

# HO unification from object language to meta language

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## ABSTRACT

Specifying and implementing a logic from scratch requires significant effort. Logical Frameworks and Higher Order Logic Programming Languages provide dedicated, high-level Meta Languages (ML) to facilitate this task in two key ways: 1) variable binding and substitution are simplified when ML binders represent object logic ones; 2) proof construction, and even proof search, is greatly simplified by leveraging the unification procedure provided by the ML. Notable examples of ML are Elf [13], Twelf [14],  $\lambda$ Prolog [10] and Isabelle [20] which have been utilized to implement various formal systems such as First Order Logic [5], Set Theory [12], Higher Order Logic [11], and even the Calculus of Constuctions [4].

The object logic we are interested in is Coq's [18] Dependent Type Theory (DTT), for which we aim to implement a unification procedure  $\approx_o$  using the ML Elpi [3], a dialect of  $\lambda$ Prolog. Elpi's equational theory comprises  $\eta\beta$  equivalence and comes equipped with a higher order unification procedure  $\approx_\lambda$  restricted to the pattern fragment [9]. We want  $\approx_o$  to be as powerful as  $\approx_\lambda$  but on the object logic DTT. Elpi also comes with an encoding for DTT that works well for meta-programming [17, 16, 7, 6]. Unfortunately this encoding, which we refer to as  $\mathcal{F}_o$ , "underuses"  $\approx_\lambda$  by restricting it to first-order unification problems only. To address this issue, we propose a better-behaved encoding,  $\mathcal{H}_o$ , demonstrate how to map unification problems in  $\mathcal{F}_o$  to related problems in  $\mathcal{H}_o$ , and illustrate how to map back the unifiers found by  $\approx_\lambda$ , effectively implementing  $\approx_o$  on top of  $\approx_\lambda$  for the encoding  $\mathcal{F}_o$ .

We apply this technique to the implementation of a type-class [19] solver for Coq [18]. Type-class solvers are proof search procedures based on unification that back-chain designated lemmas, providing essential automation to widely used Coq libraries such as Stdpp/Iris [8] and TLC [1]. These two libraries constitute our test bed.

## KEYWORDS

Logic Programming, Meta-Programming, Higher-Order Unification, Proof Automation

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## 1 INTRODUCTION

Specifying and implementing a logic from scratch requires significant effort. Logical Frameworks and Higher Order Logic Programming Languages provide dedicated, high-level Meta Languages (ML) to facilitate this task in two key ways: 1) variable binding and substitution are simplified when ML binders represent object logic ones; 2) proof construction, and even proof search, is greatly simplified by leveraging the unification procedure provided by the ML. Notable examples of ML are Elf [13], Twelf [14],  $\lambda$ Prolog [10] and Isabelle [20] which have been utilized to implement various formal systems such as First Order Logic [5], Set Theory [12], Higher Order Logic [11], and even the Calculus of Constuctions [4].

The object logic we are interested in is Coq's [18] Dependent Type Theory (DTT), and we want to code a type-class [19] solver for Coq [18] using the Coq-Elpi [17] meta programming framework. Type-class solvers are unification based proof search procedures that combine a set of designated lemmas in order to providing essential automation to widely used Coq libraries.

As the running example we take the Decide type class, from the Stdpp [8] library. The class identifies predicates equipped with a decision procedure. The following three designated lemmas (called Instances in the type-class jargon) state that: 1) the type `fin n`, of natural numbers smaller than `n` is finite; 2) the predicate `nfact n nf`, linking a natural number `n` to its prime factors `nf`, is decidable; 3) the universal closure of a predicate has a decision procedure if the predicate has and if its domain is finite.

```
Instance fin_fin n : Finite (fin n).          (* r1 *)
Instance nfact_dec n nf : Decision (nfact n nf). (* r2 *)
Instance forall_dec A P : Finite A →          (* r3 *)
  ∀x:A, Decision (P x) → Decision (∀x:A, P x).
```

Under this context of instances a type-class solver is able to prove the following statement automatically by back-chaining.

```
Check _ : Decision (forall y: fin 7, nfact y 3). (g)
```

The encoding of DTT provided by Elpi, that we will discuss at length later in section ?? and ??, is an Higher Order Abstract Syntax (HOAS) datatype `tm` featuring (among others) the following constructors:

```
type lam  tm -> (tm -> tm) -> tm.    % lambda abstraction
type app  list tm -> tm.              % n-ary application
type all  tm -> (tm -> tm) -> tm.    % forall quantifier
type con  string -> tm.              % constants
```

Following standard  $\lambda$ Prolog [10] the concrete syntax to abstract, at the meta level, an expression `e` over a variable `x` is `«x\ e»`, and square brackets denote a list of terms separated by comma. As an example we show the encoding of the Coq term `«∀y:t, nfact y 3»`:

```
all (con"t") y\ app[con"nfact", y, con"3"]
```

We now illustrate the encoding of the three instances above as higher-order logic-programming rules: capital letters denote rule parameters; `:-` separates the rule's head from the premises; `pi w\` introduces a fresh nominal constant `w` for the premise `p`.

```
finite (app[con"fin", N]). (r1)
```

```
decision (app [con"nfact", N, NF]). (r2)
```

```
decision (all A x\ app[P, x]) :- finite A, (r3)
  pi w\ decision (app[P, w]).
```

Unfortunately this translation of rule (r3) uses the predicate `P` as a first order term: for the meta language its type is `tm`. If we try to backchain the rule (r3) on the encoding of the goal (g) given below

```
decision (all (app[con"fin", con"7"]) y\
  app[con"nfact", y, con"3"]).
```

we obtain an unsolvable unification problem ( $p$ ): the two lists of terms have different lengths!

```
app[con"nfact", y, con"3"] = app[P, y] (p)
```

In this paper we study a more sophisticated encoding of Coq terms allowing us to rephrase the problematic rule (r3) as follows:

```
decision (all A x\ Pm x) :- decomp Pm P A, finite A, (r3a)
  pi x\ decision (app[P, x]).
```

Since `Pm` is an higher-order unification variable of type `tm`  $\rightarrow$  `tm`, with `x` in its scope, the unification problem ( $p'$ ) admits one solution:

```
app[con"nfact", y, con"3"] = Pm y (p')
Pm = x\ app[con"nfact", x, con"3"] % assignment for Pm
A = app[con"fin", con"7"] % assignment for A
```

After unifying the head of rule (r3a) with the goal, Elpi runs the premise `«decomp Pm A P»` that is in charge of bringing the assignment for `Pm` back to the domain `tm` of Coq terms:

```
P = lam A a\ app[con"nfact", a, con"3"]
```

This simple example is sufficient to show that the encoding we seek is not trivial and does not only concern the head of rules, but the entire sequence of unification problems that constitute the execution of a logic program. In fact the solution for `P` above generates a (Coq)  $\beta$ -redex in the second premise (the predicate under the `pi w\`):

```
decision (app[lam A (a\ app[con"nfact", a, con"3"]), w])
```

In turn this redex prevents the rule (r2) to backchain properly since the following unification problem has no solution:

```
app[lam A (a\ app[con"nfact", a, con"3"]), x] =
app[con"nfact", N, NF]
```

The root cause of the problems we sketched in the running example is that the unification procedure  $\approx_\lambda$  of the meta language is not aware of the equational theory of the object logic, even if both theories include  $\eta\beta$ -conversion and admit most general unifiers for unification problems in the pattern fragment  $\mathcal{L}_\lambda$  [9].

*Contributions.* In this paper we discuss alternative encodings of Coq in Elpi (Section ??), then we identify a minimal language  $\mathcal{F}_0$  in which the problems sketched here can be fully described. We then detail an encoding `comp` from  $\mathcal{F}_0$  to  $\mathcal{H}_0$  (the language of the meta language) and a decoding `decomp` to relate the unifiers bla

bla.. TODO citare Teyjus. The code discussed in the paper can be accessed at the URL: <https://github.com/FissoreD/paper-ho>.

## 2 PROBLEM STATEMENT

The equational theory of Coq's Dependent Type Theory is very rich. In addition to the usual  $\eta\beta$ -equivalence for functions, terms (hence types) are compared up to proposition unfolding and fix-point unrolling. Still, for efficiency and predictability reasons, most form of automatic proof search employ a unification procedure that captures a simpler one, just  $\eta\beta$ , and that solves higher-order problems restricted to the pattern fragment  $\mathcal{L}_\lambda$  [9]. We call this unification procedure  $\approx_o$ .

The equational theory of the meta language Elpi is strikingly similar, since it it comprises  $\eta\beta$  (for the meta language functions), and the unification procedure  $\approx_\lambda$  solves higher-order problems in  $\mathcal{L}_\lambda$ .

In spite of the similarity the link between  $\approx_\lambda$  and  $\approx_o$  is not trivial, since the abstraction and application term constructors the two unification procedures deal with are different. For example

$x \setminus f \ x$	$\approx_\lambda$	$f$
$\text{lam } A \ x \setminus \text{app}[\text{con} "f", x]$	$\approx_o$	$\text{con} "f"$
$\text{lam } A \ x \setminus \text{app}[\text{con} "f", x]$	$\neq_\lambda$	$\text{con} "f"$
$P \ x$	$\approx_\lambda$	$x$
$\text{app}[P, x]$	$\approx_o$	$x$
$\text{app}[P, x]$	$\neq_\lambda$	$x$

One could ignore this similarity, and “just” describe the object language unification procedure in the meta language, that is crafting a unif predicate to be used as follows in rule (r3):

```
decision X :- unif X (all A x\ app[P, x]), finite A,
  pi x\ decision (app[P, x]).
```

This choice would underuse the logic programming engine provided by the metalanguage since by removing any datum from the head of rules indexing degenerates. Moreover the unification procedure built in the meta language is likely to be faster than one implemented in it, especially if the meta language is interpreted as Elpi is.

To state precisely the problem we solve we need a  $\mathcal{F}_0$  representation of DTT terms and a  $\mathcal{H}_0$  one. We call  $=_o$  the equality over ground terms in  $\mathcal{F}_0$ ,  $=_\lambda$  the equality over ground terms in  $\mathcal{H}_0$ ,  $\approx_o$  the unification procedure we want to implement and  $\approx_\lambda$  the one provided by the meta language. TODO extend  $=_o$  and  $=_\lambda$  with reflexivity on uvars.

We write  $t_1 \approx_\lambda t_2 \mapsto \sigma$  when  $t_1$  and  $t_2$  unify with substitution  $\sigma$ ; we write  $\sigma t$  for the application of the substitution to  $t$ , and  $\sigma X = \{\sigma t \mid t \in X\}$  when  $X$  is a set; we write  $\sigma \subseteq \sigma'$  when  $\sigma$  is more general than  $\sigma'$ . We assume that the unification of our meta language is correct:

$$t_i \in \mathcal{L}_\lambda \Rightarrow t_1 \approx_\lambda t_2 \mapsto \rho \Rightarrow \rho t_1 =_\lambda \rho t_2 \quad (1)$$

$$t_i \in \mathcal{L}_\lambda \Rightarrow \rho t_1 =_\lambda \rho t_2 \Rightarrow \exists \rho', t_1 \approx_\lambda t_2 \mapsto \rho' \wedge \rho' \subseteq \rho \quad (2)$$

We illustrate a compilation  $\langle s \rangle \mapsto (t, m, l)$  that maps a term  $s$  in  $\mathcal{F}_0$  to a term  $t$  in  $\mathcal{H}_0$ , a variable mapping  $m$  and list of links  $l$ . The variable map connects unification variables in  $\mathcal{H}_0$  with variables in  $\mathcal{F}_0$  and is used to “decompile” the assignment,  $\langle \sigma, m, l \rangle^{-1} \mapsto \rho$ . Links represent problematic sub-terms which are linked to the

unification variable that stands in their place in the compiled term. These links are checked for or progress XXX improve....

We represent a logic program *run* in  $\mathcal{F}_0$  as a list *steps*  $p$  of length  $N$ . Each step is a unification problem between terms  $\mathbb{S}_{p_l}$  and  $\mathbb{S}_{p_r}$  taken from the set of all terms  $\mathbb{S}$ . The composition of these steps starting from the empty substitution  $\rho_0$  produces the final substitution  $\rho_N$ .<sup>1</sup> The initial here  $\rho_0$  is the empty substitution

$$\begin{aligned} \text{fstep}(\mathbb{S}, p, \rho) &\mapsto \rho'' \stackrel{\text{def}}{=} \rho \mathbb{S}_{p_l} \approx_o \rho \mathbb{S}_{p_r} \mapsto \rho' \wedge \rho'' = \rho \cup \rho' \\ \text{frun}(\mathbb{S}, N) &\mapsto \rho_N \stackrel{\text{def}}{=} \bigwedge_{p=1}^N \text{fstep}(\mathbb{S}, p, \rho_{p-1}) \mapsto \rho_p \end{aligned}$$

We simulate each run in  $\mathcal{F}_0$  with a run in  $\mathcal{H}_0$  as follows. Note that  $\sigma_0$  is the empty substitution.

$$\begin{aligned} \text{hstep}(\mathbb{T}, p, \sigma, \mathbb{L}) &\mapsto (\sigma'', \mathbb{L}') \stackrel{\text{def}}{=} \\ &\sigma \mathbb{T}_{p_l} \approx_\lambda \sigma \mathbb{T}_{p_r} \mapsto \sigma' \wedge \text{progress}(\mathbb{L}, \sigma \cup \sigma') \mapsto (\mathbb{L}', \sigma'') \\ \text{hrun}(\mathbb{S}, N) &\mapsto \rho_N \stackrel{\text{def}}{=} \\ &\mathbb{T} \times \mathbb{M} \times \mathbb{L}_0 = \{(t_j, m_j, l_j) \mid s_j \in \mathbb{S}, \langle s_j \rangle \mapsto (t_j, m_j, l_j)\} \\ &\bigwedge_{p=1}^N \text{hstep}(\mathbb{T}, p, \sigma_{p-1}, \mathbb{L}_{p-1}) \mapsto (\sigma_p, \mathbb{L}_p) \\ &\langle \sigma_N, \mathbb{M}, \mathbb{L}_N \rangle^{-1} \mapsto \rho_N \end{aligned}$$

Here *hstep* is made of two sub-steps: a call to  $\approx_\lambda$  (on the compiled terms) and a call to *progress* on the set of links. We claim the following:

PROPOSITION 2.1 (SIMULATION).  $\forall \mathbb{S}, \forall N$ ,

$$\text{frun}(\mathbb{S}, N) \mapsto \rho_N \Leftrightarrow \text{hrun}(\mathbb{S}, N) \mapsto \rho_N$$

That is, the two executions give the same result. Moreover:

PROPOSITION 2.2 (SIMULATION FIDELITY). *In the context of* *hrun*, *if*  $\mathbb{T} \subseteq \mathcal{L}_\lambda$  *we have that*  $\forall p \in 1 \dots N$ ,

$$\text{fstep}(\mathbb{S}, p, \rho_{p-1}) \mapsto \rho_p \Leftrightarrow \text{hstep}(\mathbb{T}, p, \sigma_{p-1}, \mathbb{L}) \mapsto (\sigma_p, \_)$$

In particular this property guarantees that a *failure* in the  $\mathcal{F}_0$  run is matched by a failure in  $\mathcal{H}_0$  at the same step. We consider this property very important from a practical point of view since it guarantees that the execution traces are strongly related and in turn this enables a user to debug a logic program in  $\mathcal{F}_0$  by looking at its execution trace in  $\mathcal{H}_0$ .

