

# HO unification from object language to meta language

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## ABSTRACT

Specifying and implementing a logic from scratch requires significant effort. Logical Frameworks and Higher Order Logic Programming Languages provide dedicated, high-level Meta Languages (ML) to facilitate this task in two key ways: 1) variable binding and substitution are simplified when ML binders represent object logic ones; 2) proof construction, and even proof search, is greatly simplified by leveraging the unification procedure provided by the ML. Notable examples of ML are Elf [13], Twelf [14],  $\lambda$ Prolog [10] and Isabelle [20] which have been utilized to implement various formal systems such as First Order Logic [5], Set Theory [12], Higher Order Logic [11], and even the Calculus of Constuctions [4].

The object logic we are interested in is Coq's [18] Dependent Type Theory (DTT), for which we aim to implement a unification procedure  $\approx_o$  using the ML Elpi [3], a dialect of  $\lambda$ Prolog. Elpi's equational theory comprises  $\eta\beta$  equivalence and comes equipped with a higher order unification procedure  $\approx_\lambda$  restricted to the pattern fragment [9]. We want  $\approx_o$  to be as powerful as  $\approx_\lambda$  but on the object logic DTT. Elpi also comes with an encoding for DTT that works well for meta-programming [17, 16, 7, 6]. Unfortunately this encoding, which we refer to as  $\mathcal{F}_o$ , "underuses"  $\approx_\lambda$  by restricting it to first-order unification problems only. To address this issue, we propose a better-behaved encoding,  $\mathcal{H}_o$ , demonstrate how to map unification problems in  $\mathcal{F}_o$  to related problems in  $\mathcal{H}_o$ , and illustrate how to map back the unifiers found by  $\approx_\lambda$ , effectively implementing  $\approx_o$  on top of  $\approx_\lambda$  for the encoding  $\mathcal{F}_o$ .

We apply this technique to the implementation of a type-class [19] solver for Coq [18]. Type-class solvers are proof search procedures based on unification that back-chain designated lemmas, providing essential automation to widely used Coq libraries such as Stdpp/Iris [8] and TLC [1]. These two libraries constitute our test bed.

## KEYWORDS

Logic Programming, Meta-Programming, Higher-Order Unification, Proof Automation

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## 1 INTRODUCTION

Specifying and implementing a logic from scratch requires significant effort. Logical Frameworks and Higher Order Logic Programming Languages provide dedicated, high-level Meta Languages (ML) to facilitate this task in two key ways: 1) variable binding and substitution are simplified when ML binders represent object logic ones; 2) proof construction, and even proof search, is greatly simplified by leveraging the unification procedure provided by the ML. Notable examples of ML are Elf [13], Twelf [14],  $\lambda$ Prolog [10] and Isabelle [20] which have been utilized to implement various formal systems such as First Order Logic [5], Set Theory [12], Higher Order Logic [11], and even the Calculus of Constuctions [4].

The object logic we are interested in is Coq's [18] Dependent Type Theory (DTT), and we want to code a type-class [19] solver for Coq [18] using the Coq-Elpi [17] meta programming framework. Type-class solvers are unification based proof search procedures that combine a set of designated lemmas in order to providing essential automation to widely used Coq libraries.

As the running example we take the Decide type class, from the Stdpp [8] library. The class identifies predicates equipped with a decision procedure. The following three designated lemmas (called Instances in the type-class jargon) state that: 1) the type `fin n`, of natural numbers smaller than `n` is finite; 2) the predicate `nfact n nf`, linking a natural number `n` to its prime factors `nf`, is decidable; 3) the universal closure of a predicate has a decision procedure if the predicate has and if its domain is finite.

```
Instance fin_fin n : Finite (fin n).          (* r1 *)
Instance nfact_dec n nf : Decision (nfact n nf). (* r2 *)
Instance forall_dec A P : Finite A →          (* r3 *)
  ∀x:A, Decision (P x) → Decision (∀x:A, P x).
```

Under this context of instances a type-class solver is able to prove the following statement automatically by back-chaining.

```
Check _ : Decision (forall y: fin 7, nfact y 3). (g)
```

The encoding of DTT provided by Elpi, that we will discuss at length later in section ?? and ??, is an Higher Order Abstract Syntax (HOAS) datatype `tm` featuring (among others) the following constructors:

```
type lam  tm -> (tm -> tm) -> tm.    % lambda abstraction
type app  list tm -> tm.              % n-ary application
type all  tm -> (tm -> tm) -> tm.    % forall quantifier
type con  string -> tm.               % constants
```

Following standard  $\lambda$ Prolog [10] the concrete syntax to abstract, at the meta level, an expression `e` over a variable `x` is `«x\ e»`, and square brackets denote a list of terms separated by comma. As an example we show the encoding of the Coq term `«∀y:t, nfact y 3»`:

```
all (con"t") y\ app[con"nfact", y, con"3"]
```

We now illustrate the encoding of the three instances above as higher-order logic-programming rules: capital letters denote rule parameters; `:-` separates the rule's head from the premises; `pi w\` introduces a fresh nominal constant `w` for the premise `p`.

```
finite (app[con"fin", N]). (r1)
```

```
decision (app [con"nfact", N, NF]). (r2)
```

```
decision (all A x\ app[P, x]) :- finite A, (r3)
  pi w\ decision (app[P, w]).
```

Unfortunately this translation of rule (r3) uses the predicate `P` as a first order term: for the meta language its type is `tm`. If we try to backchain the rule (r3) on the encoding of the goal (g) given below

```
decision (all (app[con"fin", con"7"]) y\
  app[con"nfact", y, con"3"]).
```

we obtain an unsolvable unification problem ( $p$ ): the two lists of terms have different lengths!

```
app[con"nfact", y, con"3"] = app[P, y] (p)
```

In this paper we study a more sophisticated encoding of Coq terms allowing us to rephrase the problematic rule (r3) as follows:

```
decision (all A x\ Pm x) :- decomp Pm P A, finite A, (r3a)
  pi x\ decision (app[P, x]).
```

Since `Pm` is an higher-order unification variable of type `tm`  $\rightarrow$  `tm`, with `x` in its scope, the unification problem ( $p'$ ) admits one solution:

```
app[con"nfact", y, con"3"] = Pm y (p')
Pm = x\ app[con"nfact", x, con"3"] % assignment for Pm
A = app[con"fin", con"7"] % assignment for A
```

After unifying the head of rule (r3a) with the goal, Elpi runs the premise `«decomp Pm A P»` that is in charge of bringing the assignment for `Pm` back to the domain `tm` of Coq terms:

```
P = lam A a\ app[con"nfact", a, con"3"]
```

This simple example is sufficient to show that the encoding we seek is not trivial and does not only concern the head of rules, but the entire sequence of unification problems that constitute the execution of a logic program. In fact the solution for `P` above generates a (Coq)  $\beta$ -redex in the second premise (the predicate under the `pi w\`):

```
decision (app[lam A (a\ app[con"nfact", a, con"3"]), w])
```

In turn this redex prevents the rule (r2) to backchain properly since the following unification problem has no solution:

```
app[lam A (a\ app[con"nfact", a, con"3"]), x] =
app[con"nfact", N, NF]
```

The root cause of the problems we sketched in the running example is that the unification procedure  $\approx_\lambda$  of the meta language is not aware of the equational theory of the object logic, even if both theories include  $\eta\beta$ -conversion and admit most general unifiers for unification problems in the pattern fragment  $\mathcal{L}_\lambda$  [9].

*Contributions.* In this paper we discuss alternative encodings of Coq in Elpi (Section ??), then we identify a minimal language  $\mathcal{F}_0$  in which the problems sketched here can be fully described. We then detail an encoding `comp` from  $\mathcal{F}_0$  to  $\mathcal{H}_0$  (the language of the meta language) and a decoding `decomp` to relate the unifiers bla

bla.. TODO citare Teyjus. The code discussed in the paper can be accessed at the URL: <https://github.com/FissoreD/paper-ho>.

## 2 PROBLEM STATEMENT

The equational theory of Coq's Dependent Type Theory is very rich. In addition to the usual  $\eta\beta$ -equivalence for functions, terms (hence types) are compared up to proposition unfolding and fix-point unrolling. Still, for efficiency and predictability reasons, most form of automatic proof search employ a unification procedure that captures a simpler one, just  $\eta\beta$ , and that solves higher-order problems restricted to the pattern fragment  $\mathcal{L}_\lambda$  [9]. We call this unification procedure  $\approx_o$ .

The equational theory of the meta language Elpi is strikingly similar, since it it comprises  $\eta\beta$  (for the meta language functions), and the unification procedure  $\approx_\lambda$  solves higher-order problems in  $\mathcal{L}_\lambda$ .

In spite of the similarity the link between  $\approx_\lambda$  and  $\approx_o$  is not trivial, since the abstraction and application term constructors the two unification procedures deal with are different. For example

$x \setminus f \ x$	$\approx_\lambda$	$f$
$\text{lam } A \ x \setminus \text{app}[\text{con} "f", x]$	$\approx_o$	$\text{con} "f"$
$\text{lam } A \ x \setminus \text{app}[\text{con} "f", x]$	$\neq_\lambda$	$\text{con} "f"$
$P \ x$	$\approx_\lambda$	$x$
$\text{app}[P, x]$	$\approx_o$	$x$
$\text{app}[P, x]$	$\neq_\lambda$	$x$

One could ignore this similarity, and “just” describe the object language unification procedure in the meta language, that is crafting a unif predicate to be used as follows in rule (r3):

```
decision X :- unif X (all A x\ app[P, x]), finite A,
  pi x\ decision (app[P, x]).
```

This choice would underuse the logic programming engine provided by the metalanguage since by removing any datum from the head of rules indexing degenerates. Moreover the unification procedure built in the meta language is likely to be faster than one implemented in it, especially if the meta language is interpreted as Elpi is.

To state precisely the problem we solve we need a  $\mathcal{F}_0$  representation of DTT terms and a  $\mathcal{H}_0$  one. We call  $=_o$  the equality over ground terms in  $\mathcal{F}_0$ ,  $=_\lambda$  the equality over ground terms in  $\mathcal{H}_0$ ,  $\approx_o$  the unification procedure we want to implement and  $\approx_\lambda$  the one provided by the meta language. TODO extend  $=_o$  and  $=_\lambda$  with reflexivity on uvars.

We write  $t_1 \approx_\lambda t_2 \mapsto \sigma$  when  $t_1$  and  $t_2$  unify with substitution  $\sigma$ ; we write  $\sigma t$  for the application of the substitution to  $t$ , and  $\sigma X = \{\sigma t \mid t \in X\}$  when  $X$  is a set; we write  $\sigma \subseteq \sigma'$  when  $\sigma$  is more general than  $\sigma'$ . We assume that the unification of our meta language is correct:

$$t_i \in \mathcal{L}_\lambda \Rightarrow t_1 \approx_\lambda t_2 \mapsto \rho \Rightarrow \rho t_1 =_\lambda \rho t_2 \quad (1)$$

$$t_i \in \mathcal{L}_\lambda \Rightarrow \rho t_1 =_\lambda \rho t_2 \Rightarrow \exists \rho', t_1 \approx_\lambda t_2 \mapsto \rho' \wedge \rho' \subseteq \rho \quad (2)$$

We illustrate a compilation  $\langle s \rangle \mapsto (t, m, l)$  that maps a term  $s$  in  $\mathcal{F}_0$  to a term  $t$  in  $\mathcal{H}_0$ , a variable mapping  $m$  and list of links  $l$ . The variable map connects unification variables in  $\mathcal{H}_0$  with variables in  $\mathcal{F}_0$  and is used to “decompile” the assignment,  $\langle \sigma, m, l \rangle^{-1} \mapsto \rho$ . Links represent problematic sub-terms which are linked to the

unification variable that stands in their place in the compiled term. These links are checked for or progress XXX improve....

We represent a logic program *run* in  $\mathcal{F}_0$  as a list *steps*  $p$  of length  $N$ . Each made of a unification problem between terms  $S_{p_l}$  and  $S_{p_r}$  taken from the set of all terms  $\mathcal{S}$ . The composition of these steps starting from the empty substitution  $\rho_0$  produces the final substitution  $\rho_N$ .<sup>1</sup> The initial here  $\rho_0$  is the empty substitution

$$\begin{aligned} \text{fstep}(\mathcal{S}, p, \rho) &\mapsto \rho'' \stackrel{\text{def}}{=} \rho S_{p_l} \approx_o \rho S_{p_r} \mapsto \rho' \wedge \rho'' = \rho \cup \rho' \\ \text{frun}(\mathcal{S}, N) &\mapsto \rho_N \stackrel{\text{def}}{=} \bigwedge_{p=1}^N \text{fstep}(\mathcal{S}, p, \rho_{p-1}) \mapsto \rho_p \end{aligned}$$

We simulate each run in  $\mathcal{F}_0$  with a run in  $\mathcal{H}_0$  as follows. Note that  $\sigma_0$  is the empty substitution.

$$\begin{aligned} \text{hstep}(\mathcal{T}, p, \sigma, \mathbb{L}) &\mapsto (\sigma'', \mathbb{L}') \stackrel{\text{def}}{=} \\ &\sigma \mathcal{T}_{p_l} \approx_\lambda \sigma \mathcal{T}_{p_r} \mapsto \sigma' \wedge \text{progress}(\mathbb{L}, \sigma \cup \sigma') \mapsto (\mathbb{L}', \sigma'') \\ \text{hrun}(\mathcal{S}, N) &\mapsto \rho_N \stackrel{\text{def}}{=} \\ &\mathcal{T} \times \mathbb{M} \times \mathbb{L}_0 = \{(t_j, m_j, l_j) \mid s_j \in \mathcal{S}, \langle s_j \rangle \mapsto (t_j, m_j, l_j)\} \\ &\bigwedge_{p=1}^N \text{hstep}(\mathcal{T}, p, \sigma_{p-1}, \mathbb{L}_{p-1}) \mapsto (\sigma_p, \mathbb{L}_p) \\ &\langle \sigma_N, \mathbb{M}, \mathbb{L}_N \rangle^{-1} \mapsto \rho_N \end{aligned}$$

Here *hstep* is made of two sub-steps: a call to  $\approx_\lambda$  (on the compiled terms) and a call to *progress* on the set of links. We claim the following:

PROPOSITION 2.1 (SIMULATION).  $\forall \mathcal{S}, \forall N$ ,

$$\text{frun}(\mathcal{S}, N) \mapsto \rho_N \Leftrightarrow \text{hrun}(\mathcal{S}, N) \mapsto \rho_N$$

That is, the two executions give the same result. Moreover:

PROPOSITION 2.2 (SIMULATION FIDELITY). *In the context of*  $\text{hrun}$ , *if*  $\mathcal{T} \subseteq \mathcal{L}_\lambda$  *we have that*  $\forall p \in 1 \dots N$ ,

$$\text{fstep}(\mathcal{S}, p, \rho_{p-1}) \mapsto \rho_p \Leftrightarrow \text{hstep}(\mathcal{T}, p, \sigma_{p-1}, \mathbb{L}) \mapsto (\sigma_p, \_)$$

In particular this property guarantees that a *failure* in the  $\mathcal{F}_0$  run is matched by a failure in  $\mathcal{H}_0$  at the same step. We consider this property very important from a practical point of view since it guarantees that the execution traces are strongly related and in turn this enables a user to debug a logic program in  $\mathcal{F}_0$  by looking at its execution trace in  $\mathcal{H}_0$ .

