Imperial College London

Department of Mathematics

The Uncertain Volatility Model for pricing fixed-income derivatives

Author: Phol Wehaprasirtsak (CID: 01859075)

A thesis submitted for the degree of

MSc in Mathematics and Finance, 2023-2024

Declaration

The work contained in this thesis is my own work unless otherwise stated.

Acknowledgements

I would like to express my sincere gratitude to my supervisor, Dr.Pietro Siorpaes, for providing this research opportunity and his guidance throughout my MSc research.

I would also like to thank my family for their support and my girlfriend, Hannah Petrovic, for her emotional and academic support throughout my MSc journey.

Abstract

Robust pricing is a concept of generating practical value bounds, where the actual price of the derivative lies within, based on the market information instead of relying heavily on any specific model for the dynamics of the underlying price process. This study presents the original Uncertain Volatility Model (UVM) in equity derivatives and how the ideas could develop to fixed-income product pricing. The model utilises its sub-additivity properties, from the non-linear differential equations, where ultimately imply that the price spread can be reduced by the Lagrangian UMV static hedging with other available products in the market. We have demonstrated such pricing in coupon bonds and convertible bonds, and discuss potential practical uses as well as improvements of the model.

Contents

List of Figures

List of Tables

Introduction

Financial derivatives are financial instruments whose value are derived from the value of an underlying asset, index, or rate. They are used for a variety of purposes, including hedging risk, speculating on future price movements, arbitrage, and enhancing portfolio returns.

Equity derivatives pricing has a long history of extensive research. The Black-Scholes model is one of the most famous and widely used models in finance, particularly for pricing European-style options (Black & Scholes 1973). The model provides a theoretical estimate of the price of European call and put options by assuming that the log of return of the stock prices follows a normal distribution with a constant standard deviation called volatility, σ. The model is still extensively used as a good approximation. However, due to the unpredictable movements of the derivative prices, it became clear that the volatility cannot be understood, forecast or modeled as constant or even deterministic. Incorporating heteroskedastic behavior (i.e.volatility of volatility) is essential for the risk-management of derivatives. Several attempts in this direction have been made, most notably with autoregressive models and with the use of stochastic differential equations to model volatility changes (Reider 2009, Brailsford & Faff 1996), but these methods suffer from the large number of parameter fitting (model risk). Motivated by reducing the model risk, a more robust pricing method such as the Uncertain Volatility Model (UVM) has been proposed by Avellaneda et al. (1995). 'Robust Pricing' methods rarely depend on the assumption that the real world is correctly described by any proposed models; they mainly rely on the information from the market. The aim is to produce optimal value bounds where the price of derivatives should lie within.

Fixed-income products, such as bonds, represent debt instruments that pay a fixed or variable interest rate over a specified period. Pricing these products involves identifying and predicting the interest rate term structure. In contrast to the asset price world, there is no commonly accepted model for the movement of the underlying rate in the interest rate world. The simplest approach is to price a product off a yield curve, but is ineffective for more complex products, where optionality or convexity play a role. The precise nature of the interest rate movements became more significant, so stochastic variables to model a number of 'unknown' factors were introduced. These models can be single- or multifactor-models for the movement of the short-term interest rate, or models for the movement of the whole yield curve, i.e., the Heath, Jarrow & Morton approach (HJM). All of these methods rely on the estimation of parameters, which is, again, model risk prone.

Inspired by the pioneer work on applications of UVM in interest rate contingent claims under the HJM framework Lewicki & Avellaneda (1996), a non-probabilistic uncertain model was proposed in Epstein et al. (2000). The model assumes that the interest rate and its rate of change is restricted within a bound set, but is otherwise not determined. Under this assumption, the arbitrage-free price will not be unique but a set of worst and best-case prices.

This study on the Uncertain Volatility Model (UVM) is divided into five parts: In chapter 1, we present the traditional approaches to interest rate product pricing to review some of the key definitions and methods used. In chapter 2, we will introduce Black-Scholes model and compare with the UVM and its Lagrangian approach in equity derivative as presented in Avellaneda et al. (1995), Avellaneda & Paras (1996). In chapter 3, we will present the application of UVM approaches in fixed income pricing. In chapter4, we will discuss the implementation of the UVM model using the finite difference method, and begin some analysis on linear fixed income products. In chapter 5, we will implement the numerical scheme on more a complicated product such as convertible bonds. Finally in chapter 6, we conducted an analysis to explore an improvement of the model and suggested possible areas for further research.

Chapter 1

Traditional approaches to interest rate product pricing

1.1 Continuously compound interest

To be able to compare fixed-income products, we must compute the present value (at time t) of an amount of cash C to be received at time T. The present value is the amount we would pay now for this future cash flow.
If we have a continuously-compounded short-term interest rate, r, then money invested in the bank, $M(t)$, grows exponentially according to

$$dM = rMdt.$$

When this short interest rate is a known function of time, $r(t)$, and $M(T) = C$ then we find

$$M(t) = Ce^{-\int_t^T r(\tau)d\tau}.$$

1.1.1 Zero coupon bond and its yield

The zero-coupon bond (ZCB) is a contract paying a fixed amount of £1 principal at the maturity date, T, in the future. At a time t,

$$Z(t,T) = e^{-y(T-t)},$$

where y is the yield to maturity of the ZCB. For coupon bonds with P as the principal, N number of coupons and C_i the coupon paid on date t_i, the present value is

$$V = Pe^{-y(T-t)} + \sum_{i=1}^{N} C_i e^{-y(t_i-t)}.$$

If we have a number of traded instruments then we can calculate the yield of each of their maturities. We can then interpolate between these points to construct the yield curve. This curve provides us with rates of return for all maturities. To price a fixed income product off the yield curve, we just add up the present values of all the cashflows, calculated from reading off the rate of return for the payment date of the cashflow from the yield curve and discount the cashflow at that rate.

1.2 Forward rates and bootstrapping

The main problem with the use of yield to maturity as a measure of interest rates is that it is not consistent across instruments. For example, two coupon bonds with the same maturity but different coupon structures would result in different yield curves.

One way to overcome this is to use forward rate, $F(t;T)$, which is the interest rates that are assumed to apply over given periods in the future for all instruments described by

$$F(t;T) = -\frac{\partial}{\partial T}(\log Z(t;T)).$$

1.2.1 Bootstrapping

Bootstrapping is a method used to build up the implied forward rate curve, and is applied as follows.

1. Rank the bonds according to maturity, with the shortest maturity first. Denote these as Z_i^M, where i is the position of the bond in the ranking.

2. Apply $y_1 = -\frac{\log Z_1^M}{T_1 - t}$ for the rate for discounting all instruments between the present time and the maturity date T_1 of the first bond.

3. Now $Z_2^M = e^{-y_1(T_1-t)}e^{-y_2(T_2-t)}$; therefore, $y_2 = -\frac{\log(Z_2^M/Z_1^M)}{T_1-t}$.

4. Repeat the calculations until the forward rate curve is complete.

5. Note that the curve could be made continuous using popular interpolation algorithms, such as piecewise constant gradient or cubic splines.

However, this method is still unfitting for more complex products, such as caps, floors or bond options, whose values depend more strongly on the exact nature of the underlying interest rate movements. To price these contracts, we must first construct a model for the interest rate.

1.3 Stochastic model

A popular approach to interest rate modelling is to construct a stochastic model for the movement of the short-term interest rate. We can then price a contract as the expected value of its cashflows, where we discount at this short rate, and also use the rate to value any rate-dependent cash flows, i.e.,

$$V = \sum_i \left(\mathbb{E}_{\mathbf{t}} \left[C_i(r) e^{-\int_t^{T_i} r_\tau d\tau} \right] \right),$$

where the contract has cashflows $C_i(r)$ at times T_i .

1.3.1 One-factor models

The simplest of these stochastic models are one-factor models. Many of such model have been proposed by Ho & Lee (1990), Hull & White (1993). These models assume that interest rate movements are driven by a single random factor, and have the general form

$$dr = u(r,t)dt + v(r,t)dX,$$

where u and v are some specified functions of r and t and dX is a Brownian process. There are many possible choices for the functions u and v. The most commonly used ones are Vasicek, Ho-Lee, Hull-White etc.

A pricing equation for the value of a fixed income under this model can be derived and found that the price of a contract, $V(r,t)$, satisfies

$$\frac{\partial V}{\partial t} + \frac{1}{2}v^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda v)\frac{\partial V}{\partial r} - rV = 0,$$

where λ is the market price of risk. This market price of risk is the ratio of the excess return above the risk-free rate to the level of risk inherent in a portfolio. The increase in the value of the portfolio over a time step dt is an extra λdt for each unit of risk, dX. It is necessary to introduce such a measure because the underlying process, the short rate, is not a traded quantity.The market price of risk can be viewed as an additional variable that links the random behaviour of diverse products or arguably as a big umbrella term to hide all parameter estimation problems. (Epstein et al. 2000)

1.3.2 Multi-factor model

The simplest generalisation of the one-factor stochastic model is the multi-factor model. This model makes the assumption that movements in the yield curve depend on more than one random factor. The main reasons are to have more parameters to fit market options data such as liquid caps and floors and are to allow for more realistic correlation patterns along the term structure. Brigo (2020-21)

Generally, we model the short-term interest rate, r, along with another independent variable, l, where

$$dr = udt + vdX_1,$$

and

$$dl = pdt + qdX_2.$$

Here u, v, p and q are some specified functions of $(r,\ l\ ,\ t)$, and dX_1 and dX_2 are two Brownian motion variables, with correlation ρ.

A PDE pricing equation for the value of a fixed income product under this model is derived to be of the form:

$$\frac{\partial V}{\partial t} + \frac{1}{2}v^2\frac{\partial^2 V}{\partial r^2} + \rho v q\frac{\partial^2 V}{\partial r \partial l} + \frac{1}{2}q^2\frac{\partial^2 V}{\partial l^2} + (u - \lambda_r v)\frac{\partial V}{\partial r} + (p - \lambda_l q)\frac{\partial V}{\partial l} - rV = 0,$$

where $\lambda_r(r,l,t)$ and $\lambda_l(r,l,t)$ are the market prices of risk for r and l respectively.

Similar to the one-factor model, the parameters, especially in the fixed-income world, are difficult to estimate due to instability and prone to a number of errors.

1.3.3 Heath, Jarrow & Morton (HJM)

All of the previous stochastic models have been models of the movement of one or more interest rate factors. However, Heath, Jarrow & Morton suggest a more general approach, instead of modelling the movement of the whole forward rate curve (Heath et al. 1992). Their method consistently reproduces the current yield curve, since this information is contained in the initial forward rate curve.

We assume that zero-coupon bonds evolve as a one-factor model,

$$dZ(t;T) = r(t)Z(t;T)dt + \sigma(t,T)Z(t;T)dX.$$

The evolution of the forward rate curve can, consequently, be derived,

$$dF(t;T) = \frac{\partial}{\partial T}\left(\frac{1}{2}\sigma^2(t,T) - r(t)\right)dt - \frac{\partial}{\partial T}\sigma(t,T)dX. \tag{1.3.1}$$

Since

$$r(t) = F(t;t) = F\left(t^*;t\right) + \int_{t^*}^{t} dF(\tau;t), \tag{1.3.2}$$

it can be shown, using (1.3.1) and 1.3.2 that

$$r(t) = F(t^*;t) + \int_{t^*}^{t}(\sigma(s,t)\frac{\partial \sigma}{\partial t} - \frac{\partial r(s,t)}{\partial t})ds - \int_{t^*}^{t}\frac{\partial \sigma(s,t)}{\partial t}dX(s)$$

we can also find the short rate for any time t in the future of today, t^*.

The HJM approach to modelling the whole forward rate curve is very powerful as long as Monte Carlo simulations can be used, which is not always plausible due to runtime problems. An empirical study Radhakrishnan (1998) showed that the tree option pricing model for non-Markov model such as HJM will have limited convergence rate as the tree structure grows exponentially in size with the addition of the new timesteps.