XXX permuting *hrun* does not change the final result if check does not fail eagerly

XXX if we want to apply heuristics, we can apply them in *decomp* to avoid committing to a non MGU too early

We can define  $s_1 \approx_o s_2$  by specializing the code of *hrun* to  $\mathbb{S} = \{s_1, s_2\}$  as follows:

$$\begin{aligned} s_1 \approx_o s_2 &\mapsto \rho \stackrel{\text{def}}{=} \\ &\langle s_1 \rangle \mapsto (t_1, m_1, l_1) \wedge \langle s_2 \rangle \mapsto (t_2, m_2, l_2) \\ &t_1 \approx_\lambda t_2 \mapsto \sigma' \wedge \text{progress}(\{l_1, l_2\}, \sigma') \mapsto (L, \sigma'') \wedge \\ &\langle \sigma'', \{m_1, m_2\}, L \rangle^{-1} \mapsto \rho \end{aligned}$$

<sup>1</sup>If the same rule is used multiple time in a run we just consider as many copies as needed of the terms composing the rules, with fresh unification variables each time

PROPOSITION 2.3 (PROPERTIES OF  $\approx_o$ ).

$$s_i \in \mathcal{L}_\lambda \Rightarrow s_1 \approx_o s_2 \mapsto \rho \Rightarrow \rho s_1 =_o \rho s_2 \text{ (correct)} \quad (3)$$

$$s_i \in \mathcal{L}_\lambda \Rightarrow \rho s_1 =_o \rho s_2 \Rightarrow \exists \rho', s_1 \approx_o s_2 \mapsto \rho' \wedge \rho' \subseteq \rho \text{ (complete)} \quad (4)$$

$$\rho s_1 =_o \rho s_2 \Rightarrow \rho' \subseteq \rho \Rightarrow \rho' s_i \in \mathcal{L}_\lambda \Rightarrow \rho' s_1 \approx_o \rho' s_2 \quad (5)$$

Properties (*correct*) and (*complete*) state, respectively, that in  $\mathcal{L}_\lambda$  the implementation of  $\approx_o$  is correct, complete and returns the most general unifier.

Property 2.1 states that  $\approx_o$ , hence our compilation scheme, is resilient to unification problems outside  $\mathcal{L}_\lambda$  solved by a third party. We believe this property is of practical interest since we want the user to be able to add heuristics via hand written rules to the ones obtained by our compilation scheme. A Typical example is the following problem (*q*) that is outside  $\mathcal{L}_\lambda$ :

$$\text{app } [F, \text{con} "a"] = \text{app} [\text{con} "f", \text{con} "a", \text{con} "a"] \quad (q)$$

$$F = \text{lam } x \backslash \text{app} [\text{con} "f", x, x] \quad (h)$$

Instead of rejecting it our scheme accepts it and guarantees that if (*h*) is given (after the compilation part of the scheme, as a run time hint) then ...

## 2.1 The intuition in a nutshell

A term  $s$  is compiled in a term  $t$  where every “problematic” sub term  $p$  is replaced by a fresh unification variable  $h$  and an accessory link that represent a suspended unification problem  $h \approx_\lambda p$ . As a result  $\approx_\lambda$  is “well behaved” on  $t$ , that is it does not contradict  $=_o$  as it would otherwise do on “problematic” terms. We now define “problematic” and “well behaved” more formally.

Definition 2.4 ( $\diamond \eta$ ).  $\diamond \eta = \{t \mid \exists \rho, \rho t \text{ is an eta expansion}\}$

An example of term  $t$  in  $\diamond \eta$  is  $\lambda x. \lambda y. F y x$  since the substitution  $\rho = \{F \mapsto \lambda a. \lambda b. f b a\}$  makes  $\rho t = \lambda x. \lambda y. f x y$  that is the eta long form of  $f$ . This term is problematic since its rigid part, the  $\lambda$ -abstractions, cannot justify a unification failure against, say, a constant.

Definition 2.5 ( $\overline{\mathcal{L}_\lambda}$ ).  $\overline{\mathcal{L}_\lambda} = \{X t_1 \dots t_n \mid X t_1 \dots t_n \notin \mathcal{L}_\lambda\}$ .

An example of  $t$  in  $\overline{\mathcal{L}_\lambda}$  is  $F a$  for a constant  $a$ . Note however that an oracle could provide an assignment  $\rho = \{F \mapsto \lambda x. x\}$  that makes the resulting term fall back in  $\mathcal{L}_\lambda$ .

Definition 2.6 (Subterms  $\mathcal{P}(t)$ ). The set of sub terms of  $t$  is the largest set

*subterm*  $t$  that can be obtained by the following rules.

$$\begin{aligned} t &\in \mathcal{P}(t) \\ t = f t_1 \dots t_n &\Rightarrow \mathcal{P}(t_i) \subseteq \mathcal{P}(t) \wedge f \in \mathcal{P}(t) \\ t = \lambda x. t' &\Rightarrow \mathcal{P}(t') \subseteq \mathcal{P}(t) \end{aligned}$$

We write  $\mathcal{P}(X) = \bigcup_{t \in X} \mathcal{P}(t)$  when  $X$  is a set of terms.

Definition 2.7 (Well behaved set). Given a set of terms  $X \subseteq \mathcal{H}_0$ ,

$$\mathcal{W}(X) \Leftrightarrow \forall t \in \mathcal{P}(X), t \notin (\overline{\mathcal{L}_\lambda} \cup \diamond \eta)$$

PROPOSITION 2.8 ( $\mathcal{W}$ -PRESERVATION).  $\forall \mathbb{T}, \forall \mathbb{L}, \forall p, \forall \sigma, \forall \sigma'$

$$\mathcal{W}(\sigma \mathbb{T}) \wedge \sigma \mathbb{T}_{p_l} \approx_\lambda \sigma \mathbb{T}_{p_r} \mapsto \sigma' \Rightarrow \mathcal{W}(\sigma' \mathbb{T})$$

$$\mathcal{W}(\sigma \mathbb{T}) \wedge \text{progress}(\mathbb{L}, \sigma) \mapsto (\_, \sigma') \Rightarrow \mathcal{W}(\sigma' \mathbb{T})$$

A less formal way to state 2.8 is that hstep and progress never “commit” an unneeded  $\lambda$ -abstraction in  $\sigma$  (a  $\lambda$  that could be erased by an  $\eta$ -contraction), nor put in  $\sigma$  a flexible application outside  $\mathcal{L}_\lambda$  (an application node that could be erased by a  $\beta$ -reduction).

Note that proposition 2.8 does not hold for  $\approx_o$  as a whole since decompilation can introduce (actually restore) terms in  $\diamond\eta$  or  $\overline{\mathcal{L}_\lambda}$  that were move out of the way (put in  $\mathbb{L}$ ) during compilation.

### 3 ALTERNATIVE ENCODINGS AND RELATED WORK

Paper [2] introduces semi-shallow.

Our encoding of DTT may look “semi shallow” since we use the meta-language lambda abstraction but not its application (for the terms of type  $\text{tm}$ ). A fully shallow encoding unfortunately does not fit our use case, although it would make the running example work:

```
finite (fin N).
decision (nfact N NF).
decision (all A x\ P x) :- finite A, pi x\ decision (P x).
```

There are two reasons for dismissing this encoding. The first one is that in DTT it is not always possible to adopt it since the type system of the meta language is too weak to accommodate terms with a variable arity, like the following example:

```
Fixpoint arr T n := if n is S m then T -> arr T m else T.
Definition sum n : arr nat n := ...
Check sum 2 7 8 : nat.
Check sum 3 7 8 9 : nat.
```

The second reason is the encoding for Coq is used for meta programming the system, hence it must accommodate the manipulation of terms that are now know in advance (not even defined in Coq) without using introspection primitives such as Prologs’s functor and arg.

In the literature we could find a few related encoding of DTT. TODO In [4] is related and make the discrepancy between the types of ML and DTT visible. In this case one needs 4 application nodes. Moreover the objective is an encoding of terms, proofs, not proof search. Also note the conv predicate, akin to the unif we rule out.

TODO This other paper [15] should also be cited.

None of the encodings above provide a solution to our problem.

### 4 PRELIMINARIES: $\mathcal{F}_o$ AND $\mathcal{H}_o$

In order to reason about unification we provide a description of the  $\mathcal{F}_o$  and  $\mathcal{H}_o$  languages where unification variables are first class terms, i.e. they have a concrete syntax. We keep these languages minimal, for example, we omit the all quantifier of DTT we used in the example in Section 1 together with the type notation of terms carried by the lam constructor.

```
kind fm type.          kind tm type.
type fapp list fm -> fm. type app list tm -> tm.
type flam (fm -> fm) -> fm. type lam (tm -> tm) -> tm.
type fcon string -> fm.   type con string -> tm.
type fuva addr -> fm.     type uva addr -> list tm -> tm.
```

Figure 1: The  $\mathcal{F}_o$  and  $\mathcal{H}_o$  languages

Unification variables (fuva term constructor) in  $\mathcal{F}_o$  have no explicit scope: the arguments of an higher order variable are given via

the fapp constructor. For example the term  $P \ x$  is represented as  $\text{fapp}[fuva \ N, \ x]$ , where  $N$  is a memory address and  $x$  is a bound variable.

In  $\mathcal{H}_o$  the representation of  $P \ x$  is instead  $\text{uva} \ N \ [x]$ , since unification variables come equipped with an explicit scope. We say that the unification variable occurrence  $\text{uva} \ N \ L$  is in  $\mathcal{L}_\lambda$  if and only if  $L$  is made of distinct names. The predicate to test this condition is called pattern-fragment:

```
type pattern-fragment list A -> o.
```

Natural numbers represent the memory addresses that identify unification variables in both languages. The memory and its associated operations are described below:

```
kind addr type.
type addr nat -> addr.
typeabbrev (mem A) (list (option A)).

type set? addr -> mem A -> A -> o.
type unset? addr -> mem A -> o.
type assign addr -> mem A -> A -> mem A -> o.
type new mem A -> addr -> mem A -> o.
```

If a memory cell is none, then the corresponding unification variable is not set. assign sets an unset cell to the given value, while new finds the first unused address and sets it to none.

Since in  $\mathcal{H}_o$  unification variables have a scope, their solution needs to be abstracted over it to enable the instantiation of a single solution to different scopes. This is obtained via the inctx container, and in particular via its abs binding constructor. On the contrary a solution to a  $\mathcal{F}_o$  variable is a plain term.

```
typeabbrev fsubst (mem fm).

kind inctx type -> type.
type abs (tm -> inctx A) -> inctx A.
type val A -> inctx A.
typeabbrev assignment (inctx tm).
typeabbrev subst (mem assignment).
```

We call fsubst the memory of  $\mathcal{F}_o$ , while we call subst the one of  $\mathcal{H}_o$ . Both have the invariant that they are not cyclic, TODO: explain.

```
kind arity type.
type arity nat -> arity.

kind fvariable type.
type fv addr -> fvariable.

kind hvariable type.
type hv addr -> arity -> hvariable.

kind mapping type.
type mapping fvariable -> hvariable -> mapping.
typeabbrev mmap (list mapping).
```

INVARIANT 1 (UNIFICATION VARIABLE ARITY). *Each variable  $A$  in  $\mathcal{H}_o$  has a (unique) arity  $N$  and each occurrence  $(\text{uva} \ A \ L)$  is such that  $(\text{len } L \ N)$  holds*

The compiler establishes a mapping between variables of the two languages. In order to preserve invariant 1 we store the arity of



each hvariable in the mapping and we reuse an existing mapping only if the arity matches.

TODO: add ref to section 7

```

type m-alloc fvariable -> hvariable -> mmap -> mmap ->
  subst -> subst -> o. (malloc)
m-alloc Fv Hv M M S S :- mem M (mapping Fv Hv), !.
m-alloc Fv Hv M [mapping Fv Hv|M] S S1 :- Hv = hv N _,
  alloc S N S1.

```

When a single fvariable occurs multiple times with different numbers of arguments the compiler generates multiple mappings for it, on a first approximation, and then makes the mapping bijective by introducing link- $\eta$ ; this detail is discussed in section 6.

As we mentioned in section 2.1 the compiler replaces terms in  $\diamond\eta$  and  $\overline{\mathcal{L}}_\lambda$  with fresh variables linked to the problematic terms. Each class of problematic terms has a dedicated link.

```

kind baselink type.
type link-eta tm -> tm -> baselink.
type link-beta tm -> tm -> baselink.
typeabbrev link (inctx baselink).
typeabbrev links (list link).

