HO unification from object language to meta language

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ABSTRACT

Specifying and implementing a logic from scratch requires significant effort. Logical Frameworks and Higher Order Logic Programming Languages provide dedicated, high-level Meta Languages (ML) to facilitate this task in two key ways: 1) variable binding and substitution are simplified when ML binders represent object logic ones; 2) proof construction, and even proof search, is greatly simplified by leveraging the unification procedure provided by the ML. Notable examples of ML are Elf [12], Twelf [13], λProlog [9] and Isabelle [19] which have been utilized to implement various formal systems such as First Order Logic [4], Set Theory [11], Higher Order Logic [10], and even the Calculus of Constuctions [3].

The object logic we are interested in is Coq's [17] Dependent Type Theory (DTT), for which we aim to implement a unification procedure $\simeq_o$ using the ML Elpi [2], a dialect of λProlog. Elpi's equational theory comprises $\eta\beta$ equivalence and comes equipped with a higher order unification procedure $\simeq_\lambda$ restricted to the pattern fragment [8]. We want $\simeq_o$ to be as powerful as $\simeq_\lambda$ but on the object logic DTT. Elpi also comes with an encoding for DTT that works well for meta-programming [16, 15, 6, 5]. Unfortunately this encoding, which we refer to as $\mathcal{F}_o$, "underuses" $\simeq_\lambda$ by restricting it to first-order unification problems only. To address this issue, we propose a better-behaved encoding, $\mathcal{H}_o$, demonstrate how to map unification problems in $\mathcal{F}_o$ to related problems in $\mathcal{H}_o$, and illustrate how to map back the unifiers found by $\simeq_\lambda$, effectively implementing $\simeq_o$ on top of $\simeq_\lambda$ for the encoding $\mathcal{F}_o$.

We apply this technique to the implementation of a type-class [18] solver for Coq [17]. Type-class solvers are proof search procedures based on unification that back-chain designated lemmas, providing essential automation to widely used Coq libraries such as Stdpp/Iris [7] and TLC [1]. These two libraries constitute our test bed.

KEYWORDS

Logic Programming, Meta-Programming, Higher-Order Unification, Proof Automation

ACM Reference Format:
Enrico Tassi and Davide Fissore. XXXX 2024. HO unification from object language to meta language. In *YYY*. ACM, New York, NY, USA, 7 pages. https://doi.org/ZZZZZZZZZZZZ

Conference'17, July 2017, Washington, DC, USA

ACM ISBN 978-x-xxxx-xxxx-x/YY/MM
https://doi.org/ZZZZZZZZZZZZ

1 INTRODUCTION

Specifying and implementing a logic from scratch requires significant effort. Logical Frameworks and Higher Order Logic Programming Languages provide dedicated, high-level Meta Languages (ML) to facilitate this task in two key ways: 1) variable binding and substitution are simplified when ML binders represent object logic ones; 2) proof construction, and even proof search, is greatly simplified by leveraging the unification procedure provided by the ML. Notable examples of ML are Elf [12], Twelf [13], λProlog [9] and Isabelle [19] which have been utilized to implement various formal systems such as First Order Logic [4], Set Theory [11], Higher Order Logic [10], and even the Calculus of Constuctions [3].

The object logic we are interested in is Coq's [17] Dependent Type Theory (DTT), and we want to code a type-class [18] solver for Coq [17] using the Coq-Elpi [16] meta programming framework. Type-class solvers are unification based proof search procedures that combine a set of designated lemmas in order to providing essential automation to widely used Coq libraries.

As the running example we take the `Decide` type class, from the Stdpp [7] library. The class identifies predicates equipped with a decision procedure. The following three designated lemmas (called `Instances` in the type-class jargon) state that: 1) the type `fin n`, of natural numbers smaller than `n` is finite; 2) the predicate `nfact n nf`, linking a natural number `n` to its prime factors `nf`, is decidable; 3) the universal closure of a predicate has a decision procedure if the predicate has and if its domain is finite.

```
Instance fin_fin n : Finite (fin n).                    (* r1 *)
Instance nfact_dec n nf : Decision (nfact n nf).        (* r2 *)
Instance forall_dec A P : Finite A →                    (* r3 *)
  ∀x:A, Decision (P x) → Decision (∀x:A, P x).
```

