Advanced Algebra II

Assignment 3

Solution

1. Let V_1, V_2 be *non-trivial* subspaces of V. Prove that there exits $\boldsymbol{\alpha} \in V$ such that $\boldsymbol{\alpha} \notin V_1$ and $\boldsymbol{\alpha} \notin V_2$.

   Proof. Since V_1, V_2 are non-trivial, there exist $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ such that $\boldsymbol{\alpha}_1 \notin V_1, \boldsymbol{\alpha}_2 \notin V_2$. If $\boldsymbol{\alpha}_1 \notin V_2$ or $\boldsymbol{\alpha}_2 \notin V_1$, then we are done. Otherwise, we have $\boldsymbol{\alpha}_1 \in V_2$ and $\boldsymbol{\alpha}_2 \in V_1$. In this case we claim $\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 \notin V_1$ and $\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 \notin V_2$. In fact, since $\boldsymbol{\alpha}_1 \notin V_1$ and $\boldsymbol{\alpha}_2 \in V_1$, we know $\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 \notin V_1$ (otherwise, $\boldsymbol{\alpha}_1 = (\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2) - \boldsymbol{\alpha}_2 \in V_1$, which is a contradiction). Similarly, since $\boldsymbol{\alpha}_2 \notin V_2$ and $\boldsymbol{\alpha}_1 \in V_2$, we know $\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 \notin V_2$. This completes the proof. □

2. Let V_1, V_2 be subspaces of V. Assume $V_1 \subset V_2$. Prove that $\dim V_1 = \dim V_2 \Rightarrow V_1 = V_2$.

   Proof. Assume $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_r$ is a basis of V_1, so

   $$V_1 = \mathrm{Span}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_r).$$

   By $V_1 \subset V_2$, we know $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_r$ is a linearly independent list of vectors in V_2. Since $\dim(V_2) = \dim(V_1) = r$, the list $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_r$ is also a basis for V_2. Hence,

   $$V_2 = \mathrm{Span}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_r),$$

   which implies $V_1 = V_2$. □

3. Let $A \in \mathbb{P}^{n\times n}$ be fixed. Answer the following questions:

   (1) Write $C(A) = \{B \in \mathbb{P}^{n\times n} \mid BA = AB\}$. Prove that $C(A)$ is a subspace of $\mathbb{P}^{n\times n}$.

   (2) Take $A = E$. Show $C(A)$.

   (3) Take

   $$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & n \end{bmatrix}.$$

   Show the dimension and a basis of $C(A)$.

   Proof.

   (1) Clearly, $E \in C(A)$, so $C(A) \neq \emptyset$. Let $B_1, B_2 \in \mathbb{P}^{n\times n}$ and $k \in \mathbb{P}$. We see $(B_1 + B_2)A = B_1A + B_2A = AB_1 + AB_2 = A(B_1 + B_2)$. In addition, $(kB_1)A = k(B_1A) = k(AB_1) = A(kB_1)$.

   (2) Since any matrix in $\mathbb{P}^{n\times n}$ can interchange with E, we have $C(E) = \mathbb{P}^{n\times n}$.

(3) We prove that a matrix that can interchange with A iff this matrix is a diagonal matrix (Q5, Page 133 of the textbook). In fact, let $B = (b_{ij})$. Then

$$B \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & n \end{bmatrix} = B \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & n \end{bmatrix}$$

implies that $b_{ij}j = ib_{ij}$, $1 \le i, j \le n$. Hence, $b_{ij} = 0$ when $i \ne j$. This means B is diagonal. Trivially, any diagonal matrix can interchange with A.

Therefore, $C(A)$ is the set of all diagonal matrices of order n. A basis is $M_i, 1 \le i \le n$ where M_i is the matrix with (i,i)-entry being 1 and other entries being 0. The dimension of $C(A)$ is n. (Can you find a basis for $\mathbb{P}^{n\times n}$?)

□

4. Let $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in V$. Assume $c_1\boldsymbol{\alpha} + c_2\boldsymbol{\beta} + c_3\boldsymbol{\gamma} = \mathbf{0}$, where $c_1c_3 \ne 0$. Prove $\text{Span}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \text{Span}(\boldsymbol{\beta}, \boldsymbol{\gamma})$.

Proof. By $c_3 \ne 0$,

$$\boldsymbol{\gamma} = -\frac{c_1}{c_3}\boldsymbol{\alpha} - \frac{c_2}{c_3}\boldsymbol{\beta}.$$

Then $\boldsymbol{\gamma}$ can be linearly represented by $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Similarly, by $c_1 \ne 0$,

$$\boldsymbol{\alpha} = -\frac{c_2}{c_1}\boldsymbol{\beta} - \frac{c_3}{c_1}\boldsymbol{\gamma}.$$

Then $\boldsymbol{\alpha}$ can be linearly represented by $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$. The above two displayed formulas tell us that $\{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$ and $\{\boldsymbol{\beta}, \boldsymbol{\gamma}\}$ can linearly represent each other. Hence,

$$\{\boldsymbol{\alpha}, \boldsymbol{\beta}\} \sim \{\boldsymbol{\beta}, \boldsymbol{\gamma}\},$$

i.e., $\text{Span}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \text{Span}(\boldsymbol{\beta}, \boldsymbol{\gamma})$ (see Theorem 3(1), Page 173 of the textbook). □

5. Let

$$\begin{aligned} \boldsymbol{\alpha}_1 &= (2, 1, 3, 1), \\ \boldsymbol{\alpha}_2 &= (1, 2, 0, 1), \\ \boldsymbol{\alpha}_3 &= (-1, 1, -3, 0), \\ \boldsymbol{\alpha}_4 &= (1, 1, 1, 1). \end{aligned}$$

Show the dimension and a basis of $\text{Span}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4)$. *Hint: You can use the following fact without computation: After a series of elementary row transformations, we can get*

$$A = \begin{bmatrix} 2 & 1 & -1 & 1 \\ 1 & 2 & 1 & 1 \\ 3 & 0 & -3 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -3 & -3 & -1 \\ 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{1}$$

You may need Corollary 3, Page 88 of the textbook.

