

HO unification from object language to meta language

Enrico Tassi

enrico.tassi@inria.fr

Université Côte d'Azur, Inria

France

Davide Fissore

davide.fissore@inria.fr

Université Côte d'Azur, Inria

France

ABSTRACT

Specifying and implementing a logic from scratch requires significant effort. Logical Frameworks and Higher Order Logic Programming Languages provide dedicated, high-level Meta Languages (ML) to facilitate this task in two key ways: 1) variable binding and substitution are simplified when ML binders represent object logic ones; 2) proof construction, and even proof search, is greatly simplified by leveraging the unification procedure provided by the ML. Notable examples of ML are Elf [13], Twelf [14], λ Prolog [10] and Isabelle [20] which have been utilized to implement various formal systems such as First Order Logic [5], Set Theory [12], Higher Order Logic [11], and even the Calculus of Constuctions [4].

The object logic we are interested in is Coq's [18] Dependent Type Theory (DTT), for which we aim to implement a unification procedure \approx_o using the ML Elpi [3], a dialect of λ Prolog. Elpi's equational theory comprises $\eta\beta$ equivalence and comes equipped with a higher order unification procedure \approx_λ restricted to the pattern fragment [9]. We want \approx_o to be as powerful as \approx_λ but on the object logic DTT. Elpi also comes with an encoding for DTT that works well for meta-programming [17, 16, 7, 6]. Unfortunately this encoding, which we refer to as \mathcal{F}_o , “underuses” \approx_λ by restricting it to first-order unification problems only. To address this issue, we propose a better-behaved encoding, \mathcal{H}_o , demonstrate how to map unification problems in \mathcal{F}_o to related problems in \mathcal{H}_o , and illustrate how to map back the unifiers found by \approx_λ , effectively implementing \approx_o on top of \approx_λ for the encoding \mathcal{F}_o .

We apply this technique to the implementation of a type-class [19] solver for Coq [18]. Type-class solvers are proof search procedures based on unification that back-chain designated lemmas, providing essential automation to widely used Coq libraries such as Stdpp/Iris [8] and TLC [1]. These two libraries constitute our test bed.

KEYWORDS

Logic Programming, Meta-Programming, Higher-Order Unification, Proof Automation

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1 INTRODUCTION

Specifying and implementing a logic from scratch requires significant effort. Logical Frameworks and Higher Order Logic Programming Languages provide dedicated, high-level Meta Languages (ML) to facilitate this task in two key ways: 1) variable binding and substitution are simplified when ML binders represent object logic ones; 2) proof construction, and even proof search, is greatly simplified by leveraging the unification procedure provided by the ML. Notable examples of ML are Elf [13], Twelf [14], λ Prolog [10] and Isabelle [20] which have been utilized to implement various formal systems such as First Order Logic [5], Set Theory [12], Higher Order Logic [11], and even the Calculus of Constuctions [4].

The object logic we are interested in is Coq's [18] Dependent Type Theory (DTT), and we want to code a type-class [19] solver for Coq [18] using the Coq-Elpi [17] meta programming framework. Type-class solvers are unification based proof search procedures that combine a set of designated lemmas in order to providing essential automation to widely used Coq libraries.

As the running example we take the Decide type class, from the Stdpp [8] library. The class identifies predicates equipped with a decision procedure. The following three designated lemmas (called Instances in the type-class jargon) state that: 1) the type `fin n`, of natural numbers smaller than `n` is finite; 2) the predicate `nfact n nf`, linking a natural number `n` to its prime factors `nf`, is decidable; 3) the universal closure of a predicate has a decision procedure if the predicate has and if its domain is finite.

```
Instance fin_fin n : Finite (fin n).          (* r1 *)
Instance nfact_dec n nf : Decision (nfact n nf). (* r2 *)
Instance forall_dec A P : Finite A →          (* r3 *)
  ∀x:A, Decision (P x) → Decision (∀x:A, P x).
```

Under this context of instances a type-class solver is able to prove the following statement automatically by back-chaining.

```
Check _ : Decision (forall y: fin 7, nfact y 3). (g)
```

The encoding of DTT provided by Elpi, that we will discuss at length later in section ?? and ??, is an Higher Order Abstract Syntax (HOAS) datatype `tm` featuring (among others) the following constructors:

```
type lam  tm -> (tm -> tm) -> tm.    % lambda abstraction
type app  list tm -> tm.              % n-ary application
type all  tm -> (tm -> tm) -> tm.    % forall quantifier
type con  string -> tm.               % constants
```

Following standard λ Prolog [10] the concrete syntax to abstract, at the meta level, an expression `e` over a variable `x` is `«x\ e»`, and square brackets denote a list of terms separated by comma. As an example we show the encoding of the Coq term `«∀y:t, nfact y 3»`:

```
all (con"t") y\ app[con"nfact", y, con"3"]
```

We now illustrate the encoding of the three instances above as higher-order logic-programming rules: capital letters denote rule parameters; `:-` separates the rule's head from the premises; `pi w\ p` introduces a fresh nominal constant `w` for the premise `p`.

```
finite (app[con"fin", N]).           (r1)
decision (app [con"nfact", N, NF]). (r2)
decision (all A x\ app[P, x]) :- finite A,           (r3)
  pi w\ decision (app[P, w]).
```

Unfortunately this translation of rule (r3) uses the predicate `P` as a first order term: for the meta language its type is `tm`. If we try to backchain the rule (r3) on the encoding of the goal (g) given below

```
decision (all (app[con"fin", con"7"]) y\
  app[con"nfact", y, con"3"]).
```

we obtain an unsolvable unification problem (p): the two lists of terms have different lengths!

```
app[con"nfact", y, con"3"] = app[P, y]           (p)
```

In this paper we study a more sophisticated encoding of Coq terms allowing us to rephrase the problematic rule (r3) as follows:

```
decision (all A x\ Pm x) :- link Pm P A, finite A,   (r3a)
  pi x\ decision (app[P, x]).
```

