Higher order unification for free!

Reusing the meta-language unification for the object language

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ABSTRACT

Specifying and implementing a proof system from scratch requires significant effort. Logical Frameworks and Higher Order Logic Programming Languages provide dedicated, high-level Meta Languages (ML) to facilitate this task in two key ways: 1) variable binding and substitution are simplified when ML binders represent object logic ones; 2) proof construction, and even proof search, is greatly simplified by leveraging the unification procedure provided by the ML. Notable examples of ML are Elf [13], Twelf [15], λProlog [9] and Isabelle [21] which have been utilized to implement various formal systems such as First Order Logic [3], Set Theory [12], Higher Order Logic [11], and even the Calculus of Constructions [2].

The object logic we are interested in is Coq's [19] Calculus of Inductive Constructions (CIC), for which we aim to implement a unification procedure $\simeq_o$ using the ML Elpi [1], a dialect of λProlog. Elpi's equational theory comprises $\eta\beta$ equivalence and comes equipped with a higher order unification procedure $\simeq_\lambda$ restricted to the pattern fragment [8]. We want $\simeq_o$ to be as powerful as $\simeq_\lambda$ but on the object logic CIC. Elpi also comes with an encoding for CIC that works well for meta-programming [18, 17, 6, 4]. Unfortunately this encoding, which we refer to as $\mathcal{F}_o$, "underuses" $\simeq_\lambda$ by restricting it to first-order unification problems only. To address this issue, we propose a better-behaved encoding, $\mathcal{H}_o$, demonstrate how to map unification problems in $\mathcal{F}_o$ to related problems in $\mathcal{H}_o$, and illustrate how to map back the unifiers found by $\simeq_\lambda$, effectively implementing $\simeq_o$ on top of $\simeq_\lambda$ for the encoding $\mathcal{F}_o$.

KEYWORDS

Logic Programming, Meta-Programming, Higher-Order Unification, Proof Automation

1 INTRODUCTION

Meta languages such as Elf [13], Twelf [15], λProlog [9] and Isabelle [21] have been utilized to specify various logics [3, 11, 12,

2]. The use of these meta languages facilitated this task in two key ways. The first and most well know one is that variable binding and substitution come for free. The second one is that these meta languages come equipped with some form of unification, a cornerstone in proof construction and proof search.

The object logic we are interested in is Coq's [19] Calculus of Inductive Constructions (CIC) and we want to implement a form of proof search known as type-class [20, 16] resolution. In particular we want to leverage the Elpi [18] meta programming language, a dialect of λProlog already used to extend Coq in various ways [18, 17, 6, 4]. Type-class solvers are unification based proof search procedures reminiscent of Prolog that back-chain lemmas taken from a designated database of "type class instances", hence we can expect that Elpi is a good fit for implementing such as form of automation. In this paper we focus on one aspect of this work, namely *how to reuse the higher order unification procedure of the meta language in order to implement a type-class solver for the object language*. As it turns out, re-using the unification of the meta language is not a trivial task.

We take as an example the `Decision` and `Finite` type classes from the Stdpp [7] library. The class `Decision` identifies predicates equipped with a decision procedure, while `Finite` the types whose inhabitants can be enumerated in a (finite) list. The following three designated `Instances` state that: 1) the type `fin n`, of natural numbers smaller than n, is finite; 2) the predicate `nfact n nf`, linking a natural number n to the number of its prime factors nf, is decidable; 3) the universal closure of a predicate has a decision procedure if its domain is finite and if the predicate is decidable.

```
Instance fin_fin: ∀n, Finite (fin n).        (* r1 *)
Instance nfact_dec: ∀n nf, Decision (nfact n nf). (* r2 *)
Instance forall_dec: ∀A P, Finite A →         (* r3 *)
  ∀x:A, Decision (P x) → Decision (∀x:A, P x).
```

Given this database a type-class solver is expected to prove the following statement automatically:

```
Decision (∀x: fin 7, nfact x 3)              (* g *)
```

The proof found by the solver back-chains on rule 3 (the only rule about the ∀ quantifier), and then solves the premises with rules rules 1 and 2 respectively. Note that rule 3 features a second order parameter P that stands for a function of type A → **Prop** (a predicate over A). The solver has to infer a value for P by unifying the conclusion of rule 3 with the goal, and in particular by solving the unification problem P x = nfact x 3. This higher order problem falls in the so called pattern-fragment $\mathcal{L}_\lambda$ [8] and hence admits a unique solution P = λx.nfact x 3.

In order to implement such a search in Elpi we shall describe the encoding of CIC terms and then the encoding of rules. Elpi comes

with an Higher Order Abstract Syntax [14] datatype of CIC terms, called tm, that features (among others) the following constructors:

```
type lam  tm -> (tm -> tm) -> tm.    % lambda abstraction
type app  list tm -> tm.             % n-ary application
type all  tm -> (tm -> tm) -> tm.    % forall quantifier
type con  string -> tm.              % constants
```

Following λProlog [9]'s standard syntax, the meta level binding of a variable x in an expression e is written «x\ e», and square brackets denote a list of terms separated by comma. For example the term «∀y:t, nfact y 3» is encoded as follows:

```
all (con"t") y\ app[con"nfact", y, con"3"]
```

We now illustrate the encoding of the three instances above as higher-order logic-programming rules: capital letters denote rule parameters; :- separates the rule's head from the premises and `pi w\ p` introduces a fresh nominal constant w for the premise p.

```
finite   (app [con"fin", N]).                  (r1)
decision (app [con"nfact", N, NF]).            (r2)
decision (all A x\ app [P, x]) :- finite A,    (r3)
  pi w\ decision (app [P, w]).
```

Unfortunately this intuitive encoding of rule (r3) does not work, since it uses the predicate P as a first order term: for the meta language its type is tm. If we try to back-chain the rule (r3) on the encoding of the goal g given below

```
decision (all (app [con"fin", con"7"]) y\    (g)
  app [con"nfact", y, con"3"]).
```

we obtain an unsolvable unification problem (p): the two lists of terms have different lengths!

```
app [con"nfact", y, con"3"] = app [P, y]    (p)
```

In this paper we study a more sophisticated encoding of rules that, on a first approximation, would shape (r3) as follows:

```
decision (all A x\ Pm x) :- link Pm P A, finite A,    (r3′)
  pi x\ decision (app [P, x]).
```

Since Pm is an higher-order unification variable of type tm -> tm, with x in its scope, the unification problem (p′) admits one solution:

```
app [con"nfact", y, con"3"] = Pm y    (p′)
Pm = x\ app [con"nfact", x, con"3"]    (ρ)
```

Once the head of rule (r3′) unifies with the goal (g) the premise «link Pm A P» brings the assignment (ρ) back to the domain tm of Coq terms:

```
P = lam A a\ app [con"nfact", a, con"3"]
```

This simple example is sufficient to show that the encoding we seek is not trivial and does not only concern the head of rules, but the entire sequence of unification problems that constitute the execution of a logic program. In fact the solution for P above generates a (Coq) β-redex in the second premise (the predicate under the `pi w\`). We show below the premise before and after the instantiation of P:

```
decision (app[                    P              , w])
decision (app[ lam A (a\ app [con"nfact", a, con"3"]) , w])
```

In turn this redex prevents the rule (r2) to backchain properly since the following unification problem has no solution:

```
app[ lam A (a\ app [con"nfact", a, con"3"]) , x] =    175
app[ con"nfact"                      , N, NF]          176
```

The root cause of the problems we sketched in this example is a subtle mismatch between the equational theories and unification procedures of the meta language and the object language.

The equational theory of CIC is very rich. In addition to the usual $\eta\beta$-equivalence for functions, terms (hence types) are compared up to proposition unfolding and fixpoint unrolling. Still, for efficiency and predictability reasons, most form of automatic proof search employ a unification procedure that captures a simpler one, just $\eta\beta$, and that solves higher-order problems restricted to the pattern fragment $\mathcal{L}_\lambda$. We call this unification procedure $\simeq_o$.

The equational theory of the meta language Elpi is strikingly similar, since it it comprises $\eta\beta$ (for the meta language functions), and the unification procedure $\simeq_\lambda$ solves problems in $\mathcal{L}_\lambda$ as well.

In spite of the similarity the link between $\simeq_\lambda$ and $\simeq_o$ is not trivial, since the abstraction and application term constructors the two unification procedures deal with are different. For example:

x\ f x	$\simeq_\lambda$ f	P x	$\simeq_\lambda$ x
lam A x\ app[con"f", x]	$\simeq_o$ con"f"	app[P, x]	$\simeq_o$ x
lam A x\ app[con"f", x]	$\not\simeq_\lambda$ con"f"	app[P, x]	$\not\simeq_\lambda$ x

Contributions. In this paper we identify a minimal language $\mathcal{F}_o$ in which the problems sketched in the introduction can be formally described. We detail an encoding of a logic program on $\mathcal{F}_o$ to a strongly related logic program in $\mathcal{H}_o$ (the language of the meta language) and we show that the unification procedure of the meta language $\simeq_\lambda$ can be effectively used to simulate a unification procedure $\simeq_o$ for the object language that features $\eta\beta$-conversion in the pattern-fragment.

section 2 formally states the problem and gives the intuition behind our solution. section 9 discusses alternative term encodings and related works. section 3 introduces the languages $\mathcal{F}_o$ and $\mathcal{H}_o$, section 4 describes a basic simulation of higher order logic programs. sections 5 and 6 completes its equational theory with support for η-conversion. section 7 deals with the practical necessity of "tolerating" terms outside of the pattern-fragment and discusses how heuristic can be applied. Finally section 8 discusses the implementation in Elpi.

The λProlog code discussed in the paper can be accessed at the URL: https://github.com/FissoreD/paper-ho.

2 PROBLEM STATEMENT AND SOLUTION

Even if we encountered the problem in CIC one does not need such a complex system in order to state it. The minimal setting we could find is one where a $\mathcal{F}_o$ language (for first order) with a rich(er) equational theory is phrased in a $\mathcal{H}_o$ meta language with a simpler one, and one needs to build the unification of $\mathcal{F}_o$ on top of the one of $\mathcal{H}_o$.

In order to reason about unification we provide a description of the $\mathcal{F}_o$ and $\mathcal{H}_o$ languages where unification variables are first class terms, i.e. they have a concrete syntax. We keep these languages minimal, for example, we omit the all quantifier of CIC we used in the example in Section 1 together with the type notation of terms carried by the lam constructor.

```
233   kind fm type.              kind tm  type.
234   type fapp list fm -> fm.    type app list tm -> tm.
235   type flam (fm -> fm) -> fm. type lam (tm -> tm) -> tm.
236   type fcon string -> fm.     type con string -> tm.
237   type fuva addr -> fm.       type uva addr -> list tm -> tm.
```

Figure 1: The $\mathcal{F}_o$ and $\mathcal{H}_o$ languages

Unification variables (`fuva` term constructor) in $\mathcal{F}_o$ have no explicit scope: the arguments of an higher order variable are given via the `fapp` constructor. For example the term `P x` is represented as `fapp[fuva N, x]`, where `N` is a memory address and `x` is a bound variable.

In $\mathcal{H}_o$ the representation of `P x` is instead `uva N [x]`, since unification variables come equipped with an explicit scope.

> faddr and addr are memory addresses, the details are given in bla, here they are just unique identifiers for unif variables and their types do now allow to mistake one for the other.

When we write $\mathcal{H}_o$ terms outside code blocks we follow the usual λ-calculus notation, reserving f, g, a, b for constants, x, y, z for bound variables and X, Y, Z, F, G, H for unification variables. However we need to distinguish between the "application" of a unification variable to its scope and the application of a term to a list of arguments. We write the scope of unification variables in subscript while we use juxtaposition for regular application. Here a few examples:

$f \cdot a$	`app[con "f", con "a"]`
$\lambda x.\lambda y.F_{xy}$	`lam x\ lam y\ uva F [x, y]`
$\lambda x.F_x \cdot a$	`lam x\ app[uva F [x], con "a"]`
$\lambda x.F_x \cdot x$	`lam x\ app[uva F [x], x]`

When it is clear from the context we shall use the same syntax for $\mathcal{F}_o$ terms (although we never subscripts unification variables).

2.1   eq and unif

In order to specify the two unification procedures we need to define the equational theories of $\mathcal{F}_o$ and $\mathcal{H}_o$ and define what substitutions are.

Term equality: $=_o$ vs. $=_\lambda$. We extend the equational theory over ground terms to the full languages by adding the reflexivity of unification variables (a variable is equal to itself).

The first four rules are common to both equalities and just define the usual congruence over terms, and since we use an HOAS encoding they also capture α-equivalence. In addition to that $=_o$ has rules for η and β-equivalence.

```
type (=o) fm -> fm -> o.                          (=o)
fcon X =o fcon X.
fapp A =o fapp B :- forall2 (=o) A B.
flam F =o flam G :- pi x\ x =o x => F x =o G x.
fuva N =o fuva N.
flam F =o T :-                                    (ηl)
  pi x\ beta T [x] (T' x), x =o x => F x =o T' x.
T =o flam F :-                                    (ηr)
  pi x\ beta T [x] (T' x), x =o x => T' x =o F x.
fapp [flam X | L] =o T :- beta (flam X) L R, R =o T.  (βl)
T =o fapp [flam X | L] :- beta (flam X) L R, T =o R.  (βr)
```

```
type (=λ) tm -> tm -> o.
con C =λ fcon C.
app A =λ fapp B :- forall2 (=λ) A B.
lam F =λ flam G :- pi x\ x =λ x => F x =λ G x.
uva N A =λ fuva N B :- forall2 (=λ) A B.
```

The main point in showing these equality tests is to remark how weaker $=_\lambda$ is, and to identify the four rules that need special treatment in the implementation of $\simeq_o$.

For reference, (`beta T A R`) reduces away `lam` nodes in head position in `T` whenever the list `A` provides a corresponding argument.

```
type beta fm -> list fm -> fm -> o.
beta A [] A.
beta (flam Bo) [H | L] R :- napp (Bo H) F, beta F L R.
beta (fapp A) L (fapp X) :- append A L X.
beta (fuva N) L (fapp [fuva N | L]).
beta (fcon H) L (fapp [fcon H | L]).
beta N L (fapp [N | L]) :- name N.
```

```
type napp fm -> fm -> o.
napp (fcon C) (fcon C).
napp (fuva A) (fuva A).
napp (flam F) (flam G) :- pi x\ napp (F x) (G x).
napp (fapp [fapp L1 |L2]) T :- !,
  append L1 L2 L3, napp (fapp L3) T.
napp (fapp L) (fapp L1) :- map napp L L1.
napp N N :- name N.
```

The `name` predicate holds only on nominal constants (i.e. bound variables).[1] The choice of using n-ary application, rather than binary, is to make it easy to access the application's head. The price we pay is that substituting an application in the head of an application should be amended by "flattening" `fapp` nodes, that is the job of `napp`.[2] Finally note that the cut operator is inessential, it could be removed at the cost of a verbose test on the head of `L` in the second rule about `fapp`: `L`'s head can be `fcon`, `flam` or a name.

