Advanced Algebra II

Assignment 6

Solution

1. Let $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2$ be a basis of V. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$.

(1) Construct a linear transformation σ such that

$$\sigma(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2) = (\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2)A. \tag{1}$$

(2) Is this linear transformation unique? why?

Remark: This question is to review the proof of Theorem 1, Page 191 of the textbook.

Proof. The equation (1) is equivalent to

$$\begin{aligned} \sigma(\boldsymbol{\varepsilon}_1) &= \boldsymbol{\varepsilon}_1, \\ \sigma(\boldsymbol{\varepsilon}_2) &= 2\boldsymbol{\varepsilon}_1 + 3\boldsymbol{\varepsilon}_3. \end{aligned} \tag{2}$$

What we want is to construct a linear transformation σ such that the formula (2) holds. For any vector $x_1\boldsymbol{\varepsilon}_1 + x_2\boldsymbol{\varepsilon}_2 \in V$, we define

$$\sigma(x_1\boldsymbol{\varepsilon}_1 + x_2\boldsymbol{\varepsilon}_2) = x_1 \cdot (\boldsymbol{\varepsilon}_1) + x_2 \cdot (2\boldsymbol{\varepsilon}_1 + 3\boldsymbol{\varepsilon}_3).$$

Once can check this linear transformation satisfies the formula (2).

The linear transformation σ that we defined above is unique due to Theorem 1, Page 191 of the textbook. □

2. The followings are two bases of $\mathbb{P}^3$:

$$\begin{aligned} &\boldsymbol{\varepsilon}_1 = (1, 0, 1), \quad \boldsymbol{\varepsilon}_2 = (2, 1, 0), \quad \boldsymbol{\varepsilon}_3 = (1, 1, 1); \\ &\boldsymbol{\eta}_1 = (1, 2, -1), \quad \boldsymbol{\eta}_2 = (2, 2, -1), \quad \boldsymbol{\eta}_3 = (2, -1, -1). \end{aligned}$$

Define $\sigma \in \mathcal{L}(V)$ by

$$\sigma(\boldsymbol{\varepsilon}_i) = \boldsymbol{\eta}_i, \qquad i = 1, 2, 3. \tag{3}$$

(Remark: Note that once we assign values to each $\sigma(\boldsymbol{\varepsilon}_i)$, where $\boldsymbol{\varepsilon}_i$, $i = 1, 2, 3$ is a basis of V, we have determined a linear transformation σ. Hence, to define a linear transformation σ, we don't have to write all the detail about assigning values to $\sigma(\boldsymbol{\alpha})$ for all $\boldsymbol{\alpha} \in V$.)

(1) Write down the transition matrix from the basis $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_3$ to the basis $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_3$.

Solution. Let M be the transition matrix, i.e.,

$$(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_3) = (\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_3)M. \tag{4}$$

By the above equation, the j-th column of M is the coordinate of $\boldsymbol{\eta}_j$ with respect to the basis $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_3$. Assume

$$\eta_1 = a_1\boldsymbol{\varepsilon}_1 + a_2\boldsymbol{\varepsilon}_2 + a_3\boldsymbol{\varepsilon}_3,$$

which is equivalent to

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = a_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + a_2 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + a_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Solving this equation, we get $a_1 = -2, a_2 = 1, a_3 = 1$, which is the coordinate of $\boldsymbol{\eta}_1$. Hence the first column of M is $(-2, 1, 1)^T$, where T means the transpose of a matrix. Similarly, we can compute the coordinates of $\boldsymbol{\eta}_2, \boldsymbol{\eta}_3$, which will be the 2nd and 3rd columns of M. We omit the details of computation here. Finally, we get

$$M = \begin{bmatrix} -2 & -\frac{3}{2} & \frac{3}{2} \\ 1 & \frac{3}{2} & \frac{3}{2} \\ 1 & \frac{1}{2} & -\frac{5}{2} \end{bmatrix}.$$

(2) Write down the matrix of σ with respect to the basis $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_3$.

