

HO unification from object language to meta language

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ABSTRACT

Specifying and implementing a logic from scratch requires significant effort. Logical Frameworks and Higher Order Logic Programming Languages provide dedicated, high-level Meta Languages (ML) to facilitate this task in two key ways: 1) variable binding and substitution are simplified when ML binders represent object logic ones; 2) proof construction, and even proof search, is greatly simplified by leveraging the unification procedure provided by the ML. Notable examples of ML are Elf [13], Twelf [14], λ Prolog [10] and Isabelle [20] which have been utilized to implement various formal systems such as First Order Logic [5], Set Theory [12], Higher Order Logic [11], and even the Calculus of Constuctions [4].

The object logic we are interested in is Coq's [18] Dependent Type Theory (DTT), for which we aim to implement a unification procedure \approx_o using the ML Elpi [3], a dialect of λ Prolog. Elpi's equational theory comprises $\eta\beta$ equivalence and comes equipped with a higher order unification procedure \approx_λ restricted to the pattern fragment [9]. We want \approx_o to be as powerful as \approx_λ but on the object logic DTT. Elpi also comes with an encoding for DTT that works well for meta-programming [17, 16, 7, 6]. Unfortunately this encoding, which we refer to as \mathcal{F}_o , “underuses” \approx_λ by restricting it to first-order unification problems only. To address this issue, we propose a better-behaved encoding, \mathcal{H}_o , demonstrate how to map unification problems in \mathcal{F}_o to related problems in \mathcal{H}_o , and illustrate how to map back the unifiers found by \approx_λ , effectively implementing \approx_o on top of \approx_λ for the encoding \mathcal{F}_o .

We apply this technique to the implementation of a type-class [19] solver for Coq [18]. Type-class solvers are proof search procedures based on unification that back-chain designated lemmas, providing essential automation to widely used Coq libraries such as Stdpp/Iris [8] and TLC [1]. These two libraries constitute our test bed.

KEYWORDS

Logic Programming, Meta-Programming, Higher-Order Unification, Proof Automation

ACM Reference Format:

Enrico Tassi and Davide Fissore. XXXX 2024. HO unification from object language to meta language. In *YYY*. ACM, New York, NY, USA, 15 pages. <https://doi.org/ZZZZZZZZZZZZ>

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Conference'17, July 2017, Washington, DC, USA

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ACM ISBN 978-x-xxxx-xxxx-x/YY/MM
<https://doi.org/ZZZZZZZZZZZZ>

1 INTRODUCTION

Specifying and implementing a logic from scratch requires significant effort. Logical Frameworks and Higher Order Logic Programming Languages provide dedicated, high-level Meta Languages (ML) to facilitate this task in two key ways: 1) variable binding and substitution are simplified when ML binders represent object logic ones; 2) proof construction, and even proof search, is greatly simplified by leveraging the unification procedure provided by the ML. Notable examples of ML are Elf [13], Twelf [14], λ Prolog [10] and Isabelle [20] which have been utilized to implement various formal systems such as First Order Logic [5], Set Theory [12], Higher Order Logic [11], and even the Calculus of Constuctions [4].

The object logic we are interested in is Coq's [18] Dependent Type Theory (DTT), and we want to code a type-class [19] solver for Coq [18] using the Coq-Elpi [17] meta programming framework. Type-class solvers are unification based proof search procedures that combine a set of designated lemmas in order to providing essential automation to widely used Coq libraries.

As the running example we take the Decide type class, from the Stdpp [8] library. The class identifies predicates equipped with a decision procedure. The following three designated lemmas (called Instances in the type-class jargon) state that: 1) the type `fin n`, of natural numbers smaller than `n` is finite; 2) the predicate `nfact n nf`, linking a natural number `n` to its prime factors `nf`, is decidable; 3) the universal closure of a predicate has a decision procedure if the predicate has and if its domain is finite.

```
Instance fin_fin n : Finite (fin n).          (* r1 *)
Instance nfact_dec n nf : Decision (nfact n nf). (* r2 *)
Instance forall_dec A P : Finite A →          (* r3 *)
  ∀x:A, Decision (P x) → Decision (∀x:A, P x).
```

Under this context of instances a type-class solver is able to prove the following statement automatically by back-chaining.

```
Check _ : Decision (forall y: fin 7, nfact y 3). (g)
```

The encoding of DTT provided by Elpi, that we will discuss at length later in section ?? and ??, is an Higher Order Abstract Syntax (HOAS) datatype `tm` featuring (among others) the following constructors:

```
type lam  tm -> (tm -> tm) -> tm.    % lambda abstraction
type app  list tm -> tm.              % n-ary application
type all  tm -> (tm -> tm) -> tm.    % forall quantifier
type con  string -> tm.               % constants
```

Following standard λ Prolog [10] the concrete syntax to abstract, at the meta level, an expression `e` over a variable `x` is `«x\ e»`, and square brackets denote a list of terms separated by comma. As an example we show the encoding of the Coq term `«∀y:t, nfact y 3»`:

```
all (con"t") y\ app[con"nfact", y, con"3"]
```

We now illustrate the encoding of the three instances above as higher-order logic-programming rules: capital letters denote rule parameters; `:-` separates the rule's head from the premises; `pi w\ p` introduces a fresh nominal constant `w` for the premise `p`.

```
finite (app[con"fin", N]).           (r1)
decision (app [con"nfact", N, NF]). (r2)
decision (all A x\ app[P, x]) :- finite A,           (r3)
  pi w\ decision (app[P, w]).
```

Unfortunately this translation of rule (r3) uses the predicate `P` as a first order term: for the meta language its type is `tm`. If we try to backchain the rule (r3) on the encoding of the goal (g) given below

```
decision (all (app[con"fin", con"7"]) y\
  app[con"nfact", y, con"3"]).
```

we obtain an unsolvable unification problem (p): the two lists of terms have different lengths!

```
app[con"nfact", y, con"3"] = app[P, y]           (p)
```

In this paper we study a more sophisticated encoding of Coq terms allowing us to rephrase the problematic rule (r3) as follows:

```
decision (all A x\ Pm x) :- link Pm P A, finite A, (r3a)
  pi x\ decision (app[P, x]).
```

Since `Pm` is an higher-order unification variable of type `tm -> tm`, with `x` in its scope, the unification problem (p') admits one solution:

```
app[con"nfact", y, con"3"] = Pm y           (p')
Pm = x\ app[con"nfact", x, con"3"]          % assignment for Pm
A = app[con"fin", con"7"]                   % assignment for A
```

After unifying the head of rule (r3a) with the goal, Elpi runs the premise `«link Pm A P»` that is in charge of bringing the assignment for `Pm` back to the domain `tm` of Coq terms:

```
P = lam A a\ app[con"nfact", a, con"3"]
```

This simple example is sufficient to show that the encoding we seek is not trivial and does not only concern the head of rules, but the entire sequence of unification problems that constitute the execution of a logic program. In fact the solution for `P` above generates a (Coq) β -redex in the second premise (the predicate under the `pi w\`):

```
decision (app[lam A (a\ app[con"nfact", a, con"3"]), w])
```

In turn this redex prevents the rule (r2) to backchain properly since the following unification problem has no solution:

```
app[lam A (a\ app[con"nfact", a, con"3"]), x] =
app[con"nfact", N, NF]
```

The root cause of the problems we sketched in the running example is that the unification procedure \approx_λ of the meta language is not aware of the equational theory of the object logic, even if both theories include $\eta\beta$ -conversion and admit most general unifiers for unification problems in the pattern fragment \mathcal{L}_λ [9].

Contributions. In this paper we discuss alternative encodings of Coq in Elpi (Section 2), then we identify a minimal language \mathcal{F}_0 in which the problems sketched here can be fully described. We then detail an encoding `comp` from \mathcal{F}_0 to \mathcal{H}_0 (the language of the

meta language) and a decoding `decomp` to relate the unifiers bla bla.. TODO citare Teyjus.

2 PROBLEM STATEMENT

The equational theory of Coq's Dependent Type Theory is very rich. In addition to the usual $\eta\beta$ -equivalence for functions, terms (hence types) are compared up to proposition unfolding and fix-point unrolling. Still, for efficiency and predictability reasons, most form of automatic proof search employ a unification procedure that captures a simpler one, just $\eta\beta$, and that solves higher-order problems restricted to the pattern fragment \mathcal{L}_λ [9]. We call this unification procedure \approx_o .

The equational theory of the meta language Elpi that we want to use to implement a form of proof automation is strikingly similar, since it it comprises $\eta\beta$ (for the meta language functions), and the unification procedure \approx_λ solves higher-order problems in \mathcal{L}_λ .

In spite of the similarity the link between \approx_λ and \approx_o is not trivial, since the abstraction and application term constructors the two unification procedures deal with are different. For example

$x \setminus f \ x$	\approx_λ	f
$\text{lam } A \ x \setminus \text{app}[\text{con} "f", x]$	\approx_o	$\text{con} "f"$
$\text{lam } A \ x \setminus \text{app}[\text{con} "f", x]$	\neq_λ	$\text{con} "f"$
$P \ x$	\approx_λ	x
$\text{app}[P, x]$	\approx_o	x
$\text{app}[P, x]$	\neq_λ	x

One could ignore this similarity, and “just” describe the object language unification procedure in the meta language, that is crafting a unif predicate to be used as follows in rule (r3):

```
decision X :- unif X (all A x\ app[P, x]), finite A,
  pi x\ decision (app[P, x]).
```

This choice would underuse the logic programming engine provided by the metalanguage since by removing any datum from the head of rules indexing degenerates. Moreover the unification procedure built in the meta language is likely to be faster than one implemented in it, especially if the meta language is interpreted as Elpi is.

To state precisely the problem we solve we need a \mathcal{F}_0 representation of DTT terms and a \mathcal{H}_0 one. We call $=_o$ the equality over ground terms in \mathcal{F}_0 , $=_\lambda$ the equality over ground terms in \mathcal{H}_0 , \approx_o the unification procedure we want to implement and \approx_λ the one provided by the meta language. TODO extend $=_o$ and $=_\lambda$ with reflexivity on uvars.