XXX permuting *hrun* does not change the final result if check does not fail eagerly

XXX if we want to apply heuristics, we can apply them in *decomp* to avoid committing to a non MGU too early

We can define  $s_1 \approx_o s_2$  by specializing the code of *hrun* to  $\mathcal{S} = \{s_1, s_2\}$  as follows:

$$\begin{aligned} s_1 \approx_o s_2 &\mapsto \rho \stackrel{\text{def}}{=} \\ &\langle s_1 \rangle \mapsto (t_1, m_1, l_1) \wedge \langle s_2 \rangle \mapsto (t_2, m_2, l_2) \\ &t_1 \approx_\lambda t_2 \mapsto \sigma' \wedge \text{progress}(\{l_1, l_2\}, \sigma') \mapsto (L, \sigma'') \wedge \\ &\langle \sigma'', \{m_1, m_2\}, L \rangle^{-1} \mapsto \rho \end{aligned}$$

<sup>1</sup>If the same rule is used multiple time in a run we just consider as many copies as needed of the terms composing the rules, with fresh unification variables each time

PROPOSITION 2.3 (PROPERTIES OF  $\approx_o$ ).

$$s_i \in \mathcal{L}_\lambda \Rightarrow s_1 \approx_o s_2 \mapsto \rho \Rightarrow \rho s_1 =_o \rho s_2 \text{ (correct)} \quad (3)$$

$$s_i \in \mathcal{L}_\lambda \Rightarrow \rho s_1 =_o \rho s_2 \Rightarrow \exists \rho', s_1 \approx_o s_2 \mapsto \rho' \wedge \rho' \subseteq \rho \text{ (complete)} \quad (4)$$

$$\rho s_1 =_o \rho s_2 \Rightarrow \rho' \subseteq \rho \Rightarrow \rho' s_i \in \mathcal{L}_\lambda \Rightarrow \rho' s_1 \approx_o \rho' s_2 \quad (5)$$

Properties (*correct*) and (*complete*) state, respectively, that in  $\mathcal{L}_\lambda$  the implementation of  $\approx_o$  is correct, complete and returns the most general unifier.

Property 2.1 states that  $\approx_o$ , hence our compilation scheme, is resilient to unification problems outside  $\mathcal{L}_\lambda$  solved by a third party. We believe this property is of practical interest since we want the user to be able to add heuristics via hand written rules to the ones obtained by our compilation scheme. A Typical example is the following problem (*q*) that is outside  $\mathcal{L}_\lambda$ :

$$\text{app } [F, \text{con} \text{ "a"}] = \text{app}[\text{con} \text{ "f"}, \text{con} \text{ "a"}, \text{con} \text{ "a"}] \quad (q)$$

$$F = \text{lam } x \backslash \text{app}[\text{con} \text{ "f"}, x, x] \quad (h)$$

Instead of rejecting it our scheme accepts it and guarantees that if (*h*) is given (after the compilation part of the scheme, as a run time hint) then ...

## 2.1 The intuition in a nutshell

A term *s* is compiled in a term *t* where every “problematic” sub term *p* is replaced by a fresh unification variable *h* and an accessory link that represent a suspended unification problem  $h \approx_\lambda p$ . As a result  $\approx_\lambda$  is “well behaved” on *t*, that is it does not contradict  $=_o$  as it would otherwise do on “problematic” terms. We now define “problematic” and “well behaved” more formally.

Definition 2.4 ( $\Diamond\eta$ ).  $\Diamond\eta = \{t \mid \exists \rho, t \text{ is an eta expansion}\}$

An example of term *t* in  $\Diamond\eta$  is  $\lambda x. \lambda y. F y x$  since the substitution  $\rho = \{F \mapsto \lambda a. \lambda b. f b a\}$  makes  $\rho t = \lambda x. \lambda y. f x y$  that is the eta long form of *f*. This term is problematic since its rigid part, the  $\lambda$ -abstractions, cannot justify a unification failure against, say, a constant.

Definition 2.5 ( $\overline{\mathcal{L}_\lambda}$ ).  $\overline{\mathcal{L}_\lambda} = \{X t_1 \dots t_n \mid X t_1 \dots t_n \notin \mathcal{L}_\lambda\}$ .

An example of *t* in  $\overline{\mathcal{L}_\lambda}$  is  $F a$  for a constant *a*. Note however that an oracle could provide an assignment  $\rho = \{F \mapsto \lambda x. x\}$  that makes the resulting term fall back in  $\mathcal{L}_\lambda$ .

Definition 2.6 (Subterms  $\mathcal{P}(t)$ ). The set of sub terms of *t* is the largest set  $\mathcal{P}(\sqcup)$  that can be obtained by the following rules.

$$\begin{aligned} t &\in \mathcal{P}(t) \\ t = f t_1 \dots t_n &\Rightarrow \mathcal{P}(t_i) \subseteq \mathcal{P}(t) \wedge f \in \mathcal{P}(t) \\ t = \lambda x. t' &\Rightarrow \mathcal{P}(t') \subseteq \mathcal{P}(t) \end{aligned}$$

We write  $\mathcal{P}(X) = \bigcup_{t \in X} \mathcal{P}(t)$  when *X* is a set of terms.

Definition 2.7 (Well behaved set). Given a set of terms  $X \subseteq \mathcal{H}_0$ ,

$$\mathcal{W}(X) \Leftrightarrow \forall t \in \mathcal{P}(X), t \notin (\overline{\mathcal{L}_\lambda} \cup \Diamond\eta)$$

PROPOSITION 2.8 ( $\mathcal{W}$ -PRESERVATION).  $\forall \mathcal{T}, \forall \mathbb{L}, \forall p, \forall \sigma, \forall \sigma'$

$$\mathcal{W}(\sigma \mathcal{T}) \wedge \sigma \mathcal{T}_{p_l} \approx_\lambda \sigma \mathcal{T}_{p_r} \mapsto \sigma' \Rightarrow \mathcal{W}(\sigma' \mathcal{T})$$

$$\mathcal{W}(\sigma \mathcal{T}) \wedge \text{progress}(\mathbb{L}, \sigma) \mapsto (\_, \sigma') \Rightarrow \mathcal{W}(\sigma' \mathcal{T})$$

A less formal way to state 2.8 is that hstep and progress never “commit” an unneeded  $\lambda$ -abstraction in  $\sigma$  (a  $\lambda$  that could be erased by an  $\eta$ -contraction), nor put in  $\sigma$  a flexible application outside  $\mathcal{L}_\lambda$  (an application node that could be erased by a  $\beta$ -reduction).

Note that proposition 2.8 does not hold for  $\approx_o$  as a whole since decompilation can introduce (actually restore) terms in  $\diamond\eta$  or  $\overline{\mathcal{L}_\lambda}$  that were move out of the way (put in  $\mathbb{L}$ ) during compilation.

### 3 ALTERNATIVE ENCODINGS AND RELATED WORK

Paper [2] introduces semi-shallow.

Our encoding of DTT may look “semi shallow” since we use the meta-language lambda abstraction but not its application (for the terms of type  $\text{tm}$ ). A fully shallow encoding unfortunately does not fit our use case, although it would make the running example work:

```
finite (fin N).
decision (nfact N NF).
decision (all A x\ P x) :- finite A, pi x\ decision (P x).
```

There are two reasons for dismissing this encoding. The first one is that in DTT it is not always possible to adopt it since the type system of the meta language is too weak to accommodate terms with a variable arity, like the following example:

```
Fixpoint arr T n := if n is S m then T -> arr T m else T.
Definition sum n : arr nat n := ...
Check sum 2 7 8 : nat.
Check sum 3 7 8 9 : nat.
```

The second reason is the encoding for Coq is used for meta programming the system, hence it must accommodate the manipulation of terms that are now know in advance (not even defined in Coq) without using introspection primitives such as Prologs’s functor and arg.

In the literature we could find a few related encoding of DTT. TODO In [4] is related and make the discrepancy between the types of ML and DTT visible. In this case one needs 4 application nodes. Moreover the objective is an encoding of terms, proofs, not proof search. Also note the conv predicate, akin to the unif we rule out.

TODO This other paper [15] should also be cited.

None of the encodings above provide a solution to our problem.

### 4 PRELIMINARIES: $\mathcal{F}_o$ AND $\mathcal{H}_o$

In order to reason about unification we provide a description of the  $\mathcal{F}_o$  and  $\mathcal{H}_o$  languages where unification variables are first class terms, i.e. they have a concrete syntax. We keep these languages minimal, for example, we omit the all quantifier of DTT we used in the example in Section 1 together with the type notation of terms carried by the lam constructor.

```
kind fm type.          kind tm type.
type fapp list fm -> fm. type app list tm -> tm.
type flam (fm -> fm) -> fm. type lam (tm -> tm) -> tm.
type fcon string -> fm.   type con string -> tm.
type fuva addr -> fm.     type uva addr -> list tm -> tm.
```

Figure 1: The  $\mathcal{F}_o$  and  $\mathcal{H}_o$  languages

Unification variables (fuva term constructor) in  $\mathcal{F}_o$  have no explicit scope: the arguments of an higher order variable are given via

the fapp constructor. For example the term  $P \ x$  is represented as  $\text{fapp}[fuva \ N, \ x]$ , where  $N$  is a memory address and  $x$  is a bound variable.

In  $\mathcal{H}_o$  the representation of  $P \ x$  is instead  $\text{uva} \ N \ [x]$ , since unification variables come equipped with an explicit scope. We say that the unification variable occurrence  $\text{uva} \ N \ L$  is in  $\mathcal{L}_\lambda$  if and only if  $L$  is made of distinct names. The predicate to test this condition is called pattern-fragment:

```
type pattern-fragment list A -> o.
```

Natural numbers represent the memory addresses that identify unification variables in both languages. The memory and its associated operations are described below:

```
kind addr type.
type addr nat -> addr.
typeabbrev (mem A) (list (option A)).

type set? addr -> mem A -> A -> o.
type unset? addr -> mem A -> o.
type assign addr -> mem A -> A -> mem A -> o.
type new mem A -> addr -> mem A -> o.
```

If a memory cell is none, then the corresponding unification variable is not set. assign sets an unset cell to the given value, while new finds the first unused address and sets it to none.

Since in  $\mathcal{H}_o$  unification variables have a scope, their solution needs to be abstracted over it to enable the instantiation of a single solution to different scopes. This is obtained via the inctx container, and in particular via its abs binding constructor. On the contrary a solution to a  $\mathcal{F}_o$  variable is a plain term.

```
typeabbrev fsubst (mem fm).

kind inctx type -> type.
type abs (tm -> inctx A) -> inctx A.
type val A -> inctx A.
typeabbrev assignment (inctx tm).
typeabbrev subst (mem assignment).
```

We call fsubst the memory of  $\mathcal{F}_o$ , while we call subst the one of  $\mathcal{H}_o$ . Both have the invariant that they are not cyclic, TODO: explain.

```
kind arity type.
type arity nat -> arity.

kind fvariable type.
type fv addr -> fvariable.

kind hvariable type.
type hv addr -> arity -> hvariable.

kind mapping type.
type mapping fvariable -> hvariable -> mapping.
typeabbrev mmap (list mapping).
```

INVARIANT 1 (UNIFICATION VARIABLE ARITY). *Each variable  $A$  in  $\mathcal{H}_o$  has a (unique) arity  $N$  and each occurrence  $(\text{uva} \ A \ L)$  is such that  $(\text{len } L \ N)$  holds*

The compiler establishes a mapping between variables of the two languages. In order to preserve invariant 1 we store the arity of



each hvariable in the mapping and we reuse an existing mapping only if the arity matches.

TODO: add ref to section 7

```

type m-alloc fvariable -> hvariable -> mmap -> mmap ->
  subst -> subst -> o. (malloc)
m-alloc Fv Hv M M S S :- mem M (mapping Fv Hv), !.
m-alloc Fv Hv M [mapping Fv Hv|M] S S1 :- Hv = hv N _,
  alloc S N S1.

```

When a single fvariable occurs multiple times with different numbers of arguments the compiler generates multiple mappings for it, on a first approximation, and then makes the mapping bijective by introducing link- $\eta$ ; this detail is discussed in section 6.

As we mentioned in section 2.1 the compiler replaces terms in  $\diamond\eta$  and  $\overline{\mathcal{L}}_\lambda$  with fresh variables linked to the problematic terms. Each class of problematic terms has a dedicated link.

```

kind baselink type.
type link-eta tm -> tm -> baselink.
type link-beta tm -> tm -> baselink.
typeabbrev link (inctx baselink).
typeabbrev links (list link).

```

The right hand side of a link, the problematic term, can occur under binders. To accommodate this situation the compiler wraps baselink using the inctx container (see,  $\cdot \vdash \cdot$ ).

INVARIANT 2 (LINK LEFT HAND SIDE). *The left hand side of a suspended link is a variable.*

New links are suspended by construction. If the left hand side variable is assigned during a step, then the link is considered for progress and possibly eliminated. This is discussed in section 6 and ??.