Under this context of instances a type-class solver is able to prove the following statement automatically by back-chaining.

```
Check _ : Decision (forall y: fin 7, nfact y 3).        (g)
```

The encoding of DTT provided by Elpi, that we will discuss at length later in section **??** and **??**, is an Higher Order Abstract Syntax (HOAS) datatype `tm` featuring (among others) the following constructors:

```
type lam  tm -> (tm -> tm) -> tm.     % lambda abstraction
type app  list tm -> tm.              % n-ary application
type all  tm -> (tm -> tm) -> tm.     % forall quantifier
type con  string -> tm.               % constants
```

Following standard λProlog [9] the concrete syntax to abstract, at the meta level, an expression `e` over a variable `x` is «`x\ e`», and square brackets denote a list of terms separated by comma. As an example we show the encoding of the Coq term «`∀y:t, nfact y 3`»:

```
all (con"t") y\ app[con"nfact", y, con"3"]
```

We now illustrate the encoding of the three instances above as higher-order logic-programming rules: capital letters denote rule parameters; `:-` separates the rule's head from the premises; `pi w\ p` introduces a fresh nominal constant `w` for the premise `p`.

```
finite (app[con"fin", N]).                                    (r1)
decision (app [con"nfact", N, NF]).                           (r2)
decision (all A x\ app[P, x]) :- finite A,                    (r3)
  pi w\ decision (app[P, w]).
```

Unfortunately this translation of rule ($r3$) uses the predicate `P` as a first order term for the meta language (its type is `tm`). If we try to backchain the rule ($r3$) on the encoding of the goal (g):

```
decision (all (app[con"fin", con"7"]) y\
  app[con"nfact", y, con"3"]).
```

we obtain an unsolvable unification problem (p): the two lists of terms have different lengths!

```
app[con"nfact", y, con"3"] = app[P, y]                        (p)
```

In this paper we study a more sophisticated encoding of Coq terms allowing us to rephrase the problematic rule ($r3$) as follows:

```
decision (all A x\ Pm x) :- link Pm P A, finite A,    (r3a)
  pi x\ decision (app[P, x]).
```

Since `Pm` is an higher-order unification variable with `x` in its scope, the unification problem (p') admits one solution:

```
app[con"nfact", y, con"3"] = Pm y                             (p')
Pm = x\ app[con"nfact", x, con"3"]     % assignment for Pm
A = app[con"fin", con"7"]              % assignment for A
```

After unifying the head of rule ($r3a$) with the goal, Elpi runs the premise «`link Pm A P`» that is in charge of bringing the assignment for `Pm` (that has type `tm -> tm`) back to the domain of Coq terms:

```
P = lam A a\ app[con"nfact", a, con"3"]
```

This simple example is sufficient to show that the encoding we seek is not trivial and does not only concern the head of rules, but the entire sequence of unification problems that constitute the execution of a logic program. In fact the solution for `P` above generates a (Coq) β-redex in the second premise (the predicate under the `pi w\`):

```
decision (app[lam A (a\ app[con"nfact", a, con"3"]), w])
```

In turn this redex prevents the rule ($r2$) to backchain properly since the following unification problem has no solution:

```
app[lam A (a\ app[con"nfact", a, con"3"]), x] =
app[con"nfact", N, NF]
```

The root cause of the problems we sketched in the running example is that the unification procedure $\simeq_\lambda$ of the meta language is not aware of the equational theory of the object logic, even if both theories include $\eta\beta$-conversion and admit most general unifiers for unification problems in the pattern fragment $\mathcal{L}_\lambda$ [8].

Contributions. In this paper we discuss alternative encodings of Coq in Elpi 2, then we identify a minimal language $\mathcal{F}_o$in which the problems sketched here can be fully described. We then detail an encoding `comp` from $\mathcal{F}_o$ to $\mathcal{H}_o$ (the language of the meta language) and a decoding `decomp` to relate the unifiers bla bla.. TODO citare Teyjus.