Substitution application: ρs and σt. Applying the substitution corresponds to dereferencing a term with respect to the memory. To ease the comparison we split $\mathcal{F}_o$dereferencing into a `fder` step and a `napp` one. The former step replaces references to memory cells that are set with their values, and has a corresponding operation in $\mathcal{H}_o$, namely `deref`. On the contrary `napp` has no corresponding operation in $\mathcal{H}_o$. The reasons for this asymmetry is that an `fapp` node with a flexible head is always mapped to a `uva` (as per section 4.1 and section 7), preventing nested applications to materialize.

```
type fder fsubst -> fm -> fm -> o.
fder _ (fcon C) (fcon C).
fder S (fapp A) (fapp B) :- map (fder S) A B.
fder S (flam F) (flam G) :-
  pi x\ fder S x x => fder S (F x) (G x).
fder S (fuva N) R :- set? N S T, fder S T R.
fder S (fuva N) (fuva N) :- unset? N S.
```

---

[1]Elpi provides it as a builtin, but one could implement it by systematically loading the hypothetical rule `name x` every time a nominal constant is postulated via **pi** x\
[2]Note that `napp` is an artefact of formalization of $\mathcal{F}_o$ we do in this presentation and, as we explain later, no equivalent of `napp` is needed in $\mathcal{H}_o$.

```
type fderef fsubst -> fm -> fm -> o.            (ρs)
fderef S T T2 :- fder S T T1, napp T1 T2.
```

Applying the substitution in $\mathcal{H}_o$ is very similar, with the caveat that assignments have to be moved to the current scope, i.e. renaming the abs-bound variables with the names in the scope of the unification variable occurrence.

```
type deref subst -> tm -> tm -> o.              (σt)
deref _ (con C) (con C).
deref S (app A) (app B) :- map (deref S) A B.
deref S (lam F) (lam G) :-
  pi x\ deref S x x => deref S (F x) (G x).
deref S (uva N L) R :- set? N S A,
  move A L T, deref S T R.
deref S (uva N A) (uva N B) :- unset? N S,
  map (deref S) A B.
```

Note that move strongly relies on invariant 1 that we shall define later.

```
type move assignment -> list tm -> tm -> o.
move (abs Bo) [H|L] R :- move (Bo H) L R.
move (val A) [] A.
```

Term unification: $\simeq_o$ *vs.* $\simeq_\lambda$. In this paper we assume to have an implementation of $\simeq_\lambda$ that satisfies properties 1 and 2. Although we provide an implementation in the appendix (that we used for testing purposes) we only describe its signature here. Elpi is expected to provide this brick, as well as any other implementation of λProlog.

```
type (≃λ) tm -> tm -> subst -> subst -> o.
```

We write $t_1 \simeq_\lambda t_2 \mapsto \sigma$ when t_1 and t_2 unify with substitution σ; we write σt for the application of the substitution to t, and $\sigma X = \{\sigma t \mid t \in X\}$ when X is a set; we write $\sigma \subseteq \sigma'$ when σ is more general than σ' and we use set union to denote the concatenation of two substitutions. We shall use ρ for $\mathcal{F}_o$ substitutions, and σ for the $\mathcal{H}_o$ ones. We assume that the unification of our meta language is correct and complete in $\mathcal{L}_\lambda$.

$$t_i \in \mathcal{L}_\lambda \Rightarrow t_1 \simeq_\lambda t_2 \mapsto \rho \Rightarrow \rho t_1 =_\lambda \rho t_2 \quad (1)$$

$$t_i \in \mathcal{L}_\lambda \Rightarrow \rho t_1 =_\lambda \rho t_2 \Rightarrow \exists \rho', t_1 \simeq_\lambda t_2 \mapsto \rho' \wedge \rho' \subseteq \rho \quad (2)$$

2.2 Logic program simulation

We represent a logic program *run* in $\mathcal{F}_o$ as a list *steps p* of length $\mathcal{N}$. Each step p represents a unification problem between two terms $\mathbb{P}_{p_l}$ and $\mathbb{P}_{p_r}$ taken from the set of all terms $\mathbb{P}$. The composition of these steps starting from the empty substitution ρ_0 produces the final substitution $\rho_\mathcal{N}$. [3]

$$\text{fstep}(\mathbb{P}, p, \rho) \mapsto \rho'' \stackrel{def}{=} \rho \mathbb{P}_{p_l} \simeq_o \rho \mathbb{P}_{p_r} \mapsto \rho' \wedge \rho'' = \rho \cup \rho'$$

$$\text{frun}(\mathbb{P}, \mathcal{N}) \mapsto \rho_\mathcal{N} \stackrel{def}{=} \bigwedge_{p=1}^{\mathcal{N}} \text{fstep}(\mathbb{P}, p, \rho_{p-1}) \mapsto \rho_p$$

We illustrate a compilation $\langle s \rangle \mapsto (t, m, l)$ that maps a term s in $\mathcal{F}_o$ to a term t in $\mathcal{H}_o$ and produces a variable mapping m and list of links l. The variable map connects unification variables in $\mathcal{H}_o$ with variables in $\mathcal{F}_o$ and is used to "decompile" the assignment,

---

[3]If the same rule is used multiple time in a run we just consider as many copies as needed of the terms composing the rules, with fresh unification variables each time

$\langle \sigma, m, l \rangle^{-1} \mapsto \rho$. Links are an accessory piece of information whose description is deferred to section 2.3.

We simulate each run in $\mathcal{F}_o$ with a run in $\mathcal{H}_o$ as follows.

$$\text{hstep}(\mathbb{T}, p, \sigma, \mathbb{L}) \mapsto (\sigma'', \mathbb{L}') \stackrel{def}{=}$$
$$\sigma \mathbb{T}_{p_l} \simeq_\lambda \sigma \mathbb{T}_{p_r} \mapsto \sigma' \wedge \text{progress}(\mathbb{L}, \sigma \cup \sigma') \mapsto (\mathbb{L}', \sigma'')$$
$$\text{hrun}(\mathbb{P}, \mathcal{N}) \mapsto \rho_\mathcal{N} \stackrel{def}{=}$$
$$\mathbb{T} \times \mathbb{M} \times \mathbb{L}_0 = \{(t_j, m_j, l_j) | s_j \in \mathbb{P}, \langle s_j \rangle \mapsto (t_j, m_j, l_j)\}$$
$$\bigwedge_{p=1}^{\mathcal{N}} \text{hstep}(\mathbb{T}, p, \sigma_{p-1}, \mathbb{L}_{p-1}) \mapsto (\sigma_p, \mathbb{L}_p)$$
$$\langle \sigma_\mathcal{N}, \mathbb{M}, \mathbb{L}_\mathcal{N} \rangle^{-1} \mapsto \rho_\mathcal{N}$$

Here hstep is made of two sub-steps: a call to the meta language unification and a check for progress on the set of links. We claim:

PROPOSITION 2.1 (SIMULATION). $\forall \mathbb{P}, \forall \mathcal{N}$, *if* $\mathbb{P} \subseteq \mathcal{L}_\lambda$

$$\text{frun}(\mathbb{P}, \mathcal{N}) \mapsto \rho_\mathcal{N} \Leftrightarrow \text{hrun}(\mathbb{P}, \mathcal{N}) \mapsto \rho_\mathcal{N}$$

That is, the two executions give the same result. Moreover:

PROPOSITION 2.2 (SIMULATION FIDELITY). *In the context of* hrun, *if* $\mathbb{P} \subseteq \mathcal{L}_\lambda$ *we have that* $\forall p \in 1 \ldots \mathcal{N}$,

$$\text{fstep}(\mathbb{P}, p, \rho_{p-1}) \mapsto \rho_p \Leftrightarrow \text{hstep}(\mathbb{T}, p, \sigma_{p-1}, \mathbb{L}_{p-1}) \mapsto (\sigma_p, \mathbb{L}_p)$$

In particular this property guarantees that a *failure* in the $\mathcal{F}_o$ run is matched by a failure in $\mathcal{H}_o$ *at the same step*. We consider this property very important from a practical point of view since it guarantees that the execution traces are strongly related and in turn this enables a user to debug a logic program in $\mathcal{F}_o$ by looking at its execution trace in $\mathcal{H}_o$.

We claim that hrun handles terms outside $\mathcal{L}_\lambda$ in the following sense:

PROPOSITION 2.3 (PROPERTIES OF hrun OUTSIDE $\mathcal{L}_\lambda$).

$$\exists \rho, \rho s_1 =_o \rho s_2 \Rightarrow \langle s_i \rangle \mapsto (t_i, m_i, l_i) \Rightarrow t_1 \simeq_\lambda t_2 \mapsto \sigma \quad (3)$$

$$\sigma_{p-1} \mathbb{T}_p \in \mathcal{L}_\lambda \Rightarrow \text{hstep}(\mathbb{T}, p, \sigma_{p-1}, \mathbb{L}_{p-1}) \mapsto (\sigma_p, \mathbb{L}_p) \mapsto \sigma \quad (4)$$

Property 3 states that two terms for which there is a unifier (given by an oracle, not necessarily a most general one), then the compiler generates two terms that unify in $\mathcal{H}_o$. In other words the compiler moves out of the way all problematic terms.

Property 4 says that if the two terms involved in a step re-enter $\mathcal{L}_\lambda$, then hstep succeeds. This is a typical example in which the order of the unification problems in a logic program run do matter. The simplest example is the sequence $F = \lambda x.a$ and $F a = a$: the second problem is not in $\mathcal{L}_\lambda$ and has two unifiers, namely $F = \lambda x.x$ and $F = \lambda x.a$. The first problem picks one of the two, making the second problem re-enter $\mathcal{L}_\lambda$.

2.3 The solution in a nutshell

A term s is compiled in a term t where every "problematic" sub term p is replaced by a fresh unification variable h and an accessory *link* that represent a suspended unification problem $h \simeq_\lambda p$. As a result $\simeq_\lambda$ is "well behaved" on t, in the sense that it does not contradict $=_o$ as it would otherwise do on "problematic" terms. We now define "problematic" and "well behaved" more formally.

Definition 2.4 ($\diamond \eta$). $\diamond \eta = \{t \mid \exists \rho, \rho t$ is an eta expansion$\}$

An example of term t in $\diamond \eta$ is $\lambda x.\lambda y.F \ y \ x$ since the substitution $\rho = \{F \mapsto \lambda a.\lambda b.f b a\}$ makes $\rho t = \lambda x.\lambda y.f x y$ that is the eta

long form of f. This term is problematic since its rigid part, the λ-abstractions, cannot justify a unification failure against a constant or an application.

Definition 2.5 ($\diamond\beta_0$). $\diamond\beta_0 = \{X \cdot x_1 \ldots x_n \mid x_1 \ldots x_n \text{ are distinct names}\}$

An example of term t in $\diamond\beta_0$ is the application $F \cdot x$. This term is problematic since the application node of its syntax tree cannot be used to justify a unification failure, i.e. by properly instantiating F the term head constructor may become a λ, or a constant or stay an application.

Definition 2.6 ($\overline{\mathcal{L}_\lambda}$). $\overline{\mathcal{L}_\lambda} = \{X \cdot t_1 \ldots t_n \mid X \cdot t_1 \ldots t_n \notin \mathcal{L}_\lambda\}$.

These terms are problematic for the very same reason terms in $\diamond\beta_0$ are, but cannot be handled directly by the unification of the meta language, that is only required to handle terms in $\mathcal{L}_\lambda$.

We write $\mathcal{P}(t)$ the set of sub-terms of t, and we write $\mathcal{P}(X) = \bigcup_{t \in X} \mathcal{P}(t)$ when X is a set of terms.

Definition 2.7 (Well behaved set). Given a set of terms $X \subseteq \mathcal{H}_o$,

$$\mathcal{W}(X) \Leftrightarrow \forall t \in \mathcal{P}(X), t \notin (\overline{\mathcal{L}_\lambda} \cup \diamond\eta \cup \diamond\beta_0)$$

PROPOSITION 2.8 ($\mathcal{W}$-PRESERVATION OF hstep). $\forall\mathbb{T}, \forall\mathbb{L}, \forall p, \forall\sigma, \forall\sigma'$

$$\mathcal{W}(\sigma\mathbb{T}) \wedge \sigma\mathbb{T}_{p_l} \simeq_\lambda \sigma\mathbb{T}_{p_r} \mapsto \sigma' \Rightarrow \mathcal{W}(\sigma'\mathbb{T})$$
$$\mathcal{W}(\sigma\mathbb{T}) \wedge \text{progress}(\mathbb{L}, \sigma) \mapsto (\_, \sigma') \Rightarrow \mathcal{W}(\sigma'\mathbb{T})$$

A less formal way to state 2.8 is that hstep and progress never "commit" an unneeded λ-abstraction in σ (a λ that could be erased by an η-contraction), nor put in σ an application whose head is flexible (an application node that could be erased by a β-reduction).

3 PRELIMINARIES FOR $\mathcal{F}_o$ AND $\mathcal{H}_o$

We say that the unification variable occurrence uva N L is in $\mathcal{L}_\lambda$ if and only if L is made of distinct names. The predicate to test this condition is called pattern-fragment:

```
type pattern-fragment list A -> o.
```

Natural numbers represent the memory addresses that identify unification variables in both languages. The memory and its associated operations are described below:

```
kind addr type.
type addr nat -> addr.
typeabbrev (mem A) (list (option A)).

type set? addr -> mem A -> A -> o.
type unset? addr -> mem A -> o.
type assign addr -> mem A -> A -> mem A -> o.
type new mem A -> addr -> mem A -> o.
```

If a memory cell is none, then the corresponding unification variable is not set. assign sets an unset cell to the given value, while new finds the first unused address and sets it to none.

Since in $\mathcal{H}_o$ unification variables have a scope, their solution needs to be abstracted over it to enable the instantiation of a single solution to different scopes. This is obtained via the inctx container, and in particular via its abs binding constructor. On the contrary a solution to a $\mathcal{F}_o$ variable is a plain term.

```
typeabbrev fsubst (mem fm).

kind inctx type -> type.                          (· ⊢ ·)
type abs (tm -> inctx A) -> inctx A.
type val A -> inctx A.
typeabbrev assignment (inctx tm).
typeabbrev subst (mem assignment).
```

We call fsubst the memory of $\mathcal{F}_o$, while we call subst the one of $\mathcal{H}_o$. Both have the invariant that they are not cyclic, TODO: explain.

```
kind arity type.
type arity nat -> arity.

kind fvariable type.
type fv addr -> fvariable.

kind hvariable type.
type hv addr -> arity -> hvariable.

kind mapping type.
type mapping fvariable -> hvariable -> mapping.
typeabbrev mmap (list mapping).
```

INVARIANT 1 (UNIFICATION VARIABLE ARITY). *Each variable A in $\mathcal{H}_o$ has a (unique) arity N and each occurrence* (uva A L) *is such that* (len L N) *holds*

The compiler establishes a mapping between variables of the two languages. In order to preserve invariant 1 we store the arity of each hvariable in the mapping and we reuse an existing mapping only if the arity matches.


```
type m-alloc fvariable -> hvariable -> mmap -> mmap ->
  subst -> subst -> o.                              (malloc)
m-alloc Fv Hv M M S S :- mem M (mapping Fv Hv), !.
m-alloc Fv Hv M [mapping Fv Hv|M] S S1 :- Hv = hv N _,
  alloc S N S1.
```

When a single fvariable occurs multiple times with different numbers of arguments the compiler generates multiple mappings for it, on a first approximation, and then makes the mapping bijective by introducing η-link; this detail is discussed in section 5.