Chapter 2

Pricing derivatives under UMV

Before diving into the Uncertain Volatility Modelling (UMV) framework, we cover a brief overview of some basic derivative pricing under standard Black-Scholes model

According to arbitrage pricing theory, if the market presents no arbitrage opportunities, there exists a probability measure on future scenarios such that the price of any security is the expectation of its discounted cashflows. Such a probability is known as a pricing measure. The value of derivative, V, can be mathematically represented as

$$V(t, S_t) = \mathbb{E}^{\mathbb{P}}_{(t,S_t)} \left[e^{-\int_t^T r(S_\tau,\tau)\mathrm{d}\tau} g\left(S_T\right) \right] \tag{2.0.1}$$

under probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and S_t process is defined down below.

2.1 The Black-Scholes model

Consider a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, supporting a Brownian motion $(Z_t)_{t\geq 0}$. In the Black-Scholes model, the stock price process $(S_t)_{t\geq 0}$ is the unique strong solution to the following stochastic differential equation:

$$\frac{\mathrm{d}S_t}{S_t} = r\mathrm{d}t + \sigma \mathrm{d}Z_t, \qquad S_0 > 0, \tag{2.1.1}$$

where $r \geq 0$ denotes the instantaneous risk-free interest rate and $\sigma > 0$ the instantaneous volatility.

According to the Feynman-Kac formula, it can be shown that equation 2.0.1is a solution of the following PDE:

$$\begin{cases} \frac{\partial V}{\partial t}(t, S_t) + \mu(t, S_t)\frac{\partial V}{\partial x}(t, S_t) + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2}(t, S_t) - rV(t, S_t) = 0 & , \ t < T \\ V(T, S_t) = g(S_t) & , \ t = T \end{cases} \tag{2.1.2}$$

under the following assumptions:

- No transaction cost for any trading.
- Free to lend and borrow any amount of cash with the same risk-free rate.
- The market is complete and efficient; as a result, no arbitrage exists.
- The underlying assets are fully fungible and liquid.
- The risk-free rate, r, and the volatility, σ, are constants and can be observed.
- The return of the asset follows a log-normal distribution, Equation (2.1.1).

2.1.1 Critics on Black-Scholes (BS)

The Black-Scholes Model is revolutionary in the history of option pricing. It is a good application of an arbitrage-free-pricing method, martingale measure, and risk neutral probability measure. However, most of its achievements are built on the aforementioned assumptions, while some of the assumptions do not suit the reality necessarily.

For example, there are two parameters in the Black-Scholes equation, the volatility σ and the risk-free interest rate r. Both of them are taken to be constant and observable under the BS assumptions. However, this is not always the case due to the natures of the two parameters which are not only varying but also difficult to model in practice.

Nonetheless, the BS model is being used as a reasonable 'first order' approximation to the 'true' (market) option price. For 'second order' corrections, practitioners apply various empirical tweaks to the BS model, such as using 'implied', 'local', or 'stochastic' volatilities, that should vary across stock prices or option strikes in order to reconcile the original BS formula with market prices of traded options. The implied volatility is reflected on the option price, because it is the market expectation of the future standard deviation of the return. Due to this similarity, the option trading can be viewed as volatility trading of the underlying asset. When new information appears in the market, the volatility reacts to it and change happens inevitably.
The other parameter, the interest rate (r), is facing similar problems. Although the risk-free interest rate is firmly connected to the spot rate (observable), it turns out to be unclear how the term structure will evolve for options with longer maturity. There are indeed many interest rate models with different assumptions for long-term interest rate models, but it is clear that we can not use a single value to represent the term structure.

Despite all this, Wilmott (2010) states that the BS model is widely used and widespread in finance for its simplicity and gives a price close to the market price. The Black-Scholes model performs well with some 'second order' adjustments. Those two important parameters, the volatility and the risk-free interest rate still remain as a major modelling problem in financial industry. It is clear that we can not model them as fixed values, or even a deterministic functions. Therefore, we are going to propose a generalization of the Black-Scholes model under the context of Uncertain Volatility Model in the next section.

2.2 The Uncertain Volatility Model

The Uncertain Volatility Modelling (UMV) approach was deemed as a reasonable model for managing the volatility risk. It should satisfy the following requirements Avellaneda & Paras (1996):

- Heteroskedasticity, or uncertainty in the values of forward volatilities should be taken into account by assuming that more than one arbitrage-free measure can realize current prices at any given time.

- Portfolio values should be sub-additive for the sell-side and super-additive for the buy-side due to diversification of volatility risk.

- Option prices, as well as the prices of other liquid optional instruments traded in the market, should be incorporated into the model, since they provide the information necessary to "narrow down" volatility uncertainty.

This chapter will explain the theory behind the implementation of these ideas in a model for vanilla equity options as a starting point.

The main elements in our approach are:

i Modelling volatility uncertainty by using volatility bands and,

ii Optimization over the class of admissible probabilities according to market prices of derivative instruments.

2.2.1 Model Presentation

First of all, the model of uncertain volatility has almost the same assumptions on market and assets as the Black-Scholes model, except for the volatility part. Following the practice done in Avellaneda et al. (1995) and Avellaneda & Paras (1996), we consider that the evolution of volatility will remain in a band determined from the prices of vanilla options. We consider the risky asset price to follow the dynamics below.

$$dS_t = S_t\left(\mu_t dt + \sigma_t dZ_t\right), \qquad 0 \leq t \leq T$$

where under risk-neutral measure, μ_t equals r_t, which is the risk-free interest rate and is assumed to be a constant for equity derivative pricing. The second coefficient, σ_t is the volatility that is bounded between σ_{min} and σ_{max} ,

$$\sigma_{min} \leq \sigma_t \leq \sigma_{max}.$$

Then for each pair of μ and σ that make S_t satisfy the diffusion process, we will have a unique corresponding probability measure $P(\mu, \sigma)$. Let $\mathbf{P}$ be denoted as the set of all defined equivalent martingale measure.

Now let us assume that there exists a portfolio, where payoffs can be represented by a stream of cashflows

$$F_1\left(S_{t_1}\right), F_2\left(S_{t_2}\right), F_3\left(S_{t_3}\right), \ldots, F_N\left(S_{t_N}\right)$$

at time $t_1, t_2, t_3, \ldots, t_N$ respectively. European option can be achieved by setting $N = 1$

If there are no arbitrage opportunities and our assumptions on volatility is correct, the value of this derivative should lie somewhere between the bounds

$$V^+\left(S_t, t\right) = \sup_{P \in \mathbf{P}} \mathrm{E}_t^P\left[\sum_{j=1}^{N} \mathrm{e}^{-r(t_j - t)} F_j\left(S_{t_j}\right)\right], \tag{2.2.1}$$

and

$$V^-\left(S_t, t\right) = \inf_{P \in \mathbf{P}} \mathrm{E}_t^P\left[\sum_{j=1}^{N} \mathrm{e}^{-r(t_j - t)} F_j\left(S_{t_j}\right)\right]. \tag{2.2.2}$$

Now we construct a portfolio of an option, with value $V(S_t, t)$, and hedge it with $-\Delta$ of the underlying asset. The value of this portfolio is thus

$$\Pi = V - \Delta S.$$

From Itô's lemma,

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \left(\frac{\partial V}{\partial S} - \Delta\right) dS.$$

With the volatility unknown, the choice of $\Delta = \partial V/\partial S$ eliminates the risk from price movement:

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt.$$

Under the no-arbitrage principle, the return on this portfolio should be equal to the risk-free rate:

$$d\Pi = r\Pi dt.$$

Thus we set

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt = r\left(V - S\frac{\partial V}{\partial S} \right) dt.$$

Under the vanilla Black-Scholes argument, we would be able to say that if we know V and its Greeks, then we know $d\Pi$. However, this is no longer the case since we do not know σ to be a deterministic constant. For $\sigma \in [\sigma_{min}, \sigma_{max}]$, we obtain

$$\frac{\partial V^{\pm}}{\partial t} + \frac{1}{2}S^2 \sigma_{\pm}^2 \left(\frac{\partial^2 V^{\pm}}{\partial S^2} \right) \frac{\partial^2 V^{\pm}}{\partial S^2} + r_t S \frac{\partial V^{\pm}}{\partial S} - r_t V^{\pm} = 0 \tag{2.2.3}$$

where $\sigma(x)$ is the control variable

$$\sigma_{+}^2(x) = \begin{cases} \sigma_{\max}^2 & \text{if } x \geq 0 \\ \sigma_{\min}^2 & \text{if } x < 0 \end{cases} \tag{2.2.4}$$

and

$$\sigma_{-}^2(x) = \begin{cases} \sigma_{\max}^2 & \text{if } x \leq 0 \\ \sigma_{\min}^2 & \text{if } x > 0 \end{cases} \tag{2.2.5}$$

Avellaneda et al. (1995) have shown (under delta dynamic hedging) that the two extreme solutions V^+ and V^- of the control problems (2.2.1) and (2.2.2) are obtained by solving the final-value problem (2.2.3) with σ_+ and σ_- respectively. The non-linear PDE (2.2.3) is called the Black-Scholes-Barenblatt (BSB) equation.

Since simple vanilla option values are convex in S i.e., ($\frac{\partial^2 V}{\partial S^2} > 0$), the intuitive course of actions, as a risk-averse agent that plans to delta-hedge his position in this uncertain volatility environment, would be to sell options, V^+, at the highest volatility, σ_+, and buy them, V^-, at the lowest one, σ_-, which match the results in Equations (2.2.4) and (2.2.5).

Alternatively, one can arithmetically justify the $\sigma_{\pm}(x)$ construction by looking at the Equation (2.2.3), and realise that the upper bound V^+ corresponds to when the σ takes the value σ_{min} for negative gamma, $\frac{\partial^2 V}{\partial S^2}$, or σ_{max} for positive gamma. The reverse is true for V^-.

It is notable that by changing the signs in Equation (2.2.3), we go from the upper bound to the lower bound. In other words, the problem for the V^+ and the V^- prices for long and short positions in a particular contract is mathematically equivalent to valuing a long position only. This idea will be explored more concretely in Chapter 3.

More complex derivatives and option portfolios that combine short and long option positions (and thus have mixed convexity, $\frac{\partial^2 V}{\partial S^2}$), are priced differently due to the nonlinear nature of Equation (2.2.3). The BSB equation selects the volatility path that generates the most efficient non-arbitrageable bid/offer values.

Risk-diversification

The uncertain volatility model gives us a means of quantifying the diversification of volatility risk in portfolios of derivative securities (Avellaneda et al. 1995). In fact, let two derivative products Φ and Ω with $F_j\left(S_{t_j}\right)$ and $G_i\left(S_{t_i}\right)$ cashflows, which are not identical to each other. The two products can be represented as

$$\Phi = \sum_{j=1}^{N} \mathrm{e}^{-r(t_j-t)} F_j\left(S_{t_j}\right)$$
$$\Omega = \sum_{i=1}^{N} \mathrm{e}^{-r(t_i-t)} G_i\left(S_{t_i}\right).$$

Clearly,

$$\sup_{P\in\mathbf{P}} \mathrm{E}_t^P[\Phi+\Omega] \leq \sup_{P\in\mathbf{P}} \mathrm{E}_t^P[\Phi] + \sup_{P\in\mathbf{P}} \mathrm{E}_t^P[\Omega]$$
$$\inf_{P\in\mathbf{P}} \mathrm{E}_t^P[\Phi+\Omega] \geq \inf_{P\in\mathbf{P}} \mathrm{E}_t^P[\Phi] + \inf_{P\in\mathbf{P}} \mathrm{E}_t^P[\Omega].$$

Therefore, in general, the optimal risk-averse "offer" price (V^+) of the portfolio "Φ+Ω" will be lower than the sum of the individual optimal offer prices for "Φ" and "Ω". Similarly, the 'bid' price for the "$\Phi+\Omega$" portfolio will be higher than the sum of the separate bid prices. The financial intuition for this is as follows:

- $\inf_{P\in\mathbf{P}} \mathrm{E}_t^P[\Phi]$ represent the worst-case scenario for Φ corresponding to the worst-case volatility path σ_Φ.
- $\inf_{P\in\mathbf{P}} \mathrm{E}_t^P[\Omega]$ represent the worst-case scenario for Ω corresponding to the worst-case volatility path σ_Ω.
- Since σ_Φ and σ_Ω paths are not necessarily the same, if we value the combined portfolio of Φ and Ω in a worst-case scenario, we may find that the volatility path follows some sort of 'compromise' between the two separate worst-case paths.
- Our overall portfolio has a worst-case scenario value that is higher than the two separate worst-case values added together except when the worst-case scenario volatility paths coincide.
- The opposite is true for the best-case portfolio.

