$u(x,y) = \frac{\partial}{\partial y}\psi(x,y) \qquad u(x,y) = 60\cdot x^2\cdot y^3 - 30\cdot x^4\cdot y - 6\cdot y^5$$

$$v(x,y) = -\frac{\partial}{\partial x}\psi(x,y) \qquad v(x,y) = 60\cdot x^3\cdot y^2 - 6\cdot x^5 - 30\cdot x\cdot y^4$$

$$\frac{\partial}{\partial x}v(x,y) - \frac{\partial}{\partial y}u(x,y) = 0$$

Hence flow is IRROTATIONAL

Hence $\quad u = -\frac{\partial}{\partial x}\varphi \quad$ so $\quad \varphi(x,y) = -\int u(x,y)\,dx + f(y) = 6\cdot x^5\cdot y - 20\cdot x^3\cdot y^3 + 6\cdot x\cdot y^5 + f(y)$

$v = -\frac{\partial}{\partial y}\varphi \quad$ so $\quad \varphi(x,y) = -\int v(x,y)\,dy + g(x) = 6\cdot x^5\cdot y - 20\cdot x^3\cdot y^3 + 6\cdot x\cdot y^5 + g(x)$

Comparing, the simplest velocity potential is then $\quad \varphi(x,y) = 6\cdot x^5\cdot y - 20\cdot x^3\cdot y^3 + 6\cdot x\cdot y^5$

Problem 6.78

[Difficulty: 2]

6.78 Consider the flow field presented by the potential function $\phi = x^5 - 10x^3y^2 + 5xy^4 - x^2 + y^2$. Verify that this is an incompressible flow, and obtain the corresponding stream function.

Given: Velocity potential

Find: Show flow is incompressible; Stream function

Solution:

Basic equations: Irrotationality because φ exists $u = \frac{\partial}{\partial y}\psi$ $\quad v = -\frac{\partial}{\partial x}\psi$ $\quad u = -\frac{\partial}{\partial x}\varphi$ $\quad v = -\frac{\partial}{\partial y}\varphi$

Incompressibility $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$

We have $\varphi(x,y) = x^5 - 10\cdot x^3\cdot y^2 + 5\cdot x\cdot y^4 - x^2 + y^2$

$$u(x,y) = -\frac{\partial}{\partial x}\varphi(x,y) \qquad u(x,y) = 30\cdot x^2\cdot y^2 - 5\cdot x^4 + 2\cdot x - 5\cdot y^4$$

$$v(x,y) = -\frac{\partial}{\partial y}\varphi(x,y) \qquad v(x,y) = 20\cdot x^3\cdot y - 20\cdot x\cdot y^3 - 2\cdot y$$

Hence $\frac{\partial}{\partial x}u(x,y) + \frac{\partial}{\partial y}v(x,y) = 0$ Hence flow is INCOMPRESSIBLE

Hence $u = \frac{\partial}{\partial y}\psi$ so $\psi(x,y) = \int u(x,y)\,dy + f(x) = 10\cdot x^2\cdot y^3 - 5\cdot x^4\cdot y + 2\cdot x\cdot y - y^5 + f(x)$

$v = -\frac{\partial}{\partial x}\psi$ so $\psi(x,y) = -\int v(x,y)\,dx + g(y) = 10\cdot x^2\cdot y^3 - 5\cdot x^4\cdot y + 2\cdot x\cdot y + g(y)$

Comparing, the simplest stream function is then $\psi(x,y) = 10\cdot x^2\cdot y^3 - 5\cdot x^4\cdot y + 2\cdot x\cdot y - y^5$

Problem 6.79 [Difficulty: 4]

6.79 Show by expanding and collecting real and imaginary terms that $f = z^6$ (where z is the complex number $z = x + iy$) leads to a valid velocity potential (the real part of f) and a corresponding stream function (the negative of the imaginary part of f) of an irrotational and incompressible flow. Then show that the real and imaginary parts of df/dz yield $-u$ and v, respectively.

Given: Complex function

Find: Show it leads to velocity potential and stream function of irrotational incompressible flow; Show that df/dz leads to u and v

Solution:

Basic equations: Irrotationality because φ exists $u = \frac{\partial}{\partial y}\psi$ $v = -\frac{\partial}{\partial x}\psi$ $u = -\frac{\partial}{\partial x}\varphi$ $v = -\frac{\partial}{\partial y}\varphi$

Incompressibility $\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$ Irrotationality $\frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u = 0$

$f(z) = z^6 = (x + i\cdot y)^6$ Expanding $f(z) = x^6 - 15\cdot x^4\cdot y^2 + 15\cdot x^2\cdot y^4 - y^6 + i\cdot\left(6\cdot x\cdot y^5 + 6\cdot x^5\cdot y - 20\cdot x^3\cdot y^3\right)$

We are thus to check the following

$$\varphi(x,y) = x^6 - 15\cdot x^4\cdot y^2 + 15\cdot x^2\cdot y^4 - y^6 \qquad \psi(x,y) = -\left(6\cdot x\cdot y^5 + 6\cdot x^5\cdot y - 20\cdot x^3\cdot y^3\right)$$

$$u(x,y) = -\frac{\partial}{\partial x}\varphi(x,y) \quad \text{so} \quad u(x,y) = 60\cdot x^3\cdot y^2 - 6\cdot x^5 - 30\cdot x\cdot y^4$$

$$v(x,y) = -\frac{\partial}{\partial y}\varphi(x,y) \quad \text{so} \quad v(x,y) = 30\cdot x^4\cdot y - 60\cdot x^2\cdot y^3 + 6\cdot y^5$$

An alternative derivation of u and v is

$$u(x,y) = \frac{\partial}{\partial y}\psi(x,y) \qquad u(x,y) = 60\cdot x^3\cdot y^2 - 6\cdot x^5 - 30\cdot x\cdot y^4$$

$$v(x,y) = -\frac{\partial}{\partial x}\psi(x,y) \qquad v(x,y) = 30\cdot x^4\cdot y - 60\cdot x^2\cdot y^3 + 6\cdot y^5$$

Hence $\frac{\partial}{\partial x}v(x,y) - \frac{\partial}{\partial y}u(x,y) = 0$ Hence flow is IRROTATIONAL

Hence $\frac{\partial}{\partial x}u(x,y) + \frac{\partial}{\partial y}v(x,y) = 0$ Hence flow is INCOMPRESSIBLE

Next we find $\frac{df}{dz} = \frac{d\left(z^6\right)}{dz} = 6\cdot z^5 = 6\cdot(x + i\cdot y)^5 = \left(6\cdot x^5 - 60\cdot x^3\cdot y^2 + 30\cdot x\cdot y^4\right) + i\cdot\left(30\cdot x^4\cdot y + 6\cdot y^5 - 60\cdot x^2\cdot y^3\right)$

Hence we see $\frac{df}{dz} = -u + i\cdot v$ Hence the results are verified; $u = -\mathrm{Re}\left(\frac{df}{dz}\right)$ and $v = \mathrm{Im}\left(\frac{df}{dz}\right)$

These interesting results are explained in Problem 6.113!