As we mentioned in section 2.3 the compiler replaces terms in $\diamond\eta$ and $\overline{\mathcal{L}_\lambda}$ with fresh variables linked to the problematic terms. Each class of problematic terms has a dedicated link.

```
kind baselink type.
type link-eta tm -> tm -> baselink.
type link-beta tm -> tm -> baselink.
typeabbrev link (inctx baselink).
typeabbrev links (list link).
```

The right hand side of a link, the problematic term, can occur under binders. To accommodate this situation the compiler wraps baselink using the inctx container (see, · ⊢ ·).

INVARIANT 2 (LINK LEFT HAND SIDE). *The left hand side of a suspended link is a variable.*

New links are suspended by construction. If the left hand side variable is assigned during a step, then the link is considered for

progress and possibly eliminated. This is discussed in section 5 and section 7.

Note that move strongly relies on invariant 1: the length of the arguments of all occurrences of a unification variable and the number of abstractions in its assignment have to match. In turn this grants that move never fails.

3.1 Notational conventions

When variables x and y can occur in term t we shall write t_{xy} to stress this fact.

We write $\sigma = \{ A_{xy} \mapsto y \}$ for the assignment abs x\abs y\y and $\sigma = \{ A \mapsto \lambda x.\lambda y.y \}$ for lam x\lam y\y .

When detailing examples we write links as equations between two terms under a context. The equality sign is subscripted with kind of baselink. For example $x \vdash A_x =_\beta F_x \ a$ corresponds to:

```
abs x\ val (link-beta (uva A [x]) (app[uva F [x],con "a"]))
```

4 BASIC SIMULATION OF $\mathcal{F}_o$ IN $\mathcal{H}_o$

In this section we describe a basic compilation scheme that we refine later, in the following sections. This scheme is sufficient to implement an $\simeq_o$ that respects β-conversion for terms in $\mathcal{L}_\lambda$. The extension to $\eta\beta$-conversion is described in Section 5 and the support for terms outside $\mathcal{L}_\lambda$ in Section 7.

4.1 Compilation

> E:manca beta normal in entrata

The main task of the compiler is to recognize $\mathcal{F}_o$ variables standing for functions and map them to higher order variables in $\mathcal{H}_o$. In order to bring back the substitution from $\mathcal{H}_o$ to $\mathcal{F}_o$ the compiler builds a "memory map" connecting the the kind of variables using routine (*malloc*).

The signature of the comp predicate below allows for the generation of links (suspended unification problems) that play no role in this section but play a major role in section 5 and section 7. With respect to section 2 the signature also allows for updates to the substitution.

```
type comp fm -> tm -> mmap -> mmap -> links -> links ->
  subst -> subst -> o.
comp (fcon C) (con C) M M L L S S.
comp (flam F) (lam F1) M1 M2 L1 L2 S1 S2 :-          (c_λ)
  comp-lam F F1 M1 M2 L1 L2 S1 S2.
comp (fuva A) (uva B []) M1 M2 L L S1 S2 :-
  m-alloc (fv A) (hv B (arity z)) M1 M2 S1 S2.
comp (fapp A) (app A1) M1 M2 L1 L2 S1 S2 :-          (c_@)
  fold6 comp A A1 M1 M2 L1 L2 S1 S2.
```

The code above uses that possibility in order to allocate space for the variables, i.e. sets their memory address to none (a details not worth mentioning in the previous sections).

```
type comp-lam (fm -> fm) -> (tm -> tm) ->
  mmap -> mmap -> links -> links -> subst -> subst -> o.
comp-lam F G M1 M2 L1 L3 S1 S2 :-
  pi x y\ (pi M L S\ comp x y M M L L S S) =>
    comp (F x) (G y) M1 M2 L1 (L2 y) S1 S2,
  close-links L2 L3.
```

In the code above the syntax pi x y\.. is syntactic sugar for iterated pi abstraction, as in pi x\ pi y\...

The auxiliary function close-links tests if the bound variable v really occurs in the link. If it is the case the link is wrapped into an additional abs node binding v. In this way links generated deep inside the compiled terms can be moved outside their original context of binders.

```
type close-links (tm -> links) -> links -> o.
close-links (v\[X  |L v]) [X|R] :- !, close-links L R.
close-links (v\[X v|L v]) [abs X|R] :- close-links L R.
close-links (_\[]) [].
```

Note that we could remove the first rule, whose solve purpose is to make links more readable by pruning unused context entries.

4.1.1 Compilation of terms in $\diamond\beta_0$.
The following rule is inserted just before rule ($c_@$).

```
comp (fapp [fuva A|Ag]) (uva B Ag1) M1 M2 L L S1 S2 :-
  pattern-fragment Ag, !,
    fold6 comp Ag Ag1 M1 M1 L L S1 S1,
    len Ag Arity,
    m-alloc (fv A) (hv B (arity Arity)) M1 M2 S1 S2.
```

Note that compiling Ag cannot create new mappings nor links, since Ag is made of bound variables and the hypothetical rule loaded by comp-lam (see below) grants this property. Also note that this rule generates no links.

The only detail worth discussing is the fact that the procedure updates a substitution, rather than just crafting one as presented in section 2. The reason is that the algorithm folds over a term, updating a substitution while it traverses it.

> E:explain better

4.2 Execution

A step in $\mathcal{H}_o$ consists in unifying two terms and reconsidering all links for progress. If any of the two tasks fail we say that the entire step fails, and it is at this granularity that we can relate steps in the two languages.

```
type hstep tm -> tm -> links -> links -> subst -> subst -> o.
hstep T1 T2 L1 L2 S1 S3 :-
  (T1 ≃_λ T2) S1 S2,
  progress L1 L2 S2 S3.
```

Note tha the infix notation ((A $\simeq_\lambda$ B) C D) is syntactic sugar for (($\simeq_\lambda$) A B C D).

Reconsidering links is a fixpoint, since the progress of a link can update the substitution and in turn enable another link to progress.

```
type progress links -> links -> subst -> subst -> o.
progress L L2 S1 S3 :-
  progress1 L L1 S1 S2,
  occur-check-links L1,
  if (L = L1, S1 = S2)
    (L2 = L1, S3 = S1)
    (progress L1 L2 S2 S3).
```

In the base compilation scheme progress1 is the identity on both the links and the substitution, so the fixpoint trivially terminates. Sections 5 and 7 add rules to progress1 and justify why the don't

hinder termination. For brevity we omit the code that applies the substitution S1 to all terms in $\mathbb{L}$.

Since compilation moves problematic terms out of the sigh of $\simeq_\lambda$, that procedure can only perform a partial occur check. For example the unification problem $X \simeq_\lambda f\ Y$ cannot generate a cyclic substitution alone, but should be disallowed if a $\mathbb{L}$ contains a link like $\vdash Y =_\eta \lambda z.X_z$: We don't know yet if Y will feature a lambda in head position, but we surely know it contains X, hence $f\ Y$ and that fails the occur check. The procedure occur-check-links is in charge of ensuring that each link does not represent a (suspended) unification problem doomed to fail because of occur check. This check is needed in order to guarantee proposition 2.2 (simulation fidelity).

4.3 Substitution decompilation

Decompiling the substitution requires to first force the progress of links and then allocating new unassigned variables in the substitution for $\mathcal{F}_o$ and finally decompiling all assignments. Note that invariant 2 and the occur check allows us to update the subst.

```
type decompile mmap -> links -> subst ->
  fsubst -> fsubst -> o.
decompile M1 L S F1 F3 :-
  commit-links L S S1,
  complete-mapping S1 S1 M1 M2 F1 F2,
  decompm M2 M2 S1 F2 F3.
```

Decompiling an assignment requires to turn abstractions into lamb-das. For aesthetic purposes we also eta-contract the result (not needed since $\mathcal{F}_o$ equality can do that)

```
type decompm mmap -> mmap -> subst -> fsubst -> fsubst -> o.
decompm _ [] _ F F.
decompm M [mapping (fv V) (hv H _)|MS] S F1 F3 :- set? H S A,
  deref-assmt S A A1,
  abs->lam A1 T, decomp M T T1,
  eta-contract T1 T2,
  assign V F1 T2 F2,
  decompm M MS S F2 F3.
decompm M [mapping _ (hv H _)|MS] S F1 F2 :- unset? H S,
  decompm M MS S F1 F2.
```

Finally decompiling a term is trivial, now that we have an extended mapping containing all unassigned variables $\simeq_\lambda$ may have introduced.

```
type decomp mmap -> tm -> fm -> o.
decomp _ (con C)  (fcon C).
decomp M (app A) (fapp B) :- map (decomp M) A B.
decomp M (lam F) (flam G) :-
  pi x y\ (pi M\ decomp M x y) => decomp M (F x) (G y).
decomp M (uva Hv Ag) R :-
  mem M (mapping (fv Fv) (hv Hv _)),
  map (decomp M) Ag Bg,
  beta (fuva Fv) Bg R.
```

Note that we use beta to build fapp nodes when needed (if Ag is empty no fapp node should appear).

INVARIANT 3. *TODO: dire che il mapping è bijective*

4.4 Definition of $\simeq_o$ and its properties

```
type (≃o) fm -> fm -> fsubst -> o.
(A ≃o B) F :-
  fo.beta-reduce A A',
  fo.beta-reduce B B',
  comp A' A'' [] M1 [] [] [] S1,
  comp B' B'' M1 M2 [] [] S1 S2,
  hstep A'' B'' [] [] S2 S3,
  decompm M2 M2 S3 [] F.
```

The code given so far applies to terms in $\beta\eta$-normal form where unification variables in $\mathcal{F}_o$ can occur non linearly but always with the same number of arguments, and where their arguments are distinct names (as per $\mathcal{L}_\lambda$).

LEMMA 4.1 (COMPILATION ROUND TRIP). *If* comp S T [] M [] _ [] _ *then* decomp M T S

PROOF SKETCH. trivial, since the terms are beta normal beta just builds an app. □

LEMMA 4.2. *Properties* ?? *and* ?? *hold for the implementation of* $\simeq_o$ *above*

PROOF SKETCH. In this setting $=_\lambda$ is as strong as $=_o$ on ground terms. What we have to show is that whenever two different $\mathcal{F}_o$ terms can be made equal by a substitution ρ (plus the β_l and β_r if needed) we can find this ρ by finding a σ via $\simeq_\lambda$ on the corresponding $\mathcal{H}_o$ terms and by decompiling it. If we look at the $\mathcal{F}_o$ terms, the are two interesting cases:

- fuva X $\simeq_o$ s. In this case after comp we have $Y \simeq_\lambda t$ that succeeds with $\sigma = \{Y \mapsto t\}$ and σ is decompiled to $\rho = \{Y \mapsto s\}$.
- fapp[fuva X|L] $\simeq_o$ s. In this case we have $Y_{\vec{x}} \simeq_\lambda t$ that succeeds with $\sigma = \{\vec{y} \vdash Y \mapsto t[\vec{x}/\vec{y}]\}$ that in turn is decompiled to $\rho = \{Y \mapsto \lambda\vec{y}.s[\vec{x}/\vec{y}]\}$. Thanks to β_l $(\lambda\vec{y}.s[\vec{x}/\vec{y}])\ \vec{x} =_o s$.

Since the mapping is a bijection occur check in $\mathcal{H}_o$ corresponds to occur check in $\mathcal{F}_o$. □

LEMMA 4.3. *Properties simulation (2.1) and fidelity (2.2) hold*

PROOF SKETCH. Since progress1 is trivial fstep and hstep are the same, that is in this context where input terms are $\beta\eta$-normal and we disregard η-equivalence $\simeq_\lambda$ is equivalent to $\simeq_o$. □

4.5 Limitations of by this basic scheme

The basic compilation scheme is not about to deal wit the following problem:

$$\mathbb{P} = \{\ \lambda xy.X\ y\ x \simeq_o \lambda xy.x \quad \lambda x.f\ (X\ x)\ x \simeq_o Y\ \}$$

Note that here X is used with different arities, moreover in the second problem the left hand side happens to be an eta expansion (of $f(\lambda y.y)$) only after we discover (at run time) that $X = \lambda x\lambda y.y$ (i.e. that X discards the x argument). Both problems are addressed in the next two sections.

5 HANDLING OF $\diamond\eta$

η-reduction is an equivalence relation where a term of the form $\lambda x.t \cdot x$ can be converted to t any time x does not occur as a free variable in t. We call t the η-contraction of $\lambda x.t \cdot x$.

Following the compilation scheme of section 4.1 the unification problem $\mathbb{P}$ is compiled as follows:

$$\mathbb{P} = \{ \ \lambda x.X \cdot x \ \simeq_o \ f \ \}$$
$$\mathbb{T} = \{ \ \lambda x.A_x \ \simeq_\lambda \ f \ \}$$
$$\mathbb{M} = \{ \ X \mapsto A^1 \ \}$$

While $\lambda x.X \cdot x \simeq_o f$ does admit the solution $\rho = \{ X \mapsto f \}$, the corresponding problem in $\mathbb{T}$ does not: `lam x\ uva A [x]` and `con"f"` start with different, rigid, term constructors hence $\simeq_\lambda$ fails.

In order to guarantee proposition 2.1 we detect lambdas that can disappear by eta contraction (section 5.1) and we modify the compiled terms by putting fresh unification variables in their place: the problematic term is moved from $\mathbb{T}$ to $\mathbb{L}$ (section 5.2). The compilation of the problem $\mathbb{P}$ above is refined to:

$$\mathbb{P} = \{ \ \lambda x.X \cdot x \ \simeq_o \ f \ \}$$
$$\mathbb{T} = \{ \qquad A \ \simeq_\lambda \ f \ \}$$
$$\mathbb{M} = \{ \ X \mapsto B^1 \ \}$$
$$\mathbb{L} = \{ \ \vdash A =_\eta \ \lambda x.B_x \ \}$$

As per invariant 2 the term on the left is a variable, and its right counterpart is the term in $\diamond\eta$. That term has the following property:

INVARIANT 4 (η-link rhs). *The* rhs *of any* η-link *has the shape* $\lambda x.t$ *and t is not a lambda.*

η-link are kept in the link store $\mathbb{L}$ during execution and activated when some conditions hold on lhs or rhs. Link activation is implemented by extending the `progress1` predicate (defined in section 4.2).

