Advanced Algebra II

Assignment 4

Solution

1. Let $\mathbb{P}^{m\times n}$ be the set of all $m \times n$ matrices over $\mathbb{P}$. Clearly, $\mathbb{P}^{m\times n}$ is a linear space over $\mathbb{P}$. Let $P_{m\times m}, Q_{n\times n}$ be fixed square matrices over $\mathbb{P}$. Let $\sigma : \mathbb{P}^{m\times n} \to \mathbb{P}^{m\times n}$, $A \mapsto PAQ$ be a mapping. Prove:

   (1) σ is a linear mapping.

   (2) σ is an isomorphism if P, Q are invertible.

   Proof.

   (1) Let $A, B \in \mathbb{P}^{m\times n}$ and $k \in \mathbb{P}$. We have

   $$\begin{aligned} \sigma(A+B) &= P(A+B)Q = PAQ + PBQ = \sigma(A) + \sigma(B), \\ \sigma(kA) &= P(kA)Q = k(PAQ) = k\sigma(A). \end{aligned}$$

   Hence, σ is a linear mapping.

   (2) It suffices to prove σ is a bijection. We see

   $$\begin{aligned} \sigma(A) = \sigma(B) &\Rightarrow PAQ = PBQ \\ &\Rightarrow P^{-1} \cdot PAQ = P^{-1} \cdot PBQ \\ &\Rightarrow AQ = BQ \\ &\Rightarrow AQ \cdot Q^{-1} = BQ \cdot Q^{-1} \\ &\Rightarrow A = B. \end{aligned}$$

   Thus, σ is injective. For any $B \in \mathbb{P}^{m\times n}$, we know

   $$\sigma(P^{-1}BQ^{-1}) = P(P^{-1}BQ^{-1})Q = B,$$

   which means σ is surjective. This completes the proof. □

2. Let $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n$ be a basis of V. Let $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_n$ be a list of vectors. Assume

   $$(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_n) = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n)A, \tag{1}$$

   where A is a $n \times n$ matrix. Prove that

   $$\dim\Big(\operatorname{Span}(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_n)\Big) = r(A).$$

   Hint: Build an isomorphism from V to $\mathbb{P}^n$. The linear independence will be reserved by this isomorphism.

   Proof. We know the following mapping is an isomorphism:

   $$\begin{aligned} \sigma : V &\to \mathbb{P}^n \\ \boldsymbol{\alpha} &\mapsto \text{ coordinate vector of } \boldsymbol{\alpha} \text{ with respect to the basis } \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n \end{aligned}$$

This isomorphism gives us that a list of vectors in V is linearly independent iff their coordinate vectors are linearly independent.

The formula (1) shows that the j-th column of A is exactly the coordinate vector of $\boldsymbol{\beta}_j$ with respect to the basis $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n$.

Therefore, assume $\{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_r\}$ is a maximal linearly independent subset of $\{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_n\}$. Then the coordinate vectors of $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_r$ must also be a maximal linearly independent subset of all column vectors of A. Thus,

$$r(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_n) = r(A).$$

By Theorem 3(2), Page 173 of the textbook, we know

$$\dim\Big(\operatorname{Span}(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_n)\Big) = r(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_n),$$

so

$$\dim\Big(\operatorname{Span}(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_n)\Big) = r(A).$$

□

3. Let $a \in \mathbb{P}$ be fixed. Prove that $\sigma : \mathbb{P}[x] \to \mathbb{P}[x]$, $f(x) \mapsto f(x+a)$ is a linear mapping. Compute the Kernel and Range. Is it an isomorphism?