```

The right hand side of a link, the problematic term, can occur under binders. To accommodate this situation the compiler wraps baselink using the inctx container (see,  $\cdot \vdash \cdot$ ).

INVARIANT 2 (LINK LEFT HAND SIDE). *The left hand side of a suspended link is a variable.*

New links are suspended by construction. If the left hand side variable is assigned during a step, then the link is considered for progress and possibly eliminated. This is discussed in section 6 and ??.

## 4.1 Notational conventions

When we write  $\mathcal{H}_o$  terms outside code blocks we follow the usual  $\lambda$ -calculus notation, reserving  $f, g, a, b$  for constants,  $x, y, z$  for bound variables and  $X, Y, Z, F, G, H$  for unification variables. However we need to distinguish between the “application” of a unification variable to its scope and the application of a term to a list of arguments. We write the scope of unification variables in subscript while we use juxtaposition for regular application. Here a few examples:

```

f a      app[con "f", con "a"]
 $\lambda x. \lambda y. F_{xy}$   lam x\ lam y\ uva F [x, y]
 $\lambda x. F_x a$      lam x\ app[uva F [x], con "a"]
 $\lambda x. F_x x$      lam x\ app[uva F [x], x]

```

When variables  $x$  and  $y$  can occur in term  $t$  we shall write  $t_{xy}$  to stress this fact.

We write  $\sigma = \{ A_{xy} \mapsto y \}$  for the assignment  $\text{abs } x \backslash \text{abs } y \backslash y$  and  $\sigma = \{ A \mapsto \lambda x. \lambda y. y \}$  for  $\text{lam } x \backslash \text{lam } y \backslash y$ .

When detailing examples we write links as equations between two terms under a context. The equality sign is subscripted with kind of baselink. For example  $x \vdash A_x =_\beta F_x a$  corresponds to:

```
abs x\ val (link-beta (uva A [x]) (app[uva F [x], con "a"]))
```

When it is clear from the context we shall use the same syntax for  $\mathcal{F}_o$  terms (although we never subscript unification variables).

## 4.2 Equational theory and Unification

In order to express properties ?? we need to equip  $\mathcal{F}_o$  and  $\mathcal{H}_o$  with term equality, substitution application and unification.

*Term equality:*  $=_o$  vs.  $=_\lambda$ . We extend the equational theory over ground terms to the full languages by adding the reflexivity of unification variables (a variable is equal to itself).

The first four rules are common to both equalities and just define the usual congruence over terms, and since we use an HOAS encoding they also capture  $\alpha$ -equivalence. In addition to that  $=_o$  has rules for  $\eta$  and  $\beta$ -equivalence.

```

type (=o) fm -> fm -> o. (=o)
fcon X =o fcon X.
fapp A =o fapp B :- forall2 (=o) A B.
flam F =o flam G :- pi x\ x =o x => F x =o G x.
fuva N =o fuva N.
flam F =o T :- (eta)
  pi x\ beta T [x] (T' x), x =o x => F x =o T' x.
T =o flam F :- (eta_r)
  pi x\ beta T [x] (T' x), x =o x => T' x =o F x.
fapp [flam X | L] =o T :- beta (flam X) L R, R =o T. (beta_l)
T =o fapp [flam X | L] :- beta (flam X) L R, T =o R. (beta_r)

type (=lambda) tm -> tm -> o.
con C =lambda fcon C.
app A =lambda fapp B :- forall2 (=lambda) A B.
lam F =lambda flam G :- pi x\ x =lambda x => F x =lambda G x.
uva N A =lambda fuva N B :- forall2 (=lambda) A B.

```

The main point in showing these equality tests is to remark how weaker  $=_\lambda$  is, and to identify the four rules that need special treatment in the implementation of  $=_o$ .

For reference,  $(\text{beta } T \ A \ R)$  reduces away  $\text{lam}$  nodes in head position in  $T$  whenever the list  $A$  provides a corresponding argument.

```

type beta fm -> list fm -> fm -> o.
beta A [] A.
beta (flam Bo) [H | L] R :- napp (Bo H) F, beta F L R.
beta (fapp A) L (fapp X) :- append A L X.
beta (fuva N) L (fapp [fuva N | L]).
beta (fcon H) L (fapp [fcon H | L]).
beta N L (fapp [N | L]) :- name N.

type napp fm -> fm -> o.
napp (fcon C) (fcon C).
napp (fuva A) (fuva A).
napp (flam F) (flam G) :- pi x\ napp (F x) (G x).
napp (fapp [fapp L1 | L2]) T :- !,
  append L1 L2 L3, napp (fapp L3) T.
napp (fapp L) (fapp L1) :- map napp L L1.
napp N N :- name N.

```

The name predicate holds only on nominal constants (i.e. bound variables).<sup>2</sup> The choice of using n-ary application, rather than binary, is to make it easy to access the application’s head. The price we pay is that substituting an application in the head of an application should be amended by “flattening” fapp nodes, that is the job of

<sup>2</sup>Elpi provides it as a builtin, but one could implement it by systematically loading the hypothetical rule  $\text{name } x$  every time a nominal constant is postulated via  $\text{pi } x \backslash$

napp.<sup>3</sup> Finally note that the cut operator is inessential, it could be removed at the cost of a verbose test on the head of  $L$  in the second rule about fapp:  $L$ 's head can be fcon, flam or a name.

*Substitution application:  $\rho s$  and  $\sigma t$ .* Applying the substitution corresponds to dereferencing a term with respect to the memory. To ease the comparison we split  $\mathcal{F}_0$  dereferencing into a fder step and a napp one. The former step replaces references to memory cells that are set with their values, and has a corresponding operation in  $\mathcal{H}_0$ , namely deref. On the contrary napp has no corresponding operation in  $\mathcal{H}_0$ . The reasons for this asymmetry is that an fapp node with a flexible head is always mapped to a uva (as per section 5 and section 8), preventing nested applications to materialize.

```

type fder fsubst -> fm -> fm -> o.
fder _ (fcon C) (fcon C).
fder S (fapp A) (fapp B) :- map (fder S) A B.
fder S (flam F) (flam G) :-
  pi x\ fder S x x => fder S (F x) (G x).
fder S (fuva N) R :- set? N S T, fder S T R.
fder S (fuva N) (fuva N) :- unset? N S.

type fderef fsubst -> fm -> fm -> o.           (ρs)
fderef S T T2 :- fder S T T1, napp T1 T2.

```

Applying the substitution in  $\mathcal{H}_0$  is very similar, with the caveat that assignments have to be moved to the current scope, i.e. renaming the abs-bound variables with the names in the scope of the unification variable occurrence.

```

type deref subst -> tm -> tm -> o.           (σt)
deref _ (con C) (con C).
deref S (app A) (app B) :- map (deref S) A B.
deref S (lam F) (lam G) :-
  pi x\ deref S x x => deref S (F x) (G x).
deref S (uva N L) R :- set? N S A,
  move A L T, deref S T R.
deref S (uva N A) (uva N B) :- unset? N S,
  map (deref S) A B.

```

Note that move strongly relies on invariant 1: the length of the arguments of all occurrences of a unification variable and the number of abstractions in its assignment have to match. In turn this grants that move never fails.

```

type move assignment -> list tm -> tm -> o.
move (abs Bo) [H|L] R :- move (Bo H) L R.
move (val A) [] A.

```

*Term unification:  $\approx_o$  vs.  $\approx_\lambda$ .* In this paper we assume to have an implementation of  $\approx_\lambda$  that satisfies properties 1 and 2. Although we provide an implementation in the appendix (that we used for testing purposes) we only describe its signature here. Elpi is expected to provide this brick, as well as any other implementation of  $\lambda$ Prolog.

```

type (≈λ) tm -> tm -> subst -> subst -> o.

```

The only detail worth discussing is the fact that the procedure updates a substitution, rather than just crafting one as presented

<sup>3</sup>Note that napp is an artefact of formalization of  $\mathcal{F}_0$  we do in this presentation and, as we explain later, no equivalent of napp is needed in  $\mathcal{H}_0$ .

in section 2. The reason is that the algorithm folds over a term, updating a substitution while it traverses it.

## 5 BASIC SIMULATION OF $\mathcal{F}_0$ IN $\mathcal{H}_0$

In this section we describe a basic compilation scheme that we refine later, in the following sections. This scheme is sufficient to implement an  $\approx_o$  that respects  $\beta$ -conversion for terms in  $\mathcal{L}_\lambda$ . The extension to  $\eta\beta$ -conversion is described in Section 6 and the support for terms outside  $\mathcal{L}_\lambda$  in Section 8.

### 5.1 Compilation

The main task of the compiler is to recognize  $\mathcal{F}_0$  variables standing for functions and map them to higher order variables in  $\mathcal{H}_0$ . In order to bring back the substitution from  $\mathcal{H}_0$  to  $\mathcal{F}_0$  the compiler builds a “memory map” connecting the the kind of variables using routine (*malloc*).

The signature of the comp predicate below allows for the generation of links (suspended unification problems) that play no role in this section but play a major role in section 6 and section 8. With respect to section 2 the signature also allows for updates to the substitution. The code below uses that possibility in order to allocate space for the variables, i.e. sets their memory address to none (a details not worth mentioning in the previous discussion).

```

type comp fm -> tm -> mmap -> mmap -> links -> links ->
  subst -> subst -> o.
comp (fcon C) (con C) M M L L S S.
comp (flam F) (lam F1) M1 M2 L1 L2 S1 S2 :-           (cλ)
  comp-lam F F1 M1 M2 L1 L2 S1 S2.
comp (fuva A) (uva B []) M1 M2 L L S1 S2 :-
  m-alloc (fv A) (hv B (arity z)) M1 M2 S1 S2.
comp (fapp [fuva A|Ag]) (uva B Ag1) M1 M2 L L S1 S2 :-
  pattern-fragment Ag, !,
  fold6 comp Ag Ag1 M1 M1 L L S1 S1,
  len Ag Arity,
  m-alloc (fv A) (hv B (arity Arity)) M1 M2 S1 S2.
comp (fapp A) (app A1) M1 M2 L1 L2 S1 S2 :-
  fold6 comp A A1 M1 M2 L1 L2 S1 S2.

```

This preliminary version of comp recognizes  $\mathcal{F}_0$  variables applied to a (possibly empty) duplicate free list of names. Note that compiling Ag cannot create new mappings nor links, since Ag is made of bound variables and the hypothetical rule loaded by comp-lam (see below) grants this property.

```

type comp-lam (fm -> fm) -> (tm -> tm) ->
  mmap -> mmap -> links -> links -> subst -> subst -> o.
comp-lam F G M1 M2 L1 L3 S1 S2 :-
  pi x y\ (pi M L S\ comp x y M M L L S S) =>
  comp (F x) (G y) M1 M2 L1 (L2 y) S1 S2,
  close-links L2 L3.

```

In the code above the syntax  $\pi x y\ .$  is syntactic sugar for iterated  $\pi$  abstraction, as in  $\pi x\ \pi y\ .$

The auxiliary function close-links tests if the bound variable  $v$  really occurs in the link. If it is the case the link is wrapped into an additional abs node binding  $v$ . In this way links generated deep inside the compiled terms can be moved outside their original context of binders.

```

697 type close-links (tm -> links) -> links -> o.
698 close-links (v\X |L v]) [X|R] :- !, close-links L R.
699 close-links (v\X v|L v]) [abs X|R] :- close-links L R.
700 close-links (_\[]) [].

```

Note that we could remove the first rule, whose solve purpose is to make links more readable by pruning unused context entries.

## 5.2 Execution

A step in  $\mathcal{H}_o$  consists in unifying two terms and reconsidering all links for progress. If any of the two tasks fail we say that the entire step fails, and it is at this granularity that we can relate steps in the two languages.

```

710 type hstep tm -> tm -> links -> links -> subst -> subst -> o.
711 hstep T1 T2 L1 L2 S1 S3 :-
712   (T1 ≈λ T2) S1 S2,
713   progress L1 L2 S2 S3.

```

Note that the infix notation  $((A \approx_\lambda B) C D)$  is syntactic sugar for  $((\approx_\lambda) A B C D)$ .

Reconsidering links is a fixpoint, since the progress of a link can update the substitution and in turn enable another link to progress.

```

719 type progress links -> links -> subst -> subst -> o.
720 progress L L2 S1 S3 :-
721   progress1 L L1 S1 S2,
722   occur-check-links L1,
723   if (L = L1, S1 = S2)
724     (L2 = L1, S3 = S1)
725     (progress L1 L2 S2 S3).

```

In the base compilation scheme `progress1` is the identity on both the links and the substitution, so the fixpoint trivially terminates. Sections 6 and 8 add rules to `progress1` and justify why the don't hinder termination. For brevity we omit the code that applies the substitution  $S1$  to all terms in  $\mathbb{L}$ .

Since compilation moves problematic terms out of the sigh of  $\approx_\lambda$ , that procedure can only perform a partial occur check. For example the unification problem  $X \approx_\lambda f Y$  cannot generate a cyclic substitution alone, but should be disallowed if a  $\mathbb{L}$  contains a link like  $\vdash Y \approx_\eta \lambda z.X_z$ : We don't know yet if  $Y$  will feature a lambda in head position, but we surely know it contains  $X$ , hence  $f Y$  and that fails the occur check. The procedure `occur-check-links` is in charge of ensuring that each link does not represent a (suspended) unification problem doomed to fail because of occur check. This check is needed in order to guarantee proposition 2.2 (simulation fidelity).

## 5.3 Substitution decompilation

Decompiling the substitution requires to first force the progress of links and then allocating new unassigned variables in the substitution for  $\mathcal{F}_o$  and finally decompiling all assignments. Note that invariant 2 and the occur check allows us to update the subst.

```

750 type decompile mmap -> links -> subst ->
751   fsubst -> fsubst -> o.
752 decompile M1 L S F1 F3 :-
753   commit-links L S S1,

```

```

755 complete-mapping S1 S1 M1 M2 F1 F2,
756 decomp M2 M2 S1 F2 F3.

