HO unification from object language to meta language

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ABSTRACT

Specifying and implementing a logic from scratch requires significant effort. Logical Frameworks and Higher Order Logic Programming Languages provide dedicated, high-level Meta Languages (ML) to facilitate this task in two key ways: 1) variable binding and substitution are simplified when ML binders represent object logic ones; 2) proof construction, and even proof search, is greatly simplified by leveraging the unification procedure provided by the ML. Notable examples of ML are Elf [13], Twelf [14], λProlog [10] and Isabelle [20] which have been utilized to implement various formal systems such as First Order Logic [5], Set Theory [12], Higher Order Logic [11], and even the Calculus of Constuctions [4].

The object logic we are interested in is Coq's [18] Dependent Type Theory (DTT), for which we aim to implement a unification procedure $\simeq_o$ using the ML Elpi [3], a dialect of λProlog. Elpi's equational theory comprises $\eta\beta$ equivalence and comes equipped with a higher order unification procedure $\simeq_\lambda$ restricted to the pattern fragment [9]. We want $\simeq_o$ to be as powerful as $\simeq_\lambda$ but on the object logic DTT. Elpi also comes with an encoding for DTT that works well for meta-programming [17, 16, 7, 6]. Unfortunately this encoding, which we refer to as $\mathcal{F}_o$, "underuses" $\simeq_\lambda$ by restricting it to first-order unification problems only. To address this issue, we propose a better-behaved encoding, $\mathcal{H}_o$, demonstrate how to map unification problems in $\mathcal{F}_o$ to related problems in $\mathcal{H}_o$, and illustrate how to map back the unifiers found by $\simeq_\lambda$, effectively implementing $\simeq_o$ on top of $\simeq_\lambda$ for the encoding $\mathcal{F}_o$.

We apply this technique to the implementation of a type-class [19] solver for Coq [18]. Type-class solvers are proof search procedures based on unification that back-chain designated lemmas, providing essential automation to widely used Coq libraries such as Stdpp/Iris [8] and TLC [1]. These two libraries constitute our test bed.

KEYWORDS

Logic Programming, Meta-Programming, Higher-Order Unification, Proof Automation

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1 INTRODUCTION

Specifying and implementing a logic from scratch requires significant effort. Logical Frameworks and Higher Order Logic Programming Languages provide dedicated, high-level Meta Languages (ML) to facilitate this task in two key ways: 1) variable binding and substitution are simplified when ML binders represent object logic ones; 2) proof construction, and even proof search, is greatly simplified by leveraging the unification procedure provided by the ML. Notable examples of ML are Elf [13], Twelf [14], λProlog [10] and Isabelle [20] which have been utilized to implement various formal systems such as First Order Logic [5], Set Theory [12], Higher Order Logic [11], and even the Calculus of Constuctions [4].

The object logic we are interested in is Coq's [18] Dependent Type Theory (DTT), and we want to code a type-class [19] solver for Coq [18] using the Coq-Elpi [17] meta programming framework. Type-class solvers are unification based proof search procedures that combine a set of designated lemmas in order to providing essential automation to widely used Coq libraries.

As the running example we take the `Decide` type class, from the Stdpp [8] library. The class identifies predicates equipped with a decision procedure. The following three designated lemmas (called `Instances` in the type-class jargon) state that: 1) the type `fin n`, of natural numbers smaller than `n` is finite; 2) the predicate `nfact n nf`, linking a natural number `n` to its prime factors `nf`, is decidable; 3) the universal closure of a predicate has a decision procedure if the predicate has and if its domain is finite.

```
Instance fin_fin n : Finite (fin n).                (* r1 *)
Instance nfact_dec n nf : Decision (nfact n nf).    (* r2 *)
Instance forall_dec A P : Finite A →                (* r3 *)
  ∀x:A, Decision (P x) → Decision (∀x:A, P x).
```

Under this context of instances a type-class solver is able to prove the following statement automatically by back-chaining.

```
Check _ : Decision (forall y: fin 7, nfact y 3).        (g)
```

The encoding of DTT provided by Elpi, that we will discuss at length later in section **??** and **??**, is an Higher Order Abstract Syntax (HOAS) datatype `tm` featuring (among others) the following constructors:

```
type lam  tm -> (tm -> tm) -> tm.     % lambda abstraction
type app  list tm -> tm.              % n-ary application
type all  tm -> (tm -> tm) -> tm.     % forall quantifier
type con  string -> tm.               % constants
```

Following standard λProlog [10] the concrete syntax to abstract, at the meta level, an expression `e` over a variable `x` is «`x\ e`», and square brackets denote a list of terms separated by comma. As an example we show the encoding of the Coq term «`∀y:t, nfact y 3`»:

```
all (con"t") y\ app[con"nfact", y, con"3"]
```

We now illustrate the encoding of the three instances above as higher-order logic-programming rules: capital letters denote rule parameters; `:-` separates the rule's head from the premises; `pi w\ p` introduces a fresh nominal constant `w` for the premise `p`.

```
finite (app[con"fin", N]).                                   (r1)
decision (app [con"nfact", N, NF]).                          (r2)
decision (all A x\ app[P, x]) :- finite A,                   (r3)
  pi w\ decision (app[P, w]).
```

Unfortunately this translation of rule $(r3)$ uses the predicate `P` as a first order term: for the meta language its type is `tm`. If we try to backchain the rule $(r3)$ on the encoding of the goal (g) given below

```
decision (all (app[con"fin", con"7"]) y\
  app[con"nfact", y, con"3"]).
```

we obtain an unsolvable unification problem (p): the two lists of terms have different lengths!

```
app[con"nfact", y, con"3"] = app[P, y]                       (p)
```

In this paper we study a more sophisticated encoding of Coq terms allowing us to rephrase the problematic rule $(r3)$ as follows:

```
decision (all A x\ Pm x) :- decomp Pm P A, finite A,    (r3a)
  pi x\ decision (app[P, x]).
```

Since `Pm` is an higher-order unification variable of type `tm -> tm`, with `x` in its scope, the unification problem (p') admits one solution:

```
app[con"nfact", y, con"3"] = Pm y                            (p')
Pm = x\ app[con"nfact", x, con"3"]       % assignment for Pm
A = app[con"fin", con"7"]                % assignment for A
```

After unifying the head of rule $(r3a)$ with the goal, Elpi runs the premise «`decomp Pm A P`» that is in charge of bringing the assignment for `Pm` back to the domain `tm` of Coq terms:

```
P = lam A a\ app[con"nfact", a, con"3"]
```

This simple example is sufficient to show that the encoding we seek is not trivial and does not only concern the head of rules, but the entire sequence of unification problems that constitute the execution of a logic program. In fact the solution for `P` above generates a (Coq) β-redex in the second premise (the predicate under the `pi w\`):

```
decision (app[lam A (a\ app[con"nfact", a, con"3"]), w])
```

In turn this redex prevents the rule $(r2)$ to backchain properly since the following unification problem has no solution:

```
app[lam A (a\ app[con"nfact", a, con"3"]), x] =
app[con"nfact", N, NF]
```

The root cause of the problems we sketched in the running example is that the unification procedure $\simeq_\lambda$ of the meta language is not aware of the equational theory of the object logic, even if both theories include $\eta\beta$-conversion and admit most general unifiers for unification problems in the pattern fragment $\mathcal{L}_\lambda$ [9].

Contributions. In this paper we discuss alternative encodings of Coq in Elpi (Section **??**), then we identify a minimal language $\mathcal{F}_o$ in which the problems sketched here can be fully described. We then detail an encoding `comp` from $\mathcal{F}_o$ to $\mathcal{H}_o$ (the language of the meta language) and a decoding `decomp` to relate the unifiers bla bla.. TODO citare Teyjus. The code discussed in the paper can be accessed at the URL: https://github.com/FissoreD/paper-ho.

2 PROBLEM STATEMENT

The equational theory of Coq's Dependent Type Theory is very rich. In addition to the usual $\eta\beta$-equivalence for functions, terms (hence types) are compared up to proposition unfolding and fixpoint unrolling. Still, for efficiency and predictability reasons, most form of automatic proof search employ a unification procedure that captures a simpler one, just $\eta\beta$, and that solves higher-order problems restricted to the pattern fragment $\mathcal{L}_\lambda$ [9]. We call this unification procedure $\simeq_o$.

The equational theory of the meta language Elpi is strikingly similar, since it it comprises $\eta\beta$ (for the meta language functions), and the unification procedure $\simeq_\lambda$ solves higher-order problems in $\mathcal{L}_\lambda$.

In spite of the similarity the link between $\simeq_\lambda$ and $\simeq_o$ is not trivial, since the abstraction and application term constructors the two unification procedures deal with are different. For example

`x\ f x`	$\simeq_\lambda$	`f`
`lam A x\ app[con"f", x]`	$\simeq_o$	`con"f"`
`lam A x\ app[con"f", x]`	$\not\simeq_\lambda$	`con"f"`
`P x`	$\simeq_\lambda$	`x`
`app[P, x]`	$\simeq_o$	`x`
`app[P, x]`	$\not\simeq_\lambda$	`x`

One could ignore this similarity, and "just" describe the object language unification procedure in the meta language, that is crafting a `unif` predicate to be used as follows in rule $(r3)$:

```
decision X :- unif X (all A x\ app[P, x]), finite A,
  pi x\ decision (app[P, x]).
```

This choice would underuse the logic programming engine provided by the metalanguage since by removing any datum from the head of rules indexing degenerates. Moreover the unification procedure built in the meta language is likely to be faster than one implemented in it, especially if the meta language is interpreted as Elpi is.

To state precisely the problem we solve we need a $\mathcal{F}_o$ representation of DTT terms and a $\mathcal{H}_o$ one. We call $=_o$ the equality over ground terms in $\mathcal{F}_o$, $=_\lambda$ the equality over ground terms in $\mathcal{H}_o$, $\simeq_o$ the unification procedure we want to implement and $\simeq_\lambda$ the one provided by the meta language. TODO extend $=_o$ and $=_\lambda$ with reflexivity on uvars.

We write $t_1 \simeq_\lambda t_2 \mapsto \sigma$ when t_1 and t_2 unify with substitution σ; we write σt for the application of the substitution to t, and $\sigma X = \{\sigma t | t \in X\}$ when X is a set; we write $\sigma \subseteq \sigma'$ when σ is more general than σ'. We assume that the unification of our meta language is correct:

$$t_i \in \mathcal{L}_\lambda \Rightarrow t_1 \simeq_\lambda t_2 \mapsto \rho \Rightarrow \rho t_1 =_\lambda \rho t_2 \quad (1)$$

$$t_i \in \mathcal{L}_\lambda \Rightarrow \rho t_1 =_\lambda \rho t_2 \Rightarrow \exists \rho', t_1 \simeq_\lambda t_2 \mapsto \rho' \wedge \rho' \subseteq \rho \quad (2)$$

We illustrate a compilation $\langle s \rangle \mapsto (t, m, l)$ that maps a term s in $\mathcal{F}_o$ to a term t in $\mathcal{H}_o$, a variable mapping m and list of links l. The variable map connects unification variables in $\mathcal{H}_o$ with variables in $\mathcal{F}_o$ and is used to "decompile" the assignment, $\langle \sigma, m, l \rangle^{-1} \mapsto \rho$. Links represent problematic sub-terms which are linked to the unification variable that stands in their place in the compiled term. These links are checked for or progress XXX improve....