## 4.1 Notational conventions

When we write  $\mathcal{H}_o$  terms outside code blocks we follow the usual  $\lambda$ -calculus notation, reserving  $f, g, a, b$  for constants,  $x, y, z$  for bound variables and  $X, Y, Z, F, G, H$  for unification variables. However we need to distinguish between the “application” of a unification variable to its scope and the application of a term to a list of arguments. We write the scope of unification variables in subscript while we use juxtaposition for regular application. Here a few examples:

```

f a      app[con "f", con "a"]
 $\lambda x. \lambda y. F_{xy}$  lam x\ lam y\ uva F [x, y]
 $\lambda x. F_x a$  lam x\ app[uva F [x], con "a"]
 $\lambda x. F_x x$  lam x\ app[uva F [x], x]

```

When variables  $x$  and  $y$  can occur in term  $t$  we shall write  $t_{xy}$  to stress this fact.

We write  $\sigma = \{A_{xy} \mapsto y\}$  for the assignment  $\text{abs } x \backslash \text{abs } y \backslash y$  and  $\sigma = \{A \mapsto \lambda x. \lambda y. y\}$  for  $\text{lam } x \backslash \text{lam } y \backslash y$ .

When detailing examples we write links as equations between two terms under a context. The equality sign is subscripted with kind of baselink. For example  $x \vdash A_x =_\beta F_x a$  corresponds to:

```
abs x\ val (link-beta (uva A [x]) (app[uva F [x], con "a"]))
```

When it is clear from the context we shall use the same syntax for  $\mathcal{F}_o$  terms (although we never subscript unification variables).

## 4.2 Equational theory and Unification

In order to express properties ?? we need to equip  $\mathcal{F}_o$  and  $\mathcal{H}_o$  with term equality, substitution application and unification.

*Term equality:*  $=_o$  vs.  $=_\lambda$ . We extend the equational theory over ground terms to the full languages by adding the reflexivity of unification variables (a variable is equal to itself).

The first four rules are common to both equalities and just define the usual congruence over terms, and since we use an HOAS encoding they also capture  $\alpha$ -equivalence. In addition to that  $=_o$  has rules for  $\eta$  and  $\beta$ -equivalence.

```

type (=o) fm -> fm -> o. (=o)
fcon X =o fcon X.
fapp A =o fapp B :- forall2 (=o) A B.
flam F =o flam G :- pi x\ x =o x => F x =o G x.
fuva N =o fuva N.
flam F =o T :- (eta)
  pi x\ beta T [x] (T' x), x =o x => F x =o T' x.
T =o flam F :- (eta_r)
  pi x\ beta T [x] (T' x), x =o x => T' x =o F x.
fapp [flam X | L] =o T :- beta (flam X) L R, R =o T. (beta_l)
T =o fapp [flam X | L] :- beta (flam X) L R, T =o R. (beta_r)

type (=lambda) tm -> tm -> o.
con C =lambda fcon C.
app A =lambda fapp B :- forall2 (=lambda) A B.
lam F =lambda flam G :- pi x\ x =lambda x => F x =lambda G x.
uva N A =lambda fuva N B :- forall2 (=lambda) A B.

```

The main point in showing these equality tests is to remark how weaker  $=_\lambda$  is, and to identify the four rules that need special treatment in the implementation of  $=_o$ .

For reference,  $(\text{beta } T \ A \ R)$  reduces away  $\text{lam}$  nodes in head position in  $T$  whenever the list  $A$  provides a corresponding argument.

```

type beta fm -> list fm -> fm -> o.
beta A [] A.
beta (flam Bo) [H | L] R :- napp (Bo H) F, beta F L R.
beta (fapp A) L (fapp X) :- append A L X.
beta (fuva N) L (fapp [fuva N | L]).
beta (fcon H) L (fapp [fcon H | L]).
beta N L (fapp [N | L]) :- name N.

type napp fm -> fm -> o.
napp (fcon C) (fcon C).
napp (fuva A) (fuva A).
napp (flam F) (flam G) :- pi x\ napp (F x) (G x).
napp (fapp [fapp L1 | L2]) T :- !,
  append L1 L2 L3, napp (fapp L3) T.
napp (fapp L) (fapp L1) :- map napp L L1.
napp N N :- name N.

```

The name predicate holds only on nominal constants (i.e. bound variables).<sup>2</sup> The choice of using n-ary application, rather than binary, is to make it easy to access the application’s head. The price we pay is that substituting an application in the head of an application should be amended by “flattening” fapp nodes, that is the job of

<sup>2</sup>Elpi provides it as a builtin, but one could implement it by systematically loading the hypothetical rule  $\text{name } x$  every time a nominal constant is postulated via  $\text{pi } x \backslash$

napp.<sup>3</sup> Finally note that the cut operator is inessential, it could be removed at the cost of a verbose test on the head of  $L$  in the second rule about fapp:  $L$ 's head can be fcon, flam or a name.

*Substitution application:  $\rho s$  and  $\sigma t$ .* Applying the substitution corresponds to dereferencing a term with respect to the memory. To ease the comparison we split  $\mathcal{F}_0$  dereferencing into a fder step and a napp one. The former step replaces references to memory cells that are set with their values, and has a corresponding operation in  $\mathcal{H}_0$ , namely deref. On the contrary napp has no corresponding operation in  $\mathcal{H}_0$ . The reasons for this asymmetry is that an fapp node with a flexible head is always mapped to a uva (as per section 5 and section 8), preventing nested applications to materialize.

```

type fder fsubst -> fm -> fm -> o.
fder _ (fcon C) (fcon C).
fder S (fapp A) (fapp B) :- map (fder S) A B.
fder S (flam F) (flam G) :-
  pi x\ fder S x x => fder S (F x) (G x).
fder S (fuva N) R :- set? N S T, fder S T R.
fder S (fuva N) (fuva N) :- unset? N S.

type fderef fsubst -> fm -> fm -> o.           (ρs)
fderef S T T2 :- fder S T T1, napp T1 T2.

```

Applying the substitution in  $\mathcal{H}_0$  is very similar, with the caveat that assignments have to be moved to the current scope, i.e. renaming the abs-bound variables with the names in the scope of the unification variable occurrence.

```

type deref subst -> tm -> tm -> o.           (σt)
deref _ (con C) (con C).
deref S (app A) (app B) :- map (deref S) A B.
deref S (lam F) (lam G) :-
  pi x\ deref S x x => deref S (F x) (G x).
deref S (uva N L) R :- set? N S A,
  move A L T, deref S T R.
deref S (uva N A) (uva N B) :- unset? N S,
  map (deref S) A B.

```

Note that move strongly relies on invariant 1: the length of the arguments of all occurrences of a unification variable and the number of abstractions in its assignment have to match. In turn this grants that move never fails.

```

type move assignment -> list tm -> tm -> o.
move (abs Bo) [H|L] R :- move (Bo H) L R.
move (val A) [] A.

```

*Term unification:  $\approx_o$  vs.  $\approx_\lambda$ .* In this paper we assume to have an implementation of  $\approx_\lambda$  that satisfies properties 1 and 2. Although we provide an implementation in the appendix (that we used for testing purposes) we only describe its signature here. Elpi is expected to provide this brick, as well as any other implementation of  $\lambda$ Prolog.

```

type (≈λ) tm -> tm -> subst -> subst -> o.

```

The only detail worth discussing is the fact that the procedure updates a substitution, rather than just crafting one as presented

<sup>3</sup>Note that napp is an artefact of formalization of  $\mathcal{F}_0$  we do in this presentation and, as we explain later, no equivalent of napp is needed in  $\mathcal{H}_0$ .

in section 2. The reason is that the algorithm folds over a term, updating a substitution while it traverses it.

## 5 BASIC SIMULATION OF $\mathcal{F}_0$ IN $\mathcal{H}_0$

In this section we describe a basic compilation scheme that we refine later, in the following sections. This scheme is sufficient to implement an  $\approx_o$  that respects  $\beta$ -conversion for terms in  $\mathcal{L}_\lambda$ . The extension to  $\eta\beta$ -conversion is described in Section 6 and the support for terms outside  $\mathcal{L}_\lambda$  in Section 8.

### 5.1 Compilation

The main task of the compiler is to recognize  $\mathcal{F}_0$  variables standing for functions and map them to higher order variables in  $\mathcal{H}_0$ . In order to bring back the substitution from  $\mathcal{H}_0$  to  $\mathcal{F}_0$  the compiler builds a “memory map” connecting the the kind of variables using routine (*malloc*).

The signature of the comp predicate below allows for the generation of links (suspended unification problems) that play no role in this section but play a major role in section 6 and section 8. With respect to section 2 the signature also allows for updates to the substitution. The code below uses that possibility in order to allocate space for the variables, i.e. sets their memory address to none (a details not worth mentioning in the previous discussion).

```

type comp fm -> tm -> mmap -> mmap -> links -> links ->
  subst -> subst -> o.
comp (fcon C) (con C) M M L L S S.
comp (flam F) (lam F1) M1 M2 L1 L2 S1 S2 :-           (cλ)
  comp-lam F F1 M1 M2 L1 L2 S1 S2.
comp (fuva A) (uva B []) M1 M2 L L S1 S2 :-
  m-alloc (fv A) (hv B (arity z)) M1 M2 S1 S2.
comp (fapp [fuva A|Ag]) (uva B Ag1) M1 M2 L L S1 S2 :-
  pattern-fragment Ag, !,
  fold6 comp Ag Ag1 M1 M1 L L S1 S1,
  len Ag Arity,
  m-alloc (fv A) (hv B (arity Arity)) M1 M2 S1 S2.
comp (fapp A) (app A1) M1 M2 L1 L2 S1 S2 :-
  fold6 comp A A1 M1 M2 L1 L2 S1 S2.

```

This preliminary version of comp recognizes  $\mathcal{F}_0$  variables applied to a (possibly empty) duplicate free list of names. Note that compiling Ag cannot create new mappings nor links, since Ag is made of bound variables and the hypothetical rule loaded by comp-lam (see below) grants this property.

```

type comp-lam (fm -> fm) -> (tm -> tm) ->
  mmap -> mmap -> links -> links -> subst -> subst -> o.
comp-lam F G M1 M2 L1 L3 S1 S2 :-
  pi x y\ (pi M L S\ comp x y M M L L S S) =>
  comp (F x) (G y) M1 M2 L1 (L2 y) S1 S2,
  close-links L2 L3.

```

In the code above the syntax  $\pi x y\ .$  is syntactic sugar for iterated  $\pi$  abstraction, as in  $\pi x\ \pi y\ .$

The auxiliary function close-links tests if the bound variable  $v$  really occurs in the link. If it is the case the link is wrapped into an additional abs node binding  $v$ . In this way links generated deep inside the compiled terms can be moved outside their original context of binders.

```

type close-links (tm -> links) -> links -> o.
close-links (v\ [X |L v]) [X|R] :- !, close-links L R.
close-links (v\ [X v|L v]) [abs X|R] :- close-links L R.
close-links (_\ []) [].

```

Note that we could remove the first rule, whose solve purpose is to make links more readable by pruning unused context entries.

## 5.2 Execution

A step in  $\mathcal{H}_o$  consists in unifying two terms and reconsidering all links for progress. If any of the two tasks fail we say that the entire step fails, and it is at this granularity that we can relate steps in the two languages.

```

type hstep tm -> tm -> links -> links -> subst -> subst -> o.
hstep T1 T2 L1 L2 S1 S3 :-
  (T1  $\approx_\lambda$  T2) S1 S2,
  progress L1 L2 S2 S3.

```

Note that the infix notation  $((A \approx_\lambda B) C D)$  is syntactic sugar for  $((\approx_\lambda) A B C D)$ .

Reconsidering links is a fixpoint, since the progress of a link can update the substitution and in turn enable another link to progress.

```

type progress links -> links -> subst -> subst -> o.
progress L L2 S1 S3 :-
  progress1 L L1 S1 S2,
  occur-check-links L1,
  if (L = L1, S1 = S2)
    (L2 = L1, S3 = S1)
    (progress L1 L2 S2 S3).

```

In the base compilation scheme `progress1` is the identity on both the links and the substitution, so the fixpoint trivially terminates. Sections 6 and 8 add rules to `progress1` and justify why the don't hinder termination.

Since compilation moves problematic terms out of the sigh of  $\approx_\lambda$ , that procedure can only perform a partial occur check. For example the unification problem  $X \approx_\lambda f Y$  cannot generate a cyclic substitution alone, but should be disallowed if a  $\mathbb{L}$  contains a link like  $\vdash Y =_{\eta} \lambda z.X_z$ : We don't know yet if  $Y$  will feature a lambda in head position, but we surely know it contains  $X$ , hence  $f Y$  and that fails the occur check. The procedure `occur-check-links` is in charge of ensuring that each link does not represent a (suspended) unification problem doomed to fail because of occur check. This check is needed in order to guarantee proposition 2.2 (simulation fidelity).

## 5.3 Substitution decompilation

Decompiling the substitution requires to first force the progress of links and then allocating new unassigned variables in the substitution for  $\mathcal{F}_o$  and finally decompiling all assignments. Note that invariant 2 and the occur check allows us to update the subst.

```

type decompile mmap -> links -> subst ->
  fsubst -> fsubst -> o.
decompile M1 L S F1 F3 :-
  commit-links L S S1,
  complete-mapping S1 S1 M1 M2 F1 F2,
  decomp M2 M2 S1 F2 F3.

```

TODO: What is commit-links and complete-mapping?, maybe complete-mapping can be hidden in the code rendering? Decompiling an assignment requires to turn abstractions into lambdas. For aesthetic purposes we also eta-contract the result (not needed since  $\mathcal{F}_o$  equality can do that)

```

type decomp mmap -> mmap -> subst -> fsubst -> fsubst -> o.
decomp _ [] _ F F.
decomp M [mapping (fv V) (hv H _)]MS S F1 F3 :- set? H S A,
  deref-assmt S A A1,
  abs->lam A1 T, decomp M T T1,
  eta-contract T1 T2,
  assign V F1 T2 F2,
  decomp M MS S F2 F3.
decomp M [mapping _ (hv H _)]MS S F1 F2 :- unset? H S,
  decomp M MS S F1 F2.