```

TODO: What is commit-links and complete-mapping?, maybe complete-mapping can be hidden in the code rendering? Decompiling an assignment requires to turn abstractions into lambdas. For aesthetic purposes we also eta-contract the result (not needed since  $\mathcal{F}_o$  equality can do that)

```

762 type decomp mmap -> mmap -> subst -> fsubst -> fsubst -> o.
763 decomp _ [] _ F F.
764 decomp M [mapping (fv V) (hv H _)]MS] S F1 F3 :- set? H S A,
765   deref-assmt S A A1,
766   abs->lam A1 T, decomp M T T1,
767   eta-contract T1 T2,
768   assign V F1 T2 F2,
769   decomp M MS S F2 F3.
770 decomp M [mapping _ (hv H _)]MS] S F1 F2 :- unset? H S,
771   decomp M MS S F1 F2.
772

```

Finally decompiling a term is trivial, now that we have an extended mapping containing all unassigned variables  $\approx_\lambda$  may have introduced.

```

776 type decomp mmap -> tm -> fm -> o.
777 decomp _ (con C) (fcon C).
778 decomp M (app A) (fapp B) :- map (decomp M) A B.
779 decomp M (lam F) (flam G) :-
780   pi x y\ (pi M\ decomp M x y) => decomp M (F x) (G y).
781 decomp M (uva Hv Ag) R :-
782   mem M (mapping (fv Fv) (hv Hv _)),
783   map (decomp M) Ag Bg,
784   beta (fuva Fv) Bg R.

```

Note that we use beta to build fapp nodes when needed (if Ag is empty no fapp node should appear).

INVARIANT 3. *TODO: dire che il mapping è bijective*

## 5.4 Definition of $\approx_o$ and its properties

```

791 type (≈o) fm -> fm -> fsubst -> o.
792 (A ≈o B) F :-
793   comp A A' [] M1 [] [] S1,
794   comp B B' M1 M2 [] [] S1 S2,
795   hstep A' B' [] [] S2 S3,
796   decomp M2 M2 S3 [] F.

```

The code given so far applies to terms in  $\beta\eta$ -normal form where unification variables in  $\mathcal{F}_o$  can occur non linearly but always with the same number of arguments, and where their arguments are distinct names (as per  $\mathcal{L}_\lambda$ ).

LEMMA 5.1 (COMPILATION ROUND TRIP). *If  $\text{comp } S \ T \ [] \ M \ [] \ _ \ [] \ _$  then  $\text{decomp } M \ T \ S$*

PROOF SKETCH. trivial, since the terms are beta normal beta just builds an app.  $\square$

LEMMA 5.2. *Properties (correct) and (complete) hold for the implementation of  $\approx_o$  above*

PROOF SKETCH. In this setting  $\approx_\lambda$  is as strong as  $\approx_o$  on ground terms. What we have to show is that whenever two different  $\mathcal{F}_o$  terms

can be made equal by a substitution  $\rho$  (plus the  $\beta_l$  and  $\beta_r$  if needed) we can find this  $\rho$  by finding a  $\sigma$  via  $\approx_\lambda$  on the corresponding  $\mathcal{H}_0$  terms and by decompiling it. If we look at the  $\mathcal{F}_0$  terms, there are two interesting cases:

- $\text{fuva } X \approx_o s$ . In this case after comp we have  $Y \approx_\lambda t$  that succeeds with  $\sigma = \{Y \mapsto t\}$  and  $\sigma$  is decompiled to  $\rho = \{Y \mapsto s\}$ .
- $\text{fapp}[\text{fuva } X][L] \approx_o s$ . In this case we have  $Y_{\vec{x}} \approx_\lambda t$  that succeeds with  $\sigma = \{\vec{y} \mapsto Y \mapsto t[\vec{x}/\vec{y}]\}$  that in turn is decompiled to  $\rho = \{Y \mapsto \lambda \vec{y}.s[\vec{x}/\vec{y}]\}$ . Thanks to  $\beta_l$  ( $\lambda \vec{y}.s[\vec{x}/\vec{y}]\vec{x} =_o s$ ).

Since the mapping is a bijection occur check in  $\mathcal{H}_0$  corresponds to occur check in  $\mathcal{F}_0$ .  $\square$

LEMMA 5.3. *Properties simulation (2.1) and fidelity (2.2) hold*

PROOF SKETCH. Since  $\text{progress1}$  is trivial  $\text{fststep}$  and  $\text{hstep}$  are the same, that is in this context where input terms are  $\beta\eta$ -normal and we disregard  $\eta$ -equivalence  $\approx_\lambda$  is equivalent to  $\approx_o$ .  $\square$

## 5.5 Limitations of by this basic scheme

$$\lambda xy.F \ y \ x = \lambda xy.x \quad (6)$$

$$\lambda x.f \ (F \ x) \ x = G \quad (7)$$

Note that here  $F$  is used with different arities, moreover in the second problem the left hand side happens to be an eta expansion (of  $f(\lambda y.y)$ ) only after we discover (at run time) that  $F = \lambda x\lambda y.y$  (i.e. that  $F$  discards the  $x$  argument). Both problems are addressed in the next section.

## 6 HANDLING OF $\Diamond\eta$

$\eta$ -reduction is an equivalence relation where a term of the form  $\lambda x.t \ x$  can be converted to  $t$  any time  $x$  does not occur as a free variable in  $t$ . We call  $t$  the  $\eta$ -contraction of  $\lambda x.t \ x$ .

Following the compilation scheme of section 5 the unification problem  $\mathbb{P}$  is compiled as follows:

$$\begin{aligned} \mathbb{P} &= \{ \lambda x.X \ x \approx_o f \} \\ \mathbb{T} &= \{ \lambda x.A_x \approx_\lambda f \} \\ \mathbb{M} &= \{ X \mapsto A^1 \} \end{aligned}$$

While  $\lambda x.X \ x \approx_o f$  does admit the solution  $\rho = \{X \mapsto f\}$ , the corresponding problem in  $\mathbb{T}$  does not:  $\text{lam } x \backslash \text{uva } A \ [x]$  and  $\text{con} "f"$  start with different, rigid, term constructors hence  $\approx_\lambda$  fails.

In order to guarantee proposition 2.1 we detect lambdas that can disappear by eta contraction (section 6.1) and we modify the term compiler put fresh unification variables in their place: the problematic term is moved from  $\mathbb{T}$  to  $\mathbb{L}$  (section 6.2). The compilation of the problem  $\mathbb{P}$  above is refined to:

$$\begin{aligned} \mathbb{P} &= \{ \lambda x.X \ x \approx_o f \} \\ \mathbb{T} &= \{ A \approx_\lambda f \} \\ \mathbb{M} &= \{ X \mapsto B^1 \} \\ \mathbb{L} &= \{ \vdash A =_\eta \lambda x.B_x \} \end{aligned}$$

As per invariant 2 the term on the left is a variable, and its right counterpart is the term in  $\Diamond\eta$ , and that term has the following property:

INVARIANT 4 (link- $\eta$  rhs). *The rhs of any link- $\eta$  has the shape  $\lambda x.t$  and  $t$  is not a lambda.*

link- $\eta$  are kept in the link store  $\mathbb{L}$  during execution and activated when some conditions hold on lhs or rhs. Link activation is implemented by extending the  $\text{progress1}$  predicate (defined in section 5.2).

### 6.1 Detection of $\Diamond\eta$

When compiling a term  $t$  we need to determine if any subterm  $s \in \mathcal{P}(t)$  that is of the form  $\lambda x.r$ , where  $x$  occurs in  $r$ , can be a  $\eta$ -expansion, i.e. if there exists a substitution  $\rho$  such that  $\rho(\lambda x.r) =_o s$ . The detection of lambda abstractions that can “disappear” is not as trivial as it may seem, here a few examples:

$$\begin{aligned} \lambda x.f \ (A \ x) &\in \Diamond\eta \quad \rho = \{A \mapsto \lambda x.x\} \\ \lambda x.f \ (A \ x) \ x &\in \Diamond\eta \quad \rho = \{A \mapsto \lambda x.a\} \\ \lambda x.f \ x \ (A \ x) &\notin \Diamond\eta \\ \lambda x.\lambda y.f \ (A \ x) \ (B \ y \ x) &\in \Diamond\eta \quad \rho = \{A \mapsto \lambda x.x, B \mapsto \lambda y.\lambda x.y\} \end{aligned}$$

The first two examples are easy, and show how a unification variable can expose or erase a variable and turn the resulting term in an  $\eta$ -expansion or not.

The third example shows that when a variable occurs outside the scope of a unification variable it cannot be erased and can hence prevent a term from being an  $\eta$ -expansion.

The last example shows the recursive nature of the check we need to implement. The term starts with a spine of two lambdas hence the whole term is in  $\Diamond\eta$  iff the inner term  $\lambda y.f \ (A \ x) \ (B \ y \ x)$  is in  $\Diamond\eta$  itself. If it is, it could  $\eta$ -contract to  $f \ (A \ x)$  making  $\lambda x.f \ (A \ x)$  a potential  $\eta$ -expansion.

We can now define more formally how  $\Diamond\eta$  terms are detected together with its auxiliary functions:

**Definition 6.1** (may-contract-to). A term  $s$  *may-contract-to* a name  $x$  if there exists a substitution  $\rho$  such that  $\rho s =_o x$ .

LEMMA 6.2. A  $\beta$ -normal term  $s = \lambda x_1 \dots x_n.t$  may-contract-to  $x$  only if one of the following three conditions holds:

- (1)  $n = 0$  and  $t = x$ ;
- (2)  $t$  is the application of  $x$  to a list of terms  $l$  and each  $l_i$  may-contract-to  $x_i$  (e.g.  $\lambda x_1 \dots x_n.x \ x_1 \dots x_n =_o x$ );
- (3)  $t$  is a unification variable with scope  $W$ , and for any  $v \in \{x, x_1 \dots x_n\}$ , there exists a  $w_i \in W$ , such that  $w_i$  may-contract-to  $v$  (if  $n = 0$  this is equivalent to  $x \in W$ ).

PROOF SKETCH. Since our terms are in  $\beta$ -normal form there is only one rule that can play a role (namely  $\eta_l$ ), hence if the term  $s$  is not exactly  $x$  (case 1) it can only be an  $\eta$ -expansion of it, or a unification variable that can be assigned to it, or a combination of both. If  $s$  begins with a lambda, then the lambda can only disappear by  $\eta$  contraction. In that case the term  $t$  under the spine of binders for  $x_1 \dots x_n$  can either be  $x$  itself applied to terms that can *may-contract-to* these variables (case 2), or a unification variable that can be assigned to that application (case 3).  $\square$

Note that this condition does not require the term to be in  $\mathcal{L}_\lambda$ .

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**Definition 6.3** (occurs-rigidly). A name  $x$  occurs-rigidly in a  $\beta$ -normal term  $t$ , if  $\forall \rho, x \in \mathcal{P}(\rho t)$

In other words  $x$  occurs-rigidly in  $t$  if it occurs in  $t$  outside of the scope of unification variables since an instantiation is allowed to discard  $x$  from the scope of the unification variable. Note that  $\eta$ -contraction cannot make  $x$  disappear, since the variables being erased by  $\eta$ -contraction are locally bound inside  $t$ .

We can now derive the implementation for  $\Diamond\eta$  detection:

**Definition 6.4** (maybe-eta). Given a  $\beta$ -normal term  $s = \lambda x_1 \dots x_n. t$ , maybe-eta  $s$  holds if any of the following holds:

- (1)  $t$  is a constant or variable applied to arguments  $l_1 \dots l_m$  such that  $m \geq n$  and for every  $i$  such that  $1 \leq i \leq m - n$  the term  $l_i$  may-contract-to  $x_i$ , and no  $x_i$  occurs-rigidly in  $l_1 \dots l_{m-n-1}$ ;
- (2)  $t$  is a unification variable with scope  $W$  and for each  $x_i$  there exists a  $w_j \in W$  such that  $w_j$  may-contract-to  $x_i$ .

**LEMMA 6.5** ( $\Diamond\eta$  DETECTION). If  $t$  is a  $\beta$ -normal term and maybe-eta  $t$  holds, then  $t \in \Diamond\eta$ .

**PROOF SKETCH.** Follows from definition 6.3 and lemma 6.2  $\square$

Remark that the converse of lemma 6.5 does not hold: there exists a term  $t$  satisfying the criteria (1) of definition 6.4 that is not in  $\Diamond\eta$ , i.e. there exists no substitution  $\rho$  such that  $\rho t$  is an  $\eta$ -expansion. A simple counter example is  $\lambda x. f (A x) (A x)$  since  $x$  does not occur-rigidly in the first argument of  $f$ , and the second argument of  $f$  may-contract-to  $x$ . In other words  $A x$  may either use or discard  $x$ , but our analysis does not take into account that the same term cannot have two contrasting behaviors.

As we will see in the rest of this section this is not a problem since it does not break proposition 2.1 nor proposition 2.2.

## 6.2 Compilation

The following rule is inserted just before rule  $(c_\lambda)$  from the code in section 5.

```
comp (flam F) (uva A Scope) M1 M2 L1 L3 S1 S3 :-
  maybe-eta (flam F) [], !,
  alloc S1 A S2,
  comp-lam F F1 M1 M2 L1 L2 S2 S3,
  get-scope (lam F1) Scope,
  L3 = [val (link-eta (uva A Scope) (lam F1)) | L2].
```

The rule triggers when the input term `flam F` is in  $\Diamond\eta$ . It compiles it to `lam F1` but puts the fresh variable `A` in its place. The variable sees all the names free in `lam F1`. The critical part of this rule is the creation of the `link-eta`, which relates the variable `A` with `lam F1`. This link clearly validates invariant 2.

**COROLLARY 6.6.** The rhs of any `link-eta` has exactly one lambda abstraction, hence the rule above respects invariant 4.