We represent a logic program *run* in $\mathcal{F}_o$as a list *steps* p of length $\mathcal{N}$. Each made of a unification problem between terms $\mathcal{S}_{p_l}$ and $\mathcal{S}_{p_r}$ taken from the set of all terms $\mathcal{S}$. The composition of these steps starting from the empty substitution ρ_0 produces the final substitution $\rho_{\mathcal{N}}$. [1] The initial here ρ_0 is the empty substitution

$$\begin{aligned}
&\mathrm{fstep}(\mathcal{S}, p, \rho) \mapsto \rho'' \overset{def}{=\!=} \rho\mathcal{S}_{p_l} \simeq_o \rho\mathcal{S}_{p_r} \mapsto \rho' \wedge \rho'' = \rho \cup \rho' \\
&\mathrm{frun}(\mathcal{S}, \mathcal{N}) \mapsto \rho_{\mathcal{N}} \overset{def}{=\!=} \textstyle\bigwedge_{p=1}^{\mathcal{N}} \mathrm{fstep}(\mathcal{S}, p, \rho_{p-1}) \mapsto \rho_p
\end{aligned}$$

We simulate each run in $\mathcal{F}_o$ with a run in $\mathcal{H}_o$ as follows. Note that σ_0 is the empty substitution.

$$\begin{aligned}
&\mathrm{hstep}(\mathcal{T}, p, \sigma, \mathbb{L}) \mapsto (\sigma'', \mathbb{L}') \overset{def}{=\!=} \\
&\quad \sigma\mathcal{T}_{p_l} \simeq_\lambda \sigma\mathcal{T}_{p_r} \mapsto \sigma' \wedge \mathrm{progress}(\mathbb{L}, \sigma \cup \sigma') \mapsto (\mathbb{L}', \sigma'') \\
&\mathrm{hrun}(\mathcal{S}, \mathcal{N}) \mapsto \rho_{\mathcal{N}} \overset{def}{=\!=} \\
&\quad \mathcal{T} \times \mathbb{M} \times \mathbb{L}_0 = \{(t_j, m_j, l_j) | s_j \in \mathcal{S}, \langle s_j \rangle \mapsto (t_j, m_j, l_j)\} \\
&\quad \textstyle\bigwedge_{p=1}^{\mathcal{N}} \mathrm{hstep}(\mathcal{T}, p, \sigma_{p-1}, \mathbb{L}_{p-1}) \mapsto (\sigma_p, \mathbb{L}_p) \\
&\quad \langle \sigma_{\mathcal{N}}, \mathbb{M}, \mathbb{L}_{\mathcal{N}} \rangle^{-1} \mapsto \rho_{\mathcal{N}}
\end{aligned}$$

Here hstep is made of two sub-steps: a call to $\simeq_\lambda$ (on the compiled terms) and a call to progress on the set of links. We claim the following:

Proposition 2.1 (Simulation). $\forall \mathcal{S}, \forall \mathcal{N}$,

$$\mathrm{frun}(\mathcal{S}, \mathcal{N}) \mapsto \rho_{\mathcal{N}} \Leftrightarrow \mathrm{hrun}(\mathcal{S}, \mathcal{N}) \mapsto \rho_{\mathcal{N}}$$

That is, the two executions give the same result. Moreover:

Proposition 2.2 (Simulation fidelity). *In the context of* hrun, *if* $\mathcal{T} \subseteq \mathcal{L}_\lambda$ *we have that* $\forall p \in 1 \ldots \mathcal{N}$,

$$\mathrm{fstep}(\mathcal{S}, p, \rho_{p-1}) \mapsto \rho_p \Leftrightarrow \mathrm{hstep}(\mathcal{T}, p, \sigma_{p-1}, \mathbb{L}) \mapsto (\sigma_p, \_)$$

In particular this property guarantees that a *failure* in the $\mathcal{F}_o$ run is matched by a failure in $\mathcal{H}_o$ *at the same step*. We consider this property very important from a practical point of view since it guarantees that the execution traces are strongly related and in turn this enables a user to debug a logic program in $\mathcal{F}_o$ by looking at its execution trace in $\mathcal{H}_o$.

XXX permuting hrun does not change the final result if check dooes not fail eagerly

XXX if we want to apply heuristics, we can apply them in decomp to avoid committing to a non MGU too early

We can define $s_1 \simeq_o s_2$ by specializing the code of hrun to $\mathcal{S} = \{s_1, s_2\}$ as follows:

$$\begin{aligned}
&s_1 \simeq_o s_2 \mapsto \rho \overset{def}{=\!=} \\
&\quad \langle s_1 \rangle \mapsto (t_1, m_1, l_1) \wedge \langle s_2 \rangle \mapsto (t_2, m_2, l_2) \\
&\quad t_1 \simeq_\lambda t_2 \mapsto \sigma' \wedge \mathrm{progress}(\{l_1, l_2\}, \sigma') \mapsto (L, \sigma'') \wedge \\
&\quad \langle \sigma'', \{m_1, m_2\}, L \rangle^{-1} \mapsto \rho
\end{aligned}$$

[1] If the same rule is used multiple time in a run we just consider as many copies as needed of the terms composing the rules, with fresh unification variables each time

Proposition 2.3 (Properties of $\simeq_o$).

$$s_i \in \mathcal{L}_\lambda \Rightarrow s_1 \simeq_o s_2 \mapsto \rho \Rightarrow \rho s_1 =_o \rho s_2 (correct) \tag{3}$$

$$s_i \in \mathcal{L}_\lambda \Rightarrow \rho s_1 =_o \rho s_2 \Rightarrow \exists \rho', s_1 \simeq_o s_2 \mapsto \rho' \wedge \rho' \subseteq \rho (complete) \tag{4}$$

$$\rho s_1 =_o \rho s_2 \Rightarrow \rho' \subseteq \rho \Rightarrow \rho' s_i \in \mathcal{L}_\lambda \Rightarrow \rho' s_1 \simeq_o \rho' s_2 \tag{5}$$

Properties (*correct*) and (*complete*) state, respectively, that in $\mathcal{L}_\lambda$ the implementation of $\simeq_o$ is correct, complete and returns the most general unifier. fix

Property 2.1 states that $\simeq_o$, hence our compilation scheme, is resilient to unification problems outside $\mathcal{L}_\lambda$ solved by a third party. We believe this property is of practical interest since we want the user to be able to add heuristics via hand written rules to the ones obtained by our compilation scheme. A Typical example is the following problem (q) that is outside $\mathcal{L}_\lambda$:

```
app [F, con"a"] = app[con"f", con"a", con"a"]      (q)
F = lam x\ app[con"f",x,x]                          (h)
```

Instead of rejecting it our scheme accepts it and guarantees that if (h) is given (after the compilation part of the scheme, as a run time hint) then ...

2.1 The intuition in a nutshell

A term s is compiled in a term t where every "problematic" sub term p is replaced by a fresh unification variable h and an accessory link that represent a suspended unification problem $h \simeq_\lambda p$. As a result $\simeq_\lambda$ is "well behaved" on t, that is it does not contradict $=_o$ as it would otherwise do on "problematic" terms. We now define "problematic" and "well behaved" more formally.

Definition 2.4 ($\Diamond\eta$). $\Diamond\eta = \{t \mid \exists \rho, \rho t \text{ is an eta expansion}\}$

An example of term t in $\Diamond\eta$ is $\lambda x.\lambda y.F\ y\ x$ since the substitution $\rho = \{F \mapsto \lambda a.\lambda b.fba\}$ makes $\rho t = \lambda x.\lambda y.fxy$ that is the eta long form of f. This term is problematic since its rigid part, the λ-abstractions, cannot justify a unification failure against, say, a constant.

Definition 2.5 ($\overline{\mathcal{L}_\lambda}$). $\overline{\mathcal{L}_\lambda} = \{Xt_1 \ldots t_n \mid Xt_1 \ldots t_n \notin \mathcal{L}_\lambda\}$.

An example of t in $\overline{\mathcal{L}_\lambda}$ is Fa for a constant a. Note however tha an oracle could provide an assignment $\rho = \{F \mapsto \lambda x.x\}$ that makes the resulting term fall back in $\mathcal{L}_\lambda$.

Definition 2.6 (Subterms $\mathcal{P}(t)$). The set of sub terms of t is the largest set $\mathcal{P}(\sqcup)$ that can be obtained by the following rules.

$$\begin{aligned}
&t \in \mathcal{P}(t) \\
&t = ft_1 \ldots t_n \Rightarrow \mathcal{P}(t_i) \subseteq \mathcal{P}(t) \wedge f \in \mathcal{P}(t) \\
&t = \lambda x.t' \Rightarrow \mathcal{P}(t') \subseteq \mathcal{P}(t)
\end{aligned}$$

We write $\mathcal{P}(X) = \bigcup_{t \in X} \mathcal{P}(t)$ when X is a set of terms.

Definition 2.7 (Well behaved set). Given a set of terms $X \subseteq \mathcal{H}_o$,

$$\mathcal{W}(X) \Leftrightarrow \forall t \in \mathcal{P}(X), t \notin (\overline{\mathcal{L}_\lambda} \ \cup\ \Diamond\eta)$$

Proposition 2.8 ($\mathcal{W}$-preservation). $\forall \mathcal{T}, \forall \mathbb{L}, \forall p, \forall \sigma, \forall \sigma'$

$$\begin{aligned}
&\mathcal{W}(\sigma\mathcal{T}) \wedge \sigma\mathcal{T}_{p_l} \simeq_\lambda \sigma\mathcal{T}_{p_r} \mapsto \sigma' \Rightarrow \mathcal{W}(\sigma'\mathcal{T}) \\
&\mathcal{W}(\sigma\mathcal{T}) \wedge \mathrm{progress}(\mathbb{L}, \sigma) \mapsto (_, \sigma') \Rightarrow \mathcal{W}(\sigma'\mathcal{T})
\end{aligned}$$

A less formal way to state 2.8 is that hstep and progress never "commit" an unneeded λ-abstraction in σ (a λ that could be erased by an η-contraction), nor put in σ a flexible application outside $\mathcal{L}_\lambda$ (an application node that could be erased by a β-reduction).

Note that proposition 2.8 does not hold for $\simeq_o$ as a whole since decompilation can introduce (actually restore) terms in $\Diamond\eta$ or $\overline{\mathcal{L}_\lambda}$ that were move out of the way (put in $\mathbb{L}$) during compilation.

3 ALTERNATIVE ENCODINGS AND RELATED WORK

Paper [2] introduces semi-shallow.

Our encoding of DTT may look "semi shallow" since we use the meta-language lambda abstraction but not its application (for the terms of type `tm`). A fully shallow encoding unfortunately does not fit our use case, although it would make the running example work:

```
finite (fin N).
decision (nfact N NF).
decision (all A x\ P x) :- finite A, pi x\ decision (P x).
```

There are two reasons for dismissing this encoding. The first one is that in DTT it is not always possible to adopt it since the type system of the meta language is too weak to accommodate terms with a variable arity, like the following example:

```
Fixpoint arr T n := if n is S m then T -> arr T m else T.
Definition sum n : arr nat n := ...
Check sum 2   7 8   : nat.
Check sum 3   7 8 9 : nat.
```

The second reason is the encoding for Coq is used for meta programming the system, hence it must accommodate the manipulation of terms that are now know in advance (not even defined in Coq) without using introspection primitives such as Prologs's `functor` and `arg`.

In the literature we could find a few related encoding of DTT. TODO In [4] is related and make the discrepancy between the types of ML and DTT visible. In this case one needs 4 application nodes. Moreover the objective is an encoding of terms, proofs, not proof search. Also note the conv predicate, akin to the unif we rule out.

TODO This other paper [15] should also be cited.

None of the encodings above provide a solution to our problem.

4 PRELIMINARIES: $\mathcal{F}_o$ AND $\mathcal{H}_o$

In order to reason about unification we provide a description of the $\mathcal{F}_o$ and $\mathcal{H}_o$ languages where unification variables are first class terms, i.e. they have a concrete syntax. We keep these languages minimal, for example, we omit the `all` quantifier of DTT we used in the example in Section 1 together with the type notation of terms carried by the `lam` constructor.

```
kind fm type.                      kind tm  type.
type fapp list fm -> fm.           type app list tm -> tm.
type flam (fm -> fm) -> fm.        type lam (tm -> tm) -> tm.
type fcon string -> fm.            type con string -> tm.
type fuva addr -> fm.              type uva addr -> list tm -> tm.
```

Figure 1: The $\mathcal{F}_o$ and $\mathcal{H}_o$ languages

Unification variables (`fuva` term constructor) in $\mathcal{F}_o$ have no explicit scope: the arguments of an higher order variable are given via the `fapp` constructor. For example the term `P x` is represented as `fapp[fuva N, x]`, where `N` is a memory address and `x` is a bound variable.

