**Advanced Algebra II**

**Assignment 5**

**Solution**

In this assignment, the notation 0 may denote the number 0, the vector 0 or the zero transformation, and the notation 1 may denote the number 1 or the identity transformation.

1. Let $\sigma, \tau$ be linear transformations on $\mathbb{P}[x]$ defined by

$$\sigma(f(x)) = f'(x),$$
$$\tau(f(x)) = xf(x).$$

Prove $\sigma\tau - \tau\sigma = 1$.

*Proof.* For any $f(x) \in \mathbb{P}(x)$, we have

$$\begin{aligned}(\sigma\tau - \tau\sigma)f(x) &= (\sigma\tau)f(x) - (\tau\sigma)f(x)\\ &= \sigma\Big(\tau(f(x))\Big) - \tau\Big(\sigma(f(x))\Big)\\ &= \sigma\Big(xf(x)\Big) - \tau\Big(f'(x)\Big)\\ &= f(x) + xf'(x) - xf'(x)\\ &= f(x)\\ &= \mathbb{1}(f(x)).\end{aligned}$$

Since the above is true for any $f(x) \in \mathbb{P}[x]$, we know $\sigma\tau - \tau\sigma = \mathbb{1}$. □

2. Let $\sigma$ be a linear transformation on $\mathbb{P}[x]_n$ defined by $f(x) \mapsto f(x+1) - f(x)$. Compute the matrix of $\sigma$ with respect to the basis

$$\boldsymbol{\varepsilon}_0 = 1, \quad \boldsymbol{\varepsilon}_i = \frac{x(x-1)\cdots(x-i+1)}{i!}, \quad i = 1, 2, \cdots, n-1.$$

*Proof.* We need to know the expressions of $\sigma(\boldsymbol{\varepsilon}_i)$, $i = 0, 1, \cdots, n-1$. By computation, we get

$$\sigma(\boldsymbol{\varepsilon}_0) = 1 - 1 = 0.$$

Also, note that

$$\boldsymbol{\varepsilon}_1 = x,$$

so

$$\sigma(\boldsymbol{\varepsilon}_1) = (x+1) - x = 1 = \boldsymbol{\varepsilon}_0.$$

In addition,

$$\begin{aligned}\sigma(\boldsymbol{\varepsilon}_i) &= \frac{(x+1)x(x-1)\cdots(x-i+2)}{i!} - \frac{x(x-1)\cdots(x-i+1)}{i!}\\ &= \frac{x(x-1)\cdots(x-i+2)[x+1-(x-i+1)]}{i!}\\ &= \frac{x(x-1)\cdots(x-i+2)}{(i-1)!}\\ &= \boldsymbol{\varepsilon}_{i-1}, \qquad i = 2, \cdots, n-1.\end{aligned}$$

Hence,

$$\begin{aligned}\sigma(\boldsymbol{\varepsilon}_0, \boldsymbol{\varepsilon}_1, \cdots, \boldsymbol{\varepsilon}_{n-1}) &= (\sigma(\boldsymbol{\varepsilon}_0), \sigma(\boldsymbol{\varepsilon}_1), \cdots, \sigma(\boldsymbol{\varepsilon}_{n-1})) \\ &= (0, \boldsymbol{\varepsilon}_0, \boldsymbol{\varepsilon}_2, \cdots, \boldsymbol{\varepsilon}_{n-2}) \\ &= (\boldsymbol{\varepsilon}_0, \boldsymbol{\varepsilon}_1, \cdots, \boldsymbol{\varepsilon}_{n-1}) \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \cdots & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.\end{aligned}$$

The matrix above is what we desire. □

3. (1) Let $\sigma$ be a linear transformation on $V$ satisfying

$$\sigma^{k-1}(\boldsymbol{\xi}) \neq 0, \quad \sigma^k(\boldsymbol{\xi}) = 0$$

for some $k > 0$ and some vector $\boldsymbol{\xi}$. Prove that $\boldsymbol{\xi}, \sigma(\boldsymbol{\xi}), \cdots, \sigma^{k-1}(\boldsymbol{\xi})$ are linearly independent.

*Proof.* Assume

$$k_1\boldsymbol{\xi} + k_2\sigma(\boldsymbol{\xi}) + \cdots + k_3\sigma^{k-1}(\boldsymbol{\xi}) = 0. \tag{1}$$

Let $\sigma^{k-1}$ act on both sides above. We get

$$\sigma^{k-1}\Big(k_1\boldsymbol{\xi} + k_2\sigma(\boldsymbol{\xi}) + \cdots + k_3\sigma^{k-1}(\boldsymbol{\xi})\Big) = 0,$$

which gives

$$k_1\sigma^{k-1}(\boldsymbol{\xi}) + k_2\sigma^k(\boldsymbol{\xi}) + \cdots + k_3\sigma^{2k-2}(\boldsymbol{\xi}) = 0.$$