5.1 Detection of $\diamond\eta$

When compiling a term t we need to determine if any subterm $s \in \mathcal{P}(t)$ that is of the form $\lambda x.r$, where x occurs in r, can be a η-expansion, i.e. if there exists a substitution ρ such that $\rho(\lambda x.r) =_o s$. The detection of lambda abstractions that can "disappear" is not as trivial as it may seems, here a few examples:

$$\lambda x.f \cdot (A \cdot x) \qquad\qquad \in \diamond\eta \ \ \rho = \{ A \mapsto \lambda x.x \}$$
$$\lambda x.f \cdot (A \cdot x) \cdot x \qquad\quad \in \diamond\eta \ \ \rho = \{ A \mapsto \lambda x.a \}$$
$$\lambda x.f \cdot x \cdot (A \cdot x) \qquad\quad \notin \diamond\eta$$
$$\lambda x.\lambda y.f \cdot (A \cdot x) \cdot (B \cdot y \cdot x) \in \diamond\eta \ \ \rho = \{ A \mapsto \lambda x.x, \ B \mapsto \lambda y.\lambda x.y \}$$

The first two examples are easy, and show how a unification variable can expose or erase a variable in their scope and turn the resulting term in an η-expansion or not.

The third example shows that when a variable occurs outside the scope of a unification variable it cannot be erased and can hence prevent a term from being an η-expansion.

The last example shows the recursive nature of the check we need to implement. The term starts with a spine of two lambdas hence the whole term is in $\diamond\eta$ iff the inner term $\lambda y.f \cdot (A \cdot x) \cdot (B \cdot y \cdot x)$ is in $\diamond\eta$ itself. If it is, it could η-contract to $f \cdot (A \cdot x)$ making $\lambda x.f \cdot (A \cdot x)$ a potential η-expansion.

We can now define more formally how $\diamond\eta$ terms are detected together with its auxiliary functions:

Definition 5.1 (may-contract-to). A β-normal term s *may-contract-to* a name x if there exists a substitution ρ such that $\rho s =_o x$.

LEMMA 5.2. *A β-normal term $s = \lambda x_1 \ldots x_n.t$ may-contract-to x only if one of the following three conditions holds:*

(1) *$n = 0$ and $t = x$;*
(2) *t is the application of x to a list of terms l and each l_i may-contract-to x_i (e.g. $\lambda x_1 \ldots x_n.x \cdot x_1 \ldots x_n =_o x$);*
(3) *t is a unification variable with scope W, and for any $v \in \{x, x_1 \ldots x_n\}$, there exists a $w_i \in W$, such that w_i may-contract-to v (if $n = 0$ this is equivalent to $x \in W$).*

PROOF SKETCH. Since our terms are in β-normal form there is only one rule that can play a role (namely η_l), hence if the term s is not exactly x (case 1) it can only be an η-expansion of x, or a unification variable that can be assigned to x, or a combination of both. If s begins with a lambda, then the lambda can only disappear by η contraction. In that case the term t is under the spine of binders $x_1 \ldots x_n$, t can either be x applied to terms that can *may-contract-to* these variables (case 2), or a unification variable that can be assigned to that application (case 3). $\square$

Definition 5.3 (occurs-rigidly). A name x *occurs-rigidly* in a β-normal term t, if $\forall\rho, x \in \mathcal{P}(\rho t)$

In other words x occurs-rigidly in t if it occurs in t outside of the scope of unification variables since an instantiation is allowed to discard x from the scope of the unification variable. Note that η-contraction cannot make x disappear, since the variables being erased by η-contraction are locally bound inside t.

We can now derive the implementation for $\diamond\eta$ detection:

Definition 5.4 (maybe-eta). Given a β-normal term $s = \lambda x_1 \ldots x_n.t$, *maybe-eta s* holds if any of the following holds:

(1) *t is a constant or a name applied to the arguments $l_1 \ldots l_m$ such that $m \geq n$ and for every i such that $m - n < i \leq m$ the term l_i may-contract-to x_i, and no x_i occurs-rigidly in $l_1 \ldots l_{m-n}$;*
(2) *t is a unification variable with scope W and for each x_i there exists a $w_j \in W$ such that w_j may-contract-to x_i.*

LEMMA 5.5 ($\diamond\eta$ DETECTION). *If t is a β-normal term and maybe-eta t holds, then $t \in \diamond\eta$.*

PROOF SKETCH. Follows from definition 5.3 and lemma 5.2 $\square$

Remark that the converse of lemma 5.5 does not hold: there exists a term t satisfying the criteria (1) of definition 5.4 that is not in $\diamond\eta$, i.e. there exists no substitution ρ such that ρt is an η-expansion. A simple counter example is $\lambda x.f \cdot (A \cdot x) \cdot (A \cdot x)$ since x does not *occur-rigidly* in the first argument of f, and the second argument of f *may-contract-to* x. In other words $A \cdot x$ may either use or discard x, but our analysis does not take into account that *the same term* cannot have two contrasting behaviors.

As we will see in the rest of this section this is not a problem since it does not break proposition 2.1 nor proposition 2.2.

5.2 Compilation and decompilation

Compilation. The following rule is inserted just before rule (c_λ) from the code in section 4.1.

```
comp (flam F) (uva A Scope) M1 M2 L1 L3 S1 S3 :-
  maybe-eta (flam F) [], !,
    alloc S1 A S2,
    comp-lam F F1 M1 M2 L1 L2 S2 S3,
    get-scope (lam F1) Scope,
    L3 = [val (link-eta (uva A Scope) (lam F1)) | L2].
```

The rule triggers when the input term `flam F` is in $\Diamond\eta$. It compiles `flam F` to `lam F1` but puts the fresh variable `A` in its place. This variable sees all the names free in `lam F1`. The critical part of this rule is the creation of the η-link, which relates the variable `A` with `lam F1`. This link clearly validates invariant 2.

COROLLARY 5.6. *The rhs of any η-link has exactly one lambda abstraction, hence the rule above respects invariant 4.*

PROOF SKETCH. By contradiction, suppose that the rule above is triggered and that the rhs of the link is $\lambda x.\lambda y.t_{xy}$. If *maybe-eta $\lambda y.t_{xy}$* holds the recursive call to comp (made by comp-lam) must have put a fresh variable in its place, so this case is impossible. Otherwise, if *maybe-eta $\lambda y.t_{xy}$* does not hold, also *maybe-eta $\lambda x.\lambda y.t_{xy}$* does not hold, contradicting the assumption that the rule triggered. □

Decompilation. Decompilation of η-link is performed by adding new rules to the commit-link predicate. In particular, given $\Gamma \vdash X =_\eta t$, we can note that this unification never fails, since X is a flexible term and no other η-link has X has lhs (by definition 5.9). The link is remove from $\mathbb{L}$ and commit-links terminates.

5.3 Progress

η-link are meant to delay the unification of "problematic" terms until we know for sure if the term has to be η-contracted or not.

Definition 5.7 (progress-η-left). A link $\Gamma \vdash X =_\eta T$ is removed from $\mathbb{L}$ when X becomes rigid. Let $y \in \Gamma$, there are two cases:

(1) if $X = a$ or $X = y$ or $X = f\, a_1 \ldots a_n$ we unify the η-expansion of X with T, that is we run $\lambda x.X\, x \simeq_\lambda T$
(2) if $X = \lambda x.t$ we run $X \simeq_\lambda T$.

Definition 5.8 (progress-η-right). A link $\Gamma \vdash X =_\eta T$ is removed from $\mathbb{L}$ when either 1) *maybe-eta T* does not hold (anymore) or 2) by η-contracting T to T', T' is a term not starting with the lam constructor. In the first case, X is unified with T and in the second one, X is unified with T' (under the context Γ).

There is a third case in which a link is removed from $\mathbb{L}$, namely when the lhs is assigned to a variable that is the lhs of another η-link.

Definition 5.9 (progress-η-deduplicate). A link $\Gamma \vdash X_{\vec{s}} =_\eta T$ is removed from $\mathbb{L}$ when another link $\Delta \vdash X_{\vec{r}} =_\eta T'$ is in $\mathbb{L}$. By invariant 1 the length of $\vec{s}$ and $\vec{r}$ is the same hence we can move the term T' from Δ to Γ by renaming its bound variables, i.e. $T'' = T'[\vec{r}/\vec{s}]$. We then run $T \simeq_\lambda T''$ (under the context Γ).

> D:Below the proof of proposition 2.8, ho usato 3 lemmi ausiliari, forse si può compattare in una prova più piccola?

LEMMA 5.10. *Given a η-link l, the unification done by* progress-η-left *is between terms in* $\mathcal{W}$

PROOF SKETCH. Let σ be the substitution, such that $\mathcal{W}(\sigma)$. lhs $\in \sigma$, therefore $\mathcal{W}(\text{lhs})$. By definition 5.7, if 1) lhs is a name, a constant of an application, then, lhs is unified with the η-reduced term t obtain from rhs. By corollary 5.6, rhs has one lambda, therefore $\mathcal{W}(t)$. Otherwise, 2) lhs has lam as functor, rhs should not be an η-expansion ans, so, $\mathcal{W}(\text{rhs})$. In both cases, unification is performed between terms in $\mathcal{W}$. □

LEMMA 5.11. *Given a η-link l, the unification done by* progress-η-right *is between terms in* $\mathcal{W}$.

PROOF SKETCH. lhs is variable, and, by definition 5.8, rhs is either no more a $\Diamond\eta$, i.e. rhs is not a η-expansion and, so, $\mathcal{W}(\text{rhs})$. Otherwise, rhs can reduce to a term which cannot be a η-expansion, and, so, $\mathcal{W}(\text{rhs})$. In both cases, unification is done between terms in $\mathcal{W}$. □

LEMMA 5.12. *Given a η-link l, the unification done by* progress-η-deduplicate *is between terms in* $\mathcal{W}$.

PROOF. Trivial, since the unification is done between unification variables, which are by definition in $\mathcal{W}$. □

LEMMA 5.13. *Proposition 2.8 holds, i.e., given a substitution σ and a η-link l, after the activation of l, $\mathcal{W}(\sigma)$ holds.*

PROOF SKETCH. By lemmas 5.10 to 5.12, every unification performed by the activation of a η-link is performed between terms in $\mathcal{W}$, therefore, the substitution remains $\mathcal{W}$.

> D:Bisogna aggiungere un lemma nella section 2.3 che dice che unificare due termini in $\mathcal{W}$, in una σ, tale che $\mathcal{W}(\sigma)$, non invalida $\mathcal{W}$

□

LEMMA 5.14. progress *terminates.*

PROOF SKETCH. Rules definitions 5.7 and 5.8 and definition 5.9 remove one link from $\mathbb{L}$, hence they cannot be applied indefinitely. Moreover each rule only relies on terminating operations such as $\simeq_\lambda$, η-contraction, η-expansion, relocation (a recursive copy of a finite term). □

> D:Proove simulation fidelity, dicendo che *progress-η-right* è inutile

LEMMA 5.15 (FIDELITY WITH η-link). *Let $\mathbb{T}$ a unification problem and σ a substitution. The introduction of η-link guarantees proposition 2.2*

PROOF SKETCH. If 1) a η-link is activated by *progress-η-left*, then unification is done between a $\Diamond\eta$ term t_1 and a term t_2 (with $\mathcal{W}(t_2)$). This activation performs a unification which succeeds iff the original problem in $\mathcal{F}_o$ succeeds. If 2) a η-link is activated by *progress-η-deduplicate*, then the unification is done between two $\Diamond\eta$ terms, and again this unification succeeds iff it succeeds in $\mathcal{F}_o$. Finally, if 3) a η-link is activated by *progress-η-right*, the unification, done between a variable and a term, always succeeds, this is what we expect to guarantee fidelity, since we are essentially removing

a hole added by the compilation in the place of a $\diamond\eta$ subterm. In all the cases fidelity is respected. □

Example of progress-η-left. The example at the beginning of section 5, once $\sigma = \{ A \mapsto f \}$, triggers this rule since the link becomes $\vdash f =_\eta \lambda x.B_x$ and the lhs is a constant. In turn the rule runs $\lambda x.f \cdot x \simeq_\lambda \lambda x.B_x$, resulting in $\sigma = \{ A \mapsto f \ ; \ B_x \mapsto f \}$. Decompilation the generates $\rho = \{ X \mapsto f \}$, since X is mapped to B and f is the η-contracted version of $\lambda x.f \cdot x$.

Example of progress-η-deduplicate. A very basic example of η-link deduplication, is given below:

$$\mathbb{P} = \{ \ \lambda x.(X \cdot x) \simeq_o \lambda x.(Y \cdot x) \ \}$$
$$\mathbb{T} = \{ \qquad A \simeq_\lambda C \qquad \}$$
$$\mathbb{M} = \{ \ X \mapsto B^1 \qquad Y \mapsto D^1 \ \}$$
$$\mathbb{L} = \{ \ \vdash A =_\eta \lambda x.B_x \qquad \vdash C =_\eta \lambda x.D_x \ \}$$

The result of $A \simeq_\lambda C$ is that the two η-link share the same lhs. By unifying the two rhs we get $\sigma = \{ A \mapsto C, B \mapsto D \}$. In turn, given the map $\mathbb{M}$, this second assignment is decompiled to $\rho = \{ X \mapsto Y \}$ as expected.

We delay at the end of next section an example of η-link progression due to *progress-η-right*

6 ENFORCING INVARIANT 1

We report here the problem given in section 4.5 where X is used with two different arities and the output of the compilation does not respect invariant 3 (merging the two mappings for s would break invariant 1). In this section we explain how to replace the duplicate mapping with some η-link in order to restore the invariants.

$$\mathbb{P} = \{ \ \lambda x.\lambda y.(X \cdot y \cdot x) \simeq_o \lambda x.\lambda y.x \qquad \lambda x.(f \cdot (X \cdot x) \cdot x) \simeq_o Y \ \}$$
$$\mathbb{T} = \{ \qquad A \simeq_\lambda \lambda x.\lambda y.x \qquad\qquad D \simeq_\lambda F \ \}$$
$$\mathbb{M} = \{ \ X \mapsto E^1 \qquad Y \mapsto F^0 \qquad X \mapsto C^2 \ \}$$
$$\mathbb{L} = \left\{ \begin{array}{ll} \vdash D =_\eta \lambda x.(f \cdot E_x \cdot x) & \vdash A =_\eta \lambda x.B_x \\ x \vdash B_x =_\eta \lambda y.C_{yx} & \end{array} \right\}$$

We see that the maybe-eta as identified $\lambda xy.X \cdot y \cdot x$ and $\lambda x.f \cdot (X \cdot x) \cdot x$ and the compiler has replaced them with A and D respectively. However, the mapping $\mathbb{M}$ breaks invariant 3: the $\mathcal{F}_o$ variable X is mapped to two different $\mathcal{H}_o$ variables. To address this problem we adjust the compiler's output with a `map-deduplication` procedure.

Definition 6.1 (align-arity). Given two mappings $m_1 : X \mapsto A^m$ and $m_2 : X \mapsto C^n$ where $m < n$ and $d = n - m$, *align-arity* m_1 m_2 generates the following d links, one for each i such that $0 \le i < d$,

$$x_0 \ldots x_{m+i} \vdash B^i_{x_0 \ldots x_{m+i}} =_\eta \lambda x_{m+i+1}.B^{i+1}_{x_0 \ldots x_{m+i+1}}$$

where B^i is a fresh variable of arity $m + i$, and $B^0 = A$ as well as $B^d = C$.