Proof. We first prove σ is linear. In fact, for $f(x), g(x) \in \mathbb{P}[x]$ and $k \in \mathbb{P}$, we have

$$\begin{aligned} \sigma(f(x)+g(x)) &= f(x+a) + g(x+a) = \sigma(f(x)) + \sigma(g(x)), \\ \sigma(k(f(x))) &= kf(x+a) = k\sigma(f(x)). \end{aligned}$$

Note that

$$\begin{aligned} \ker\sigma &= \{f(x) \in \mathbb{P}[x] \mid \sigma(f(x)) = 0\} \\ &= \{f(x) \in \mathbb{P}[x] \mid f(x+a) = 0\} \\ &= \{f(x) \in \mathbb{P}[x] \mid f(x) = 0\}. \end{aligned}$$

The last equality is due to $f(x) = 0 \Leftrightarrow f(x+a) = 0$. In addition, we see

$$\operatorname{rang}\sigma = \{f(x+a) \mid f(x) \in \mathbb{P}[x]\}. \tag{2}$$

We claim that when $f(x)$ runs through all polynomials in $\mathbb{P}[x]$, $f(x+a)$ also runs through all polynomials in $\mathbb{P}[x]$, i.e., $\{f(x+a) \mid f(x) \in \mathbb{P}[x]\} = \mathbb{P}[x]$. To explain precisely, let $g(x) \in \mathbb{P}[x]$ be arbitrary. Write $h(x) = g(x-a)$. Then we see $g(x) = h(x+a)$ with $h(x) \in \mathbb{P}[x]$, which means that

$$g(x) \in \{f(x+a) \mid f(x) \in \mathbb{P}[x]\}\ .$$

Thus,

$$\mathbb{P}[x] \subset \{f(x+a) \mid f(x) \in \mathbb{P}[x]\}.$$

Trivially,

$$\mathbb{P}[x] \supset \{f(x+a) \mid f(x) \in \mathbb{P}[x]\}.$$

Hence,

$$\mathbb{P}[x] = \{f(x+a) \mid f(x) \in \mathbb{P}[x]\}.$$

Together with the identity (2), it gives

$$\operatorname{rang}\sigma = \mathbb{P}[x].$$

□

4. Prove the following mappings are linear mappings, and compute the Kernel and Range of the mappings.

(1) $\sigma : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x$;

(2) $\sigma : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x - y$;

(3) $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $(x, y) \mapsto (x + y, x - y, 2x + 3y)$.

Proof.

(1) We see

$$\sigma\Big((x_1, y_1) + (x_2, y_2)\Big) = \sigma\Big((x_1 + x_2, y_1 + y_2)\Big) = x_1 + x_2 = \sigma\Big((x_1, y_1)\Big) + \sigma\Big((x_2, y_2)\Big).$$

$$\sigma\Big(k(x_1, y_1)\Big) = \sigma\Big((kx_1, ky_1)\Big) = kx_1 = k\sigma\Big((x_1, y_1)\Big).$$

In addition,

$$\begin{aligned} \ker \sigma &= \Big\{(x, y) \in \mathbb{R}^2 \;\Big|\; \sigma\Big((x, y)\Big) = \mathbf{0}\Big\} \\ &= \Big\{(x, y) \in \mathbb{R}^2 \;\Big|\; x = 0\Big\} \\ &= \Big\{(0, y) \;\Big|\; y \in \mathbb{R}\Big\}. \end{aligned}$$

Also,

$$\begin{aligned} \operatorname{range} \sigma &= \Big\{\sigma\Big((x, y)\Big) \;\Big|\; (x, y) \in \mathbb{R}^2\Big\} \\ &= \Big\{x \;\Big|\; (x, y) \in \mathbb{R}^2\Big\} \\ &= \mathbb{R}. \end{aligned}$$

This means σ is a surjection.