2 PROBLEM STATEMENT

The equational theory of Coq's Dependent Type Theory is very rich. In addition to the usual $\eta\beta$-equivalence for functions, terms (hence types) are compared up to definition unfolding and fixpoint unrolling. Still, for efficiency and predictability reasons, most form of automatic proof search employ a unification procedure that captures a simpler one, just $\eta\beta$, and that solves higher-order problems restricted to the pattern fragment $\mathcal{L}_\lambda$ [8]. We call this unification procedure $\simeq_o$.

The equational theory of the meta language Elpi that we want to use to implement a form of proof automation is strikingly similar, since it it comprises $\eta\beta$ (for the meta language functions), and the unification procedure $\simeq_\lambda$ solves higher-order problems in $\mathcal{L}_\lambda$.

In spite of the similarity the link between $\simeq_\lambda$ and $\simeq_o$ is not trivial, since the abstraction and application term constructors the two unification procedures deal with are different. For example

```
x\ f x                          ≃λ   f
lam A x\ app[con"f", x]         ≃o   con"f"
lam A x\ app[con"f", x]         ≄λ   con"f"
P x                             ≃λ   x
app[P, x]                       ≃o   x
app[P, x]                       ≄λ   x
```

One could ignore this similarity, and "just" describe the object language unification procedure in the meta language, that is crafting a `unif` predicate to be used as follows in rule ($r3$):

```
decision X :- unif X (all A x\ app[P, x]), finite A,
  pi x\ decision (app[P, x]).
```

This choice would underuse the logic programming engine provided by the metalanguage since by removing any datum from the head of rules indexing degenerates. Moreover the unification procedure built in the meta language is likely to be faster than one implemented in it, especially if the meta language is interpreted as Elpi is.

To state precisely the problem we solve we need a $\mathcal{F}_o$ representation of DTT terms and a $\mathcal{H}_o$ one. We call $=_o$ the equality over ground terms in $\mathcal{F}_o$, $=_\lambda$ the equality over ground terms in $\mathcal{H}_o$, $\simeq_o$ the unification procedure we want to implement and $\simeq_\lambda$ the one provided by the meta language. TODO extend $=_o$ and $=_\lambda$ with reflexivity on uvars.

We write $t_1 \simeq_\lambda t_2 \mapsto \sigma$ when t_1 and t_2 unify with substitution σ, we write σt for the application of the substitution to t, $\sigma \subseteq \sigma'$ when σ is more general than σ', and we assume that the unification of our meta language is correct:

$$t_i \in \mathcal{L}_\lambda \Rightarrow t_1 \simeq_\lambda t_2 \mapsto \rho \Rightarrow \rho t_1 =_\lambda \rho t_2 \quad (1)$$

$$t_i \in \mathcal{L}_\lambda \Rightarrow \rho t_1 =_\lambda \rho t_2 \Rightarrow \exists \rho', t_1 \simeq_\lambda t_2 \mapsto \rho' \wedge \rho' \subseteq \rho \quad (2)$$

We illustrate a compilation $\langle s \rangle \mapsto (t, l)$ that maps a term s in $\mathcal{F}_o$ to a term t in $\mathcal{H}_o$ and a list of links l. The links connect unification variables in $\mathcal{H}_o$ with variables in $\mathcal{F}_o$ and are used to decompile the assignment, $\langle \sigma, l \rangle^{-1} \mapsto \rho$.

Given

$$\langle s_1 \rangle \mapsto (t_1, l_1) \wedge \langle s_2 \rangle \mapsto (t_2, l_2)$$

we define

$$s_1 \simeq_o s_2 \mapsto \rho \stackrel{def}{=} t_1 \simeq_\lambda t_2 \mapsto \sigma \wedge \langle \sigma, l_1 + l_2 \rangle^{-1} \mapsto \rho$$

Where $l_1 + l_2$ is the list concatenation of links.

We write $s \in \mathcal{L}_\lambda$ if all unif variables in s are applied to distinct bound variables.

$$s_i \in \mathcal{L}_\lambda \Rightarrow s_1 \simeq_o s_2 \mapsto \rho \Rightarrow \rho s_1 =_o \rho s_2 \quad (3)$$

$$s_i \in \mathcal{L}_\lambda \Rightarrow \rho s_1 =_o \rho s_2 \Rightarrow \exists \rho', s_1 \simeq_o s_2 \mapsto \rho' \wedge \rho' \subseteq \rho \quad (4)$$

$$\rho s_1 =_o \rho s_2 \Rightarrow \rho' \subseteq \rho \Rightarrow \rho' s_i \in \mathcal{L}_\lambda \Rightarrow \rho' s_1 \simeq_o \rho' s_2 \quad (5)$$

Properties 3 and 4 state, respectively, that in $\mathcal{L}_\lambda$ the implementation of $\simeq_o$ is correct, complete and returns the most general unifier.