By $\sigma^k(\boldsymbol{\xi}) = 0$, we have for $n \geq k$, $\sigma^n(\boldsymbol{\xi}) = \sigma^{n-k}(\sigma^k(\boldsymbol{\xi})) = \sigma^{n-k}(0) = 0$. Hence, the above displayed formula implies

$$k_1\sigma^{k-1}(\boldsymbol{\xi}) = 0.$$

Due to $\sigma^{k-1}(\boldsymbol{\xi}) \neq 0$, we get $k_1 = 0$. Now by (1), we have

$$k_2\sigma(\boldsymbol{\xi}) + \cdots + k_3\sigma^{k-1}(\boldsymbol{\xi}) = 0.$$

Let $\sigma^{k-2}$ act on the both sides above. Similarly, we can deduce $k_2 = 0$. Repeating this process gives

$$k_1 = k_2 = \cdots = k_n = 0,$$

as desired. □

(2) Let $\dim(V) = k$ and $\sigma$ be the same as in (1). Prove that there exists a basis such that the matrix of $\sigma$ with respect to this basis is

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

*Proof.* By the conclusion of Q(3), we know $\boldsymbol{\xi}, \sigma(\boldsymbol{\xi}), \cdots, \sigma^{k-1}(\boldsymbol{\xi})$ is a basis. Hence,

$$\begin{aligned}
\sigma(\boldsymbol{\xi}, \sigma(\boldsymbol{\xi}), \cdots, \sigma^{k-1}(\boldsymbol{\xi})) &= (\sigma(\boldsymbol{\xi}), \sigma^2(\boldsymbol{\xi}), \cdots, \sigma^k(\boldsymbol{\xi})) \\
&= (\sigma(\boldsymbol{\xi}), \sigma^2(\boldsymbol{\xi}), \cdots, 0) \\
&= (\boldsymbol{\xi}, \sigma(\boldsymbol{\xi}), \cdots, \sigma^{k-1}(\boldsymbol{\xi})) \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.
\end{aligned}$$

□

4. Let $\sigma, \tau \in \mathcal{L}(V)$ satisfying $\sigma^2 = \sigma$, $\tau^2 = \tau$. Prove $(\sigma+\tau)^2 = \sigma + \tau \Rightarrow \sigma\tau = 0$.

*Proof.* Note that

$$\begin{aligned}
(\sigma+\tau)^2 &= \sigma^2 + \sigma\tau + \tau\sigma + \tau^2 \qquad \text{(Why } (\sigma+\tau)^2 = \sigma^2 + 2\sigma\tau + \tau^2 \text{ is not correct?)} \\
&= \sigma + \sigma\tau + \tau\sigma + \tau \\
&= \sigma + \tau + \sigma\tau + \tau\sigma.
\end{aligned}$$

Together with $(\sigma+\tau)^2 = \sigma + \tau$, this implies

$$\sigma\tau + \tau\sigma = 0,$$

so

$$\sigma\tau = -\tau\sigma$$

With the above equality, we have

$$\sigma\tau = \sigma^2\tau = \sigma \cdot \sigma\tau = \sigma \cdot (-\tau\sigma) = (-\sigma\tau) \cdot \sigma = \tau\sigma \cdot \sigma = \tau\sigma^2 = \tau\sigma.$$

The above two displayed formula give

$$\sigma\tau = -\sigma\tau,$$

so

$$2\sigma\tau = 0,$$

which gives

$$\sigma\tau = 0.$$

□

5. Let $V$ be a linear space with $\dim(V) = n$.

(1) Prove $\dim(\mathcal{L}(V)) = n^2$.

*Proof.* Fix a basis of $V$. We have an isomorphism between $\mathcal{L}(V)$ and $\mathbb{P}^{n\times n}$. It suffices to prove $\dim(\mathbb{P}^{n\times n}) = n^2$. Let $M_{ij}$ be the $n$ by $n$ matrix with the $(i,j)$-entry being 1 and other entries being all 0. It is easy to prove $M_{ij}$, $1 \le i, j \le n$ is linearly independent and any matrix can be linearly represented by them, so $M_{ij}$, $1 \le i, j \le n$ is a basis of the vector space $\mathbb{P}^{n\times n}$. This means $\dim(\mathbb{P}^{n\times n}) = n^2$. □

(2) Let $\sigma \in \mathcal{L}(V)$. Assume $\sigma\tau = \tau\sigma$ for any $\tau \in \mathcal{L}(V)$. Prove $\sigma$ must be a scalar transformation. *Hint: Map $\sigma, \tau$ to their corresponding matrices and use the conclusion of Q7(3), Page 134 of the textbook.*

*Proof.* Fix a basis of $V$. We have an isomorphism between $\mathcal{L}(V)$ and $\mathbb{P}^{n\times n}$. Let $A, B$ be the matrices of $\sigma, \tau$ with respect to this basis, respectively.

Note that this isomorphism not only preserves addition and scalar multiplication, it also preserves multiplication of two linear transformations. (i.e., $\sigma\tau \mapsto AB$. See Theorem 2, Page 192 of the textbook). On the matrix side, by Q7(3), Page 134 of the textbook, we know only scalar matrices can interchange with all matrices, so on the linear transformation side, only scalar transformations can interchange with all linear transformations. □

6. Let $\sigma$ be a linear transformation on $V$ over $\mathbb{P}$ and $\dim(V) = n$. Prove

(1) There exists a polynomial $f(x) \in \mathbb{P}[x]$ of degree $\leq n^2$ such that $f(\sigma) = 0$.