```

Finally decompiling a term is trivial, now that we have an extended mapping containing all unassigned variables  $\approx_\lambda$  may have introduced.

```

type decomp mmap -> tm -> fm -> o.
decomp _ (con C) (fcon C).
decomp M (app A) (fapp B) :- map (decomp M) A B.
decomp M (lam F) (flam G) :-
  pi x y\ (pi M\ decomp M x y) => decomp M (F x) (G y).
decomp M (uva Hv Ag) R :-
  mem M (mapping (fv Fv) (hv Hv _)),
  map (decomp M) Ag Bg,
  beta (fuva Fv) Bg R.

```

Note that we use beta to build fapp nodes when needed (if Ag is empty no fapp node should appear).

INVARIANT 3. *TODO: dire che il mapping è bijective*

## 5.4 Definition of $\approx_o$ and its properties

```

type ( $\approx_o$ ) fm -> fm -> fsubst -> o.
(A  $\approx_o$  B) F :-
  comp A A' [] M1 [] [] S1,
  comp B B' M1 M2 [] [] S1 S2,
  hstep A' B' [] [] S2 S3,
  decomp M2 M2 S3 [] F.

```

The code given so far applies to terms in  $\beta\eta$ -normal form where unification variables in  $\mathcal{F}_o$  can occur non linearly but always with the same number of arguments, and where their arguments are distinct names (as per  $\mathcal{L}_\lambda$ ).

LEMMA 5.1 (COMPILATION ROUND TRIP). *If  $\text{comp } S \ T \ [] \ M \ [] \ _ \ [] \ _$  then  $\text{decomp } M \ T \ S$*

PROOF SKETCH. trivial, since the terms are beta normal beta just builds an app.  $\square$

LEMMA 5.2. *Properties (correct) and (complete) hold for the implementation of  $\approx_o$  above*

PROOF SKETCH. In this setting  $\approx_\lambda$  is as strong as  $\approx_o$  on ground terms. What we have to show is that whenever two different  $\mathcal{F}_o$  terms can be made equal by a substitution  $\rho$  (plus the  $\beta_l$  and  $\beta_r$  if needed) we can find this  $\rho$  by finding a  $\sigma$  via  $\approx_\lambda$  on the corresponding  $\mathcal{H}_o$

terms and by decompiling it. If we look at the  $\mathcal{F}_0$  terms, there are two interesting cases:

- $\text{fuva } X \approx_o s$ . In this case after comp we have  $Y \approx_\lambda t$  that succeeds with  $\sigma = \{Y \mapsto t\}$  and  $\sigma$  is decompiled to  $\rho = \{Y \mapsto s\}$ .
- $\text{fapp}[\text{fuva } X[L] \approx_o s]$ . In this case we have  $Y_{\vec{x}} \approx_\lambda t$  that succeeds with  $\sigma = \{\vec{y} \mapsto Y \mapsto t[\vec{x}/\vec{y}]\}$  that in turn is decompiled to  $\rho = \{Y \mapsto \lambda \vec{y}.s[\vec{x}/\vec{y}]\}$ . Thanks to  $\beta_l (\lambda \vec{y}.s[\vec{x}/\vec{y}]) \vec{x} \approx_o s$ .

Since the mapping is a bijection occur check in  $\mathcal{H}_0$  corresponds to occur check in  $\mathcal{F}_0$ .  $\square$

LEMMA 5.3. *Properties simulation (2.1) and fidelity (2.2) hold*

PROOF SKETCH. Since  $\text{progress1}$  is trivial  $\text{fstep}$  and  $\text{hstep}$  are the same, that is in this context where input terms are  $\beta\eta$ -normal and we disregard  $\eta$ -equivalence  $\approx_\lambda$  is equivalent to  $\approx_o$ .  $\square$

## 5.5 Limitations of by this basic scheme

$$\lambda xy.F y x = \lambda xy.x \quad (6)$$

$$\lambda x.f x (F x) x = G \quad (7)$$

Note that here  $F$  is used with different arities, moreover in the second problem the left hand side happens to be an eta expansion (of  $f(\lambda y.y)$ ) only after we discover (at run time) that  $F = \lambda x \lambda y.y$  (i.e. that  $F$  discards the  $x$  argument). Both problems are addressed in the next section.

## 6 HANDLING OF $\diamond\eta$

$\eta$ -reduction is an equivalence relation where a term of the form  $\lambda x.t x$  can be converted to  $t$  any time  $x$  does not occur as a free variable in  $t$ . We call  $\lambda x.t x$  the  $\eta$ -expansion of  $t$ .

**Entry pb:**  $\mathcal{P}_1 : \text{flam } x \backslash \text{fapp } [\text{fuva } X, x] \approx_o \text{fcon} "f"$

**Compil:**  $\mathcal{T}_1 : \lambda x.X'_x \approx_\lambda f$

**Mapping:**  $m_1 : X \mapsto X'^1$

Following the implementation of comp given in section 5, the unification problem  $\mathcal{P}_1$  given above, is compiled into the unification problem  $\mathcal{T}_1$ . We can note that all the terms appearing in  $\mathcal{P}_1$  are unifiable by  $\approx_o$ , but, despite this,  $\approx_\lambda$  is not able to solve  $\mathcal{T}_1$ : its left and right hand sides have both different rigid head:  $\lambda x.A'_x$  corresponds to the  $\mathcal{H}_0$  term  $\text{lam } x \backslash \text{app}[\text{con} "f", x]$  and  $f$  corresponds to  $\text{con} "f"$ . This failure is motivated by the fact that  $\text{flam } x \backslash \text{fapp } [\text{fuva } A, x]$  is a term belonging to  $\diamond\eta$ .

In order to guarantee proposition 2.1, we need to modify the way terms are compiled. The goal is to recognize every  $\diamond\eta$  subterm  $t$  and replace it with fresh  $\mathcal{H}_0$  variables  $v$ . This connection between the variable  $v$  and the subterm  $t$  is stored in what we call  $\text{link-}\eta$  which is an object with the following type

**type**  $\text{link-}\eta \text{ tm} \rightarrow \text{tm} \rightarrow \text{baselink}$

where, as sketched in section 4, the term on the left hand side (lhs) is linked with its left counterpart (rhs).

$\text{link-}\eta$  are added in the link store ( $\mathbb{L}$ ) and activated when special conditions are satisfied on lhs or rhs. These link activations are managed by extending the  $\text{progress1}$  predicate (see section 5.2). We claim that  $\text{link-}\eta$  progression does not contradict invariant 2 and we add the following invariant:

INVARIANT 4 ( $\text{link-}\eta$  rhs). *The rhs of any  $\text{link-}\eta$  in  $\mathbb{L}$  has the shape  $\lambda x.t_x$  where  $t_x$  is a  $\diamond\eta$  term and  $x$  is free in  $t$ .*

In the next three subsections we explain how we detect  $\diamond\eta$  terms, how we compile them and how the generated  $\text{link-}\eta$  are activated during the progress. Moreover, we provide justification for why invariants 2 and 4 remain true.

### 6.1 Detection of $\diamond\eta$

Compiling a term  $t$  forces us to determine if there exists a  $t' \in \mathcal{P}(t)$ , such that  $t' = \lambda x.t'_x$ , for any term  $t''$ , can be a  $\eta$ -expansion, i.e. under a given substitution  $\sigma$ , we have  $\sigma(\lambda x.t'_x) = t'$ . This  $\diamond\eta$  detection is not a trivial operation as it may seems.

$$\lambda x.f A_x \quad (8) \quad \lambda x.\lambda y.f A_x B_{yx} \quad (10)$$

$$\lambda x.f x A_x \quad (9) \quad \lambda x.f A_x x \quad (11)$$

Term	Status	Evidence
$\lambda x.f A_x$	$\in \diamond\eta$	$\sigma = \{A_x \mapsto x\}$
$\lambda x.f x A_x$	$\notin \diamond\eta$	
$\lambda x.\lambda y.f A_x B_{yx}$	$\in \diamond\eta$	$\sigma = \{A_x \mapsto x, B_{yx} \mapsto y\}$
$\lambda x.f A_x x$	$\in \diamond\eta$	$\sigma = \{A_x \mapsto a\}$

In the examples above, the first term is a  $\diamond\eta$  since  $A_x$  can reduce to  $x$  by setting  $A_x = \lambda x.x$ , the second one is not a  $\diamond\eta$  since it exists no substitution for  $A_x$  such that  $A_x$  reduces to  $x$  and  $x$  is not free in the subterm  $f x$ . It is a bit more complicated to determine if eq. 10 a  $\diamond\eta$  since, we have a spine of lambdas, this means that the whole term is a  $\diamond\eta$ , if the inner  $\lambda$ -term  $\lambda y.f A_x B_{yx}$  is an  $\eta$ -expansion and, if so, we need to verify if the  $\eta$ -reduced subterm, i.e.  $f A_x$ , can make  $\lambda x.f A_x$  an  $\eta$ -expansion. Indeed, eq. 10 is a  $\diamond\eta$  under the substitution  $\sigma = \{A \mapsto \lambda x.x, B \mapsto \lambda x.\lambda y.x\}$ . Lastly, eq. 11 is a  $\diamond\eta$  even if  $x$  occurs in the subterm  $t' = f A_x$ , since it exists a substitution  $\sigma = \{A \mapsto \lambda x.a\}$ , for any constant  $a$ , such that  $\sigma t' = f a$  is a term where  $x$  does not appears as a free variable.

We can now define more formally how  $\diamond\eta$  terms are detected together with its auxiliary functions:

**Definition 6.1 (reduce-to).** A term  $t$  reduce to a name  $x$ , if  $\exists \sigma, \sigma t = x$ . In particular, for any term  $t$ ,  $\lambda x_1 \dots x_n.t_{x_1 \dots x_n}$  reduces to a bound variable  $x$  if one of the three following cases is satisfied: 1)  $n = 0$  and  $t = x$ ; 2)  $t$  is the application of  $x$  to a list of terms  $l$  and each  $l_i$  reduces to  $x_i$  (e.g.  $\lambda x_1 \dots x_n.x_{x_1 \dots x_n} = x$ ); 3)  $t$  is a unification variable in  $\mathcal{L}_\lambda$  with scope  $s$ , and for any  $v \in \{x, x_1 \dots x_n\}$ , there exists a  $s_i \in s$ , such that  $s_i$  reduces to  $v$ .

**Definition 6.2 (occurs-rigidly).** A name  $x$  occurs rigidly in a term  $t$ , if  $\forall \sigma, x \in \mathcal{P}(\sigma t)$

In other words  $x$  occurs-rigidly in  $t$  if it occurs in  $t$  outside of the scope of unification variables since their instantiations are allowed to discard their scope.

We can now derive the implementation for  $\diamond\eta$  detection:

**Definition 6.3 ( $\diamond\eta$  detection).** A term  $\lambda x_1 \dots x_n.t$  where each  $x_i$  occurs in  $t$  is a  $\diamond\eta$  if either: 1)  $t$  is a constant applied to arguments  $l_1 \dots l_m$  such that  $m \geq n$  and every  $l_{m-n+i}$  reduces-to  $x_i$  and no  $x_i$  occurs-rigidly in  $l_{1 \dots m-n-1}$ ; or 2)  $t$  is a unification variable with scope  $s$  and for each  $x_i$  there exists a  $s_j \in s$  such that  $s_j$  reduces-to  $x_i$ .



As a final remark, the  $\diamond\eta$  detection defined just before is an over-approximation, in the sense that there exists some terms  $t$  considered as  $\diamond\eta$ , such that for all substitution  $\sigma$ ,  $\sigma t$  is not an  $\eta$ -expansion. A small example is  $\lambda x.f A_x A_x$  which is considered a  $\diamond\eta$ . This is because, we suppose at the same time that the second  $A_x$  can reduce to  $x$ , to the first  $A_x$  is not reducible to  $A_x$ . This is however not considered as a problem, since, adding holes to the compiled term does not break proposition 2.1, since each hole is connected to a  $\text{link-}\eta$  which is activated exactly when the hole is instantiated the  $\text{link-}\eta$ , as explain below is activated.

## 6.2 Compilation

Thanks to the maybe-eta predicate, we can detect “ $\eta$ -problematic” terms and, consequently replace them with fresh  $\mathcal{H}_o$  unification variables at compilation time. The code below illustrate how this relation is used to for term compilation.

```
comp (flam F) (uva A Scope) M1 M2 L1 L3 S1 S3 :-
  maybe-eta (flam F) [], !,
  alloc S1 A S2,
  comp-lam F F1 M1 M2 L1 L2 S2 S3,
  get-scope (lam F1) Scope,
  L3 = [val (link-eta (uva A Scope) (lam F1)) | L2].
```

This rule, to be inserted just before rule  $(c_\lambda)$  from the code in section 5, verifies if the  $\mathcal{H}_o$  term  $t$  received in entry is a  $\diamond\eta$ . Let  $\lambda t'$  be the compiled version of  $t$ , then the fresh variable  $A$  returned as the new  $\mathcal{H}_o$  term as in scope all the free names in  $t'$ . The critical part of this compilation is the creation of the  $\text{link-}\eta$ , which links the variable  $A$  with  $t$ . This link creation enforce invariant 2 and invariant 4, since lhs is a trivially a variable and the rhs is a term  $t$  starting with the  $\text{lam}$  constructor where  $x$  is free in  $t$  otherwise  $t$  would not have been detected as a  $\diamond\eta$ .

**COROLLARY 6.4.** *The rhs of any  $\text{link-}\eta$  has exactly one lambda abstraction.*

**PROOF SKETCH.** By contradiction, suppose that a  $\text{link-}\eta$   $l$  has  $t$  as rhs and  $t = \lambda x.\lambda y.t'_{xy}$ . Two cases are to be analyses: 1)  $\lambda y.t'_{xy}$  is a  $\diamond\eta$ , then, by construction, rhs would have been replaced with a the  $\eta$ -expansion of fresh variable  $v$ , which is a contradiction, since  $\lambda y.t'_{xy} \neq \lambda x.v_x$ ; 2)  $\lambda y.t'_{xy}$  is not an a  $\diamond\eta$ , then neither  $t$  is, which is a contradiction since rhs is always a  $\diamond\eta$  by construction.  $\square$

## 6.3 Progress

$\text{link-}\eta$  are meant to delay the unification of “problematic” terms. In the following, we call  $\mathbb{L}$  the list of suspended links.