The intuition is that we η-expand the occurrence of the variable with lower arity to match the higher arity. Since each η-link can add exactly one lambda, we need as many links as the difference between the two arities.

Definition 6.2 (map-deduplication). Forall mappings $m_1, m_2 \in \mathbb{M}$ such that $m_1 : X \mapsto A^m$ and $m_2 : X \mapsto C^n$ and $m < n$ we remove m_1 from $\mathbb{M}$ and add to $\mathbb{L}$ the result of *align-arity* m_1 m_2.

If we look back the example give at the beginning of this section, we can deduplicate $X \mapsto E^1, X \mapsto C^2$ by removing the first mapping and adding the auxiliary η-link: $x \vdash E_x =_\eta \lambda y.C_{xy}$. After deduplication the compiler output is as follows:

$$\mathbb{P} = \{ \ \lambda x.\lambda y.(X \cdot y \cdot x) \simeq_o \lambda x.\lambda y.x \qquad \lambda x.(f \cdot (X \cdot x) \cdot x) \simeq_o Y \ \}$$
$$\mathbb{T} = \{ \qquad A \simeq_\lambda \lambda x.\lambda y.x \qquad\qquad D \simeq_\lambda F \ \}$$
$$\mathbb{M} = \{ \ Y \mapsto F^0 \qquad X \mapsto C^2 \ \}$$
$$\mathbb{L} = \left\{ \begin{array}{ll} x \vdash E_x =_\eta \lambda y.C_{xy} & \vdash D =_\eta \lambda x.(f \cdot E_x \cdot x) \\ \vdash A =_\eta \lambda x.B_x & x \vdash B_x =_\eta \lambda y.C_{yx} \end{array} \right\}$$

In this example, $\mathbb{T}_1$ assigns A which triggers $\mathbb{L}_3$ and then $\mathbb{L}_4$ by definition 5.7. C_{yx} is therefore assigned to x (the second variable of its scope). We finally see the *progress-η-right* of $\mathbb{L}_1$: its rhs is now $\lambda y.y$ (C_{xy} gives y). Since it is no more in $\diamond\eta$, $\lambda y.y$ is unified with E_x. Moreover, $\mathbb{L}_2$ is also triggered due to definition 5.8: $\lambda x.(f \cdot (\lambda y.y) \cdot x)$ is η-reducible to $f \cdot (\lambda y.y)$ which is a term not starting with the `lam` constructor.

7 HANDLING OF $\overline{\mathcal{L}_\lambda}$

> D:I've rewritten it, it is clearer?

Until now, we have only dealt we unification of terms in $\mathcal{L}_\lambda$. However, we want the unification relation to be more robust so that it can work with terms in $\overline{\mathcal{L}_\lambda}$. In general, unification in $\overline{\mathcal{L}_\lambda}$ admits more then one solution and committing one of them in the substitution does not guarantee prop. **??**. For instance, $X \cdot a \simeq_o a$ is a unification problem admits two different substitutions: $\rho_1 = \{X \mapsto \lambda x.x\}$ and $\rho_2 = \{X \mapsto \lambda\_.a\}$. Prefer one over the other may break future unifications.

It is the case that, given a list of unification problems, $\mathbb{P}_1 \ldots \mathbb{P}_n$ with $\mathbb{P}_n$ in $\overline{\mathcal{L}_\lambda}$, the resolution of $\bigwedge_{i=0}^{n-1} \mathbb{P}_i$ gives a partial substitution ρ, such that $\rho\mathbb{P}_n$ falls again in $\mathcal{L}_\lambda$.

$$\mathbb{P} = \{ \ X \simeq_o \lambda x.Y \qquad (X \cdot a) \simeq_o a \ \}$$
$$\mathbb{T} = \{ \ A \simeq_\lambda \lambda x.B \qquad (A \cdot a) \simeq_\lambda a \ \}$$
$$\mathbb{M} = \{ \ Y \mapsto B^0 \qquad X \mapsto A^0 \ \}$$

In the example above, we see that $\mathbb{P}_1$ instantiates X so that $\mathbb{P}_2$, can be solved in $\mathcal{L}_\lambda$.

> E:it is even a ground term, there is no unification left to perform actually

> D:i don't understand the note

On the other hand, we see that, $\simeq_\lambda$ can't solve the compiled problems $\mathbb{T}$. In fact, the resolution of $\mathbb{T}_1$ gives the substitution $\sigma = \{A \mapsto \lambda x.B\}$, but the dereferencing of $\mathbb{T}_2$ gives the non-unifiable problem $(\lambda x.B) \cdot a \simeq_\lambda a$.

To address this unification problem, term compilation should capture the terms t in $\overline{\mathcal{L}_\lambda}$ and replace them with fresh variables X. The variables X and the terms t are linked through a β-link.

β-link guarantees invariant 2 and the term on the rhs has the following property:

> D:Is it clearer?

INVARIANT 5 (β-link rhs). *The rhs of any β-link has the shape $X_{s_1 \ldots s_n} \cdot t_1 \ldots t_m$ such that X is a unification variable with scope $s_1 \ldots s_n$ and $t_1 \ldots t_m$ is a list of terms. This is equivalent to* `app[uva X S | L]` *where* `S` $= s_1 \ldots s_n$ *and* `L` $= t_1 \ldots t_m$.

Lemma 7.1 (β-link with rigid lhs). *If the* lhs *of a* β-link *is instantiated to a rigid term and its rhs counterpart is still in* $\overline{\mathcal{L}_\lambda}$*, the original unification problem is not in* $\mathcal{L}_\lambda$ *and the unification fails.*

Proof sketch. Given $X \cdot t_1 \ldots t_n \simeq_\lambda t$ where t is a rigid term and $t_1 \ldots t_n$ is not in $\mathcal{L}_\lambda$. By construction, $X \cdot t_1 \ldots t_n$ is replaced with a variable Y, and the β-link $\Gamma \vdash Y =_\beta X \cdot t_1 \ldots t_n$ is created. The unification instantiates Y to t, making the lhs of the link a rigid term, while rhs is still in $\overline{\mathcal{L}_\lambda}$. The original problem is in fact outside $\mathcal{L}_\lambda$. □

7.1 Compilation and decompilation

Detection of $\overline{\mathcal{L}_\lambda}$ is quite simple to implement in the compiler, since it is sufficient to detect applications with flexible head and argument that are not in $\mathcal{L}_\lambda$.

```
comp (fapp [fuva A|Ag]) (uva B Scope) M1 M3 L1 L3 S1 S4 :- !,
  pattern-fragment-prefix Ag Pf Extra,
  len Pf Arity,
  alloc S1 B S2,
  m-alloc (fv A) (hv C (arity Arity)) M1 M2 S2 S3,
  fold6 comp Pf    Pf1   M2 M2 L1 L1 S3 S3,
  fold6 comp Extra Extra1 M2 M3 L1 L2 S3 S4,
  Beta = app [uva C Pf1 | Extra1],
  get-scope Beta Scope,
  L3 = [val (link-beta (uva B Scope) Beta) | L2].
```

The list `Ag` is split into the list `Pf` and `Extra` such that append `Pf Extra Ag` and `Pf` is the largest prefix of `Ag` such that `Pf` is in $\mathcal{L}_\lambda$. The rhs of the β-link is the application of a fresh variable `C` having in scope all the free variables appearing in the compiled version of `Pf` and `Extra`. The variable `B`, returned has the compiled term, is a fresh variable having in scope all the free variables occurring in `Pf1` and `Extra1`

Invariant 6. *The* rhs *of a* β-link *has the shape* $X_{s_1 \ldots s_n} \cdot t_1 \ldots t_m$.

Corollary 7.2. *Let* $X_{s_1 \ldots s_n} \cdot t_1 \ldots t_m$ *be the* rhs *of a* β-link*, then* $m > 0$.

Proof sketch. Assume we have a β-link, by contradiction, if $m = 0$, then the original $\mathcal{F}_o$ term has the shape fapp[fuva M | Ag] where `Ag` is a list of distinct names (i.e. the list `Extra` is empty). This case is however captured by rule (c_λ) (from section 4.1) and no β-link is produced which contradicts our initial assumption. □

Corollary 7.3. *Let* $X_{s_1 \ldots s_n} \cdot t_1 \ldots t_m$ *be the* rhs *of a* β-link*, then* t_1 *either appears in* $s_1 \ldots s_n$ *or it is not a name.*

Proof sketch. By construction, the lists $s_1 \ldots s_n$ and $t_1 \ldots t_m$ are built by splitting the list `Ag` from the original term fapp [fuva A|Ag]. $s_1 \ldots s_n$ is the longest prefix of the compiled terms in `Ag` which is in $\mathcal{L}_\lambda$. Therefore, by definition of $\mathcal{L}_\lambda$, t_1 must appear in $s_1 \ldots s_n$, otherwise $s_1 \ldots s_n$ is not the longest prefix in $\mathcal{L}_\lambda$, or it is a term with a constructor of tm as functor. □

> E:Dire che maybe eta fa il detect anche su termini che non sono il llambda, oppure dirlo in section of maybeeta + dare un esempio?

Decompilation. During progress, as claimed in invariant 5, the decompilation can only have β-link with not instantiated lhs. In this case, lhs is unified with rhs.

> D:not really sure of this, we can have $F\, a = \lambda x.Gx$. In this case when do we fail: for sure in decompile. But to respect fidelity, we should fail immediately: we have a β-link and a η-link with same lhs

7.2 Progress

The activation of a β-link is performed when its rhs falls under $\mathcal{L}_\lambda$ under a given substitution.

Definition 7.4 (progress-beta-$\mathcal{L}_\lambda$). Given a substitution σ and a β-link $\Gamma \vdash T =_\beta X_{s_1 \ldots s_n} \cdot t_1 \ldots t_m$ such that σt_1 is a name, say t, and $t \notin s_1 \ldots s_n$. If $m = 0$, then the β-link is removed and lhs is unified with $X_{s_1 \ldots s_n}$. If $m > 0$, then the β-link is replaced by a refined version $\Gamma \vdash T =_\beta Y_{s_1 \ldots s_n,t} \cdot t_2 \ldots t_m$ with reduced list of arguments and Y being a fresh variable. Moreover, the new link $\Gamma \vdash X_{s_1 \ldots s_n} =_\eta \lambda x.Y_{s_1 \ldots s_n,x}$ is added to $\mathbb{L}$.

> D:L'ho ri-for-mu-lato

Definition 7.5 (progress-beta-rigid-head). A link $\Gamma \vdash X =_\beta X_{s_1 \ldots s_n} \cdot t_1 \ldots t_m$ is removed from $\mathbb{L}$ if $X_{s_1 \ldots s_n}$ is instantiated to a term t and the β-reduced term t' obtained from the application of t to $l_1 \ldots l_m$ is in $\mathcal{L}_\lambda$. Moreover, X is unified to t.

Definition 7.6 (progress-beta-dedup). Given a β-link l_1 and second link $l_2 \in \mathbb{L}$, such that they share the same lhs. In this case, the two rhs are unified and a l_2 is removed from $\mathbb{L}$.

> D:added the def below to respect fidelity

Lemma 7.7. progress *terminates*

Proof sketch. Let l a β-link in the store $\mathbb{L}$. If l is activated by *progress-beta-rigid-head*, then it disappears from $\mathbb{L}$ and progress terminates. Otherwise, the rhs of l is made by a variable applied to m arguments. At each activation of *progress-beta-$\mathcal{L}_\lambda$*, l is replaced by a new β-link l^1 having $m - 1$ arguments. At the mth iteration, the β-link l^m has no more arguments and is removed from $\mathbb{L}$. Note that at the mth iteration, m new η-link have been added to $\mathbb{L}$, however, by lemma 5.14, the algorithm terminates. Finally *progress-beta-dedup* also guarantees termination since it makes a unification $\mathbb{U}$ and if $\mathbb{U}$ fails, then progress terminates and if $\mathbb{U}$ succeeds, the recursive calls to progress have a specialized β-link ...

> E:funziona. per essere più precisi io parlerei di ordine lessicografico (tipico ordine ben fondato usato per dimostrare terminazione). Nl nostro caso è la tripla (argomenti extra dei beta, numero di beta, numero di eta).

> D:da finire

□

Corollary 7.8. *Given a* β-link*, the variables occurring in its* rhs *are in* $\mathcal{L}_\lambda$.

> D:is it clearer?

Proof sketch. By construction, the rhs of β-link has the shape $X_{s_1 \ldots s_n} \cdot t_1 \ldots t_m$, $s_1 \ldots s_n$ is in $\mathcal{L}_\lambda$ and all the terms $t_1 \ldots t_n$ are in $\mathcal{L}_\lambda$, too. If a β-link is triggered by *progress-beta-rigid-head*, then, by definition 7.5, that link is removed by $\mathbb{L}$, and the property is satisfied. If the η-link is activated by *progress-beta-$\mathcal{L}_\lambda$*, then, by

definition 7.4, the new β-link as a variable as a scope which is still in $\mathcal{L}_\lambda$.

LEMMA 7.9 (FIDELITY WITH β-link). *The introduction of β-link guarantees proposition 2.2*

PROOF SKETCH. Let $\mathbb{T}$ a unification problem and σ a substitution such that $\mathbb{T} \in \overline{\mathcal{L}_\lambda}$. If $\sigma\mathbb{T}$ is in $\mathcal{L}_\lambda$, then by definitions 7.4 and 7.5, the β-link associated to the subterm of $\mathbb{T}$ have been solved and removed. The unification is done between terms in $\mathcal{L}_\lambda$ and by lemma 5.15 fidelity is guaranteed. If $\sigma\mathbb{T}$ is in $\overline{\mathcal{L}_\lambda}$, then, by lemma 7.1, the unification fails, as per the corresponding unification in $\mathcal{F}_o$. □

Example of progress-beta-$\mathcal{L}_\lambda$. Consider the β-link below:

$$\mathbb{P} = \{ \; X \simeq_o \lambda x.x \quad \lambda x.(Y \cdot (X \cdot x)) \simeq_o f \; \}$$
$$\mathbb{T} = \{ \; A \simeq_\lambda \lambda x.x \qquad\qquad B \simeq_\lambda f \; \}$$
$$\mathbb{M} = \{ \; Y \mapsto D^0 \quad X \mapsto A^0 \; \}$$
$$\mathbb{L} = \left\{ \begin{array}{ll} \vdash A =_\eta \lambda x.E_x & \vdash B =_\eta \lambda x.C_x \\ x \vdash C_x =_\beta (D \cdot E_x) \end{array} \right\}$$

Initially the β-link rhs is a variable D applied to the E_x. The first unification problem results in $\sigma = \{A \mapsto \lambda x.x\}$. In turn this instantiation triggers $\mathbb{L}_1$ by *progress-η-left* and E_x is assigned to x. Under this substitution the β-link becomes $x \vdash C_x =_\beta (D \cdot x)$, and by *progress-beta-$\mathcal{L}_\lambda$* it is replaced with the link: $\vdash E =_\eta \lambda x.D_x$, while C_x is unified with D_x. The second unification problem assigns f to B, that in turn activates the second η-link (f is assigned to C), and then all the remaining links are solved. The final $\mathcal{H}_o$ substitution is $\sigma = \{A \mapsto \lambda x.x, B \mapsto f, C_x \mapsto (f \cdot x), D \mapsto f, E_x \mapsto x, F_x \mapsto C_x\}$ and is decompiled into $\rho = \{X \mapsto \lambda x.x, Y \mapsto f\}$.