We write $t_1 \approx_\lambda t_2 \mapsto \sigma$ when t_1 and t_2 unify with substitution σ ; we write σt for the application of the substitution to t , and $\sigma X = \{\sigma t \mid t \in X\}$ when X is a set; we write $\sigma \subseteq \sigma'$ when σ is more general than σ' . We assume that the unification of our meta language is correct:

$$t_i \in \mathcal{L}_\lambda \Rightarrow t_1 \approx_\lambda t_2 \mapsto \rho \Rightarrow \rho t_1 =_\lambda \rho t_2 \quad (1)$$

$$t_i \in \mathcal{L}_\lambda \Rightarrow \rho t_1 =_\lambda \rho t_2 \Rightarrow \exists \rho', t_1 \approx_\lambda t_2 \mapsto \rho' \wedge \rho' \subseteq \rho \quad (2)$$

We illustrate a compilation $\langle s \rangle \mapsto (t, m, l)$ that maps a term s in \mathcal{F}_0 to a term t in \mathcal{H}_0 , a variable mapping m and list of links l . The variable map connects unification variables in \mathcal{H}_0 with variables in \mathcal{F}_0 and is used to “decompile” the assignment, $\langle \sigma, m, l \rangle^{-1} \mapsto \rho$. Links represent problematic sub-terms which are linked to the

unification variable that stands in their place in the compiled term. These links are checked for or progress XXX improve....

We represent a logic program *run* in \mathcal{F}_0 as a list *steps* p of length N . Each made of a unification problem between terms S_{p_l} and S_{p_r} taken from the set of all terms \mathcal{S} . The composition of these steps starting from the empty substitution ρ_0 produces the final substitution ρ_N .¹ The initial here ρ_0 is the empty substitution

$$\begin{aligned} \text{fstep}(\mathcal{S}, p, \rho) &\mapsto \rho' \stackrel{\text{def}}{=} \rho S_{p_l} \simeq_o \rho S_{p_r} \mapsto \rho' \wedge \rho'' = \rho \cup \rho' \\ \text{frun}(\mathcal{S}, N) &\mapsto \rho_N \stackrel{\text{def}}{=} \bigwedge_{p=1}^N \text{fstep}(\mathcal{S}, p, \rho_{p-1}) \mapsto \rho_p \end{aligned}$$

We simulate each run in \mathcal{F}_0 with a run in \mathcal{H}_0 as follows. Note that σ_0 is the empty substitution.

$$\begin{aligned} \text{hstep}(\mathcal{T}, p, \sigma, \mathbb{L}) &\mapsto \sigma'' \stackrel{\text{def}}{=} \\ &\sigma \mathcal{T}_{p_l} \simeq_\lambda \sigma \mathcal{T}_{p_r} \mapsto \sigma' \wedge \text{check}(\mathbb{L}, \sigma \cup \sigma') \mapsto \sigma'' \\ \text{hrun}(\mathcal{S}, N) &\mapsto \rho_N \stackrel{\text{def}}{=} \\ &\mathcal{T} \times \mathbb{M} \times \mathbb{L} = \{(t_j, m_j, l_j) \mid s_j \in \mathcal{S}, \langle s_j \rangle \mapsto (t_j, m_j, l_j)\} \\ &\bigwedge_{p=1}^N \text{hstep}(\mathcal{T}, p, \sigma_{p-1}, \mathbb{L}) \mapsto \sigma_p \\ &\langle \sigma_N, \mathbb{M}, \mathbb{L} \rangle^{-1} \mapsto \rho_N \end{aligned}$$

Here *hstep* is made of two sub-steps: a call to \simeq_λ (on the compiled terms) and a call to *check* on the set of links. We claim the following:

PROPOSITION 2.1 (SIMULATION). $\forall \mathcal{S}, \forall N$

$$\text{frun}(\mathcal{S}, N) \mapsto \rho_N \Leftrightarrow \text{hrun}(\mathcal{S}, N) \mapsto \rho_N$$

That is, the two executions give the same result. Moreover:

PROPOSITION 2.2 (SIMULATION FIDELITY). *In the context of hrun , we have that $\forall p \in 1 \dots N$*

$$\text{fstep}(\mathcal{S}, p, \rho_{p-1}) \mapsto \rho_p \Leftrightarrow \text{hstep}(\mathcal{T}, p, \sigma_{p-1}, \mathbb{L}) \mapsto \sigma_p$$

In particular this property guarantees that a *failure* in the \mathcal{F}_0 run is matched by a failure in \mathcal{H}_0 at the same step. We consider this property very important from a practical point of view since it guarantees that the execution traces are strongly related and in turn this enables a user to debug a logic program in \mathcal{F}_0 by looking at its execution trace in \mathcal{H}_0 .

We can define $s_1 \simeq_o s_2$ by specializing the code of *hrun* to $\mathcal{S} = \{s_1, s_2\}$ as follows:

$$\begin{aligned} s_1 \simeq_o s_2 &\mapsto \rho \stackrel{\text{def}}{=} \\ &\langle s_1 \rangle \mapsto (t_1, m_1, l_1) \wedge \langle s_2 \rangle \mapsto (t_2, m_2, l_2) \\ &t_1 \simeq_\lambda t_2 \mapsto \sigma' \wedge \text{check}(\{l_1, l_2\}, \sigma') \mapsto \sigma'' \wedge \\ &\langle \sigma'', \{m_1, m_2\}, \{l_1, l_2\} \rangle^{-1} \mapsto \rho \end{aligned}$$

PROPOSITION 2.3 (PROPERTIES OF \simeq_o).

$$s_i \in \mathcal{L}_\lambda \Rightarrow s_1 \simeq_o s_2 \mapsto \rho \Rightarrow \rho s_1 =_o \rho s_2 \quad (3)$$

$$s_i \in \mathcal{L}_\lambda \Rightarrow \rho s_1 =_o \rho s_2 \Rightarrow \exists \rho', s_1 \simeq_o s_2 \mapsto \rho' \wedge \rho' \subseteq \rho \quad (4)$$

$$\rho s_1 =_o \rho s_2 \Rightarrow \rho' \subseteq \rho \Rightarrow \rho' s_i \in \mathcal{L}_\lambda \Rightarrow \rho' s_1 \simeq_o \rho' s_2 \quad (5)$$

Properties 3 and 4 state, respectively, that in \mathcal{L}_λ the implementation of \simeq_o is correct, complete and returns the most general unifier.

Property 5 states that \simeq_o , hence our compilation scheme, is resilient to unification problems outside \mathcal{L}_λ solved by a third party.

¹If the same rule is used multiple time in a run we just consider as many copies as needed of the terms composing the rules, with fresh unification variables each time

We believe this property is of practical interest since we want the user to be able to add heuristics via hand written rules to the ones obtained by our compilation scheme. A Typical example is the following problem (q) that is outside \mathcal{L}_λ :

$$\begin{aligned} \text{app}[F, \text{con}"a"] &= \text{app}[\text{con}"f", \text{con}"a", \text{con}"a"] \quad (q) \\ F &= \text{lam } x \backslash \text{app}[\text{con}"f", x, x] \quad (h) \end{aligned}$$

Instead of rejecting it our scheme accepts it and guarantees that if (h) is given (after the compilation part of the scheme, as a run time hint) then ...

2.1 The intuition in a nutshell

A term s is compiled in a term t where any “problematic” subterm p is replaced by a fresh unification variable h and an accessory link that represent a suspended unification problem $h \simeq_\lambda p$. As a result \simeq_λ is “well behaved” on t , meaning it does not contradict $=_o$ (as it would do on “problematic” terms). We now define “problematic” and “well behaved” more formally.

Definition 2.4 ($\diamond\eta$). $\diamond\eta = \{t \mid \exists \rho, \rho t \text{ is an eta expansion}\}$

An example of term t in $\diamond\eta$ is $\lambda x. \lambda y. F y x$ since the substitution $\rho = \{F \mapsto \lambda a. \lambda b. fba\}$ makes $\rho t = \lambda x. \lambda y. fxy$ that is the eta long form of f .

Definition 2.5 ($\diamond\beta$). $\diamond\beta = \{Xt_1 \dots t_n \mid Xt_1 \dots t_n \notin \mathcal{L}_\lambda\}$.

An example of t in $\diamond\beta$ is Fa for a constant a . Note however that an oracle could provide an assignment $\rho = \{F \mapsto \lambda x. x\}$ that makes the resulting term fall outside of $\diamond\beta$.

Definition 2.6 (Subterm $\mathcal{P}(t)$).

$$\begin{aligned} t &\in \mathcal{P}(t) \\ t &= ft_1 \dots t_n \Rightarrow \mathcal{P}(t_i) \subseteq \mathcal{P}(t) \wedge f \in \mathcal{P}(t) \\ t &= \lambda x. t' \Rightarrow \mathcal{P}(t') \subseteq \mathcal{P}(t) \end{aligned}$$

We write $\mathcal{P}(X) = \bigcup_{t \in X} \mathcal{P}(t)$.

Definition 2.7 (Well behaved set). Given a set of terms $X \subseteq \mathcal{H}_0$,

$$\mathcal{W}(X) \Leftrightarrow \forall t \in \mathcal{P}(X), t \notin (\diamond\beta \cup \diamond\eta)$$

PROPOSITION 2.8 (\mathcal{W} -PRESERVATION). $\forall \mathcal{T}, \forall \mathbb{L}, \forall p, \forall \sigma, \forall \sigma'$

$$\mathcal{W}(\sigma \mathcal{T}) \wedge \text{hstep}(\mathcal{T}, p, \sigma, \mathbb{L}) \mapsto \sigma' \Rightarrow \mathcal{W}(\sigma' \mathcal{T})$$

A less formal way to state 2.8 is that *hstep* never “commits” an unneeded λ -abstraction in σ (a λ that could be erased by an η -contraction), nor puts in σ a flexible application outside \mathcal{L}_λ (an application node that could be erased by a β -reduction).