In other words, the solutions of the BSB equation yield smaller bid/ask spread, reflecting the benefit of pricing portfolio of combined options. Note that having tighter bid/ask spread is beneficial as a market maker because it increases the market share due to the price being supposedly closer to the fair value.

The starting point of the uncertain volatility model consists of an environment with a multiplicity of pricing measures. As shown above, the existence of derivative products in the market offers the possibility of further risk-diversification, resulting in narrower bounds for the prices of OTC derivative products than those corresponding to delta-hedging with the underlying security alone. The range between the upper and lower bounds will be progressively narrowed if the number of derivative products available for hedging increases and the market becomes more complete.

It is noteworthy that dynamically complete models, or even models that incorporate a random volatility with known statistics, cannot distinguish between hedges that use derivatives and those that use only the underlying stock. Therefore, unlike the present model, they cannot be used to construct hedges that involve derivative securities. Risk-diversification through derivatives is a natural and important tool in derivative market-making, precisely because of the uncertain nature of volatility (Avellaneda et al. 1995).

Critics on Delta hedging

Delta hedging leads to a risk-elimination strategy that can be used in practice. There are, however, a few problems on both the practical and theoretical side. In practice, hedging must be done at discrete times and is costly. Some options with discontinuous payoffs such as barrier options would require one to buy or sell a large number of the underlying stock. Theoretically, Delta hedging alone leaves the holder very exposed to the model risk, since we never know the parameter values accurately and even the model itself can just be inaccurate. One way to alleviate these problems is by employing "static hedging" by other market-traded derivatives in addition to Delta hedging of the residual payoff.

Static hedging

The concept of static hedging relies on the fact that the value of a portfolio of contracts is not necessarily equal to the sum of their individual values. Hence, the value of a contract depends on what else it is priced with. The hedging amount remains the same throughout; thus, it can reduce the hedging cost.

2.3 The Lagrangian Uncertain Volatility Model

In this section, we will see how to construct efficiently hedging portfolios using market-traded option derivatives in addition to the underlying asset. Following the practice in the paper by Avellaneda & Paras (1996), we will introduce the Lagrangian Uncertain Volatility Model (known as λ-UVM). The main construction is

$$(\text{Hedge Contract})_{\text{worst-case}} = (\text{Contract} + \text{Hedging Instrument})_{\text{worst-case}} - \text{Cost of Hedge}.$$

The goal is to optimize λ values, the hedging quantities, such that the hedged portfolios return the highest worst- and lowest best- prices. Equivalently, it gives us the smallest bid/ask spread. Similar to previously, we assume that the market value of a hedging instrument is contained within the best- and worst-case scenario prices for the instrument, obtained from our model. This is due to the no arbitrage assumption.

2.3.1 Formulation of the optimization problem

Let us suppose that we want to hedge a long position on a derivative security at time t. We denote Φ the discounted payoff of this derivative and

$$F_1\left(S_{t_1}\right), F_2\left(S_{t_2}\right), \ldots, F_N\left(S_{t_N}\right)$$

be its cashflows at the settlement dates $t_1 < t_2 < \ldots < t_N$:

$$\Phi = \sum_{i=1}^{N} \mathrm{e}^{-r(t_i - t)} F_i\left(S_{t_i}\right).$$

Furthermore, let us suppose there are M options (mainly Calls or Puts) in the market which we can use to hedge our position. We denote their payoffs respectively at the maturity dates $t'_1, \dots, t'_m$ by

$$G_1\left(S_{t'_1}\right), G_2\left(S_{t'_2}\right), \dots, G_M\left(S_{t'_M}\right).$$

These options are available in the market at the following prices: $C_1, C_2, \dots, C_M$.
The main aim is to find the number of option contracts to long or short in order to statically hedge the position. If we denote them by $\lambda_1, \lambda_2, \dots, \lambda_M$, we can define the discounted payoff of the hedging portfolio Ψ:

$$\Psi = \sum_{j=1}^{M} \lambda_j \mathrm{e}^{-r\left(t'_j - t\right)} G_j\left(S_{t'_j}\right).$$

Then, we define the expected discounted residual liability L^+as the maximum payoff loss of the trading strategy:

$$L^+ = \sup_{P\in\mathcal{P}} \mathbb{E}_t^P[\Phi - \Psi] = \sup_{P\in\mathcal{P}} \mathbb{E}_t^P\left[\sum_{i=1}^{N} \mathrm{e}^{-r(t_i - t)} F_i\left(S_{t_i}\right) - \sum_{j=1}^{M} \mathrm{e}^{-r\left(t'_j - t\right)} \lambda_j G_j\left(S_{t'_j}\right)\right]. \quad (2.3.1)$$

Finally, the total cost of the statically hedged portfolio, defined as $V^+\left(t, S_t, \lambda_1, \dots, \lambda_M\right)$, is obtained by adding the above worst-case liability to the cost of the options:

$$\begin{aligned} V^+\left(t, S_t, \lambda_1, \dots, \lambda_M\right) &= L^+ + \sum_{i=1}^{M} \lambda_i C_i \\ &= \sup_{P\in\mathcal{P}} \mathbb{E}_t^P[\Phi - \Psi] + \sum_{i=1}^{M} \lambda_i C_i \\ &= \sup_{P\in\mathcal{P}} \mathbb{E}_t^P\left[\sum_{i=1}^{N} \mathrm{e}^{-r(t_i - t)} F_i\left(S_{t_i}\right) - \sum_{j=1}^{M} \mathrm{e}^{-r\left(t'_j - t\right)} \lambda_j G_j\left(S_{t'_j}\right)\right] + \sum_{i=1}^{M} \lambda_i C_i \end{aligned}$$

Notice that the first term of this equation is similar to the upper bound (2.2.1) seen in the standard UVM pricing except that now the payoff incorporates the hedging option cashflows.

The optimal hedging strategy is the one that minimizes the total hedging cost $V^+\left(t, S_t, \lambda_1, \dots, \lambda_M\right)$; let us call this optimal solution $\widetilde{V}^+\left(t, S_t, \lambda_1, \dots, \lambda_M\right)$. Thus, we are left with the following optimization problem:

$$\widetilde{V}^+\left(t, S_t, \lambda_1, \dots, \lambda_M\right) = \inf_{\lambda_1, \dots, \lambda_M} V^+\left(t, S_t, \lambda_1, \dots, \lambda_M\right) \quad (2.3.2)$$

$\widetilde{V}^+$is therefore the worst-case optimal hedge of an offer price of the derivative security. It acts as an upper bound of the derivative price. Indeed, there is equality only when the volatility follows the best-case scenario. Such an optimization is called the λ-UVM of an upper price.

The steps of deriving the lower bound price of the derivative security are analogous to those for the upper bound. Since we want to hedge a long position on the derivative, the expected discounted residual L^- liability is the infimum of the cashflows :

$$L^- = \inf_{P\in\mathcal{P}} \mathbb{E}_t^P[\Phi - \Psi] = \inf_{P\in\mathcal{P}} \mathbb{E}_t^P\left[\sum_{i=1}^{N} \mathrm{e}^{-r(t_i - t)} F_i\left(S_{t_i}\right) - \sum_{j=1}^{M} \mathrm{e}^{-r\left(t'_j - t\right)} \lambda_j G_j\left(S_{t'_j}\right)\right]. \quad (2.3.3)$$

Similarly, the total cost of the hedging strategy $V^-(t, S_t, \lambda_1, \ldots, \lambda_M)$ is:

$$V^-(t, S_t, \lambda_1, \ldots, \lambda_M) = \inf_{P \in \mathcal{P}} \mathbb{E}_t^P \left[\sum_{i=1}^{N} \mathrm{e}^{-r(t_i - t)} F_i(S_{t_i}) - \sum_{j=1}^{M} \mathrm{e}^{-r\left(t'_j - t\right)} \lambda_j G_j \left(S_{t'_j}\right) \right] + \sum_{i=1}^{M} \lambda_i C_i. \tag{2.3.4}$$

Keeping in mind that the optimal solution is the one which maximizes the above total cost, let us denote it $\tilde{V}^-(t, S_t, \lambda_1, \ldots, \lambda_M)$. The optimization problem for the lower bound price therefore becomes:

$$\tilde{V}^-(t, S_t, \lambda_1, \ldots, \lambda_M) = \sup_{\lambda_1, \ldots, \lambda_M} V^-(t, S_t, \lambda_1, \ldots, \lambda_M). \tag{2.3.5}$$

We restrict the number of short and long positions to some boundaries. This is mainly due to trading limits constraints and speeding up the optimization. For every i we impose:

$$\Lambda_i^- \leq \lambda_i \leq \Lambda_i^+, \tag{2.3.6}$$

for some constants $\Lambda_i^+ > 0$ and $\Lambda_i^- < 0$.

2.3.2 Bid/offer spreads

In reality, the existence of bid/offer spread brings a slight complication to consider. We will assume that options can be purchased at the offer price and sold at the bid price for the amounts specified by the $\Lambda_i^\pm$constraints in (2.3.6). We write

$$C_i^{(b)} = \text{ bid price for the } i^{th} \text{ option}$$

and

$$C_i^{(o)} = \text{ offer price for the } i^{th} \text{ option,}$$

for

$$i = 1, 2, \ldots, M.$$

We consider the cost of acquiring a portfolio $(\lambda_1, \lambda_2, \ldots, \lambda_M)$ is computed as explained in section 2.3.1, but with C_i replaced by $C_i^{(b)}$ if $\lambda_i < 0$ or $C_i^{(o)}$ if $\lambda_i > 0$. Accordingly, this cost is given by

$$\sum_{i=1}^{M} \left[\lambda_i \left(\frac{C_i^{(o)} + C_i^{(b)}}{2} \right) + |\lambda_i| \left(\frac{C_i^{(o)} - C_i^{(b)}}{2} \right) \right]. \tag{2.3.7}$$

The total hedging cost for the derivative product with cashflows $F_j\left(S_{t_j}\right)$ is therefore,

$$V^{(b/o)}(S_t, t; \lambda_1, \ldots, \lambda_M) = \sup_{P \in \mathcal{P}} \mathbf{E}^P \left\{ \sum_{j=1}^{N} e^{-r(t_j - t)} F_j\left(S_{t_j}\right) - \sum_{i=1}^{M} \lambda_i e^{-r\left(t'_i - t\right)} G_i \left(S_{t'_i}\right) \right\}$$
$$+ \sum_{i=1}^{M} \left[\lambda_i \left(\frac{C_i^{(o)} + C_i^{(b)}}{2} \right) + |\lambda_i| \left(\frac{C_i^{(o)} - C_i^{(b)}}{2} \right) \right].$$

Chapter 3

Applications in interest rate products

From the issues with interest rate modelling as mentioned in the introduction, the properties of an ideal interest rate model mentioned in (Wilmott 1998, page 510) are as follows:

- Using few factors to model the interest rate to prevent model risk
- Easy to price many products and robust in general
- Rational strategy for Delta hedging
- Insensitivity of results to hard-to-measure parameters, such as volatilities and correlations.

The non-probabilistic interest rate model delivers all of these and more. This model requires as few assumptions as possible about the process underlying the movement of interest rates. As a result, dynamic Delta hedging plays no important role, no fitting is necessary, nor does any market price of risk term[1] appear. The only hedging will be entirely static.