Example of progress-beta-rigid-head. We can take the example provided in section 7. The problem is compiled into:

$$\mathbb{P} = \{ \; X \simeq_o \lambda x.Y \quad (X \cdot a) \simeq_o a \; \}$$
$$\mathbb{T} = \{ \; A \simeq_\lambda \lambda x.B \qquad C \simeq_\lambda a \; \}$$
$$\mathbb{M} = \{ \; Y \mapsto B^0 \quad X \mapsto A^0 \; \}$$
$$\mathbb{L} = \{ \; \vdash C =_\beta (A \cdot a) \; \}$$

The first unification problems is solved by the substitution $\sigma = \{A \mapsto \lambda x.B\}$. The β-link becomes $\vdash C =_\beta ((\lambda x.B) \cdot a)$ whose rhs can be β-reduced to B. B is in $\mathcal{L}_\lambda$ and is unified with C. The resolution of the second unification problem gives the final substitution $\sigma = \{A \mapsto \lambda x.B, B \mapsto C, C \mapsto a\}$ which is decompiled into $\rho = \{X \mapsto \lambda x.a, Y \mapsto a\}$.

7.3 Relaxing lemma β-link with rigid lhs

Working with terms in $\mathcal{L}_\lambda$ is sometime too restrictive. There exists systems such as λProlog [10], Abella [5], which delay the resolution of $\overline{\mathcal{L}_\lambda}$ unification problems if the substitution is not able to put them in $\mathcal{L}_\lambda$.

$$\mathbb{P} = \{ \; (X \cdot a) \simeq_o a \quad X \simeq_o \lambda x.Y \; \}$$

In the example above, $\mathbb{P}_1$ is in $\overline{\mathcal{L}_\lambda}$ and the object language cannot solve it, and, by proposition 2.2, the meta language neither. However, we can be more permissive, and relax lemma 7.1. This modification is quite simple to manage: we are introducing a new $\overline{\mathcal{L}_\lambda}$ progress rule, say *progress-beta-$\overline{\mathcal{L}_\lambda}$*, by which, if lhs is rigid and rhs is flexible, the considered β-link is kept in the store and no progression is

done[4]. *progress-beta-$\overline{\mathcal{L}_\lambda}$* makes occur-check-links partial, since the check is possible only on links with a variable on the lhs. This means that we can have two links $\vdash X =_\beta Y \cdot a$ and $\vdash f \cdot X =_\beta Y \cdot a$ where the occur check does not throw an error. Note however, that the decompilation of the two links will force the unification of X to $Y \cdot a$ and then the unification of $f \cdot (Y \cdot a)$ to $Y \cdot a$, which fails by the occur check of $\simeq_\lambda$.

A second strategy to deal with problem that are in $\overline{\mathcal{L}_\lambda}$ is to make some approximation. This is the case for example of the unification algorithm of Coq used in its type class solver [16]. The approximation consists in forcing a choice (among the others) when the unification problem is in $\overline{\mathcal{L}_\lambda}$. For instance, in $X \cdot a \cdot b = Y \cdot b$, the last argument of the two terms is the same, therefore Y is assigned to Xa. Note that this is of course an approximation, since $\sigma = \{X = \lambda x.Y, Y = \_\}$ is another valid substitution for the original problem. This approximation can be easily introduced in our unification procedure, by adding new custom β-link progress rules.

The commit-link predicate can be extended to add heuristics if during the decompilation phase β-link remain. For example, the same approximation explained above can be delayed and applied only if the terms in $\lozenge\beta_0$ never falls in $\mathcal{L}_\lambda$ after the execution of all the unification problems. We want to point out, that we call this *approximation*, since we are making a choice among all the possible unifiers and therefore, we can pick the wrong one.

8 ACTUAL IMPLEMENTATION IN ELPI

In this paper we show a minimized example. The full code is there. But we also have to code things in Coq-Elpi.

The main difference between the presentation in the previous sections and the actual implementation for Coq is that the main loop hrun is replaced by the one of Prolog that chains calls to the unification procedure. In order implement the store of links we resort to Elpi's CLP engine and use constraints (suspended goals) to represent links, and constraint handling rules to implement progress operations involving more than one link.

about the progress of 1 link:

```
link-eta L R :- suspend-condition L R Holes, !,
  declare_constraint (link-eta L R) Holes.
link-eta L R :-
  progress. % e.g. L = R.
```

about the progress of 2 links:

```
constraint link-eta {
  rule (N1 ▹ G1 ?- link-eta (uvar X LX1) T1) % match
    / (N2 ▹ G2 ?- link-eta (uvar X LX2) T2) % remove
    | (relocate LX1 LX2 T2 T2')              % condition
   <=> (N1 ▹ G1 ?- T1 = T2').                % new goal
}
```

Remark how the invariant about uvar arity makes this easy, since LX1 and LX2 have the same length. Also note that N1 only contains the names of the first link (while relocate runs in the disjoint union) and Elpi ensures that T2' can live in N1.

---

[4]This new rule trivially guarantees the termination of progress

9   OTHER ENCODINGS AND RELATED WORK

One could ignore the similarity between $\simeq_o$ and $\simeq_\lambda$ and "just" describe the object language unification procedure in the meta language by crafting a `unif` routine and using it as follows in rule $(r3)$:

```
decision X :- unif X (all A x\ app [P, x]), finite A,
  pi x\ decision (app [P, x]).
```

This choice would underuse the logic programming engine provided by the meta language since, by removing any datum from the head of rules, indexing degenerates. Moreover the unification procedure `unif` programmed in the meta language is likely to be an order of magnitude slower than one that is built-in.

Another possibility is to avoid having the application and abstraction nodes in the syntax tree, and use the ones of meta language, as in the following:

```
finite (fin N).
decision (nfact N NF).
decision (all A x\ P x) :- finite A, pi x\ decision (P x).
```

There are two reasons for dismissing this encoding. The first one is that in CIC it is not always possible to adopt it since the type system of the meta language is too weak to accommodate for the one of the object language. In CIC the lambda abstraction has to carry a type in order to make type checking decidable. Moreover CIC allows for functions with a variable arity, like the following example:

```
Fixpoint arr T n := if n is S m then T -> arr T m else T.
Definition sum n : arr nat n := ...
Check sum 2   7 8   : nat.
Check sum 3   7 8 9 : nat.
```

The type system of the λProlog is too stringent to accept this terms. The second reason is that the CIC encoding provided by Elpi is used for meta programming (extending) the Coq system, hence it must accommodate the manipulation of terms that are now know in advance (not even defined in Coq) without using introspection primitives such as Prologs's `functor` and `arg`. In this sense constants have to live in an open world, like the `string` data type used in the examples so far.

In the literature we could find related encoding of the Calculus of Constructions [2]. The goal of that work was to exhibit a logic program performing proof checking in CC and hence relate the proof system of intuitionistic higher-order logic (that animates λProlog programs) with the Calculus of Constructions. The encoding is hence tailored toward a different goal, and utilizes three relations to represent the equational theory of CC. Section 6 contains a discussion about the use of the unification procedure of the meta language in presence of non ground goals, but the authors do not aim at exploiting it to the degree we want.

sucks

cite isabelle's TC, that are baked in

10   CONCLUSION

Benefits: less work, reuse efficient ho unif (3x faster), indexing,
  Future: tabling and static analysis (reuse for ML again).
  Very little is Coq specific. Applies to all OL that are not a subsystem of HOL, or for ML that are used for meta programming.

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APPENDIX

This appendix contains the entire code described in this paper. The code can also be accessed at the URL: https://github.com/FissoreD/paper-ho

Note that `(a infix b) c d` de-sugars to `(infix) a b c d`.

Explain builtin name (can be implemented by loading name after each pi)

11 THE MEMORY

```
kind addr type.
type addr nat -> addr.
typeabbrev (mem A) (list (option A)).

type set? addr -> mem A -> A -> o.
set? (addr A) Mem Val :- get A Mem Val.

type unset? addr -> mem A -> o.
unset? Addr Mem :- not (set? Addr Mem _).

type assign-aux nat -> mem A -> A -> mem A -> o.
assign-aux z (none :: L) Y (some Y :: L).
assign-aux (s N) (X :: L) Y (X :: L1) :- assign-aux N L Y L1.

type assign addr -> mem A -> A -> mem A -> o.
assign (addr A) Mem1 Val Mem2 :- assign-aux A Mem1 Val Mem2.

type get nat -> mem A -> A -> o.
get z (some Y :: _) Y.
get (s N) (_ :: L) X :- get N L X.

type alloc-aux nat -> mem A -> mem A -> o.
alloc-aux z [] [none] :- !.
alloc-aux z L L.
alloc-aux (s N) [] [none | M] :- alloc-aux N [] M.
alloc-aux (s N) [X | L] [X | M] :- alloc-aux N L M.

type alloc addr -> mem A -> mem A -> o.
alloc (addr A as Ad) Mem1 Mem2 :- unset? Ad Mem1,
  alloc-aux A Mem1 Mem2.

type new-aux mem A -> nat -> mem A -> o.
new-aux [] z [none].
new-aux [A | As] (s N) [A | Bs] :- new-aux As N Bs.

type new mem A -> addr -> mem A -> o.
new Mem1 (addr Ad) Mem2 :- new-aux Mem1 Ad Mem2.
```

12 THE OBJECT LANGUAGE

```
kind fm type.
type fapp list fm -> fm.
type flam (fm -> fm) -> fm.
type fcon string -> fm.
type fuva addr -> fm.

typeabbrev fsubst (mem fm).
```

```
type fder fsubst -> fm -> fm -> o.
fder _ (fcon C) (fcon C).
fder S (fapp A) (fapp B) :- map (fder S) A B.
fder S (flam F) (flam G) :-
  pi x\ fder S x x => fder S (F x) (G x).
fder S (fuva N) R :- set? N S T, fder S T R.
fder S (fuva N) (fuva N) :- unset? N S.
```

```
type fderef fsubst -> fm -> fm -> o.                    (ρs)
fderef S T T2 :- fder S T T1, napp T1 T2.
```

```
type (=_o) fm -> fm -> o.                               (=_o)
fcon X =_o fcon X.
fapp A =_o fapp B :- forall2 (=_o) A B.
flam F =_o flam G :- pi x\ x =_o x => F x =_o G x.
fuva N =_o fuva N.
flam F =_o T :-                                         (η_l)
  pi x\ beta T [x] (T' x), x =_o x => F x =_o T' x.
T =_o flam F :-                                         (η_r)
  pi x\ beta T [x] (T' x), x =_o x => T' x =_o F x.
fapp [flam X | L] =_o T :- beta (flam X) L R, R =_o T.  (β_l)
T =_o fapp [flam X | L] :- beta (flam X) L R, T =_o R.  (β_r)
```

```
type extend-subst fm -> fsubst -> fsubst -> o.
extend-subst (fuva N) S S' :- mem.alloc N S S'.
extend-subst (flam F) S S' :-
  pi x\ (pi S\extend-subst x S S) => extend-subst (F x) S S'.
extend-subst (fcon _) S S.
extend-subst (fapp L) S S1 :- fold extend-subst L S S1.
```

```
type beta fm -> list fm -> fm -> o.
beta A [] A.
beta (flam Bo) [H | L] R :- napp (Bo H) F, beta F L R.
beta (fapp A) L (fapp X) :- append A L X.
beta (fuva N) L (fapp [fuva N | L]).
beta (fcon H) L (fapp [fcon H | L]).
beta N L (fapp [N | L]) :- name N.
```

```
type napp fm -> fm -> o.
napp (fcon C) (fcon C).
napp (fuva A) (fuva A).
napp (flam F) (flam G) :- pi x\ napp (F x) (G x).
napp (fapp [fapp L1 |L2]) T :- !,
  append L1 L2 L3, napp (fapp L3) T.
napp (fapp L) (fapp L1) :- map napp L L1.
napp N N :- name N.
```

```
type beta-reduce fm -> fm -> o.
beta-reduce (uvar _ _) _ :- halt "Passed uvar to beta-reduce".
beta-reduce A A :- name A.
beta-reduce (fcon A) (fcon A).
beta-reduce (fuva A) (fuva A).
beta-reduce (flam A) (flam B) :-
  pi x\ beta-reduce (A x) (B x).
beta-reduce (fapp [flam B | L]) T2 :- !,
```

```
1741      beta (flam B) L T1, beta-reduce T1 T2.
1742    beta-reduce (fapp L) (fapp L1) :-
1743      map beta-reduce L L1.
1744
1745    type mk-app fm -> list fm -> fm -> o.
1746    mk-app T L S :- beta T L S.
1747
1748    type eta-contract fm -> fm -> o.
1749    eta-contract (fcon X) (fcon X).
1750    eta-contract (fapp L) (fapp L1) :- map eta-contract L L1.
1751    eta-contract (flam F) T :- eta-contract-aux [] (flam F) T.
1752    eta-contract (flam F) (flam F1) :-
1753      pi x\ eta-contract x x => eta-contract (F x) (F1 x).
1754    eta-contract (fuva X) (fuva X).
1755    eta-contract X X :- name X.
1756
1757    type eta-contract-aux list fm -> fm -> fm -> o.
1758    eta-contract-aux L (flam F) T :-
1759      pi x\ eta-contract-aux [x|L] (F x) T. % also checks H Prefix does not keep x
1760    eta-contract-aux L (fapp [H|Args]) T :-
1761      rev L LRev, append Prefix LRev Args,
1762      if (Prefix = []) (T = H) (T = fapp [H|Prefix]).
1763
```

