**Advanced Algebra II**

**Assignment 1**

**Solution**

1. (1) Assume

$$\begin{aligned} f: X_1 \times X_2 &\to Y \\ (x_1, x_2) &\mapsto y. \end{aligned}$$

Write down the value of $a \oplus b$.

**Solution :**

$$a \oplus b = f((a, b)).$$

(2) Assume

$$\begin{aligned} f: \mathbb{Z} \times 2\mathbb{Z} &\to \mathbb{Q} \\ (x_1, x_2) &\mapsto |x_1 - x_2|. \end{aligned}$$

Write down the values of $a \oplus b$ and $1 \oplus 6$. (Recall $xA = \{xa \mid a \in A\}$.)

**Solution :**

$$a \oplus b = |a - b|; \qquad 1 \oplus 6 = |1 - 6| = 5.$$

(3) Assume we have an addition $a \oplus b = (a - b)^2$ for any $a \in \mathbb{Q}, b \in \mathbb{Z}$. Write down a suitable binary mapping that induces this addition.

**Solution :**

$$\begin{aligned} f: \mathbb{Q} \times \mathbb{Z} &\to \mathbb{Q} \\ (x_1, x_2) &\mapsto (x_1 - x_2)^2. \end{aligned}$$

(4) Assume we have a multiplication $a \otimes b = (a - b)^3 + 2i$ for any $a \in \mathbb{Q}, b \in \mathbb{Z}$. Write down a suitable binary mapping that induces this multiplication.

**Solution :**

$$\begin{aligned} f: \mathbb{Q} \times \mathbb{Z} &\to \mathbb{C} \\ (x_1, x_2) &\mapsto (x_1 - x_2)^3 + 2i. \end{aligned}$$

2. Determine if the following sets $V$ are linear spaces over $\mathbb{R}$. *(Hint: Do not forget to check if "$\oplus$" and "$\otimes$" are closed in $V$ or not.)*

   (1) Let $n \geq 1$ be fixed. Let

   $$V = \{f(x) \mid f(x) \in \mathbb{R}[x],\ \partial(f) = n\}.$$

   The operations "$\oplus$" and "$\otimes$" are usual operations for polynomials.

   **Solution :** No. The additive identity 0 is not included in $V$.

(2) Let $A_{n\times n}$ be a real matrix and be fixed. Let

$$V = \{f(A) \mid f(x) \in \mathbb{R}[x]\}.$$

The operations "$\oplus$" and "$\otimes$" are usual operations for matrices. *(Hint: If $g(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then $g(A) = a_n A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 E$. In addition, note that $g(A)$ is a matrix, not a polynomial.)*

**Solution :** Yes. Note the addition $\oplus$ is closed in $V$. In fact, for $f(A), g(A) \in V$, we write $h(x) = f(x) + g(x)$. Then $f(A) \oplus g(A) = f(A) + g(A) = h(A) \in V$. Other details of proof are omitted.

(3) Let

$$V = \text{ the set of real symmetric matrices of order } n.$$

The operations "$\oplus$" and "$\otimes$" are usual operations for matrices.

**Solution :** Yes. The addition is closed. In fact, for any $A, B \in V$, we know $A^T = A, B^T = B$, so $(A + B)^T = A^T + B^T = A + B$, which means $A + B \in V$. Other details of proof are omitted.

(4) Let $\boldsymbol{\alpha}$ $(\neq \mathbf{0})$ be a vector (in a geometric sense) on the plane. Let

$$V = \text{ the set of vectors that are not parallel to } \boldsymbol{\alpha}.$$

The operations "$\oplus$" and "$\otimes$" are usual operations for vectors (in a geometric sense).

**Solution :** No. The zero vector is not included in $V$. (The zero vector is considered to be parallel to every other vector.)

(5) Let

$$V = \{(a, b) \mid a, b \in \mathbb{R}\}.$$

The operations are as follows:

$$\begin{aligned} (a_1, b_1) \oplus (a_2, b_2) &= (a_1 + a_2, b_1 + b_2 + a_1 a_2), \\ k \otimes (a_1, b_1) &= \left(ka_1, kb_1 + \frac{k(k-1)}{2} a_1^2\right). \end{aligned} \tag{1}$$

Here the addition and multiplication on the right-hand side of (1) are usual operations for numbers. You should write down all the details for this question.