In $\mathcal{H}_o$ the representation of `P x` is instead `uva N [x]`, since unification variables come equipped with an explicit scope. We say that the unification variable occurrence `uva N L` is in $\mathcal{L}_\lambda$ if and only if `L` is made of distinct names. The predicate to test this condition is called `pattern-fragment`:

```
type pattern-fragment list A -> o.
```

Natural numbers represent the memory addresses that identify unification variables in both languages. The memory and its associated operations are described below:

```
kind addr type.
type addr nat -> addr.
typeabbrev (mem A) (list (option A)).

type set? addr -> mem A -> A -> o.
type unset? addr -> mem A -> o.
type assign addr -> mem A -> A -> mem A -> o.
type new mem A -> addr -> mem A -> o.
```

If a memory cell is `none`, then the corresponding unification variable is not set. `assign` sets an unset cell to the given value, while `new` finds the first unused address and sets it to `none`. is new used?

Since in $\mathcal{H}_o$ unification variables have a scope, their solution needs to be abstracted over it to enable the instantiation of a single solution to different scopes. This is obtained via the `inctx` container, and in particular via its `abs` binding constructor. On the contrary a solution to a $\mathcal{F}_o$variable is a plain term.

```
typeabbrev fsubst (mem fm).

kind inctx type -> type.                                (· ⊢ ·)
type abs (tm -> inctx A) -> inctx A.
type val A -> inctx A.
typeabbrev assignment (inctx tm).
typeabbrev subst (mem assignment).
```

We call `fsubst` the memory of $\mathcal{F}_o$, while we call `subst` the one of $\mathcal{H}_o$. Both have the invariant that they are not cyclic, TODO: explain.

```
kind arity type.
type arity nat -> arity.

kind fvariable type.
type fv addr -> fvariable.

kind hvariable type.
type hv addr -> arity -> hvariable.

kind mapping type.
type mapping fvariable -> hvariable -> mapping.
typeabbrev mmap (list mapping).
```

Invariant 1 (Unification variable arity). *Each variable* `A` *in* $\mathcal{H}_o$ *has a (unique) arity* `N` *and each occurrence* `(uva A L)` *is such that* `(len L N)` *holds*

The compiler establishes a mapping between variables of the two languages. In order to preserve invariant 1 we store the arity of each hvariable in the mapping and we reuse an existing mapping only if the arity matches.

TODO: add ref to section 7

```
type m-alloc fvariable -> hvariable -> mmap -> mmap ->
  subst -> subst -> o.                                (malloc)
m-alloc Fv Hv M M S S :- mem M (mapping Fv Hv), !.
m-alloc Fv Hv M [mapping Fv Hv|M] S S1 :- Hv = hv N _,
  alloc S N S1.
```

When a single fvariable occurs multiple times with different numbers of arguments the compiler generates multiple mappings for it, on a first approximation, and then makes the mapping bijective by introducing link-η; this detail is discussed in section 6.

As we mentioned in section 2.1 the compiler replaces terms in $\Diamond\eta$ and $\overline{\mathcal{L}_\lambda}$ with fresh variables linked to the problematic terms. Each class of problematic terms has a dedicated link.

```
kind baselink type.
type link-eta  tm -> tm -> baselink.
type link-beta tm -> tm -> baselink.
typeabbrev link (inctx baselink).
typeabbrev links (list link).
```

The right hand side of a link, the problematic term, can occur under binders. To accommodate this situation the compiler wraps baselink using the inctx container (see, $\cdot \vdash \cdot$).

Invariant 2 (Link left hand side). *The left hand side of a suspended link is a variable.*

New links are suspended by construction. If the left hand side variable is assigned during a step, then the link is considered for progress and possibly eliminated. This is discussed in section 6 and ??.

4.1 Notational conventions

When we write $\mathcal{H}_o$ terms outside code blocks we follow the usual λ-calculus notation, reserving f, g, a, b for constants, x, y, z for bound variables and X, Y, Z, F, G, H for unification variables. However we need to distinguish between the "application" of a unification variable to its scope and the application of a term to a list of arguments. We write the scope of unification variables in subscript while we use juxtaposition for regular application. Here a few examples:

$f\ a$	`app[con "f", con "a"]`
$\lambda x.\lambda y.F_{xy}$	`lam x\ lam y\ uva F [x, y]`
$\lambda x.F_x\ a$	`lam x\ app[uva F [x], con "a"]`
$\lambda x.F_x\ x$	`lam x\ app[uva F [x], x]`

When variables x and y can occur in term t we shall write t_{xy} to stress this fact.

We write $\sigma = \{A_{xy} \mapsto y\}$ for the assignment `abs x\abs y\y` and $\sigma = \{A \mapsto \lambda x.\lambda y.y\}$ for `lam x\lam y\y` .

When detailing examples we write links as equations between two terms under a context. The equality sign is subscripted with kind of baselink. For example $x \vdash A_x =_\beta F_x\ a$ corresponds to:

```
abs x\ val (link-beta (uva A [x]) (app[uva F [x],con "a"]))
```

When it is clear from the context we shall use the same syntax for $\mathcal{F}_o$ terms (although we never subscripts unification variables).

4.2 Equational theory and Unification

In order to express properties ?? we need to equip $\mathcal{F}_o$ and $\mathcal{H}_o$ with term equality, substitution application and unification.

Term equality: $=_o$ *vs.* $=_\lambda$. We extend the equational theory over ground terms to the full languages by adding the reflexivity of unification variables (a variable is equal to itself).

The first four rules are common to both equalities and just define the usual congruence over terms, and since we use an HOAS encoding they also capture α-equivalence. In addition to that $=_o$ has rules for η and β-equivalence.

```
type (=o) fm -> fm -> o.                              (=o)
fcon X =o fcon X.
fapp A =o fapp B :- forall2 (=o) A B.
flam F =o flam G :- pi x\ x =o x => F x =o G x.
fuva N =o fuva N.
flam F =o T :-                                        (ηl)
  pi x\ beta T [x] (T' x), x =o x => F x =o T' x.
T =o flam F :-                                        (ηr)
  pi x\ beta T [x] (T' x), x =o x => T' x =o F x.
fapp [flam X | L] =o T :- beta (flam X) L R, R =o T.  (βl)
T =o fapp [flam X | L] :- beta (flam X) L R, T =o R.  (βr)

type (=λ) tm -> tm -> o.
con C =λ fcon C.
app A =λ fapp B :- forall2 (=λ) A B.
lam F =λ flam G :- pi x\ x =λ x => F x =λ G x.
uva N A =λ fuva N B :- forall2 (=λ) A B.
```

The main point in showing these equality tests is to remark how weaker $=_\lambda$ is, and to identify the four rules that need special treatment in the implementation of $\simeq_o$.

For reference, (`beta T A R`) reduces away `lam` nodes in head position in `T` whenever the list `A` provides a corresponding argument.

```
type beta fm -> list fm -> fm -> o.
beta A [] A.
beta (flam Bo) [H | L] R :- napp (Bo H) F, beta F L R.
beta (fapp A) L (fapp X) :- append A L X.
beta (fuva N) L (fapp [fuva N | L]).
beta (fcon H) L (fapp [fcon H | L]).
beta N L (fapp [N | L]) :- name N.

type napp fm -> fm -> o.
napp (fcon C) (fcon C).
napp (fuva A) (fuva A).
napp (flam F) (flam G) :- pi x\ napp (F x) (G x).
napp (fapp [fapp L1 |L2]) T :- !,
  append L1 L2 L3, napp (fapp L3) T.
napp (fapp L) (fapp L1) :- map napp L L1.
napp N N :- name N.
```

The `name` predicate holds only on nominal constants (i.e. bound variables).[2] The choice of using n-ary application, rather than binary, is to make it easy to access the application's head. The price we pay is that substituting an application in the head of an application should be amended by "flattening" `fapp` nodes, that is the job of

[2] Elpi provides it as a builtin, but one could implement it by systematically loading the hypothetical rule `name x` every time a nominal constant is postulated via `pi x\`

napp. [3] Finally note that the cut operator is inessential, it could be removed at the cost of a verbose test on the head of `L` in the second rule about `fapp`: `L`'s head can be `fcon`, `flam` or a name.

Substitution application: ρs and σt. Applying the substitution corresponds to dereferencing a term with respect to the memory. To ease the comparison we split $\mathcal{F}_o$dereferencing into a `fder` step and a `napp` one. The former step replaces references to memory cells that are set with their values, and has a corresponding operation in $\mathcal{H}_o$, namely `deref`. On the contrary `napp` has no corresponding operation in $\mathcal{H}_o$. The reasons for this asymmetry is that an `fapp` node with a flexible head is always mapped to a `uva` (as per section 5 and section 8), preventing nested applications to materialize.

```
type fder fsubst -> fm -> fm -> o.
fder _ (fcon C) (fcon C).
fder S (fapp A) (fapp B) :- map (fder S) A B.
fder S (flam F) (flam G) :-
  pi x\ fder S x x => fder S (F x) (G x).
fder S (fuva N) R :- set? N S T, fder S T R.
fder S (fuva N) (fuva N) :- unset? N S.

type fderef fsubst -> fm -> fm -> o.                    (ρs)
fderef S T T2 :- fder S T T1, napp T1 T2.
```

Applying the substitution in $\mathcal{H}_o$ is very similar, with the caveat that assignments have to be moved to the current scope, i.e. renaming the `abs`-bound variables with the names in the scope of the unification variable occurrence.

```
type deref subst -> tm -> tm -> o.                      (σt)
deref _ (con C) (con C).
deref S (app A) (app B) :- map (deref S) A B.
deref S (lam F) (lam G) :-
  pi x\ deref S x x => deref S (F x) (G x).
deref S (uva N L) R :- set? N S A,
  move A L T, deref S T R.
deref S (uva N A) (uva N B) :- unset? N S,
  map (deref S) A B.
```

Note that move strongly relies on invariant 1: the length of the arguments of all occurrences of a unification variable and the number of abstractions in its assignment have to match. In turn this grants that `move` never fails.

```
type move assignment -> list tm -> tm -> o.
move (abs Bo) [H|L] R :- move (Bo H) L R.
move (val A)  [] A.
```

Term unification: $\simeq_o$ vs. $\simeq_\lambda$. In this paper we assume to have an implementation of $\simeq_\lambda$ that satisfies properties 1 and 2. Although we provide an implementation in the appendix (that we used for testing purposes) we only describe its signature here. Elpi is expected to provide this brick, as well as any other implementation of λProlog.

```
type (≃λ) tm -> tm -> subst -> subst -> o.
```

The only detail worth discussing is the fact that the procedure updates a substitution, rather than just crafting one as presented in section 2. The reason is that the algorithm folds over a term, updating a substitution while it traverses it. explain better

5 BASIC SIMULATION OF $\mathcal{F}_o$ IN $\mathcal{H}_o$

In this section we describe a basic compilation scheme that we refine later, in the following sections. This scheme is sufficient to implement an $\simeq_o$ that respects β-conversion for terms in $\mathcal{L}_\lambda$. The extension to $\eta\beta$-conversion is described in Section 6 and the support for terms outside $\mathcal{L}_\lambda$ in Section 8.

5.1 Compilation

The main task of the compiler is to recognize $\mathcal{F}_o$ variables standing for functions and map them to higher order variables in $\mathcal{H}_o$. In order to bring back the substitution from $\mathcal{H}_o$ to $\mathcal{F}_o$ the compiler builds a "memory map" connecting the the kind of variables using routine (*malloc*). manca beta normal in entrata

The signature of the `comp` predicate below allows for the generation of links (suspended unification problems) that play no role in this section but play a major role in section 6 and section 8. With respect to section 2 the signature also allows for updates to the substitution. The code below uses that possibility in order to allocate space for the variables, i.e. sets their memory address to `none` (a details not worth mentioning in the previous discussion).

```
type comp fm -> tm -> mmap -> mmap -> links -> links ->
  subst -> subst -> o.
comp (fcon C) (con C) M M L L S S.
comp (flam F) (lam F1) M1 M2 L1 L2 S1 S2 :-            (cλ)
  comp-lam F F1 M1 M2 L1 L2 S1 S2.
comp (fuva A) (uva B []) M1 M2 L L S1 S2 :-
  m-alloc (fv A) (hv B (arity z)) M1 M2 S1 S2.
comp (fapp [fuva A|Ag]) (uva B Ag1) M1 M2 L L S1 S2 :-
  pattern-fragment Ag, !,
    fold6 comp Ag Ag1 M1 M1 L L S1 S1,
    len Ag Arity,
    m-alloc (fv A) (hv B (arity Arity)) M1 M2 S1 S2.
comp (fapp A) (app A1) M1 M2 L1 L2 S1 S2 :-
  fold6 comp A A1 M1 M2 L1 L2 S1 S2.
```

This preliminary version of `comp` recognizes $\mathcal{F}_o$ variables applied to a (possibly empty) duplicate free list of names. Note that compiling `Ag` cannot create new mappings nor links, since `Ag` is made of bound variables and the hypothetical rule loaded by `comp-lam` (see below) grants this property.

```
type comp-lam (fm -> fm) -> (tm -> tm) ->
  mmap -> mmap -> links -> links -> subst -> subst -> o.
comp-lam F G M1 M2 L1 L3 S1 S2 :-
  pi x y\ (pi M L S\ comp x y M M L L S S) =>
    comp (F x) (G y) M1 M2 L1 (L2 y) S1 S2,
  close-links L2 L3.
```

In the code above the syntax `pi x y\..` is syntactic sugar for iterated `pi` abstraction, as in `pi x\ pi y\..`

The auxiliary function `close-links` tests if the bound variable `v` really occurs in the link. If it is the case the link is wrapped into an additional `abs` node binding `v`. In this way links generated deep inside the compiled terms can be moved outside their original context of binders.