Since `Pm` is an higher-order unification variable of type `tm -> tm`, with `x` in its scope, the unification problem (p') admits one solution:

```
app[con"nfact", y, con"3"] = Pm y           (p')
Pm = x\ app[con"nfact", x, con"3"]           % assignment for Pm
A = app[con"fin", con"7"]                     % assignment for A
```

After unifying the head of rule (r3a) with the goal, Elpi runs the premise `«link Pm A P»` that is in charge of bringing the assignment for `Pm` back to the domain `tm` of Coq terms:

```
P = lam A a\ app[con"nfact", a, con"3"]
```

This simple example is sufficient to show that the encoding we seek is not trivial and does not only concern the head of rules, but the entire sequence of unification problems that constitute the execution of a logic program. In fact the solution for `P` above generates a (Coq) β -redex in the second premise (the predicate under the `pi w\`):

```
decision (app[lam A (a\ app[con"nfact", a, con"3"]), w])
```

In turn this redex prevents the rule (r2) to backchain properly since the following unification problem has no solution:

```
app[lam A (a\ app[con"nfact", a, con"3"]), x] =
app[con"nfact", N, NF]
```

The root cause of the problems we sketched in the running example is that the unification procedure \approx_λ of the meta language is not aware of the equational theory of the object logic, even if both theories include $\eta\beta$ -conversion and admit most general unifiers for unification problems in the pattern fragment \mathcal{L}_λ [9].

Contributions. In this paper we discuss alternative encodings of Coq in Elpi (Section 2), then we identify a minimal language \mathcal{F}_0 in which the problems sketched here can be fully described. We then detail an encoding `comp` from \mathcal{F}_0 to \mathcal{H}_0 (the language of the

meta language) and a decoding `decomp` to relate the unifiers bla bla.. TODO citare Teyjus.

2 PROBLEM STATEMENT

The equational theory of Coq's Dependent Type Theory is very rich. In addition to the usual $\eta\beta$ -equivalence for functions, terms (hence types) are compared up to proposition unfolding and fix-point unrolling. Still, for efficiency and predictability reasons, most form of automatic proof search employ a unification procedure that captures a simpler one, just $\eta\beta$, and that solves higher-order problems restricted to the pattern fragment \mathcal{L}_λ [9]. We call this unification procedure \approx_o .

The equational theory of the meta language Elpi that we want to use to implement a form of proof automation is strikingly similar, since it it comprises $\eta\beta$ (for the meta language functions), and the unification procedure \approx_λ solves higher-order problems in \mathcal{L}_λ .

In spite of the similarity the link between \approx_λ and \approx_o is not trivial, since the abstraction and application term constructors the two unification procedures deal with are different. For example

| | | |
|---|-------------------|-----------------|
| $x \backslash f \ x$ | \approx_λ | f |
| $\text{lam } A \ x \backslash \text{app}[\text{con}"f", x]$ | \approx_o | $\text{con}"f"$ |
| $\text{lam } A \ x \backslash \text{app}[\text{con}"f", x]$ | \neq_λ | $\text{con}"f"$ |
| $P \ x$ | \approx_λ | x |
| $\text{app}[P, x]$ | \approx_o | x |
| $\text{app}[P, x]$ | \neq_λ | x |

One could ignore this similarity, and “just” describe the object language unification procedure in the meta language, that is crafting a unif predicate to be used as follows in rule (r3):

```
decision X :- unif X (all A x\ app[P, x]), finite A,
  pi x\ decision (app[P, x]).
```

This choice would underuse the logic programming engine provided by the metalanguage since by removing any datum from the head of rules indexing degenerates. Moreover the unification procedure built in the meta language is likely to be faster than one implemented in it, especially if the meta language is interpreted as Elpi is.

To state precisely the problem we solve we need a \mathcal{F}_0 representation of DTT terms and a \mathcal{H}_0 one. We call $=_o$ the equality over ground terms in \mathcal{F}_0 , $=_\lambda$ the equality over ground terms in \mathcal{H}_0 , \approx_o the unification procedure we want to implement and \approx_λ the one provided by the meta language. TODO extend $=_o$ and $=_\lambda$ with reflexivity on uvars.

We write $t_1 \approx_\lambda t_2 \mapsto \sigma$ when t_1 and t_2 unify with substitution σ , we write σt for the application of the substitution to t , $\sigma \subseteq \sigma'$ when σ is more general than σ' , and we assume that the unification of our meta language is correct:

$$t_i \in \mathcal{L}_\lambda \Rightarrow t_1 \approx_\lambda t_2 \mapsto \rho \Rightarrow \rho t_1 =_\lambda \rho t_2 \quad (1)$$

$$t_i \in \mathcal{L}_\lambda \Rightarrow \rho t_1 =_\lambda \rho t_2 \Rightarrow \exists \rho', t_1 \approx_\lambda t_2 \mapsto \rho' \wedge \rho' \subseteq \rho \quad (2)$$

We illustrate a compilation $\langle s \rangle \mapsto (t, m, l)$ that maps a term s in \mathcal{F}_0 to a term t in \mathcal{H}_0 , a variable mapping m and list of links l . The variable map connects unification variables in \mathcal{H}_0 with variables in \mathcal{F}_0 and is used to “decompile” the assignment, $\langle \sigma, m, l \rangle^{-1} \mapsto \rho$. Links represent problematic sub-terms which are linked to the unification variable that stands in their place in the compiled term. These links are checked for or progress XXX improve....