Property 5 states that $\simeq_o$ is resilient to problems outside $\mathcal{L}_\lambda$ if a third party provides a (partial) solution for its problem. Since we are interested in using $\simeq_o$ in a proof search procedure, made of a sequence of unification problems, not necessarily in $\mathcal{L}_\lambda$. In practice it is often the case the order in which these problems are stated matters. A Typical example is the following problem

```
app [F, con"a"] = app[F, con"a", con"a"]
```

preceded by

```
F = lam x\app[const f,x,x]
```

becomes solvable in DTT trivially, since the term is ground (hence in $\mathcal{L}_\lambda$), but is one substitutes F in the LHS does not find, structurally, the RHS hence $\simeq_\lambda$ would fail (since $\simeq_\lambda$ does not know about the β rule of DTT). Our compiler takes care of making property 5 hold, see section XXX.

Property 4 is also relevant to use $\simeq_o$ for logic programming. In particular we want failures to occur as early as possible, so want the decomp phase to take place immediately after $\simeq_\lambda$, and fail if need be. This becomes particularly important since compile may introduce two ho variables for the same fo one, leaving the task of unifying the solutions to decomp.

3 ALTERNATIVE ENCODINGS AND RELATED WORK

Our encoding of DTT may look "semi shallow" since we we use the meta language lambda abstraction but not its application (for the terms of type `tm`). A fully shallow encoding unfortunately does not fit our use case, although it would make the running example work:

```
finite (fin N).
decision (nfact N NF).
decision (all A x\ P x) :- finite A, pi x\ decision (P x).
```

There are two reasons for dismissing this encoding. The first one is that in DTT it is not always possible to adopt it since the type system of the meta language is too weak to accommodate terms with a variable arity, like the following example:

```
Fixpoint arr T n := if n is S m then T -> arr T m else T.
Definition sum n : arr nat n := ...
Check sum 2   7 8   : nat.
Check sum 3   7 8 9 : nat.
```

The second reason is the encoding for Coq is used for meta programming the system, hence it must accommodate the manipulation of terms that are now know in advance (not even defined in Coq) without using introspection primitives such as Prologs's `functor` and `arg`.

In the literature we could find a few related encoding of DTT. TODO In [3] is related and make the discrepancy between the types of ML and DTT visible. In this case one needs 4 application nodes. Moreover the objective is an encoding of terms, proofs, not proof search. Also note the conv predicate, akin to the unif we rule out.

TODO This other paper [14] should also be cited.

None of the encodings above provide a solution to our problem.

4 LANGUAGES DESCRIPTION

In order to reason about unification of the terms of the objet language within the meta language, we start by formally describing the two languages. Employing meta-programming for this purpose, we present in fig. 1 a new type for the terms of the OL.

```
kind tm type.
type app list tm -> tm.
type lam (tm -> tm) -> tm.
type con string -> tm.
```

Figure 1: Common terms

This encoding is very similar to the one introduced in section 1, except for the `all` constructor. We explicitly neglect it since blabla. Moreover, since we are working with unification variables we need to introduce a new constructor.

say we do not care about types

In the case of the OL, variables have no scope. For example, the subterm `P x` from the instance forall_dec has `P` an higher order variables with type `A → Prop` and `x` as bound variable. However, at the meta level, the translation of `P x` becomes `app[P, x]`, that is, `P` cannot reference `x`. In our encoding, we represent the variables of the OL in the following way:

```
type fo_uv nat -> tm.
```

In particular, a variable of the OL is identified by a integer refereeing a memory address. This constructor is prefixed with `fo` since the variable, having no scope, is a first-order variable. Finally, the memory of the OL is depicted by the following type abbreviation:

```
typeabbrev fo_subst list (option tm).
```

which is a list of optional terms. If a cell is `none`, then the variable corresponding to this cell is not instantiated.

