Spring 21 Math 7/8680: Project 2

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Project Problem

An experiment consists of performing independent $Bernoulli(p)$ trials. Suppose the trials yield Y successes, but the experimenter lost track of the number N of the trials performed. Assume that p has a $Beta(\alpha, \beta)$ prior distribution, $N|\lambda$ has a $Poisson(\lambda)$ prior distribution, and λ has a $Gamma(a, b)$ hyperprior distribution.

1. Write down fully conditional distributions for p, N, and λ and write an R code for implementing Gibbs sampling to compute the posterior distributions of these parameters.
2. With $Y = 15$, $\alpha = 0.2$, $\beta = 4$, $a = 0.5$, $b = 2$, estimate these parameters using squared error loss function.

Project Problem

3. Repeat the above estimation with loss function

$$\frac{1}{\theta}\left(\theta - d\right)^2$$

4. A second experiment consisting of independent Bernoulli trials was performed under different conditions. If p' is the probability of success under these new conditions and $Y' = 4$, but the experimenter again failed to record the number of trials, calculate the posterior probability that $p > \sqrt{p'}$.

Solution

1. Let $X_1, X_2, ..., X_N$ be i.i.d $Ber(p)$ trials.
   Then, number of success, $Y = \sum_{i=1}^{N} X_i \sim Bin(N, p)$
   Priors: $p \sim Beta(\alpha, \beta)$; $N|\lambda \sim Po(\lambda)$; $\lambda \sim Gamma(a, b)$.

   Conditional density of (p, N, λ) given Y is:

$$\tau(p, N, \lambda|Y) \propto f(Y|N, p)\tau(p)\tau(N|\lambda)\tau(\lambda)$$

$$\propto \binom{N}{Y} p^Y (1-p)^{N-Y} \frac{\lambda^N e^{-\lambda}}{N!} \lambda^{a-1} e^{-b\lambda} p^{\alpha-1} (1-p)^{\beta-1}$$

$$= \frac{N!}{(N-Y)!Y!} p^{Y+\alpha-1} (1-p)^{N-Y+\beta-1} \frac{\lambda^{N+a-1} e^{-\lambda(b+1)}}{N!}$$

Solution

$$\propto \frac{1}{(N-Y)!} p^{Y+\alpha-1}(1-p)^{N-Y+\beta-1}\lambda^{N+a-1}e^{-\lambda(b+1)}$$

Fully conditional posteriors of N, p, and λ:

$$\tau(N|p,\lambda,Y) \propto \frac{1}{(N-Y)!}(1-p)^N\lambda^N$$

$$\propto \frac{(\lambda(1-p))^{N-Y}}{(N-Y)!}e^{-\lambda(1-p)}$$

$$\sim Poisson(\lambda(1-p)) + Y$$

Solution

$$\tau(p|N,\lambda,Y) \propto p^{Y+\alpha-1}(1-p)^{N-Y+\beta-1}$$

$$\sim Beta(Y+\alpha, N-Y+\beta)$$

$$\tau(\lambda|p,N,Y) \propto \lambda^{N+a-1}e^{-\lambda(b+1)}$$

$$\sim Gamma(N+a, b+1)$$

Solution

Gibbs sampling process:

Initialize $N^{(0)}$

$$1^{st}\ iteration \begin{cases} p^{(1)}\ sampled\ from\ \sim Beta(\alpha + Y, N^{(0)} + \beta - Y) \\ \lambda^{(1)}\ sampled\ from\ \sim Gamma(N^{(0)} + a, b + 1) \\ N^{(1)}\ sampled\ from\ \sim Poisson(\lambda^{(1)}(1 - p^{(1)})) + Y \end{cases}$$