We represent a logic program *run* in \mathcal{F}_0 as a list *steps* p of length N . Each made of a unification problem between terms S_{p_l} and S_{p_r} taken from the set of all terms \mathcal{S} . The composition of these steps starting from the empty substitution ρ_0 produces the final substitution ρ_N .¹ The initial here ρ_0 is the empty substitution

$$\begin{aligned} \text{fstep}(\mathcal{S}, p, \rho) &\mapsto \rho'' \stackrel{\text{def}}{=} \rho S_{p_l} \approx_o \rho S_{p_r} \mapsto \rho' \wedge \rho'' = \rho \cup \rho' \\ \text{frun}(\mathcal{S}, N) &\mapsto \rho_N \stackrel{\text{def}}{=} \bigwedge_{p=1}^N \text{fstep}(\mathcal{S}, p, \rho_{p-1}) \mapsto \rho_p \end{aligned}$$

We simulate each run in \mathcal{F}_0 with a run in \mathcal{H}_0 as follows. Note that σ_0 is the empty substitution.

$$\begin{aligned} \text{hstep}(\mathcal{T}, p, \sigma, \mathbb{L}) &\mapsto \sigma'' \stackrel{\text{def}}{=} \\ &\sigma \mathcal{T}_{p_l} \approx_\lambda \sigma \mathcal{T}_{p_r} \mapsto \sigma' \wedge \text{check}(\mathbb{L}, \sigma \cup \sigma') \mapsto \sigma'' \\ \text{hrun}(\mathcal{S}, N) &\mapsto \rho_N \stackrel{\text{def}}{=} \\ &\mathcal{T} \times \mathbb{M} \times \mathbb{L} = \{(t_j, m_j, l_j) | s_j \in \mathcal{S}, \langle s_j \rangle \mapsto (t_j, m_j, l_j)\} \\ &\bigwedge_{p=1}^N \text{hstep}(\mathcal{T}, p, \sigma_{p-1}, \mathbb{L}) \mapsto \sigma_p \\ &\langle \sigma_N, \mathbb{M}, \mathbb{L} \rangle^{-1} \mapsto \rho_N \end{aligned}$$

Here *hstep* is made of two sub-steps: a call to \approx_λ (on the compiled terms) and a call to check on the set of links. We claim the following:

PROPOSITION 2.1 (SIMULATION). $\forall \mathcal{S}, \forall N$

$$\text{frun}(\mathcal{S}, N) \mapsto \rho_N \Leftrightarrow \text{hrun}(\mathcal{S}, N) \mapsto \rho_N$$

That is, the two executions give the same result. Moreover:

PROPOSITION 2.2 (SIMULATION FIDELITY). *In the context of* *hrun*, *we have that* $\forall p \in 1 \dots N$

$$\text{fstep}(\mathcal{S}, p, \rho_{p-1}) \mapsto \rho_p \Leftrightarrow \text{hstep}(\mathcal{T}, p, \sigma_{p-1}, \mathbb{L}) \mapsto \sigma_p$$

In particular this property guarantees that a *failure* in the \mathcal{F}_0 run is matched by a failure in \mathcal{H}_0 at the same step. We consider this property very important from a practical point of view since it guarantees that the execution traces are strongly related and in turn this enables a user to debug a logic program in \mathcal{F}_0 by looking at its execution trace in \mathcal{H}_0 .

We can define $s_1 \approx_o s_2$ by specializing the code of *hrun* to $\mathcal{S} = \{s_1, s_2\}$ as follows:

$$\begin{aligned} s_1 \approx_o s_2 &\mapsto \rho \stackrel{\text{def}}{=} \\ &\langle s_1 \rangle \mapsto (t_1, m_1, l_1) \wedge \langle s_2 \rangle \mapsto (t_2, m_2, l_2) \\ &t_1 \approx_\lambda t_2 \mapsto \sigma' \wedge \text{check}(\{l_1, l_2\}, \sigma') \mapsto \sigma'' \wedge \\ &\langle \sigma'', \{m_1, m_2\}, \{l_1, l_2\} \rangle^{-1} \mapsto \rho \end{aligned}$$

PROPOSITION 2.3 (PROPERTIES OF \approx_o).

$$s_i \in \mathcal{L}_\lambda \Rightarrow s_1 \approx_o s_2 \mapsto \rho \Rightarrow \rho s_1 =_o \rho s_2 \quad (3)$$

$$s_i \in \mathcal{L}_\lambda \Rightarrow \rho s_1 =_o \rho s_2 \Rightarrow \exists \rho', s_1 \approx_o s_2 \mapsto \rho' \wedge \rho' \subseteq \rho \quad (4)$$

$$\rho s_1 =_o \rho s_2 \Rightarrow \rho' \subseteq \rho \Rightarrow \rho' s_i \in \mathcal{L}_\lambda \Rightarrow \rho' s_1 \approx_o \rho' s_2 \quad (5)$$

Properties 3 and 4 state, respectively, that in \mathcal{L}_λ the implementation of \approx_o is correct, complete and returns the most general unifier.

Property 5 states that \approx_o , hence our compilation scheme, is resilient to unification problems outside \mathcal{L}_λ solved by a third party. We believe this property is of practical interest since we want the user to be able to add heuristics via hand written rules to the ones

¹If the same rule is used multiple time in a run we just consider as many copies as needed of the terms composing the rules, with fresh unification variables each time

obtained by our compilation scheme. A Typical example is the following problem (q) that is outside \mathcal{L}_λ :

$$\begin{aligned} \text{app} [F, \text{con} "a"] &= \text{app} [\text{con} "f", \text{con} "a", \text{con} "a"] \quad (q) \\ F &= \text{lam } x \backslash \text{app} [\text{con} "f", x, x] \quad (h) \end{aligned}$$

Instead of rejecting it our scheme accepts it and guarantees that if (h) is given (after the compilation part of the scheme, as a run time hint) then ...

2.1 The intuition in a nutshell

A term s is compiled in a term t where any “problematic” subterm tp is replaced by a unification variable h and an accessory link, that represent a suspended unification problem $h \approx_\lambda tp$. As a result \approx_λ is well behaved on t , that is it captures $=_o$. We now define “problematic” formally.

Definition 2.4 ($\diamond \eta$). $\diamond \eta = \{t | \exists \rho, \rho t \text{ is an eta expansion}\}$

An example of term t in $\diamond \eta$ is $\lambda x. \lambda y. F y x$ since the substitution $\rho = \{F \mapsto \lambda a. \lambda b. fba\}$ makes $\rho t = \lambda x. \lambda y. fxy$ that is the eta long form of f .