Note that proposition 2.8 does not hold for \simeq_o since decompilation can introduce (actually restore) terms in $\diamond\eta$ or $\diamond\beta$ that were move out of the way (put in \mathbb{L}) by compilation.

3 ALTERNATIVE ENCODINGS AND RELATED WORK

Paper [2] introduces semi-shallow.

Our encoding of DTT may look “semi shallow” since we use the meta-language lambda abstraction but not its application (for the terms of type tm). A fully shallow encoding unfortunately does not fit our use case, although it would make the running example work:

```

349   finite (fin N).
350   decision (nfact N NF).
351   decision (all A x\ P x) :- finite A, pi x\ decision (P x).

```

There are two reasons for dismissing this encoding. The first one is that in DTT it is not always possible to adopt it since the type system of the meta language is too weak to accommodate terms with a variable arity, like the following example:

```

357   Fixpoint arr T n := if n is S m then T -> arr T m else T.
358   Definition sum n : arr nat n := ...
359   Check sum 2 7 8 : nat.
360   Check sum 3 7 8 9 : nat.

```

The second reason is the encoding for Coq is used for meta programming the system, hence it must accommodate the manipulation of terms that are now known in advance (not even defined in Coq) without using introspection primitives such as Prologs's functor and arg.

In the literature we could find a few related encoding of DTT. TODO In [4] is related and make the discrepancy between the types of ML and DTT visible. In this case one needs 4 application nodes. Moreover the objective is an encoding of terms, proofs, not proof search. Also note the conv predicate, akin to the unif we rule out.

TODO This other paper [15] should also be cited.

None of the encodings above provide a solution to our problem.

4 PRELIMINARIES: \mathcal{F}_0 AND \mathcal{H}_0

In order to reason about unification we provide a description of the \mathcal{F}_0 and \mathcal{H}_0 languages where unification variables are first class terms, i.e. they have a concrete syntax. We keep these languages minimal, for example, we omit the all quantifier of DTT we used in the example in Section 1 together with the type notation of terms carried by the lam constructor.

```

383   kind fm type.           kind tm type.
384   type fapp list fm -> fm. type app list tm -> tm.
385   type flam (fm -> fm) -> fm. type lam (tm -> tm) -> tm.
386   type fcon string -> fm.   type con string -> tm.
387   type fuva nat -> fm.      type uva nat -> list tm -> tm.

```

Figure 1: \mathcal{F}_0 and \mathcal{H}_0 language

In the case of \mathcal{F}_0 unification variables fuva have no explicit scope: the arguments of an higher order unification variable are via the fapp constructor. For example in the statement of the instance forall_dec the term $P\ x$ is represented as `fapp[fuva N, x]`, where N is a memory address and x is a bound variable.

In \mathcal{H}_0 the representation of $P\ x$ is instead `uva N [x]`. We say that the unification variable `uva N L` is in \mathcal{L}_λ iff `distinct L` holds.

```

397   type distinct list A -> o.
398   distinct [].
399   distinct [X|XS] :- name X, not(mem X XS), distinct XS.

```

The name builtin predicate tests if a term is a bound variable.² The compiler ?? needs to support terms outside \mathcal{L}_λ for practical reasons, so we don't assume all our terms are in \mathcal{L}_λ but rather test. **what??**

²one could always load name x for every x under a pi and get rid of the name builtin

In both languages unification variables are identified by a natural number, which can be seen as a memory address. The memory and its associated operations are described below:

```

407   typeabbrev memory A (list (option A)).
408   type set?   nat -> memory A -> A -> o.
409   type unset? nat -> memory A -> o.
410   type assign nat -> memory A -> A -> memory A -> o.

```

If a memory cell is none, then the corresponding unification variable is not set. `assign` sets an unset cell to the given value.

Since in \mathcal{H}_0 unification variables have a scope, their solution needs to be abstracted over it in order to enable the instantiation of a single solution to different scopes. On the contrary solutions to \mathcal{F}_0 variables are plain terms.

```

421   typeabbrev fsubst (memory ftm).
422   kind assmt type.
423   type abs (tm -> assmt) -> assmt.
424   type val tm -> assmt.
425   typeabbrev subst (memory assmt).

```

We call `fsubst` the memory of \mathcal{F}_0 , while we call `subst` the one of \mathcal{H}_0 . Both have the invariant that they are not cyclic, TODO explain. Other invariant: the terms in `ho_subst` never contains eta and beta expansion

4.1 Equational theory and Unification

here we give the functions/signatures to express the properties 3-5 in the problem statement

Together with the description of the terms of the language, we need some auxiliary functions to perform operations like term equality, unification, dereferencing. There predicates are supposed to be implemented in the OL and the ML following respectively their specification. In the following few paragraphs we give a brief proposition of them.

Term dereferencing: ρs and σt . Since in our encoding we explicitly carry a substitution we need to define the operation that applies it to a term. Its proposition takes care to normalize (flatten) applications, for example it turns `(app [app [con"f",con"a"],con"b"])` into `(app [con"f",con"a",con"b"])`.

dereference variables of the two languages. This is particularly useful to check if two terms in the OL (resp. in the ML) are equal. The constructors representing rigid terms, namely `app`, `lam` and `con`, make the dereferencing procedure to recurse over the their subterms. The code below display this behavior for the dereferencing performed by the OL:

```

454   type fder fsubst -> fm -> fm -> o.
455   fder S (fapp A) (fapp B) :- map (fder S) A B.
456   fder S (flam F) (flam G) :-
457     pi x\ fder S x x => fder S (F x) (G x).
458   fder _ (fcon C) (fcon C).
459   fder S (fuva N) R :- set? N S T, fder S T R.
460   fder S (fuva N) (fuva N) :- unset? N S.
461
462   type fderef fsubst -> fm -> fm -> o.
463   fderef S T R :- fder S T T', napp T' R.

```

(ρs)


```

465 type (=λ) tm -> tm -> o.
466 app A =λ fapp B :- map (=λ) A B.
467 lam F =λ flam G :- pi x\ x =λ x => F x =λ G x.
468 con C =λ fcon C.
469 uva N A =λ fuva N B :- map (=λ) A B.

```

Figure 2: Equal predicate ML

```

473 type napp fm -> fm -> o.

```

TODO explain napp. We use the cut operator to keep the code compact. It is possible to rewrite the rule for application traversal so that it is mutually exclusive with the first one, but that requires a rather verbose analysis of the head of A.

The corresponding code for \mathcal{H}_0 is similar, we only show the last two rules that differ in a substantial way:

```

480 type deref subst -> tm -> tm -> o. (σt)
481 deref S (app A) (app B) :- map (deref S) A B.
482 deref S (lam F) (lam G) :-
483   pi x\ deref S x x => deref S (F x) (G x).
484 deref _ (con C) (con C).
485 deref S (uva N A) R :- set? N S F, move F A T, deref S T R.
486 deref S (uva N A) (uva N B) :- unset? N S, map (deref S) A B.
487 type move assignment -> list tm -> tm -> o.
488 move (abs Bo) [H|L] R :- move (Bo H) L R.
489 move (val A) [] A :- !.
490 move (val (uva N A)) L (uva N X) :- std.append A L X.

```

TODO: no need to napp, see the beta section. Note that when the substitution S maps a unification variable N to an assignment F we we

TODO: invariant: variables in subst are always fully applied, and length of scope is the arity of the HO variable.

Important!!! A different reasoning is to be addressed to the variables of the ML. Firstly, a meta variable cannot appear in the app node as the first element of the list, we will explain why in section 5

```

500 type (=o) ftm -> ftm -> o. (=o)
501 fapp A =o fapp B :- map (=o) A B.
502 flam F =o flam G :- pi x\ x =o x => F x =o G x.
503 fcon C =o fcon C.
504 fuva N =o fuva N.
505 flam F =o T :- (ηl)
506   pi x\ beta T [x] (R x), x =o x => F x =o R x.
507 T =o flam F :- (ηr)
508   pi x\ beta T [x] (R x), x =o x => R x =o F x.
509 fapp [flam X | L] =o T :- beta (flam X) L R, R =o T. (βl)
510 T =o fapp [flam X | L] :- beta (flam X) L R, T =o R. (βr)

```

Term equality: $=_o$ vs. $=_\lambda$. We can test if two terms are equal following the equational theory of the language being considered. In ?? we provide an implementation of the $=_o$ predicate. The first four rules check if the two terms are equal regarding the structure of the current node, that is, two terms are equal if they have same head and if recursively each subterm is two by two equal. Moreover, since the theory of the OL accepts η - and β -equivalence, then we implement the corresponding rules.

The equality relation for the ML, accepts $\eta\beta$ -equivalence between terms of the ML. Recall that $\text{abs } x \backslash f \ x$, is a valid η expansion of

the function f and that $\text{lam } x \backslash \text{app}[f, x]$ is not that equivalent to f at meta level. However, since we are interested in using the unification procure of the ML, by eq. (1), we can use the $=_\lambda$ relation to test, when needed if two terms are equal in the ML.

Term unification: \approx_o vs. \approx_λ . The last but not least important relation we should take care of before presenting our full algorithm aiming to unify terms of the OL in the ML and provide the substitution produced in the ML to the OL, is term unification. This procedure is a more powerful version of the equal predicate, since unification checks if two terms can be equal by assigning unification variables. In our representation, variable assignment (or refinement) is performed by modifying the corresponding substitution mapping. We will not give an implementation of \approx_o , since we are giving an implementation of it using our algorithm, see ??.

```

538 type (≈λ) tm -> tm -> subst -> subst -> o.

```

On the other hand, unification in the ML needs to be defined. In ??, we give an implementation of \approx_λ but that is actually what our meta language provides as a builtin.