On the other hand, the terms of the ML have exactly the same shape has the terms of fig. 1. However, since the ML is an Higher Order Programming Language, unification unification variables must have a scope. We represent them in the following way:

```
type ho_uv nat -> list tm -> tm.
```

Of course some attention should be payed when dealing with `ho_uv`, since we have to certify that an `uv i` remain in the pattern fragment, that is, the list of term in the scope of `i` is always a list of distinct names.

Another important implementation should be provided in order to represent the lambda abstractions and the applications of the ML. As outlined in section 2, «`x\ f x`» has not the same semantic as «`lam x\ app[f, x]`», since, even though, the two terms represent the same concept, the former is the lambda abstraction of `f`

applied to the binder in ML, whereas the latter is the same term representation but wrt the OL. We already have defined the nodes app and lam for the OL, therefore, in the following code snippet, we give their representation in the ML.

```
kind assmt type.
type abs (tm -> assmt) -> assmt.
type val tm -> assmt.
```

In particular, the node abs stands for the lambda abstraction. The node val contains terms of the object language. The node for variable application is not really necessary, since, using the syntax of the ML, it is defined by putting two terms of the ML side by side. For example, the terms «x\ f x» becomes «abs x\ f x».

The memory of the ML, is defined with the following type abbreviation:

```
typeabbrev ho_subst list (option tm).
```

with the invariant that the tm inside the cell never contains the node fo_uv. Note that, reciprocally, the fo_subst does not contain any node of the form ho_uv.

4.1 Auxiliary relations

Together with the description of the terms of the language, we need some auxiliary functions to perform operations like term equality, unification, dereferencing. There predicates are supposed to be implemented in the OL and the ML following respectively their specification. In the following few paragraphs we give a brief definition of them.

Term dereferencing: ρs vs. σt. Since in our encoding we explicitly drag substitution mappings, we need to dereference variables of the two languages. This is particularly useful to check if two terms in the OL (resp. in the ML) are equal. The constructors representing rigid terms, namely app, lam and con, make the dereferencing procedure to recurse over the their subterms. The code below display this behavior for the dereferencing performed by the OL:

```
type fo_deref fo_subst -> tm -> tm -> o.
fo_deref S (app L1) (app L2) :- forall2 (fo_deref S) L1 L2.
fo_deref S (lam F1) (lam F2) :-
  pi x\ fo_deref S x x => fo_deref S (F1 x) (F2 x).
fo_deref _ (con X) (con X).
```

A very similar implementation for the dereferencing performed by the ML, given in the appendix, can be encoded respecting the signature:

```
type ho_deref ho_subst -> tm -> tm -> o.
```

More attention should be payed when the dereferencing encounters a subterm that is unification variable. The following new rules should be added to the previous code snipper to complete the dereferencing of variables of the OL:

```
fo_deref S (fov N) R :- assigned? N S T, fo_deref S T R.
fo_deref S (app [fov N|L]) R :-
  assigned? N S T, beta T L R', fo_deref S R' R.
fo_deref _ (fov N) (fov N).
```

In particular, 1) if the variable fov N is assigned to a term T in a substitution mapping S, we return the result of dereferencing T; 2) if the current term is the application of a variable fov N assigned to a list of terms L and uv N is assigned to a term T, we return the result of dereferencing the beta reduction of T to L. This is due to the semantics of the unification variables within the OL; 3) finally, if uv N is not assigned, then the same variable is returned.

```
type (=o) tm -> tm -> o.
% congruence
app L1 =o app L2 :- forall2 (=o) L1 L2.
lam F1 =o lam F2 :- pi x\ x =o x => F1 x =o F2 x.
con X =o con X.
fov N =o fov N.
% eta
lam F =o T :- pi x\ beta T [x] (T' x),                    (η1)
  x =o x => F x =o T' x.
T =o lam F :- pi x\ beta T [x] (T' x),                    (η2)
  x =o x => T' x =o F x.
% beta
app [lam X | TL] =o T :- beta (lam X) TL T', T' =o T.     (β1)
T =o app [lam X | TL] :- beta (lam X) TL T', T =o T'.     (β2)
```

Figure 2: Equal predicate

A different reasoning is to be addressed to the variables of the ML. Firstly, a meta variable cannot appear in the app node as the first element of the list, we will explain why in section 5; secondly, variables in the ML have a scope which should be taken into account. The following few lines of code outline the rules of dereferencing for the ML:

```
ho_deref H (hov N L) R :- assigned? N H T,
  ho_beta T L R', ho_deref H R' R.
ho_deref H (hov X L) (hov X L1) :-
  forall2 (ho_deref H) L L1.
```

where, 1) if the variable hov N L has an assignment T, we return the dereferencing of the beta reduction T to the scope L.