Definition 2.5 ($\diamond \beta$). $\diamond \beta = \{X t_1 \dots t_n | t \notin \mathcal{L}_\lambda\}$.

An example of t in $\diamond \beta$ is Fa for a constant a . Note however that an oracle could provide an assignment $\rho = \{F \mapsto \lambda x. x\}$ that makes the resulting term fall outside of $\diamond \beta$.

Definition 2.6 (Subterm $\mathcal{P}(t)$).

$$\begin{aligned} t &\in \mathcal{P}(t) \\ t &= f t_1 \dots t_n \Rightarrow \mathcal{P}(t_i) \subseteq \mathcal{P}(t) \wedge f \in \mathcal{P}(t) \\ t &= \lambda x. t' \Rightarrow \mathcal{P}(t') \subseteq \mathcal{P}(t) \end{aligned}$$

We write $\mathcal{P}(X) = \bigcup_{t \in X} \mathcal{P}(t)$.

Definition 2.7 (Normal form). Given a set of terms $X \subseteq \mathcal{H}_0$,

$$\text{normal}(X) = \mathcal{P}(X) \cap (\diamond \beta \cup \diamond \eta) = \emptyset$$

We write $\sigma X = \{\sigma t | t \in X\}$.

PROPOSITION 2.8 (NORMAL FORM PRESERVATION). $\forall \mathcal{T}, \forall \mathbb{L}, \forall p, \forall \sigma, \forall \sigma'$

$$\text{normal}(\sigma \mathcal{T}) \wedge \text{hstep}(\mathcal{T}, p, \sigma, \mathbb{L}) \mapsto \sigma' \Rightarrow \text{normal}((\sigma \cup \sigma') \mathcal{T})$$

In particular this guarantees that is we start from normal terms we never introduce eta-long or non-beta-normal terms in σ' .

Note that proposition 2.8 does not hold for \approx_o since decompilation can introduce (actually restore) terms in $\diamond \eta$ or $\diamond \beta$.

3 ALTERNATIVE ENCODINGS AND RELATED WORK

Paper [2] introduces semi-shallow.

Our encoding of DTT may look “semi shallow” since we use the meta-language lambda abstraction but not its application (for the terms of type tm). A fully shallow encoding unfortunately does not fit our use case, although it would make the running example work:

```
finite (fin N).
decision (nfact N NF).
decision (all A x \ P x) :- finite A, pi x \ decision (P x).
```

There are two reasons for dismissing this encoding. The first one is that in DTT it is not always possible to adopt it since the type system of the meta language is too weak to accommodate terms with a variable arity, like the following example:

```
Fixpoint arr T n := if n is S m then T -> arr T m else T.
Definition sum n : arr nat n := ...
Check sum 2 7 8 : nat.
Check sum 3 7 8 9 : nat.
```

The second reason is the encoding for Coq is used for meta programming the system, hence it must accommodate the manipulation of terms that are now known in advance (not even defined in Coq) without using introspection primitives such as Prolog's functor and arg.

In the literature we could find a few related encoding of DTT. TODO In [4] is related and make the discrepancy between the types of ML and DTT visible. In this case one needs 4 application nodes. Moreover the objective is an encoding of terms, proofs, not proof search. Also note the conv predicate, akin to the unif we rule out.

TODO This other paper [15] should also be cited.

None of the encodings above provide a solution to our problem.

4 PRELIMINARIES: \mathcal{F}_0 AND \mathcal{H}_0

In order to reason about unification we provide a description of the \mathcal{F}_0 and \mathcal{H}_0 languages where unification variables are first class terms, i.e. they have a concrete syntax. We keep these languages minimal, for example we omit the all quantifier of DTT we used in the example in Section 1.

```
kind fm type.          kind tm type.
type fapp list fm -> fm. type app list tm -> tm.
type flam (fm -> fm) -> fm. type lam (tm -> tm) -> tm.
type fcon string -> fm.   type con string -> tm.
type fuva nat -> fm.      type uva nat -> list tm -> tm.
```

Figure 1: \mathcal{F}_0 and \mathcal{H}_0 language

In the case of \mathcal{F}_0 unification variables fuva have no explicit scope: Unification variables standing for functions are applied their arguments via the fapp constructor. For example in the statement of the instance forall_dec the term $P\ x$ is represented as fapp[fuva N, x], where N is of type nat and x is a bound variable.

In \mathcal{H}_0 the representation of $P\ x$ is instead uva N [x]. We say that the unification variable uva N L is in \mathcal{L}_λ iff distinct L holds.

```
type distinct list A -> o.
distinct [].
distinct [X|XS] :- name X, not(mem X XS), distinct XS.
```

The name builtin predicate tests if a term is a bound variable.² The compiler ?? needs to support terms outside \mathcal{L}_λ for practical reasons, so we don't assume all out terms are in \mathcal{L}_λ but rather test.

In both languages unification variables are identified by a natural number, which can be seen as a memory address. The memory and its associated operations are described below:

```
typeabbrev memory A (list (option A)).
type set? nat -> memory A -> A -> o.
```

²one could always load name x for every x under a pi and get rid of the name builtin

```
type unset? nat -> memory A -> o.
type assign nat -> memory A -> A -> memory A -> o.
```

If a memory cell is none, then the corresponding unification variable is not set. assign sets an unset cell to the given value.

Since in \mathcal{H}_0 unification variables have a scope, their solution needs to be abstracted over it in order to enable the instantiation of a single solution to different scopes. On the contrary solutions to \mathcal{F}_0 variables are plain terms.

```
typeabbrev fsubst (memory ftm).
kind assmt type.
type abs (tm -> assmt) -> assmt.
type val tm -> assmt.
typeabbrev subst (memory assmt).
```

We call fsubst the memory of \mathcal{F}_0 , while we call subst the one of \mathcal{H}_0 . Both have the invariant that they are not cyclic, TODO explain.