This predicate has four arguments, the two terms t_1 and t_2 to unify, the old substitution map ρ_1 , and the new substitution map ρ_2 , with the invariant $\rho_1 \subseteq \rho_2$. The first three rules unify terms with same rigid heads, and call the unification relation on the sub-terms. If t_1 (resp. t_2) is an assigned variables, t_1 is dereferenced to t'_1 (resp. t'_2) and the unification is called between t'_1 and t_2 (resp. t_1 and t'_2). If both terms are unification variables, we test that their arguments are in the pattern fragment, we allocate a new variable w in ρ_1 such that w is the pruning of the arguments of t_1 and t_2 , we assign both t_1 and t_2 to w and return the new mapping ρ_2 containing all the new variable assignment. Finally, if only one of the two terms is an unification variable v , after having verified that v does not occur in the other term t , we bind v to t and return the new substitution mapping.

OLD

A key property needed in unification is being able to verify if two terms are equal wrt a given equational theory. This relation allow to compare terms under a certain substitution mapping, so that any time a variable v is assigned in a subterm, a dereferencing of v is performed. After variable dereferencing, the test for equality is continued on the new-created subterm.

The base equality function over terms can be defined as follows:

The solution we are proposing aim to overcome these unification issues by 1) compiling the terms t and u of the OL into an internal version t' and u' in the ML; 2) unifying t' and u' at the meta level instantiating meta variables; 3) decompiling the meta variable into terms of the OL; 4) assigning the variables of the OL with the decompiled version of their corresponding meta variables. We claim that t and u unify if and only if t' and u' unify and that the substitution in the object language is the same as the one returned by the ML.

In the following section we explain how we deal with term (de)compilation and links between unification variables.

5 BASIC COMPILATION \mathcal{F}_o TO \mathcal{H}_o

In this section we describe a basic compilation scheme that we refine later, in the following sections.

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The objective of the compilation is to recognize the higher-order variables available in \mathcal{H}_o when expressed in a first order way in \mathcal{F}_o . The compiler also generates a list of links that are used to bring back the substitution from \mathcal{H}_o to \mathcal{F}_o and allocates in the memory a cell for each variable.

```

kind link type.
type link nat -> nat -> nat -> subst. % link Fo Ho Arity
typeabbrev links list link.
type comp fm -> tm -> links -> links -> subst -> subst -> o.
comp (fcon X) (con X) L L S S.
comp (flam F) (lam G) K L R S :- pi x y\
  (pi A S\ comp x y L L S S) => comp (F x) (G y) K L R S.
comp (fuva M) (uva N [I]) K [link M N z|K] R S :- new R N S.
comp (fapp[fuva M|A]) (uva N B) K L R S :- distinct A, !,
  fold4 comp A B K K R R,
  new R N S, len A Arity,
  L = [link N M Arity | K].
comp (fapp A) (app B) K L R S :- fold4 comp A B K L R S.

```

Note that link carries the arity (number of expected arguments) of the variable.

```

type solve-links links -> links -> subst -> subst -> o.
solve-links L L S S.

```

Then decomp

```

type decompile links -> subst -> fsubst -> o.
decompile L S O :-
  map (\r\ r = none) S O1, % allocate empty fsubst
  (pi N X\ knil N X :- mem L (link X N _) ; N = X) =>
  decomp L S L O1 O.
type knil nat -> nat -> o.

type decomp1 links -> subst -> fsubst -> o.
decomp1 S [I] [I].
decomp1 S [link _ N _|L] O P :- unset? N S X,
  decomp1 S L O P.
decomp1 S [link M N _|L] O P :- set? N S X,
  decomp-assignment S X T, assign M O (some T) O1,
  decomp1 S L O1 P.

type decomp-assignment subst -> assignment -> fm -> o.
decomp-assignment S (abs F) (flam G) :-
  pi x y\ decomp-tm S x y => decomp-assignment S (F x) (G y).
decomp-assignment S (val T) T1 :- decomp S T T1.

type decomp subst -> tm -> fm.
decomp _ (con C) (fcon C).
decomp S (app A) (app B) :- map (decomp S) A B.
decomp S (lam F) (flam G) :-
  pi x y\ decomp S x y => decomp S (F x) (G y).
decomp S (uva N A) R :- set? N S F,
  move F A T, decomp S T R.
decomp S (uva N A) R :- unset? N S,
  map (decomp S) A B, knil N M, napp (fapp[fuva M|B]) R.

```

Now unif

```

type (≈o) fm -> fm -> subst -> subst -> o.
(X ≈o Y) S S1 :-
  fderef S X X0, fderef S Y Y0,
  comp X0 X1 [I] S0 [I] L0,
  comp Y0 Y1 S0 S1 L0 L1,
  (X1 ≈λ Y1) [I] HS0,
  solve-links L1 L2 HS0 HS1,
  decompile L2 HS1 S1.

```

5.1 Prolog simulation

Allows us to express the properties. we take all terms involved in a search (if a rule is used twice we simply take a copy of it), we compile all of them, and then we pick the unification problems among these terms and step through them.

```

type pick list A -> (pair nat nat) -> (pair A A) -> o.
pick L (pr X Y) (pr TX TY) :- nth X L TX, nth Y L TY.

type prolog-fo list fm -> list (pair nat nat) -> subst -> o.
prolog-fo Terms Problems S :-
  map (pick Terms) Problems FoProblems,
  fold4 (≈o) FoProblems [I] S.

type step-ho (pair tm tm) -> links -> links -> subst -> subst -> o.
step-ho (pr X Y) L0 L1 S0 S2 :-
  (X1 ≈λ Y1) S0 S1,
  solve-links L0 L1 S1 S2.

type prolog-ho list fm -> list (pair nat nat) -> subst -> o.
prolog-ho Terms Problems S :-
  fold4 comp Terms HoTerms [I] L0 [I] HS0,
  map (pick HoTerms) Problems HoProblems,
  fold4 step-ho HoProblems L0 L HS0 HS,
  decompile L HS S.

```

the property is that if a step for Fo succeeds then the Ho one does, and if Fo fails then the Ho fails ()

5.2 Example

OK

```

Terms [ flam x\ fapp[fcon"g",fapp[fuva z, x]]
      , flam x\ fapp[fcon"g", fcon"a"] ]
Problems = [ pr z (s z) ] % λx.g(Fx) = λx.ga
lam x\ app[con"g",uva z [x]] ≈o lam x\ app[con"g", con"a"]
link z z (s z)
HS = [some (abs x\con"a")]
S = [some (flam x\fcon a)]

```

KO

```

Terms [ flam x\ fapp[fcon"g",fapp[fuva z, x]]
      , flam x\ fapp[fcon"g", fcon"a"] ]
Problems = [ pr 0 1 % A = λx.x
            , pr 2 3 ] % Aa = a
lam x\ app[con"g",uva z [x]] ≈o lam x\ app[con"g", con"a"]
link z z (s z)
HS = [some (abs x\con"a")]
S = [some (flam x\fcon a)]

```

TODO
link
TODO
nuove
subst
TODO:
code
unif

```

697 lam x\ app[f, app[X, x]] = Y,
698 lam x\ x[] = X.

```

TODO: Goal: $s_1 \approx_o s_2$ is compiled into $t_1 \approx_\lambda t_2$

TODO: What is done: uvars fo_{uv} of OL are replaced into uvars ho_{uv} of the ML

TODO: Each fo_{uv} is linked to an ho_{uv} of the OL

TODO: Example needing the compiler v0 (tra l'altro lo scope è ignorato):

```

705 lam x\ app[con"g", app[uv 0, x]]  $\approx_o$  lam x\ app[con"g", c"a"]

```

TODO: Links used to instantiate vars of elpi

TODO: After all links, the solution in links are compacted and given to coq

TODO: It is not so simple, see next sections (multi-vars, eta, beta)

The compilation step is meant to recover the higher-order variables of the OL, expressed in a first order way, by replacing them with higher-order variables in the ML. In particular, every time a variable of the OL is encountered in the original term, it is replaced with a meta variable, and if the OL variable is applied to a list of distinct names L , then this list becomes the scope of the variable. For all the other constructors of tm , the same term constructor is returned and its arguments are recursively compiled. The predicate in charge for term compilation is:

```

721 type comp tm -> tm -> links -> links -> subst -> subst -> o.

```

where, we take the term of the OL, produce the term of the ML, take a list of link and produce a list of new links, take a substitution and return a new substitution.

In particular, due to programming constraints, we need to drag the old subst and return a new one extended, if needed, with the new declared meta-variables.

The following code

```

730 kind link type.
731 type link nat -> nat -> nat -> subst.

```

defines a link, which is a relation between to variables indexes, the first being the index of a OL variable and the second being the index of a ML variable. The third integer is the number of term in the scope of the two variables, or equivalently, in a typed language, their arity.

As an example, let's study the following unification problem (a slightly modified version from section 1):

```

740 lam x\ app[c"decision", app[c"nfact", x, c"3"]]  $\approx_o$ 
741 lam x\ app [c"decision", app[uv 0, x]]

```

we have the main unification problem where the nested app nodes have lists of different lengths making the unification to fail. The compilation of these terms produces a new unification problem with the following shape:

```

747 lam x\ app[c"decision", app[c"nfact", x, c"3"]]  $\approx_\lambda$ 
748 lam x\ app [c"decision", uv 1 [x]]

```

The main difference is the replacement of the subterm $app[uv\ 0, x]$ of the OL with the subterm $uv\ 0\ [x]$. Variable indexes are chosen by the ML, that is, the index 0 for that unification variable of the OL term has not the sam meaning of the index 0 in the ML. There exists

two different substitution mapping, one for the OL and one for the ML and the indexes of variable point to the respective substitution.

decomp che mappa abs verso lam **TODO: An other example:**

```

758 lam x\ app[f, app[X, x]] = Y, (lam x\ x) = X.