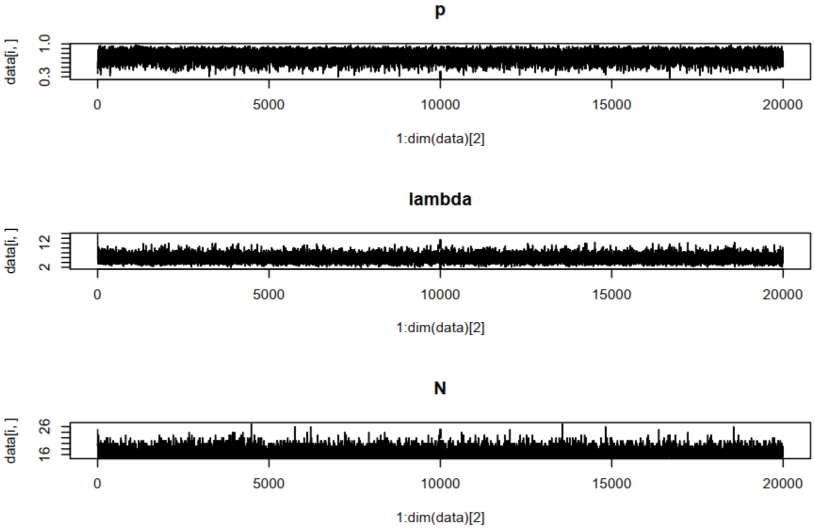
$$\vdots$$

$$K^{th}\ iteration \begin{cases} p^{(k)}\ sampled\ from\ \sim Beta(\alpha + Y, N^{(k-1)} + \beta - Y) \\ \lambda^{(k)}\ sampled\ from\ \sim Gamma(N^{(k-1)} + a, b + 1) \\ N^{(k)}\ sampled\ from\ \sim Poisson(\lambda^{(k)}(1 - p^{(k)})) + Y \end{cases}$$

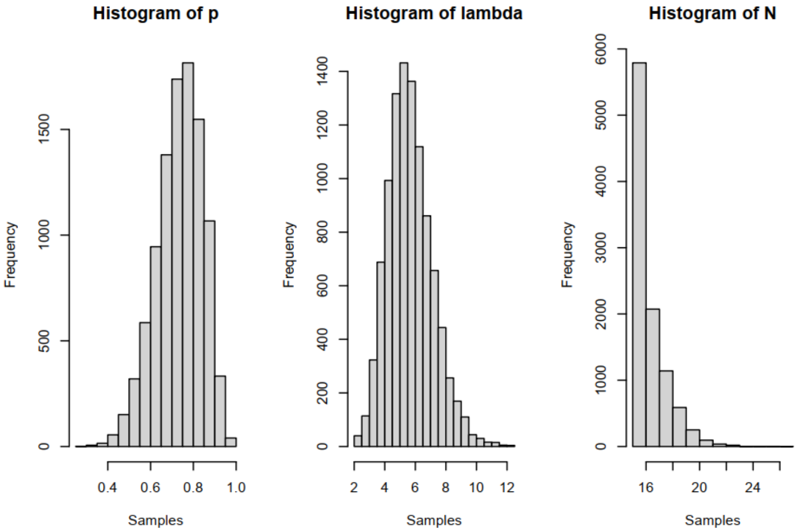
R Code for Gibbs sampling

```
# It is assumed that Y, alpha, beta, a, and b are known
# Letter 'l' refers to lambda

k <- 20000 # total iterations
T <- 10000 # burn in


N <- N # Initialize

my_mat <- matrix(data = (k*3)*NA, nrow = k, ncol = 3) # empty matrix
colnames(my_mat) <-  c("p", "lambda", "N") # parameters as column names

for (i in 1:k) {
  p <- rbeta(1, shape1 = Y + alpha, shape2 = N - Y + beta) # sample p
  l <- rgamma(1, shape = N + a, rate = b + 1) # sample lambda
  N <- rpois(1, l*(1-p)) + Y # sample N


  my_mat[i, ] <- c(p,l,N) # fill the matrix row by row

}
```