**PROOF SKETCH.** By contradiction, suppose that the rule above triggered and that the rhs of the link is  $\lambda x. \lambda y. t_{xy}$ . If maybe-eta  $\lambda y. t_{xy}$  holds the recursive call to `comp` (made by `comp-lam`) must have put a fresh variable in its place, so this case is impossible. Otherwise, if maybe-eta  $\lambda y. t_{xy}$  does not hold, also maybe-eta  $\lambda x. \lambda y. t_{xy}$  does not hold, contradicting the assumption that the rule triggered.  $\square$

TODO: W preservation proposition 2.8

## 6.3 Progress

`link-eta` are meant to delay the unification of “problematic” terms until we know for sure if the head lambda has to be  $\eta$ -contracted or not.

**Definition 6.7** (progress-eta-left). A link  $\Gamma \vdash X =_\eta T$  is removed from  $\mathbb{L}$  when  $X$  becomes rigid. There are two cases:

- (1) if  $X = a$  or  $X = y$  or  $X = f a_1 \dots a_n$  we unify the  $\eta$ -expansion of the  $X$  with  $T$ , that is we run  $\lambda x. X x \approx_\lambda T$  (under the context  $\Gamma$ )
- (2) if  $X = \lambda x. t$  we run  $X \approx_\lambda T$ .

**Definition 6.8** (progress-eta-right). A link  $\Gamma \vdash X =_\eta T$  is removed from  $\mathbb{L}$  when maybe-eta  $T$  does not hold (anymore) and by  $\eta$ -contracting  $T$  to  $T'$  (if possible, else  $T' = T$ ) and executing  $X \approx_\lambda T'$  (under the context  $\Gamma$ ).

There is a third case in which a link is removed from  $\mathbb{L}$ , namely when the lhs is assigned to another variable that is the lhs of another `link-eta`.

**Definition 6.9** (progress-eta-deduplicate). A link  $\Gamma \vdash X_{\vec{s}} =_\eta T$  is removed from  $\mathbb{L}$  when another link  $\Delta \vdash X_{\vec{r}} =_\eta T'$  is in  $\mathbb{L}$ . By invariant 1 the length of  $\vec{s}$  and  $\vec{r}$  is the same hence we can move the term  $T'$  from  $\Delta$  to  $\Gamma$  by renaming its bound variables, i.e.  $T'' = T'[\vec{r}/\vec{s}]$ . We then run  $T \approx_\lambda T''$  (under the context  $\Gamma$ ).

TODO: prove proposition 2.8: we never commit a  $\Diamond\eta$  term in  $\sigma$  since we run  $\approx_\lambda$  only when we know that the terms are no more  $\Diamond\eta$ , and when lhs is no more a variable or rhs is no more a  $\Diamond\eta$ , the link is removed from  $\mathbb{L}$ .

**LEMMA 6.10.** progress terminates.

**PROOF SKETCH.** Rules definitions 6.7 and 6.8 and definition 6.9 remove one link from  $\mathbb{L}$ , hence they cannot be applied indefinitely. Moreover each rule only relies on terminating operations such as  $\approx_\lambda$ ,  $\eta$ -contraction,  $\eta$ -expansion, relocation (a recursive copy of a finite term).  $\square$

**Example of progress-eta-left.** The example at the beginning of section 6, once  $\sigma = \{ A \mapsto f \}$ , triggers this rule since the link becomes  $\vdash f =_\eta \lambda x. B_x$  and the lhs is a constant. In turn the rule runs  $\lambda x. f x \approx_\lambda \lambda x. B_x$ , resulting in  $\sigma = \{ A \mapsto f ; B_x \mapsto f \}$ . Decompile the generates  $\rho = \{ X \mapsto f \}$ , since  $X$  is mapped to  $B$  and  $f$  is the  $\eta$ -contracted version of  $\lambda x. f x$ .

**Example of progress-eta-right.** A second example, showing the activation of a link when the rhs is no more a  $\Diamond\eta$ , is given in section 7, since we need to work with variables used with different arities. This example represent the run of the unification problems proposed at section 5.5

**Example of progress-eta-deduplicate.** A very basic example of `link-eta` deduplication, is given below:

$$\begin{aligned} \mathbb{P} &= \{ \lambda x. (X x) \approx_o \lambda x. (Y x) \} \\ \mathbb{T} &= \{ A \approx_\lambda C \} \\ \mathbb{M} &= \{ X \mapsto B^1 \quad Y \mapsto D^1 \} \\ \mathbb{L} &= \{ \vdash A =_\eta \lambda x. B_x \quad \vdash C =_\eta \lambda x. D_x \} \end{aligned}$$

The result of  $A \approx_\lambda C$  is that the two link- $\eta$  share the same lhs. By unifying the two rhs we get  $\sigma = \{A \mapsto C, B \mapsto D\}$ . In turn, given the map  $\mathbb{M}$ , this second assignment is decompiled to  $\rho = \{X \mapsto Y\}$  as expected.

TODO: we can have  $\lambda x.F_x$  in the substitution if we know that  $F$  does not reduce to  $T_x$  where  $x$  is not free in  $T$ .

## 7 ENFORCING INVARIANT 1

In section 5.5, we have given two unification problems to be run one after the other. In the following table we present the entry problem, its compiled version, with the corresponding list of mapping and links.

$$\begin{aligned} \mathbb{P} &= \{ \lambda x.\lambda y.(X y x) \approx_o \lambda x.\lambda y.x \quad \lambda x.(f (X x) x) \approx_o Y \} \\ \mathbb{T} &= \{ \quad \quad \quad A \approx_\lambda \lambda x.\lambda y.x \quad \quad \quad D \approx_\lambda F \} \\ \mathbb{M} &= \left\{ \begin{array}{l} X \mapsto E^1 \quad Y \mapsto F^0 \\ X \mapsto C^2 \end{array} \right\} \\ \mathbb{L} &= \left\{ \begin{array}{l} \vdash D =_\eta \lambda x.(f E_x x) \quad \vdash A =_\eta \lambda x.B_x \\ x \vdash B_x =_\eta \lambda y.C_{yx} \end{array} \right\} \end{aligned}$$

We see that the maybe-eta as detected  $\lambda xy.F y x$  and  $\lambda x.f (F x) x$  and replaced them with respectively the  $\mathcal{H}_0$  vars  $X$  and  $Z$ .  $X$  is linked with  $\lambda x.Y_x$ ,  $Y$  has arity 1 and is  $\eta$ -linked with  $\lambda y.H y x$  and  $Z$  is linked to the term  $\lambda x.f G_x x$ . However, the mapping returned by the compilation, does not breaks invariant 3: the  $\mathcal{F}_0$  variable  $F$  is mapped to two different  $\mathcal{H}_0$  variables. To address this problem and enforce invariant 3, we clean the mapping with a second phase after the compilation. This phase is called map-deduplication.

Before formally defining this procedure, we need to define some auxiliary relations. Let  $\mathbb{M}$  be the list of mapping and  $\langle m_1, m_2 \rangle \in \mathbb{M}$  such that the arity of the  $\mathcal{H}_0$  variable in  $m_1$  is smaller than the one in  $m_2$ . Let  $X$  (resp.  $Y$ ) the  $\mathcal{H}_0$  variable of  $m_1$  (resp.  $m_2$ ) and  $n = ar(m_1) - ar(m_2)$ . We also let  $A^i$  be a fresh  $\mathcal{H}_0$  variable. We define the make-eta-link relation taking two mappings  $\langle m_1, m_2 \rangle$  and returning the following list of link link- $\eta$ :  $\forall i \in [1..n]$ ,

$$\left\{ \begin{array}{ll} \vdash X =_\eta \lambda x.A_x^1 & \text{if } i = 1 \\ x_1 \dots x_{i-1} \vdash A_{x_1 \dots x_{i-1}}^{i-1} =_\eta \lambda x_i.A_{x_1 \dots x_i}^i & \text{if } 1 < i < n \\ x_1 \dots x_{i-1} \vdash A_{x_1 \dots x_{i-1}}^{i-1} =_\eta \lambda x_i.Y_{x_1 \dots x_i} & \text{if } i = n \end{array} \right.$$

More concretely, we are saying that for any two mappings, we build as many link- $\eta$  as the difference of the arities between the two mappings. This links are constructed in such a way that the  $\mathcal{H}_0$  variable  $v$  with lowest arity is linked to a fresh variable eta-expanded variable  $A^1$  having the scope of  $v$ . This variable  $A^1$  is then linked to an  $\eta$ -expanded fresh variable  $A^2$  with same scope of  $A^1$  and so on. The last link is built between the  $A^{n-1}$  (where  $n$  is the difference of arities between the two mappings) and the  $\mathcal{H}_0$  variable  $u$  with higher arity in the two mappings being considered.

**Definition 7.1 (map-deduplication).** For all mappings  $\langle m_1, m_2 \rangle \in \mathbb{M}$ , sharing the same  $\mathcal{F}_0$  variable, the list of link- $\eta$   $L$  is created thanks to make-eta-link $m_1 m_2 L$  and is added to  $\mathbb{L}$ . Then  $m_1$  is removed from  $\mathbb{M}$ .

If we take back the example give at the beginning of this section, we can deduplicate  $F \mapsto G^1, F \mapsto H^2$  by removing the first mapping and adding the auxiliary link- $\eta$ :  $x \vdash G_x =_\eta \lambda y.H_{xy}$ .

The complete problem to run for resolution is now:

$$\begin{aligned} \mathbb{P} &= \{ \lambda x.\lambda y.(X y x) \approx_o \lambda x.\lambda y.x \quad \lambda x.(f (X x) x) \approx_o Y \} \\ \mathbb{T} &= \{ \quad \quad \quad A \approx_\lambda \lambda x.\lambda y.x \quad \quad \quad D \approx_\lambda F \} \\ \mathbb{M} &= \left\{ \begin{array}{l} Y \mapsto F^0 \quad X \mapsto C^2 \end{array} \right\} \\ \mathbb{L} &= \left\{ \begin{array}{l} x \vdash E_x =_\eta \lambda y.C_{xy} \quad \vdash D =_\eta \lambda x.(f E_x x) \\ \vdash A =_\eta \lambda x.B_x \quad x \vdash B_x =_\eta \lambda y.C_{yx} \end{array} \right\} \end{aligned}$$

After unification of the two terms,  $X$  is assigned to  $\lambda x.\lambda y.x$ . This assignment makes  $l_2$  to progress since the lhs is materialized and by unification, between  $X$  and  $\lambda x.Y_x$ ,  $Y_x$  is instantiate to  $\lambda y.x$ . Once  $Y_x$  is instantiated,  $l_1$  can progress, and set  $H_{xy}$  to  $x$ . After all these progresses,  $l_1$  and  $l_2$  are remove from  $\mathbb{L}$  and the progress fixpoint terminates. Next, the second unification problem is run, and  $Z$  is set to  $f(\lambda x.x)$ . This unification wakes up  $l_3$  and since  $Z$  starts with the app node, the  $\eta$ -expanded version of  $Z$  is unified with  $\lambda x.f G_x x$  and  $G_x$  is set to  $x$ . As last step, the last link is progressed and the final  $\mathcal{H}_0$  substitution is  $\{X \mapsto \lambda x.\lambda y.x, Y_x \mapsto \lambda y.x, G_{yx} \mapsto y, Z \mapsto f \lambda x.x, H_x \mapsto \lambda y.y\}$ .

The decompilation phase is only charged, in this example to solve the mappings, since no suspended links remain. The only mapping in the list is  $F \mapsto H^2$ , which will assign the  $F$  variable in  $\mathcal{F}_0$  to  $\lambda xy.y$ .

TODO: dire che preserviamo l'invariante che tutte le variable sono fully-applied

## 8 HANDLING OF $\overline{\mathcal{L}}_\lambda$

TODO: say that maybe-eta also work in not(lambda)

All the previous sections we have dealt with terms in  $\mathcal{L}_\lambda$ , however, it is often possible to work with terms in  $\overline{\mathcal{L}}_\lambda$  and wish to unify them. There are situation, for example, where the oracle has given which  $\beta$ -reduction problems  $(\overline{\mathcal{L}}_\lambda)$  appears any time we deal with a subterm  $t = Xt_1 \dots t_n$ , where  $X$  is flexible and the list  $[t_1 \dots t_n]$  in not in  $\mathcal{L}_\lambda$ . This unification problem is not solvable without loss of generality, since there is not a most general unifier. If we take back the example given in section 2.1, the unification  $Fa = a$  admits two solutions for  $F$ :  $\rho_1 = \{F \mapsto \lambda x.x\}$  and  $\rho_2 = \{F \mapsto \lambda_.a\}$ . Despite this, it is possible to work with  $\overline{\mathcal{L}}_\lambda$  if an oracle provides a substitution  $\rho$  such that  $\rho t$  falls again in the  $\mathcal{L}_\lambda$ .

On the other hand, the  $\approx_\lambda$  is not designed to understand how the  $\beta$ -redexes work in the object language. Therefore, even if we know that  $F$  is assigned to  $\lambda x.x$ ,  $\approx_\lambda$  is not able to unify  $Fa$  with  $a$ . On the other hand, the problem  $Fa = G$  is solvable by  $\approx_\lambda$ , but the final result is that  $G$  is assigned to  $(\lambda x.x)a$  which breaks the invariant saying that the substitution of the meta language does not generate terms outside  $\mathcal{W}$  (Property 2.8).

The solution to this problem is to modify the compiler such that any sub-term  $t$  considered as a potential  $\beta$ -redex is replaced with a hole  $h$  and a new dedicated link, called link- $\beta$ .

**type link-beta tm -> tm -> link.**

This link carries two terms, the former representing the variable  $h$  for the new created hole and the latter containing the subterm  $t$ . As for the link- $\eta$ , we will call  $h$  and  $t$  respectively the left hand side (*lhs*) and the right hand side (*rhs*) of the link- $\beta$ .

## 8.1 Compilation

Detection of  $\overline{\mathcal{L}}_\lambda$ . TODO: ...