```

6 USE OF MULTIVARS

Se il termine iniziale è della forma

```

763 app[con"xxx", (lam x\ lam y\ Y y x), (lam x\ f)]
764 =
765 app[con"xxx", X, X]

```

allora se non uso due X diverse non ho modo di recuperare il quoziente che mi manca.

a sto punto consideriamo liste di problemi e così da eliminare sta xxx senza perdita di generalità (e facciamo problemi più corti, e modellizziamo anche la sequenza)

6.1 Problems with η

TODO: The following goal necessita v1 (lo scope è usato):

```

774 X = lam x\ lam y\ Y y x, X = lam x\ f

```

TODO: The snd unif pb, we have to unif lam x\ lam y\ Y x y with lam x\ f

TODO: It is not doable, with the same elpi var

Invarianti: A destra della eta abbiamo sempre un termine che comincia per $\lambda x.bla$

La deduplicate eta:

- viene chiamata che della forma [variable] -> [eta1] e
- ↪ [variable] -> [eta2]
- (a destra non c'è mai un termine con testa rigida)
- i due termini a dx vengono unificati con la unif e uno
- ↪ dei due link viene buttato
- NOTA!! A dx abbiamo sempre un termine della forma lam
- ↪ x.VAR x!!!
- Altrimenti il link sarebbe stato risolto!!
- dopo l'unificazione rimane un link [variabile] -> [etaX]
- nella progress-eta, se a sx abbiamo una costante o
- ↪ un'app, allora eta-espandiamo
- di uno per poter unificare con il termine di dx.

6.2 Problems with β

β -reduction problems ($\diamond\beta$) appears any time we deal with a subterm $t = Xt_1 \dots t_n$, where X is flexible and the list $[t_1 \dots t_n]$ in not in \mathcal{L}_λ . This unification problem is not solvable without loss of generality, since there is not a most general unifier. If we take back the example given in section 2.1, the unification $Fa = a$ admits two solutions for F : $\rho_1 = \{F \mapsto \lambda x.x\}$ and $\rho_2 = \{F \mapsto \lambda _{.}a\}$. Despite this, it is possible to work with $\diamond\beta$ if an oracle provides a substitution ρ such that ρt falls again in the \mathcal{L}_λ .

On the other hand, the \approx_λ is not designed to understand how the β -redexes work in the object language. Therefore, even if we know that F is assigned to $\lambda x.x$, \approx_λ is not able to unify Fa with a . On the other hand, the problem $Fa = G$ is solvable by \approx_λ , but the final result is that G is assigned to $(\lambda x.x)a$ which breaks the invariant saying that the substitution of the meta language does not generate terms outside \mathcal{W} (Property 2.8).

integer
or
nat?

The solution to this problem is to modify the compiler such that any sub-term t considered as a potential β -redex is replaced with a hole h and a new dedicated link, called $\text{link-}\beta$.

```
type link-beta tm -> tm -> link.
```

This link carries two terms, the former representing the variable h for the new created hole and the latter containing the subterm t . As for the $\text{link-}\eta$, we will call h and t respectively the left hand side (LHS) and the right hand side (RHS) of the $\text{link-}\beta$.

A subterm is in $\diamond\beta$ if it has the shape $\text{fapp}[f, \text{uva } N \mid L]$ and distinct L does not hold. In that case, L is split in two sublists PF and NPF such that former is the longest prefix of L such that distinct PF holds. NPF is the list such that $\text{append } PF \ NPF \ L$. The LHS is set to a new variable named M with PF in scope whereas the RHS is given by the term $\text{app}[\text{uva } N' \ PF \mid NPF]$ where the \mathcal{H}_0 variable identified by N' is mapped to the \mathcal{F}_0 variable named N .

After its creation, a $\text{link-}\beta$ remain suspended until the head of the RHS is instantiated by the oracle (see eq. (5)). In this case the RHS is β -reduced to a new term t . t is either a term in \mathcal{L}_λ , in which case t is unified with the LHS, otherwise, the link remain suspended and no progress is performed. Another way to wake a $\text{link-}\beta$ up is when the LHS is a term T and RHS has the shape $\text{app}[\text{uva } N \ PF \mid NPF]$ and some of the arguments in the NPF list become names. This is possible after the resolution of other links. In this case, the list L obtained by the concatenation between PF and NPF is split again in to lists PF' and NPF' . If PF is not that same as PF' , then we can 1) remove the current $\text{link-}\beta$, 2) create a new $\text{link-}\beta$ between T and $\text{app}[\text{uva } N' \ PF' \mid NPF']$ and 3) create a new $\text{link-}\eta$ between the variables N and N' .

An example justifying this last link manipulation is given by the following unification problem:

```
f = flam x\ fapp[F, (X x), a] % f = λx.F(Xx)a
```

under the substitution $\rho = \{X \mapsto \lambda x.x\}$.

The links generated from this unification problem are:

```
X ↦ X1; F ↦ X2 % The mappings
⊢ X0 =η= x\ `X3 x'
x ⊢ X3 x =β= X2 `X1 x' a
```

where the first link is a $\text{link-}\eta$ between the variable $X0$, representing the right side of the unification problem (it is a $\diamond\eta$) and $X3$; and a $\text{link-}\beta$ between the variable $X3$ and the subterm $c0 \ X2 \ 'X1 \ c0' \ a$ (it is a $\diamond\beta$). The substitution tells that $x \vdash X1 \ x = x$.

We can now represent the hrun execution from this configuration which will, at first, dereference all the links, and then try to solve them. The only link being modified is the second one, which is set to $X3 \ x =\beta= X2 \ x \ a$. The RHS of the link has now a variable which is partially in the PF , we can therefore remove the original $\text{link-}\beta$ and replace it with the following couple on links:

```
⊢ X1 =η= x\ `X4 x'
x ⊢ X3 x =β= x\ `X4 x' a
```

By these links we say that $X1$ is now η -linked to a fresh variable $X4$ with arity one. This new variable is used in the new $\text{link-}\beta$ where the name x is in its scope. This allows

6.3 Tricky examples

```
triple ok (@lam x\ @app[@f, @app[@X, x]]) @Y,
triple ok @X (@lam x\ x),
triple ok @Y @f

% @ok1 22 [
%   triple ok (@lam x\ @lam y\ @app[@Y, y, x]) @X,
%   triple ok (@lam x\ @f) @X,
% ].
```

7 FIRST ORDER APPROXIMATION

TODO: Coq can solve this: $f \ 1 \ 2 = x \ 2$, by setting X to $f \ 1$

TODO: We can re-use part of the algo for β given before

8 UNIF ENCODING IN REAL LIFE

TODO: Il ML presentato qui è esattamente elpi

TODO: Il OL presentato qui è esattamente coq

TODO: Come implementiamo tutto ciò nel solver

9 RESULTS: STDPP AND TLC

TODO: How may rule are we solving?

TODO: Can we do some perf test

10 CONCLUSION

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APPENDIX

Note that (a infix b) c d de-sugars to (infix) a b c d.

11 THE MEMORY

```

kind address type.
type addr nat -> address.

typeabbrev (mem A) (list (option A)).

type get nat -> mem A -> A -> o.
get z (some Y :: _) Y.
get (s N) (_ :: L) X :- get N L X.

type alloc-aux nat -> mem A -> mem A -> o.
alloc-aux z [] [none] :- !.
alloc-aux z L L.
alloc-aux (s N) [] [none | M] :- alloc-aux N [] M.
alloc-aux (s N) [X | L] [X | M] :- alloc-aux N L M.

type alloc address -> mem A -> mem A -> o.
alloc (addr A as Ad) Mem1 Mem2 :- unset? Ad Mem1,
  alloc-aux A Mem1 Mem2.

type new-aux mem A -> nat -> mem A -> o.
new-aux [] z [none].
new-aux [A | As] (s N) [A | Bs] :- new-aux As N Bs.

type new mem A -> address -> mem A -> o.
new Mem1 (addr Ad) Mem2 :- new-aux Mem1 Ad Mem2.

type set? address -> mem A -> A -> o.
set? (addr A) Mem Val :- get A Mem Val.

type unset? address -> mem A -> o.
unset? Addr Mem :- not (set? Addr Mem _).

type assign-aux nat -> mem A -> A -> mem A -> o.
assign-aux z (none :: L) Y (some Y :: L).
assign-aux (s N) (X :: L) Y (X :: L1) :- assign-aux N L Y L1.

type assign address -> mem A -> A -> mem A -> o.
assign (addr A) Mem1 Val Mem2 :- assign-aux A Mem1 Val Mem2.

```

12 THE OBJECT LANGUAGE

```

type (=_o) ftm -> ftm -> o. (=_o)
fapp A =_o fapp B :- map (=_o) A B.
flam F =_o flam G :- pi x\ x =_o x => F x =_o G x.
fcon C =_o fcon C.
fuva N =_o fuva N.
flam F =_o T :- (ηl)
  pi x\ beta T [x] (R x), x =_o x => F x =_o R x.
T =_o flam F :- (ηr)
  pi x\ beta T [x] (R x), x =_o x => R x =_o F x.
fapp [flam X | L] =_o T :- beta (flam X) L R, R =_o T. (βl)
T =_o fapp [flam X | L] :- beta (flam X) L R, T =_o R. (βr)

```

```

type beta fm -> list fm -> fm -> o.
beta A [] A.
beta (flam F) [H | L] R :- subst F H B,
  beta B L R. % since F could be x\app[x|_] and H be lam _
beta (fapp A) L (fapp X) :- append A L X.
beta (fuva N) L (fapp [fuva N | L]).
beta (fcon H) L (fapp [fcon H | L]).

type subst (fm -> fm) -> fm -> fm -> o.
subst F H B :- napp (F H) B. % since (F H) may generate (app[app|_])

type napp fm -> fm -> o.
napp (fcon C) (fcon C).
napp (flam F) (flam G) :- pi x\ napp x x => napp (F x) (G x).
napp (fapp [fapp L|M] R) :- !, append L M N, napp (fapp N) R.
napp (fapp [X] R) :- !, napp X R.
napp (fapp A) (fapp B) :- map napp A B.
napp (fuva N) (fuva N).