4.1 Equational theory and Unification

here we give the functions/signatures to express the properties 3-5 in the problem statement

Together with the description of the terms of the language, we need some auxiliary functions to perform operations like term equality, unification, dereferencing. There predicates are supposed to be implemented in the OL and the ML following respectively their specification. In the following few paragraphs we give a brief proposition of them.

Term dereferencing: ρs and σt . Since in our encoding we explicitly carry a substitution we need to define the operation that applies it to a term. Its proposition takes care to normalize (flatten) applications, for example it turns (app [app [con "f", con "a"], con "b"]) into (app [con "f", con "a", con "b"]).

dereference variables of the two languages. This is particularly useful to check if two terms in the OL (resp. in the ML) are equal. The constructors representing rigid terms, namely app, lam and con, make the dereferencing procedure to recurse over their subterms. The code below display this behavior for the dereferencing performed by the OL:

```
type fder fsubst -> fm -> fm -> o.
fder S (fapp A) (fapp B) :- map (fder S) A B.
fder S (flam F) (flam G) :-
  pi x\ fder S x x => fder S (F x) (G x).
fder _ (fcon C) (fcon C).
fder S (fuva N) R :- set? N S T, fder S T R.
fder S (fuva N) (fuva N) :- unset? N S.
```

```
type fderef fsubst -> fm -> fm -> o. (ρs)
fderef S T R :- fder S T T', napp T' R.
```

```
type napp fm -> fm -> o.
```

TODO explain napp. We use the cut operator to keep the code compact. It is possible to rewrite the rule for application traversal so that it is mutually exclusive with the first one, but that requires a rather verbose analysis of the head of A.

The corresponding code for \mathcal{H}_0 is similar, we only show the last two rules that differ in a substantial way:


```

465 type (=λ) tm -> tm -> o.
466 app A =λ fapp B :- map (=λ) A B.
467 lam F =λ flam G :- pi x\ x =λ x => F x =λ G x.
468 con C =λ fcon C.
469 uva N A =λ fuva N B :- map (=λ) A B.

```

Figure 2: Equal predicate ML

```

471 type deref subst -> tm -> tm -> o. (σt)
472 deref S (app A) (app B) :- map (deref S) A B.
473 deref S (lam F) (lam G) :-
474   pi x\ deref S x x => deref S (F x) (G x).
475 deref _ (con C) (con C).
476 deref S (uva N A) R :- set? N S F, move F A T, deref S T R.
477 deref S (uva N A) (uva N B) :- unset? N S, map (deref S) A B.
478 type move assignment -> list tm -> tm -> o.
479 move (abs Bo) [H|L] R :- move (Bo H) L R.
480 move (val A) [] A :- !.
481 move (val (uva N A)) L (uva N X) :- std.append A L X.

```

TODO: no need to napp, see the beta section. Note that when the substitution S maps a unification variable N to an assignment F we

Important!!! A different reasoning is to be addressed to the variables of the ML. Firstly, a meta variable cannot appear in the app node as the first element of the list, we will explain why in section 5

```

486 type (=o) ftm -> ftm -> o. (=o)
487 fapp A =o fapp B :- map (=o) A B.
488 flam F =o flam G :- pi x\ x =o x => F x =o G x.
489 fcon C =o fcon C.
490 fuva N =o fuva N.
491 flam F =o T :- (ηl)
492   pi x\ beta T [x] (R x), x =o x => F x =o R x.
493 T =o flam F :- (ηr)
494   pi x\ beta T [x] (R x), x =o x => R x =o F x.
495 fapp [flam X | L] =o T :- beta (flam X) L R, R =o T. (βl)
496 T =o fapp [flam X | L] :- beta (flam X) L R, T =o R. (βr)

```

Term equality: $=_o$ vs. $=_\lambda$. We can test if two terms are equal following the equational theory of the language being considered. In ?? we provide an implementation of the $=_o$ predicate. The first four rules check if the two terms are equal regarding the structure of the current node, that is, two terms are equal if they have same head and if recursively each subterm is two by two equal. Moreover, since the theory of the OL accepts η - and β -equivalence, then we implement the corresponding rules.

The equality relation for the ML, accepts $\eta\beta$ -equivalence between terms of the ML. Recall that $\text{abs } x \backslash f \ x$, is a valid η expansion of the function f and that $\text{lam } x \backslash \text{app}[f, x]$ is not that equivalent to f at meta level. However, since we are interested in using the unification procure of the ML, by eq. (1), we can use the \approx_λ relation to test, when needed if two terms are equal in the ML.

Term unification: \approx_o vs. \approx_λ . The last but not least important relation we should take care of before presenting our full algorithm aiming to unify terms of the OL in the ML and provide the substitution produced in the ML to the OL, is term unification. This procedure is a more powerful version of the equal predicate, since unification checks if two terms can be equal be assigning unification

variables. In our representation, variable assignment (or refinement) is performed by modifying the corresponding substitution mapping. We will not give an implementation of \approx_o , since we are giving an implementation of it using our algorithm, see ??.

```

523 type (≈λ) tm -> tm -> subst -> subst -> o.
524
525
526

```

On the other hand, unification in the ML needs to be defined. In fig. 5, we give an implementation of \approx_λ but that is actually what our meta language provides as a builtin.

This predicate has four arguments, the two terms t_1 and t_2 to unify, the old substitution map ρ_1 , and the new substitution map ρ_2 , with the invariant $\rho_1 \subseteq \rho_2$. The first three rules unify terms with same rigid heads, and call the unification relation on the sub-terms. If t_1 (resp. t_2) is an assigned variables, t_1 is dereferenced to t'_1 (resp. t'_2) and the unification is called between t'_1 and t_2 (resp. t_1 and t'_2). If both terms are unification variables, we test that their arguments are in the pattern fragment, we allocate a new variable w in ρ_1 such that w is the pruning of the arguments of t_1 and t_2 , we assign both t_1 and t_2 to w and return the new mapping ρ_2 containing all the new variable assignment. Finally, if only one of the two terms is an unification variable v , after having verified that v does not occur in the other term t , we bind v to t and return the new substitution mapping.