Problem 6.80

(Difficulty: 2)

6.80 Consider the flow field represented by the velocity potential $\phi = Ax + Bx^2 - By^2$, where $A = 1\, m \cdot s^{-1}$, $B = 1\, m^{-1} \cdot s^{-1}$, and the coordinates are measured in meters. Obtain expressions for the velocity field and the stream function. Calculate the pressure difference between the origin and point $(x, y) = (1,2)$.

Assumptions: The flow is ideal

Solution:

The velocity field is determined from the velocity potential:

$$u = -\frac{\partial \phi}{\partial x} = -A - 2Bx$$

$$v = -\frac{\partial \phi}{\partial y} = 2By$$

$$\vec{V} = -(A + 2Bx)\hat{\imath} + 2By\hat{\jmath}$$

From the definition of the stream function,

$$u = \frac{\partial \Psi}{\partial y}$$

$$v = -\frac{\partial \Psi}{\partial x}$$

Thus

$$\frac{\partial \Psi}{\partial y} = u = -(A + 2Bx)$$

$$\Psi = \int -(A + 2Bx)dy = -(A + 2Bx)y + f(x)$$

$$\frac{\partial \Psi}{\partial x} = -v = \frac{\partial f(x)}{\partial x} - 2By = -2By$$

$$\frac{\partial f(x)}{\partial x} = 0$$

$$f(x) = constant$$

$$\Psi = -(Ay + 2Bxy)$$

Since we have the following:

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 2B - 2B = 0$$

The flow is irrotational and the Bernoulli equation will be applied between any two points in the flow field:

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

Assume that $\rho = constant$ and $z_1 = z_2$,

$$\vec{V}(0,0) = -A\hat{\imath} = -\hat{\imath}\ \frac{m}{s}$$

$$V_{0,0} = 1\ \frac{m}{s}$$

$$\vec{V}(1,2) = -(A + 2B)\hat{\imath} + 4By\hat{\jmath} = -3\hat{\imath} + 4\hat{\jmath}\ \frac{m}{s}$$

$$V_{1,2} = 5\ \frac{m}{s}$$

So we have:

$$p_1 - p_2 = \frac{\rho}{2}(V_2^2 - V_1^2)$$

Assume the fluid is water we have for the pressure difference:

$$p_1 - p_2 = \frac{1}{2} \times 999\ \frac{kg}{m^3} \times (25 - 1)\ \frac{m^2}{s^2} = 12\ kPa$$

Problem 6.81

(Difficulty: 3)

6.81 An incompressible flow field is characterized by the stream function $\Psi = 3Ax^2y - Ay^3$, where $A = 1\ m^{-1} \cdot s^{-1}$. Show that this flow field is irrotational. Derive the velocity potential for the flow. Plot the streamlines and potential lines, and visually verify that they are orthogonal. (Hint: Use the Excel workbook of Example 6.10)

Assumption: The flow is ideal

Solution: For a 2-D incompressible, irrotational flow we have the definition of the stream function

$$\nabla^2 \Psi = 0$$

Thus

$$\nabla^2 \Psi = \frac{\partial^2(3Ax^2y - Ay^3)}{\partial x^2} + \frac{\partial^2(3Ax^2y - Ay^3)}{\partial y^2} = 6Ay - 6Ay = 0$$

The velocity field is given by:

$$\vec{V} = u\hat{\imath} + v\hat{\jmath}$$

$$u = \frac{\partial \Psi}{\partial y} = 3Ax^2 - 3Ay^2 = 3A(x^2 - y^2)$$

$$v = -\frac{\partial \Psi}{\partial x} = -6Axy$$

$$\vec{V} = 3A(x^2 - y^2)\hat{\imath} - 6Axy\hat{\jmath}$$

The velocity potential is defined such that:

$$u = -\frac{\partial \phi}{\partial x}$$

$$v = -\frac{\partial \phi}{\partial y}$$

Then we have:

$$\phi = -\int u dx + f(y) = -\int 3A(x^2 - y^2) dx + f(y) = -Ax^3 + 3Axy^2 + f(y)$$

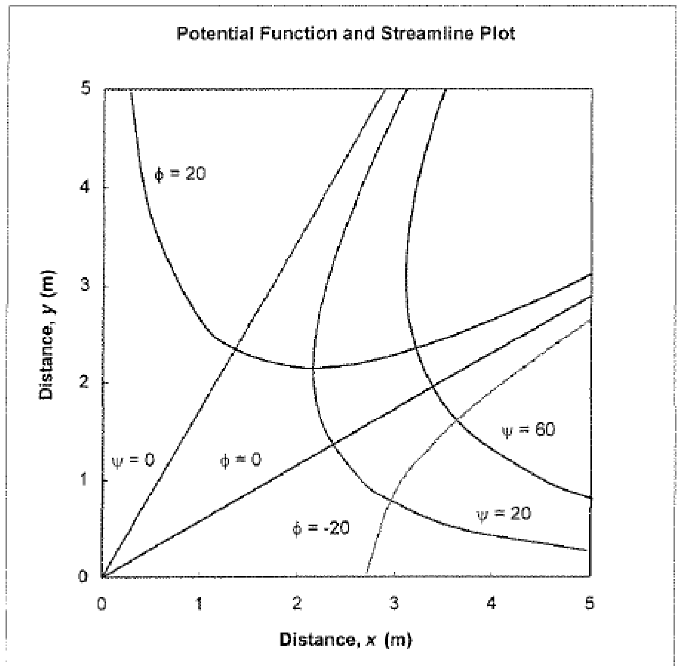
$$\phi = -\int v dy + g(x) = -\int -6Axy dy + g(x) = 3Axy^2 + g(x)$$