[3] Note that napp is an artefact of formalization of $\mathcal{F}_o$ we do in this presentation and, as we explain later, no equivalent of napp is needed in $\mathcal{H}_o$.

```
type close-links (tm -> links) -> links -> o.
close-links (v\[X  |L v]) [X|R] :- !, close-links L R.
close-links (v\[X v|L v]) [abs X|R] :- close-links L R.
close-links (_\[]) [].
```

Note that we could remove the first rule, whose solve purpose is to make links more readable by pruning unused context entries.

5.2 Execution

A step in $\mathcal{H}_o$ consists in unifying two terms and reconsidering all links for progress. If any of the two tasks fail we say that the entire step fails, and it is at this granularity that we can relate steps in the two languages.

```
type hstep tm -> tm -> links -> links -> subst -> subst -> o.
hstep T1 T2 L1 L2 S1 S3 :-
  (T1 ≃λ T2) S1 S2,
  progress L1 L2 S2 S3.
```

Note tha the infix notation `((A ≃λ B) C D)` is syntactic sugar for `((≃λ) A B C D)`.

Reconsidering links is a fixpoint, since the progress of a link can update the substitution and in turn enable another link to progress.

```
type progress links -> links -> subst -> subst -> o.
progress L L2 S1 S3 :-
  progress1 L L1 S1 S2,
  occur-check-links L1,
  if (L = L1, S1 = S2)
     (L2 = L1, S3 = S1)
     (progress L1 L2 S2 S3).
```

In the base compilation scheme `progress1` is the identity on both the links and the substitution, so the fixpoint trivially terminates. Sections 6 and 8 add rules to `progress1` and justify why the don't hinder termination.

Since compilation moves problematic terms out of the sigh of $\simeq_\lambda$, that procedure can only perform a partial occur check. For example the unification problem $X \simeq_\lambda f\ Y$ cannot generate a cyclic substitution alone, but should be disallowed if a $\mathbb{L}$ contains a link like $\vdash Y =_\eta \lambda z.X_z$: We don't know yet if Y will feature a lambda in head position, but we surely know it contains X, hence $f\ Y$ and that fails the occur check. The procedure `occur-check-links` is in charge of ensuring that each link does not represent a (suspended) unification problem doomed to fail because of occur check. This check is needed in order to guarantee proposition 2.2 (simulation fidelity).

5.3 Substitution decompilation

Decompiling the substitution requires to first force the progress of links and then allocating new unassigned variables in the substitution for $\mathcal{F}_o$ and finally decompiling all assignments. Note that invariant 2 and the occur check allows us to update the subst.

```
type decompile mmap -> links -> subst ->
  fsubst -> fsubst -> o.
decompile M1 L S F1 F3 :-
  commit-links L S S1,
  complete-mapping S1 S1 M1 M2 F1 F2,
  decompm M2 M2 S1 F2 F3.
```

TODO: What is commit-links and complete-mapping?, maybe complete-mapping can be hidden in the code rendering? Decompiling an assignment requires to turn abstractions into lambdas. For aestetic purposes we also eta-contract the result (not needed since $\mathcal{F}_o$ equality can do that)

```
type decompm mmap -> mmap -> subst -> fsubst -> fsubst -> o.
decompm _ [] _ F F.
decompm M [mapping (fv V) (hv H _)|MS] S F1 F3 :- set? H S A,
  deref-assmt S A A1,
  abs->lam A1 T, decomp M T T1,
  eta-contract T1 T2,
  assign V F1 T2 F2,
  decompm M MS S F2 F3.
decompm M [mapping _ (hv H _)|MS] S F1 F2 :- unset? H S,
  decompm M MS S F1 F2.
```

Finally decompiling a term is trivial, now that we have an extended mapping containing all unassigned variables $\simeq_\lambda$ may have introduced.

```
type decomp mmap -> tm -> fm -> o.
decomp _ (con C)  (fcon C).
decomp M (app A) (fapp B) :- map (decomp M) A B.
decomp M (lam F) (flam G) :-
  pi x y\ (pi M\ decomp M x y) => decomp M (F x) (G y).
decomp M (uva Hv Ag) R :-
  mem M (mapping (fv Fv) (hv Hv _)),
  map (decomp M) Ag Bg,
  beta (fuva Fv) Bg R.
```

Note that we use beta to build fapp nodes when needed (if Ag is empty no `fapp` node should appear).

Invariant 3. *TODO: dire che il mapping è bijective*

5.4 Definition of $\simeq_o$ and its properties

```
type (≃o) fm -> fm -> fsubst -> o.
(A ≃o B) F :-
  comp A A' [] M1 [] [] [] S1,
  comp B B' M1 M2 [] [] S1 S2,
  hstep A' B' [] [] S2 S3,
  decompm M2 M2 S3 [] F.
```

The code given so far applies to terms in $\beta\eta$-normal form where unification variables in $\mathcal{F}_o$ can occur non linearly but always with the same number of arguments, and where their arguments are distinct names (as per $\mathcal{L}_\lambda$).

Lemma 5.1 (Compilation round trip). *If* `comp S T [] M [] _ [] _` *then* `decomp M T S`

Proof sketch. trivial, since the terms are beta normal beta just builds an app. □

Lemma 5.2. *Properties* (*correct*) *and* (*complete*) *hold for the implementation of* $\simeq_o$ *above*

Proof sketch. In this setting $=_\lambda$ is as strong as $=_o$ on ground terms. What we have to show is that whenever two different $\mathcal{F}_o$ terms can be made equal by a substitution ρ (plus the β_l and β_r if needed) we can find this ρ by finding a σ via $\simeq_\lambda$ on the corresponding $\mathcal{H}_o$

terms and by decompiling it. If we look at the $\mathcal{F}_o$ terms, the are two interesting cases:

- `fuva X` $\simeq_o$ `s`. In this case after `comp` we have $Y \simeq_\lambda t$ that succeeds with $\sigma = \{Y \mapsto t\}$ and σ is decompiled to $\rho = \{Y \mapsto s\}$.
- `fapp[fuva X|L]` $\simeq_o$ `s`. In this case we have $Y_{\vec{x}} \simeq_\lambda t$ that succeeds with $\sigma = \{\vec{y} \vdash Y \mapsto t[\vec{x}/\vec{y}]\}$ that in turn is decompiled to $\rho = \{Y \mapsto \lambda \vec{y}.s[\vec{x}/\vec{y}]\}$. Thanks to $\beta_l\ (\lambda \vec{y}.s[\vec{x}/\vec{y}])\ \vec{x} =_o s$.

Since the mapping is a bijection occur check in $\mathcal{H}_o$ corresponds to occur check in $\mathcal{F}_o$. □

Lemma 5.3. *Properties simulation (2.1) and fidelity (2.2) hold*

Proof sketch. Since `progress1` is trivial fstep and hstep are the same, that is in this context where input terms are $\beta\eta$-normal and we disregard η-equivalence $\simeq_\lambda$ is equivalent to $\simeq_o$. □

5.5 Limitations of by this basic scheme

$$\lambda xy.F\cdot y\cdot x = \lambda xy.x \tag{6}$$

$$\lambda x.f\cdot(F\cdot x)\cdot x = G \tag{7}$$

Note that here F is used with different arities, moreover in the second problem the left hand side happens to be an eta expansion (of $f(\lambda y.y)$) only after we discover (at run time) that $F = \lambda x \lambda y.y$ (i.e. that F discards the x argument). Both problems are addressed in the next section.

6 HANDLING OF $\Diamond\eta$

η-reduction is an equivalence relation where a term of the form $\lambda x.t\cdot x$ can be converted to t any time x does not occur as a free variable in t. We call $\lambda x.t\cdot x$ the η-expansion of t.

Entry pb: $\mathcal{P}_1$: `flam x\ fapp [fuva X, x]` $\simeq_o$ `fcon"f"`
Compil: $\mathcal{T}_1 : \lambda x.X'_x \simeq_\lambda f$
Mapping: m_1 : `X` $\mapsto X'^1$

Following the implementation of `comp` given in section 5, the unification problem $\mathcal{P}_1$ given above, is compiled into the unification problem $\mathcal{T}_1$. We can note that all the terms appearing in $\mathcal{P}_1$ are unifiable by $\simeq_o$, but, despite this, $\simeq_\lambda$, is not able to solve $\mathcal{T}_1$: its left and right hand sides have both different rigid head: $\lambda x.A'_x$ corresponds to the $\mathcal{H}_o$ term `lam x\ app[con "f", x]` and f corresponds to `con"f"`. This failure is motivated by the fact that `flam x\ fapp [fuva A, x]` is a term belonging to $\Diamond\eta$.

In order to guarantee proposition 2.1, we need to modify the way terms are compiled. The goal is to recognize every $\Diamond\eta$ subterm t and replace it with fresh $\mathcal{H}_o$ variables v. This connection between the variable v and the subterm t is stored in what we call `link-`η which is an object with the following type

```
type link-eta tm -> tm -> baselink
```

where, as sketched in section 4, the term on the left hand side (lhs) is linked with its left counterpart (rhs).

`link-`η are added in the link store ($\mathbb{L}$) and activated when special conditions are satisfied on lhs or rhs. These link activations are managed by extending the `progress1` predicate (see section 5.2). We claim that `link-`η progression does not contradict invariant 2 and we add the following invariant:

Invariant 4 (`link-`η rhs). *The* rhs *of any* `link-`η *in* $\mathbb{L}$ *has the shape* $\lambda x.t_x$ *where* t_x *is a* $\Diamond\eta$ *term and* x *is free in* t.

In the next three subsections we explain how we detect $\Diamond\eta$ terms, how we compile them and how the generated `link-`η are activated during the `progress`. Moreover, we provide justification for why invariants 2 and 4 remain true.

6.1 Detection of $\Diamond\eta$

Compiling a term t forces us to determine if there exists a $t' \in \mathcal{P}(t)$, such that $t' = \lambda x.t''_x$, for any term t'', can be a η-expansion, i.e. under a given substitution σ, we have $\sigma(\lambda x.t''_x) = t'$. This $\Diamond\eta$ detection is not a trivial operation as it may seems.

$$\lambda x.f\cdot A_x \quad (8) \qquad \lambda x.\lambda y.f\cdot A_x\cdot B_{yx} \quad (10)$$

$$\lambda x.f\cdot x\cdot A_x \quad (9) \qquad \lambda x.f\cdot A_x\cdot x \quad (11)$$

Term	Status	Evidence
$\lambda x.f\cdot A_x$	$\in \Diamond\eta$	$\sigma = \{\ A_x \mapsto x\ \}$
$\lambda x.f\cdot x\cdot A_x$	$\notin \Diamond\eta$	
$\lambda x.\lambda y.f\cdot A_x\cdot B_{yx}$	$\in \Diamond\eta$	$\sigma = \{\ A_x \mapsto x,\ B_{yx} \mapsto y\ \}$
$\lambda x.f\cdot A_x\cdot x$	$\in \Diamond\eta$	$\sigma = \{\ A_x \mapsto a\ \}$

In the examples above, the first term is a $\Diamond\eta$ since A_x can reduce to x by setting $A_x = \lambda x.x$, the second one is not a $\Diamond\eta$ since it exists no substitution for A_x such that A_x reduces to x and x is not free in the subterm $f\cdot x$. It is a bit more complicated to determine if eq. 10 a $\Diamond\eta$ since, we have a spine of lambdas, this means that the whole term is a $\Diamond\eta$, if the inner λ-term $\lambda y.f\cdot A_x\cdot B_y x$, is an η-expansion and, if so, we need to verify if the η-reduced subterm, i.e. $f\cdot A_x$, can make $\lambda x.f\cdot A_x$ an η-expansion. Indeed, eq. 10 is a $\Diamond\eta$ under the substitution $\sigma = \{A \mapsto \lambda x.x, B \mapsto \lambda x.\lambda y.x\}$. Lastly, eq. 11 is a $\Diamond\eta$ even if x occurs in the subterm $t' = f\cdot A_x$, since it exists a substitution $\sigma = \{A \mapsto \lambda x.a\}$, for any constant a, such that $\sigma t' = f\cdot a$ is a term where x does not appears as a free variable.