Term equality: $=_o$ vs. $=_\lambda$. We can test if two terms are equal following the equational theory of the language being considered. In fig. 2 we provide an implementation of the $=_o$ predicate. The first four rules check if the two terms are equal regarding the structure of the current node, that is, two terms are equal if they have same head and if recursively each subterm is two by two equal. Moreover, since the theory of the OL accepts η- and β-equivalence, then we implement the corresponding rules.
The equality relation for the ML, accepts $\eta\beta$-equivalence between terms of the ML. Recall that abs x\ f x, is a valid η expansion of the function f and that lam x\ app[f, x] is not that equivalent to f at meta level. However, since we are interested in using the unification procure of the ML, by eq. (1), we can use the $\simeq_\lambda$ relation to test, when needed if two terms are equal in the ML.

Term unification: $\simeq_o$ vs. $\simeq_\lambda$. The last but not least important relation we should take care of before presenting our full algorithm aiming to unify terms of the OL in the ML and provide the substitution produced in the ML to the OL, is term unification. This procedure is a more powerful version of the equal predicate, since unification checks if two terms can be equal be assigning unification variables. In our representation, variable assignment (or refinement) is performed by modifying the corresponding substitution mapping.

```
type unif tm -> tm -> ho_subst -> ho_subst ->  o.
% Base cases
unif (app L1) (app L2) S S1 :- fold2 unif L1 L2 S S1.
unif (lam F1) (lam F2) S S1 :-
  pi x\ (pi S\ unif x x S S) => unif (F1 x) (F2 x) S S1.
unif (con X) (con X) S S.
% Assigned variables
unif (hov N Args) T S S1 :-
  assigned-deref N S F, beta-abs F Args T1, unif T1 T S S1.
unif T (hov N Args) S S1 :-
  assigned-deref N S F, beta-abs F Args T1, unif T T1 S S1.
% Non-assigned variabless
unif (hov N Args1) (hov M Args2) S S3 :-
  pattern-fragment Args1, pattern-fragment Args2,
  new S W S1, prune W Args1 Args2 Ass,
  assign N S1 Ass S2, assign M S2 Ass S3.
unif (hov N Args) T S S1 :- pattern-fragment Args,
  not (T = hov _ _), not_occ N S T,
  bind Args T T1, assign N S T1 S1.
unif T (hov N Args) S S1 :- pattern-fragment Args,
  not (T = hov _ _), not_occ N S T,
   bind Args T T1, assign N S T1 S1.
```

Figure 3: The $\simeq_\lambda$ predicate

We will not give an implementation of $\simeq_o$, since we are giving an implementation of it using our algorithm, see **??**.

On the other hand, unification in the ML needs to be defined. In fig. 3, we give an implementation of $\simeq_\lambda$. This predicate has four arguments, the two terms t_1 and t_2 to unify, the old substitution map ρ_1, and the new substitution map ρ_2, with the invariant $\rho_1 \subseteq \rho_2$. The first three rules unify terms with same rigid heads, and call the unification relation on the sub-terms. If t_1 (resp. t_2) is an assigned variables, t_1 is dereferenced to t_1' (resp. t_2') and the unification is called between t_1' and t_2 (resp. t_1 and t_2'). If both terms are unification variables, we test that their arguments are in the pattern fragment, we allocate a new variable w in ρ_1 such that w is the pruning of the arguments of t_1 and t_2, we assign both t_1 and t_2 to w and return the new mapping ρ_2 containing all the new variable assignment. Finally, if only one of the two terms is an unification variable v, after having verified that v does not occur in the other term t, we bind v to t and return the new substitution mapping.

OLD

A key property needed in unification is being able to verify if two terms are equal wrt a given equational theory. This relation allow to compare terms under a certain substitution mapping, so that any time a variable v is assigned in a subterm, a dereferencing of v is performed. After variable dereferencing, the test for equality is continued on the new-created subterm.