In order to activate a  $\text{link-}\eta$   $l$ , we need to extend the progress1 predicate by adding new rules. After passing under all the abs constructors of  $l$ , there are two cases making a  $\text{link-}\eta$  to progress, 1) lhs is instantiated to a rigid term 2) rhs is no more a  $\diamond\eta$  or it is a term which can be reduced to a term with rigid head. If lhs is instantiated to a rigid term  $t$ , by proposition 2.8, we know that  $t$  does not contain any  $\diamond\eta$ . Let  $t'$  the right hand side, if  $t$  is a constant or a function application, then,  $t'$ , which by construction has  $\text{lam}$  as head, should be an  $\eta$ -expansion. We are therefore allowed to unify  $\lambda x.t x$  (the  $\eta$ -expanded version of  $t$ ) with  $t'$ . Finally, if  $t$  is a term with  $\text{lam}$  as head, then it is not an  $\eta$ -expansion and therefore,  $t$  can be unified with  $t'$ .

The second way to activate a  $\text{link-}\eta$  is when the rhs is no more a  $\diamond\eta$  or rhs can be  $\eta$ -reduced to a term  $t$  with rigid head. In both cases, lhs is unified with  $t'$ .

Once a  $\text{link-}\eta$  is activated, it can be removed from  $\mathbb{L}$ , otherwise, the link is kept for a further iteration of progress. Note that this link progression enforce proposition 2.8 and invariants 2 and 4: we never commit a term in the  $\mathcal{H}_o$  substitution, since we make unification only when we know that the terms are no more  $\diamond\eta$ , and when lhs is no more a variable or rhs is no more a  $\diamond\eta$ , the link is removed from  $\mathbb{L}$ .

**Entry pb:**  $\mathcal{P}_1 : \text{flam } x \backslash \text{fapp } [\text{fuva } X, x] \approx_o \text{fcon } "f"$

**Compil:**  $\mathcal{T}_1 : A \approx_\lambda f$

**Mapping:**  $m_1 : X \mapsto X'^1$

**Links:**  $l_1 : A =_\eta \lambda x.X'_x$

The example above shows the new compilation of the unification problem given at the beginning of section 6. This time, we see that the the left hand side  $t$  of  $\mathcal{P}_1$  has been detected as a  $\diamond\eta$  and replaced with the fresh variable  $A$ . Moreover,  $\mathbb{L}$  contains the link  $l_1$  connecting  $A$  with  $\lambda x.X'_x$ , the compiled version of  $t$ . After the resolution of  $\mathcal{T}_1$ ,  $A$  is assigned to  $f$ . Therefore lhs of  $l_1$  is a term with  $\text{con}$  as constructor. This means that rhs of  $l_1$  is a  $\eta$ -expansion and therefore we can unify  $\lambda x.f x$  with rhs, which instantiate  $X'_x$  to  $f x$ .  $l_1$  is removed from  $\mathbb{L}$  which is now empty, progress terminate, and the decompilation, will instantiate the  $X$  variable to  $f$  (which is the  $\eta$ -contracted version of  $\lambda x.f x$ ).

A second example, showing the activation of a link when the rhs is no more a  $\diamond\eta$ , is given in section 7, since we need to work with variables used with different arities. This example represent the run of the unification problems proposed at section 5.5

Another, way to progress  $\text{link-}\eta$ , that we call  $\text{link-}\eta$  deduplication, is when  $\mathbb{L}$  contains two  $\text{link-}\eta$   $l_1$  and  $l_2$  with a lhs having the same variable address. Let the lhs of  $l_1$  be  $\text{uva } U R$  and the lhs of  $l_2$  be  $\text{uva } V S$ , then, by invariant 1,  $R$  and  $S$  have same scope. Let  $t$  be the term obtained by replacing all each name  $S_i$  in the rhs of  $l_1$  with  $R_i$ ,  $t$  is unified with the rhs of  $l_2$  and one of the two links between  $l_1$  and  $l_2$  is removed from  $\mathbb{L}$ .

A very basic example of  $\text{link-}\eta$  deduplication, is given below:

**Entry pb:**  $\mathcal{P}_1 : \text{flam } x \backslash \text{fapp } [\text{fuva } X, x] == \text{flam } x \backslash \text{fapp } [\text{fuva } X, x]$

**Compil:**  $\mathcal{T}_1 : E0 == E2$

**Mapping:**  $m_1 : X \mapsto X'^1$

**Links:**  $l_1 : E2 =_\eta \lambda x.X'_x \quad | \quad l_2 : E0 =_\eta \lambda x.X'_x$

TODO: explain

**LEMMA 6.5.** *For all list of links  $\mathbb{L}$  and  $S$ , progress  $\mathbb{L} \_ S \_ \text{terminates}$*

**PROOF SKETCH.** The addition of rules for progress1 complicates the function progress. We can note, however, that they do not prevent the termination of progress. 1) If a link is activated it is removed from  $\mathbb{L}$  and the recursive call to progress will have a smaller list of links to recurse on. Moreover, link activation only runs terminating instructions (such as unification). 2) If a link is deduplicated, the termination of progress is still guaranteed since again we reduce  $\mathbb{L}$  and the instructions run by link deduplications are all terminating. 3) If a link is neither activated nor deduplicated,

i.e. it remains suspended, then  $\mathbb{L}$  remains unchanged like the substitution; therefore, if  $(L = L1, S1 = S2)$  succeeds and progress terminates.  $\square$

TODO: we can have  $\lambda x.F_x$  in the substitution if we know that  $F$  does not reduce to  $T_x$  where  $x$  is not free in  $T$ .

## 7 ENFORCING INVARIANT 1

In section 5.5, we have given two unification problems to be run one after the other. In the following table we present the entry problem, its compiled version, with the corresponding list of mapping and links.

**Entry pb:**  $\mathcal{P}_1 : \text{flam } x \backslash \text{flam } y \backslash \text{fapp } [\text{fuva } X, y, x] == \text{flam } x \backslash \text{flam } y \backslash \text{fapp } [\text{fcon} "f", \text{fapp } [\text{fuva } X, x], x] == \text{fuva } x$   
 $\mathcal{P}_2 : \text{flam } x \backslash \text{fapp } [\text{fcon} "f", \text{fapp } [\text{fuva } X, x], x] == \text{fuva } x$   
**Compil:**  $\mathcal{T}_1 : E0 == \lambda x.\lambda y.x$  |  $\mathcal{T}_2 : E3 == E5$   
**Mapping:**  $m_1 : Y \mapsto E5^0$  |  $m_2 : X \mapsto E2^2$   
**Links:**  $l_1 : c0 \vdash E4_{c0} =_\eta$  |  $l_2 : \vdash E3 =_\eta$   
 $\lambda x.E2_{c0x}$  |  $\lambda x.(f E4_x x)$   
 $l_3 : \vdash E0 =_\eta \lambda x.E1_x$  |  $l_4 : c0 \vdash E1_{c0} =_\eta$   
 $\lambda x.E2_{xc0}$

We see that the maybe-eta as detected  $\lambda xy.F y x$  and  $\lambda x.f (F x) x$  and replaced them with respectively the  $\mathcal{H}_0$  vars  $X$  and  $Z$ .  $X$  is linked with  $\lambda x.Y_x$ ,  $Y$  has arity 1 and is  $\eta$ -linked with  $\lambda y.H y x$  and  $Z$  is linked to the term  $\lambda x.f G_x x$ . However, the mapping returned by the compilation, does not breaks invariant 3: the  $\mathcal{F}_0$  variable  $F$  is mapped to two different  $\mathcal{H}_0$  variables. To address this problem and enforce invariant 3, we clean the mapping with a second phase after the compilation. This phase is called map-deduplication.

Before formally defining this procedure, we need to define some auxiliary relations. Let  $\mathbb{M}$  be the list of mapping and  $\langle m_1, m_2 \rangle \in \mathbb{M}$  such that the arity of the  $\mathcal{H}_0$  variable in  $m_1$  is smaller than the one in  $m_2$ . Let  $X$  (resp.  $Y$ ) the  $\mathcal{H}_0$  variable of  $m_1$  (resp.  $m_2$ ) and  $n = ar(m_1) - ar(m_2)$ . We also let  $A^i$  be a fresh  $\mathcal{H}_0$  variable. We define the make-eta-link relation taking two mappings  $\langle m_1, m_2 \rangle$  and returning the following list of link  $\text{link-}\eta$ :  $\forall i \in [1..n]$ ,

$$\left\{ \begin{array}{ll} \vdash X =_\eta \lambda x.A_x^1 & \text{if } i = 1 \\ x_1 \dots x_{i-1} \vdash A_{x_1 \dots x_{i-1}}^{i-1} =_\eta \lambda x_i.A_{x_1 \dots x_i}^i & \text{if } 1 < i < n \\ x_1 \dots x_{i-1} \vdash A_{x_1 \dots x_{i-1}}^{i-1} =_\eta \lambda x_i.Y_{x_1 \dots x_i} & \text{if } i = n \end{array} \right.$$

More concretely, we are saying that for any two mappings, we build as many  $\text{link-}\eta$  as the difference of the arities between the two mappings. This links are constructed in such a way that the  $\mathcal{H}_0$  variable  $v$  with lowest arity is linked to a fresh variable eta-expanded variable  $A^1$  having the scope of  $v$ . This variable  $A^1$  is then linked to a an  $\eta$ -expanded fresh variable  $A^2$  with same scope of  $A^1$  and so on. The last link is built between the  $A^{n-1}$  (where  $n$  is the difference of arities between the two mappings) and the  $\mathcal{H}_0$  variable  $u$  with higher arity in the two mappings being considered.

**Definition 7.1 (map-deduplication).** For all mappings  $\langle m_1, m_2 \rangle \in \mathbb{M}$ , sharing the same  $\mathcal{F}_0$  variable, the list of  $\text{link-}\eta$   $L$  is created thanks to  $\text{make-eta-link}_{m_1 m_2} L$  and is added to  $\mathbb{L}$ . Then  $m_1$  is removed from  $\mathbb{M}$ .

If we take back the example give at the beginning of this section, we can deduplicate  $F \mapsto G^1, F \mapsto H^2$  by removing the first mapping and adding the auxiliary  $\text{link-}\eta$ :  $x \vdash G_x =_\eta \lambda y.H_{xy}$ .

The complete problem to run for resolution is now:

**Compil:**  $\mathcal{T}_1 : X = \lambda x.\lambda y.x$  |  $\mathcal{T}_2 : Z = f (\lambda x.x)$

**Mapping:**  $m_1 : F \mapsto H^2$

**Links:**  $l_1 : x \vdash Y_x =_\eta \lambda y.H_{yx}$  |  $l_2 : \vdash X =_\eta \lambda x.Y_x$   
 $l_3 : \vdash Z =_\eta \lambda x.f G_x x$  |  $l_4 : x \vdash G_x =_\eta \lambda y.H_x y$

After unification of the two terms,  $X$  is assigned to  $\lambda x.\lambda y.x$ . This assignment makes  $l_2$  to progress since the lhs is materialized and by unification, between  $X$  and  $\lambda x.Y_x$ ,  $Y_x$  is instantiate to  $\lambda y.x$ . Once  $Y_x$  is instantiated,  $l_1$  can progress, and set  $H_{xy}$  to  $x$ . After all these progresses,  $l_1$  and  $l_2$  are remove from  $\mathbb{L}$  and the progress fixpoint terminates. Next, the second unification problem is run, and  $Z$  is set to  $f (\lambda x.x)$ . This unification wakes up  $l_3$  and since  $Z$  starts with the app node, the  $\eta$ -expanded version of  $Z$  is unified with  $\lambda x.f G_x x$  and  $G_x$  is set to  $x$ . As last step, the last link is progressed and the final  $\mathcal{H}_0$  substitution is  $\{X \mapsto \lambda x.\lambda y.x, Y_x \mapsto \lambda y.x, G_{yx} \mapsto y, Z \mapsto f \lambda x.x, H_x \mapsto \lambda y.y\}$ .

The decompilation phase is only charged, in this example to solve the mappings, since no suspended links remain. The only mapping in the list is  $F \mapsto H^2$ , which will assign the  $F$  variable in  $\mathcal{F}_0$  to  $\lambda xy.y$ .

TODO: dire che preserviamo l'invariante che tutte le variable sono fully-applied

## 8 HANDLING OF $\overline{\mathcal{L}_\lambda}$

TODO: say that maybe-eta also work in  $\text{not}(\text{llambda})$

All the previous sections we have dealt with terms in  $\mathcal{L}_\lambda$ , however, it is often possible to work with terms in  $\overline{\mathcal{L}_\lambda}$  and wish to unify them. There are situation, for example, where the oracle has given which  $\beta$ -reduction problems ( $\overline{\mathcal{L}_\lambda}$ ) appears any time we deal with a subterm  $t = X t_1 \dots t_n$ , where  $X$  is flexible and the list  $[t_1 \dots t_n]$  in not in  $\mathcal{L}_\lambda$ . This unification problem is not solvable without loss of generality, since there is not a most general unifier. If we take back the example given in section 2.1, the unification  $F a = a$  admits two solutions for  $F$ :  $\rho_1 = \{F \mapsto \lambda x.x\}$  and  $\rho_2 = \{F \mapsto \lambda_.a\}$ . Despite this, it is possible to work with  $\overline{\mathcal{L}_\lambda}$  if an oracle provides a substitution  $\rho$  such that  $\rho t$  falls again in the  $\mathcal{L}_\lambda$ .

On the other hand, the  $\approx_\lambda$  is not designed to understand how the  $\beta$ -redexes work in the object language. Therefore, even if we know that  $F$  is assigned to  $\lambda x.x$ ,  $\approx_\lambda$  is not able to unify  $F a$  with  $a$ . On the other hand, the problem  $F a = G$  is solvable by  $\approx_\lambda$ , but the final result is that  $G$  is assigned to  $(\lambda x.x)a$  which breaks the invariant saying that the substitution of the meta language does not generate terms outside  $\mathcal{W}$  (Property 2.8).

The solution to this problem is to modify the compiler such that any sub-term  $t$  considered as a potential  $\beta$ -redex is replaced with a hole  $h$  and a new dedicated link, called  $\text{link-}\beta$ .

**type link-beta**  $\text{tm} \rightarrow \text{tm} \rightarrow \text{link}$ .

This link carries two terms, the former representing the variable  $h$  for the new created hole and the latter containing the subterm  $t$ . As for the  $\text{link-}\eta$ , we will call  $h$  and  $t$  respectively the left hand side ( $lhs$ ) and the right hand side ( $rhs$ ) of the  $\text{link-}\beta$ .