Compilation with link- $\beta$ . In order to build a link- $\beta$ , we need to adapt the compiler so that it can recognize these “problematic” subterms. The following code snippet illustrate such behavior, we suppose the rule to be added just after ??.

```
comp (fapp [fuva A|Ag]) (uva C Scope) M1 M3 L1 L3 S1 S4 :- !,
  pattern-fragment-prefix Ag Pf Extra,
  fold6 comp Pf Scope1 M1 M1 L1 L1 S1 S1,
  fold6 comp Extra Extra1 M1 M2 L1 L2 S1 S2,
  len Pf Arity,
  m-alloc (fv A) (hv B (arity Arity)) M2 M3 S2 S3,
  Beta = app [uva B Scope1 | Extra1],
  get-scope Beta Scope,
  alloc S3 C S4,
  L3 = [eval-link-beta (uva C Scope) Beta | L2].
```

A term is  $\overline{\mathcal{L}}_\lambda$  if it has the shape `fapp[fuva A|Ag]` and distinct `Ag` does not hold. In that case, `Ag` is split in two sublist `Pf` and `Extra` such that former is the longest prefix of `Ag` such that distinct `Pf` holds. `Extra` is the list such that append `Pf Extra Ag`. Next important step is to compile recursively the terms of these lists and allocate a memory adress `B` from the substitution in order to map the  $\mathcal{F}_0$  variable `fuva A` to the  $\mathcal{H}_0$  variable `uva B`. The link- $\beta$  to return in the end is given by the term `Beta = app[uva B Scope1 | Extra1]` constituting the *rhs*, and a fresh variable `C` having in scope all the free variables occurring in `Beta` (this is *lhs*). We point out that the *rhs* is intentionally built as an *uva* where `Extra1` are not in scope, since by invariant, we want all the variables appearing in  $\mathcal{H}_0$  to be in  $\mathcal{L}_\lambda$ .

## 8.2 Progress

Once created, there exist two main situations waking up a suspended link- $\beta$ . The former is strictly connected to the definition of  $\beta$ -redex and occurs when the head of *rhs* is materialized by the oracle (see proposition 2.1). In this case *rhs* is safely  $\beta$ -reduced to a new term  $t'$  and the result can be unified with *lhs*. In this scenario the link- $\beta$  has accomplished its goal and can be removed from  $\mathbb{L}$ .

The second circumstance making the link- $\beta$  to progress is the instantiation of the variables in the `Extra1` making the corresponding arguments to reduce to names. In this case, we want to take the list `Scope1` and append to it the largest prefix of `Extra1` in a new variable `Scope2` such that `Scope2` remains in  $\mathcal{L}_\lambda$ ; we call `Extra2` the suffix of `Extra1` such that the concatenation of `Scope1` and `Extra1` is the same as the concatenation of `Scope2` and `Extra2`. Finally, two cases should be considered: 1) `Extra2` is the empty list, *lhs* and *rhs* can be unified: we have two terms in  $\mathcal{L}_\lambda$ ; otherwise 2) the link- $\beta$  in question is replaced with a refined version where the *rhs* is `app[uva C Scope2 | Extra2]` and a new link- $\eta$  is added between the *lhs* and the new-added variable `C`.

An example justifying this second link manipulation is given by the following unification problem:

```
f = flam x\ fapp[F, fapp[A, x]].
```

The compilation of these terms produces the new unification problem:  $f = X0$

We obtain the mappings  $F \mapsto F^0$ ,  $A \mapsto A^1$  and the links:

$$c0 \vdash X3_{c0} =_\beta X2 X1_{c0} \quad (8)$$

$$\vdash X0 =_\eta \lambda c0.X3_{c0} \quad (9)$$

where the first link is a link- $\eta$  between the variable `X0`, representing the right side of the unification problem (it is a  $\diamond\eta$ ) and `X3`; and a link- $\beta$  between the variable `X3` and the subterm  $\lambda x.X1_x a$  (it is a  $\overline{\mathcal{L}}_\lambda$ ). The substitution tells that  $x \vdash X1_x = x$ .

We can now represent the hrn execution from this configuration which will, at first, dereference all the links, and then try to solve them. The only link being modified is the second one, which is set to  $x \vdash X3 =_\beta X2 x a$ . The *rhs* of the link has now a variable which is partially in the PF, we can therefore remove the original link- $\beta$  and replace it with the following couple on links:

$$\vdash X1 =_\eta x \setminus X4 \ x'$$

$$x \vdash X3 \ x =_\beta x \setminus X4 \ x' \ a$$

By these links we say that `X1` is now  $\eta$ -linked to a fresh variable `X4` with arity one. This new variable is used in the new link- $\beta$  where the name `x` is in its scope. This allows

## 8.3 Tricky examples

```
triple ok (@lam x\ @app[@f, @app[@X, x]]) @Y,
triple ok @X (@lam x\ x),
triple ok @Y @f
```

```
% @ok1 22 [
%   triple ok (@lam x\ @lam y\ @app[@Y, y, x]) @X,
%   triple ok (@lam x\ @f) @X,
% ].
```

## 9 FIRST ORDER APPROXIMATION

**TODO:** Coq can solve this:  $f \ 1 \ 2 = x \ 2$ , by setting  $X$  to  $f \ 1$

**TODO:** We can re-use part of the algo for  $\beta$  given before

## 10 UNIF ENCODING IN REAL LIFE

**TODO:** Il ML presentato qui è esattamente elpi

**TODO:** Il OL presentato qui è esattamente coq

**TODO:** Come implementiamo tutto ciò nel solver

## 11 RESULTS: STDPP AND TLC

**TODO:** How may rule are we solving?

**TODO:** Can we do some perf test

## 12 CONCLUSION

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## APPENDIX

This appendix contains the entire code described in this paper. The code can also be accessed at the URL: <https://github.com/FissoreD/paper-ho>

Note that (a infix b) c d de-sugars to (infix) a b c d.

Explain builtin name (can be implemented by loading name after each pi)

## 13 THE MEMORY

```

kind addr type.
type addr nat -> addr.
typeabbrev (mem A) (list (option A)).

type set? addr -> mem A -> A -> o.
set? (addr A) Mem Val :- get A Mem Val.

type unset? addr -> mem A -> o.
unset? Addr Mem :- not (set? Addr Mem _).

type assign-aux nat -> mem A -> A -> mem A -> o.
assign-aux z (none :: L) Y (some Y :: L).
assign-aux (s N) (X :: L) Y (X :: L1) :- assign-aux N L Y L1.

type assign addr -> mem A -> A -> mem A -> o.
assign (addr A) Mem1 Val Mem2 :- assign-aux A Mem1 Val Mem2.

type get nat -> mem A -> A -> o.
get z (some Y :: _) Y.
get (s N) (_ :: L) X :- get N L X.

type alloc-aux nat -> mem A -> mem A -> o.
alloc-aux z [] [none] :- !.
alloc-aux z L L.
alloc-aux (s N) [] [none | M] :- alloc-aux N [] M.
alloc-aux (s N) [X | L] [X | M] :- alloc-aux N L M.

type alloc addr -> mem A -> mem A -> o.
alloc (addr A as Ad) Mem1 Mem2 :- unset? Ad Mem1,
  alloc-aux A Mem1 Mem2.

type new-aux mem A -> nat -> mem A -> o.
new-aux [] z [none].
new-aux [A | As] (s N) [A | Bs] :- new-aux As N Bs.

type new mem A -> addr -> mem A -> o.
new Mem1 (addr Ad) Mem2 :- new-aux Mem1 Ad Mem2.

```

## 14 THE OBJECT LANGUAGE

```

kind fm type.
type fapp list fm -> fm.
type flam (fm -> fm) -> fm.
type fcon string -> fm.
type fuva addr -> fm.

typeabbrev fsubst (mem fm).

```

```

type fder fsubst -> fm -> fm -> o.
fder _ (fcon C) (fcon C).
fder S (fapp A) (fapp B) :- map (fder S) A B.
fder S (flam F) (flam G) :-
  pi x\ fder S x x => fder S (F x) (G x).
fder S (fuva N) R :- set? N S T, fder S T R.
fder S (fuva N) (fuva N) :- unset? N S.

type fderef fsubst -> fm -> fm -> o. (ρs)
fderef S T T2 :- fder S T T1, napp T1 T2.

```

```

type (=o) fm -> fm -> o. (=o)
fcon X =o fcon X.
fapp A =o fapp B :- forall2 (=o) A B.
flam F =o flam G :- pi x\ x =o x => F x =o G x.
fuva N =o fuva N.
flam F =o T :- (ηl)
  pi x\ beta T [x] (T' x), x =o x => F x =o T' x.
T =o flam F :- (ηr)
  pi x\ beta T [x] (T' x), x =o x => T' x =o F x.
fapp [flam X | L] =o T :- beta (flam X) L R, R =o T. (βl)
T =o fapp [flam X | L] :- beta (flam X) L R, T =o R. (βr)

```

```

type extend-subst fm -> fsubst -> fsubst -> o.
extend-subst (fuva N) S S' :- mem.alloc N S S'.
extend-subst (flam F) S S' :-
  pi x\ (pi S\ extend-subst x S S) => extend-subst (F x) S S'.
extend-subst (fcon _) S S.
extend-subst (fapp L) S S1 :- fold extend-subst L S S1.

type beta fm -> list fm -> fm -> o.
beta A [] A.
beta (flam Bo) [H | L] R :- napp (Bo H) F, beta F L R.
beta (fapp A) L (fapp X) :- append A L X.
beta (fuva N) L (fapp [fuva N | L]).
beta (fcon H) L (fapp [fcon H | L]).
beta N L (fapp [N | L]) :- name N.

```

```

type napp fm -> fm -> o.
napp (fcon C) (fcon C).
napp (fuva A) (fuva A).
napp (flam F) (flam G) :- pi x\ napp (F x) (G x).
napp (fapp [fapp L1 [L2]] T) :- !,
  append L1 L2 L3, napp (fapp L3) T.
napp (fapp L) (fapp L1) :- map napp L L1.
napp N N :- name N.

```

```

type mk-app fm -> list fm -> fm -> o.
mk-app T L S :- beta T L S.

```

```

type eta-contract fm -> fm -> o.
eta-contract (fcon X) (fcon X).
eta-contract (fapp L) (fapp L1) :- map eta-contract L L1.
eta-contract (flam F) T :- eta-contract-aux [] (flam F) T.
eta-contract (flam F) (flam F1) :-

```

```

1509   pi x\ eta-contract x x => eta-contract (F x) (F1 x).
1510 eta-contract (fuva X) (fuva X).
1511 eta-contract X X :- name X.
1512
1513 type eta-contract-aux list fm -> fm -> fm -> o.
1514 eta-contract-aux L (flam F) T :-
1515   pi x\ eta-contract-aux [x|L] (F x) T. % also checks H Prefix does
1516 eta-contract-aux L (fapp [H|Args]) T :-
1517   rev L LRev, append Prefix LRev Args,
1518   if (Prefix = []) (T = H) (T = fapp [H|Prefix]).
1519
1520
1521
1522 kind inctx type -> type.
1523 type abs (tm -> inctx A) -> inctx A.
1524 type val A -> inctx A.
1525 typeabbrev assignment (inctx tm).
1526 typeabbrev subst (mem assignment).
1527
1528 kind tm type.
1529 type app list tm -> tm.
1530 type lam (tm -> tm) -> tm.
1531 type con string -> tm.
1532 type uva addr -> list tm -> tm.
1533
1534 type ( $\approx_\lambda$ ) tm -> tm -> subst -> subst -> o.
1535 (con C  $\approx_\lambda$  con C) S S.
1536 (app L1  $\approx_\lambda$  app L2) S S1 :- fold2 ( $\approx_\lambda$ ) L1 L2 S S1.
1537 (lam F1  $\approx_\lambda$  lam F2) S S1 :-
1538   pi x\ (pi S\ (x  $\approx_\lambda$  x) S S) => (F1 x  $\approx_\lambda$  F2 x) S S1.
1539 (uva N Args  $\approx_\lambda$  T) S S1 :-
1540   set? N S F,!, move F Args T1, (T1  $\approx_\lambda$  T) S S1.
1541 (T  $\approx_\lambda$  uva N Args) S S1 :-
1542   set? N S F,!, move F Args T1, (T  $\approx_\lambda$  T1) S S1.
1543 (uva M A1  $\approx_\lambda$  uva N A2) S1 S2 :- !,
1544   pattern-fragment A1, pattern-fragment A2,
1545   prune! M A1 N A2 S1 S2.
1546 (uva N Args  $\approx_\lambda$  T) S S1 :- not_occ N S T, pattern-fragment Args,
1547   bind T Args T1, assign N S T1 S1.
1548 (T  $\approx_\lambda$  uva N Args) S S1 :- not_occ N S T, pattern-fragment Args,
1549   bind T Args T1, assign N S T1 S1.
1550
1551 type prune! addr -> list tm -> addr ->
1552   list tm -> subst -> subst -> o.
1553 /* no pruning needed */
1554 prune! N A N A S S :- !.
1555 prune! M A N A S1 S2 :- !, bind (uva M A) A Ass,
1556   assign N S1 Ass S2.
1557 /* prune different arguments */
1558 prune! N A1 N A2 S1 S3 :- !,
1559   new S1 W S2, prune-same-variable W A1 A2 [] Ass,
1560   assign N S2 Ass S3.
1561 /* prune to the intersection of scopes */
1562 prune! N A1 M A2 S1 S4 :- !,
1563   new S1 W S2, prune-diff-variables W A1 A2 Ass1 Ass2,
1564   assign N S2 Ass1 S3,
1565   assign M S3 Ass2 S4.
1566