3.1 Non-probabilistic Interest Rate Modelling

Epstein & Wilmott (1999), inspired by Avellaneda et al. (1995) and Avellaneda & Paras (1996), modelled a short-term interest rate and price a contract in a worst/best- case scenario, using the short rate as the discount rate. The model is stochastic but not in the Brownian motion sense. There will be no probability statements except to assign zero probabilities to certain events. For instance, there is a zero probability that the interest rate will move outside of a certain range and there is a zero probability that the interest rate will grow or decay faster than a certain rate. We remark that this model does not appear to replicate the locally unbounded growth seen in the traditional stochastic models for the short rate and, to some extent, observed in practice. The lack of Brownian motion evolution of r can be justified for modelling a long-term behaviour for which we are less concerned about the very short-term movements. (Wilmott 1998, p. 510).

The resulting problem becomes non-linear and the value of a contract then depends on what it is hedged with. Epstein & Wilmott (1999) generate the 'Yield Envelope' (an alternative to the yield curve). At a maturity for which there are no traded instruments, a spread for the yield is obtained.

[1]The market price of risk is commonly denoted as λ. It is widely used when pricing interest rate products under one/two factor interest rate modelling. The derivation for one factor stochastic intrest rate model can be found in (Wilmott 1998, page 427-430)

3.1.1 Model presentation

Similar to the UMV for equity derivative, the best and worst case prices have slightly different form from Equations (2.2.1) and (2.2.2) due to the rate is now not a constant. We write

$$V^{+}\left(S_t, t\right) = \sup_{P \in \mathbf{P}} \mathrm{E}_t^P \left[\sum_{j=1}^{N} \mathrm{e}^{-\int_t^{t_j} r(\tau) d\tau} F_j\left(S_{t_j}\right) \right] \tag{3.1.1}$$

and

$$V^{-}\left(S_t, t\right) = \inf_{P \in \mathbf{P}} \mathrm{E}_t^P \left[\sum_{j=1}^{N} \mathrm{e}^{-\int_t^{t_j} r(\tau) d\tau} F_j\left(S_{t_j}\right) \right]. \tag{3.1.2}$$

Now we construct a portfolio an option, with value $V(S_t, t)$, and the spot interest rate is modelled to be in the range (r_{min}, r_{max}), and cannot increase or decrease at a speed outside the range (c_{min}, c_{max}).

$$r_{min} \leq r \leq r_{max} \tag{3.1.3}$$

and

$$c_{min} \leq \frac{dr}{dt} \leq c_{max} \tag{3.1.4}$$

Let $V(r, t)$ be the value of our contract when the short-term interest rate is r and the time is t. The change in the value of this contract during a timestep dt is considered.

Using Taylor's theorem to expand the value of the contract for small changes in its arguments, we find that

$$V(r + dr, t + dt) = V(r, t) + \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial t} dt + o(dr, dt).$$

Note that there is no second r-derivative term because the process is not Brownian so the quadratic variation is not the same as that of Weiner processes . We want to investigate dV such that the return on the portfolio is set equal to the risk-free rate under the worst-case assumption for the lower bound price. Hence,

$$\min_{dr}(dV) = \min_{dr} \left(\frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial t} dt \right).$$

Since the rate of change of r is bounded, we find that

$$\min_{dr}(dV) = \left(c_{-}\left(\frac{\partial V}{\partial r} \right) \frac{\partial V}{\partial r} + \frac{\partial V}{\partial t} \right) dt,$$

where

$$c_{-}(x) = \begin{cases} c_{min} & \text{if } x \geq 0 \\ c_{max} & \text{if } x < 0 \end{cases}. \tag{3.1.5}$$

Therefore, the portfolio always earns the risk-free rate of interest in the worst case price, V^{-}. The best-case price, V^{+} is given by similar derivation with

$$c_{+}(x) = \begin{cases} c_{max} & \text{if } x \geq 0 \\ c_{min} & \text{if } x < 0 \end{cases}. \tag{3.1.6}$$

In summary, interest rate derivative best/worst-case values under the non-probabilistic interest rate modelling can be solved from

$$\frac{\partial V^{\pm}}{\partial t} + c_{\pm}\left(\frac{\partial V^{\pm}}{\partial r}\right)\frac{\partial V^{\pm}}{\partial r} - rV^{\pm} = 0 \tag{3.1.7}$$

,where corresponding $c_{\pm}$ has the form as described in equations (3.1.5) and (3.1.6).

The best-case result can be further rationalised by realising that the best case for the contract holder is equivalent to the worst case for the writer. Mathematically, the best-case value can be represented as follows

$$V_{\text{best-case}} = -(-V)_{\text{worst-case}}. \tag{3.1.8}$$

That is, the value of a long contract in a best-case scenario can be obtained either from solving equation (3.1.7) with c_+ of (3.1.6) form or solving for a short contract with (3.1.5) and take the negative of the result.

This is the equation to be solved. It is a first-order non-linear hyperbolic partial differential equation, with known final data $V(r,T)$. This is the value of the final cash flow in the contract. In the case of Brownian motion, it is natural to have a convexity term, $\frac{\partial^2 V}{\partial r^2}$. But in this world, the rate of change of r is bounded and the convexity term disappears in the limit $dr, dt \to 0$.

We can solve this equation to value a contract with cashflows $F_j(r)$ at times T_j, for $j = 1, 2, \ldots, N$. We apply the last cashflow as final data for the PDE,

$$V\left(r, T_N\right) = C_N(r),$$

where $C_1, \ldots, C_N$ are the payoffs at $T_1, \ldots, T_N$. Then, we solve backwards in time from maturity, T_N, to the present day, t_0.
Since the initial short rate is known, this solution contains the current worst-case price for the contract, $V(r,t)$. In the absence of arbitrage opportunities, V is everywhere continuous except at cash flow dates. If there is a cash flow $F_j(r)$ at time T_j, then a no-arbitrage assumption gives us that over the cash flow date,

$$V\left(r, T_{j^-}\right) = V\left(r, T_{j^+}\right) + C_j(r).$$

3.1.2 Consequence of our non-linear model

The derived first-order, nonlinear, hyperbolic partial differential equation will be solved numerically in the following Chapter. Details of the numerical solution of the problem and its construction can be found in Chapter 4.

It is not surprising that the best we can do is to find bounds for the value of a contract since our uncertain interest rate model places bounds on the short-term interest rate. Finding a spread for prices is not necessarily a disadvantage of the model. After all, the market itself has such a property (the bid-offer spread). In some sense, spreads are therefore a more realistic result than a single price. However, it becomes a disadvantage when the spreads are so large that the results become meaningless. As a result, static hedging, inspired from Avellaneda et al. (1995) and Avellaneda & Paras (1996), will be employed.

By static hedging, we can generate an optimal static hedge to reduce the interest rate risk. This would not be possible with a linear model calibrated to the yield curve. This is because the price of a contract would be invariant under the addition of hedging instruments if they had originally been used to construct the yield curve. Hence, a further use of the model is to construct static hedges for portfolios, regardless of which interest rate model we choose to finally price them.(Epstein 1999)

3.1.3 Interest Rate Uncertainty Bands

This model does not capture the Brownian nature of the spot rate over short time scales. To accommodate this we make the following simple modification to the model. We introduce the new variable r' which is to be constrained by (3.1.3) and let the real spot interest rate, still denoted by r, move within a prescribed distance of r '. That is

$$|r - r'| \leq \varepsilon,$$

which gives

$$\frac{\partial V}{\partial t} + c\left(\frac{\partial V}{\partial r}\right)\frac{\partial V}{\partial r} - (r + e(V))V = 0 \qquad .$$

for

$$e(x) = \begin{cases} \epsilon & \text{if } x \geq 0 \\ -\epsilon & \text{if } x < 0 \end{cases} \qquad .$$

3.2 Static Hedging for interest rate products

We now demonstrate how to use the fixed income products available in the market as hedging instruments to reduce the exposure of our contract to changes in the interest rate. We will perform an optimisation on our hedged contract, using the same philosophy inspired by Avellaneda & Paras (1996) .

3.2.1 Formulation of optimisation

Similar to Section 2.3, the main formulation is still

$$(\text{Hedge Contract})_{\text{worst-case}} = (\text{Contract} + \text{Hedging Instrument})_{\text{worst-case}} - \text{Cost of Hedge}.$$

We consider a fixed income products as a series of coupons, C_i, being hedged by m zero coupon bonds, D_j, publicly traded in the market, and each has a known market price P_j, for $1 \leq j \leq m$. The cost of setting up this static hedge is equal to the current market value of the hedging instruments,

$$\sum_{j=1}^{m} \lambda_j P_j$$

The marginal value of our hedged contract, V, is therefore the value of the overall portfolio minus the cost of the static hedge,

$$V = \text{ value (original contract } + \text{ hedging instruments)} - \sum_{j=1}^{m} \lambda_j P_j$$

There will be an optimal static hedge, for which we obtain the maximum possible worst-case scenario value for V. To find this, we evaluate the hedged portfolio value as a function dependent on $\lambda_1, \ldots, \lambda_m$

$$\text{Hedged Portfolio Value} = \inf_{P \in \mathcal{P}} \mathbb{E}_t^P \left[\sum_{i=1}^{N} C_i \mathrm{e}^{-\int_t^{T_i} \bar{r}(\tau) d\tau} + \sum_{j=1}^{N} \lambda_j D_j \mathrm{e}^{-\int_{t'_j}^{T_j} \bar{r}(\tau) d\tau} \right] \qquad (3.2.1)$$

We then maximise the value of the portfolio with respect to the hedge quantities, λ_j :

$$V^{-}(t,\lambda_1,\ldots,\lambda_m) = \max_{\lambda_1,\ldots,\lambda_m}\left(\inf_{P\in\mathcal{P}}\mathbb{E}_t^P\left[\sum_{i=1}^{N} C_i \mathrm{e}^{-\int_t^{T_i}\bar{r}(\tau)d\tau} + \sum_{j=1}^{N}\lambda_j D_j \mathrm{e}^{-\int_{t'_j}^{T_j}\bar{r}(\tau)d\tau}\right]\right) - \max_{\lambda_1,\ldots,\lambda_m}\left(\sum_{j=1}^{m}\lambda_j P_j\right) \quad (3.2.2)$$

Similarly, there will be an optimal static hedge, for which we obtain the minimum possible best-case scenario value for V. To find this, we minimise the value of the portfolio with respect to the hedge quantities, λ_j :

$$V^{+}(t,S_t,\lambda_1,\ldots,\lambda_m) = \min_{\lambda_1,\ldots,\lambda_m}\left(\sup_{P\in\mathcal{P}}\mathbb{E}_t^P\left[\sum_{i=1}^{N} C_i \mathrm{e}^{-\int_t^{T_i}\bar{r}(\tau)d\tau} + \sum_{j=1}^{N}\lambda_j D_j \mathrm{e}^{-\int_{t'_j}^{T_j}\bar{r}(\tau)d\tau}\right]\right) - \min_{\lambda_1,\ldots,\lambda_m}\left(\sum_{j=1}^{m}\lambda_j P_j\right) \quad (3.2.3)$$

Note that $\bar{r}(t)$ will depend implicitly on the value of λ. The particular evolution of the interest rate in a worst case scenario for the portfolio will depend on the exact makeup of the portfolio, and consequently on the amount, λ, of the hedging bond contained in the portfolio.

To include a bid-offer spread in the market price for a hedging instrument, we simply make P_j dependent on the sign of λ_j. If $\lambda_j > 0$ then we are long the bond and the market price is the offer price. If $\lambda_j < 0$ then we are short the bond and the market price is the bid price, i.e.

$$P_j(\lambda_j) = \begin{cases} P_j^{+} & \text{if } \lambda_j > 0 \\ P_j^{-} & \text{if } \lambda_j < 0 \end{cases} \quad (3.2.4)$$

where $P_j^{+}$is the offer price and $P_j^{-}$is the bid price.