### 8.1 Compilation

*Detection of  $\overline{\mathcal{L}_\lambda}$ .* TODO: ...

*Compilation with link- $\beta$ .* In order to build a  $\text{link-}\beta$ , we need to adapt the compiler so that it can recognize these "problematic"

subterms. The following code snippet illustrate such behavior, we suppose the rule to be added just after ??.

```
comp (fapp [fuva A|Ag]) (uva C Scope) M1 M3 L1 L3 S1 S4 :- !,
  pattern-fragment-prefix Ag Pf Extra,
  fold6 comp Pf Scope1 M1 M1 L1 L1 S1 S1,
  fold6 comp Extra Extra1 M1 M2 L1 L2 S1 S2,
  len Pf Arity,
  m-alloc (fv A) (hv B (arity Arity)) M2 M3 S2 S3,
  Beta = app [uva B Scope1 | Extra1],
  get-scope Beta Scope,
  alloc S3 C S4,
  L3 = [eval-link-beta (uva C Scope) Beta | L2].
```

A term is  $\overline{\mathcal{L}}_\lambda$  if it has the shape  $\text{fapp}[fuva A|Ag]$  and  $\text{distinct } Ag$  does not hold. In that case,  $Ag$  is split in two sublist  $Pf$  and  $Extra$  such that former is the longest prefix of  $Ag$  such that  $\text{distinct } Pf$  holds.  $Extra$  is the list such that  $\text{append } Pf Extra Ag$ . Next important step is to compile recursively the terms of these lists and allocate a memory address  $B$  from the substitution in order to map the  $\mathcal{F}_0$  variable  $fuva A$  to the  $\mathcal{H}_0$  variable  $uva B$ . The  $\text{link-}\beta$  to return in the end is given by the term  $Beta = \text{app}[uva B Scope1 | Extra1]$  constituting the  $rhs$ , and a fresh variable  $C$  having in scope all the free variables occurring in  $Beta$  (this is  $lhs$ ). We point out that the  $rhs$  is intentionally built as an  $uva$  where  $Extra1$  are not in scope, since by invariant, we want all the variables appearing in  $\mathcal{H}_0$  to be in  $\mathcal{L}_\lambda$ .

## 8.2 Progress

Once created, there exist two main situations waking up a suspended  $\text{link-}\beta$ . The former is strictly connected to the definition of  $\beta$ -redex and occurs when the head of  $rhs$  is materialized by the oracle (see proposition 2.1). In this case  $rhs$  is safely  $\beta$ -reduced to a new term  $t'$  and the result can be unified with  $lhs$ . In this scenario the  $\text{link-}\beta$  has accomplished its goal and can be removed from  $\mathbb{L}$ .

The second circumstance making the  $\text{link-}\beta$  to progress is the instantiation of the variables in the  $Extra1$  making the corresponding arguments to reduce to names. In this case, we want to take the list  $Scope1$  and append to it the largest prefix of  $Extra1$  in a new variable  $Scope2$  such that  $Scope2$  remains in  $\mathcal{L}_\lambda$ ; we call  $Extra2$  the suffix of  $Extra1$  such that the concatenation of  $Scope1$  and  $Extra1$  is the same as the concatenation of  $Scope2$  and  $Extra2$ . Finally, two cases should be considered: 1)  $Extra2$  is the empty list,  $lhs$  and  $rhs$  can be unified: we have two terms in  $\mathcal{L}_\lambda$ ; otherwise 2) the  $\text{link-}\beta$  in question is replaced with a refined version where the  $rhs$  is  $\text{app}[uva C Scope2 | Extra2]$  and a new  $\text{link-}\eta$  is added between the  $lhs$  and the new-added variable  $C$ .

An example justifying this second link manipulation is given by the following unification problem:

```
f = flam x\ fapp[F, fapp[A, x]].
```

The compilation of these terms produces the new unification problem:  $f = X0$

We obtain the mappings  $F \mapsto \mathbf{F}^0, A \mapsto \mathbf{A}^1$  and the links:

$$c0 \vdash X3_{c0} =_\beta X2 X1_{c0} \quad (12)$$

$$\vdash X0 =_\eta \lambda c0. X3_{c0} \quad (13)$$

where the first link is a  $\text{link-}\eta$  between the variable  $X0$ , representing the right side of the unification problem (it is a  $\diamond\eta$ ) and  $X3$ ; and

a  $\text{link-}\beta$  between the variable  $X3$  and the subterm  $\lambda x. X1_x a$  (it is a  $\overline{\mathcal{L}}_\lambda$ ). The substitution tells that  $x \vdash X1_x = x$ .

We can now represent the hrun execution from this configuration which will, at first, dereference all the links, and then try to solve them. The only link being modified is the second one, which is set to  $x \vdash X3 =_\beta X2 x a$ . The  $rhs$  of the link has now a variable which is partially in the PF, we can therefore remove the original  $\text{link-}\beta$  and replace it with the following couple on links:

```
\vdash X1 =_\eta x\ `X4 x'
x \vdash X3 x =_\beta x\ `X4 x' a
```

By these links we say that  $X1$  is now  $\eta$ -linked to a fresh variable  $x4$  with arity one. This new variable is used in the new  $\text{link-}\beta$  where the name  $x$  is in its scope. This allows

## 8.3 Tricky examples

```
triple ok (@lam x\ @app[@f, @app[@x, x]]) @Y,
triple ok @X (@lam x\ x),
triple ok @Y @f

% @okl 22 [
%   triple ok (@lam x\ @lam y\ @app[@Y, y, x]) @X,
%   triple ok (@lam x\ @f) @X,
% ].
```

## 9 FIRST ORDER APPROXIMATION

**TODO: Coq can solve this:  $f \ 1 \ 2 = x \ 2$ , by setting  $X$  to  $f \ 1$**

**TODO: We can re-use part of the algo for  $\beta$  given before**

## 10 UNIF ENCODING IN REAL LIFE

**TODO: Il ML presentato qui è esattamente elpi**

**TODO: Il OL presentato qui è esattamente coq**

**TODO: Come implementiamo tutto ciò nel solver**

## 11 RESULTS: STDPP AND TLC

**TODO: How may rule are we solving?**

**TODO: Can we do some perf test**

## 12 CONCLUSION

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## APPENDIX

This appendix contains the entire code described in this paper. The code can also be accessed at the URL: <https://github.com/FissoreD/paper-ho>

Note that (a infix b) c d de-sugars to (infix) a b c d.

Explain builtin name (can be implemented by loading name after each pi)

## 13 THE MEMORY

```

kind addr type.
type addr nat -> addr.
typeabbrev (mem A) (list (option A)).

type set? addr -> mem A -> A -> o.
set? (addr A) Mem Val :- get A Mem Val.

type unset? addr -> mem A -> o.
unset? Addr Mem :- not (set? Addr Mem _).

type assign-aux nat -> mem A -> A -> mem A -> o.
assign-aux z (none :: L) Y (some Y :: L).
assign-aux (s N) (X :: L) Y (X :: L1) :- assign-aux N L Y L1.

type assign addr -> mem A -> A -> mem A -> o.
assign (addr A) Mem1 Val Mem2 :- assign-aux A Mem1 Val Mem2.

type get nat -> mem A -> A -> o.
get z (some Y :: _) Y.
get (s N) (_ :: L) X :- get N L X.

type alloc-aux nat -> mem A -> mem A -> o.
alloc-aux z [] [none] :- !.
alloc-aux z L L.
alloc-aux (s N) [] [none | M] :- alloc-aux N [] M.
alloc-aux (s N) [X | L] [X | M] :- alloc-aux N L M.

type alloc addr -> mem A -> mem A -> o.
alloc (addr A as Ad) Mem1 Mem2 :- unset? Ad Mem1,
  alloc-aux A Mem1 Mem2.

type new-aux mem A -> nat -> mem A -> o.
new-aux [] z [none].
new-aux [A | As] (s N) [A | Bs] :- new-aux As N Bs.

type new mem A -> addr -> mem A -> o.
new Mem1 (addr Ad) Mem2 :- new-aux Mem1 Ad Mem2.

```

## 14 THE OBJECT LANGUAGE

```

kind fm type.
type fapp list fm -> fm.
type flam (fm -> fm) -> fm.
type fcon string -> fm.
type fuva addr -> fm.

typeabbrev fsubst (mem fm).

```

```

type fder fsubst -> fm -> fm -> o.
fder _ (fcon C) (fcon C).
fder S (fapp A) (fapp B) :- map (fder S) A B.
fder S (flam F) (flam G) :-
  pi x\ fder S x x => fder S (F x) (G x).
fder S (fuva N) R :- set? N S T, fder S T R.
fder S (fuva N) (fuva N) :- unset? N S.

type fderef fsubst -> fm -> fm -> o. (ρs)
fderef S T T2 :- fder S T T1, napp T1 T2.

```

```

type (=o) fm -> fm -> o. (=o)
fcon X =o fcon X.
fapp A =o fapp B :- forall2 (=o) A B.
flam F =o flam G :- pi x\ x =o x => F x =o G x.
fuva N =o fuva N.
flam F =o T :- (ηl)
  pi x\ beta T [x] (T' x), x =o x => F x =o T' x.
T =o flam F :- (ηr)
  pi x\ beta T [x] (T' x), x =o x => T' x =o F x.
fapp [flam X | L] =o T :- beta (flam X) L R, R =o T. (βl)
T =o fapp [flam X | L] :- beta (flam X) L R, T =o R. (βr)

```

```

type extend-subst fm -> fsubst -> fsubst -> o.
extend-subst (fuva N) S S' :- mem.alloc N S S'.
extend-subst (flam F) S S' :-
  pi x\ (pi S\ extend-subst x S S) => extend-subst (F x) S S'.
extend-subst (fcon _) S S.
extend-subst (fapp L) S S1 :- fold extend-subst L S S1.

type beta fm -> list fm -> fm -> o.
beta A [] A.
beta (flam Bo) [H | L] R :- napp (Bo H) F, beta F L R.
beta (fapp A) L (fapp X) :- append A L X.
beta (fuva N) L (fapp [fuva N | L]).
beta (fcon H) L (fapp [fcon H | L]).
beta N L (fapp [N | L]) :- name N.

```

```

type napp fm -> fm -> o.
napp (fcon C) (fcon C).
napp (fuva A) (fuva A).
napp (flam F) (flam G) :- pi x\ napp (F x) (G x).
napp (fapp [fapp L1 [L2]] T) :- !,
  append L1 L2 L3, napp (fapp L3) T.
napp (fapp L) (fapp L1) :- map napp L L1.
napp N N :- name N.

```

```

type mk-app fm -> list fm -> fm -> o.
mk-app T L S :- beta T L S.

```

```

type eta-contract fm -> fm -> o.
eta-contract (fcon X) (fcon X).
eta-contract (fapp L) (fapp L1) :- map eta-contract L L1.
eta-contract (flam F) T :- eta-contract-aux [] (flam F) T.
eta-contract (flam F) (flam F1) :-

```

```

1509   pi x\ eta-contract x x => eta-contract (F x) (F1 x).
1510 eta-contract (fuva X) (fuva X).
1511 eta-contract X X :- name X.
1512
1513 type eta-contract-aux list fm -> fm -> fm -> o.
1514 eta-contract-aux L (flam F) T :-
1515   pi x\ eta-contract-aux [x|L] (F x) T. % also checks H Prefix does
1516 eta-contract-aux L (fapp [H|Args]) T :-
1517   rev L LRev, append Prefix LRev Args,
1518   if (Prefix = []) (T = H) (T = fapp [H|Prefix]).
1519
1520
1521 15 THE META LANGUAGE
1522 kind inctx type -> type.
1523 type abs (tm -> inctx A) -> inctx A.
1524 type val A -> inctx A.
1525 typeabbrev assignment (inctx tm).
1526 typeabbrev subst (mem assignment).
1527
1528 kind tm type.
1529 type app list tm -> tm.
1530 type lam (tm -> tm) -> tm.
1531 type con string -> tm.
1532 type uva addr -> list tm -> tm.
1533
1534 type ( $\approx_\lambda$ ) tm -> tm -> subst -> subst -> o.
1535 (con C  $\approx_\lambda$  con C) S S.
1536 (app L1  $\approx_\lambda$  app L2) S S1 :- fold2 ( $\approx_\lambda$ ) L1 L2 S S1.
1537 (lam F1  $\approx_\lambda$  lam F2) S S1 :-
1538   pi x\ (pi S\ (x  $\approx_\lambda$  x) S S) => (F1 x  $\approx_\lambda$  F2 x) S S1.
1539 (uva N Args  $\approx_\lambda$  T) S S1 :-
1540   set? N S F,!, move F Args T1, (T1  $\approx_\lambda$  T) S S1.
1541 (T  $\approx_\lambda$  uva N Args) S S1 :-
1542   set? N S F,!, move F Args T1, (T  $\approx_\lambda$  T1) S S1.
1543 (uva M A1  $\approx_\lambda$  uva N A2) S1 S2 :- !,
1544   pattern-fragment A1, pattern-fragment A2,
1545   prune! M A1 N A2 S1 S2.
1546 (uva N Args  $\approx_\lambda$  T) S S1 :- not_occ N S T, pattern-fragment Args,
1547   bind T Args T1, assign N S T1 S1.
1548 (T  $\approx_\lambda$  uva N Args) S S1 :- not_occ N S T, pattern-fragment Args,
1549   bind T Args T1, assign N S T1 S1.
1550
1551 type prune! addr -> list tm -> addr ->
1552   list tm -> subst -> subst -> o.
1553 /* no pruning needed */
1554 prune! N A N A S S :- !.
1555 prune! M A N A S1 S2 :- !, bind (uva M A) A Ass,
1556   assign N S1 Ass S2.
1557 /* prune different arguments */
1558 prune! N A1 N A2 S1 S3 :- !,
1559   new S1 W S2, prune-same-variable W A1 A2 [] Ass,
1560   assign N S2 Ass S3.
1561 /* prune to the intersection of scopes */
1562 prune! N A1 M A2 S1 S4 :- !,
1563   new S1 W S2, prune-diff-variables W A1 A2 Ass1 Ass2,
1564   assign N S2 Ass1 S3,
1565   assign M S3 Ass2 S4.
1566