13 THE META LANGUAGE

```
1766    kind inctx type -> type.                              (· ⊢ ·)
1767    type abs (tm -> inctx A) -> inctx A.
1768    type val A -> inctx A.
1769    typeabbrev assignment (inctx tm).
1770    typeabbrev subst (mem assignment).
1771
1772    kind tm  type.
1773    type app list tm -> tm.
1774    type lam (tm -> tm) -> tm.
1775    type con string -> tm.
1776    type uva addr -> list tm -> tm.
1777
1778    type (≃λ) tm -> tm -> subst -> subst -> o.
1779    (con C ≃λ con C) S S.
1780    (app L1 ≃λ app L2) S S1 :- fold2 (≃λ) L1 L2 S S1.
1781    (lam F1 ≃λ lam F2) S S1 :-
1782      pi x\ (pi S\ (x ≃λ x) S S) => (F1 x ≃λ F2 x) S S1.
1783    (uva N Args ≃λ T) S S1 :-
1784      set? N S F,!, move F Args T1, (T1 ≃λ T) S S1.
1785    (T ≃λ uva N Args) S S1 :-
1786      set? N S F,!, move F Args T1, (T ≃λ T1) S S1.
1787    (uva M A1 ≃λ uva N A2) S1 S2 :- !,
1788      pattern-fragment A1, pattern-fragment A2,
1789      prune! M A1 N A2 S1 S2.
1790    (uva N Args ≃λ T) S S1 :- not_occ N S T, pattern-fragment Args,
1791      bind T Args T1, assign N S T1 S1.
1792    (T ≃λ uva N Args) S S1 :- not_occ N S T, pattern-fragment Args,
1793      bind T Args T1, assign N S T1 S1.
1794
1795    type prune! addr -> list tm -> addr ->
1796              list tm -> subst -> subst -> o.
1797    /* no pruning needed */
1798
```

```
1799    prune! N A  N A  S S :- !.
1800    prune! M A  N A  S1 S2 :- !, bind (uva M A) A Ass,
1801      assign N S1 Ass S2.
1802    /* prune different arguments */
1803    prune! N A1 N A2 S1 S3 :- !,
1804      new S1 W S2, prune-same-variable W A1 A2 [] Ass,
1805      assign N S2 Ass S3.
1806    /* prune to the intersection of scopes */
1807    prune! N A1 M A2 S1 S4 :- !,
1808      new S1 W S2, prune-diff-variables W A1 A2 Ass1 Ass2,
1809      assign N S2 Ass1 S3,
1810      assign M S3 Ass2 S4.
1811
1812    type prune-same-variable addr -> list tm -> list tm ->
1813                             list tm -> assignment -> o.
1814    prune-same-variable N [] [] ACC (val (uva N Args)) :-
1815      rev ACC Args.
1816    prune-same-variable N [X|XS] [X|YS] ACC (abs F) :-
1817      pi x\ prune-same-variable N XS YS [x|ACC] (F x).
1818    prune-same-variable N [_|XS] [_|YS] ACC (abs F) :-
1819      pi x\ prune-same-variable N XS YS ACC (F x).
1820
1821    type permute list nat -> list tm -> list tm -> o.
1822    permute [] _ [].
1823    permute [P|PS] Args [T|TS] :-
1824      nth P Args T,
1825      permute PS Args TS.
1826
1827    type build-perm-assign addr -> list tm -> list bool ->
1828                   list nat -> assignment -> o.
1829    build-perm-assign N ArgsR [] Perm (val (uva N PermutedArgs)) :-
1830      rev ArgsR Args, permute Perm Args PermutedArgs.
1831    build-perm-assign N Acc [tt|L] Perm (abs T) :-
1832      pi x\ build-perm-assign N [x|Acc] L Perm (T x).
1833    build-perm-assign N Acc [ff|L] Perm (abs T) :-
1834      pi x\ build-perm-assign N Acc L Perm (T x).
1835
1836    type keep list A -> A -> bool -> o.
1837    keep L A tt :- mem L A, !.
1838    keep _ _ ff.
1839
1840    type prune-diff-variables addr -> list tm -> list tm ->
1841                      assignment -> assignment -> o.
1842    prune-diff-variables N Args1 Args2 Ass1 Ass2 :-
1843      map (keep Args2) Args1 Bits1,
1844      map (keep Args1) Args2 Bits2,
1845      filter Args1 (mem Args2) ToKeep1,
1846      filter Args2 (mem Args1) ToKeep2,
1847      map (index ToKeep1) ToKeep1 IdPerm,
1848      map (index ToKeep1) ToKeep2 Perm21,
1849      build-perm-assign N [] Bits1 IdPerm Ass1,
1850      build-perm-assign N [] Bits2 Perm21 Ass2.
1851
1852    type beta tm -> list tm -> tm -> o.
1853    beta A [] A :- !.
1854    beta (lam Bo) [H | L] R :- beta (Bo H) L R1, beta-aux R1 R.
1855    beta (app A) L (app X) :- append A L X.
1856
```

```
1857    beta (con H) L (app [con H | L]).
1858    beta X L (app[X|L]) :- name X.
1859
1860    type beta-aux tm -> tm -> o.
1861    beta-aux (app [HD|TL]) R :- !, beta HD TL R.
1862    beta-aux A A.
1863
1864    /* occur check for N before crossing a functor */
1865    type not_occ  addr -> subst -> tm -> o.
1866    not_occ N S (uva M Args) :- set? M S F,
1867      move F Args T, not_occ N S T.
1868    not_occ N S (uva M Args) :- unset? M S, not (M = N),
1869      forall1 (not_occ_aux N S) Args.
1870    not_occ _ _ (con _).
1871    not_occ N S (app L) :- not_occ_aux N S (app L).
1872    /* Note: lam is a functor for the meta language! */
1873    not_occ N S (lam L) :- pi x\ not_occ_aux N S (L x).
1874    not_occ _ _ X :- name X.
1875    /* finding N is ok */
1876    not_occ N _ (uva N _).
1877
1878    /* occur check for X after crossing a functor */
1879    type not_occ_aux addr -> subst -> tm -> o.
1880    not_occ_aux N S (uva M _) :- unset? M S, not (N = M).
1881    not_occ_aux N S (uva M Args) :- set? M S F,
1882      move F Args T, not_occ_aux N S T.
1883    not_occ_aux N S (app L) :- forall1 (not_occ_aux N S) L.
1884    not_occ_aux N S (lam F) :- pi x\ not_occ_aux N S (F x).
1885    not_occ_aux _ _ (con _).
1886    not_occ_aux _ _ X :- name X.
1887    /* finding N is ko, hence no rule */
1888
1889    /* copy T T' vails if T contains a free variable, i.e. it
1890       performs scope checking for bind */
1891    type copy tm -> tm -> o.
1892    copy (con C)   (con C).
1893    copy (app L)   (app L') :- map copy L L'.
1894    copy (lam T)   (lam T') :- pi x\ copy x x => copy (T x) (T' x).
1895    copy (uva A L) (uva A L') :- map copy L L'.
1896
1897    type bind tm -> list tm -> assignment -> o.
1898    bind T [] (val T') :- copy T T'.
1899    bind T [X | TL] (abs T') :- pi x\ copy X x => bind T TL (T' x).
1900
1901    type deref subst -> tm -> tm -> o.                    (σt)
1902    deref _ (con C) (con C).
1903    deref S (app A) (app B) :- map (deref S) A B.
1904    deref S (lam F) (lam G) :-
1905      pi x\ deref S x x => deref S (F x) (G x).
1906    deref S (uva N L) R :- set? N S A,
1907      move A L T, deref S T R.
1908    deref S (uva N A) (uva N B) :- unset? N S,
1909      map (deref S) A B.
1910
1911    type move assignment -> list tm -> tm -> o.
1912    move (abs Bo) [H|L] R :- move (Bo H) L R.
1913    move (val A)  [] A.
1914
```

```
1915
1916
1917    type deref-assmt subst -> assignment -> assignment -> o.
1918    deref-assmt S (abs T) (abs R) :- pi x\ deref-assmt S (T x) (R x).
1919    deref-assmt S (val T) (val R) :- deref S T R.
1920
1921
```

14   THE COMPILER

```
1923    kind arity type.
1924    type arity nat -> arity.
1925
1926    kind fvariable type.
1927    type fv addr -> fvariable.
1928
1929    kind hvariable type.
1930    type hv addr -> arity -> hvariable.
1931
1932    kind mapping type.
1933    type mapping fvariable -> hvariable -> mapping.
1934    typeabbrev mmap (list mapping).
1935
1936    typeabbrev scope (list tm).
1937    typeabbrev inctx ho.inctx.
1938    kind baselink type.
1939    type link-eta  tm -> tm -> baselink.
1940    type link-beta tm -> tm -> baselink.
1941    typeabbrev link (inctx baselink).
1942    typeabbrev links (list link).
1943
1944    macro @val-link-eta T1 T2 :- ho.val (link-eta T1 T2).
1945    macro @val-link-beta T1 T2 :- ho.val (link-beta T1 T2).
1946
1947
1948    type get-lhs link -> tm -> o.
1949    get-lhs (val (link-beta A _)) A.
1950    get-lhs (val (link-eta A _)) A.
1951
1952    type get-rhs link -> tm -> o.
1953    get-rhs (val (link-beta _ A)) A.
1954    get-rhs (val (link-eta _ A)) A.
1955
1956
1957    type occurs-rigidly fm -> fm -> o.
1958    occurs-rigidly N N.
1959    occurs-rigidly _ (fapp [fuva _|_]) :- !, fail.
1960    occurs-rigidly N (fapp L) :- exists (occurs-rigidly N) L.
1961    occurs-rigidly N (flam B) :- pi x\ occurs-rigidly N (B x).
1962
1963    type reducible-to list fm -> fm -> fm -> o.
1964    reducible-to _ N N :- !.
1965    reducible-to L N (fapp[fuva _|Args]) :- !,
1966      forall1 (x\ exists (reducible-to [] x) Args) [N|L].
1967    reducible-to L N (flam B) :- !,
1968      pi x\ reducible-to [x | L] N (B x).
1969    reducible-to L N (fapp [N|Args]) :-
1970      last-n {len L} Args R,
1971      forall2 (reducible-to []) R {rev L}.
1972
```

```
1973
1974  type maybe-eta fm -> list fm -> o.                          (◇η)
1975  maybe-eta (fapp[fuva _|Args]) L :- !,
1976    forall1 (x\ exists (reducible-to [] x) Args) L, !.
1977  maybe-eta (flam B) L :- !, pi x\ maybe-eta (B x) [x | L].
1978  maybe-eta (fapp [T|Args]) L :- (name T; T = fcon _),
1979    split-last-n {len L} Args First Last,
1980    none (x\ exists (y\ occurs-rigidly x y) First) L,
1981    forall2 (reducible-to []) {rev L} Last.
1982
1983
1984  type locally-bound tm -> o.
1985  type get-scope-aux tm -> list tm -> o.
1986  get-scope-aux (con _) [].
1987  get-scope-aux (uva _ L) L1 :-
1988    forall2 get-scope-aux L R,
1989    flatten R L1.
1990  get-scope-aux (lam B) L1 :-
1991    pi x\ locally-bound x => get-scope-aux (B x) L1.
1992  get-scope-aux (app L) L1 :-
1993    forall2 get-scope-aux L R,
1994    flatten R L1.
1995  get-scope-aux X [X] :- name X, not (locally-bound X).
1996  get-scope-aux X [] :- name X, (locally-bound X).
1997
1998  type names1 list tm -> o.
1999  names1 L :-
2000    names L1,
2001    new_int N,
2002    if (1 is N mod 2) (L1 = L) (rev L1 L).
2003
2004  type get-scope tm -> list tm -> o.
2005  get-scope T Scope :-
2006    get-scope-aux T ScopeDuplicata,
2007    undup ScopeDuplicata Scope.
2008  type rigid fm -> o.
2009  rigid X :- not (X = fuva _).
2010
2011  type comp-lam (fm -> fm) -> (tm -> tm) ->
2012    mmap -> mmap -> links -> links -> subst -> subst -> o.
2013  comp-lam F G M1 M2 L1 L3 S1 S2 :-
2014    pi x y\ (pi M L S\ comp x y M M L L S S) =>
2015      comp (F x) (G y) M1 M2 L1 (L2 y) S1 S2,
2016    close-links L2 L3.
2017
2018  type close-links (tm -> links) -> links -> o.
2019  close-links (v\[X  |L v]) [X|R] :- !, close-links L R.
2020  close-links (v\[X v|L v]) [abs X|R] :- close-links L R.
2021  close-links (_\[]) [].
2022  type comp fm -> tm -> mmap -> mmap -> links -> links ->
2023    subst -> subst -> o.
2024  comp (fcon C) (con C) M M L L S S.
2025  comp (flam F) (uva A Scope) M1 M2 L1 L3 S1 S3 :-
2026    maybe-eta (flam F) [], !,
2027      alloc S1 A S2,
2028      comp-lam F F1 M1 M2 L1 L2 S2 S3,
2029      get-scope (lam F1) Scope,
2030
2031      L3 = [val (link-eta (uva A Scope) (lam F1)) | L2].
2032  comp (flam F) (lam F1) M1 M2 L1 L2 S1 S2 :-            (c_λ)
2033    comp-lam F F1 M1 M2 L1 L2 S1 S2.
2034  comp (fuva A) (uva B []) M1 M2 L L S1 S2 :-
2035    m-alloc (fv A) (hv B (arity z)) M1 M2 S1 S2.
2036  comp (fapp [fuva A|Ag]) (uva B Ag1) M1 M2 L L S1 S2 :-
2037    pattern-fragment Ag, !,
2038      fold6 comp Ag Ag1 M1 M1 L L S1 S1,
2039      len Ag Arity,
2040      m-alloc (fv A) (hv B (arity Arity)) M1 M2 S1 S2.
2041  comp (fapp [fuva A|Ag]) (uva B Scope) M1 M3 L1 L3 S1 S4 :- !,
2042    pattern-fragment-prefix Ag Pf Extra,
2043    len Pf Arity,
2044    alloc S1 B S2,
2045    m-alloc (fv A) (hv C (arity Arity)) M1 M2 S2 S3,
2046    fold6 comp Pf    Pf1    M2 M2 L1 L1 S3 S3,
2047    fold6 comp Extra Extra1 M2 M3 L1 L2 S3 S4,
2048    Beta = app [uva C Pf1 | Extra1],
2049    get-scope Beta Scope,
2050    L3 = [val (link-beta (uva B Scope) Beta) | L2].
2051  comp (fapp A) (app A1) M1 M2 L1 L2 S1 S2 :-            (c_@)
2052    fold6 comp A A1 M1 M2 L1 L2 S1 S2.
2053
2054  type alloc mem A -> addr -> mem A -> o.
2055  alloc S N S1 :- mem.new S N S1.
2056
2057  type compile-terms-diagnostic
2058    triple diagnostic fm fm ->
2059    triple diagnostic tm tm ->
2060    mmap -> mmap ->
2061    links -> links ->
2062    subst -> subst -> o.
2063  compile-terms-diagnostic (triple D FO1 FO2) (triple D HO1 HO2) M1 M3 L1
2064    fo.beta-reduce FO1 FO1',
2065    fo.beta-reduce FO2 FO2',
2066    comp FO1' HO1 M1 M2 L1 L2 S1 S2,
2067    comp FO2' HO2 M2 M3 L2 L3 S2 S3.
2068
2069  type compile-terms
2070    list (triple diagnostic fm fm) ->
2071    list (triple diagnostic tm tm) ->
2072    mmap -> links -> subst -> o.
2073  compile-terms T H M L S :-
2074    fold6 compile-terms-diagnostic T H [] M_ [] L_ [] S_,
2075    print-compil-result T H L_ M_,
2076    deduplicate-map M_ M S_ S L_ L.
2077
2078  type make-eta-link-aux nat -> addr -> addr ->
2079    list tm -> links -> subst -> subst -> o.
2080  make-eta-link-aux z Ad1 Ad2 Scope1 L H1 H1  :-
2081    rev Scope1 Scope, eta-expand (uva Ad2 Scope) T1,
2082    L = [val (link-eta (uva Ad1 Scope) T1)].
2083  make-eta-link-aux (s N) Ad1 Ad2 Scope1 L H1 H3 :-
2084    rev Scope1 Scope, alloc H1 Ad H2,
2085    eta-expand (uva Ad Scope) T2,
2086    (pi x\ make-eta-link-aux N Ad Ad2 [x|Scope1] (L1 x) H2 H3),
2087    close-links L1 L2,
2088
18
```

```
2089      L = [val (link-eta (uva Ad1 Scope) T2) | L2].

2091    type make-eta-link nat -> nat -> addr -> addr ->
2092         list tm -> links -> subst -> subst -> o.
2093    make-eta-link (s N) z Ad1 Ad2 Vars L H H1 :-
2094      make-eta-link-aux N Ad2 Ad1 Vars L H H1.
2095    make-eta-link z (s N) Ad1 Ad2 Vars L H H1 :-
2096      make-eta-link-aux N Ad1 Ad2 Vars L H H1.
2097    make-eta-link (s N) (s M) Ad1 Ad2 Vars Links H H1 :-
2098      (pi x\ make-eta-link N M Ad1 Ad2 [x|Vars] (L x) H H1),
2099      close-links L Links.

2101    type deduplicate-map mmap -> mmap ->
2102         subst -> subst -> links -> links -> o.
2103    deduplicate-map [] [] H H L L.
2104    deduplicate-map [(mapping (fv O) (hv M (arity LenM)) as X1) | Map1]
2105      take-list Map1 (mapping (fv O) (hv M' (arity LenM'))) _, !,
2106      std.assert! (not (LenM = LenM')) "Deduplicate map, there is a bug"
2107      print "arity-fix links:" {ppmapping X1} "~!~" {ppmapping (mapping (fv O) (hv M' (arity LenM')))},
2108      make-eta-link LenM LenM' M M' [] New H1 H2,
2109      print "new eta link" {pplinks New},
2110      append New L1 L2,
2111      deduplicate-map Map1 Map2 H2 H3 L2 L3.
2112    deduplicate-map [A|As] [A|Bs] H1 H2 L1 L2 :-
2113      deduplicate-map As Bs H1 H2 L1 L2, !.
2114    deduplicate-map [A|_] _ H _ _ _ :-
2115      halt "deduplicating mapping error" {ppmapping A} {ho.ppsubst H}.
```