We can now define more formally how $\Diamond\eta$ terms are detected together with its auxiliary functions:

Definition 6.1 (reduce-to). A term t reduce to a name x, if $\exists\sigma, \sigma t = x$. In particular, for any term t, $\lambda x_1 \ldots x_n.t_{x_1 \ldots x_n}$ reduces to a bound variable x if one of the three following cases is satisfied: 1) $n = 0$ and $t = x$; 2) t is the application of x to a list of terms l and each l_i reduces to x_i (e.g. $\lambda x_1 \ldots x_n.x_{x_1 \ldots x_n} = x$) ; 3) t is a unification variable in $\mathcal{L}_\lambda$ with scope s, and for any $v \in \{x, x_1 \ldots x_n\}$, there exists a $s_i \in s$, such that s_i reduces to v

Definition 6.2 (occurs-rigidly). A name x occurs rigidly in a term t, if $\forall\sigma, x \in \mathcal{P}(\sigma t)$

In other words x *occurs-rigidly* in t if it occurs in t outside of the scope of unification variables since theirs instantiations are allowed to discard their scope.

We can now derive the implementation for $\Diamond\eta$ detection:

Definition 6.3 ($\Diamond\eta$ detection). A term $\lambda x_1 \ldots x_n.t$ where each x_i occurs in t is a $\Diamond\eta$ if either: 1) t is a constant applied to arguments $l_1 \ldots l_m$ such that $m \geq n$ and every l_{m-n+i} reduces-to x_i and no x_i occurs-rigidly in $l_{1 \ldots m-n-1}$; or 2) t is a unification variable with scope s and for each x_i there exists a $s_j \in s$ such that s_j reduces-to x_i.

As a final remark, the $\diamond\eta$ detection defined just before is an over-approximation, in the sense that there exists some terms t considered as $\diamond\eta$, such that forall substitution σ, σt is not an η-expansion. A small example is $\lambda x.f \cdot A_x \cdot A_x$ which is considered a $\diamond\eta$. Thi is because, we suppose at the same time that the second A_x can reduce to x, to the first A_x is not reducible to A_x. This is however not considered as a problem, since, adding holes to the compiled term does not break proposition 2.1, since each hole is connected to a `link-`η which is activated exactly when the hole is instantiated the `link-`η, as explain below is activated.

6.2 Compilation

Thanks to the `maybe-eta` predicate, we can detect "η-problematic" terms and, consequently replace them with fresh $\mathcal{H}_o$ unification variables at compilation time. The code below illustrate how this relation is used to for term compilation.

```
comp (flam F) (uva A Scope) M1 M2 L1 L3 S1 S3 :-
  maybe-eta (flam F) [], !,
    alloc S1 A S2,
    comp-lam F F1 M1 M2 L1 L2 S2 S3,
    get-scope (lam F1) Scope,
    L3 = [val (link-eta (uva A Scope) (lam F1)) | L2].
```

This rule, to be inserted just before rule (c_λ) from the code in section 5, verifies if the $\mathcal{F}_o$ term t received in entry is a $\diamond\eta$. Let $\lambda t'$ be the compiled version of t, then the fresh variable `A` returned as the new $\mathcal{H}_o$ term as in scope all the free names in t'. The critical part of this compilation is the creation of the `link-`η, which links the variable `A` with t. This link creation enforce invariant 2 and invariant 4, since lhs is a trivially a variable and the rhs is a term t starting with the `lam` constructor where x is free in t otherwise t would not have been detected as a $\diamond\eta$.

Corollary 6.4. *The* rhs *of any* `link-`η *has exactly one lambda abstraction.*

Proof sketch. By contradiction, suppose that a `link-`η l has t as rhs and $t = \lambda x.\lambda y.t'_{xy}$. Two cases are to be analyses: 1) $\lambda y.t'_{xy}$ is a $\diamond\eta$, then, by construction, rhs would have been replaced with a the η-expansion of fresh variable v, which is a contradiction, since $\lambda y.t'_{xy} \neq \lambda x.v_x$; 2) $\lambda y.t'_{xy}$ is not an a $\diamond\eta$, then neither t is, which is a contradiction since rhs is always a $\diamond\eta$ by construction. ◻

6.3 Progress

`link-`η are meant to delay the unification of "problematic" terms. In the following, we call $\mathbb{L}$ the list of suspended links.

In order to activate a `link-`η l, we need to extend the `progress1` predicate by adding new rules. After passing under all the `abs` constructors of l, there are two cases making a `link-`η to progress, 1) lhs is instantiated to a rigid term 2) rhs is no more a $\diamond\eta$ or it is a term which can be reduced to a term with rigid head. If lhs is instantiated to a rigid term t, by proposition 2.8, we know that t does not contain any $\diamond\eta$. Let t' the right hand side, if t is a constant or a function application, then, t', which by construction has `lam` as head, should be an η-expansion. We are therefore allowed to unify $\lambda x.t \cdot x$ (the η-expanded version of t) with t'. Finally, if t is a term with `lam` as head, then it is not an η-expansion and therefore, t can be unified with t'.

The second way to activate a `link-`η is when the rhs is no more a $\diamond\eta$ or rhs can be η-reduced to a term t with rigid head. In both cases, lhs is unified with t'.

Once a `link-`η is activated, it can be removed from $\mathbb{L}$, otherwise, the link is kept for a further iteration of `progress`. Note that this link progression enforce proposition 2.8 and invariants 2 and 4: we never commit a term in the $\mathcal{H}_o$ substitution, since we make unification only when we know that the terms are no more $\diamond\eta$, and when lhs is no more a variable or rhs is no more a $\diamond\eta$, the link is removed from $\mathbb{L}$.

Entry pb: $\mathcal{P}_1$: `flam x\ fapp [fuva X, x]` $\simeq_o$ `fcon"f"`
Compil: $\mathcal{T}_1 : A \simeq_\lambda f$
Mapping: $m_1 : X \mapsto X'^1$
Links: $l_1 \mathrel{:\!\vdash} A =_\eta \lambda x.X'_x$

The example above shows the new compilation of the unification problem given at the beginning of section 6. This time, we see that the the left hand side t of $\mathcal{P}_1$ has been detected as a $\diamond\eta$ and replaced with the fresh variable A. Moreover, $\mathbb{L}$ contains the link l_1 connecting A with $\lambda x.X'_x$, the compiled version of t. After the resolution of $\mathcal{T}_1$, A is assigned to f. Therefore lhs of l_1 is a term with `con` as constructor. This means that rhs of l_1 is a η-expansion and therefore we can unify $\lambda x.fx$ with rhs, which instantiate X'_x to $f \cdot x$. l_1 is removed from $\mathbb{L}$ which is now empty, `progress` terminate, and the decompilation, will instantiate the X variable to f (which is the η-contracted version of $\lambda x.f \cdot x$).

A second example, showing the activation of a link when the rhs is no more a $\diamond\eta$, is given in section 7, since we need to work with variables used with different arities.

TODO: example for case 2: $\lambda x.\lambda y.F \cdot y \cdot x = G, F = \lambda x.\lambda y.a$

A second way to progress `link-`η, that we call `link-`η deduplication, is when $\mathbb{L}$ contains two `link-`η l_1 and l_2 with a lhs having the same variable address. Let the lhs of l_1 be `uva U R` and the lhs of l_2 be `uva V S`, then, by invariant 1, `R` and `S` have same scope. Let t be the term obtained by replacing all each name `S`$_i$ in the rhs of l_1 with `R`$_i$, t is unified with the rhs of l_2 and one of the two links between l_1 and l_2 is removed from $\mathbb{L}$.

TODO: example for this: $\lambda x.\lambda y.F \cdot y \cdot x = X, \lambda x.\lambda y.F \cdot y \cdot x = Y$

Lemma 6.5. *Forall list of links* `L` *and* `S`, `progress L _ S _` *terminates*

Proof sketch. The addition of rules for `progres1` complicates the function `progress`. We can note, however, that they do not prevent the termination of `progress`. 1) If a link is activated it is removed from $\mathbb{L}$ and the recursive call to `progress` will have a smaller list of links to recurse on. Moreover, link activation only runs terminating instructions (such as unification). 2) If a link is deduplicated, the termination of `progress` is still guaranteed since again we reduce $\mathbb{L}$ and the instructions run by link deduplications are all terminating. 3) If a link is neither activated nor deduplicated, i.e. it remains suspended, then $\mathbb{L}$ remains unchanged like the substitution; therefore, `if (L = L1, S1 = S2)` succeeds and `progress` terminates. ◻

TODO: we can have $\lambda x.F_x$ in the substitution if we know that F does not reduce to Tx where x is not free in T.

7 ENFORCING INVARIANT 1

In section 5.5, we have given two unification problems to be run one after the other. In the following table we present the entry problem, its compiled version, with the corresponding list of mapping and links.

Entry pb: $\mathcal{P}_1$: `flam x\ flam y\ fapp [fuva X, y, x] == flam x\ flam y\ x`
$\mathcal{P}_2$: `flam x\ fapp [fcon"f", fapp [fuva X, x], x] == fuva Y`

Compil:	$\mathcal{T}_1 : E0 == \lambda x.\lambda y.x$	$\mathcal{T}_2 : E3 == E5$
Mapping:	$m_1 : Y \mapsto E5^0$	$m_2 : X \mapsto E2^2$
Links:	$l_1 : c0 \vdash E4_{c0} =_\eta \lambda x.E2_{c0x}$	$l_2 : \vdash E3 =_\eta \lambda x.(f\ E4_x\ x)$
	$l_3 :\vdash E0 =_\eta \lambda x.E1_x$	$l_4 : c0 \vdash E1_{c0} =_\eta \lambda x.E2_{xc0}$

We see that the `maybe-eta` as detected $\lambda xy.F \cdot y \cdot x$ and $\lambda x.f \cdot (F \cdot x) \cdot x$ and replaced them with respectively the $\mathcal{H}_o$ vars X and Z. X is linked with $\lambda x.Y_x$, Y has arity 1 and is η-linked with $\lambda y.H \cdot y \cdot x$ and Z is linked to the term $\lambda x.f \cdot G_x\ x$. However, the mapping returned by the compilation, does not breaks invariant 3: the $\mathcal{F}_o$ variable F is mapped to two different $\mathcal{H}_o$ variables. To address this problem and enforce invariant 3, we clean the mapping with a second phase after the compilation. This phase is called `map-deduplication`.

