Equilibrium in the Logistic Growth Model

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```
suppressPackageStartupMessages({
  library(tidyverse)
})
```

The model

The logistic growth model is given by:

$$N_{t+1} = N_t + rN_t\left(1 - \frac{N_t}{K}\right) - cN_t$$

Where N_t denotes population time (biomass or abundance) at time t, r represents population growth rate, K the carrying capacity and c a catch rate. From these parameters, r has to be a number different from zero ($r \neq 0$), c has to lie between 0 and 1 ($c\exists(0,1)$), and K is a positive number ($K > 0$).

Equilibrium without harvesting

When $c = 0$, we have a population without any havesting, and so there are two possible equilibrium points, which we derive as follows:

Let N^* denote the equilibrium density or biomass, so that the equation above becomes:

$$N^* = N^* + rN^*\left(1 - \frac{N^*}{K}\right)$$

Note then, that as time progresses, N^* remains the same. Also, note that the $-cN_t$ term has been dropped to ignore harvest for now.

From above, we see that both sides are the same. Just as $a = a$ can be rewritten as $0 = a - $, we rewrite the above equation as:

$$N^* + rN^*\left(1 - \frac{N^*}{K}\right) - N^* = 0$$

The single N^* terms cancel each other out, leaving us with:

$$rN^*\left(1 - \frac{N^*}{K}\right) = 0$$

We have the multiplication of rN^* and $1 - \frac{N^*}{K}$, so one of them has to be zero for their product to be zero. Having $rN^* = 0$ is a trivial solution because since $r \neq 0$, it implies that population $N = 0$. (This is like saying that if there's no population, it's stable equilibrium is 0… wel of course). An alternative is that $1 - \frac{N^*}{K} = 0$, so we'll go with that.

$$
\begin{aligned}
1 - \frac{N^*}{K} &= 0 \\
-\frac{N^*}{K} &= -1 \\
\frac{N^*}{K} &= 1 \\
N^* &= K
\end{aligned}
$$

Out math suggests that the equilibrium of the model lies at $N^* = 0$ and at $N^* = K$, which can be shown in the following simulations:

```
logistic_model <- function(r = 0.5, K = 100, N0 = 10, nsteps = 50, c = 0.5, p = 10){
  # Define vector of time, N, and C
  time <- c(0:nsteps)
  N <- numeric(length = length(time))

  # Assign initial population size
  N[1] <- N0

  # Start for loop to simulate population
  for (t in 1:nsteps) {
    N[t+1] <- N[t] + (r * N[t] * (1 - (N[t] / K))) - c * N[t]
  }
  # End for loop

  # Create data.frame with simulation results
  simul <- data.frame(time, N) %>%
    mutate(C = c * N,
           R = p * C)