15   THE PROGRESS FUNCTION

```
2119    macro @one :- s z.

2121    type contract-rigid list ho.tm -> ho.tm -> ho.tm -> o.
2122    contract-rigid L (ho.lam F) T :-
2123      pi x\ contract-rigid [x|L] (F x) T. % also checks H Prefix does not
2124    contract-rigid L (ho.app [H|Args]) T :-
2125      rev L LRev, append Prefix LRev Args,
2126      if (Prefix = []) (T = H) (T = ho.app [H|Prefix]).

2128    type progress-eta-link ho.tm -> ho.tm -> ho.subst -> ho.subst -> links
2129    progress-eta-link (ho.app _ as T) (ho.lam x\ _ as T1) H H1 [] :- !, not (T1 = ho.uva _ _), !, fail.
2130      ({eta-expand T @one} ==l T1) H H1.
2131    progress-eta-link (ho.con _ as T) (ho.lam x\ _ as T1) H H1 [] :- !
2132      ({eta-expand T @one} ==l T1) H H1.
2133    progress-eta-link (ho.lam _ as T) T1 H H1 [] :- !,
2134      (T ==l T1) H H1.
2135    progress-eta-link (ho.uva _ _ as X) T H H1 [] :-
2136      contract-rigid [] T T1, !, (X ==l T1) H H1.
2137    progress-eta-link (ho.uva Ad _ as T1) T2 H H [@val-link-eta T1 T2] :- !,
2138      if (ho.not_occ Ad H T2) true fail.

2140    type is-in-pf ho.tm -> o.
2141    is-in-pf (ho.app [ho.uva _ _ | _]) :- !, fail.
2142    is-in-pf (ho.lam B) :- !, pi x\ is-in-pf (B x).
2143    is-in-pf (ho.con _).
2144    is-in-pf (ho.app L) :- forall1 is-in-pf L.
2145    is-in-pf N :- name N.
```

```
2147    is-in-pf (ho.uva _ L) :- pattern-fragment L.

2149    type arity ho.tm -> nat -> o.
2150    arity (ho.con _) z.
2151    arity (ho.app L) A :- len L A.

2153    type occur-check-err ho.tm -> ho.tm -> ho.subst -> o.
2154    occur-check-err (ho.con _) _ _ :- !.
2155    occur-check-err (ho.app _) _ _ :- !.
2156    occur-check-err (ho.lam _) _ _ :- !.
2157    occur-check-err (ho.uva Ad _) T S :-
2158      not (ho.not_occ Ad S T).

2160    type progress-beta-link-aux ho.tm -> ho.tm ->
2161         ho.subst -> ho.subst -> links -> o.
2162    progress-beta-link-aux T1 T2 S1 S2 [] :- is-in-pf T2, !,
2163      (T1 ==l T2) S1 S2.
2164    progress-beta-link-aux T1 T2 S S [@val-link-beta T1 T2] :- !.

2166    type progress-beta-link ho.tm -> ho.tm -> ho.subst ->
2167         ho.subst -> links -> o.
2168    progress-beta-link T (ho.app[ho.uva V Scope | L] as T2) S S2 [@val-link-
2169      arity T Arity, len L ArgsNb, ArgsNb >n Arity, !,
2170      minus ArgsNb Arity Diff, mem.new S V1 S1,
2171      eta-expand (ho.uva V1 Scope) Diff T1,
2172      ((ho.uva V Scope) ==l T1) S1 S2.

2173    progress-beta-link (ho.uva _ _ as T) (ho.app[ho.uva Ad1 Scope1 | L1] as
2175      append Scope1 L1 Scope1L,
2176      pattern-fragment-prefix Scope1L Scope2 L2,
2177      not (Scope1 = Scope2), !,
2178      mem.new S1 Ad2 S2,
2179      len Scope1 Scope1Len,
2180      len Scope2 Scope2Len,
2181      make-eta-link Scope1Len Scope2Len Ad1 Ad2 [] LinkEta S2 S3,
2182      if (L2 = []) (NewLinks = LinkEta, T2 = ho.uva Ad2 Scope2)
2183        (T2 = ho.app [ho.uva Ad2 Scope2 | L2],
2184         NewLinks = [@val-link-beta T T2 | LinkEta]).

2186    progress-beta-link T1 (ho.app[ho.uva _ _ | _] as T2) _ _ _ :-
2187
2188
2189    progress-beta-link (ho.uva _ _ as T) (ho.app[ho.uva _ _ | _] as T2) S1 _
2190      occur-check-err T T2 S1, !, fail.

2192    progress-beta-link T1 (ho.app[ho.uva _ _ | _] as T2) H H [@val-link-beta
2193
2194    progress-beta-link T1 (ho.app [Hd | Tl]) S1 S2 B :-
2195      ho.beta Hd Tl T3,
2196      progress-beta-link-aux T1 T3 S1 S2 B.

2198    type solve-link-abs link -> links -> ho.subst -> ho.subst -> o.
2199    solve-link-abs (ho.abs X) R H H1 :-
2200      pi x\ ho.copy x x => (pi S\ ho.deref S x x) =>
2201        solve-link-abs (X x) (R' x) H H1,
2202      close-links R' R.
```

```
solve-link-abs (@val-link-eta A B) NewLinks S S1 :- !,
  progress-eta-link A B S S1 NewLinks.

solve-link-abs (@val-link-beta A B) NewLinks S S1 :- !,
  progress-beta-link A B S S1 NewLinks.

type take-link link -> links -> link -> links -> o.
take-link A [B|XS] B XS :- link-abs-same-lhs A B, !.
take-link A [L|XS] B [L|YS] :- take-link A XS B YS.

type link-abs-same-lhs link -> link -> o.
link-abs-same-lhs (ho.abs F) B :-
  pi x\ link-abs-same-lhs (F x) B.
link-abs-same-lhs A (ho.abs G) :-
  pi x\ link-abs-same-lhs A (G x).
link-abs-same-lhs (@val-link-eta (ho.uva N _) _) (@val-link-eta (ho.uva N y\) _).

type same-link-eta link -> link -> ho.subst -> ho.subst -> o.
same-link-eta (ho.abs F) B H H1 :- !, pi x\ same-link-eta (F x) B H H1.
same-link-eta A (ho.abs G) H H1 :- !, pi x\ same-link-eta A (G x) H H1.
same-link-eta (@val-link-eta (ho.uva N S1) A)
              (@val-link-eta (ho.uva N S2) B) H H1 :-
  std.map2 S1 S2 (x\y\r\ r = ho.copy x y) Perm,
  Perm => ho.copy A A',
  (A' ==l B) H H1.

type progress1 links -> links -> ho.subst -> ho.subst -> o.
progress1 [] [] X X.
progress1 [A|L1] [A|L3] S S2 :- take-link A L1 B L2, !,
  same-link-eta A B S S1,
  progress1 L2 L3 S1 S2.
progress1 [L0|L1] L3 S S2 :- deref-link S L0 L,
  solve-link-abs L R S S1, !,
  progress1 L1 L2 S1 S2, append R L2 L3.
```

16 THE DECOMPILER

```
type abs->lam ho.assignment -> ho.tm -> o.
abs->lam (ho.abs T) (ho.lam R)  :- !, pi x\ abs->lam (T x) (R x).
abs->lam (ho.val A) A.

type commit-links-aux link -> ho.subst -> ho.subst -> o.
commit-links-aux (@val-link-eta T1 T2) H1 H2 :-
  ho.deref H1 T1 T1', ho.deref H1 T2 T2',
  (T1' ==l T2') H1 H2.
commit-links-aux (@val-link-beta T1 T2) H1 H2 :-
  ho.deref H1 T1 T1', ho.deref H1 T2 T2',
  (T1' ==l T2') H1 H2.
commit-links-aux (ho.abs B) H H1 :-
  pi x\ commit-links-aux (B x) H H1.

type commit-links links -> links -> ho.subst -> ho.subst -> o.
commit-links [] [] H H.
commit-links [Abs | Links] L H H2 :-
  commit-links-aux Abs H H1, !, commit-links Links L H1 H2.

type decompl-subst map -> map -> ho.subst ->
```

```
    fo.fsubst -> fo.fsubst -> o.
decompl-subst _ [A|_] _ _ _ :- fail.
decompl-subst _ [] _ F F.
decompl-subst Map [mapping (fv VO) (hv VM _)|Tl] H F F2 :-
  mem.set? VM H T, !,
  ho.deref-assmt H T TTT,
  abs->lam TTT T', tm->fm Map T' T1,
  fo.eta-contract T1 T2, mem.assign VO F T2 F1,
  decompl-subst Map Tl H F1 F2.
decompl-subst Map [mapping _ (hv VM _)|Tl] H F F2 :-
  mem.unset? VM H, decompl-subst Map Tl H F F2.

type tm->fm map -> ho.tm -> fo.fm -> o.
tm->fm _ (ho.con C)  (fo.fcon C).
tm->fm L (ho.lam B1) (fo.flam B2) :-
  pi x y\ x y => tm->fm L (B1 x) (B2 y).
tm->fm L (ho.app L1) T :- map (tm->fm L) L1 [Hd|Tl],
  fo.mk-app Hd Tl T.
tm->fm L (ho.uva VM TL) T :- mem L (mapping (fv VO) (hv VM _)),
  map (tm->fm L) TL T1, fo.mk-app (fo.fuva VO) T1 T.

type add-new-map-aux ho.subst -> list ho.tm -> map ->
     map ->  fo.fsubst -> fo.fsubst -> o.
add-new-map-aux _ [] _ [] S S.
add-new-map-aux H [T|Ts] L L2 S S2 :-
  add-new-map H T L L1 S S1,
  add-new-map-aux H Ts L1 L2 S1 S2.

type add-new-map ho.subst -> ho.tm -> map ->
     map ->  fo.fsubst -> fo.fsubst -> o.
add-new-map _ (ho.uva N _) Map [] F1 F1 :-
  mem Map (mapping _ (hv N _)), !.
add-new-map H (ho.uva N L) Map [Map1 | MapL] F1 F3 :-
  mem.new F1 M F2,
  len L Arity, Map1 = mapping (fv M) (hv N (arity Arity)),
  add-new-map H (ho.app L) [Map1 | Map] MapL F2 F3.
add-new-map H (ho.lam B) Map NewMap F1 F2 :-
  pi x\ add-new-map H (B x) Map NewMap F1 F2.
add-new-map H (ho.app L) Map NewMap F1 F3 :-
  add-new-map-aux H L Map NewMap F1 F3.
add-new-map _ (ho.con _) _ [] F F :- !.
add-new-map _ N _ [] F F :- name N.

type complete-mapping-under-ass ho.subst -> ho.assignment ->
  map -> map ->  fo.fsubst -> fo.fsubst -> o.
complete-mapping-under-ass H (ho.val Val) Map1 Map2 F1 F2 :-
  add-new-map H Val Map1 Map2 F1 F2.
complete-mapping-under-ass H (ho.abs Abs) Map1 Map2 F1 F2 :-
  pi x\ complete-mapping-under-ass H (Abs x) Map1 Map2 F1 F2.

type complete-mapping ho.subst -> ho.subst ->
  map -> map -> fo.fsubst -> fo.fsubst -> o.
complete-mapping _ [] L L F F.
complete-mapping H [none | Tl] L1 L2 F1 F2 :-
  complete-mapping H Tl L1 L2 F1 F2.
complete-mapping H [some T0 | Tl] L1 L3 F1 F3 :-
  ho.deref-assmt H T0 T,
```

```
complete-mapping-under-ass H T L1 L2 F1 F2,
  append L1 L2 LAll,
  complete-mapping H Tl LAll L3 F2 F3.

type decompile map -> links -> ho.subst ->
  fo.fsubst -> fo.fsubst -> o.
decompile Map1 L HO FO FO2 :-
  commit-links L L1_ HO HO1, !,
  complete-mapping HO1 HO1 Map1 Map2 FO FO1,
  decompl-subst Map2 Map2 HO1 FO1 FO2.
```

17 AUXILIARY FUNCTIONS

```
type fold4 (A -> A1 -> B -> B -> C -> C -> o) -> list A ->
  list A1 -> B -> B -> C -> C -> o.
fold4 _ [] [] A A B B.
fold4 F [X|XS] [Y|YS] A A1 B B1 :- F X Y A A0 B B0,
  fold4 F XS YS A0 A1 B0 B1.

type len list A -> nat -> o.
len [] z.
len [_|L] (s X) :- len L X.
```