```

13 THE META LANGUAGE

```

typeabbrev subst list (option assignment).

kind inctx type -> type.
type abs (tm -> inctx A) -> inctx A.
type val A -> inctx A.

typeabbrev assignment (inctx tm).

kind tm type.
type app list tm -> tm.
type lam (tm -> tm) -> tm.
type con string -> tm.
type uva address -> list tm -> tm.

type (==l) tm -> tm -> subst -> subst -> o.
% congruence
((app L1) ==l (app L2)) S S1 :- fold2 (==l) L1 L2 S S1.
((lam F1) ==l (lam F2)) S S1 :-
  pi x\ copy x x => ((F1 x) ==l (F2 x)) S S1.
((con X) ==l (con X)) S S.
% set variables
((uva N Args) ==l T) S S1 :-
  mem.set? N S F,!, move F Args T1, (T1 ==l T) S S1.
(T ==l (uva N Args)) S S1 :-
  mem.set? N S F,!, move F Args T1, (T ==l T1) S S1.
% flex-flex
((uva M A1) ==l (uva N A2)) S1 S2 :- !,
  pattern-fragment A1, pattern-fragment A2,
  prune! M A1 N A2 S1 S2.
((uva N Args) ==l T) S S1 :- not_occ N S T, pattern-fragment Args,
  bind T Args T1, mem.assign N S T1 S1.
% variable assignment
(T ==l (uva N Args)) S S1 :- not_occ N S T, pattern-fragment Args,
  bind T Args T1, mem.assign N S T1 S1.
(N ==l N) S S :- name N.

```

```

1161
1162 % Note: We suppose the scopes to always be in PF
1163 type prune! address -> list ho.tm -> address ->
1164     list ho.tm -> subst -> subst -> o.
1165 prune! N A N A S S :- !.
1166 prune! M A N A S1 S2 :- !, bind (uva M A) A Ass,
1167     mem.assign N S1 Ass S2.
1168 prune! N A1 N A2 S1 S3 :- !,
1169     std.assert!(len A1 {len A2}) "Not typechecking", !,
1170     mem.new S1 W S2, prune-same-variable W A1 A2 [] Ass,
1171     mem.assign N S2 Ass S3.
1172 prune! N A1 M A2 S1 S4 :- !,
1173     mem.new S1 W S2, prune-diff-variables W A1 A2 Ass1 Ass2,
1174     mem.assign N S2 Ass1 S3,
1175     mem.assign M S3 Ass2 S4.
1176
1177 type prune-same-variable address -> list tm -> list tm ->
1178     list tm -> assignment -> o.
1179 prune-same-variable N [] [] ACC (val (uva N Args)) :-
1180     std.rev ACC Args.
1181 prune-same-variable N [X|XS] [X|YS] ACC (abs F) :-
1182     pi x\ prune-same-variable N XS YS [x|ACC] (F x).
1183 prune-same-variable N [_|XS] [_|YS] ACC (abs F) :-
1184     pi x\ prune-same-variable N XS YS ACC (F x).
1185
1186 type prune-build-ass1 address -> list tm ->
1187     list bool -> assignment -> o.
1188 prune-build-ass1 N Acc [] (val (uva N Args)) :-
1189     std.rev Acc Args.
1190 prune-build-ass1 N Acc [tt|L] (abs T) :-
1191     pi x\ prune-build-ass1 N [x|Acc] L (T x).
1192 prune-build-ass1 N Acc [ff|L] (abs T) :-
1193     pi x\ prune-build-ass1 N Acc L (T x).
1194
1195 type build-order list nat -> list tm -> list tm -> o.
1196 build-order L T R :-
1197     len L Len, list-init Len z
1198     (p\r\ sigma Index Elt\ index L p Index, nth Index T r) R.
1199
1200 type prune-build-ass2 address -> list tm -> list bool ->
1201     list nat -> assignment -> o.
1202 prune-build-ass2 N Acc [] Pos (val (uva N Args)) :-
1203     std.rev Acc Acc', build-order Pos Acc' Args.
1204 prune-build-ass2 N Acc [tt|L] Pos (abs T) :-
1205     pi x\ prune-build-ass2 N [x|Acc] L Pos (T x).
1206 prune-build-ass2 N Acc [ff|L] Pos (abs T) :-
1207     pi x\ prune-build-ass2 N Acc L Pos (T x).
1208
1209 type keep list A -> A -> bool -> o.
1210 keep L A tt :- mem L A, !.
1211 keep _ _ ff.
1212
1213 type prune-diff-variables address -> list tm -> list tm ->
1214     assignment -> assignment -> o.
1215 prune-diff-variables N Args1 Args2 Ass1 Ass2 :-
1216     std.map Args1 (keep Args2) Bits1,
1217     prune-build-ass1 N [] Bits1 Ass1,
1218
1219     std.map Args2 (keep Args1) Bits2,
1220     std.filter Args1 (mem Args2) ToKeep1,
1221     std.filter Args2 (mem Args1) ToKeep2,
1222     std.map ToKeep2 (index ToKeep1) Pos,
1223     prune-build-ass2 N [] Bits2 Pos Ass2.
1224
1225 type move assignment -> list tm -> tm -> o.
1226 move (abs Bo) [H|L] R :- move (Bo H) L R.
1227 move (val A) [] A :- !.
1228 move (val (uva N A)) L (uva N X) :- std.append A L X.
1229
1230 type beta tm -> list tm -> tm -> o.
1231 beta A [] A.
1232 beta (lam Bo) [H | L] R :- beta (Bo H) L R.
1233 beta (app A) L (app X) :- std.append A L X.
1234 beta (uva N A) L (uva N A') :- std.append A L A'.
1235 beta (con H) L (app [con H | L]).
1236
1237 type not_occ_aux address -> subst -> tm -> o.
1238 not_occ_aux N S (uva M _) :- mem.unset? M S, not (N = M).
1239 not_occ_aux N S (uva M Args) :- mem.set? M S F,
1240     move F Args T, not_occ_aux N S T.
1241 not_occ_aux N S (app L) :- forall1 (not_occ_aux N S) L.
1242 not_occ_aux N S (lam F) :- pi x\ not_occ_aux N S (F x).
1243 not_occ_aux _ _ (con _).
1244 not_occ_aux _ _ X :- name X.
1245
1246 type not_occ address -> subst -> tm -> o.
1247 not_occ N _ (uva N _).
1248 not_occ N S (uva M Args) :- mem.set? M S F,
1249     move F Args T, not_occ N S T.
1250 not_occ N S (uva M Args) :- mem.unset? M S,
1251     std.forall Args (not_occ_aux N S).
1252 not_occ _ _ (con _).
1253 not_occ N S (app L) :- not_occ_aux N S (app L).
1254 not_occ N S (lam L) :- pi x\ not_occ N S (L x).
1255 not_occ _ _ X :- name X.
1256
1257 type copy tm -> tm -> o.
1258 copy (app L) (app L') :- forall2 copy L L'.
1259 copy (lam T) (lam T') :- pi x\ copy x x => copy (T x) (T' x).
1260 copy (uva N L) (uva N L') :- forall2 copy L L'.
1261 copy (con C) (con C).
1262 copy N N :- name N.
1263
1264 type bind tm -> list tm -> assignment -> o.
1265 bind T [] (val T') :- copy T T'.
1266 bind T [X | TL] (abs T') :- pi x\ copy X x => bind T TL (T' x).
1267
1268 type deref subst -> tm -> tm -> o.
1269 deref S X _ :- (var S; var X), halt "flex deref".
1270 deref H (uva N L) X :- mem.set? N H T,
1271     move T L X', !, deref H X' X.
1272 deref H (app L) (app L1) :- forall2 (deref H) L L1.
1273 deref _ (con X) (con X).
1274 deref H (uva X L) (uva X L1) :- mem.unset? X H,
1275     forall2 (deref H) L L1.
1276

```

```

1277   deref H (lam F) (lam G)   :- pi x\ deref H (F x) (G x).
1278   deref _ N      N          :- name N.
1279

```

```

1280   type deref-assmt subst -> assignment -> assignment -> o.
1281   deref-assmt S (abs T) (abs R) :- pi x\ deref-assmt S (T x) (R x).
1282   deref-assmt S (val T) (val R) :- deref S T R.
1283

```

14 THE COMPILER

```

1286   type use-binder fo.fm -> fo.fm -> o.
1287   use-binder N N.
1288   use-binder N (fo.fapp L) :- exists (use-binder N) L.
1289   use-binder N (fo.flam B) :- pi x\ use-binder N (B x).
1290
1291   type maybe-eta fo.fm -> fo.fm -> list fo.fm -> o.
1292   maybe-eta N (fo.fapp[fo.fuva _|Args]) _ :- !,
1293     exists (x\ maybe-eta-of [] N x) Args, !.
1294   maybe-eta N (fo.flam B) L :- !, pi x\ maybe-eta N (B x) [x | L].
1295   maybe-eta _ (fo.fapp [fo.fcon _|Args]) L :-
1296     split-last-n {len L} Args First Last,
1297     forall1 (x\ forall1 (y\ not (use-binder x y)) First) L,
1298     forall2 (maybe-eta-of []) {std.rev L} Last.
1299

```

```

1300   type maybe-eta-of list fo.fm -> fo.fm -> fo.fm -> o.
1301   maybe-eta-of _ N N :- !.
1302   maybe-eta-of L N (fo.fapp[fo.fuva _|Args]) :- !,
1303     % Head is flex -> if N is in Args, we can have eta
1304     forall1 (x\ exists (maybe-eta-of [] x) Args) [N|L].
1305   maybe-eta-of L N (fo.flam B) :- !,
1306     pi x\ maybe-eta-of [x | L] N (B x).
1307   maybe-eta-of L N (fo.fapp [N|Args]) :-
1308     % Head is rigid -> to be an eta we consider also L
1309     last-n {len L} Args R,
1310     forall2 (maybe-eta-of []) R {std.rev L}.
1311

```

```

1312   type locale ho.tm -> o.
1313   type get-scope-aux ho.tm -> list ho.tm -> o.
1314   get-scope-aux (ho.con _) [].
1315   get-scope-aux (ho.uva _ L) L1 :-
1316     forall2 get-scope-aux L R,
1317     flatten R L1.
1318   get-scope-aux (ho.lam B) L1 :-
1319     pi x\ locale x => get-scope-aux (B x) L1.
1320   get-scope-aux (ho.app L) L1 :-
1321     forall2 get-scope-aux L R,
1322     flatten R L1.
1323   get-scope-aux X [X] :- name X, not (locale X).
1324   get-scope-aux X [] :- name X, (locale X).
1325