Trace Plots of p, λ and N



Histogram Plots of p, λ and N



Solution

2. Estimate of parameters, p, λ and N using squared error loss function.

   Bayes estimator of parameter θ, $d_{B_\theta}(y)$ for squared error loss function is given as:

   $$d_{B_\theta}(y) = E(\theta|Y = y)$$

   From Monte Carlo Method, given large number of samples $\theta^{(1)}, \theta^{(2)}, .., \theta^{(S)}$ the sample mean converges to the above expectation value.

   $$\frac{1}{S}\sum_{i=1}^{S}\theta^{(i)} \to E(\theta|Y = y)$$

   $$d_{B_\theta}(y) = E(\theta|Y = y) \approx \frac{1}{S}\sum_{i=1}^{S}\theta^{(i)}$$

Solution

S =10,000 number of samples are taken from 20,000 iterations followed by 10,000 burn-in periods.

$$d_{B_N}(y) \approx \frac{1}{S} \sum_{i=1}^{S} N^{(i)}$$

$$d_{B_\lambda}(y) \approx \frac{1}{S} \sum_{i=1}^{S} \lambda^{(i)}$$

$$d_{B_p}(y) \approx \frac{1}{S} \sum_{i=1}^{S} p^{(i)}$$

	p	lambda	N
Bayes Estimate	0.736	5.667	16

Solution

3. Estimate of parameters p, λ and N using loss function of:

$$\frac{1}{\theta}(\theta - d)^2$$

Bayes Estimator for weighted squared error loss function of parameter θ, $d_{B_\theta}(y)$, is given as:

$$d_{B_\theta}(y) = \frac{E(\theta w(\theta)|Y = y)}{E(w(\theta)|Y = y)},\ where\, w(\theta) = \frac{1}{\theta}$$

$$= \frac{E(\theta \frac{1}{\theta}|Y = y)}{E(\frac{1}{\theta}|Y = y)} = \frac{E(1|Y = y)}{E(\frac{1}{\theta}|Y = y)} = \frac{1}{E(\frac{1}{\theta}|Y = y)}$$

Solution

From Monte Carlo Approximation Method, given large number of samples $\theta^{(1)}, \theta^{(2)}, ..., \theta^{(S)}$ from posterior distribution $\tau(\theta|Y)$, we have:

$$\frac{1}{S}\sum_{i=1}^{S} g(\theta^{(i)}) \rightarrow E(g(\theta)|Y = y)$$

$$E(w(\theta)|Y = y) = \begin{cases} \int_{\Omega} w(\theta)\tau(\theta|Y)d\theta \;\; cont\ \theta \\ \sum_{\theta\in\Omega} w(\theta)\tau(\theta|Y) \;\; disc\ \theta \end{cases} \quad \approx \frac{1}{S}\sum_{i=1}^{S} w(\theta^{(i)})$$

Solution

S =10,000 number of samples are taken from 20,000 iterations followed by 10,000 burn-in periods.

$$d_{B_N}(y) \approx \frac{1}{\frac{1}{S}\sum_{i=1}^{S} w(N^{(i)})}$$

$$d_{B_\lambda}(y) \approx \frac{1}{\frac{1}{S}\sum_{i=1}^{S} w(\lambda^{(i)})}$$

$$d_{B_p}(y) \approx \frac{1}{\frac{1}{S}\sum_{i=1}^{S} w(p^{(i)})}$$

R Code for Bayes Estimator with weighted squared error loss

```r
w_bayes <- function(theta){ # function for wighted loss estimates
  1/(sum(1/theta)/length(theta))
}

w_p <- w_bayes(samp_p) # p
w_l <- w_bayes(samp_l) # lambda
w_N <- w_bayes(samp_N) # N

pander(matrix(data = c(round(w_p, 3), round(w_l, 3), as.integer(w_N)),
              nrow = 1, ncol = 3, dimnames = list("Weighted Bayes Estimate", c("p", "lambda", "N"))))
```