Proof. The key point is to find a maximal linearly independent subset for $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4$. By the formula (1) and Corollary 3, Page 88 of the textbook, the 1st, 2nd, 4th columns of A, which are $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_4$, are a maximal linearly independent subset of $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4$, so

$$\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_4\} \sim \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4\}.$$

This means

$$\mathrm{Span}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_4) = \mathrm{Span}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4).$$

Also, $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_4$ is linearly independent. Therefore, $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_4$ is a basis of $\mathrm{Span}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4)$ and the dimension is 3. □

6. Let W denote the solution space of $A\boldsymbol{x} = \mathbf{0}$, where A is the matrix defined in Q5. Show the dimension and a basis of W.

   Solution. By the formula (1), $A\boldsymbol{x} = \mathbf{0}$ is equivalent to

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 = 0, \\ -3x_2 - 3x_3 - x_4 = 0, \\ \frac{1}{3}x_4 = 0. \end{cases}$$

   We can see x_3 is a free variable. Taking $x_3 = 1$, then $x_4 = 0, x_2 = -1, x_1 = 1$. Hence, the fundamental set of solutions only consists of $\eta_1 = (1, -1, 1, 0)$. Note that the fundamental set of solutions is linearly independent and it spans the all solutions of the system $A\boldsymbol{x} = \mathbf{0}$, so the fundamental set of solutions is a basis of the solution space. Hence, η_1 is a basis and the dimension is 1.

7. Let V_1, V_2 be solutions spaces of $x_1 + x_2 + \cdots + x_n = 0$ and $x_1 = x_2 = \cdots = x_n$, respectively. Prove $\mathbb{P}^n = V_1 \oplus V_2$.

   Proof. We first prove $\mathbb{P}^n = V_1 + V_2$, and then prove it is a direct sum. Let $(x_1, x_2, \cdots, x_n)$ be any vector in $\mathbb{P}^n$. Then we can write it as

$$\begin{aligned} &(x_1, x_2, \cdots, x_n) \\ &= \left(x_1 - \frac{1}{n}\sum_{i=1}^n x_i,\ x_2 - \frac{1}{n}\sum_{i=1}^n x_i,\ \cdots,\ x_n - \frac{1}{n}\sum_{i=1}^n x_i\right) + \left(\frac{1}{n}\sum_{i=1}^n x_i,\ \frac{1}{n}\sum_{i=1}^n x_i,\ \cdots,\ \frac{1}{n}\sum_{i=1}^n x_i\right). \end{aligned}$$

   Clearly, the first term on the right hand side of the above equality is a vector in V_1, and the second term is a vector in V_2. Hence, $\mathbb{P}^n = V_1 + V_2$.

   Let $(x_1, x_2, \cdots, x_n) \in V_1 \cap V_2$ be any vector. By $(x_1, x_2, \cdots, x_n) \in V_2$, all x_i, $i = 1, 2, \cdots, n$ are equal, so assume $x_i = a$, $i = 1, 2, \cdots, n$. By $(x_1, x_2, \cdots, x_n) \in V_1$, we know $na = 0$, which means $a = 0$. Therefore, $V_1 \cap V_2 = \{0\}$, which proves that the sum $V_1 + V_2$ is indeed a direct sum. □

8. Assume $\dim(V) = n$. Prove that there exist subspaces $V_1, V_2, \cdots, V_n$ with $\dim(V_i) = 1$, $i = 1, 2, \cdots, n$ such that

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_n.$$

Proof. Let $\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, \cdots, \boldsymbol{\epsilon}_n$ be a basis of V. Write

$$V_1 = \mathrm{Span}(\boldsymbol{\epsilon}_1), \quad V_2 = \mathrm{Span}(\boldsymbol{\epsilon}_2), \quad \cdots, \quad V_n = \mathrm{Span}(\boldsymbol{\epsilon}_n).$$

Clearly,

$$V = V_1 + V_2 + \cdots + V_n$$

since any vector $\boldsymbol{\alpha} \in V$ can be written as a form of $\boldsymbol{\alpha} = k_1\boldsymbol{\epsilon}_1 + k_2\boldsymbol{\epsilon}_2 + \cdots k_n\boldsymbol{\epsilon}_n$.

It suffices to prove the above sum is a direct sum. In fact, clearly, $\dim(V) = n$. Trivially, we know $\dim(\mathrm{Span}(\boldsymbol{\epsilon}_i)) = 1$ for any $1 \leq i \leq n$. Thus,

$$\dim(V) = \dim(\mathrm{Span}(\boldsymbol{\epsilon}_1)) + \dim(\mathrm{Span}(\boldsymbol{\epsilon}_2)) + \cdots + \dim(\mathrm{Span}(\boldsymbol{\epsilon}_n)),$$

which means it is a direct sum. $\square$