```

```

1326   type get-scope ho.tm -> list ho.tm -> o.
1327   get-scope T Scope :- names N,
1328     get-scope-aux T ScopeDuplicata,
1329     std.filter N (mem ScopeDuplicata) Scope.
1330

```

```

1331   type close-links (ho.tm -> links) -> links -> o.
1332   close-links (_[]) [].
1333   close-links (v\[L|XS v]) [L|YS] :- !, close-links XS YS.
1334

```

```

1335   close-links (v\[L v]|XS v]) [ho.abs L|YS] :- !,
1336   close-links XS YS.
1337

```

```

1338   type comp-lam (fo.fm -> fo.fm) -> (ho.tm -> ho.tm) ->
1339   mappings -> mappings -> links -> links -> ho.subst ->
1340   ho.subst -> o.
1341   comp-lam F F1 Mappings1 Mappings2 L L2 S S1 :-
1342     pi x y\ (pi M L S\ comp x y M M L L S S) =>
1343     comp (F x) (F1 y) Mappings1 Mappings2 L (L1 y) S S1,
1344     close-links L1 L2.
1345

```

```

1346   type comp fo.fm -> ho.tm -> mappings -> mappings -> links ->
1347   links -> ho.subst -> ho.subst -> o.
1348   comp (fo.fcon X) (ho.con X) Map1 Map1 L1 L1 S1 S1.
1349   comp (fo.flam F) (ho.uva E0 Scope) Map1 Map2 L1 L3 S1 S3 :-
1350     (pi x\ maybe-eta x (F x) [x]), !,
1351     mem.new S1 E0 S2,
1352     comp-lam F F1 Map1 Map2 L1 L2 S2 S3,
1353     (pi x\ locale x => get-scope (F1 x) Scope),
1354     NewLink = ho.val(link-eta (ho.uva E0 Scope) (ho.lam F1)),
1355     L3 = [NewLink|L2].
1356

```

```

1356   comp (fo.flam F) (ho.lam F1) Map1 Map2 L1 L2 S1 S2 :-
1357     comp-lam F F1 Map1 Map2 L1 L2 S1 S2.
1358   comp (fo.fuva N) (ho.uva M []) Map1 Map2 L L S S1 :-
1359     alloc Map1 Map2 (fv N) (hv M (arity z)) S S1.
1360   comp (fo.fapp [fo.fuva N|Ag]) (ho.uva M Scope) Map1 Map2 L1 L1 S1 S2 :-
1361     pattern-fragment Ag, !,
1362     fold6 comp Ag Scope Map1 Map1 L1 L1 S1 S1,
1363     len Ag Arity,
1364     alloc Map1 Map2 (fv N) (hv M (arity Arity)) S1 S2.
1365   comp (fo.fapp [fo.fuva N|Ag]) (ho.uva M' Scope) Map1 Map3 L1 L3 S1 S4 :-
1366     pattern-fragment-prefix Ag PF NPF,
1367     fold6 comp PF PF1 Map1 Map1 L1 L1 S1 S1,
1368     fold6 comp NPF NPF1 Map1 Map2 L1 L2 S1 S2,
1369     len PF Arity,
1370     alloc Map2 Map3 (fv N) (hv M (arity Arity)) S2 S3,
1371     get-scope (ho.app {std.append PF1 NPF1}) Scope,
1372     mem.new S3 M' S4,
1373     L3 = [@val-link-beta (ho.uva M' Scope) (ho.app[ho.uva M PF1|NPF1])],
1374     comp (fo.fapp A) (ho.app A1) Map1 Map2 L1 L2 S1 S2 :-
1375     fold6 comp A A1 Map1 Map2 L1 L2 S1 S2.
1376

```

```

1377   type make-eta-link-aux nat -> address -> address ->
1378   list ho.tm -> links -> ho.subst -> ho.subst -> o.
1379   make-eta-link-aux z Ad1 Ad2 Scope1 L H1 H1 :-
1380     std.rev Scope1 Scope, eta-expand (ho.uva Ad2 Scope) @one T1,
1381     L = [@val-link-eta (ho.uva Ad1 Scope) T1].
1382   make-eta-link-aux (s N) Ad1 Ad2 Scope1 L H1 H3 :-
1383     std.rev Scope1 Scope, mem.new H1 Ad H2,
1384     eta-expand (ho.uva Ad Scope) @one T2,
1385     (pi x\ make-eta-link-aux N Ad Ad2 [x|Scope1] (L1 x) H2 H3),
1386     close-links L1 L2,
1387     L = [@val-link-eta (ho.uva Ad1 Scope) T2 | L2].
1388

```

```

1389   type make-eta-link nat -> nat -> address -> address ->
1390   list ho.tm -> links -> ho.subst -> ho.subst -> o.
1391   make-eta-link (s N) z Ad1 Ad2 Vars L H H1 :-
1392

```



```

1393   make-eta-link-aux N Ad2 Ad1 Vars L H H1.
1394   make-eta-link z (s N) Ad1 Ad2 Vars L H H1 :-
1395     make-eta-link-aux N Ad1 Ad2 Vars L H H1.
1396   make-eta-link (s N) (s M) Ad1 Ad2 Vars Links H H1 :-
1397     (pi x\ make-eta-link N M Ad1 Ad2 [x|Vars] (L x) H H1),
1398     close-links L Links.
1399
1400   type deduplicate-mappings mappings -> mappings ->
1401     ho.subst -> ho.subst -> links -> links -> o.
1402   deduplicate-mappings [] [] H H L L.
1403   deduplicate-mappings [(mapping (fv 0) (hv M (arity LenM)) as X1) |
1404     take-list Map1 (mapping (fv 0) (hv M' (arity LenM')) _) , !,
1405     make-eta-link LenM LenM' M M' [] New H1 H2,
1406     std.append New L1 L2,
1407     deduplicate-mappings Map1 Map2 H2 H3 L2 L3.
1408   deduplicate-mappings [A|As] [A|Bs] H1 H2 L1 L2 :-
1409     deduplicate-mappings As Bs H1 H2 L1 L2, !.

```

15 THE PROGRESS FUNCTION

```

1413   macro @one :- s z.
1414
1415   type contract-rigid list ho.tm -> ho.tm -> ho.tm -> o.
1416   contract-rigid L (ho.lam F) T :-
1417     pi x\ contract-rigid [x|L] (F x) T. % also checks H Prefix does not
1418   contract-rigid L (ho.app [H|Args]) T :-
1419     std.rev L LRev, std.appendR Prefix LRev Args,
1420     if (Prefix = []) (T = H) (T = ho.app [H|Prefix]).
1421
1422   type progress-eta-link ho.tm -> ho.tm -> ho.subst -> ho.subst -> links -> links -> o.
1423   progress-eta-link (ho.app _ as T) (ho.lam x\ _ as T1) H H1 [] :- !, not (T1 = ho.uva _ _), !, fail.
1424     ({eta-expand T @one} ==1 T1) H H1.
1425   progress-eta-link (ho.con _ as T) (ho.lam x\ _ as T1) H H1 [] :- !, not (T1 = ho.uva _ _), !, fail.
1426     ({eta-expand T @one} ==1 T1) H H1.
1427   progress-eta-link (ho.lam _ as T) T1 H H1 [] :- !,
1428     (T ==1 T1) H H1.
1429   progress-eta-link (ho.uva _ _ as X) T H H1 [] :-
1430     contract-rigid [] T T1, !, (X ==1 T1) H H1.
1431   progress-eta-link (ho.uva Ad _ as T1) T2 H H [eval-link-eta T1 T2] :- !,
1432     if (ho.not_occ Ad H T2) true fail.
1433
1434   type is-in-pf ho.tm -> o.
1435   is-in-pf (ho.app [ho.uva _ _ | _]) :- !, fail.
1436   is-in-pf (ho.lam B) :- !, pi x\ is-in-pf (B x).
1437   is-in-pf (ho.con _) .
1438   is-in-pf (ho.app L) :- forall1 is-in-pf L.
1439   is-in-pf N :- name N.
1440   is-in-pf (ho.uva _ L) :- pattern-fragment L.
1441
1442   type arity ho.tm -> nat -> o.
1443   arity (ho.con _) z.
1444   arity (ho.app L) A :- len L A.
1445
1446   type occur-check-err ho.tm -> ho.tm -> ho.subst -> o.
1447   occur-check-err (ho.con _) _ _ :- !.
1448   occur-check-err (ho.app _) _ _ :- !.
1449   occur-check-err (ho.lam _) _ _ :- !.