Before formally defining this procedure, we need to define some auxiliary relations. Let $\mathbb{M}$ be the list of mapping and $\langle m_1, m_2 \rangle \in \mathbb{M}$ such that the arity of the $\mathcal{H}_o$ variable in m_1 is smaller than the one in m_2. Let X (resp. Y) the $\mathcal{H}_o$ variable of m_1 (resp. m_2) and $n = ar(m_1) - ar(m_2)$. We also let A^i be a fresh $\mathcal{H}_o$ variable. We define the `make-eta-link` relation taking two mappings $\langle m_1, m_2 \rangle$ and returning the following list of link `link-`η: $\forall i \in [1..n]$,

$$\begin{cases} \vdash X =_\eta \lambda x.A^1_x & if\ i = 1 \\ x_1 \dots x_{i-1} \vdash A^{i-1}_{x_1 \dots x_{i-1}} =_\eta \lambda x_i.A^i_{x_1 \dots x_i} & if\ 1 < i < n \\ x_1 \dots x_{i-1} \vdash A^{i-1}_{x_1 \dots x_{i-1}} =_\eta \lambda x_i.Y_{x_1 \dots x_i} & if\ i = n \end{cases}$$

More concretely, we are saying that for any two mappings, we build as many `link-`η as the difference of the arities between the two mappings. This links are constructed in such a way that the $\mathcal{H}_o$ variable v with lowest arity is linked to a fresh variable eta-expended variable A^1 having the scope of v. This variable A^1 is then linked to a an η-expanded fresh variable A^2 with same scope of A^1 and so on. The last link is built between the A^{n-1} (where n is the difference of arities between the two mappings) and the $\mathcal{H}_o$ variable u with higher arity in the two mappings being considered.

Definition 7.1 (map-deduplication). Forall mappings $\langle m_1, m_2 \rangle \in \mathbb{M}$, sharing the same $\mathcal{F}_o$ variable, the list of `link-`η L is created thanks to `make-eta-link`$m_1\ m_2\ L$ and is added to $\mathbb{L}$. Then m_1 is removed from $\mathbb{M}$.

If we take back the example give at the beginning of this section, we can deduplicate $F \mapsto G^1, F \mapsto H^2$ by removing the first mapping and adding the auxiliary `link-`η: $x \vdash G_x =_\eta \lambda y.H_{xy}$.

The complete problem to run for resolution is now:

Compil:	$\mathcal{T}_1 : X = \lambda x.\lambda y.x$	$\mathcal{T}_2 : Z = f \cdot (\lambda x.x)$
Mapping:	$m_1 : F \mapsto H^2$	
Links:	$l_1 : x \vdash Y_x =_\eta \lambda y.H_{yx}$	$l_2 : \ \vdash X =_\eta \lambda x.Y_x$
	$l_3 : \ \vdash Z =_\eta \lambda x.f \cdot G_x\ x$	$l_4 : x \vdash G_x =_\eta \lambda y.H_{x\ y}$

After unification of the two terms, X is assigned to $\lambda x.\lambda y.x$. This assignment makes l_2 to progress since the lhs is materialized and by unification, between X and $\lambda x.Y_x$, Y_x is instantiate to $\lambda y.x$. Once Y_x is instantiated, l_1 can progress, and set H_{xy} to x. After all these progresses, l_1 and l_2 are remove from $\mathbb{L}$ and the `progress` fixpoint terminates. Next, the second unification problem is run, and Z is set to $f \cdot (\lambda x.x)$. This unification wakes up l_3 and since Z starts with the `app` node, the η-expanded version of Z is unified with $\lambda x.f \cdot G_x\ x$ and G_x is set to x. As last step, the last link is progressed and the final $\mathcal{H}_o$ substitution is $\{X \mapsto \lambda x.\lambda y.x, Y_x \mapsto \lambda y.x, G_{yx} \mapsto y, Z \mapsto f\ \lambda x.x, H_x \mapsto \lambda y.y\}$.

The decompilation phase is only charged, in this example to solve the mappings, since no suspended links remain. The only mapping in the list is $F \mapsto H^2$, which will assign the F variable in $\mathcal{F}_o$ to $\lambda xy.y$

TODO: dire che preserviamo l'invariante che tutte le variable sono fully-applied

8 HANDLING OF $\overline{\mathcal{L}_\lambda}$

TODO: say that maybe-eta also work in not(llambda)

All the previous sections we have dealt with terms in $\mathcal{L}_\lambda$, however, it is often possible to work with terms in $\overline{\mathcal{L}_\lambda}$ and wish to unify them. There are situtation, for example, where the oracle has given which β-reduction problems ($\overline{\mathcal{L}_\lambda}$) appears any time we deal with a subterm $t = Xt_1 \dots t_n$, where X is flexible and the list $[t_1 \dots t_n]$ in not in $\mathcal{L}_\lambda$. This unification problem is not solvable without loss of generality, since there is not a most general unifier. If we take back the example given in section 2.1, the unification $Fa = a$ admits two solutions for F: $\rho_1 = \{F \mapsto \lambda x.x\}$ and $\rho_2 = \{F \mapsto \lambda\_.a\}$. Despite this, it is possible to work with $\overline{\mathcal{L}_\lambda}$ if an oracle provides a substitution ρ such that ρt falls again in the $\mathcal{L}_\lambda$.

On the other hand, the $\simeq_\lambda$ is not designed to understand how the β-redexes work in the object language. Therefore, even if we know that F is assigned to $\lambda x.x$, $\simeq_\lambda$ is not able to unify Fa with a. On the other hand, the problem $Fa = G$ is solvable by $\simeq_\lambda$, but the final result is that G is assigned to $(\lambda x.x)a$ which breaks the invariant saying that the substitution of the meta language does not generate terms outide $\mathcal{W}$ (Property 2.8).

The solution to this problem is to modify the compiler such that any sub-term t considered as a potential β-redex is replaced with a hole h and a new dedicated link, called `link-`β.

```
type link-beta tm -> tm -> link.
```

This link carries two terms, the former representing the variable h for the new created hole and the latter containing the subterm t. As for the `link-`η, we will call h and t respectively the left hand side (lhs) and the right hand side (rhs) of the `link-`β.

8.1 Compilation

Detection of $\overline{\mathcal{L}_\lambda}$. TODO: ...

Compilation with `link-`β. In order to build a `link-`β, we need to adapt the compiler so that it can recognize these "problematic" subterms. The following code snippet illustrate such behavior, we suppose the rule to be added just after **??**.

```
comp (fapp [fuva A|Ag]) (uva C Scope) M1 M3 L1 L3 S1 S4 :- !,
  pattern-fragment-prefix Ag Pf Extra,
  fold6 comp Pf     Scope1 M1 M1 L1 L1 S1 S1,
  fold6 comp Extra Extra1 M1 M2 L1 L2 S1 S2,
```

```
    len Pf Arity,
    m-alloc (fv A) (hv B (arity Arity)) M2 M3 S2 S3,
    Beta = app [uva B Scope1 | Extra1],
    get-scope Beta Scope,
    alloc S3 C S4,
    L3 = [@val-link-beta (uva C Scope) Beta | L2].
```

A term is $\overline{\mathcal{L}_\lambda}$ if it has the shape `fapp[fuva A|Ag]` and `distinct Ag` does not hold. In that case, `Ag` is split in two sublist `Pf` and `Extra` such that former is the longest prefix of `Ag` such that `distinct Pf` holds. `Extra` is the list such that `append Pf Extra Ag`. Next important step is to compile recursively the terms of these lists and allocate a memory adress `B` from the substitution in order to map the $\mathcal{F}_o$variable `fuva A` to the $\mathcal{H}_o$ variable `uva B`. The `link-`β to return in the end is given by the term `Beta = app[uva B Scope1 | Extra1]` constituting the *rhs*, and a fresh variable `C` having in scope all the free variables occurring in `Beta` (this is *lhs*). We point out that the *rhs* is intentionally built as an `uva` where `Extra1` are not in scope, since by invariant, we want all the variables appearing in $\mathcal{H}_o$ to be in $\mathcal{L}_\lambda$.

8.2 Progress

Once created, there exist two main situations waking up a suspended `link-`β. The former is strictly connected to the definition of β-redex and occurs when the head of *rhs* is materialized by the oracle (see proposition 2.1). In this case *rhs* is safely β-reduced to a new term t' and the result can be unified with *lhs*. In this scenario the `link-`β has accomplished its goal and can be removed from $\mathbb{L}$.

explain why

The second circumstance making the `link-`β to progress is the instantiation of the variables in the `Extra1` making the corresponding arguments to reduce to names. In this case, we want to take the list `Scope1` and append to it the largest prefix of `Extra1` in a new variable `Scope2` such that `Scope2` remains in $\mathcal{L}_\lambda$; we call `Extra2` the suffix of `Extra1` such that the concatenation of `Scope1` and `Extra1` is the same as the concatenation of `Scope2` and `Extra2`. Finally, two cases should be considered: 1) `Extra2` is the empty list, *lhs* and rhs can be unified: we have two terms in $\mathcal{L}_\lambda$; otherwise 2) the `link-`β in question is replaced with a refined version where the *rhs* is `app[uva C Scope2 | Extra2]` and a new `link-`η is added between the *lhs* and the new-added variable `C`.

dire che si adatta bene nelle approximation

An example justifying this second link manipulation is given by the following unification problem:

```
f = flam x\ fapp[F, fapp[A, x]].
```

The compilation of these terms produces the new unification problem: $f = X0$

We obtain the mappings $F \mapsto \mathbf{F}^0, A \mapsto \mathbf{A}^1$ and the links:

$$c0 \vdash X3_{c0} =_\beta X2\ X1_{c0} \tag{12}$$

$$\vdash X0 =_\eta \lambda c0.X3_{c0} \tag{13}$$

where the first link is a `link-`η between the variable `X0`, representing the right side of the unification problem (it is a $\Diamond\eta$) and `X3`; and a `link-`β between the variable `X3` and the subterm $\lambda x.X1_x\ a$ (it is a $\overline{\mathcal{L}_\lambda}$). The substitution tells that $x \vdash X1_x = x$.

We can now represent the hrun execution from this configuration which will, at first, dereference all the links, and then try to solve them. The only link being modified is the second one, which is set to $x \vdash X3 =_\beta X2xa$. The *rhs* of the link has now a variable which is partially in the PF, we can therefore remove the original `link-`β and replace it with the following couple on links:

```
  ⊢ X1   =η= x\ `X4 x'
x ⊢ X3 x =β= x\ `X4 x' a
```

By these links we say that `X1` is now η-linked to a fresh variable `X4` with arity one. This new variable is used in the new `link-`β where the name `x` is in its scope. This allows

8.3 Tricky examples

```
triple ok (@lam x\ @app[@f, @app[@X, x]]) @Y,
triple ok @X (@lam x\ x),
triple ok @Y @f

% @okl 22 [
%   triple ok (@lam x\ @lam y\ @app[@Y, y, x]) @X,
%   triple ok (@lam x\ @f) @X,
% ].
```

9 FIRST ORDER APPROXIMATION

TODO: Coq can solve this: `f 1 2 = X 2`, by setting X to f 1
TODO: We can re-use part of the algo for β given before

10 UNIF ENCODING IN REAL LIFE

TODO: Il ML presentato qui è esattamente elpi
TODO: Il OL presentato qui è esattamente coq
TODO: Come implementatiamo tutto ciò nel solver

11 RESULTS: STDPP AND TLC

TODO: How may rule are we solving?
TODO: Can we do some perf test

12 CONCLUSION

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APPENDIX

This appendix contains the entire code described in this paper. The code can also be accessed at the URL: https://github.com/FissoreD/paper-ho

Note that `(a infix b) c d` de-sugars to `(infix) a b c d`.