Solution. Let

$$\sigma(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_3) = (\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_3)N, \tag{5}$$

and N is what we want. Note $\sigma(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_3) = (\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_3)$ due to equation (3), so

$$(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_3) = (\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_3)N.$$

By the formula (4), it means

$$N = M.$$

(3) Write down the matrix of σ with respect to the basis $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_3$.

Solution. Let

$$\sigma(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_3) = (\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_3)Q, \tag{6}$$

where Q is what we want. The equation (5) tells us

$$\sigma\Big(\sigma(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_3)\Big) = \sigma\Big((\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_3)N\Big),$$

which is

$$\sigma\Big(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_3\Big) = \sigma\Big(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_3\Big)N,$$

which again is

$$\sigma\Big(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_3\Big) = \Big(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_3\Big)N.$$

In the above, we used (3) twice. Comparing the above equation with (6), we get

$$Q = N = M.$$

(4) Check if the above two matrices are similar to each other?

Solution. Since

$$N = P,$$

certainly they are similar.

3. Prove the following two matrices are similar.

$$\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \lambda_{i_1} & & & \\ & \lambda_{i_2} & & \\ & & \ddots & \\ & & & \lambda_{i_n} \end{bmatrix},$$

where $i_1 i_2 \cdots i_n$ is a permutation of $1, 2, \cdots, n$.

Proof. We can assume $\lambda_i \in \mathbb{P}$, where $\mathbb{P}$ is a number field. Let A, B denote the above two matrices, respectively. Let V be a n-dimension vector space over $\mathbb{P}$. Let $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \cdots, \boldsymbol{\varepsilon}_n$ be a basis of V. Let σ be the linear transformation defined by

$$\sigma(\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \cdots, \boldsymbol{\varepsilon}_n) = (\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \cdots, \boldsymbol{\varepsilon}_n)A.$$

This equation is equivalent to

$$\sigma(\boldsymbol{\varepsilon}_i) = \lambda_i \boldsymbol{\varepsilon}_i, \qquad i = 1, 2, \cdots, n.$$

Clearly, $\boldsymbol{\varepsilon}_{i_1}, \boldsymbol{\varepsilon}_{i_2}, \cdots, \boldsymbol{\varepsilon}_{i_n}$ is also a basis since it is just a permutation of $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \cdots, \boldsymbol{\varepsilon}_n$. Note that

$$\sigma(\boldsymbol{\varepsilon}_{i_j}) = \lambda_{i_j} \boldsymbol{\varepsilon}_{i_j}, \qquad j = 1, 2, \cdots, n,$$

so

$$\sigma(\boldsymbol{\varepsilon}_{i_1}, \boldsymbol{\varepsilon}_{i_2}, \cdots, \boldsymbol{\varepsilon}_{i_n}) = (\boldsymbol{\varepsilon}_{i_1}, \boldsymbol{\varepsilon}_{i_2}, \cdots, \boldsymbol{\varepsilon}_{i_n})B.$$

Because A, B are two matrices of σ with respect to different bases, we conclude that they are similar. □

4. Prove:

(1) A is invertible $\Rightarrow$ AB and BA are similar.

Proof. Note

$$A^{-1} \cdot AB \cdot A = BA,$$

so AB is similar to B. □

(2) If A is similar to B and C is similar to D, then

$$\begin{bmatrix} A & O \\ O & C \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} B & O \\ O & D \end{bmatrix}$$

are similar.

Proof. Recall the fact that if P, Q are invertible,

$$\begin{bmatrix} P & O \\ O & Q \end{bmatrix}^{-1} = \begin{bmatrix} P^{-1} & O \\ O & Q^{-1} \end{bmatrix}.$$