(2) We see

$$\begin{aligned} \sigma\Big((x_1, y_1) + (x_2, y_2)\Big) &= \sigma\Big((x_1 + x_2, y_1 + y_2)\Big) = x_1 + x_2 - (y_1 + y_2) = (x_1 - y_1) + (x_2 - y_2) \\ &= \sigma\Big((x_1, y_1)\Big) + \sigma\Big((x_2, y_2)\Big). \end{aligned}$$

$$\sigma\Big(k(x_1, y_1)\Big) = \sigma\Big((kx_1, ky_1)\Big) = kx_1 - ky_1 = k(x_1 - y_1) = k\sigma\Big((x_1, y_1)\Big).$$

In addition,

$$\begin{aligned} \ker \sigma &= \Big\{(x, y) \in \mathbb{R}^2 \;\Big|\; \sigma\Big((x, y)\Big) = \mathbf{0}\Big\} \\ &= \Big\{(x, y) \in \mathbb{R}^2 \;\Big|\; x - y = 0\Big\} \\ &= \Big\{(x, x) \in \mathbb{R}^2 \;\Big|\; x \in \mathbb{R}\Big\}. \end{aligned}$$

Also,

$$\begin{aligned} \operatorname{range} \sigma &= \Big\{\sigma\Big((x, y)\Big) \;\Big|\; (x, y) \in \mathbb{R}^2\Big\} \\ &= \Big\{x - y \;\Big|\; (x, y) \in \mathbb{R}^2\Big\} \\ &= \mathbb{R}. \end{aligned}$$

This means σ is a surjection. The last equality above is due to the fact that as x, y both run through all real numbers, $x - y$ also runs through all real numbers. More precisely, let $z \in \mathbb{R}$ be any real number. Then $z = z - 0$, where $z, 0 \in \mathbb{R}$, so $z \in \left\{ x - y \;\middle|\; (x, y) \in \mathbb{R}^2 \right\}$.

(3) We see

$$
\begin{aligned}
\sigma\Big((x_1, y_1) + (x_2, y_2)\Big) &= \sigma\Big((x_1 + x_2, y_1 + y_2)\Big) \\
&= \Big(x_1 + x_2 + y_1 + y_2,\ x_1 + x_2 - (y_1 + y_2),\ 2(x_1 + x_2) + 3(x_1 + x_2)\Big) \\
&= \Big(x_1 + y_1,\ x_1 - y_1,\ 2x_1 + 3x_1\Big) + \Big(x_2 + y_2,\ x_2 - y_2,\ 2x_2 + 3x_2\Big) \\
&= \sigma\Big((x_1, y_1)\Big) + \sigma\Big((x_2, y_2)\Big). \\
\sigma\Big(k(x_1, y_1)\Big) &= \sigma\Big((kx_1, ky_1)\Big) = \Big(kx_1 + ky_1,\ kx_1 - ky_1,\ 2(kx_1) + 3(kx_1)\Big) \\
&= k(x_1 + y_1,\ x_1 - y_1,\ 2x_1 + 3x_1) \\
&= k\sigma\Big((x_1, y_1)\Big).
\end{aligned}
$$

In addition,

$$
\begin{aligned}
\ker \sigma &= \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; \sigma\Big((x, y)\Big) = \mathbf{0} \right\} \\
&= \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; (x + y, x - y, 2x + 3y) = (0, 0, 0) \right\} \\
&= \text{the solution space of the system of equations } \begin{cases} x + y = 0, \\ x - y = 0, \\ 2x + 3y = 0. \end{cases} \\
&= \{(0, 0)\} \\
&= \{\mathbf{0}\}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\operatorname{range} \sigma &= \left\{ \sigma\Big((x, y)\Big) \;\middle|\; (x, y) \in \mathbb{R}^2 \right\} \\
&= \left\{ (x + y, x - y, 2x + 3y) \;\middle|\; (x, y) \in \mathbb{R}^2 \right\} \\
&= \left\{ (a, b, c) \;\middle|\; a = x + y,\ b = x - y,\ c = 2x + 3y,\ x, y \in \mathbb{R} \right\} \\
&= \left\{ (a, b, c) \;\middle|\; c = \frac{5}{2}a - \frac{1}{2}b \right\}.
\end{aligned}
$$

$\square$