```

(· ⊢ ·)

```

1567 type prune-same-variable addr -> list tm -> list tm ->
1568   list tm -> assignment -> o.
1569 prune-same-variable N [] [] ACC (val (uva N Args)) :-
1570   rev ACC Args.
1571 prune-same-variable N [X|XS] [X|YS] ACC (abs F) :-
1572   pi x\ prune-same-variable N XS YS [x|ACC] (F x).
1573 prune-same-variable N [_|XS] [_|YS] ACC (abs F) :-
1574   pi x\ prune-same-variable N XS YS ACC (F x).
1575
1576 type permute list nat -> list tm -> list tm -> o.
1577 permute [] _ [].
1578 permute [P|PS] Args [T|TS] :-
1579   nth P Args T,
1580   permute PS Args TS.
1581
1582 type build-perm-assign addr -> list tm -> list bool ->
1583   list nat -> assignment -> o.
1584 build-perm-assign N ArgsR [] Perm (val (uva N PermutedArgs)) :-
1585   rev ArgsR Args, permute Perm Args PermutedArgs.
1586 build-perm-assign N Acc [tt|L] Perm (abs T) :-
1587   pi x\ build-perm-assign N [x|Acc] L Perm (T x).
1588 build-perm-assign N Acc [ff|L] Perm (abs T) :-
1589   pi x\ build-perm-assign N Acc L Perm (T x).
1590
1591 type keep list A -> A -> bool -> o.
1592 keep L A tt :- mem L A, !.
1593 keep _ _ ff.
1594
1595 type prune-diff-variables addr -> list tm -> list tm ->
1596   assignment -> assignment -> o.
1597 prune-diff-variables N Args1 Args2 Ass1 Ass2 :-
1598   map (keep Args2) Args1 Bits1,
1599   map (keep Args1) Args2 Bits2,
1600   filter Args1 (mem Args2) ToKeep1,
1601   filter Args2 (mem Args1) ToKeep2,
1602   map (index ToKeep1) ToKeep1 IdPerm,
1603   map (index ToKeep1) ToKeep2 Perm21,
1604   build-perm-assign N [] Bits1 IdPerm Ass1,
1605   build-perm-assign N [] Bits2 Perm21 Ass2.
1606
1607 type beta tm -> list tm -> tm -> o.
1608 beta A [] A.
1609 beta (lam Bo) [H | L] R :- beta (Bo H) L R.
1610 beta (app A) L (app X) :- append A L X.
1611 beta (con H) L (app [con H | L]).
1612 beta X L (app [X|L]) :- name X.
1613
1614 /* occur check for N before crossing a functor */
1615 type not_occ addr -> subst -> tm -> o.
1616 not_occ N S (uva M Args) :- set? M S F,
1617   move F Args T, not_occ N S T.
1618 not_occ N S (uva M Args) :- unset? M S, not (M = N),
1619   forall1 (not_occ_aux N S) Args.
1620 not_occ _ _ (con _).
1621 not_occ N S (app L) :- not_occ_aux N S (app L).
1622 /* Note: lam is a functor for the meta language! */
1623

```

```

1625 not_occ N S (lam L) :- pi x\ not_occ_aux N S (L x).
1626 not_occ _ _ X :- name X.
1627 /* finding N is ok */
1628 not_occ N _ (uva N _).
1629
1630 /* occur check for X after crossing a functor */
1631 type not_occ_aux addr -> subst -> tm -> o.
1632 not_occ_aux N S (uva M _) :- unset? M S, not (N = M).
1633 not_occ_aux N S (uva M Args) :- set? M S F,
1634   move F Args T, not_occ_aux N S T.
1635 not_occ_aux N S (app L) :- forall1 (not_occ_aux N S) L.
1636 not_occ_aux N S (lam F) :- pi x\ not_occ_aux N S (F x).
1637 not_occ_aux _ _ (con _).
1638 not_occ_aux _ _ X :- name X.
1639 /* finding N is ko, hence no rule */
1640
1641 /* copy T T' fails if T contains a free variable, i.e. it
1642   performs scope checking for bind */
1643 type copy tm -> tm -> o.
1644 copy (con C) (con C).
1645 copy (app L) (app L') :- map copy L L'.
1646 copy (lam T) (lam T') :- pi x\ copy x x => copy (T x) (T' x).
1647 copy (uva A L) (uva A L') :- map copy L L'.
1648
1649 type bind tm -> list tm -> assignment -> o.
1650 bind T [] (val T') :- copy T T'.
1651 bind T [X | TL] (abs T') :- pi x\ copy X x => bind T TL (T' x).
1652
1653 type deref subst -> tm -> tm -> o. (σt)
1654 deref _ (con C) (con C).
1655 deref S (app A) (app B) :- map (deref S) A B.
1656 deref S (lam F) (lam G) :-
1657   pi x\ deref S x x => deref S (F x) (G x).
1658 deref S (uva N L) R :- set? N S A,
1659   move A L T, deref S T R.
1660 deref S (uva N A) (uva N B) :- unset? N S,
1661   map (deref S) A B.
1662
1663 type move assignment -> list tm -> tm -> o.
1664 move (abs Bo) [H|L] R :- move (Bo H) L R.
1665 move (val A) [] A.
1666
1667
1668 type deref-assmt subst -> assignment -> assignment -> o.
1669 deref-assmt S (abs T) (abs R) :- pi x\ deref-assmt S (T x) (R x).
1670 deref-assmt S (val T) (val R) :- deref S T R.

```

## 16 THE COMPILER

```

1674 kind arity type.
1675 type arity nat -> arity.
1676
1677 kind fvariable type.
1678 type fv addr -> fvariable.
1679
1680 kind hvariable type.
1681 type hv addr -> arity -> hvariable.

```

```

kind mapping type.
type mapping fvariable -> hvariable -> mapping.
typeabbrev mmap (list mapping).

typeabbrev scope (list tm).
typeabbrev inctx ho.inctx.
kind baselink type.
type link-eta tm -> tm -> baselink.
type link-beta tm -> tm -> baselink.
typeabbrev link (inctx baselink).
typeabbrev links (list link).

macro @val-link-eta T1 T2 :- ho.val (link-eta T1 T2).
macro @val-link-beta T1 T2 :- ho.val (link-beta T1 T2).

type occurs-rigidly fm -> fm -> o.
occurs-rigidly N N.
occurs-rigidly _ (fapp [fuva _|_]) :- !, fail.
occurs-rigidly N (fapp L) :- exists (occurs-rigidly N) L.
occurs-rigidly N (flam B) :- pi x\ occurs-rigidly N (B x).

type reducible-to list fm -> fm -> fm -> o.
reducible-to _ N N :- !.
reducible-to L N (fapp [fuva _|Args]) :- !,
  forall1 (x\ exists (reducible-to [] x) Args) [N|L].
reducible-to L N (flam B) :- !,
  pi x\ reducible-to [x | L] N (B x).
reducible-to L N (fapp [N|Args]) :-
  last-n {len L} Args R,
  forall2 (reducible-to []) R {rev L}.

type maybe-eta fm -> list fm -> o. (◇η)
maybe-eta (fapp [fuva _|Args]) L :- !,
  forall1 (x\ exists (reducible-to [] x) Args) L, !.
maybe-eta (flam B) L :- !, pi x\ maybe-eta (B x) [x | L].
maybe-eta (fapp [fcon _|Args]) L :-
  split-last-n {len L} Args First Last,
  none (x\ exists (y\ occurs-rigidly x y) First) L,
  forall2 (reducible-to []) {rev L} Last.

type locally-bound tm -> o.
type get-scope-aux tm -> list tm -> o.
get-scope-aux (con _) [].
get-scope-aux (uva _ L) L1 :-
  forall2 get-scope-aux L R,
  flatten R L1.
get-scope-aux (lam B) L1 :-
  pi x\ locally-bound x => get-scope-aux (B x) L1.
get-scope-aux (app L) L1 :-
  forall2 get-scope-aux L R,
  flatten R L1.
get-scope-aux X [X] :- name X, not (locally-bound X).
get-scope-aux X [] :- name X, (locally-bound X).

```

```

1741
1742 type names1 list tm -> o.
1743 names1 L :-
1744   names L1,
1745   new_int N,
1746   if (1 is N mod 2) (L1 = L) (rev L1 L).
1747
1748 type get-scope tm -> list tm -> o.
1749 get-scope T Scope :-
1750   get-scope-aux T ScopeDuplicata,
1751   undup ScopeDuplicata Scope.
1752 type rigid fm -> o.
1753 rigid X :- not (X = fuva _).
1754
1755 type comp-lam (fm -> fm) -> (tm -> tm) ->
1756   mmap -> mmap -> links -> links -> subst -> subst -> o.
1757 comp-lam F G M1 M2 L1 L3 S1 S2 :-
1758   pi x y\ (pi M L S\ comp x y M M L L S S) =>
1759     comp (F x) (G y) M1 M2 L1 (L2 y) S1 S2,
1760     close-links L2 L3.
1761
1762 type close-links (tm -> links) -> links -> o.
1763 close-links (v\ [X |L v]) [X|R] :- !, close-links L R.
1764 close-links (v\ [X v|L v]) [abs X|R] :- close-links L R.
1765 close-links (_\ []) [].
1766 type comp fm -> tm -> mmap -> mmap -> links -> links ->
1767   subst -> subst -> o.
1768 comp (fcon C) (con C) M M L L S S.
1769 comp (flam F) (uva A Scope) M1 M2 L1 L3 S1 S3 :-
1770   maybe-eta (flam F) [], !,
1771   alloc S1 A S2,
1772   comp-lam F F1 M1 M2 L1 L2 S2 S3,
1773   get-scope (lam F1) Scope,
1774   L3 = [val (link-eta (uva A Scope) (lam F1)) | L2].
1775 comp (flam F) (lam F1) M1 M2 L1 L2 S1 S2 :- (cλ)
1776   comp-lam F F1 M1 M2 L1 L2 S1 S2.
1777 comp (fuva A) (uva B []) M1 M2 L L S1 S2 :-
1778   m-alloc (fv A) (hv B (arity z)) M1 M2 S1 S2.
1779 comp (fapp [fuva A|Ag]) (uva B Ag1) M1 M2 L L S1 S2 :-
1780   pattern-fragment Ag, !,
1781   fold6 comp Ag Ag1 M1 M1 L L S1 S1,
1782   len Ag Arity,
1783   m-alloc (fv A) (hv B (arity Arity)) M1 M2 S1 S2.
1784 comp (fapp [fuva A|Ag]) (uva C Scope) M1 M3 L1 L3 S1 S4 :- !,
1785   pattern-fragment-prefix Ag Pf Extra,
1786   fold6 comp Pf Scope1 M1 M1 L1 L1 S1 S1,
1787   fold6 comp Extra Extra1 M1 M2 L1 L2 S1 S2,
1788   len Pf Arity,
1789   m-alloc (fv A) (hv B (arity Arity)) M2 M3 S2 S3,
1790   Beta = app [uva B Scope1 | Extra1],
1791   get-scope Beta Scope,
1792   alloc S3 C S4,
1793   L3 = [eval-link-beta (uva C Scope) Beta | L2].
1794 comp (fapp A) (app A1) M1 M2 L1 L2 S1 S2 :-
1795   fold6 comp A A1 M1 M2 L1 L2 S1 S2.
1796
1797 type alloc mem A -> addr -> mem A -> o.
1798
1799 alloc S N S1 :- mem.new S N S1.
1800
1801 type compile-terms-diagnostic
1802   triple diagnostic fm fm ->
1803   triple diagnostic tm tm ->
1804   mmap -> mmap ->
1805   links -> links ->
1806   subst -> subst -> o.
1807 compile-terms-diagnostic (triple D F01 F02) (triple D H01 H02) M0 M3 L1
1808   comp F01 H01 M1 M2 L1 L2 S1 S2,
1809   comp F02 H02 M2 M3 L2 L3 S2 S3.
1810
1811 type compile-terms
1812   list (triple diagnostic fm fm) ->
1813   list (triple diagnostic tm tm) ->
1814   mmap -> links -> subst -> o.
1815 compile-terms T H M L S :-
1816   fold6 compile-terms-diagnostic T H [] M_ [] L_ [] S_,
1817   deduplicate-map M_ M S_ S L_ L.
1818
1819 type make-eta-link-aux nat -> addr -> addr ->
1820   list tm -> links -> subst -> subst -> o.
1821 make-eta-link-aux z Ad1 Ad2 Scope1 L H1 H1 :-
1822   rev Scope1 Scope, eta-expand (uva Ad2 Scope) T1,
1823   L = [val (link-eta (uva Ad1 Scope) T1)].
1824 make-eta-link-aux (s N) Ad1 Ad2 Scope1 L H1 H3 :-
1825   rev Scope1 Scope, alloc H1 Ad H2,
1826   eta-expand (uva Ad Scope) T2,
1827   (pi x\ make-eta-link-aux N Ad Ad2 [x|Scope1] (L1 x) H2 H3),
1828   close-links L1 L2,
1829   L = [val (link-eta (uva Ad1 Scope) T2) | L2].
1830
1831 type make-eta-link nat -> nat -> addr -> addr ->
1832   list tm -> links -> subst -> subst -> o.
1833 make-eta-link (s N) z Ad1 Ad2 Vars L H H1 :-
1834   make-eta-link-aux N Ad2 Ad1 Vars L H H1.
1835 make-eta-link z (s N) Ad1 Ad2 Vars L H H1 :-
1836   make-eta-link-aux N Ad1 Ad2 Vars L H H1.
1837 make-eta-link (s N) (s M) Ad1 Ad2 Vars Links H H1 :-
1838   (pi x\ make-eta-link N M Ad1 Ad2 [x|Vars] (L x) H H1),
1839   close-links L Links.
1840
1841 type deduplicate-map mmap -> mmap ->
1842   subst -> subst -> links -> links -> o.
1843 deduplicate-map [] [] H H L L.
1844 deduplicate-map [(mapping (fv O) (hv M (arity LenM))) as X1] | Map1 Map2
1845   take-list Map1 (mapping (fv O) (hv M' (arity LenM'))), !,
1846   std.assert! (not (LenM = LenM')) "Deduplicate map, there is a bug",
1847   print "arity-fix links:" {ppmapping X1} "~!~" {ppmapping (mapping (fv
1848   make-eta-link LenM LenM' M M' [] New H1 H2,
1849   print "new eta link" {pplinks New},
1850   append New L1 L2,
1851   deduplicate-map Map1 Map2 H2 H3 L2 L3.
1852 deduplicate-map [A|As] [B|Bs] H1 H2 L1 L2 :-
1853   deduplicate-map As Bs H1 H2 L1 L2, !.
1854 deduplicate-map [A|_] _ H _ _ :-
1855   halt "deduplicating mapping error" {ppmapping A} {ho.ppsubst H}.