```

```

1451   occur-check-err (ho.uva Ad _) T S :-
1452     not (ho.not_occ Ad S T).
1453
1454   type progress-beta-link-aux ho.tm -> ho.tm ->
1455     ho.subst -> ho.subst -> links -> o.
1456   progress-beta-link-aux T1 T2 S1 S2 [] :- is-in-pf T2, !,
1457     (T1 ==1 T2) S1 S2.
1458   progress-beta-link-aux T1 T2 S S [eval-link-beta T1 T2] :- !.
1459
1460   type progress-beta-link ho.tm -> ho.tm -> ho.subst ->
1461     ho.subst -> links -> o.
1462   progress-beta-link T (ho.app [ho.uva V Scope | L] as T2) S S2 [eval-link-
1463     arity T Arity, len L ArgsNb, ArgsNb > n Arity, !,
1464     minus ArgsNb Arity Diff, mem.new S V1 S1,
1465     eta-expand (ho.uva V1 Scope) Diff T1,
1466     ((ho.uva V Scope) ==1 T1) S1 S2.
1467
1468   progress-beta-link (ho.uva _ _ as T) (ho.app [ho.uva Ad1 Scope1 | L1] as
1469     std.append Scope1 L1 Scope1L,
1470     pattern-fragment-prefix Scope1L Scope2 L2,
1471     not (Scope1 = Scope2), !,
1472     mem.new S1 Ad2 S2,
1473     len Scope1 Scope1Len,
1474     len Scope2 Scope2Len,
1475     make-eta-link Scope1Len Scope2Len Ad1 Ad2 [] LinkEta S2 S3,
1476     if (L2 = []) (NewLinks = LinkEta, T2 = ho.uva Ad2 Scope2)
1477     (T2 = ho.app [ho.uva Ad2 Scope2 | L2],
1478     NewLinks = [eval-link-beta T T2 | LinkEta]).
1479
1480   progress-beta-link T1 (ho.app [ho.uva _ _ | _] as T2) _ _ _ :-
1481     progress-beta-link (ho.uva _ _ as T) (ho.app [ho.uva _ _ | _] as T2) S1
1482     occur-check-err T T2 S1, !, fail.
1483
1484   progress-beta-link T1 (ho.app [ho.uva _ _ | _] as T2) H H [eval-link-beta
1485     progress-beta-link T1 (ho.app [Hd | T1]) S1 S2 B :-
1486     ho.lam beta Hd T1 T3,
1487     progress-beta-link-aux T1 T3 S1 S2 B.
1488
1489   type solve-link-abs link -> links -> ho.subst -> ho.subst -> o.
1490   solve-link-abs (ho.abs X) R H H1 :-
1491     pi x\ ho.copy x x => (pi S\ ho.deref S x x) =>
1492       solve-link-abs (X x) (R' x) H H1,
1493       close-links R' R.
1494
1495   solve-link-abs (@eval-link-eta A B) NewLinks S S1 :- !,
1496     progress-eta-link A B S S1 NewLinks.
1497
1498   solve-link-abs (@eval-link-beta A B) NewLinks S S1 :- !,
1499     progress-beta-link A B S S1 NewLinks.
1500
1501   type take-link link -> links -> link -> links -> o.
1502   take-link A [B|XS] B XS :- link-abs-same-lhs A B, !.
1503   take-link A [L|XS] B [L|YS] :- take-link A XS B YS.

```

```

1509 type link-abs-same-lhs link -> link -> o.
1510 link-abs-same-lhs (ho.abs F) B :-
1511   pi x\ link-abs-same-lhs (F x) B.
1512 link-abs-same-lhs A (ho.abs G) :-
1513   pi x\ link-abs-same-lhs A (G x).
1514 link-abs-same-lhs (@val-link-eta (ho.uva N _) _) (@val-link-eta (ho.uva N y) tm) => fm _ x y => tm->fm L (B1 x) (B2 y).
1515
1516 type same-link-eta link -> link -> ho.subst -> ho.subst -> o.
1517 same-link-eta (ho.abs F) B H H1 :- !, pi x\ same-link-eta (F x) B H H1 => fm L (ho.uva VM TL) T :- mem L (mapping (fv V0) (hv VM _)),
1518 same-link-eta A (ho.abs G) H H1 :- !, pi x\ same-link-eta A (G x) H H1 => fm L (ho.uva VM TL) T :- mem L (mapping (fv V0) (hv VM _)),
1519 same-link-eta (@val-link-eta (ho.uva N S1) A)
1520   (@val-link-eta (ho.uva N S2) B) H H1 :-
1521   std.map2 S1 S2 (x\y\r\ r = ho.copy x y) Perm,
1522   Perm => ho.copy A A',
1523   (A' ==l B) H H1.
1524
1525 type solve-links links -> links -> ho.subst -> ho.subst -> o.
1526 solve-links [] [] X X.
1527 solve-links [A|L1] [A|L3] S S2 :- take-link A L1 B L2, !,
1528   same-link-eta A B S S1,
1529   solve-links L2 L3 S1 S2.
1530 solve-links [L0|L1] L3 S S2 :- deref-link S L0 L,
1531   solve-link-abs L R S S1, !,
1532   solve-links L1 L2 S1 S2, std.append R L2 L3.
1533
1534 16 THE DECOMPILER
1535
1536 type abs->lam ho.assignment -> ho.tm -> o.
1537 abs->lam (ho.abs T) (ho.lam R) :- !, pi x\ abs->lam (T x) (R x).
1538 abs->lam (ho.val A) A.
1539
1540 type commit-links-aux link -> ho.subst -> ho.subst -> o.
1541 commit-links-aux (@val-link-eta T1 T2) H1 H2 :-
1542   ho.deref H1 T1 T1', ho.deref H1 T2 T2',
1543   (T1' ==l T2') H1 H2.
1544 commit-links-aux (@val-link-beta T1 T2) H1 H2 :-
1545   ho.deref H1 T1 T1', ho.deref H1 T2 T2',
1546   (T1' ==l T2') H1 H2.
1547 commit-links-aux (ho.abs B) H H1 :-
1548   pi x\ commit-links-aux (B x) H H1.
1549
1550 type commit-links links -> links -> ho.subst -> ho.subst -> o.
1551 commit-links [] [] H H.
1552 commit-links [Abs | Links] L H H2 :-
1553   commit-links-aux Abs H H1, !, commit-links Links L H1 H2.
1554
1555 type decomp-subst mappings -> mappings -> ho.subst ->
1556   fo.subst -> fo.subst -> o.
1557 decomp-subst _ [A|_] _ _ _ :- fail.
1558 decomp-subst _ [] _ F F.
1559 decomp-subst Map [mapping (fv V0) (hv VM _)]T1] H F F2 :-
1560   mem.set? VM H T, !,
1561   ho.deref-assmt H T TTT,
1562   abs->lam TTT T', tm->fm Map T' T1,
1563   fo.eta-contract T1 T2, mem.assign V0 F T2 F1,
1564   decomp-subst Map T1 H F1 F2.
1565 decomp-subst Map [mapping _ (hv VM _)]T1] H F F2 :-
1566   mem.unset? VM H, decomp-subst Map T1 H F F2.
1567
1568 type tm->fm mappings -> ho.tm -> fo.fm -> o.
1569 tm->fm _ (ho.con C) (fo.fcon C).
1570 tm->fm L (ho.lam B1) (fo.flam B2) :-
1571   tm->fm L (ho.app L1) T :- forall2 (tm->fm L) L1 [Hd|T1],
1572   fo.mk-app Hd T1 T.
1573 tm->fm L (ho.app L1) T :- forall2 (tm->fm L) L1 [Hd|T1],
1574   fo.mk-app Hd T1 T.
1575 tm->fm L (ho.uva VM TL) T :- mem L (mapping (fv V0) (hv VM _)),
1576   forall2 (tm->fm L) TL T1, fo.mk-app (fo.fuva V0) T1 T.
1577
1578 type add-new-mappings-aux ho.subst -> list ho.tm -> mappings ->
1579   mappings -> fo.subst -> fo.subst -> o.
1580 add-new-mappings-aux _ [] _ [] S S.
1581 add-new-mappings-aux H [T|Ts] L L2 S S2 :-
1582   add-new-mappings H T L L1 S S1,
1583   add-new-mappings-aux H Ts L1 L2 S1 S2.
1584
1585 type add-new-mappings ho.subst -> ho.tm -> mappings ->
1586   mappings -> fo.subst -> fo.subst -> o.
1587 add-new-mappings _ (ho.uva N _) Map [] F1 F1 :-
1588   mem Map (mapping _ (hv N _)), !.
1589 add-new-mappings H (ho.uva N L) Map [Map1 | MapL] F1 F3 :-
1590   mem.new F1 M F2,
1591   len L Arity, Map1 = mapping (fv M) (hv N (arity Arity)),
1592   add-new-mappings H (ho.app L) [Map1 | MapL] MapL F2 F3.
1593 add-new-mappings H (ho.lam B) Map NewMap F1 F2 :-
1594   pi x\ add-new-mappings H (B x) Map NewMap F1 F2.
1595 add-new-mappings H (ho.app L) Map NewMap F1 F3 :-
1596   add-new-mappings-aux H L Map NewMap F1 F3.
1597 add-new-mappings _ (ho.con _) _ [] F F :- !.
1598 add-new-mappings _ N _ [] F F :- name N.
1599
1600 type complete-mapping-under-ass ho.subst -> ho.assignment ->
1601   mappings -> mappings -> fo.subst -> fo.subst -> o.
1602 complete-mapping-under-ass H (ho.val Val) Map1 Map2 F1 F2 :-
1603   add-new-mappings H Val Map1 Map2 F1 F2.
1604 complete-mapping-under-ass H (ho.abs Abs) Map1 Map2 F1 F2 :-
1605   pi x\ complete-mapping-under-ass H (Abs x) Map1 Map2 F1 F2.
1606
1607 type complete-mapping ho.subst -> ho.subst ->
1608   mappings -> mappings -> fo.subst -> fo.subst -> o.
1609 complete-mapping _ [] L L F F.
1610 complete-mapping H [none | T1] L1 L2 F1 F2 :-
1611   complete-mapping H T1 L1 L2 F1 F2.
1612 complete-mapping H [some T0 | T1] L1 L3 F1 F3 :-
1613   ho.deref-assmt H T0 T,
1614   complete-mapping-under-ass H T L1 L2 F1 F2,
1615   std.append L1 L2 LAll,
1616   complete-mapping H T1 LAll L3 F2 F3.
1617
1618 type decompile mappings -> links -> ho.subst ->
1619   fo.subst -> fo.subst -> o.
1620 decompile Map1 L H0 F0 F02 :-
1621   commit-links L L1_ H0 H01, !,
1622   complete-mapping H01 H01 Map1 Map2 F0 F01,
1623   decomp-subst Map2 Map2 H01 F01 F02.
1624

```

17 AUXILIARY FUNCTIONS

```

type fold4 (A -> A1 -> B -> B -> C -> C -> o) -> list A ->
  list A1 -> B -> B -> C -> C -> o.
fold4 _ [] [] A A B B.
fold4 F [X|XS] [Y|YS] A A1 B B1 :- F X Y A A0 B B0,
  fold4 F XS YS A0 A1 B0 B1.

type len list A -> nat -> o.
len [] z.
len [_|L] (s X) :- len L X.

```