	p	lambda	N
Weighted Bayes Estimate	0.718	5.363	16

Solution

4. The Gibbs sampling process is performed again in a similar experiment as part (1) under different conditions, $Y' = 4$, number of successes and p', probability of success.
   We are interested in calculating posterior probability that $p > \sqrt{p^{'}}$
   This Gibbs Sampling process gives us new samples: $p'^{(1)}, p'^{(2)}, ..., p'^{(S)}$. Having $p^{(1)}, p^{(2)}, ..., p^{(S)}$ from part(1), we calculate:

$$P(p > \sqrt{p'} | Y = y, Y' = y')$$

Solution

From Monte Carlo Approximation, given samples of two estimates: ${p'}^{(1)}, {p'}^{(2)}, ..., {p'}^{(S)}$ and $p^{(1)}, p^{(2)}, ..., p^{(S)}$, define a new variable as follows:

$$I(p^{(i)} > \sqrt{p'^{(i)}}) = \begin{cases} 1 & p^{(i)} > \sqrt{p'^{(i)}} \\ 0 & Otherwise, \end{cases}$$

Then the posterior probability that $p > \sqrt{p'}$ can be approximated as:

$$P(p > \sqrt{p'} | Y = y, Y' = y') \approx \frac{1}{S} \sum_{i=1}^{S} I(p^{(i)} > \sqrt{p'^{(i)}})$$

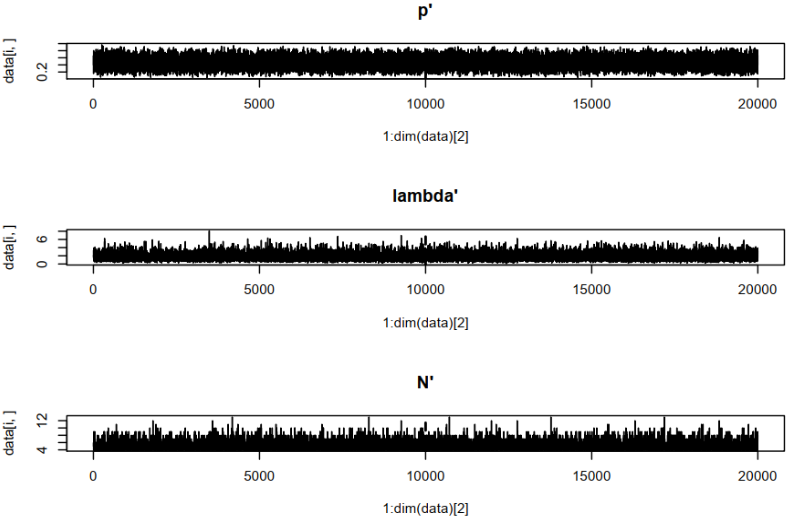
$P(p > \sqrt{p'} | Y = y, Y' = y')$ is computed to be 0.635

R Code for calculating posterior probability that $p > \sqrt{p'}$

```
# Part (d)
Y_prime <- 4

N <- 10


my_mat2 <- matrix(data = (k*3)*NA, nrow = k, ncol = 3)
colnames(my_mat2) <-  c("p", "lambda", "N")

for (i in 1:k) {
  p <- rbeta(1, shape1 = Y_prime + alpha, shape2 = N - Y_prime + beta)
  l <- rgamma(1, shape = N + a, rate = b + 1)
  N <- rpois(1, l*(1-p)) + Y_prime


  my_mat2[i, ] <- c(p,l,N)

}
```

```
for (i in 1:1000) {
  if(samp_p[i] > sqrt(samp_p2[i])){
    I[i] <- 1
  } else {
    I[i] <- 0
  }
}

prob <- sum(I)/length(I)

trace.plot(t(my_mat2), BurnIn = 10000)
```

Trace Plots of p', λ' and N'



Q & A