$$g(x) = -Ax^3$$

$$f(y) = 0$$

$$\phi = -Ax^3 + 3Axy^2$$

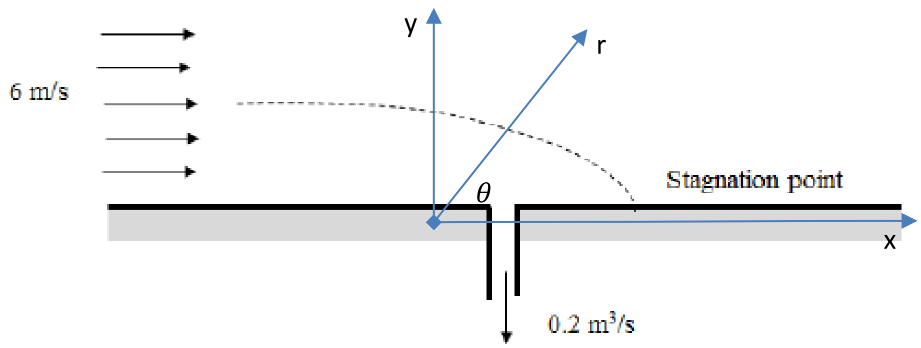
The plot is shown as:



Problem 6.82

(Difficulty: 2)

6.82 Consider an air flow over a flat wall with an upstream velocity of 6 $\frac{m}{s}$. There is a narrow slit through which air is drawn in at a flow rate of 0.2 $\frac{m^3}{s}$ per meter of width. Represent the flow as a combination of a uniform flow and a sink. Determine the location of the stagnation point. Sketch the dividing line between the air that enters slit and the air that continues downstream.



Find: The stagnation point and sketch the streamlines

Assumption: The flow is incompressible, steady, and frictionless

Solution: Apply the potential flow methods. We set the location of the sink as the origin of the coordinate. For this flow, we can apply the superposition of the uniform flow and sink.

The velocity for the uniform flow is:

$$u = U$$

$$v = 0$$

The velocity field for the sink flow is:

$$u = -\frac{q}{2\pi r}\cos\theta$$

$$v = -\frac{q}{2\pi r}\sin\theta$$

The combined flow velocity is:

$$u = U - \frac{q}{2\pi r}\cos\theta$$

$$v = -\frac{q}{2\pi r}\sin\theta$$

The velocity vector is

$$\boldsymbol{V} = \left(U - \frac{q}{2\pi r}\cos\theta\right) i + \left(-\frac{q}{2\pi r}\sin\theta\right) j$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

For the stagnation point we have:

$$u = U - \frac{q}{2\pi r}\cos\theta = 0$$

$$v = -\frac{q}{2\pi r}\sin\theta = 0$$

So we have:

$$\theta = 0$$

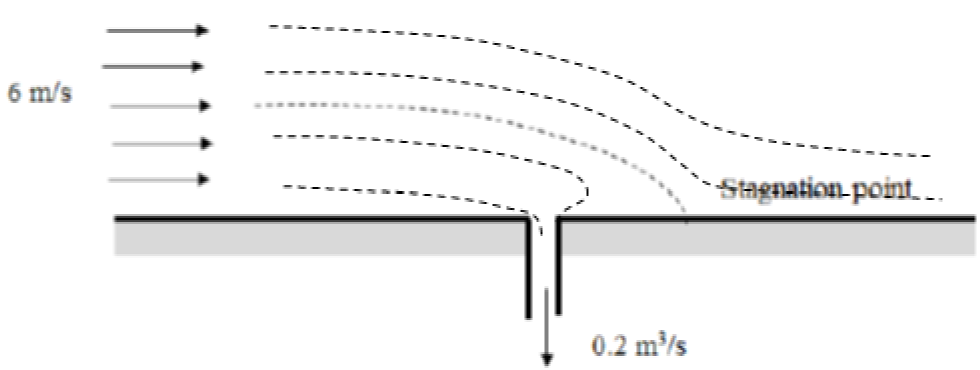
$$U - \frac{q}{2\pi r} = 0$$

The value of the radius is

$$r = x = \frac{q}{2\pi U} = \frac{0.2\ \frac{m^2}{s}}{2\pi \times 6\ \frac{m}{s}} = 0.0053\ m$$

The stagnation point is at the $(0.0053\ m, 0\ m)$.

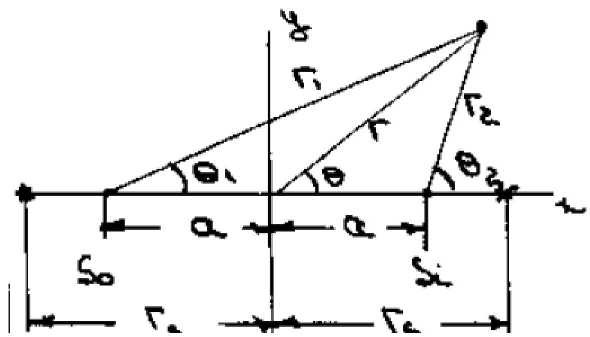
The streamlines are sketched below



Problem 6.83

(Difficulty: 2)

6.83 A source with a strength of $q = 3\pi\ \frac{m^2}{s}$ and a sink with a strength of $q = \pi\ \frac{m^2}{s}$ are located on the $x = -1\ m$ and $x = 1\ m$ respectively. Determine the stream function and velocity potential for the combined flow and sketch the streamlines.



Find: The stream function and velocity potential.

Assumption: The flow is incompressible, steady, and frictionless

Solution: Apply the potential flow methods.

