Advanced Algebra II

Assignment 2

Solution

1. Let V be the vector space of all real functions over $\mathbb{R}$. Prove that $1, \cos^2 t, \cos 2t$ are linearly dependent.

   Proof. Note that
   $$\cos 2t = 2\cos^2 t - 1,$$
   which means
   $$1 + (-2) \cdot \cos^2 t + 1 \cdot \cos 2t = 0.$$
   This completes the proof. □

2. Let $\mathbb{P}[x]$ be a vector space over $\mathbb{P}$. Assume $f_1(x), f_2(x), f_3(x)$ are co-prime, and any two of them are not co-prime. Prove that they are linearly independent.

   Proof. Assume
   $$k_1 f_1 + k_2 f_2 + k_3 f_3 = 0.$$
   If there exist k_i such that $k_i \neq 0$, then we will prove there is contradiction. Without loss of generality, we assume $k_1 \neq 0$. Then
   $$f_1 = -\frac{k_2}{k_1} f_2 - \frac{k_3}{k_1} f_3.$$
   Since any two of f_1, f_2, f_3 are not co-prime, we assume $d(x) = (f_2, f_3)$. Note $\partial(d(x)) \geq 1$. By the above equality, we know $d(x) | f_1$. This implies that $d(x)$ is a common divisor of f_1, f_2, f_3, which is a contradiction, because f_1, f_2, f_3 are co-prime. Therefore, we have $k_1 = k_2 = k_3 = 0$, which means f_1, f_2, f_3 are linearly independent. □

3. Find the dimension of the following vector spaces and find a basis.

   (1) The vector space $\mathbb{P}^{n\times n}$ over $\mathbb{P}$. (The notation $\mathbb{P}^{n\times n}$ is the set of all $n \times n$ matrices with entries in $\mathbb{P}$.)

   Solution :
   Let N_{ij} be denote the matrix with the (i,j)-entry being 1 and other entries being 0. Then N_{ij}, $1 \leq i, j \leq n$ is a basis.

   In fact, you can easily prove that N_{ij}, $1 \leq i, j \leq n$ are linear independent, and each matrix can be linear represented by N_{ij}, $1 \leq i, j \leq n$, and .

   The dimension is clearly n^2.

   (2) Let
   $$V = \mathbb{R}_{>0}.$$
   The operations are as follows:
   $$a \oplus b = ab,$$
   $$k \otimes a = a^k,$$

for any $k \in \mathbb{R}$ and any $a, b \in V$.

(Hint: What is the additive identity of V?)

Solution :
The additive identity is 1 since $a \oplus 1 = a \cdot 1 = a$ for any a.

The element e is a basis. In fact, e itself is linear independent, and for any a, $a = e^{\ln a} = (\ln a) \otimes e$ (This means a can be linearly represented by e).

The dimension is 1.

4. Let $\mathbb{P}[x]_n$ be a vector space. Prove that

$$f_i = (x - a_1) \cdots (x - a_{i-1})(x - a_{i+1}) \cdots (x - a_n), \qquad i = 1, 2, \cdots, n$$

is a basis. Here $a_1, a_2, \cdots, a_n$ are pair-wisely distinct.

Proof. Recall

$$\mathbb{P}[x]_n = \{f(x) \in \mathbb{P}[x] \mid \partial(f) < n\}.$$

Observe that for each f_i, we have $f_i(a_j) = 0$ when $j \neq i$, and $f_i(a_j) \neq 0$ when $j = i$ since $a_1, \cdots a_n$ are pair-wisely distinct.

Now assume

$$k_1 f_1(x) + k_2 f_2(x) + \cdots k_n f_n(x) = 0.$$

Take $x = a_1$. Then we have

$$k_1 f_1(a_1) + k_2 f_2(a_1) + \cdots + k_n f_n(a_1) = 0.$$

By our previous observation, we know

$$f_1(a_1) \neq 0, \qquad f_i(a_1) = 0, \quad i = 2, \cdots, n.$$

The above two facts imply

$$k_1 f_1(a_1) = 0.$$

Since $f_1(a_1) \neq 0$, we know $k_1 = 0$.

Similarly, in the above process, we can take $x = a_j$, $j = 2, 3, \cdots, n$, which lead to $k_2 = k_3 = \cdots = k_n = 0$. This concludes the proof. □

5. (1) In $\mathbb{P}^3$, find the transition matrix from the basis $\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, \boldsymbol{\epsilon}_3$ to the basis $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_3$.

$$\boldsymbol{\epsilon}_1 = (1,0,0), \quad \boldsymbol{\epsilon}_2 = (1,1,0), \quad \boldsymbol{\epsilon}_3 = (1,1,1);$$
$$\boldsymbol{\eta}_1 = (1,1,0), \quad \boldsymbol{\eta}_2 = (0,0,1), \quad \boldsymbol{\eta}_3 = (3,2,1).$$

Solution :
Observe

$$\begin{aligned} \boldsymbol{\eta}_1 &= 0\boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_2 + 0\boldsymbol{\epsilon}_3; \\ \boldsymbol{\eta}_2 &= 0\boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_2 + \boldsymbol{\epsilon}_3; \\ \boldsymbol{\eta}_3 &= \boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_2 + \boldsymbol{\epsilon}_3. \end{aligned}$$

Write $\mathcal{B} = \{\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, \boldsymbol{\epsilon}_3\}$, and $\mathcal{B}' = \{\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_3\}$. Hence,

$$M_{\mathcal{B}\to\mathcal{B}'} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Note that we have

$$(\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, \boldsymbol{\epsilon}_3) = (\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_3) M_{\mathcal{B}\to\mathcal{B}'}.$$

(2) Find the coordinate of $\boldsymbol{\alpha} = (5, 4, 2)$ with respect to the bases $\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, \boldsymbol{\epsilon}_3$ and $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_3$, respectively. Also, show the relation between these two coordinates.

Solution :
It is easy to check

$$\begin{aligned} \boldsymbol{\alpha} &= \boldsymbol{\epsilon}_1 + 2\boldsymbol{\epsilon}_2 + 2\boldsymbol{\epsilon}_3, \\ \boldsymbol{\alpha} &= 2\boldsymbol{\eta}_1 + \boldsymbol{\eta}_2 + \boldsymbol{\eta}_3. \end{aligned}$$

Hence the coordinate of $\boldsymbol{\alpha}$ with respect to the basis $\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2, \boldsymbol{\epsilon}_3$ is $(1, 2, 2)$, and the the coordinate of $\boldsymbol{\alpha}$ with respect to the basis $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_3$ is $(2, 1, 1)$.

The relation between these two coordinates is

$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$