3.3 Applications of the model

There are a number of ways one can use the theory and techniques explained so far assuming that the movements of the interest rate conform to our constraints

1. **Identifying arbitrage opportunities :** We have the spread for the possible price of a contract, bounded by the Best- and worst-scenario prices. If we find a contract whose value lies outside of these bounds, then we have identified an arbitrage opportunity.

2. **Setting bid/ask prices as a market maker:** We cannot lose money on any deal by setting our bid price at the low end and our offer price at the high end of the spread range. This technique is particularly appropriate in the OTC contracts business where spreads are usually much higher because of the often exotic nature and illiquidity of the product.

Chapter 4

Numerical implementation using finite difference method

4.1 Finite Difference Method

Following the practice done in Epstein (1999), we discretise the solution space

$$0 \le t \le T, r_{min} \le r \le r_{max}$$

using a grid of m space steps, Δr apart, and n time steps, Δt apart, where

$$\Delta r = \frac{(r_{max} - r_{min})}{m} \text{ and } \Delta t = T/n.$$

A general point on the grid has position

$$(r, t) = (r_{min} + i\Delta r, j\Delta t),$$

where

$$0 \le i \le m \text{ and } 0 \le j \le n.$$

This grid is shown in Figure 4.1. We approximate the solution V at a gridpoint U, where,

$$V(r_{min} + i\Delta r, j\Delta t) \approx U_i^j.$$

We then solve backwards in time from expiry, using our chosen numerical scheme. We place final data at expiry,

$$U_i^n = C_N(r_{min} + i\Delta r),$$

where $C_1, \ldots, C_N$ are the payoffs at $T_1, \ldots, T_N$.
At a cashflow date,

$$V\left(r, T_{j^-}\right) = V\left(r, T_{j^+}\right) + C_j(r),$$

which we discretise by including

$$U_i^{j_c} = U_i^{j_c} + C_j(r_{min} + i\Delta r)$$

in our scheme at the cashflow timestep, j_c, where $j_c = T_j/\Delta t$.

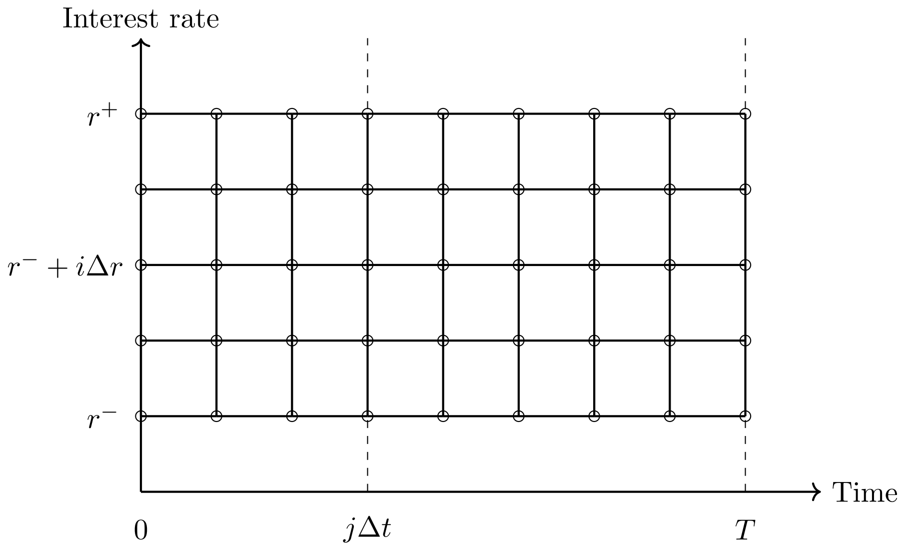


Figure 4.1: The discretised solution space.

Explicit finite difference scheme

In the explicit finite-difference scheme, we approximate the partial derivatives in Equation (3.1.7), using Taylor series expansions near the points of interest (Morton & Mayers 2005).

We approximate V_t using a backwards difference,

$$V_t(r,t) \approx \frac{U_i^j - U_i^{j-1}}{\Delta t}.$$

, and V_r using an upwind scheme,

$$V_r(r,t) \approx \begin{cases} \frac{U_{i+1}^j - U_i^j}{\Delta r} & \text{if } c > 0 \\ \frac{U_i^j - U_{i-1}^j}{\Delta r} & \text{if } c < 0 \end{cases} \tag{4.1.1}$$

Upwind differencing is particularly useful for first-order hyperbolic partial differential equations (PDEs), where upwind schemes are often preferred over the central difference schemes because we can ensure stability even though it may offer higher discretization error (Morton & Mayers 2005).

This gives us the numerical scheme:

$$\frac{U_i^j - U_i^{j-1}}{\Delta t} + c\left(r, V_r\right) \frac{1}{\Delta r} \left\{ \begin{array}{ll} U_{i+1}^j - U_i^j & \text{if } c > 0 \\ U_i^j - U_{i-1}^j & \text{if } c < 0 \end{array} \right\} - \left(r_{min} + i\Delta r\right) U_i^j = 0. \tag{4.1.2}$$

Rearranging this equation, we have

$$U_i^{j-1} = \left(1 - \left(r_{min} + i\Delta r\right)\Delta t\right) U_i^j + c\left(r, V_r\right) \frac{\Delta t}{\Delta r} \begin{cases} U_{i+1}^j - U_i^j & \text{if } c > 0 \\ U_i^j - U_{i-1}^j & \text{if } c \leq 0. \end{cases} \tag{4.1.3}$$

The scheme is shown in Figure 4.2.

The upwind scheme is stable if the Courant–Friedrichs–Lewy condition (CFL) is satisfied. This is a necessary condition for the convergence of any finite difference scheme for a first-order hyperbolic PDE (Smith 1985). Our scheme satisfies this condition as long as

$$\left|c\left(r, V_r\right)\right| \Delta t \leq \Delta r. \tag{4.1.4}$$

Next, we need to describe the discretisation of $c(r, V_r)$ in our equation, which turns out to be more involved than it may seem due to the reasons to be discussed in the following section.

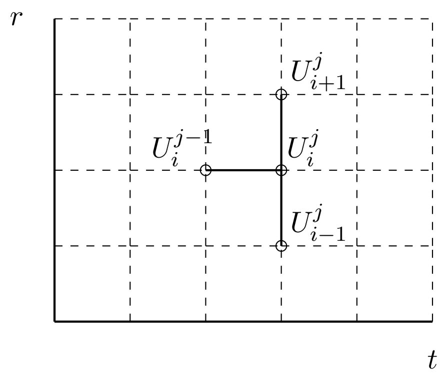


Figure 4.2: The explicit finite difference scheme

4.2 Discretisation of $c(r, V_r)$

From the description of $c(r, V_r)$ in Equation 3.1.5 for the worst-case pricing, the central difference is an attractive choice to approximate V_r in $c(r, V_r)$ due to its relatively small discretisation error compared to the explicit/implicit scheme counterparts (see Appendix A.1).

The discretised version is :

$$c\left(r, V_r\right) = \begin{cases} c_{max} & \text{if } U_{i+1}^{j} - U_{i-1}^{j} < 0 \\ c_{min} & \text{if } U_{i+1}^{j} - U_{i-1}^{j} > 0 \end{cases}. \tag{4.2.1}$$

However, the function (4.2.1) alone cannot capture the qualitative behaviour of our model near a maximum or minimum region ($V_r = 0$)(Epstein 1999).
Instead we must address this issue depending on the U_{i+1}^{j}, U_i^j and U_{i-1}^{j} values, so that we minimise $c\left(r, V_r\right) V_r$ at each point (worst-case assumption), given that we will use the upwind scheme for the latter V_r term.

If we are at a minimum, i.e.,

$$U_i^j \leq U_{i-1}^{j} \text{ and } U_i^j \leq U_{i+1}^{j},$$

then to minimise $c\left(r, V_r\right) V_r$ under the upwind scheme, we choose

$$c\left(r, V_r\right) = 0. \tag{4.2.2}$$

Note that this has to be the case because of the (4.1.1) constraint.

- $V_r(r, t)$ takes the forward difference value (a positive quantity) if $c > 0$. Hence, $c\left(r, V_r\right) V_r$ can be minimised if $c = 0^+$ since ultimately $c \in [c_{min}, c_{max}]$.
- Conversely, $V_r(r, t)$ takes the backward difference value (a negative quantity) if $c < 0$. Hence, $c\left(r, V_r\right) V_r$ can be minimised if $c = 0^-$.
- Thus, $c\left(r, V_r\right) = 0$.

If we are at a maximum, i.e.,

$$U_i^j \geq U_{i-1}^{j} \text{ and } U_i^j \geq U_{i+1}^{j},$$

then to minimise cV_r, we choose

$$c\left(r, V_r\right) = \begin{cases} c_{max} & \text{if } c_{max}\left(U_{i+1}^{j} - U_i^j\right) \leq c_{min}\left(U_i^j - U_{i-1}^{j}\right) \\ c_{min} & \text{if } c_{max}\left(U_{i+1}^{j} - U_i^j\right) > c_{min}\left(U_i^j - U_{i-1}^{j}\right) \end{cases} \tag{4.2.3}$$

so as to minimise $c\left(r, V_r\right) V_r$.

Modification at the boundaries

As commonly practiced among finite difference differential solvers, we must manually overwrite at the boundaries, in this case the first and last entry of U_0^{j-1} and U_m^{j-1} can only have a one-sided scheme for the space derivative. At the lower boundary,

$$V_r\left(r_{min},t\right)\approx\frac{U_1^j-U_0^j}{\Delta r}$$

and choose

$$c\left(r_{min},V_r\right)=\begin{cases}c_{max} & \text{if } V_r\left(r_{min},t\right)<0\\ 0 & \text{if } V_r\left(r_{min},t\right)\geq 0\end{cases}\tag{4.2.4}$$

because $c\left(r_{min},V_r\right)$ cannot go below 0 since the backward difference does not exist at r_{min}. At the upper boundary,

$$V_r\left(r_{max},t\right)\approx\frac{U_m^j-U_{m-1}^j}{\Delta r}$$

and choose

$$c\left(r_{max},V_r\right)=\begin{cases}0 & \text{if } V_r\left(r_{max},t\right)\leq 0\\ c_{min} & \text{if } V_r\left(r_{max},t\right)>0\end{cases}\tag{4.2.5}$$

because, again, $c\left(r_{min},V_r\right)$ cannot go beyond 0 since the forward difference does not exist at r_{min}.

4.3 The Zero Coupon Bond Pricing

In this chapter, we will use the zero-coupon bond (ZCB) as a case example to demonstrate the pricing and hedging of a contract and compare the results to other known approaches. To price a zero-coupon bond under our model, we solve our partial differential equation (3.1.7) with a final condition

$$V(r,T)=P,$$

where T is the maturity and P the principal of the bond.
To price a coupon bond, we would just add each coupon in as a jump condition. For example, to include a coupon of size $c\%$ at time T_j, we would add the condition

$$V\left(r,T_{j^-}\right)=cP+V\left(r,T_{j^+}\right).$$

4.3.1 Result presentation of the worst/best-case pricing

The zero-coupon bond with 10 year maturity was priced under the following conditions:

⇒The interest rate bounds for these figures are 3% and 20% and the growth rate is bounded by -4% pa and 4% pa. The bond specifications were chosen to be the same as in (Epstein 1999, page 76) so that the results can be easily compared.

By solving the (4.1.3), using the explicit upwind scheme as outlined in 4.1, we were able to obtain the envelope of worst/best-case prices of the 10 year zero-coupon bond as shown in Figures 4.3 and 4.4 (The best-case price is the negative of worst-case price of the negative ZCB as explained in Equation (3.1.8)).

In addition to the aforementioned constraints of the interest rate path, the time steps, n, and rate grid step, m, were chosen to be 1000 and 100 respectively, such that the CFL condition (4.1.4) is satisfied to ensure stability of the numerical scheme.

Figure 4.3: Worst/best prices of the unhedged zero coupon bond at different times before maturity ($T = 10$), where the current risk-free rate is 0.05.