We set the location of the sink as the origin. Then:

$$a = \pm 1$$

The stream function for the combined flow is:

$$\Psi = \frac{q_1}{2\pi}\theta_1 - \frac{q_2}{2\pi}\theta_2 = \frac{3}{2}\theta_1 - \frac{1}{2}\theta_2$$

Or in terms of x and y

$$\Psi = \frac{3}{2}\tan^{-1}\left(\frac{y}{x+1}\right) - \frac{1}{2}\tan^{-1}\left(\frac{y}{x-1}\right)$$

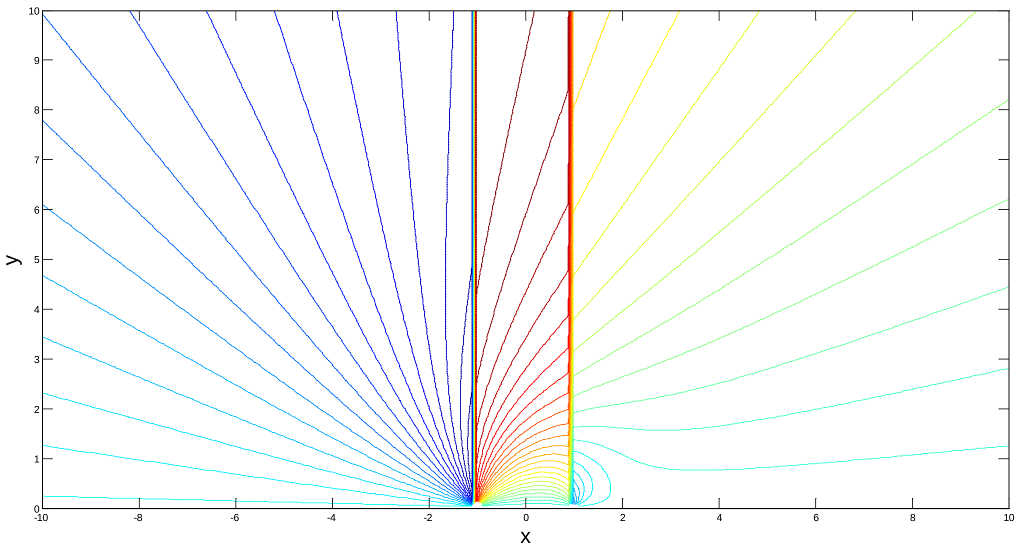
The velocity potential for the combined flow:

$$\phi = -\frac{q_1}{2\pi}\ln r_1 + \frac{q_2}{2\pi}\ln r_2 = -\frac{3}{2}\ln r_1 + \frac{1}{2}\ln r_2$$

Or in terms of x and y

$$\phi = -\frac{3}{2}\ln\sqrt{(x+1)^2+y^2}+\frac{1}{2}\ln\sqrt{(x-1)^2+y^2}$$

The streamlines are shown as:



Problem 6.84

(Difficulty: 3)

6.84 The velocity distribution in a two-dimensional, steady, inviscid flow field in the xy plane is $\vec{V} = \int (Ax+B)\,\hat{\imath} + (C-Ay)\hat{\jmath}$, where $A = 3\ s^{-1}$, $B = 6\ m/s$, $C = 4\ m/s$, and the coordinates are measured in meters. The body force distribution is $\vec{B} = -g\hat{k}$ and the density is $825\ kg/m^3$. Does this represent a possible incompressible flow field? Plot a few streamlines in the upper half plane. Find the stagnation point(s) of the flow field. Is the flow irrotational? If so, obtain the potential function. Evaluate the pressure difference between the origin and point $(x, y, z) = (2,2,2)$.

Assumption: The flow is ideal

Solution: use the continunity expression and the definition of irrotational flow

For incompressible flow, the continuity expression is

$$\nabla \cdot \vec{V} = 0$$

For this flow:

$$\nabla \cdot \vec{V} = \frac{\partial (Ax+B)}{\partial x} + \frac{\partial (C-Ay)}{\partial y} = A - A = 0$$

Velocity field represents possible incompressible flow.

At the stagnation point,

$$u = v = 0$$

$$Ax + B = 0$$

$$x = -\frac{B}{A} = -\frac{6\,\frac{m}{s}}{3\ s^{-1}} = -2\ m$$

$$C - Ay = 0$$

$$y = \frac{C}{A} = \frac{4\,\frac{m}{s}}{3\ s^{-1}} = \frac{4}{3}\ m$$

The stagnation point is at $(x, y) = \left(-2, \frac{4}{3}\right) m$.

The fluid rotation (for a 2-D flow) is given by:

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

For this flow,

$$\omega_z = \frac{1}{2}(0 - 0) = 0$$

The flow is irrotational. Then we have:

$$\vec{V} = -\nabla\phi$$

$$u = -\frac{\partial\phi}{\partial x}$$

$$v = -\frac{\partial\phi}{\partial y}$$

$$\phi = \int -u dx + f(y) = -\int (Ax + B)dx + f(y) = -\frac{Ax^2}{2} - Bx + f(y)$$

And

$$\phi = \int -v dy + g(x) = -\int (C - Ay)dy + g(x) = \frac{Ay^2}{2} - Cy + g(x)$$

$$g(x) = -\frac{Ax^2}{2} - Bx$$

$$f(y) = \frac{Ay^2}{2} - Cy$$

$$\phi = -\frac{Ax^2}{2} - Bx + \frac{Ay^2}{2} - Cy = \frac{A}{2}(y^2 - x^2) - Bx - Cy$$

Since the flow is irrotational we can apply the Bernoulli equation between any two points in the flow field,

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

At point 1 (0,0,0), $\overrightarrow{V_1} = B\hat{\imath} + C\hat{\jmath} = 6\hat{\imath} + 4\hat{\jmath}\ \frac{m}{s}$.

$$V_1^2 = 52\ \frac{m^2}{s^2}$$

At point 2 (2,2,2),

$$\overrightarrow{V_2} = (3 \times 2 + 6)\hat{\imath} + (4 - 3 \times 2)\hat{\jmath}\ \frac{m}{s} = 12\hat{\imath} - 2\hat{\jmath}\ \frac{m}{s}$$

$$V_2^2 = 148\ \frac{m^2}{s^2}$$

$$p_1 - p_2 = \frac{\rho}{2}(V_2^2 - V_1^2) + \rho g(z_2 - z_1)$$

$$p_1 - p_2 = \frac{825\ \frac{kg}{m^3}}{2} \times \left(148\ \frac{m^2}{s^2} - 52\ \frac{m^2}{s^2}\right) + 825\ \frac{kg}{m^3} \times 9.81\ \frac{m}{s^2} \times 2\ m = 55.8\ kPa$$

The stream function is defined such that:

$$u = \frac{\partial \Psi}{\partial y}$$

$$v = -\frac{\partial \Psi}{\partial x}$$

$$\Psi = \int u dy + f(x) = \int (Ax + B)dy + f(x) = Axy + By + f(x)$$

We also have:

$$\Psi = \int -v dx + g(y) = -\int (C - Ay)dx + g(y) = -Cx + Axy + g(y)$$

Equating the two expressions for Ψ, we have:

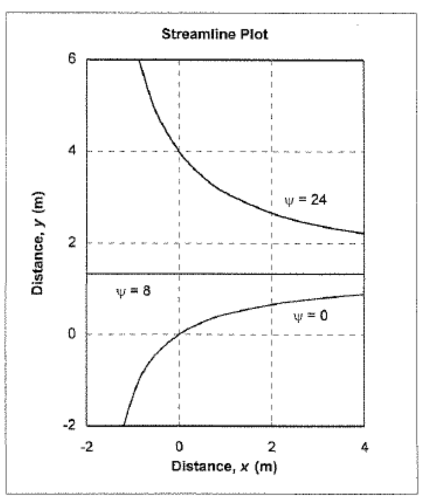
$$f(x) = -Cx$$

$$g(y) = By$$

$$\Psi = Axy + By - Cx$$

The stagnation streamline goes through the stagnation point $\left(-2, \frac{4}{3}\right)$, then

$$\Psi_{stag} = 3\ s^{-1} \times (-2m) \times \frac{4}{3}\ m + 6\ \frac{m}{s} \times \frac{4}{3}\ m - 4\ \frac{m}{s} \times (-2m) = 8\ \frac{m^2}{s}$$



Problem 6.85

(Difficulty: 2)

6.85 Consider the flow past a circular cylinder, of radius a, used in Example 6.11. Show that $V_r = 0$ along the lines $(r,\theta) = (r, \pm\pi/2)$. Plot V_θ/U versus radius for $r \geq a$, along the line $(r,\theta) = (r,\pi/2)$. Find the distance beyond which the influence of the cylinder is less than 1 percent of U.

Solution: Use the potential flow relations

From example Problem 6.11, we have

$$\vec{V} = \left(-\frac{\Lambda\cos\theta}{r^2} + U\cos\theta\right)\hat{\imath}_r + \left(-\frac{\Lambda\sin\theta}{r^2} - U\sin\theta\right)\hat{\imath}_\theta$$

Then the radial velocity is

$$V_r = \left(-\frac{\Lambda}{r^2} + U\right)\cos\theta$$

For $\theta = \pm\frac{\pi}{2}$,

$$\cos\theta = 0$$

$$V_r = 0$$

$$V_\theta = -\left(\frac{\Lambda}{r^2} + U\right)\sin\theta$$

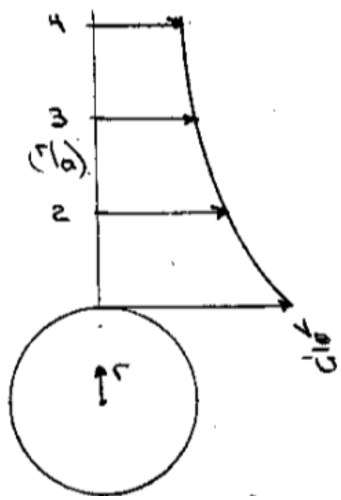
$$\frac{\Omega}{U} = a^2$$

$$V_\theta = -\left(\frac{a^2}{r^2} + 1\right)\mathrm{U}\sin\theta$$

For $\theta = \frac{\pi}{2}$,

$$\frac{V_\theta}{U} == -\left(\frac{a^2}{r^2} + 1\right)$$

The velocity profile is



$$\vec{V} = U\cos\theta\left(1 - \frac{a^2}{r^2}\right)\hat{\imath} - U\sin\theta\left(\frac{a^2}{r^2} + 1\right)\hat{\jmath}$$

For $\theta = \frac{\pi}{2}$,

$$\frac{V}{U} = 1 + \frac{a^2}{r^2}$$

If $\frac{V}{U} = 1.01$, then

$$\frac{a^2}{r^2} = 0.01$$

$$\frac{a}{r} = 0.1$$

$$\frac{V}{U} < 1\, for\; r > 10a$$

Problem 6.86

(Difficulty: 2)

6.86 The flow in a corner with an angle α can be described in radial coordinates by the stream function as $\Psi = Ar^{\frac{\pi}{a}} \sin\frac{\pi\theta}{a}$. Determine the velocity potential for the flow and plot streamlines for flow for $a = 60$ degrees.

Find: The velocity potential.

Assumption: The flow is incompressible, steady, and frictionless

Solution: Apply the potential flow methods.

We have the radial and tangential velocities in terms of the stream function and velocity potential as:

$$V_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} = -\frac{\partial \phi}{\partial r}$$

$$V_\theta = -\frac{\partial \Psi}{\partial r} = -\frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

So we have for this problem using the stream function:

$$V_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} = \frac{1}{r} A r^{\frac{\pi}{a}} \frac{\pi}{a} \cos\frac{\pi\theta}{a} = \frac{A\pi}{a}\cos\frac{\pi\theta}{a} r^{\frac{\pi-a}{a}}$$

The derivative of the velocity potential with respect to r is then

$$\frac{\partial \phi}{\partial r} = -\frac{A\pi}{a}\cos\frac{\pi\theta}{a} r^{\frac{\pi-a}{a}}$$

Integrating with respect to r

$$\phi = -\frac{A\pi}{a}\frac{a}{\pi}\cos\frac{\pi\theta}{a} r^{\frac{\pi}{a}} + f(\theta) = -A\cos\frac{\pi\theta}{a} r^{\frac{\pi}{a}} + f(\theta)$$

The derivative of the velocity potential with respect to θ is then

$$\frac{\partial \phi}{\partial \theta} = \frac{A\pi}{a} r^{\frac{\pi}{a}} \sin\frac{\pi\theta}{a} + \frac{df(\theta)}{d\theta}$$

We also have for the tangential velocity:

$$V_\theta = -\frac{\partial \Psi}{\partial r} = -\frac{A\pi}{a} r^{\frac{\pi-a}{a}} \sin\frac{\pi\theta}{a}$$

And the derivative with respect to θ is

$$\frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{A\pi}{a} r^{\frac{\pi-a}{a}} \sin\frac{\pi\theta}{a}$$

Or

$$\frac{\partial\phi}{\partial\theta} = \frac{A\pi}{a} r^{\frac{\pi}{a}} \sin\frac{\pi\theta}{a}$$