```

```

1567
1568 type prune-same-variable addr -> list tm -> list tm ->
1569   list tm -> assignment -> o.
1570 prune-same-variable N [] [] ACC (val (uva N Args)) :-
1571   rev ACC Args.
1572 prune-same-variable N [X|XS] [X|YS] ACC (abs F) :-
1573   pi x\ prune-same-variable N XS YS [x|ACC] (F x).
1574 prune-same-variable N [_|XS] [_|YS] ACC (abs F) :-
1575   pi x\ prune-same-variable N XS YS ACC (F x).
1576
1577 type permute list nat -> list tm -> list tm -> o.
1578 permute [] _ [].
1579 permute [P|PS] Args [T|TS] :-
1580   nth P Args T,
1581   permute PS Args TS.
1582
1583 type build-perm-assign addr -> list tm -> list bool ->
1584   list nat -> assignment -> o.
1585 build-perm-assign N ArgsR [] Perm (val (uva N PermutedArgs)) :-
1586   rev ArgsR Args, permute Perm Args PermutedArgs.
1587 build-perm-assign N Acc [tt|L] Perm (abs T) :-
1588   pi x\ build-perm-assign N [x|Acc] L Perm (T x).
1589 build-perm-assign N Acc [ff|L] Perm (abs T) :-
1590   pi x\ build-perm-assign N Acc L Perm (T x).
1591
1592 type keep list A -> A -> bool -> o.
1593 keep L A tt :- mem L A, !.
1594 keep _ _ ff.
1595
1596 type prune-diff-variables addr -> list tm -> list tm ->
1597   assignment -> assignment -> o.
1598 prune-diff-variables N Args1 Args2 Ass1 Ass2 :-
1599   map (keep Args2) Args1 Bits1,
1600   map (keep Args1) Args2 Bits2,
1601   filter Args1 (mem Args2) ToKeep1,
1602   filter Args2 (mem Args1) ToKeep2,
1603   map (index ToKeep1) ToKeep1 IdPerm,
1604   map (index ToKeep1) ToKeep2 Perm21,
1605   build-perm-assign N [] Bits1 IdPerm Ass1,
1606   build-perm-assign N [] Bits2 Perm21 Ass2.
1607
1608 type beta tm -> list tm -> tm -> o.
1609 beta A [] A.
1610 beta (lam Bo) [H | L] R :- beta (Bo H) L R.
1611 beta (app A) L (app X) :- append A L X.
1612 beta (con H) L (app [con H | L]).
1613 beta X L (app[X|L]) :- name X.
1614
1615 /* occur check for N before crossing a functor */
1616 type not_occ addr -> subst -> tm -> o.
1617 not_occ N S (uva M Args) :- set? M S F,
1618   move F Args T, not_occ N S T.
1619 not_occ N S (uva M Args) :- unset? M S, not (M = N),
1620   forall1 (not_occ_aux N S) Args.
1621 not_occ _ _ (con _).
1622 not_occ N S (app L) :- not_occ_aux N S (app L).
1623 /* Note: lam is a functor for the meta language! */
1624

```

```

1625 not_occ N S (lam L) :- pi x\ not_occ_aux N S (L x).
1626 not_occ _ _ X :- name X.
1627 /* finding N is ok */
1628 not_occ N _ (uva N _).
1629
1630 /* occur check for X after crossing a functor */
1631 type not_occ_aux addr -> subst -> tm -> o.
1632 not_occ_aux N S (uva M _) :- unset? M S, not (N = M).
1633 not_occ_aux N S (uva M Args) :- set? M S F,
1634   move F Args T, not_occ_aux N S T.
1635 not_occ_aux N S (app L) :- forall1 (not_occ_aux N S) L.
1636 not_occ_aux N S (lam F) :- pi x\ not_occ_aux N S (F x).
1637 not_occ_aux _ _ (con _).
1638 not_occ_aux _ _ X :- name X.
1639 /* finding N is ko, hence no rule */
1640
1641 /* copy T T' fails if T contains a free variable, i.e. it
1642   performs scope checking for bind */
1643 type copy tm -> tm -> o.
1644 copy (con C) (con C).
1645 copy (app L) (app L') :- map copy L L'.
1646 copy (lam T) (lam T') :- pi x\ copy x x => copy (T x) (T' x).
1647 copy (uva A L) (uva A L') :- map copy L L'.
1648
1649 type bind tm -> list tm -> assignment -> o.
1650 bind T [] (val T') :- copy T T'.
1651 bind T [X | TL] (abs T') :- pi x\ copy X x => bind T TL (T' x).
1652
1653 type deref subst -> tm -> tm -> o. (σt)
1654 deref _ (con C) (con C).
1655 deref S (app A) (app B) :- map (deref S) A B.
1656 deref S (lam F) (lam G) :-
1657   pi x\ deref S x x => deref S (F x) (G x).
1658 deref S (uva N L) R :- set? N S A,
1659   move A L T, deref S T R.
1660 deref S (uva N A) (uva N B) :- unset? N S,
1661   map (deref S) A B.
1662
1663 type move assignment -> list tm -> tm -> o.
1664 move (abs Bo) [H|L] R :- move (Bo H) L R.
1665 move (val A) [] A.
1666
1667
1668 type deref-assmt subst -> assignment -> assignment -> o.
1669 deref-assmt S (abs T) (abs R) :- pi x\ deref-assmt S (T x) (R x).
1670 deref-assmt S (val T) (val R) :- deref S T R.

```

## 16 THE COMPILER

```

1674 kind arity type.
1675 type arity nat -> arity.
1676
1677 kind fvariable type.
1678 type fv addr -> fvariable.
1679
1680 kind hvariable type.
1681 type hv addr -> arity -> hvariable.

```

```

1683 kind mapping type.
1684 type mapping fvariable -> hvariable -> mapping.
1685 typeabbrev mmap (list mapping).
1686
1687 typeabbrev scope (list tm).
1688 typeabbrev inctx ho.inctx.
1689 kind baselink type.
1690 type link-eta tm -> tm -> baselink.
1691 type link-beta tm -> tm -> baselink.
1692 typeabbrev link (inctx baselink).
1693 typeabbrev links (list link).
1694
1695 macro @val-link-eta T1 T2 :- ho.val (link-eta T1 T2).
1696 macro @val-link-beta T1 T2 :- ho.val (link-beta T1 T2).
1697
1698
1699 type occurs-rigidly fm -> fm -> o.
1700 occurs-rigidly N N.
1701 occurs-rigidly _ (fapp [fuva _|_]) :- !, fail.
1702 occurs-rigidly N (fapp L) :- exists (occurs-rigidly N) L.
1703 occurs-rigidly N (flam B) :- pi x\ occurs-rigidly N (B x).
1704
1705 type reducible-to list fm -> fm -> fm -> o.
1706 reducible-to _ N N :- !.
1707 reducible-to L N (fapp [fuva _|Args]) :- !,
1708   forall1 (x\ exists (reducible-to [] x) Args) [N|L].
1709 reducible-to L N (flam B) :- !,
1710   pi x\ reducible-to [x | L] N (B x).
1711 reducible-to L N (fapp [N|Args]) :-
1712   last-n {len L} Args R,
1713   forall2 (reducible-to []) R {rev L}.
1714
1715 type maybe-eta fm -> list fm -> o. (◇η)
1716 maybe-eta (fapp [fuva _|Args]) L :- !,
1717   forall1 (x\ exists (reducible-to [] x) Args) L, !.
1718 maybe-eta (flam B) L :- !, pi x\ maybe-eta (B x) [x | L].
1719 maybe-eta (fapp [fcon _|Args]) L :-
1720   split-last-n {len L} Args First Last,
1721   none (x\ exists (y\ occurs-rigidly x y) First) L,
1722   forall2 (reducible-to []) {rev L} Last.
1723
1724 type locally-bound tm -> o.
1725 type get-scope-aux tm -> list tm -> o.
1726 get-scope-aux (con _) [].
1727 get-scope-aux (uva _ L) L1 :-
1728   forall2 get-scope-aux L R,
1729   flatten R L1.
1730 get-scope-aux (lam B) L1 :-
1731   pi x\ locally-bound x => get-scope-aux (B x) L1.
1732 get-scope-aux (app L) L1 :-
1733   forall2 get-scope-aux L R,
1734   flatten R L1.
1735 get-scope-aux X [X] :- name X, not (locally-bound X).
1736 get-scope-aux X [] :- name X, (locally-bound X).

```

```

1741
1742 type names1 list tm -> o.
1743 names1 L :-
1744   names L1,
1745   new_int N,
1746   if (1 is N mod 2) (L1 = L) (rev L1 L).
1747
1748 type get-scope tm -> list tm -> o.
1749 get-scope T Scope :-
1750   get-scope-aux T ScopeDuplicata,
1751   undup ScopeDuplicata Scope.
1752 type rigid fm -> o.
1753 rigid X :- not (X = fuva _).
1754
1755 type comp-lam (fm -> fm) -> (tm -> tm) ->
1756   mmap -> mmap -> links -> links -> subst -> subst -> o.
1757 comp-lam F G M1 M2 L1 L3 S1 S2 :-
1758   pi x y\ (pi M L S\ comp x y M M L L S S) =>
1759     comp (F x) (G y) M1 M2 L1 (L2 y) S1 S2,
1760     close-links L2 L3.
1761
1762 type close-links (tm -> links) -> links -> o.
1763 close-links (v\[X |L v]) [X|R] :- !, close-links L R.
1764 close-links (v\[X v|L v]) [abs X|R] :- close-links L R.
1765 close-links (_\[ ]) [ ].
1766 type comp fm -> tm -> mmap -> mmap -> links -> links ->
1767   subst -> subst -> o.
1768 comp (fcon C) (con C) M M L L S S.
1769 comp (flam F) (uva A Scope) M1 M2 L1 L3 S1 S3 :-
1770   maybe-eta (flam F) [ ], !,
1771   alloc S1 A S2,
1772   comp-lam F F1 M1 M2 L1 L2 S2 S3,
1773   get-scope (lam F1) Scope,
1774   L3 = [val (link-eta (uva A Scope) (lam F1)) | L2].
1775 comp (flam F) (lam F1) M1 M2 L1 L2 S1 S2 :-
1776   comp-lam F F1 M1 M2 L1 L2 S1 S2.
1777 comp (fuva A) (uva B [ ]) M1 M2 L L S1 S2 :-
1778   m-alloc (fv A) (hv B (arity z)) M1 M2 S1 S2.
1779 comp (fapp [fuva A|Ag]) (uva B Ag1) M1 M2 L L S1 S2 :-
1780   pattern-fragment Ag, !,
1781   fold6 comp Ag Ag1 M1 M1 L L S1 S1,
1782   len Ag Arity,
1783   m-alloc (fv A) (hv B (arity Arity)) M1 M2 S1 S2.
1784 comp (fapp [fuva A|Ag]) (uva C Scope) M1 M3 L1 L3 S1 S4 :- !,
1785   pattern-fragment-prefix Ag Pf Extra,
1786   fold6 comp Pf Scope1 M1 M1 L1 L1 S1 S1,
1787   fold6 comp Extra Extra1 M1 M2 L1 L2 S1 S2,
1788   len Pf Arity,
1789   m-alloc (fv A) (hv B (arity Arity)) M2 M3 S2 S3,
1790   Beta = app [uva B Scope1 | Extra1],
1791   get-scope Beta Scope,
1792   alloc S3 C S4,
1793   L3 = [eval-link-beta (uva C Scope) Beta | L2].
1794 comp (fapp A) (app A1) M1 M2 L1 L2 S1 S2 :-
1795   fold6 comp A A1 M1 M2 L1 L2 S1 S2.
1796
1797 type alloc mem A -> addr -> mem A -> o.
1798
1799 alloc S N S1 :- mem.new S N S1.
1800
1801 type compile-terms-diagnostic
1802   triple diagnostic fm fm ->
1803   triple diagnostic tm tm ->
1804   mmap -> mmap ->
1805   links -> links ->
1806   subst -> subst -> o.
1807 compile-terms-diagnostic (triple D F01 F02) (triple D H01 H02) M0 M3 L1
1808   comp F01 H01 M1 M2 L1 L2 S1 S2,
1809   comp F02 H02 M2 M3 L2 L3 S2 S3.
1810
1811 type compile-terms
1812   list (triple diagnostic fm fm) ->
1813   list (triple diagnostic tm tm) ->
1814   mmap -> links -> subst -> o.
1815 compile-terms T H M L S :-
1816   fold6 compile-terms-diagnostic T H [ ] M_ [ ] L_ [ ] S_,
1817   print-compil-result T H L_ M_,
1818   deduplicate-map M_ M S_ S L_ L.
1819
1820 type make-eta-link-aux nat -> addr -> addr ->
1821   list tm -> links -> subst -> subst -> o.
1822 make-eta-link-aux z Ad1 Ad2 Scope1 L H1 H1 :-
1823   rev Scope1 Scope, eta-expand (uva Ad2 Scope) T1,
1824   L = [val (link-eta (uva Ad1 Scope) T1)].
1825 make-eta-link-aux (s N) Ad1 Ad2 Scope1 L H1 H3 :-
1826   rev Scope1 Scope, alloc H1 Ad H2,
1827   eta-expand (uva Ad Scope) T2,
1828   (pi x\ make-eta-link-aux N Ad Ad2 [x|Scope1] (L1 x) H2 H3),
1829   close-links L1 L2,
1830   L = [val (link-eta (uva Ad1 Scope) T2) | L2].
1831
1832 type make-eta-link nat -> nat -> addr -> addr ->
1833   list tm -> links -> subst -> subst -> o.
1834 make-eta-link (s N) z Ad1 Ad2 Vars L H H1 :-
1835   make-eta-link-aux N Ad2 Ad1 Vars L H H1.
1836 make-eta-link z (s N) Ad1 Ad2 Vars L H H1 :-
1837   make-eta-link-aux N Ad1 Ad2 Vars L H H1.
1838 make-eta-link (s N) (s M) Ad1 Ad2 Vars Links H H1 :-
1839   (pi x\ make-eta-link N M Ad1 Ad2 [x|Vars] (L x) H H1),
1840   close-links L Links.
1841
1842 type deduplicate-map mmap -> mmap ->
1843   subst -> subst -> links -> links -> o.
1844 deduplicate-map [ ] [ ] H H L L.
1845 deduplicate-map [(mapping (fv 0) (hv M (arity LenM))) as X1] | Map1 Map2
1846   take-list Map1 (mapping (fv 0) (hv M' (arity LenM'))), !,
1847   std.assert! (not (LenM = LenM')) "Deduplicate map, there is a bug",
1848   print "arity-fix links:" {ppmapping X1} "~!~" {ppmapping (mapping (fv
1849   make-eta-link LenM LenM' M M' [ ] New H1 H2,
1850   print "new eta link" {pplinks New},
1851   append New L1 L2,
1852   deduplicate-map Map1 Map2 H2 H3 L2 L3.
1853 deduplicate-map [A|As] [A|Bs] H1 H2 L1 L2 :-
1854   deduplicate-map As Bs H1 H2 L1 L2, !.
1855
1856