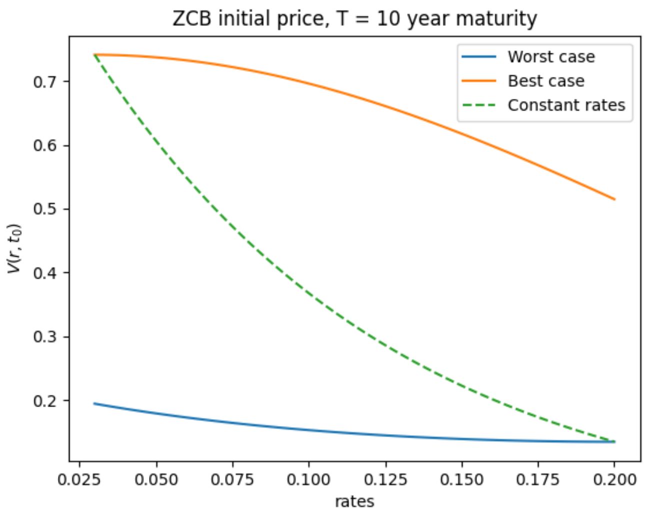


Figure 4.4: Worst/best prices of the unhedged zero coupon bond against different current risk-free rates before maturity ($T = 10$).

Figure 4.3 shows behaviours that we would expect of ZCB price, where the price goes down the more it is away from the maturity date because the bond is more heavily discounted. The worst- and best- case envelope contains the price from constant rate path price as it should encompass every possible interest rate path given that $r \in [r_{min}, r_{max}]$ and $c(r, V_r) \in [c_{min}, c_{max}]$. Nonetheless, the result definitely needs improvement for it to become of any use as the spread of 0.56 at $t = 0$ in Figure 4.3 for a zero-coupon bond price would not occur in practice. The results can be massively improved after implementing static hedging, which will be shown in the next section.

Figure 4.4 shows the current price of a 10 year-ZCB with different initial risk-free rates. The dashed line shows the price at $t = 0$ for ZCB with different constant risk-free rates, which again stays within the worst/best case scenarios. The intersects at r_{min} and r_{max} make intuitive sense as the constant $r_{min/max}$ is indeed the best and worst case prices.

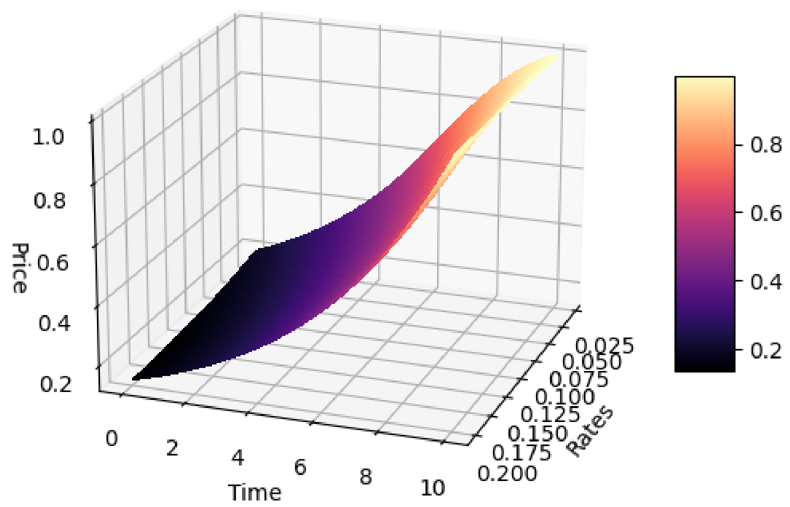


Figure 4.5: Zero-coupon bond value in a worst-case scenario.

Figure 4.5 shows the ZCB worst-case price at different times and rates within the specified range $\in (0.03, 0.2)$ before maturity date at $T = 10$. The surface has the same shape as shown in (Epstein 1999, page 75).
Lastly, we investigate the stability condition by varying $n = 100$, such that the condition (4.1.4) is not satisfied. We observe unstable values as shown in Figure 4.6.

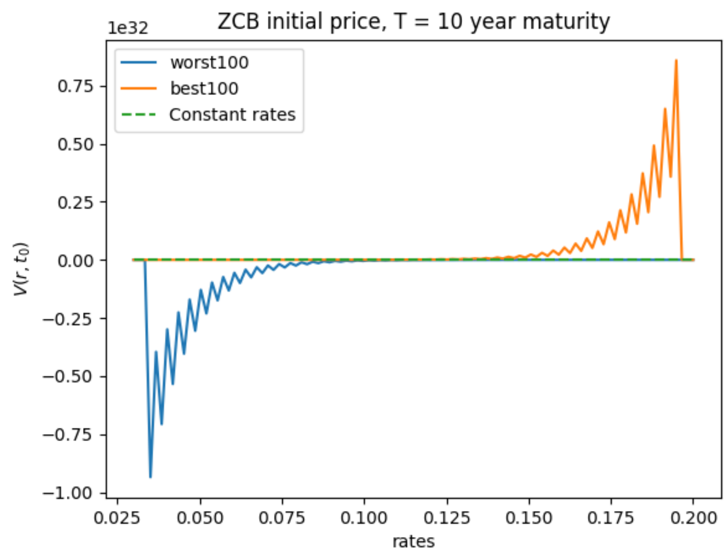


Figure 4.6: 10-year Zero-coupon bond value in a worst/best-case scenario with $n = 100$, resulting in unstable finite differenc scheme.

All in all, we conclude that the Python code (attached in Appendix A.2) and our analysis works out as intended so far.

4.3.2 Hedging with different ZCB bonds

The relevant mathematical equations have been explained in Section 3.2. The optimization, carrying out in Python can be summarised as follows:

1. Determine the expected worst- and best-case prices of the hedged portfolio using Equation 3.2.1 (by solving the classical PDE equations (3.1.7)).
2. Maximise/minimise equations (3.2.2) and (3.2.3) over the hedging quantities, $\lambda_1, \ldots, \lambda_m \in [\Lambda_{min}, \Lambda_{max}]$, under quasi-Newton routine (Avellaneda & Paras 1996, page 15) using `scipy.optimize`.
3. We omit the bid-offer spreads to simplify the arguments involved as the results from their inclusion would not affect the discussion.
4. We assume that the market is highly liquid, allowing hedging quantities to be bought in increments as precise as three decimal places.

Following the numerical values presented in (Epstein 1999, page 83-84), we will consider an example:

Pricing a 4 year zero-coupon bond, while hedging against other publicly tradable zero coupon bonds listed in Table 4.1. The spot short-term interest rate is 6%. The interest rate bounds are 3% and 20% and the growth rate is bounded by -4% pa and 4% pa.

Hedging bond	Maturity (yrs)	Market price
Z_1	0.5	0.970
Z_2	1	0.933
Z_3	2	0.868
Z_4	3	0.805
Z_5	5	0.687
Z_6	7	0.579

Table 4.1: The zero-coupon bonds with which we hedge.

The optimal static hedges for the worst- and best-case valuations are shown in Table 4.2. The results of the valuation for the 4 year bond, with and without hedging, are shown in Table 4.3.

We note that the hedge quantities are different (although similar) for worst and best case scenarios, and that not all hedging instruments are used in the optimal hedging strategy. The different hedge quantities reflect different ways of positioning oneself such that the 4-year zero coupon bond can be viewed as certain prices. For example, if a trader sees the 4-year bond being traded under the worst price, he/she can profit from arbitrage by longing the portfolio using the hedge quantities for the worst case (given the interest rates and the change of rates stay within the chosen ranges).

Local truncation error

The truncated error is essentially the difference between the PDE equation (3.1.7) and the discretised approximation (4.1.2).
From the Taylor's expansions given in AppendixA.1, one can deduce that the truncated error by the proposed upwind finite difference scheme is $O(\Delta t) + O(\Delta r)$ because the

scheme uses either forward or backward schemes for $\frac{\partial U}{\partial t}$ and $\frac{\partial U}{\partial r}$ approximations in Equation (4.1.2). As a result, the obtained hedging quantities , λs, are slightly different from the results shown in the literature Epstein (1999). The reason is likely due to the different choices in m and n, which directly determine the grid sizes Δt and Δr; hence, leading to different truncation errors.

Hedging bond	Literature worst case λs	Literature best case λs	Obtained worst case λs	Obtained best case λs
Z_1	0.000	0.002	0.000	0.043
Z_2	-0.004	-0.004	-0.002	-0.040
Z_3	0.169	0.117	0.168	0.125
Z_4	-0.699	-0.653	-0.709	-0.641
Z_5	-0.468	-0.481	-0.451	-0.489
Z_6	0.020	0.000	0.011	0.000

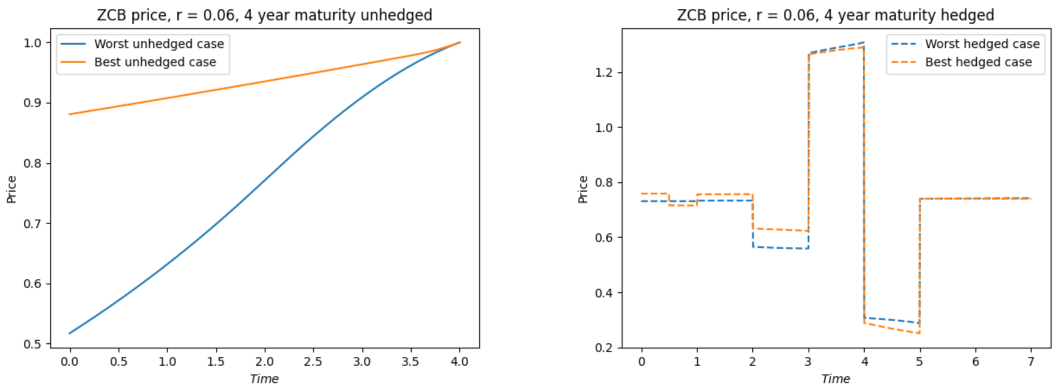
Table 4.2: The optimal static hedges for a 4 year zero-coupon bond. The obtained best/-worst cases hedge quantities, λ, are presented here in comparison with the results from the literature.

Now we will compare the result from pricing the 4-year zero coupon bond with and without hedging shown in Figure 4.7 and Table 4.3.

	Worst price	Best price
No hedge	0.575	0.877
Literature	0.730	0.758
Our result	0.731	0.758

Table 4.3: Values of a 4 year zero-coupon bond with and without hedging (current rate of 0.06).

Table 4.3 shows considerable improvement of the worst/best prices for our result after hedging with 6 other available bonds in the market, very small difference between the obtained and literature prices.



(a) Unhedged 4-year ZCB

(b) Hedged 4-year ZCB

Figure 4.7: Values of a 4 year zero-coupon bond over time with and without hedging (initial risk free rate of 0.06).

Figure 4.7 compares the nature of our worst/best-case pricing under the non-probabilistic approach. Figure 4.7(a) shows the same trend as presented before, where the discounted price steadily decreases the longer it is from maturity with the largest spread at the purchasing time (t_0). Figure 4.7(b) exhibits a smaller spread throughout the time frame, displaying the benefits from risk diversification. The prices do have alarmingly large jumps especially around the time of cash flows of the hedging component bonds. The drawdown after the maturity of the 4-year ZCB is due to the discounted cashflow of the bonds with later maturity being unable to catch up with the initial cost of static hedging, which can be dealt with by closing your positions of the 5 & 7-year ZCB after the maturity of 4-year bond or wait for all the hedging bonds to payoff.

Chapter 5

More sophisticated products

In this Chapter, we will look into how the UMV model under a non-probabilistic approach can be used to price bonds with early exercise option, specifically "convertible bonds".

5.1 Convertible bond

A convertible bond gives the holder the option to convert or exchange it for a predetermined number of shares in the issuing company. When issued, they act just like regular corporate coupon bearing bonds. The conversion ratio, q, determines how many shares can be converted from each bond.