Explain builtin name (can be implemented by loading name after each pi)

13 THE MEMORY

```
kind addr type.
type addr nat -> addr.
typeabbrev (mem A) (list (option A)).

type set? addr -> mem A -> A -> o.
set? (addr A) Mem Val :- get A Mem Val.

type unset? addr -> mem A -> o.
unset? Addr Mem :- not (set? Addr Mem _).

type assign-aux nat -> mem A -> A -> mem A -> o.
assign-aux z (none :: L) Y (some Y :: L).
assign-aux (s N) (X :: L) Y (X :: L1) :- assign-aux N L Y L1.

type assign addr -> mem A -> A -> mem A -> o.
assign (addr A) Mem1 Val Mem2 :- assign-aux A Mem1 Val Mem2.

type get nat -> mem A -> A -> o.
get z (some Y :: _) Y.
get (s N) (_ :: L) X :- get N L X.

type alloc-aux nat -> mem A -> mem A -> o.
alloc-aux z [] [none] :- !.
alloc-aux z L L.
alloc-aux (s N) [] [none | M] :- alloc-aux N [] M.
alloc-aux (s N) [X | L] [X | M] :- alloc-aux N L M.

type alloc addr -> mem A -> mem A -> o.
alloc (addr A as Ad) Mem1 Mem2 :- unset? Ad Mem1,
  alloc-aux A Mem1 Mem2.

type new-aux mem A -> nat -> mem A -> o.
new-aux [] z [none].
new-aux [A | As] (s N) [A | Bs] :- new-aux As N Bs.

type new mem A -> addr -> mem A -> o.
new Mem1 (addr Ad) Mem2 :- new-aux Mem1 Ad Mem2.
```

14 THE OBJECT LANGUAGE

```
kind fm type.
type fapp list fm -> fm.
type flam (fm -> fm) -> fm.
type fcon string -> fm.
type fuva addr -> fm.

typeabbrev fsubst (mem fm).

type fder fsubst -> fm -> fm -> o.
fder _ (fcon C) (fcon C).
fder S (fapp A) (fapp B) :- map (fder S) A B.
fder S (flam F) (flam G) :-
  pi x\ fder S x x => fder S (F x) (G x).
fder S (fuva N) R :- set? N S T, fder S T R.
fder S (fuva N) (fuva N) :- unset? N S.

type fderef fsubst -> fm -> fm -> o.                    (ρs)
fderef S T T2 :- fder S T T1, napp T1 T2.


type (=_o) fm -> fm -> o.                               (=_o)
fcon X =_o fcon X.
fapp A =_o fapp B :- forall2 (=_o) A B.
flam F =_o flam G :- pi x\ x =_o x => F x =_o G x.
fuva N =_o fuva N.
flam F =_o T :-                                         (η_l)
  pi x\ beta T [x] (T' x), x =_o x => F x =_o T' x.
T =_o flam F :-                                         (η_r)
  pi x\ beta T [x] (T' x), x =_o x => T' x =_o F x.
fapp [flam X | L] =_o T :- beta (flam X) L R, R =_o T.  (β_l)
T =_o fapp [flam X | L] :- beta (flam X) L R, T =_o R.  (β_r)

type extend-subst fm -> fsubst -> fsubst -> o.
extend-subst (fuva N) S S' :- mem.alloc N S S'.
extend-subst (flam F) S S' :-
  pi x\ (pi S\extend-subst x S S) => extend-subst (F x) S S'.
extend-subst (fcon _) S S.
extend-subst (fapp L) S S1 :- fold extend-subst L S S1.

type beta fm -> list fm -> fm -> o.
beta A [] A.
beta (flam Bo) [H | L] R :- napp (Bo H) F, beta F L R.
beta (fapp A) L (fapp X) :- append A L X.
beta (fuva N) L (fapp [fuva N | L]).
beta (fcon H) L (fapp [fcon H | L]).
beta N L (fapp [N | L]) :- name N.

type napp fm -> fm -> o.
napp (fcon C) (fcon C).
napp (fuva A) (fuva A).
napp (flam F) (flam G) :- pi x\ napp (F x) (G x).
napp (fapp [fapp L1 |L2]) T :- !,
  append L1 L2 L3, napp (fapp L3) T.
napp (fapp L) (fapp L1) :- map napp L L1.
napp N N :- name N.

type mk-app fm -> list fm -> fm -> o.
mk-app T L S :- beta T L S.

type eta-contract fm -> fm -> o.
eta-contract (fcon X) (fcon X).
eta-contract (fapp L) (fapp L1) :- map eta-contract L L1.
eta-contract (flam F) T :- eta-contract-aux [] (flam F) T.
eta-contract (flam F) (flam F1) :-
```

```
  pi x\ eta-contract x x => eta-contract (F x) (F1 x).
eta-contract (fuva X) (fuva X).
eta-contract X X :- name X.

type eta-contract-aux list fm -> fm -> fm -> o.
eta-contract-aux L (flam F) T :-
  pi x\ eta-contract-aux [x|L] (F x) T. % also checks H Prefix does not see x
eta-contract-aux L (fapp [H|Args]) T :-
  rev L LRev, append Prefix LRev Args,
  if (Prefix = []) (T = H) (T = fapp [H|Prefix]).
```

15 THE META LANGUAGE

```
kind inctx type -> type.                              (· ⊢ ·)
type abs (tm -> inctx A) -> inctx A.
type val A -> inctx A.
typeabbrev assignment (inctx tm).
typeabbrev subst (mem assignment).

kind tm  type.
type app list tm -> tm.
type lam (tm -> tm) -> tm.
type con string -> tm.
type uva addr -> list tm -> tm.

type (≃λ) tm -> tm -> subst -> subst -> o.
(con C ≃λ con C) S S.
(app L1 ≃λ app L2) S S1 :- fold2 (≃λ) L1 L2 S S1.
(lam F1 ≃λ lam F2) S S1 :-
  pi x\ (pi S\ (x ≃λ x) S S) => (F1 x ≃λ F2 x) S S1.
(uva N Args ≃λ T) S S1 :-
  set? N S F,!, move F Args T1, (T1 ≃λ T) S S1.
(T ≃λ uva N Args) S S1 :-
  set? N S F,!, move F Args T1, (T ≃λ T1) S S1.
(uva M A1 ≃λ uva N A2) S1 S2 :- !,
  pattern-fragment A1, pattern-fragment A2,
  prune! M A1 N A2 S1 S2.
(uva N Args ≃λ T) S S1 :- not_occ N S T, pattern-fragment Args,
  bind T Args T1, assign N S T1 S1.
(T ≃λ uva N Args) S S1 :- not_occ N S T, pattern-fragment Args,
  bind T Args T1, assign N S T1 S1.

type prune! addr -> list tm -> addr ->
            list tm -> subst -> subst -> o.
/* no pruning needed */
prune! N A  N A  S S :- !.
prune! M A  N A  S1 S2 :- !, bind (uva M A) A Ass,
  assign N S1 Ass S2.
/* prune different arguments */
prune! N A1 N A2 S1 S3 :- !,
  new S1 W S2, prune-same-variable W A1 A2 [] Ass,
  assign N S2 Ass S3.
/* prune to the intersection of scopes */
prune! N A1 M A2 S1 S4 :- !,
  new S1 W S2, prune-diff-variables W A1 A2 Ass1 Ass2,
  assign N S2 Ass1 S3,
  assign M S3 Ass2 S4.
```

```
type prune-same-variable addr -> list tm -> list tm ->
                         list tm -> assignment -> o.
prune-same-variable N [] [] ACC (val (uva N Args)) :-
  rev ACC Args.
prune-same-variable N [X|XS] [X|YS] ACC (abs F) :-
  pi x\ prune-same-variable N XS YS [x|ACC] (F x).
prune-same-variable N [_|XS] [_|YS] ACC (abs F) :-
  pi x\ prune-same-variable N XS YS ACC (F x).

type permute list nat -> list tm -> list tm -> o.
permute [] _ [].
permute [P|PS] Args [T|TS] :-
  nth P Args T,
  permute PS Args TS.

type build-perm-assign addr -> list tm -> list bool ->
                       list nat -> assignment -> o.
build-perm-assign N ArgsR [] Perm (val (uva N PermutedArgs)) :-
  rev ArgsR Args, permute Perm Args PermutedArgs.
build-perm-assign N Acc [tt|L] Perm (abs T) :-
  pi x\ build-perm-assign N [x|Acc] L Perm (T x).
build-perm-assign N Acc [ff|L] Perm (abs T) :-
  pi x\ build-perm-assign N Acc L Perm (T x).

type keep list A -> A -> bool -> o.
keep L A tt :- mem L A, !.
keep _ _ ff.

type prune-diff-variables addr -> list tm -> list tm ->
                          assignment -> assignment -> o.
prune-diff-variables N Args1 Args2 Ass1 Ass2 :-
  map (keep Args2) Args1 Bits1,
  map (keep Args1) Args2 Bits2,
  filter Args1 (mem Args2) ToKeep1,
  filter Args2 (mem Args1) ToKeep2,
  map (index ToKeep1) ToKeep1 IdPerm,
  map (index ToKeep1) ToKeep2 Perm21,
  build-perm-assign N [] Bits1 IdPerm Ass1,
  build-perm-assign N [] Bits2 Perm21 Ass2.

type beta tm -> list tm -> tm -> o.
beta A [] A.
beta (lam Bo) [H | L] R :- beta (Bo H) L R.
beta (app A) L (app X) :- append A L X.
beta (con H) L (app [con H | L]).
beta X L (app[X|L]) :- name X.

/* occur check for N before crossing a functor */
type not_occ  addr -> subst -> tm -> o.
not_occ N S (uva M Args) :- set? M S F,
  move F Args T, not_occ N S T.
not_occ N S (uva M Args) :- unset? M S, not (M = N),
  forall1 (not_occ_aux N S) Args.
not_occ _ _ (con _).
not_occ N S (app L) :- not_occ_aux N S (app L).
/* Note: lam is a functor for the meta language! */
```

```
not_occ N S (lam L) :- pi x\ not_occ_aux N S (L x).
not_occ _ _ X :- name X.
/* finding N is ok */
not_occ N _ (uva N _).

/* occur check for X after crossing a functor */
type not_occ_aux addr -> subst -> tm -> o.
not_occ_aux N S (uva M _) :- unset? M S, not (N = M).
not_occ_aux N S (uva M Args) :- set? M S F,
  move F Args T, not_occ_aux N S T.
not_occ_aux N S (app L) :- forall1 (not_occ_aux N S) L.
not_occ_aux N S (lam F) :- pi x\ not_occ_aux N S (F x).
not_occ_aux _ _ (con _).
not_occ_aux _ _ X :- name X.
/* finding N is ko, hence no rule */

/* copy T T' vails if T contains a free variable, i.e. it
   performs scope checking for bind */
type copy tm -> tm -> o.
copy (con C)   (con C).
copy (app L)   (app L') :- map copy L L'.
copy (lam T)   (lam T') :- pi x\ copy x x => copy (T x) (T' x).
copy (uva A L) (uva A L') :- map copy L L'.

type bind tm -> list tm -> assignment -> o.
bind T [] (val T') :- copy T T'.
bind T [X | TL] (abs T') :- pi x\ copy X x => bind T TL (T' x).

type deref subst -> tm -> tm -> o.                        (σt)
deref _ (con C) (con C).
deref S (app A) (app B) :- map (deref S) A B.
deref S (lam F) (lam G) :-
  pi x\ deref S x x => deref S (F x) (G x).
deref S (uva N L) R :- set? N S A,
  move A L T, deref S T R.
deref S (uva N A) (uva N B) :- unset? N S,
  map (deref S) A B.

type move assignment -> list tm -> tm -> o.
move (abs Bo) [H|L] R :- move (Bo H) L R.
move (val A)  [] A.


type deref-assmt subst -> assignment -> assignment -> o.
deref-assmt S (abs T) (abs R) :- pi x\ deref-assmt S (T x) (R x).
deref-assmt S (val T) (val R) :- deref S T R.
```