```



## 17 THE PROGRESS FUNCTION

```

1857 deduplicate-map [A|_] _ H _ _ :-
1858   halt "deduplicating mapping error" {ppmapping A} {ho.ppsubst H}.
1859
1860
1861
1862 macro @one :- s z.
1863
1864 type contract-rigid list ho.tm -> ho.tm -> ho.tm -> o.
1865 contract-rigid L (ho.lam F) T :-
1866   pi x\ contract-rigid [x|L] (F x) T. % also checks H Prefix does not make eta-link
1867 contract-rigid L (ho.app [H|Args]) T :-
1868   rev L LRev, append Prefix LRev Args,
1869   if (Prefix = []) (T = H) (T = ho.app [H|Prefix]).
1870
1871 type progress-eta-link ho.tm -> ho.tm -> ho.subst -> ho.subst -> link -> link -> o.
1872 progress-eta-link (ho.app _ as T) (ho.lam x\ _ as T1) H H1 [] :- !, not (T1 = ho.uva _ _), !, fail.
1873   ({eta-expand T @one} ==1 T1) H H1.
1874 progress-eta-link (ho.con _ as T) (ho.lam x\ _ as T1) H H1 [] :- !
1875   ({eta-expand T @one} ==1 T1) H H1.
1876 progress-eta-link (ho.lam _ as T) T1 H H1 [] :- !,
1877   (T ==1 T1) H H1.
1878 progress-eta-link (ho.uva _ _ as X) T H H1 [] :-
1879   contract-rigid [] T T1, !, (X ==1 T1) H H1.
1880 progress-eta-link (ho.uva Ad _ as T1) T2 H H [eval-link-eta T1 T2] :- !,
1881   if (ho.not_occ Ad H T2) true fail.
1882
1883 type is-in-pf ho.tm -> o.
1884 is-in-pf (ho.app [ho.uva _ _ | _]) :- !, fail.
1885 is-in-pf (ho.lam B) :- !, pi x\ is-in-pf (B x).
1886 is-in-pf (ho.con _).
1887 is-in-pf (ho.app L) :- forall1 is-in-pf L.
1888 is-in-pf N :- name N.
1889 is-in-pf (ho.uva _ L) :- pattern-fragment L.
1890
1891 type arity ho.tm -> nat -> o.
1892 arity (ho.con _) z.
1893 arity (ho.app L) A :- len L A.
1894
1895 type occur-check-err ho.tm -> ho.tm -> ho.subst -> o.
1896 occur-check-err (ho.con _) _ _ :- !.
1897 occur-check-err (ho.app _) _ _ :- !.
1898 occur-check-err (ho.lam _) _ _ :- !.
1899 occur-check-err (ho.uva Ad _) T S :-
1900   not (ho.not_occ Ad S T).
1901
1902 type progress-beta-link-aux ho.tm -> ho.tm ->
1903   ho.subst -> ho.subst -> links -> o.
1904 progress-beta-link-aux T1 T2 S1 S2 [] :- is-in-pf T2, !,
1905   (T1 ==1 T2) S1 S2.
1906 progress-beta-link-aux T1 T2 S S [eval-link-beta T1 T2] :- !.
1907
1908 type progress-beta-link ho.tm -> ho.tm -> ho.subst ->
1909   ho.subst -> links -> o.
1910 progress-beta-link T (ho.app [ho.uva V Scope | L] as T2) S S2 [eval-link-beta T T2] :- !,
1911   arity T Arity, len L ArgsNb, ArgsNb >n Arity, !,
1912   minus ArgsNb Arity Diff, mem.new S V1 S1,
1913   eta-expand (ho.uva V1 Scope) Diff T1,
1914   ((ho.uva V Scope) ==1 T1) S1 S2.
1915
1916 progress-beta-link (ho.uva _ _ as T) (ho.app [ho.uva Ad1 Scope1 | L1] as T2) S1 S2 :- !,
1917   append Scope1 L1 Scope1L,
1918   pattern-fragment-prefix Scope1L Scope2 L2,
1919   not (Scope1 = Scope2), !,
1920   mem.new S1 Ad2 S2,
1921   len Scope1 Scope1Len,
1922   len Scope2 Scope2Len,
1923   make-eta-link Scope1Len Scope2Len Ad1 Ad2 [] LinkEta S2 S3,
1924   if (L2 = []) (NewLinks = LinkEta, T2 = ho.uva Ad2 Scope2)
1925   (T2 = ho.app [ho.uva Ad2 Scope2 | L2],
1926    NewLinks = [eval-link-beta T T2 | LinkEta]).
1927
1928 progress-beta-link T1 (ho.app [ho.uva _ _ | _] as T2) _ _ _ :- !,
1929   progress-beta-link (ho.uva _ _ as T) (ho.app [ho.uva _ _ | _] as T2) S1 S2 :- !,
1930   occur-check-err T T2 S1, !, fail.
1931
1932 progress-beta-link T1 (ho.app [ho.uva _ _ | _] as T2) H H [eval-link-beta T T2] :- !,
1933   progress-beta-link T1 (ho.app [Hd | T1]) S1 S2 B :- !,
1934   ho.lam beta Hd T1 T3,
1935   progress-beta-link-aux T1 T3 S1 S2 B.
1936
1937 type solve-link-abs link -> links -> ho.subst -> ho.subst -> o.
1938 solve-link-abs (ho.abs X) R H H1 :-
1939   pi x\ ho.copy x x => (pi S\ ho.deref S x x) =>
1940     solve-link-abs (X x) (R' x) H H1,
1941     close-links R' R.
1942
1943 solve-link-abs (@eval-link-eta A B) NewLinks S S1 :- !,
1944   progress-eta-link A B S S1 NewLinks.
1945
1946 solve-link-abs (@eval-link-beta A B) NewLinks S S1 :- !,
1947   progress-beta-link A B S S1 NewLinks.
1948
1949 type take-link link -> links -> link -> links -> o.
1950 take-link A [B|XS] B XS :- link-abs-same-lhs A B, !.
1951 take-link A [L|XS] B [L|YS] :- take-link A XS B YS.
1952
1953 type link-abs-same-lhs link -> link -> o.
1954 link-abs-same-lhs (ho.abs F) B :-
1955   pi x\ link-abs-same-lhs (F x) B.
1956 link-abs-same-lhs A (ho.abs G) :-
1957   pi x\ link-abs-same-lhs A (G x).
1958 link-abs-same-lhs (@eval-link-eta (ho.uva N _) _) (@eval-link-eta (ho.uva N _) _) :- !.
1959
1960 type same-link-eta link -> link -> ho.subst -> ho.subst -> o.
1961 same-link-eta (ho.abs F) B H H1 :- !, pi x\ same-link-eta (F x) B H H1.
1962 same-link-eta A (ho.abs G) H H1 :- !, pi x\ same-link-eta A (G x) H H1.
1963 same-link-eta (@eval-link-eta (ho.uva N S1) A) (@eval-link-eta (ho.uva N S2) B) H H1 :- !,
1964   std.map2 S1 S2 (x\y\r\ r = ho.copy x y) Perm,
1965   Perm => ho.copy A A',
1966   (A' ==1 B) H H1.
1967
1968
1969
1970
1971
1972

```

```

1973
1974 type progress1 links -> links -> ho.subst -> ho.subst -> o.
1975 progress1 [] [] X X.
1976 progress1 [A|L1] [A|L3] S S2 :- take-link A L1 B L2, !,
1977     same-link-eta A B S S1,
1978     progress1 L2 L3 S1 S2.
1979 progress1 [L0|L1] L3 S S2 :- deref-link S L0 L,
1980     solve-link-abs L R S S1, !,
1981     progress1 L1 L2 S1 S2, append R L2 L3.
1982

```

## 18 THE DECOMPILER

```

1983
1984
1985 type abs->lam ho.assignment -> ho.tm -> o.
1986 abs->lam (ho.abs T) (ho.lam R) :- !, pi x\ abs->lam (T x) (R x).
1987 abs->lam (ho.val A) A.
1988
1989 type commit-links-aux link -> ho.subst -> ho.subst -> o.
1990 commit-links-aux (@val-link-eta T1 T2) H1 H2 :-
1991     ho.deref H1 T1 T1', ho.deref H1 T2 T2',
1992     (T1' ==1 T2') H1 H2.
1993 commit-links-aux (@val-link-beta T1 T2) H1 H2 :-
1994     ho.deref H1 T1 T1', ho.deref H1 T2 T2',
1995     (T1' ==1 T2') H1 H2.
1996 commit-links-aux (ho.abs B) H H1 :-
1997     pi x\ commit-links-aux (B x) H H1.
1998
1999 type commit-links links -> links -> ho.subst -> ho.subst -> o.
2000 commit-links [] [] H H.
2001 commit-links [Abs | Links] L H H2 :-
2002     commit-links-aux Abs H H1, !, commit-links Links L H1 H2.
2003
2004 type decomp-subst map -> map -> ho.subst ->
2005     fo.fsubst -> fo.fsubst -> o.
2006 decomp-subst _ [A|_] _ _ :- fail.
2007 decomp-subst _ [] _ F F.
2008 decomp-subst Map [mapping (fv V0) (hv VM _)|T1] H F F2 :-
2009     mem.set? VM H T, !,
2010     ho.deref-assmt H T TTT,
2011     abs->lam TTT T', tm->fm Map T' T1,
2012     fo.eta-contract T1 T2, mem.assign V0 F T2 F1,
2013     decomp-subst Map T1 H F1 F2.
2014 decomp-subst Map [mapping _ (hv VM _)|T1] H F F2 :-
2015     mem.unset? VM H, decomp-subst Map T1 H F F2.
2016
2017 type tm->fm map -> ho.tm -> fo.fm -> o.
2018 tm->fm _ (ho.con C) (fo.fcon C).
2019 tm->fm L (ho.lam B1) (fo.flam B2) :-
2020     pi x y\ tm->fm _ x y => tm->fm L (B1 x) (B2 y).
2021 tm->fm L (ho.app L1) T :- map (tm->fm L) L1 [Hd|T1],
2022     fo.mk-app Hd T1 T.
2023 tm->fm L (ho.uva VM TL) T :- mem L (mapping (fv V0) (hv VM _)),
2024     map (tm->fm L) TL T1, fo.mk-app (fo.fuva V0) T1 T.
2025
2026 type add-new-map-aux ho.subst -> list ho.tm -> map ->
2027     map -> fo.fsubst -> fo.fsubst -> o.
2028 add-new-map-aux _ [] _ [] S S.
2029 add-new-map-aux H [T|Ts] L L2 S S2 :-

```

```

2030
2031 add-new-map H T L L1 S S1,
2032 add-new-map-aux H Ts L1 L2 S1 S2.
2033
2034 type add-new-map ho.subst -> ho.tm -> map ->
2035     map -> fo.fsubst -> fo.fsubst -> o.
2036 add-new-map _ (ho.uva N _) Map [] F1 F1 :-
2037     mem Map (mapping _ (hv N _)), !.
2038 add-new-map H (ho.uva N L) Map [Map1 | MapL] F1 F3 :-
2039     mem.new F1 M F2,
2040     len L Arity, Map1 = mapping (fv M) (hv N (arity Arity)),
2041     add-new-map H (ho.app L) [Map1 | MapL] MapL F2 F3.
2042 add-new-map H (ho.lam B) Map NewMap F1 F2 :-
2043     pi x\ add-new-map H (B x) Map NewMap F1 F2.
2044 add-new-map H (ho.app L) Map NewMap F1 F3 :-
2045     add-new-map-aux H L Map NewMap F1 F3.
2046 add-new-map _ (ho.con _) _ [] F F :- !.
2047 add-new-map _ N _ [] F F :- name N.
2048
2049 type complete-mapping-under-ass ho.subst -> ho.assignment ->
2050     map -> map -> fo.fsubst -> fo.fsubst -> o.
2051 complete-mapping-under-ass H (ho.val Val) Map1 Map2 F1 F2 :-
2052     add-new-map H Val Map1 Map2 F1 F2.
2053 complete-mapping-under-ass H (ho.abs Abs) Map1 Map2 F1 F2 :-
2054     pi x\ complete-mapping-under-ass H (Abs x) Map1 Map2 F1 F2.
2055
2056 type complete-mapping ho.subst -> ho.subst ->
2057     map -> map -> fo.fsubst -> fo.fsubst -> o.
2058 complete-mapping _ [] L L F F.
2059 complete-mapping H [none | T1] L1 L2 F1 F2 :-
2060     complete-mapping H T1 L1 L2 F1 F2.
2061 complete-mapping H [some T0 | T1] L1 L3 F1 F3 :-
2062     ho.deref-assmt H T0 T,
2063     complete-mapping-under-ass H T L1 L2 F1 F2,
2064     append L1 L2 LAll,
2065     complete-mapping H T1 LAll L3 F2 F3.
2066
2067 type decompile map -> links -> ho.subst ->
2068     fo.fsubst -> fo.fsubst -> o.
2069 decompile Map1 L H0 F0 F02 :-
2070     commit-links L L1_ H0 H01, !,
2071     complete-mapping H01 H01 Map1 Map2 F0 F01,
2072     decomp-subst Map2 Map2 H01 F01 F02.
2073

```

## 19 AUXILIARY FUNCTIONS

```

2074
2075 type fold4 (A -> A1 -> B -> B -> C -> C -> o) -> list A ->
2076     list A1 -> B -> B -> C -> C -> o.
2077 fold4 _ [] [] A A B B.
2078 fold4 F [X|XS] [Y|YS] A A1 B B1 :- F X Y A A0 B B0,
2079     fold4 F XS YS A0 A1 B0 B1.
2080
2081 type len list A -> nat -> o.
2082 len [] z.
2083 len [_|L] (s X) :- len L X.
2084

```