**Solution :** Yes. Firstly, the operations are closed in $V$. Secondly, the additive identity is $(0, 0)$, and the additive inverse of $(a, b)$ is $(-a, a^2 - b)$. The multiplicative identity is 1. Other details of proof omitted. One can check P157 of the Solution Book. The proof also has been shown during the lecture.

(6) Let

$$V = \text{ the set of all vectors on the plane.}$$

The "$\oplus$" is the usual addition for vectors (in a geometric sense), and "$\otimes$" is as follows:

$$k \otimes \boldsymbol{\alpha} = \mathbf{0},$$

for any $k \in \mathbb{R}$ and any $\boldsymbol{\alpha} \in V$.

**Solution :** No. Let $\boldsymbol{\beta} \neq \mathbf{0}$. If $V$ is a vector space, then $1 \otimes \boldsymbol{\beta} = \boldsymbol{\beta}$ which contradicts to the fact that $k \otimes \boldsymbol{\alpha} = \mathbf{0}$ for any $k \in \mathbb{R}$ and any $\boldsymbol{\alpha} \in V$.

(7) Let
$$V = \text{ the set of all vectors on the plane.}$$
The "$\oplus$" is the usual addition for vectors (in a geometric sense), and "$\otimes$" is as follows:
$$k \otimes \boldsymbol{\alpha} = \boldsymbol{\alpha},$$
for any $k \in \mathbb{R}$ and any $\boldsymbol{\alpha} \in V$.

**Solution :** No. If $V$ is a linear space, we know $0 \otimes \boldsymbol{\alpha} = \mathbf{0}$ for any $\boldsymbol{\alpha} \in V$ (Page 165 of the textbook, the third property), but here $k \otimes \boldsymbol{\alpha} = \boldsymbol{\alpha}$, which will be a contradiction when we choose $\boldsymbol{\alpha}$ not to be $\mathbf{0}$ and choose $k = 0$.

(8) Let
$$V = \mathbb{R}_{>0}.$$
The operations are as follows:
$$a \oplus b = ab,$$
$$k \otimes a = a^k,$$
for any $k \in \mathbb{R}$ and any $a, b \in V$.

**Solution :** Yes. The additive identity is 1, and the multiplicative identity is 1. The additive inverse of $a$ is $\frac{1}{a}$. Other details are omitted. The (Correction: In the first version of Assignment 1, $V$ is defined to be $\mathbb{R}_{\geq 0}$. In that case, $V$ is not a vector space.)

(9) Let
$$V = \mathbb{R}_{>0}.$$
The operations are as follows:
$$a \oplus b = -ab,$$
$$k \otimes a = a^k,$$
for any $k \in \mathbb{R}$ and any $a, b \in V$.

**Solution :** No. Note $1 \oplus 2 = -2 \in V$, so the addition is not closed in $V$.

3. Let $V$ be a linear space over $\mathbb{R}$. Prove $k(\boldsymbol{\alpha} - \boldsymbol{\beta}) = k\boldsymbol{\alpha} - k\boldsymbol{\beta}$.

*Proof.* We know $\boldsymbol{\alpha} - \boldsymbol{\beta} = \boldsymbol{\alpha} + (-\boldsymbol{\beta})$, where $-\boldsymbol{\beta}$ is the additive inverse of $\boldsymbol{\beta}$. Hence,
$$k(\boldsymbol{\alpha} - \boldsymbol{\beta}) = k[\boldsymbol{\alpha} + (-\boldsymbol{\beta})] = k\boldsymbol{\alpha} + k(-\boldsymbol{\beta}) = k\boldsymbol{\alpha} + ((-1)k)\boldsymbol{\beta} = k\boldsymbol{\alpha} + (-1)(k\boldsymbol{\beta}) = k\boldsymbol{\alpha} - k\boldsymbol{\beta}.$$
The second identity is due the distributive law for addition, and the third identity is due to $-\boldsymbol{\beta} = (-1)\boldsymbol{\beta}$. □