The share price is modelled using a lognormal random walk,

$$dS = \mu S dt + \sigma S dX,$$

and forms a Black-Scholes hedged portfolio, $\Pi = V - \Delta S$
because the convertible bond, $V = V(r, t, S)$, has the share price as the underlying asset, Delta hedging is assumed. The increment $d\Pi$ has the form

$$d\Pi = dV(r, t, S) - \Delta dS - D\Delta S dt.$$

Note that $-D\Delta S dt$ comes from the continuously paid dividend yield Ddt per price $\times$ share, and the portfolio Π consists of $-\Delta$ shares. Itô's lemma gives

$$d\Pi = V_t dt + V_S dS + V_r dr + \frac{1}{2}\sigma^2 S^2 V_{SS} dt - \Delta dS - D\Delta S dt,$$

where V_x is denoted as $\frac{\partial V}{\partial x}$. We choose $\Delta = V_S$ to eliminate the leading order randomness from the share price movements (dS term) and value the portfolio in a worst case scenario. Under this worst case assumption, the change in the value of our portfolio is

$$\min_{dr} d\Pi = \min_{dr} \left(V_t dt + V_r dr + \frac{1}{2}\sigma^2 S^2 V_{SS} dt - DSV_S dt \right).$$

We require that, in the worst case, our portfolio always earns the risk-free rate. This gives us that

$$\min_{dr} d\Pi = r\Pi dt.$$

Hence, the pricing equation for the bond is

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + (r - D)SV_S + c(r, V_r) V_r - rV = 0, \tag{5.1.1}$$

where $V(r,t,S)$ is the bond price, and

$$c(r,X) = \begin{cases} c_{max} & \text{if } X < 0 \\ c_{min} & \text{if } X > 0 \end{cases}. \tag{5.1.2}$$

This is the worst-case scenario value for the contract. The best value would be given by the solution of Equation (5.1.1) with reversal of the inequalities in Equation (5.1.2).

Optimal conversion into q of the stocks is assured by insisting that

$$V(r,t,S) \geq qS \tag{5.1.3}$$

at all times that conversion is permitted, where q is the conversion ratio. The final condition at maturity of the bond, T, is that the bond value is equal to the principal, assumed to be 1 , plus the last coupon (C_T)

$$V(r,T,S) = 1 + C_T.$$

Across each coupon date, t_i, the bond falls by the amount of the coupon (C_j), and,

$$V\left(r, t_{j^-}, S\right) = V\left(r, t_{j^+}, S\right) + C_j.$$

This completes the specification of the convertible bond model under the risk-neutral measure for the asset and the worst-case scenario for the interest rate.

5.2 Finite Difference Scheme

Similar to Section 4.1 in Chapter 4, the non-linear PDE (5.1.1) can be discretised as follows:

First, we discretise the solution space

$$r_{min} \leq r \leq r_{max}, 0 \leq t \leq T \text{ and } s_{min} \leq s \leq s_{max},$$

where

$$\Delta r = \frac{(r_{max} - r_{min})}{m}, \Delta t = T/n \text{ and } \Delta s = \frac{(s_{max} - s_{min})}{m}.$$

A general point on the grid has position

$$(r_i, t_j, s_k) = (r_{min} + i\Delta r, j\Delta t, s_{min} + i\Delta s)$$

where

$$0 \leq i \leq m, 0 \leq j \leq n \text{ and } 0 \leq k \leq m.$$

$$V\left(r_i, t_j, s_k\right) \approx U_{ik}^j.$$

The m and n was chosen to be 100 and 1000 respectively. For interest rate discretisation, we can rationalise by comparing with the CFL condition (4.1.4), which will ensure the stability of the scheme as far as r_k is concerned. Following the Black-Scholes PDE stability analysis done in Jeong et al. (2018), the Peclet stability condition was used,

$$\Delta s < \frac{\sigma^2 s_k}{r_i}. \tag{5.2.1}$$

The number of grids for the price, s_k, was chosen to be 100 as it passes the stability condition (5.2.1) for most $(r,t,s) \in [r_{min}, r_{max}] \times [0,T] \times [s_{min}, s_{max}]$ except where $\lim_{s_k \to 0} s_k$.

Next, we will discretise the Equation (5.1.1). All derivatives with respect to S are approximated by central differences because they are more accurate than one-sided (upwind or downwind) schemes for approximating derivatives (refer to AppendixA.1). The partial differential equation exhibits diffusion in the S direction, so there will be no instability due to these discretisations given that Equation (5.2.1) is satisfied. Meanwhile, the U_r is still approximated by the upwind scheme like previously. We obtain

$$\begin{aligned}\frac{U_{ik}^{j-1}-U_{ik}^{j}}{\Delta t} = \frac{\sigma^2 s_k^2}{2}\frac{U_{i(k+1)}^{j}-2U_{ik}^{j}+U_{i(k-1)}^{j}}{\Delta s^2} + (r_i - D)s_k\frac{U_{i(k+1)}^{j}-U_{i(k-1)}^{j}}{2\Delta s} \\ + c\left(U_r\right)\frac{1}{\Delta r}\left\{\begin{array}{ll} U_{(i+1)k}^{j}-U_{ik}^{j} & \text{if } c>0 \\ U_{ik}^{j}-U_{(i-1)k}^{j} & \text{if } c\leq 0\end{array}\right\} - r_i U_{ik}^{j}, \quad (5.2.2)\end{aligned}$$

which can be rearranged to

$$\begin{aligned}U_{ik}^{j-1} = \left[1 - r_i\Delta t - \frac{\sigma^2\Delta t s_k^2}{\Delta s^2}\right]U_{ik}^{j} + \left[\frac{\sigma^2\Delta t s_k^2}{2\Delta s^2} + \frac{(r_i - D)s_k\Delta t}{2\Delta s}\right]U_{i(k+1)}^{j} \\ + \left[\frac{\sigma^2\Delta t s_k^2}{2\Delta s^2} - \frac{(r_i - D)S_k\Delta t}{2\Delta s}\right]U_{i(k-1)}^{j} + c\left(U_r\right)\frac{\Delta t}{\Delta r}\left\{\begin{array}{ll} U_{(i+1)k}^{j}-U_{ik}^{j} & \text{if } c>0 \\ U_{ik}^{j}-U_{(i-1)k}^{j} & \text{if } c\leq 0\end{array}\right\}. \quad (5.2.3)\end{aligned}$$

Then, like most finite difference PDE solver, boundaries conditions need to be separately addressed. The intuition is: When the asset value is low, s_{min}, there is little reason to convert the bond and so it behaves like a simple, non-convertible, coupon-bearing bond. When the stock price is high, s_{max}, the option to convert gives the bond a value closer to the value of the relevant quantity of the underlying asset (Epstein et al. 2000).

The specification of $c(r_i, V_{r_i})$ will be treated in the same fashion as in section 4.2.
To summarise,

1. Since $c(r, V_r)$ depends on s_k implicitly, we have to create a 2D array, representing the $c^j[i, k]$ values at each grid point U_{ik}^j.
2. Calculate the values at the boundaries c^j[0,:] and c^j[-1,:] first as they can only be one way of the finite difference (forward and backward difference respectively).
3. Calculate c^j[1:-1, :] by checking whether U_{ik}^j is a minimum, maximum or neither, and assign values accordingly.

Mathematically, the conversion problem is similar to the early exercise problem of an American option. The implementation of the constraint (5.1.3) is

$$V(r_i, t_j, S_k) = \max(U_{ik}^j, qS).$$

5.3 Result presentation

We consider the pricing of a convertible bond under the UVM non-probabilistic approach compared with the case where interest rate is constant as a benchmark.

Example: The underlying asset has current value 100, the volatility is 15% and the dividend yield is 4%. Note that we are not questioning the accuracy of these asset price parameters.

We value a convertible bond with a maturity of 25th November 2001 (where today is 14th May 1998). The bond has principal 1, can be converted into 0.01 of the asset and pays a coupon of 3% every six months until expiry. The spot short-term interest rate is 7%. The bond specification was chosen the be the same as in Epstein (1999) for ease of comparison.

For a constant interest rate of 7%, the convertible values were calculated from Equation (5.1.1) except $dr = 0$. Therefore, the discretised PDE becomes the same as Equation (5.2.3) without the last term which is Δr dependent.

$$U_{ik}^{j-1} = \left[1 - \frac{\sigma^2 \Delta t s_k^2}{\Delta s^2} - \Delta t r_i\right] U_{ik}^{j} + \left[\frac{\sigma^2 \Delta t s_k^2}{2\Delta s^2} + \frac{(r_i - D) S_k \Delta t}{2\Delta s}\right] U_{i(k+1)}^{j} + \left[\frac{\sigma^2 \Delta t s_k^2}{2\Delta s^2} - \frac{(r_i - D) S_k \Delta t}{2\Delta s}\right] U_{i(k-1)}^{j}.$$

It turns out the value at t_0 for such convertible bond is 1.120, and the value stays within the worst/best-case price envelope.

	Obtained CB Value	Literature Value
Worst-case	1.069	1.072
Best-case	1.178	1.191
Constant rate	1.120	1.131

Table 5.1: Values of a 3.57 year zero-coupon bond with and without hedging (current rate of 0.07 and $s_0 = 100$).

The obtained values are consistently lower than the values presented in the literature. The reasons are likely due to the differences in rounding conventions when dealing with decimals of the time indices from the coupon dates and more importantly different grid sizes, resulting in different truncation errors.

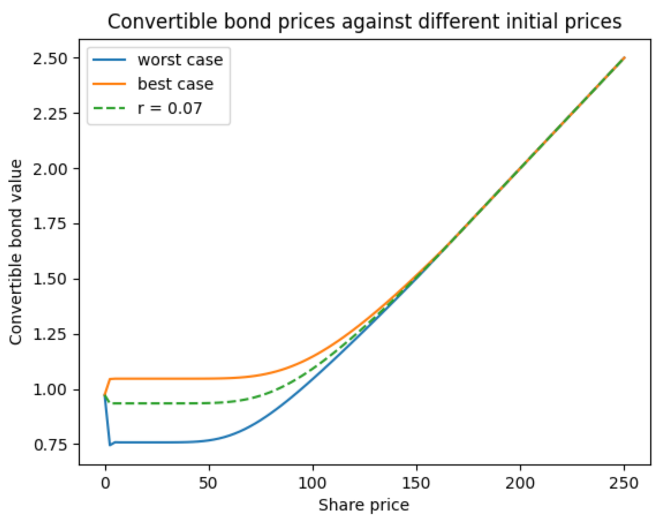


Figure 5.1: Convertible bond values with varying initial share prices, s_0, and a constant interest rate, 0.07, at t_0 compared with the worst/best prices.

Figure 5.1 shows the worst/best-case prices under the specification. The shape of the curves is similar to that of an American call option as expected. At large S_0, conversion will be exercised. At smaller S_0, the discounted price shape of convertible bonds can be

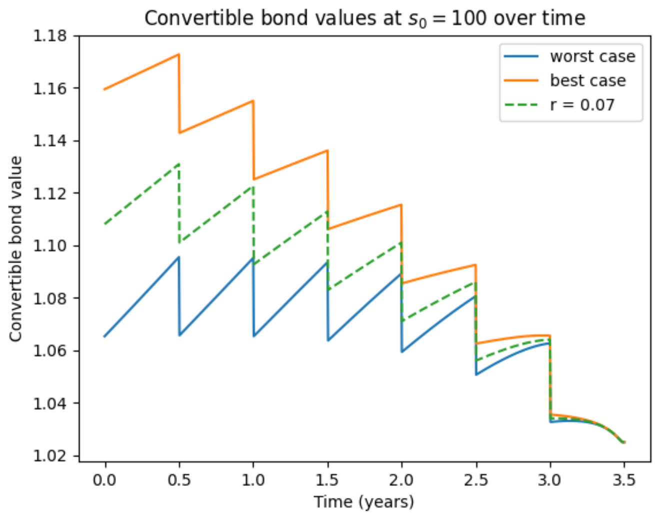


Figure 5.2: Convertible bond values with varying initial share prices,s_0, and constant interest rate , 0.07, at t_0 compared with the worst/best prices

observed while there were noticeable kinks around s_{min} as expected from the stability condition (5.2.1). In reality, this is not a major problem as we generally do not concern ourselves with $S_0 = s_{min}$ or any regions very far away from the strike price since contracts with such S_0 rarely exist.