Equating the two expressions for the derivative:

$$\frac{A\pi}{a} r^{\frac{\pi}{a}} \sin\frac{\pi\theta}{a} + \frac{df(\theta)}{d\theta} = \frac{A\pi}{a} r^{\frac{\pi}{a}} \sin\frac{\pi\theta}{a}$$

Therefore

$$\frac{df(\theta)}{d\theta} = 0$$

Which yields

$$f(\theta) = constant$$

The velocity potential is then:

$$\phi = -A\cos\frac{\pi\theta}{a} r^{\frac{\pi}{a}} + constant$$

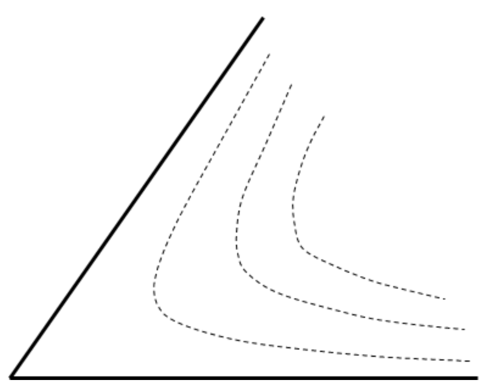
The velocity field for $a = 60 = \frac{\pi}{3}$ degrees is:

$$\Psi = Ar^{\frac{\pi}{a}} \sin\frac{\pi\theta}{a} = Ar^3 \sin 3\theta$$

$$V_r = \frac{A\pi}{a}\cos\frac{\pi\theta}{a} r^{\frac{\pi-a}{a}} = \frac{A\pi}{\frac{\pi}{3}}\cos\frac{\pi\theta}{\frac{\pi}{3}} r^{\frac{\pi-\frac{\pi}{3}}{\frac{\pi}{3}}} = 3Ar^2\cos 3\theta$$

$$V_\theta = -\frac{A\pi}{\frac{\pi}{3}} r^{\frac{\pi-\frac{\pi}{3}}{\frac{\pi}{3}}} \sin\frac{\pi\theta}{\frac{\pi}{3}} = -3Ar^2 \sin 3\theta$$

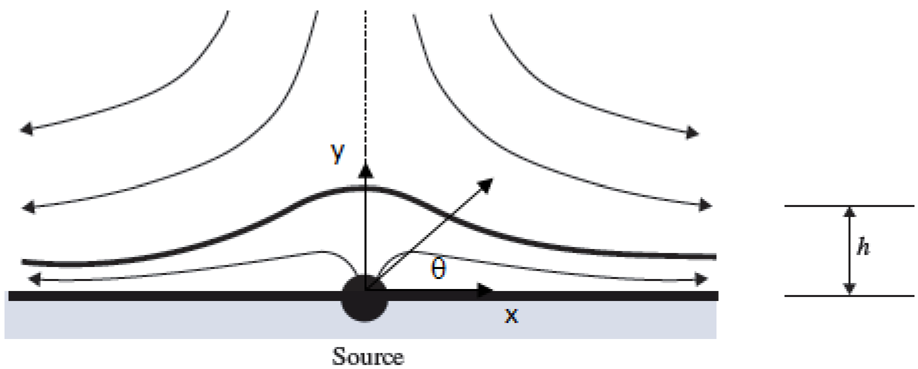
The streamlines are shown as:



Problem 6.87

(Difficulty: 2)

6.87 Consider the two-dimension flow against a flat plate that is characterized by the stream function $\Psi = Axy$. Superimpose a plane source of strength B placed in the origin. Determine the relation between the height of the stagnation point h, the constant A, and the strength B. Sketch streamlines for the flow and identify the streamline that divides the two flows.



Find: The stagnation point properties.

Assumption: The flow is incompressible, steady, and frictionless

Solution: Apply the potential flow methods.

For the plane source of strength B we have:

$$\Psi = \frac{B}{2\pi}\theta$$

The stream function for the combined flow is:

$$\Psi = Axy + \frac{B}{2\pi}\theta = Axy + \frac{B}{2\pi}\tan^{-1}\left(\frac{y}{x}\right)$$

The velocity field is calculated using the stream function as:

$$u = \frac{\partial \Psi}{\partial y} = Ax + \frac{B}{2\pi}\left(\frac{x}{x^2+y^2}\right)$$

$$v = -\frac{\partial \Psi}{\partial x} = -Ay + \frac{B}{2\pi}\left(\frac{y}{x^2+y^2}\right)$$

For the stagnation point we have:

$$u = v = 0$$

For $x = 0$ we have:

$$u = 0$$

Then

$$v = -Ah + \frac{B}{2\pi}\left(\frac{h}{h^2}\right) = 0$$

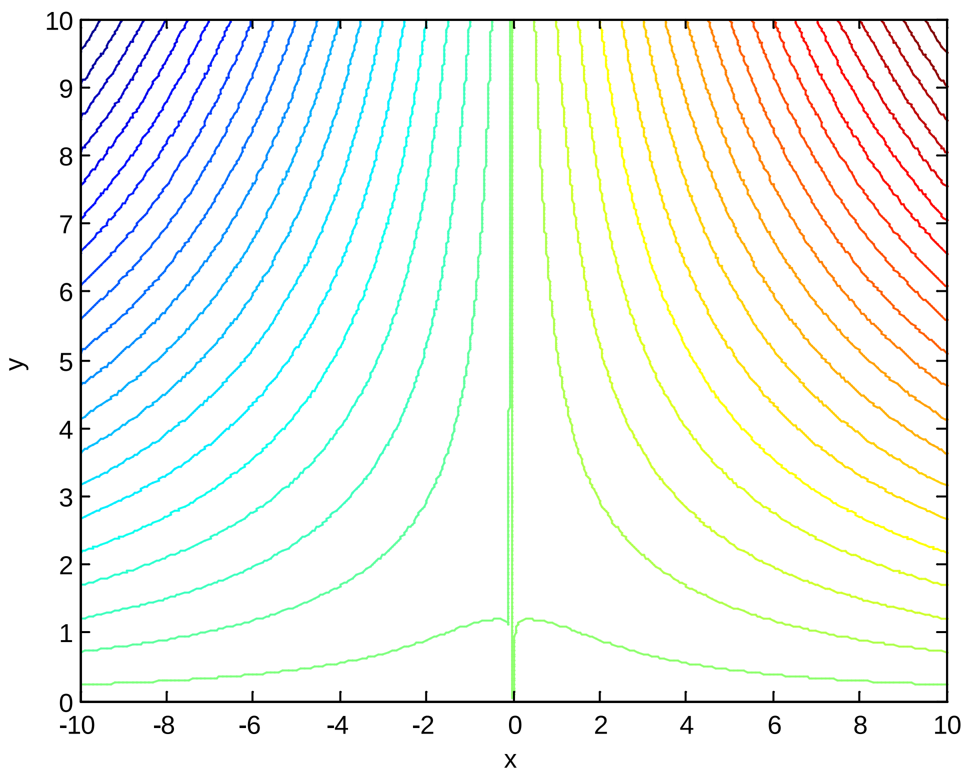
So the relation between the height of the stagnation point and strength B and constant A is:

$$A = \frac{B}{2\pi}\left(\frac{1}{h^2}\right)$$

For the streamline pass the stagnation point we have:

$$\Psi = Axy + \frac{B}{2\pi}\tan^{-1}\left(\frac{y}{x}\right) = \frac{B}{2\pi}\tan^{-1}(\infty) = \frac{B}{4}$$

The streamline is shown as (assume A=B=0.1):



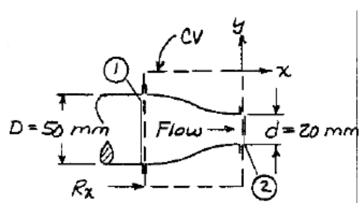
The streamline divides the two flows can be seen from the figure.

Problem 6.88

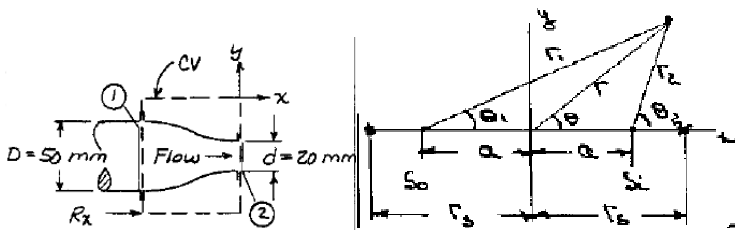
(Difficulty: 3)

6.88 A source and a sink with strengths of equal magnitude, $q = 3\pi\ m^2/s$, are placed on the x axis at $x = -a$ and $x = a$, respectively. A uniform flow, with speed $U = 20\ m/s$, in the positive x direction, is added to obtain the flow past a Rankine body. Obtain the stream function, velocity potential, and velocity field for the combined flow. Find the value of $\Psi = constant$ on the stagnation streamline. Locate the stagnation points if $a = 0.3\ m$.

Assumptions: The flow is ideal.



Solution: The flow picture is shown below:



We have the following for the stream function:

$$\Psi = \Psi_{So} + \Psi_{Si} + \Psi_{Uf} = \frac{q}{2\pi}\theta_1 - \frac{q}{2\pi}\theta_2 + Uy$$

$$\Psi = \frac{q}{2\pi}(\theta_1 - \theta_2) + Ur\sin\theta$$

$$\Phi = \Phi_{So} + \Phi_{Si} + \Phi_{Uf} = -\frac{q}{2\pi}\ln r_1 + \frac{q}{2\pi}\ln r_2 - Ux$$

$$\Phi = \frac{q}{2\pi}\ln\frac{r_2}{r_1} - Ur\cos\theta$$

$$u = u_{So} + u_{Si} + u_{Uf} = \frac{q}{2\pi r_1}\cos\theta_1 - \frac{q}{2\pi r_2}\cos\theta_2 + U$$

$$v = v_{So} + v_{Si} + v_{Uf} = \frac{q}{2\pi r_1}\sin\theta_1 - \frac{q}{2\pi r_2}\sin\theta_2$$

$$\vec{V} = u\hat{\imath} + v\hat{\jmath} = \left\{\frac{q}{2\pi}\left(\frac{\cos\theta_1}{r_1} - \frac{\cos\theta_2}{r_2}\right) + U\right\}\hat{\imath} + \frac{q}{2\pi}\left(\frac{\sin\theta_1}{r_1} - \frac{\sin\theta_2}{r_2}\right)\hat{\jmath}$$

At stagnation point, $\vec{V} = 0, y = 0, \theta_1 = \theta_2 = 0$.

$$r_2 = r_s - a$$

$$r_1 = r_s + a$$

$$u = 0 = \frac{q}{2\pi}\left(\frac{1}{r_s + a} - \frac{1}{r_s - a}\right) + U = \frac{q}{2\pi}\left[\frac{(r_s - a) - (r_s + a)}{r_s^2 - a^2}\right] + U$$

$$0 = \frac{-qa}{\pi(r_s^2 - a^2)} + U$$

$$(r_s^2 - a^2) = \frac{qa}{\pi U}$$

$$r_s = \left(a^2 + \frac{qa}{\pi U}\right)^{\frac{1}{2}} = a\left(1 + \frac{q}{\pi U a}\right)^{\frac{1}{2}}$$

For $a = 0.3\ m$,

$$r = 0.3\ m \times \left(1 + \frac{3\pi\ \frac{m^2}{s}}{\pi \times 20\ \frac{m}{s} \times 0.3\ m}\right)^{\frac{1}{2}} = 0.367\ m$$

Stagnation points located at $\theta = 0, \pi, r = 0.367\ m$.