OLD

A key property needed in unification is being able to verify if two terms are equal wrt a given equational theory. This relation allow to compare terms under a certain substitution mapping, so that any time a variable v is assigned in a subterm, a dereferencing of v is performed. After variable dereferencing, the test for equality is continued on the new-created subterm.

The base equality function over terms can be defined as follows:

The solution we are proposing aim to overcome these unification issues by 1) compiling the terms t and u of the OL into an internal version t' and u' in the ML; 2) unifying t' and u' at the meta level instantiating meta variables; 3) decompiling the meta variable into terms of the OL; 4) assigning the variables of the OL with the decompiled version of their corresponding meta variables. We claim that t and u unify if and only if t' and u' unify and that the substitution in the object language is the same as the one returned by the ML.

In the following section we explain how we deal with term (de)compilation and links between unification variables.

5 BASIC COMPILATION \mathcal{F}_o TO \mathcal{H}_o

In this section we describe a basic compilation scheme that we refine later, in the following sections.

The objective of the compilation is to recognize the higher-order variables available in \mathcal{H}_o when expressed in a first order way in \mathcal{F}_o . The compiler also generates a list of links that are used to bring back the substitution from \mathcal{H}_o to \mathcal{F}_o and allocates in the memory a cell for each variable.

```

527 kind link type.
528 type link nat -> nat -> subst. % link Fo Ho Arity
529 typeabbrev links list link.
530 type comp fm -> tm -> links -> links -> subst -> subst -> o.
531 comp (fcon X) (con X) L L S S.

```

same
or
⊇
or
⊆

```

581 comp (flam F) (lam G) K L R S :- pi x y\
582   (pi A S\ comp x y L L S S) => comp (F x) (G y) K L R S.
583 comp (fuva M) (uva N []) K [link M N z|K] R S :- new R N S.
584 comp (fapp[fuva M|A]) (uva N B) K L R S :- distinct A, !,
585   fold4 comp A B K K R R,
586   new R N S, len A Arity,
587   L = [link N M Arity | K].
588 comp (fapp A) (app B) K L R S :- fold4 comp A B K L R S.
589

```

Note that link carries the arity (number of expected arguments) of the variable.

```

591 type solve-links links -> links -> subst -> subst -> o.
592 solve-links L L S S.

```

Then decomp

```

595 type decompile links -> subst -> fsubst -> o.
596 decompile L S O :-
597   map (\r\l = none) S O1, % allocate empty fsubst
598   (pi N X\ knil N X :- mem L (link X N _) ; N = X) =>
599   decomp L S L O1 O.
600 type knil nat -> nat -> o.
601
602 type decomp1 links -> subst -> fsubst -> o.
603 decomp1 S [] [].
604 decomp1 S [link _ N _|L] O P :- unset? N S X,
605   decomp1 S L O P.
606 decomp1 S [link M N _|L] O P :- set? N S X,
607   decomp-assignment S X T, assign M O (some T) O1,
608   decomp1 S L O1 P.
609
610 type decomp-assignment subst -> assignment -> fm -> o.
611 decomp-assignment S (abs F) (flam G) :-
612   pi x y\ decomp-tm S x y => decomp-assignment S (F x) (G y).
613 decomp-assignment S (val T) T1 :- decomp S T T1.

```

```

614 type decomp subst -> tm -> fm.
615 decomp _ (con C) (fcon C).
616 decomp S (app A) (app B) :- map (decomp S) A B.
617 decomp S (lam F) (flam G) :-
618   pi x y\ decomp S x y => decomp S (F x) (G y).
619 decomp S (uva N A) R :- set? N S F,
620   move F A T, decomp S T R.
621 decomp S (uva N A) R :- unset? N S,
622   map (decomp S) A B, knil N M, napp (fapp[fuva M|B]) R.

```

Now unif

```

623 type (≈o) fm -> fm -> subst -> subst -> o.
624 (X ≈o Y) S S1 :-
625   fderef S X X0, fderef S Y Y0,
626   comp X0 X1 [] S0 [] L0,
627   comp Y0 Y1 S0 S1 L0 L1,
628   (X1 ≈λ Y1) [] HS0,
629   solve-links L1 L2 HS0 HS1,
630   decompile L2 HS1 S1.

```

(norm)
(compile)

(unify)
(link)
(decompile)

5.1 Prolog simulation

Allows us to express the properties. we take all terms involved in a search (if a rule is used twice we simply take a copy of it), we compile all of them, and then we pick the unification problems among these terms and step trough them.

```

631 type pick list A -> (pair nat nat) -> (pair A A) -> o.
632 pick L (pr X Y) (pr TX TY) :- nth X L TX, nth Y L TY.

```

```

633 type prolog-fo list fm -> list (pair nat nat) -> subst -> o.
634 prolog-fo Terms Problems S :-
635   map (pick Terms) Problems FoProblems,
636   fold4 (≈o) FoProblems [] S.

```

```

637 type step-ho (pair tm tm) -> links -> links -> subst -> subst -> o.
638 step-ho (pr X Y) L0 L1 S0 S2 :-
639   (X1 ≈λ Y1) S0 S1,
640   solve-links L0 L1 S1 S2.

```

```

641 type prolog-ho list fm -> list (pair nat nat) -> subst -> o.
642 prolog-ho Terms Problems S :-
643   fold4 comp Terms HoTerms [] L0 [] HS0,
644   map (pick HoTerms) Problems HoProblems,
645   fold4 step-ho HoProblems L0 L HS0 HS,
646   decompile L HS S.