  return(simul)
}
```

```
  rbind(logistic_model(N = 0, c = 0),
        logistic_model(N = 75, c = 0),
        logistic_model(N = 1, c = 0)) %>%
  mutate(model = c(rep("N0 = 0", 51),
                   rep("N0 = 75", 51),
                   rep("N0 = 1", 51))) %>%
  ggplot(mapping = aes(x = time, y = N)) +
  geom_line() +
  geom_point(color = "steelblue", size = 2) +
  theme_bw() +
  facet_wrap(~model, ncol = 2)
```

Equilibrium with harvesting

With harvesting, our model becomes:

$$
N_{t+1} = N_t + rN_t\left(1 - \frac{N_t}{K}\right) - cN_t
$$

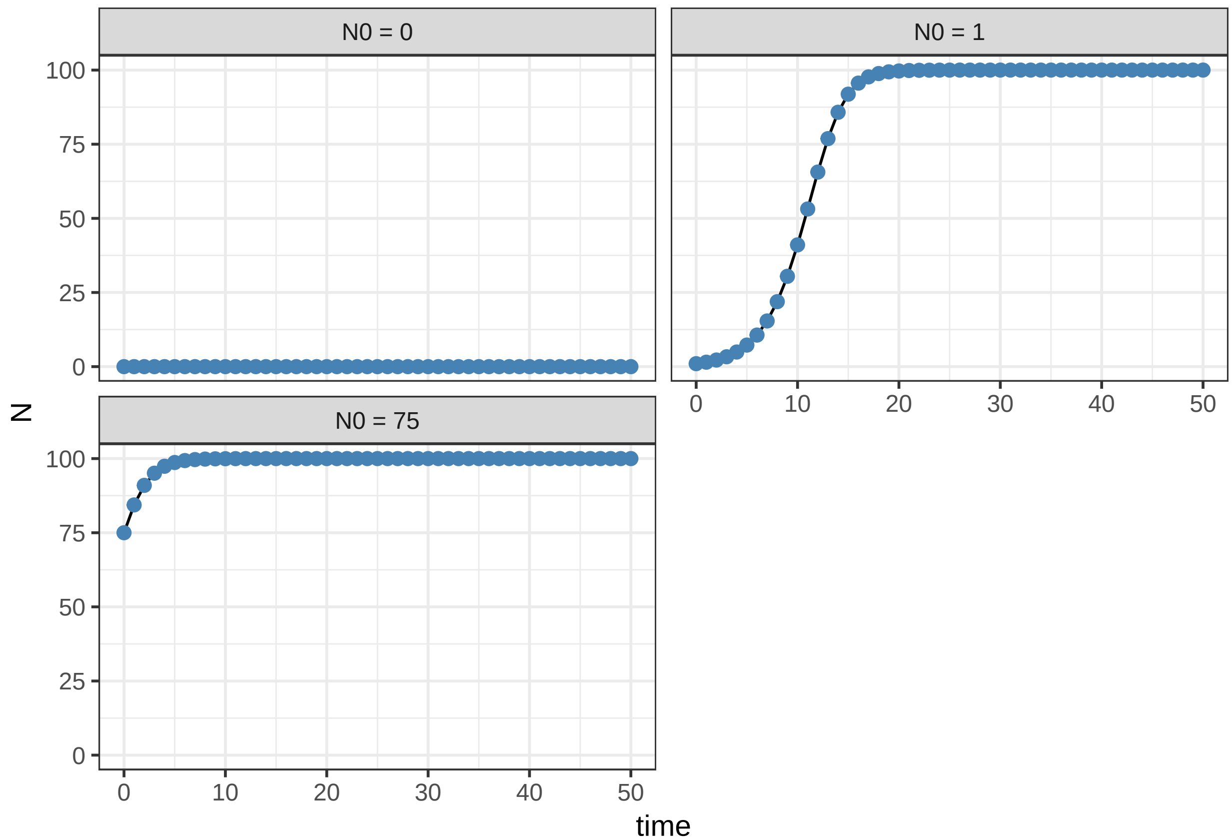

Figure 1: Three simulations of a logistic population growth model. The top-left graph shows a population at stable equilibrium where N0 = 0 (this is the trivial solution). The other two plots show how, independent of N0, they both reach an equilibrium at K = 100. Notice how N0 = 1 takes more time to reach equilibrium than N0 = 75

From above, we know that We can use N^* to denote the equilibrium population size at any timestep, thus rewritting the equation as:

$$N^* = N^* + rN^*\left(1 - \frac{N^*}{K}\right) - cN^*$$

From above, we know that the N^* cancel each other out, and thus we obtain:

$$rN^*\left(1 - \frac{N^*}{K}\right) - cN^* = 0$$

We can separate the terms (growth and catches) to obtain:

$$cN^* = rN^*\left(1 - \frac{N^*}{K}\right)$$

Dividing both sides by rN^*:

$$\frac{cN^*}{rN^*} = 1 - \frac{N^*}{K}$$

The N^* on the left side cancel each other out, and so we obtain:

$$\frac{c}{r} = 1 - \frac{N^*}{K}$$

We substract 1 from both sides:

$$\frac{c}{r} - 1 = -\frac{N^*}{K}$$

Mutliply both sides by -1:

$$1 - \frac{c}{r} = \frac{N^*}{K}$$

And multiply both sides by K to obtain:

$$K\left(1 - \frac{c}{r}\right) = N^*$$

Thus when we allow harvesting of the population (*i.e.* $c > 0$) the equilibrium is denoted by $N^* = K\left(1 - \frac{c}{r}\right)$. Intuitively, when $c = 0$, this becomes just $N^* = K$:

$$\begin{aligned}
N^* &= K\left(1 - \frac{c}{r}\right)\\
N^* &= K\left(1 - \frac{0}{r}\right)\\
N^* &= K\left(1 - 0\right)\\
N^* &= K
\end{aligned}$$

We can simulate a case with $c = 0$, where the population converges to K, and a case with $c = 0.2$, where the population converges to

$$
\begin{aligned}
N^* &= K\left(1 - \frac{c}{r}\right) \\
&= 100\left(1 - \frac{0.2}{0.5}\right) \\
&= 100\,(1 - 0.4) \\
&= 100(0.6) \\
&= 60
\end{aligned}
$$

Note that both simulations use $r = 0.5$

```
rbind(logistic_model(N = 20, c = 0),
      logistic_model(N = 20, c = 0.2)) %>%
mutate(model = c(rep("c = 0", 51),
                 rep("c = 0.2", 51))) %>%
ggplot(mapping = aes(x = time, y = N)) +
geom_line() +
geom_point(color = "steelblue", size = 2) +
theme_bw() +
facet_wrap(~model, ncol = 2) +
geom_hline(yintercept = c(100, 60), linetype = "dashed", size = 1)
```