*Proof.* By the conclusion of Q5(1), we have $\dim(\mathcal{L}(V)) = n^2$. Then any $n^2 + 1$ vectors in $\mathcal{L}(V)$ must be linearly dependent. Hence,

$$\mathbb{1}, \sigma, \sigma^2, \cdots, \sigma^{n^2}$$

must be linearly dependent. Hence, there exist $a_0, a_1, \cdots, a_{n^2}$ such that

$$a_0\mathbb{1} + a_1\sigma + \cdots + a_{n^2}\sigma^{n^2} = 0.$$

Take

$$f(x) = a_{n^2}x^{n^2} + \cdots + a_1x + a_0.$$

Clearly,

$$f(\sigma) = 0,$$

as desired.

*Remark: The Cayley-Hamilton theorem (the theorem on Page 202 and the corollary on Page 203 of the textbook) tells us that there exist a polynomial $f(x)$ of degree $n$ such that $f(\sigma) = 0$, which is a stronger result than the conclusion of this question.* □

(2) $f(\sigma) = 0, g(\sigma) = 0 \Rightarrow d(\sigma) = 0$, where $d(x) = (f, g)$.

*Proof.* Since $d(x) = (f, g)$, by Theorem 2, Page 9 of the textbook, we have

$$d(x) = u(x)f(x) + v(x)g(x).$$

Hence,

$$d(\sigma) = u(\sigma)f(\sigma) + v(\sigma)g(\sigma) = 0.$$

□

(3) $\sigma$ is invertable **iff** there exists a polynomial $f(x) \in \mathbb{P}[x]$, with constant term $\neq 0$, such that $f(\sigma) = 0$.

*Proof.* ($\Rightarrow$) By Q6(3), we know there exist $f(x)$ of degree $\leq n^2$ such that $f(\sigma) = 0$. Write

$$f(x) = a_{n^2}x^{n^2} + \cdots + a_1 x + a_0.$$

We have

$$f(\sigma) = a_{n^2}\sigma^{n^2} + \cdots + a_1\sigma + a_0\mathbb{1} = 0. \tag{2}$$

If $a_0 = 0$, the we are done. Otherwise, we check if $a_1 = 0$. We assume $a_r$ is first non-zero coefficient in $a_0, a_1, a_2, \cdots, a_{n^2}$.

We claim $r \neq n^2$. In fact, if $r = n^2$, then $a_0 = a_1 = a_2 = \cdots = a_{n^2-1} = 0$ and $a_{n^2} \neq 0$. Inserting these into (2) gives

$$a_{n^2}\sigma^{n^2} = 0.$$

Since $a_{n^2} \neq 0$, it implies $\sigma^{n^2} = 0$, where 0 denotes zero transformation. Then $\sigma$ could not invertible, which is a contradiction.

In the above we have proved $r < n^2$. Note $a_r \neq 0$ and $a_j = 0$ for all $0 \leq j \leq r-1$. Hence,

$$f(x) = a_{n^2}x^{n^2} + \cdots + a_r x^r.$$

Define

$$g(x) = \frac{f(x)}{x^r} = a_{n^2}x^{n^2-r} + \cdots + a_{r+1}x + a_r.$$

We will prove $g(\sigma) = 0$, so $g(x)$ is what we want (since $a_r \neq 0$). In fact, since $f(x) = x^r g(x)$, we have

$$\sigma^r g(\sigma) = f(\sigma) = 0.$$

Because $\sigma$ is invertible, so $\sigma^r$ is also invertible. (Compare this with the fact for matrices, this is, $A$ is invertible, then $A^r$ is invertible). Multiplying both sides in the above equality by $\sigma^{-r}$ and using the fact $\sigma^{-1}\sigma = \mathbb{1}$, we have

$$g(\sigma) = \sigma^{-r} \cdot 0 = 0,$$

as desired. (Note here 0 all denote the zero transformation.)

($\Leftarrow$) Let $f(x) = a_m x^m + \cdots + a_1 x + a_0$ be the polynomial with $a_0 \neq 0$ such that $f(\sigma) = 0$. Then

$$a_m\sigma^m + \cdots + a_1\sigma + a_0\mathbb{1} = 0,$$

so

$$a_m\sigma^m + \cdots + a_1\sigma = -a_0\mathbb{1}.$$

Since $a_0 \neq 0$,

$$-\frac{a_m}{a_0}\sigma^m - \cdots - \frac{a_1}{a_0}\sigma = \mathbb{1}.$$

Taking one $\sigma$ out gives

$$\sigma\left(-\frac{a_m}{a_0}\sigma^{m-1} - \cdots - \frac{a_1}{a_0}\right) = \mathbb{1}$$

This means $\sigma$ is invertible. (In the textbook, it says that $\sigma$ is invertible if there exist $\tau$ such that $\sigma\tau = \tau\sigma = \mathbb{1}$. You can try to prove that if there exist $\tau$ such that $\sigma\tau = \mathbb{1}$, then $\sigma$ is invertible.) $\square$