```



## 17 THE PROGRESS FUNCTION

```
macro @one :- s z.
```

```
type contract-rigid list ho.tm -> ho.tm -> ho.tm -> o.
```

```
contract-rigid L (ho.lam F) T :-
```

```
  pi x\ contract-rigid [x|L] (F x) T. % also checks H Prefix does not see x
```

```
contract-rigid L (ho.app [H|Args]) T :-
```

```
  rev L LRev, append Prefix LRev Args,  
  if (Prefix = []) (T = H) (T = ho.app [H|Prefix]).
```

```
type progress-eta-link ho.tm -> ho.tm -> ho.subst -> ho.subst -> links -> o.
```

```
progress-eta-link (ho.app _ as T) (ho.lam x\ _ as T1) H H1 [] :- !,  
  ({eta-expand T @one} ==1 T1) H H1.
```

```
progress-eta-link (ho.con _ as T) (ho.lam x\ _ as T1) H H1 [] :- !,  
  ({eta-expand T @one} ==1 T1) H H1.
```

```
progress-eta-link (ho.lam _ as T) T1 H H1 [] :- !,  
  (T ==1 T1) H H1.
```

```
progress-eta-link (ho.uva _ _ as X) T H H1 [] :-
```

```
  contract-rigid [] T T1, !, (X ==1 T1) H H1.
```

```
progress-eta-link (ho.uva Ad _ as T1) T2 H H1 [eval-link-eta T1 T2] :-
```

```
  if (ho.not_occ Ad H T2) true fail.
```

```
type is-in-pf ho.tm -> o.
```

```
is-in-pf (ho.app [ho.uva _ _ | _]) :- !, fail.
```

```
is-in-pf (ho.lam B) :- !, pi x\ is-in-pf (B x).
```

```
is-in-pf (ho.con _) .
```

```
is-in-pf (ho.app L) :- forall1 is-in-pf L.
```

```
is-in-pf N :- name N.
```

```
is-in-pf (ho.uva _ L) :- pattern-fragment L.
```

```
type arity ho.tm -> nat -> o.
```

```
arity (ho.con _) z.
```

```
arity (ho.app L) A :- len L A.
```

```
type occur-check-err ho.tm -> ho.tm -> ho.subst -> o.
```

```
occur-check-err (ho.con _) _ _ :- !.
```

```
occur-check-err (ho.app _) _ _ :- !.
```

```
occur-check-err (ho.lam _) _ _ :- !.
```

```
occur-check-err (ho.uva Ad _) T S :-
```

```
  not (ho.not_occ Ad S T).
```

```
type progress-beta-link-aux ho.tm -> ho.tm ->
```

```
  ho.subst -> ho.subst -> links -> o.
```

```
progress-beta-link-aux T1 T2 S1 S2 [] :- is-in-pf T2, !,
```

```
  (T1 ==1 T2) S1 S2.
```

```
progress-beta-link-aux T1 T2 S S [eval-link-beta T1 T2] :- !.
```

```
type progress-beta-link ho.tm -> ho.tm -> ho.subst ->
```

```
  ho.subst -> links -> o.
```

```
progress-beta-link T (ho.app [ho.uva V Scope | L] as T2) S S2 [eval-link-beta T1 T2] :-
```

```
  arity T Arity, len L ArgsNb, ArgsNb >n Arity, !,
```

```
  minus ArgsNb Arity Diff, mem.new S V1 S1,
```

```
  eta-expand (ho.uva V1 Scope) Diff T1,
```

```
  ((ho.uva V Scope) ==1 T1) S1 S2.
```

```
progress-beta-link (ho.uva _ _ as T) (ho.app [ho.uva Ad1 Scope1 | L] as T2) S1 S2 [eval-link-beta T1 T2] :- !,
```

```
  append Scope1 L1 Scope1L,
```

```
  pattern-fragment-prefix Scope1L Scope2 L2,
```

```
  not (Scope1 = Scope2), !,
```

```
  mem.new S1 Ad2 S2,
```

```
  len Scope1 Scope1Len,
```

```
  len Scope2 Scope2Len,
```

```
  make-eta-link Scope1Len Scope2Len Ad1 Ad2 [] LinkEta S2 S3,
```

```
  if (L2 = []) (NewLinks = LinkEta, T2 = ho.uva Ad2 Scope2)
```

```
  (T2 = ho.app [ho.uva Ad2 Scope2 | L2],
```

```
  NewLinks = [eval-link-beta T T2 | LinkEta]).
```

```
progress-beta-link T1 (ho.app [ho.uva _ _ | _] as T2) _ _ _ :-
```

```
  not (T1 = ho.uva _ _), !, fail.
```

```
progress-beta-link (ho.uva _ _ as T) (ho.app [ho.uva _ _ | _] as T2) S1 S2 [eval-link-beta T1 T2] :-
```

```
  occur-check-err T T2 S1, !, fail.
```

```
progress-beta-link T1 (ho.app [ho.uva _ _ | _] as T2) H H1 [eval-link-beta T1 T2] :-
```

```
progress-beta-link T1 (ho.app [Hd | T1]) S1 S2 B :-
```

```
  ho.beta Hd T1 T3,
```

```
  progress-beta-link-aux T1 T3 S1 S2 B.
```

```
type solve-link-abs link -> links -> ho.subst -> ho.subst -> o.
```

```
solve-link-abs (ho.abs X) R H H1 :-
```

```
  pi x\ ho.copy x x => (pi S\ ho.deref S x x) =>
```

```
    solve-link-abs (X x) (R' x) H H1,
```

```
  close-links R' R.
```

```
solve-link-abs (@eval-link-eta A B) NewLinks S S1 :- !,
```

```
  progress-eta-link A B S S1 NewLinks.
```

```
solve-link-abs (@eval-link-beta A B) NewLinks S S1 :- !,
```

```
  progress-beta-link A B S S1 NewLinks.
```

```
type take-link link -> links -> link -> links -> o.
```

```
take-link A [B|XS] B XS :- link-abs-same-lhs A B, !.
```

```
take-link A [L|XS] B [L|YS] :- take-link A XS B YS.
```

```
type link-abs-same-lhs link -> link -> o.
```

```
link-abs-same-lhs (ho.abs F) B :-
```

```
  pi x\ link-abs-same-lhs (F x) B.
```

```
link-abs-same-lhs A (ho.abs G) :-
```

```
  pi x\ link-abs-same-lhs A (G x).
```

```
link-abs-same-lhs (@eval-link-eta (ho.uva N _) _) (@eval-link-eta (ho.uva N S1) A) :-
```

```
type same-link-eta link -> link -> ho.subst -> ho.subst -> o.
```

```
same-link-eta (ho.abs F) B H H1 :- !, pi x\ same-link-eta (F x) B H H1.
```

```
same-link-eta A (ho.abs G) H H1 :- !, pi x\ same-link-eta A (G x) H H1.
```

```
same-link-eta (@eval-link-eta (ho.uva N S1) A)
```

```
  (@eval-link-eta (ho.uva N S2) B) H H1 :-
```

```
  std.map2 S1 S2 (x\y\r\ r = ho.copy x y) Perm,
```

```
  Perm => ho.copy A A',
```

```
  (A' ==1 B) H H1.
```

```
type progress1-links -> links -> ho.subst -> ho.subst -> o.
```

```
progress1 [] [] X X.
```

```

1973 progress1 [A|L1] [A|L3] S S2 :- take-link A L1 B L2, !,
1974   same-link-eta A B S S1,
1975   progress1 L2 L3 S1 S2.
1976 progress1 [L0|L1] L3 S S2 :- deref-link S L0 L,
1977   solve-link-abs L R S S1, !,
1978   progress1 L1 L2 S1 S2, append R L2 L3.

```

## 18 THE DECOMPIER

```

1982 type abs->lam ho.assignment -> ho.tm -> o.
1983 abs->lam (ho.abs T) (ho.lam R) :- !, pi x\ abs->lam (T x) (R x).
1984 abs->lam (ho.val A) A.

```

```

1986 type commit-links-aux link -> ho.subst -> ho.subst -> o.
1987 commit-links-aux (@val-link-eta T1 T2) H1 H2 :-
1988   ho.deref H1 T1 T1', ho.deref H1 T2 T2',
1989   (T1' ==1 T2') H1 H2.
1990 commit-links-aux (@val-link-beta T1 T2) H1 H2 :-
1991   ho.deref H1 T1 T1', ho.deref H1 T2 T2',
1992   (T1' ==1 T2') H1 H2.
1993 commit-links-aux (ho.abs B) H H1 :-
1994   pi x\ commit-links-aux (B x) H H1.

```

```

1996 type commit-links links -> links -> ho.subst -> ho.subst -> o.
1997 commit-links [] [] H H.
1998 commit-links [Abs | Links] L H H2 :-
1999   commit-links-aux Abs H H1, !, commit-links Links L H1 H2.

```

```

2001 type decomp1-subst map -> map -> ho.subst ->
2002   fo.fsubst -> fo.fsubst -> o.
2003 decomp1-subst _ [A|_] _ _ :- fail.
2004 decomp1-subst _ [] _ F F.
2005 decomp1-subst Map [mapping (fv V0) (hv VM _)|T1] H F F2 :-
2006   mem.set? VM H T, !,
2007   ho.deref-assmt H T TTT,
2008   abs->lam TTT T', tm->fm Map T' T1,
2009   fo.eta-contract T1 T2, mem.assign V0 F T2 F1,
2010   decomp1-subst Map T1 H F1 F2.
2011 decomp1-subst Map [mapping _ (hv VM _)|T1] H F F2 :-
2012   mem.unset? VM H, decomp1-subst Map T1 H F F2.

```

```

2014 type tm->fm map -> ho.tm -> fo.fm -> o.
2015 tm->fm _ (ho.con C) (fo.fcon C).
2016 tm->fm L (ho.lam B1) (fo.flam B2) :-
2017   pi x y\ tm->fm _ x y => tm->fm L (B1 x) (B2 y).
2018 tm->fm L (ho.app L1) T :- map (tm->fm L) L1 [Hd|T1],
2019   fo.mk-app Hd T1 T.
2020 tm->fm L (ho.uva VM TL) T :- mem L (mapping (fv V0) (hv VM _)),
2021   map (tm->fm L) TL T1, fo.mk-app (fo.fuva V0) T1 T.

```

```

2023 type add-new-map-aux ho.subst -> list ho.tm -> map ->
2024   map -> fo.fsubst -> fo.fsubst -> o.
2025 add-new-map-aux _ [] _ S S.
2026 add-new-map-aux H [T|Ts] L L2 S S2 :-
2027   add-new-map H T L L1 S S1,
2028   add-new-map-aux H Ts L1 L2 S1 S2.

```

```

2031 type add-new-map ho.subst -> ho.tm -> map ->
2032   map -> fo.fsubst -> fo.fsubst -> o.
2033 add-new-map _ (ho.uva N _) Map [] F1 F1 :-
2034   mem Map (mapping _ (hv N _)), !.
2035 add-new-map H (ho.uva N L) Map [Map1 | MapL] F1 F3 :-
2036   mem.new F1 M F2,
2037   len L Arity, Map1 = mapping (fv M) (hv N (arity Arity)),
2038   add-new-map H (ho.app L) [Map1 | Map] MapL F2 F3.
2039 add-new-map H (ho.lam B) Map NewMap F1 F2 :-
2040   pi x\ add-new-map H (B x) Map NewMap F1 F2.
2041 add-new-map H (ho.app L) Map NewMap F1 F3 :-
2042   add-new-map-aux H L Map NewMap F1 F3.
2043 add-new-map _ (ho.con _) _ [] F F :- !.
2044 add-new-map _ N _ [] F F :- name N.

```

```

2046 type complete-mapping-under-ass ho.subst -> ho.assignment ->
2047   map -> map -> fo.fsubst -> fo.fsubst -> o.
2048 complete-mapping-under-ass H (ho.val Val) Map1 Map2 F1 F2 :-
2049   add-new-map H Val Map1 Map2 F1 F2.
2050 complete-mapping-under-ass H (ho.abs Abs) Map1 Map2 F1 F2 :-
2051   pi x\ complete-mapping-under-ass H (Abs x) Map1 Map2 F1 F2.
2052
2053 type complete-mapping ho.subst -> ho.subst ->
2054   map -> map -> fo.fsubst -> fo.fsubst -> o.
2055 complete-mapping _ [] L L F F.
2056 complete-mapping H [none | T1] L1 L2 F1 F2 :-
2057   complete-mapping H T1 L1 L2 F1 F2.
2058 complete-mapping H [some T0 | T1] L1 L3 F1 F3 :-
2059   ho.deref-assmt H T0 T,
2060   complete-mapping-under-ass H T L1 L2 F1 F2,
2061   append L1 L2 Lall,
2062   complete-mapping H T1 Lall L3 F2 F3.

```

```

2064 type decompile map -> links -> ho.subst ->
2065   fo.fsubst -> fo.fsubst -> o.
2066 decompile Map1 L H0 F0 F02 :-
2067   commit-links L L1_ H0 H01, !,
2068   complete-mapping H01 H01 Map1 Map2 F0 F01,
2069   decomp1-subst Map2 Map2 H01 F01 F02.

```

## 19 AUXILIARY FUNCTIONS

```

2071 type fold4 (A -> A1 -> B -> B -> C -> C -> o) -> list A ->
2072   list A1 -> B -> B -> C -> C -> o.
2073 fold4 _ [] [] A A B B.
2074 fold4 F [X|XS] [Y|YS] A A1 B B1 :- F X Y A A0 B B0,
2075   fold4 F XS YS A0 A1 B0 B1.
2076
2077 type len list A -> nat -> o.
2078 len [] z.
2079 len [_|L] (s X) :- len L X.

```