Since $A \sim B$, $C \sim D$, there exists T_1, T_2 such that $T_1^{-1}AT_1 = B$, $T_2^{-1}CT_2 = D$. Hence,

$$\begin{aligned}
\begin{bmatrix} T_1 & O \\ O & T_2 \end{bmatrix}^{-1} \begin{bmatrix} A & O \\ O & B \end{bmatrix} \begin{bmatrix} T_1 & O \\ O & T_2 \end{bmatrix} &= \begin{bmatrix} T_1^{-1} & O \\ O & T_2^{-1} \end{bmatrix} \begin{bmatrix} A & O \\ O & B \end{bmatrix} \begin{bmatrix} T_1 & O \\ O & T_2 \end{bmatrix} \\
&= \begin{bmatrix} T_1^{-1}AT_1 & O \\ O & T_2^{-1}BT_2 \end{bmatrix} \\
&= \begin{bmatrix} A & O \\ O & B \end{bmatrix},
\end{aligned}$$

as desired. □

5. Let V be a linear space over $\mathbb{C}$. Let $\sigma \in \mathcal{L}(V)$. The matrix of σ with respect to a basis $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_3$ is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Find all eigenvalues of σ and eigenvectors corresponding to each of them.

Solution. Let A denote the above matrix. Let

$$f(\lambda) = |\lambda E - A|,$$

that is

$$f(\lambda) = \begin{vmatrix} \lambda & 0 & -1 \\ 0 & \lambda - 1 & 0 \\ -1 & 0 & \lambda \end{vmatrix} = \lambda^2(\lambda - 1) - (\lambda - 1) = (\lambda - 1)^2(\lambda + 1).$$

Let $f(\lambda) = 0$ and then we get $\lambda = 1, 1, -1$.

For $\lambda = 1$, we need to solve

$$(E - A)\boldsymbol{x} = \mathbf{0},$$

i.e.,

$$\begin{aligned} x_1 - x_3 &= 0, \\ -x_1 + x_3 &= 0. \end{aligned}$$

The fundamental set of solutions is

$$(0, 1, 0), \quad (1, 0, 1).$$

Then vectors in V corresponding to these two coordinates are

$$\boldsymbol{\varepsilon}_2, \quad \boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_3.$$

Hence,

$$k_1\boldsymbol{\varepsilon}_2 + k_2(\boldsymbol{\varepsilon}_1 + \boldsymbol{\varepsilon}_3), \qquad k_1, k_2 \in \mathbb{P}$$

are all eigenvectors of σ corresponding to $\lambda = 1$.

For $\lambda = -1$, we need to solve

$$(-E - A)\boldsymbol{x} = \boldsymbol{0},$$

i.e.,

$$\begin{aligned} -x_1 - x_3 &= 0, \\ -2x_2 &= 0, \\ -x_1 - x_3 &= 0. \end{aligned}$$

The fundamental set of solutions is

$$(1, 0, -1).$$

Then vector in V corresponding to this coordinate are

$$\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_3.$$

Hence,

$$k(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_3), \qquad k \in \mathbb{P}$$

are all eigenvectors of σ corresponding to $\lambda = -1$.

6. Let $\sigma \in \mathcal{L}(V)$ and σ be invertible. Prove

(1) The eigenvalues of σ must not be 0.

Proof. Fix a basis and let A be the matrix of σ with respect to this basis. Since σ is invertible, A is also invertible.

Let

$$f(\lambda) = |\lambda E - A|$$

be the characteristic polynomial of A (which is also the characteristic polynomial of σ).

If an eigenvalue of σ is 0, then 0 is a root of $f(\lambda)$, that is, $f(0) = 0$. Hence

$$|0E - A| = 0,$$

which implies

$$|A| = 0.$$

Then A is not invertible, which is a contradiction. $\square$

(2) If λ is an eigenvalue of σ, then $\frac{1}{\lambda}$ is an eigenvalue of σ^{-1}.

Proof. Note $\lambda \neq 0$ by the conclusion of Q6(1). Since λ is an eigenvalue of σ, there exist a non-zero vector $\boldsymbol{\alpha}$ such that

$$\sigma(\boldsymbol{\alpha}) = \lambda\boldsymbol{\alpha}.$$

Since σ is invertible, letting σ^{-1} act on both sides of the above, we obtain

$$\boldsymbol{\alpha} = \sigma^{-1}(\lambda\boldsymbol{\alpha}).$$

Because σ^{-1} is also a linear transformation, we have

$$\boldsymbol{\alpha} = \lambda\sigma^{-1}(\boldsymbol{\alpha}),$$

which is

$$\sigma^{-1}(\boldsymbol{\alpha}) = \lambda^{-1}\boldsymbol{\alpha}.$$

$\square$