Figure 5.2 shows the convertible bond value over time. The sudden drops in values of the bonds represents when it passes the coupon dates. The constant rate path stays within the worst/best-case envelope throughout the period with largest spread at initial time, t_0. All in all, we conclude that the Python code for this part has been correct so far as the results make intuitive sense and agree with literature results up to 2 decimal places.

5.4 Optimal Static hedging

To price the hedged convertible bond, we must keep track of two separate quantities, which are the value of the hedging instruments and the overall value of the portfolio (Epstein 1999, page 128). We consider two functions, Π_0 and Π_1:

$$\Pi_0(S, r, t) = \text{the hedging instruments,}$$
$$\Pi_1(S, r, t) = \text{the hedged convertible bond before the conversion.}$$

To price Π_0, we solve Equation (5.1.1) with the appropriate final and jump conditions to represent all of the cashflows for the hedging instruments. Since there is no S dependence in these hedging cashflows, the discretised equation is reduced to that of ZCBs like Equation (4.1.3).

To price Π_1, we then solve Equation (5.1.1) with the appropriate final and jump conditions to represent all of the cashflows for the hedging instruments and the convertible bond (before conversion). Since we will only convert if it is optimal to do so, we have the additional constraint that

$$\Pi_1(S, r, t) \geq nS + \Pi_0(S, r, t),$$

i.e., we convert when the value of the portfolio including the convertible bond is less than the value of the portfolio of hedging instruments plus the assets that we would receive if we were to convert. The constraint for conversion option is explained schematically in Figure 5.3.

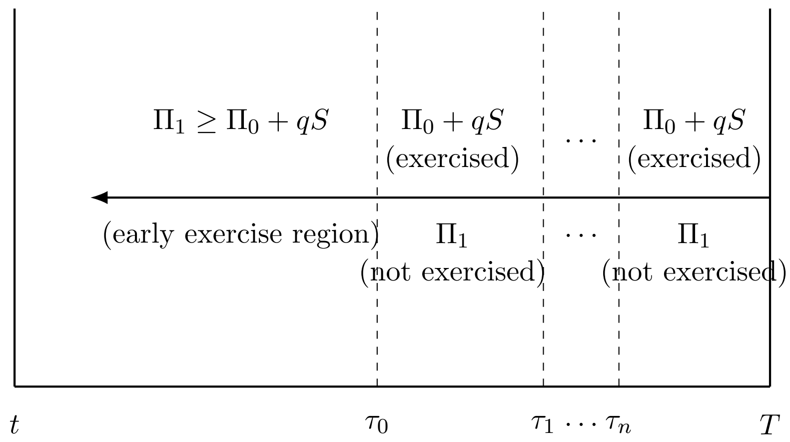


Figure 5.3: Schematic diagram for the valuation of interest rate products with early exercise option, in this case convertible bond. The τ_i, where $i \in [0, n]$ represents n early stopping times. The valuation was calculated based on the assumption that the conversion can happen at any point on $t \in [0, T]$.

We hedge the convertible with the 6 month and the 1, 2 and 5 year zero-coupon bonds as shown on Table 5.2. The optimal worst-case hedge for these bonds is shown in Table 5.2.

Bond Maturity	Bond Yield	Bond Price	Worst-case hedge
6 m	0.07447	0.963	0.283
1 yr	0.07016	0.932	0.053
2 yr	0.06631	0.876	-0.421
5 yr	0.06224	0.733	-0.290

Table 5.2: Optimal static hedge for convertible bond.

The ZCB prices in Table 5.2 were calculated from the bond yields using,

$$Y = -\frac{\log Z}{T - t}.$$

It turns out that the worst/best-case prices are both smaller and bigger than the values obtained from unhedged convertible values. That is, it seems that the $\lambda-$hedging using the ZCBs presented in Table 5.2 was not able to tighten the spread in this case.

Chapter 6

Extension to the model

In this chapter we consider potential improvements on the UMV for interest rate products. A few suggestions have been made in (Wilmott 1998, Chapter 40), such as crash modelling, modelling interest rate trend from economic cycle and liquidity problems, although only analysis on the effect that liquidity has on the prices will be presented in this thesis.

6.1 Liquidity

The effect of liquidity on our model will be investigated by using the fact that the bigger the ZCB spread, the lower liquidity the market has.

Following the idea suggested in Epstein (1999), we will consider an example of pricing a 4-year ZCB bond hedging against other publicly tradable zero coupon bonds listed on Table 6.1. The spot short-term interest rate is 6%. The interest rate bounds are 3% and 20% and the growth rate is bounded by -4% pa and 4% pa just like in Section 4.3.2.

Hedging bond	Maturity (yrs)	Market price
Z_1	0.5	0.970
Z_2	1	0.933
Z_3	2	0.868
Z_4	3	0.805
Z_5	5	0.687
Z_6	7	0.579

Table 6.1: The zero-coupon bonds with which we hedge.

To try and gain some insight into the effects concerned, we will make only one particular instrument illiquid. We consider different liquidities for the 5 year bond, Z_5, increasing the bid-offer spread from 0.687 - 0.687 to 0.647 - 0.727 and then to 0.607 - 0.767. The bid-ask prices can be incorportaed into the optimization as outlined in Section 3.2.

We omit the bid-ask spread for all contracts except for Z_5 that has a varying bid-ask spread. This is to simplify the analysis and the result already serves well enough as a proof of concept.

Bid-ask spread of Z_5	$0.687 - 0.687$	$0.667 - 0.707$	$0.607 - 0.767$
No hedge	0.575	0.575	0.575
Optimal hedge on worst-case	0.731	0.707	0.684

Table 6.2: Worst-case values of the 4 year-ZCB with illiquid hedge.

From Table 6.2, it is clear that as the 5 year bond becomes less liquid, the worst-case price decreases reflecting the inability to buy and sell near the spot prices. As a result of illiquid market, the hedger suffers from worse bid-ask prices.

Hedging bond	$0.687 - 0.687$ bid-ask	$0.667 - 0.707$ bid-ask	$0.607 - 0.767$ bid-ask
Z_1	0.000	0.001	0.001
Z_2	-0.002	-0.010	-0.010
Z_3	0.168	0.10	0.010
Z_4	-0.709	-0.600	-0.601
Z_5	-0.451	-0.401	-0.401
Z_6	0.002	0.001	0.000

Table 6.3: The illiquid optimal static hedges for the 4 year zero-coupon bond worst-case price.

As the instrument becomes more illiquid, we would expect the static hedge to adjust to include less of the illiquid instrument and larger quantities of more liquid instruments. We can observe this change in Table 6.3.

All these results make intuitive sense as having wider bid-ask spread while the quoted price, assumed to to be the average of the bid/ask prices, remain constant means that the hedgers have to buy(sell) the bonds with higher(lower) prices. Therefore, the worst-case scenario price should go down as observed.

6.2 Areas for further research

1. Parameter estimation for the uncertainty band width, ϵ mentioned in section 3.1.3
   Further work could be analysing the price bounds as a result of varying the bandwidth.

2. Crash modelling
   Since the interest rate can be significantly impacted by external economic and political events, it has sudden jumps and crashes in nature. To improve the interest rate path movements in our model, we may want to include the possibility of jumps and crashes

3. Credit Valuation Adjustment(CVA)
   Credit risk is a financial valuation method that takes into account the credit risk associated with a financial instrument. Under the non-probabilistic approach, where the rate path is not governed by a known probability distribution, the framework for calculating CVA will have to be revised.

Conclusion

In this study, we have introduced the Uncertain Volatility Model and its application under the non-probabilistic approach in pricing fixed-income products. By analysing the worst/best-cases for the non-linear differential equations of the product value, we were able to obtain the price envelope which encompasses all the arbitrage-free price for coupon and convertible bonds. Price spread reduction was achieved through static hedging using other available instruments in the market. Lastly, a few potential improvements of the approach were discussed.

Appendix A

Technical Proofs

A.1 Derivation of the truncation error for different numerical schemes

From Taylor's expansion,

$$f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)h^2}{2}$$

$$f(x_0 - h) = f(x_0) - f'(x_0)h + \frac{f''(x_0)h^2}{2}$$

Forward difference scheme:

$$\frac{f(x_0 + h) - f(x_0)}{h} - f'(x_0) = O(h)$$

Backward difference scheme:

$$\frac{f(x_0) - f(x_0 - h)}{h} - f'(x_0) = O(h)$$

Central difference scheme:

$$\frac{f(x_0 + h) - f(x_0 - h)}{2h} - f'(x_0) = O(h^2)$$

2^{nd} order finite difference scheme:

$$\frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2} - f''(x_0) = O(h^2)$$

A.2 Python Script Links

Links for Python scripts:

1. Code for Chapter 4 and 6: ZCB bond scripts is here.
2. Code for Chapter 5: Convertible bond scripts is here.

Bibliography

Avellaneda, M., Levy, A. & Parás, A. (1995), 'Pricing and hedging derivative securities in markets with uncertain volatilities', *Applied Mathematical Finance* **2**(2), 73–88.

Avellaneda, M. & Paras, A. (1996), 'Managing the volatility risk of portfolios of derivative securities: the lagrangian uncertain volatility model', *Applied Mathematical Finance* **3**(1), 21–52.

Black, F. & Scholes, M. (1973), 'The pricing of options and corporate liabilities', *Journal of Political Economy* **81**(3), 637–654.

Brailsford, T. J. & Faff, R. W. (1996), 'An evaluation of volatility forecasting techniques', *Journal of Banking & Finance* **20**(3), 419–438.

Brigo, D. (2020-21), 'Lecture slides in interest rate models with credit risk, collateral,funding liquidity risk and multiple curves (math97114)'.

Epstein, D. (1999), Uncertain interest rate modelling. PhD thesis.

Epstein, D., Haber, R. & Wilmott, P. (2000), 'Pricing and hedging convertible bonds under non-probabilistic interest rates', *Journal of Derivatives* **7**(4), 31–40.

Epstein, D. & Wilmott, P. (1999), *The yield envelope: price ranges for fixed income products*, Oxford Financial Research Centre.

Heath, D., Jarrow, R. & Morton, A. (1992), 'Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation', *Econometrica: Journal of the Econometric Society* pp. 77–105.

Ho, T. S. & Lee, S. B. (1990), 'Interest rate futures options and interest rate options', *Financial Review* **25**(3), 345–370.

Hull, J. & White, A. (1993), 'One-factor interest-rate models and the valuation of interest-rate derivative securities', *Journal of financial and quantitative analysis* **28**(2), 235–254.

Jeong, D., Yoo, M. & Kim, J. (2018), 'Finite difference method for the black–scholes equation without boundary conditions', *Computational Economics* **51**, 961–972.

Lewicki, P. & Avellaneda, M. (1996), 'Pricing interest rate contingent claims in markets with uncertain volatilities', *http://papers. ssrn. com/sol3/papers. cfm* .

Morton, K. W. & Mayers, D. F. (2005), *Numerical solution of partial differential equations: an introduction*, Cambridge university press.

Radhakrishnan, A. (1998), 'An empirical study of the convergence properties of the non-recombining hjm forward rate tree in pricing interest rate derivatives', *Available at SSRN 123170* .

Reider, R. (2009), 'Volatility forecasting i: Garch models', *New York* **19**.

Smith, G. D. (1985), *Numerical solution of partial differential equations: finite difference methods*, Oxford university press.

Wilmott, P. (1998), *Derivatives: the theory and practice of financial engineering*, John Wiley & Son Ltd.

Wilmott, P. (2010), *Frequently asked questions in quantitative finance*, John Wiley & Sons.