```

the proprty is that if a step for Fo succeeds then the Ho one does, and if Fo fails then the Ho fails ()

5.2 Example

OK

```

647 Terms [ flam x\ fapp[fcon"g",fapp[fuva z, x]]
648   , flam x\ fapp[fcon"g", fcon"a"] ]
649 Problems = [ pr z (s z) ] % λx.g(Fx) = λx.ga
650 lam x\ app[con"g",uva z [x]] ≈o lam x\ app[con"g", con"a"]
651 link z z (s z)
652 HS = [some (abs x\con"a")]
653 S = [some (flam x\fcon a)]

```

KO

```

654 Terms [ flam x\ fapp[fcon"g",fapp[fuva z, x]]
655   , flam x\ fapp[fcon"g", fcon"a"] ]
656 Problems = [ pr 0 1 % A = λx.x
657   , pr 2 3 ] % Aa = a
658 lam x\ app[con"g",uva z [x]] ≈o lam x\ app[con"g", con"a"]
659 link z z (s z)
660 HS = [some (abs x\con"a")]
661 S = [some (flam x\fcon a)]
662 lam x\ app[f, app[X, x]] = Y,
663 lam x\ x[] = X.

```

TODO: Goal: $s_1 \approx_o s_2$ is compiled into $t_1 \approx_\lambda t_2$

TODO: What is done: uvars fo_{uv} of OL are replaced into uvars ho_{uv} of the ML

TODO: Each fo_{uv} is linked to an ho_{uv} of the OL

TODO: Example needing the compiler v0 (tra l'altro lo scope è ignorato):

```

664 lam x\ app[con"g",app[uv 0, x]] ≈o lam x\ app[con"g", c"a"]

```

TODO:
link
TODO:
nuove
subst
TODO:
code
unif

TODO: Links used to instantiate vars of elpi
TODO: After all links, the solution in links are compacted and given to coq
TODO: It is not so simple, see next sections (multi-vars, eta, beta)

The compilation step is meant to recover the higher-order variables of the OL, expressed in a first order way, by replacing them with higher-order variables in the ML. In particular, every time a variable of the OL is encountered in the original term, it is replaced with a meta variable, and if the OL variable is applied to a list of distinct names L , then this list becomes the scope of the variable. For all the other constructors of tm , the same term constructor is returned and its arguments are recursively compiled. The predicate in charge for term compilation is:

```
type comp tm -> tm -> links -> links -> subst -> subst -> o.
```

where, we take the term of the OL, produce the term of the ML, take a list of link and produce a list of new links, take a substitution and return a new substitution.

In particular, due to programming constraints, we need to drag the old subst and return a new one extended, if needed, with the new declared meta-variables.

The following code

```
kind link type.
type link nat -> nat -> nat -> subst.
```

defines a link, which is a relation between to variables indexes, the first being the index of a OL variable and the second being the index of a ML variable. The third integer is the number of term in the scope of the two variables, or equivalently, in a typed language, their arity.

As an example, let's study the following unification problem (a slightly modified version from section 1):

```
lam x\ app[c"decision", app[c"nfact", x, c"3"]] ≈o
lam x\ app [c"decision", app[uv 0, x]]
```

we have the main unification problem where the nested app nodes have lists of different lengths making the unification to fail. The compilation of these terms produces a new unification problem with the following shape:

```
lam x\ app[c"decision", app[c"nfact", x, c"3"]] ≈λ
lam x\ app [c"decision", uv 1 [x]]
```

The main difference is the replacement of the subterm $app[uv\ 0, x]$ of the OL with the subterm $uv\ 0\ [x]$. Variable indexes are chosen by the ML, that is, the index 0 for that unification variable of the OL term has not the same meaning of the index 0 in the ML. There exists two different substitution mapping, one for the OL and one for the ML and the indexes of variable point to the respective substitution.

decomp che mappa abs verso lam **TODO: An other example:**

```
lam x\ app[f, app[X, x]] = Y, (lam x\ x) = X.
```

6 USE OF MULTIVARS

Se il termine iniziale è della forma

```
app[con"xxx", (lam x\ lam y\ Y y x), (lam x\ f)]
=
app[con"xxx", X, X]
```

allora se non uso due X diverse non ho modo di recuperare il quoziente che mi manca.

a sto punto consideriamo liste di problemi e così da eliminare sta xxx senza perdita di generalità (e facciamo problemi più corti, e modellizziamo anche la sequenza)

6.1 Problems with η

TODO: The following goal necessita v1 (lo scope è usato):

```
X = lam x\ lam y\ Y y x, X = lam x\ f
```

TODO: The snd unif pb, we have to unif lam x\ lam y\ Y x y with lam x\ f

TODO: It is not doable, with the same elpi var

6.2 Problems with β

TODO: The following goal: $X = lam\ x\ x$, $app[X, 3] = 3$

TODO: We use links-beta

6.3 Tricky examples

```
triple ok (@lam x\ @app[@f, @app[@X, x]]) @Y,
triple ok @X (@lam x\ x),
triple ok @Y @f
```

```
% @ok1 22 [
%   triple ok (@lam x\ @lam y\ @app[@Y, y, x]) @X,
%   triple ok (@lam x\ @f) @X,
% ].
```

7 FIRST ORDER APPROXIMATION

TODO: Coq can solve this: $f\ 1\ 2 = X\ 2$, by setting X to $f\ 1$

TODO: We can re-use part of the algo for β given before

8 UNIF ENCODING IN REAL LIFE

TODO: Il ML presentato qui è esattamente elpi

TODO: Il OL presentato qui è esattamente coq

TODO: Come implementiamo tutto ciò nel solver

9 RESULTS: STDPP AND TLC

TODO: How may rule are we solving?