Since we have:

$$\Psi = \frac{q}{2\pi}\theta_1 - \frac{q}{2\pi}\theta_2 + Uy$$

$$\theta_1 = \theta_2$$

$$y = 0$$

At stagnation we have:

$$\Psi_{stag} = 0$$

Problem 6.89

(Difficulty: 3)

6.89 A flow field is formed by combining a uniform flow in the positive x direction, with $U = 10\ m/s$, and a counter-clockwise vortex, with strength $K = 16\pi\ m^2/s$, located at the origin. Obtain the stream function, velocity potential, and velocity field for the combined flow. Locate the stagnation point(s) for the flow. Plot the streamlines and potential lines. (Hint: use the excel workbook of example 6.10)

Solution: Use the relations for stream funcition and velocity potential in radial coordinates

The stream and potential functions are the sums of that for uniform flow (uf) and a vortex (v)

$$\Psi = \Psi_{u \cdot f} + \Psi_v = Uy - \frac{k}{2\pi}\ln r = Ur\sin\theta - \frac{k}{2\pi}\ln r$$

$$\phi = \phi_{u \cdot f} + \phi_v = -Ux - \frac{k}{2\pi}\theta = -Ur\cos\theta - \frac{k}{2\pi}\theta$$

$$V_r = -\frac{\partial \phi}{\partial r} = U\cos\theta$$

$$V_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta} = -U\sin\theta + \frac{k}{2\pi r}$$

$$\bar{V} = U\cos\theta\,\hat{e}_r + \left(\frac{k}{2\pi r} - U\sin\theta\right)\hat{e}_\theta$$

At stagnation point, $\bar{V} = 0$

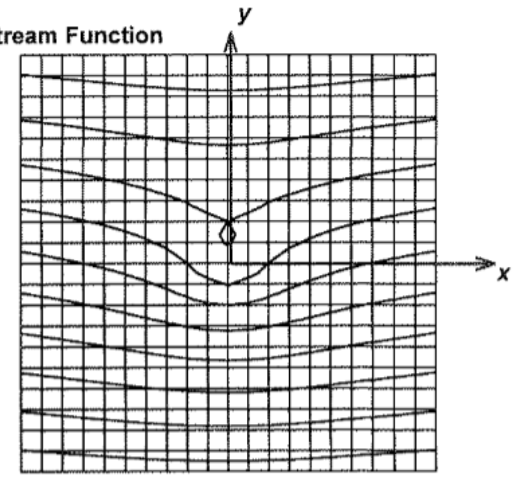
$$V_r = 0\ \ at\ \theta = \pm\frac{\pi}{2}$$

$$V_\theta = 0\ on\ r = \frac{k}{2\pi U\sin\theta}$$

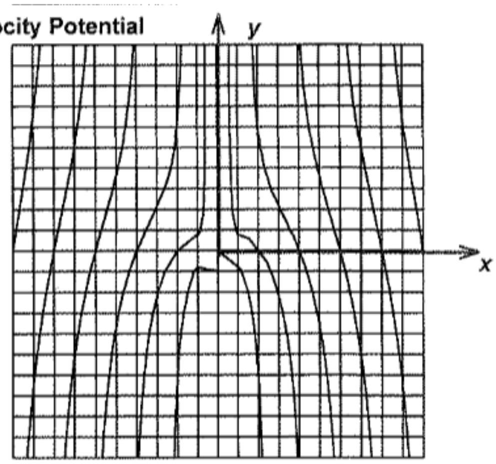
$$\bar{V} = 0\ at\ (r, \theta) = \frac{k}{2\pi U}, \frac{\pi}{2}$$

Use Excel, the stream function and velocity potential can be plotted. The data below was obatained using the workbook for example problem 6.10. Note the orthogonality of Ψ and ϕ.

Stream Function



Velocity Potential



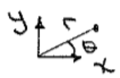
Problem 6.90

(Difficulty: 3)

6.90 Consider the flow field formed by combining a uniform flow in the positive x direction and a source located at the origin. Obtain expressions for the stream function, velocity potential, and velocity field for the combined flow. If $U = 25\ m/s$, determine the source strength if the stagnation point is located at $x = -1\ m$. Plot the streamlines and potential lines. (Hint: Use the Excel workbook of Example 6.10).

Solution: Use the relations for stream function and velocity potential.

In radial coordinates



The stream and potential functions are the sums of that for uniform flow (uf) and a source (so)

$$\Psi = \Psi_{u \cdot f} + \Psi_{so} = Uy + \frac{q}{2\pi}\theta = Ur\sin\theta + \frac{q}{2\pi}\theta$$

$$\phi = \phi_{u \cdot f} + \phi_{so} = -Ux - \frac{q}{2\pi}\ln r = -Ur\cos\theta - \frac{q}{2\pi}\ln r$$

$$u = u_{u \cdot f} + u_{so}$$

The uniform flow and source flow velocities are

$$u_{u \cdot f} = U$$

$$u_{so} = V_r\cos\theta = \frac{q}{2\pi r}\frac{x}{r}$$

The velocity is then

$$u = U + \frac{q}{2\pi}\frac{x}{r^2}$$

We also have:

$$v = v_{u \cdot f} + v_{so}$$

$$v_{u \cdot f} = 0$$

$$v_{so} = V_r\sin\theta = \frac{q}{2\pi r}\frac{y}{r}$$

$$v = \frac{q}{2\pi}\frac{y}{r^2}$$

$$\bar{V} = \left\{U + \frac{q}{2\pi}\frac{x}{(x^2+y^2)}\right\}\hat{\imath} + \frac{q}{2\pi}\frac{y}{(x^2+y^2)}\hat{\jmath}$$

At the stagnation point, $\bar{V} = 0$, $x = -1.0\ m$, $y = 0\ (v = 0)$,

For $u = 0 = U + \frac{q}{2\pi}\frac{x}{(x^2)}$, then

$$q = -2\pi U x_{stag}$$

$$q = -2\pi \times 25\frac{m}{s} \times (-1.0\ m) = 50\ \pi\ \frac{m^2}{s}$$

At the stagnation piont, $\theta = \pi$,

$$\Psi_{stag} = \frac{q}{2\pi}\pi = \frac{q}{2}$$

The equation of the stagnation steramline is then:

$$\frac{q}{2} = Ur\sin\theta + \frac{q}{2\pi}\theta$$

and

$$r = \frac{q(\pi-\theta)}{2\pi U\sin\theta}$$

At $\theta = \frac{\pi}{2}$,

$$r = \frac{q}{4U} = 50\ \pi\ \frac{m^2}{s} \times \frac{1}{4} \times \frac{s}{25\ m} = \frac{\pi}{2}\ m$$

For downstream, $\theta \to 0$ and the y coordinate of the body:

$$y = r\sin\theta = \frac{q(\pi-\theta)}{2\pi U}$$

approaches

$$\frac{q}{2U} = \frac{50\ \pi}{2\times 25} = \pi\ m$$

Use Excel, the stream function and velocity potential can be plotted. The data below was obatained using the workbook for example problem 6.10. Note the orthogonality of Ψ and ϕ.