The base equality function over terms can be defined as follows:

The solution we are proposing aim to overcome these unification issues by 1) compiling the terms t and u of the OL into an internal version t' and u' in the ML; 2) unifying t' and u' at the meta level instantiating meta variables; 3) decompiling the meta variable into terms of the OL; 4) assigning the variables of the OL with the decompiled version of their corresponding meta variables. We claim that t and u unify if and only if t' and u' unify and that the substitution in the object language is the same as the one returned by the ML.

same or $\supseteq$ or $\subseteq$

In the following section we explain how we deal with term (de)compilation and links between unification variables.

5 COMPILATION

TODO: Goal: $s_1 \simeq_o s_2$ is compiled into $t_1 \simeq_\lambda t_2$
TODO: What is done: uvars `fo_uv` of OL are replaced into uvars `ho_uv` of the ML
TODO: Each `fo_uv` is linked to an `ho_uv` of the OL
TODO: Example: `lam x\ app[uv 0, x]` $\simeq_\lambda$ `lam x\ c"f"`
TODO: Links used to instantiate vars of elpi
TODO: After all links, the solution in links are compacted and given to coq
TODO: It is not so simple, see next sections (multi-vars, eta, beta)

The compilation step is meant to recover the higher-order variables of the OL, expressed in a first order way, by replacing them with higher-order variables in the ML. In particular, every time a variable of the OL is encountered in the original term, it is replaced with a meta variable, and if the OL variable is applied to a list of distinct names L, then this list becomes the scope of the variable. For all the other constructors of `tm`, the same term constructor is returned and its arguments are recursively compiled. The predicate in charge for term compilation is:

```
type comp tm -> tm -> links -> links -> subst -> subst -> o.
```

where, we take the term of the OL, produce the term of the ML, take a list of link and produce a list of new links, take a substitution and return a new substitution.

In particular, due to programming constraints, we need to drag the old subst and return a new one extended, if needed, with the new declared meta-variables.

The following code

```
kind link type.
type link nat -> nat -> nat -> subst.
```

defines a link, which is a relation between to variables indexes, the first being the index of a OL variable and the second being the index of a ML variable. The third integer is the number of term in the scope of the two variables, or equivalently, in a typed language, their arity.

integer or nat?

As an example, let's study the following unification problem (a slightly modified version from section 1):

```
lam x\ app[c"decision", app[c"nfact", x, c"3"]] ≃o
  lam x\ app [c"decision", app[uv 0, x]]
```

we have the main unification problem where the nested app nodes have lists of different lengths making the unification to fail. The compilation of these terms produces a new unification problem with the following shape:

```
lam x\ app[c"decision", app[c"nfact", x, c"3"]] ≃λ
  lam x\ app [c"decision", uv 1 [x]]
```

The main difference is the replacement of the subterm `app[uv 0, x]` of the OL with the subterm `uv 0 [x]`. Variable indexes are chosen by the ML, that is, the index `0` for that unification variable of the OL term has not the sam meaning of the index `0` in the ML. There exists

two different substitution mapping, one for the OL and one for the ML and the indexes of variable point to the respective substitution.

6 USE OF MULTIVARS

6.1 Problems with η

TODO: The following goal: `X = lam x\ lam y\ Y x y, X = lam x\ f`
TODO: The snd unif pb, we have to unif `lam x\ lam y\ Y x y` **with** `lam x\ f`
TODO: It is not doable, with the same elpi var

6.2 Problems with β

TODO: The following goal: `X = lam x\ x, app[X, 3] = 3`
TODO: We use links-beta

7 FIRST ORDER APPROXIMATION

TODO: Coq can solve this: `f 1 2 = X 2`**, by setting X to f 1**
TODO: We can re-use part of the algo for β given before

8 UNIF ENCODING IN REAL LIFE

TODO: Il ML presentato qui è esattamente elpi
TODO: Il OL presentato qui è esattamente coq
TODO: Come implementatiamo tutto ciò nel solver

9 RESULTS: STDPP AND TLC

TODO: How may rule are we solving?
TODO: Can we do some perf test

10 CONCLUSION

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