TODO: Can we do some perf test

10 CONCLUSION

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APPENDIX

Note that $(a \text{ infix } b) \text{ c d}$ de-sugars to $(\text{infix}) \text{ a b c d}$.

```

1045  type ( $=_o$ ) f $\tau$ m  $\rightarrow$  f $\tau$ m  $\rightarrow$  o.                                ( $=_o$ )
1046  fapp A  $=_o$  fapp B :- map ( $=_o$ ) A B.
1047  flam F  $=_o$  flam G :-  $\pi$  x \ x  $=_o$  x  $\Rightarrow$  F x  $=_o$  G x.
1048  fcon C  $=_o$  fcon C.
1049  fuva N  $=_o$  fuva N.
1050  flam F  $=_o$  T :-                                              ( $\eta_l$ )
1051     $\pi$  x \ beta T [x] (R x), x  $=_o$  x  $\Rightarrow$  F x  $=_o$  R x.
1052  T  $=_o$  flam F :-                                              ( $\eta_r$ )
1053     $\pi$  x \ beta T [x] (R x), x  $=_o$  x  $\Rightarrow$  R x  $=_o$  F x.
1054  fapp [flam X | L]  $=_o$  T :- beta (flam X) L R, R  $=_o$  T. ( $\beta_l$ )
1055  T  $=_o$  fapp [flam X | L] :- beta (flam X) L R, T  $=_o$  R. ( $\beta_r$ )
1056
1057  type beta fm  $\rightarrow$  list fm  $\rightarrow$  fm  $\rightarrow$  o.
1058  beta A [] A.
1059  beta (flam F) [H | L] R :- subst F H B,
1060    beta B L R. % since F could be x\app[x|_] and H be lam _
1061  beta (fapp A) L (fapp X) :- append A L X.
1062  beta (fuva N) L (fapp [fuva N | L]).
1063  beta (fcon H) L (fapp [fcon H | L]).
1064
1065  type subst (fm  $\rightarrow$  fm)  $\rightarrow$  fm  $\rightarrow$  fm  $\rightarrow$  o.
1066  subst F H B :- napp (F H) B. % since (F H) may generate (app[app _|_])
1067
1068  type napp fm  $\rightarrow$  fm  $\rightarrow$  o.
1069  napp (fcon C) (fcon C).
1070  napp (flam F) (flam G) :-  $\pi$  x \ napp x x  $\Rightarrow$  napp (F x) (G x).
1071  napp (fapp[fapp L|M]) R :- !, append L M N, napp (fapp N) R.
1072  napp (fapp[X]) R :- !, napp X R.
1073  napp (fapp A) (fapp B) :- map napp A B.
1074  napp (fuva N) (fuva N).

```

Figure 3: Full implementation of the $=_o$ predicate for \mathcal{F}_o

```

1161 type ( $\approx_\lambda$ ) tm -> tm -> subst -> subst -> o.
1162 % Congruence
1163 (app A  $\approx_\lambda$  app B) R S :- fold2 ( $\approx_\lambda$ ) A B R S.
1164 (lam F  $\approx_\lambda$  lam G) R S :- pi x\ (pi S\ (x  $\approx_\lambda$  x) S S) => (F x  $\approx_\lambda$  G x) R S.
1165  $\approx_\lambda$  (con C) (con C) S S.
1166 % deref
1167 (uva N A  $\approx_\lambda$  T) R S :- set? N S F, move F A T1, (T1  $\approx_\lambda$  T) R S.
1168 (T  $\approx_\lambda$  uva N A) R S :- set? N S F, move F A T1, (T  $\approx_\lambda$  T1) R S.
1169 % flex-flex
1170 (uva N A  $\approx_\lambda$  uva M B) S S3 :- unset? M, unset? N,
1171   distinct A, distinct B,
1172   new S W S1, prune W Args1 B Ass,
1173   assign N S1 Ass S2, assign M S2 Ass S3.
1174 % assignment
1175 (uva N A  $\approx_\lambda$  T) R S :- distinct A, not (T = uva _ _), not_occ N S T,
1176   bind A T T1, assign N S T1 S1.
1177 (T  $\approx_\lambda$  uva N A) R S :- distinct A, not (T = uva _ _), not_occ N S T,
1178   bind A T T1, assign N S T1 S1.
1179
1180 type distinct list A -> o.
1181 distinct [].
1182 distinct [X|XS] :- name X, not(mem X XS),
1183 distinct XS.
1184
1185 typeabbrev memory A (list (option A)).
1186 type set? nat -> memory A -> A -> o.
1187 set? N S T :- nth N S (some T).
1188 type unset? nat -> memory A -> o.
1189 unset? N S :- nth N S none.
1190 type assign nat -> memory A -> A -> memory A -> o.
1191 assign z [none|M] T [some T|M].
1192 assign (s N) [X|M] T [X|M1] :- assign N M T M1.
1193 kind nat type.
1194 type z nat.
1195 type s nat -> nat.
1196 type nth nat -> list A -> A -> o.
1197 nth z [X|_] X.
1198 nth (s N) [_|L] X :- nth N L X.
1199
1200 type new memory A -> nat -> memory A -> o.
1201 new [] z [none].
1202 new [X|XS] (s N) [X|YS] :- new XS N YS.
1203
1204 type prune .
1205 type move .
1206 type beta.
1207 type bind.
1208 type not_occ.
1209 TODO
1210
1211 type fold2 (A -> A1 -> B -> B -> o) -> list A -> list A1 -> B -> B -> o.
1212 fold2 _ [] [] A A.
1213 fold2 F [X|XS] [Y|YS] A A1 :- F X Y A A0, fold2 F XS YS A0 A1.

```

Figure 4: Implementation of the \approx_λ predicate for \mathcal{H}_0

```

1277  type fold4 (A -> A1 -> B -> B -> C -> C -> o) -> list A -> list A1 -> B -> B -> C -> C -> o.
1278  fold4 _ [] [] A A B B.
1279  fold4 F [X|XS] [Y|YS] A A1 B B1 :- F X Y A A0 B B0, fold4 F XS YS A0 A1 B0 B1.
1280
1281  type len list A -> nat -> o.
1282  len [] z.
1283  len [_|L] (s X) :- len L X.

```

Figure 5: Implementation of the compiler