16 THE COMPILER

```
kind arity type.
type arity nat -> arity.

kind fvariable type.
type fv addr -> fvariable.

kind hvariable type.
type hv addr -> arity -> hvariable.

kind mapping type.
type mapping fvariable -> hvariable -> mapping.
typeabbrev mmap (list mapping).

typeabbrev scope (list tm).
typeabbrev inctx ho.inctx.
kind baselink type.
type link-eta  tm -> tm -> baselink.
type link-beta tm -> tm -> baselink.
typeabbrev link (inctx baselink).
typeabbrev links (list link).

macro @val-link-eta T1 T2 :- ho.val (link-eta T1 T2).
macro @val-link-beta T1 T2 :- ho.val (link-beta T1 T2).



type occurs-rigidly fm -> fm -> o.
occurs-rigidly N N.
occurs-rigidly _ (fapp [fuva _|_]) :- !, fail.
occurs-rigidly N (fapp L) :- exists (occurs-rigidly N) L.
occurs-rigidly N (flam B) :- pi x\ occurs-rigidly N (B x).

type reducible-to list fm -> fm -> fm -> o.
reducible-to _ N N :- !.
reducible-to L N (fapp[fuva _|Args]) :- !,
  forall1 (x\ exists (reducible-to [] x) Args) [N|L].
reducible-to L N (flam B) :- !,
  pi x\ reducible-to [x | L] N (B x).
reducible-to L N (fapp [N|Args]) :-
  last-n {len L} Args R,
  forall2 (reducible-to []) R {rev L}.

type maybe-eta fm -> list fm -> o.                   (◇η)
maybe-eta (fapp[fuva _|Args]) L :- !,
  forall1 (x\ exists (reducible-to [] x) Args) L, !.
maybe-eta (flam B) L :- !, pi x\ maybe-eta (B x) [x | L].
maybe-eta (fapp [fcon _|Args]) L :-
  split-last-n {len L} Args First Last,
  none (x\ exists (y\ occurs-rigidly x y) First) L,
  forall2 (reducible-to []) {rev L} Last.


type locally-bound tm -> o.
type get-scope-aux tm -> list tm -> o.
get-scope-aux (con _) [].
get-scope-aux (uva _ L) L1 :-
  forall2 get-scope-aux L R,
  flatten R L1.
get-scope-aux (lam B) L1 :-
  pi x\ locally-bound x => get-scope-aux (B x) L1.
get-scope-aux (app L) L1 :-
  forall2 get-scope-aux L R,
  flatten R L1.
get-scope-aux X [X] :- name X, not (locally-bound X).
get-scope-aux X [] :- name X, (locally-bound X).
```

```
type names1 list tm -> o.
names1 L :-
  names L1,
  new_int N,
  if (1 is N mod 2) (L1 = L) (rev L1 L).

type get-scope tm -> list tm -> o.
get-scope T Scope :-
  get-scope-aux T ScopeDuplicata,
  undup ScopeDuplicata Scope.
type rigid fm -> o.
rigid X :- not (X = fuva _).

type comp-lam (fm -> fm) -> (tm -> tm) ->
  mmap -> mmap -> links -> links -> subst -> subst -> o.
comp-lam F G M1 M2 L1 L3 S1 S2 :-
  pi x y\ (pi M L S\ comp x y M M L L S S) =>
    comp (F x) (G y) M1 M2 L1 (L2 y) S1 S2,
  close-links L2 L3.

type close-links (tm -> links) -> links -> o.
close-links (v\[X  |L v]) [X|R] :- !, close-links L R.
close-links (v\[X v|L v]) [abs X|R] :- close-links L R.
close-links (_\[]) [].
type comp fm -> tm -> mmap -> mmap -> links -> links ->
  subst -> subst -> o.
comp (fcon C) (con C) M M L L S S.
comp (flam F) (uva A Scope) M1 M2 L1 L3 S1 S3 :-
  maybe-eta (flam F) [], !,
    alloc S1 A S2,
    comp-lam F F1 M1 M2 L1 L2 S2 S3,
    get-scope (lam F1) Scope,
    L3 = [val (link-eta (uva A Scope) (lam F1)) | L2].
comp (flam F) (lam F1) M1 M2 L1 L2 S1 S2 :-            (c_λ)
  comp-lam F F1 M1 M2 L1 L2 S1 S2.
comp (fuva A) (uva B []) M1 M2 L L S1 S2 :-
  m-alloc (fv A) (hv B (arity z)) M1 M2 S1 S2.
comp (fapp [fuva A|Ag]) (uva B Ag1) M1 M2 L L S1 S2 :-
  pattern-fragment Ag, !,
    fold6 comp Ag Ag1 M1 M1 L L S1 S1,
    len Ag Arity,
    m-alloc (fv A) (hv B (arity Arity)) M1 M2 S1 S2.
comp (fapp [fuva A|Ag]) (uva C Scope) M1 M3 L1 L3 S1 S4 :- !,
  pattern-fragment-prefix Ag Pf Extra,
  fold6 comp Pf    Scope1 M1 M1 L1 L1 S1 S1,
  fold6 comp Extra Extra1 M1 M2 L1 L2 S1 S2,
  len Pf Arity,
  m-alloc (fv A) (hv B (arity Arity)) M2 M3 S2 S3,
  Beta = app [uva B Scope1 | Extra1],
  get-scope Beta Scope,
  alloc S3 C S4,
  L3 = [@val-link-beta (uva C Scope) Beta | L2].
comp (fapp A) (app A1) M1 M2 L1 L2 S1 S2 :-
  fold6 comp A A1 M1 M2 L1 L2 S1 S2.

type alloc mem A -> addr -> mem A -> o.
alloc S N S1 :- mem.new S N S1.

type compile-terms-diagnostic
  triple diagnostic fm fm ->
  triple diagnostic tm tm ->
  mmap -> mmap ->
  links -> links ->
  subst -> subst -> o.
compile-terms-diagnostic (triple D FO1 FO2) (triple D HO1 HO2) M1 M3 L1
  comp FO1 HO1 M1 M2 L1 L2 S1 S2,
  comp FO2 HO2 M2 M3 L2 L3 S2 S3.

type compile-terms
  list (triple diagnostic fm fm) ->
  list (triple diagnostic tm tm) ->
  mmap -> links -> subst -> o.
compile-terms T H M L S :-
  fold6 compile-terms-diagnostic T H [] M_ [] L_ [] S_,
  deduplicate-map M_ M S_ S L_ L.

type make-eta-link-aux nat -> addr -> addr ->
  list tm -> links -> subst -> subst -> o.
make-eta-link-aux z Ad1 Ad2 Scope1 L H1 H1  :-
  rev Scope1 Scope, eta-expand (uva Ad2 Scope) T1,
  L = [val (link-eta (uva Ad1 Scope) T1)].
make-eta-link-aux (s N) Ad1 Ad2 Scope1 L H1 H3 :-
  rev Scope1 Scope, alloc H1 Ad H2,
  eta-expand (uva Ad Scope) T2,
  (pi x\ make-eta-link-aux N Ad Ad2 [x|Scope1] (L1 x) H2 H3),
  close-links L1 L2,
  L = [val (link-eta (uva Ad1 Scope) T2) | L2].

type make-eta-link nat -> nat -> addr -> addr ->
      list tm -> links -> subst -> subst -> o.
make-eta-link (s N) z Ad1 Ad2 Vars L H H1 :-
  make-eta-link-aux N Ad2 Ad1 Vars L H H1.
make-eta-link z (s N) Ad1 Ad2 Vars L H H1 :-
  make-eta-link-aux N Ad1 Ad2 Vars L H H1.
make-eta-link (s N) (s M) Ad1 Ad2 Vars Links H H1 :-
  (pi x\ make-eta-link N M Ad1 Ad2 [x|Vars] (L x) H H1),
  close-links L Links.

type deduplicate-map mmap -> mmap ->
    subst -> subst -> links -> links -> o.
deduplicate-map [] [] H H L L.
deduplicate-map [(mapping (fv O) (hv M (arity LenM)) as X1) | Map1] Map2
  take-list Map1 (mapping (fv O) (hv M' (arity LenM'))) _, !,
  std.assert! (not (LenM = LenM')) "Deduplicate map, there is a bug",
  print "arity-fix links:" {ppmapping X1} "~!~" {ppmapping (mapping (fv
  make-eta-link LenM LenM' M M' [] New H1 H2,
  print "new eta link" {pplinks New},
  append New L1 L2,
  deduplicate-map Map1 Map2 H2 H3 L2 L3.
deduplicate-map [A|As] [A|Bs] H1 H2 L1 L2 :-
  deduplicate-map As Bs H1 H2 L1 L2, !.
deduplicate-map [A|_] _ H _ _ _ :-
  halt "deduplicating mapping error" {ppmapping A} {ho.ppsubst H}.
```

17 THE PROGRESS FUNCTION

```
macro @one :- s z.

type contract-rigid list ho.tm -> ho.tm -> ho.tm -> o.
contract-rigid L (ho.lam F) T :-
  pi x\ contract-rigid [x|L] (F x) T. % also checks H Prefix does not see x
contract-rigid L (ho.app [H|Args]) T :-
  rev L LRev, append Prefix LRev Args,
  if (Prefix = []) (T = H) (T = ho.app [H|Prefix]).

type progress-eta-link ho.tm -> ho.tm -> ho.subst -> ho.subst -> links -> o.
progress-eta-link (ho.app _ as T) (ho.lam x\ _ as T1) H H1 [] :- !,
  ({eta-expand T @one} ==l T1) H H1.
progress-eta-link (ho.con _ as T) (ho.lam x\ _ as T1) H H1 [] :- !,
  ({eta-expand T @one} ==l T1) H H1.
progress-eta-link (ho.lam _ as T) T1 H H1 [] :- !,
  (T ==l T1) H H1.
progress-eta-link (ho.uva _ _ as X) T H H1 [] :-
  contract-rigid [] T T1, !, (X ==l T1) H H1.
progress-eta-link (ho.uva Ad _ as T1) T2 H H [@val-link-eta T1 T2] :- !,
  if (ho.not_occ Ad H T2) true fail.

type is-in-pf ho.tm -> o.
is-in-pf (ho.app [ho.uva _ _ | _]) :- !, fail.
is-in-pf (ho.lam B) :- !, pi x\ is-in-pf (B x).
is-in-pf (ho.con _).
is-in-pf (ho.app L) :- forall1 is-in-pf L.
is-in-pf N :- name N.
is-in-pf (ho.uva _ L) :- pattern-fragment L.

type arity ho.tm -> nat -> o.
arity (ho.con _) z.
arity (ho.app L) A :- len L A.

type occur-check-err ho.tm -> ho.tm -> ho.subst -> o.
occur-check-err (ho.con _) _ _ :- !.
occur-check-err (ho.app _) _ _ :- !.
occur-check-err (ho.lam _) _ _ :- !.
occur-check-err (ho.uva Ad _) T S :-
  not (ho.not_occ Ad S T).

type progress-beta-link-aux ho.tm -> ho.tm ->
        ho.subst -> ho.subst -> links -> o.
progress-beta-link-aux T1 T2 S1 S2 [] :-  is-in-pf T2, !,
  (T1 ==l T2) S1 S2.
progress-beta-link-aux T1 T2 S S [@val-link-beta T1 T2] :- !.

type progress-beta-link ho.tm -> ho.tm -> ho.subst ->
      ho.subst -> links -> o.
progress-beta-link T (ho.app[ho.uva V Scope | L] as T2) S S2 [@val-link-beta T T2] :-
  arity T Arity, len L ArgsNb, ArgsNb >n Arity, !,
  minus ArgsNb Arity Diff, mem.new S V1 S1,
  eta-expand (ho.uva V1 Scope) Diff T1,
  ((ho.uva V Scope) ==l T1) S1 S2.

progress-beta-link (ho.uva _ _ as T) (ho.app[ho.uva Ad1 Scope1 | L1] as T1) S1 S3 NewLinks :-
  append Scope1 L1 Scope1L,
  pattern-fragment-prefix Scope1L Scope2 L2,
  not (Scope1 = Scope2), !,
  mem.new S1 Ad2 S2,
  len Scope1 Scope1Len,
  len Scope2 Scope2Len,
  make-eta-link Scope1Len Scope2Len Ad1 Ad2 [] LinkEta S2 S3,
  if (L2 = []) (NewLinks = LinkEta, T2 = ho.uva Ad2 Scope2)
    (T2 = ho.app [ho.uva Ad2 Scope2 | L2],
    NewLinks = [@val-link-beta T T2 | LinkEta]).

progress-beta-link T1 (ho.app[ho.uva _ _ | _] as T2) _ _ _ :-
  not (T1 = ho.uva _ _), !, fail.

progress-beta-link (ho.uva _ _ as T) (ho.app[ho.uva _ _ | _] as T2) S1 _
  occur-check-err T T2 S1, !, fail.

progress-beta-link T1 (ho.app[ho.uva _ _ | _] as T2) H H [@val-link-beta

progress-beta-link T1 (ho.app [Hd | Tl]) S1 S2 B :-
  ho.beta Hd Tl T3,
  progress-beta-link-aux T1 T3 S1 S2 B.

type solve-link-abs link -> links -> ho.subst -> ho.subst -> o.
solve-link-abs (ho.abs X) R H H1 :-
  pi x\ ho.copy x x => (pi S\ ho.deref S x x) =>
    solve-link-abs (X x) (R' x) H H1,
  close-links R' R.

solve-link-abs (@val-link-eta A B) NewLinks S S1 :- !,
  progress-eta-link A B S S1 NewLinks.

solve-link-abs (@val-link-beta A B) NewLinks S S1 :- !,
  progress-beta-link A B S S1 NewLinks.

type take-link link -> links -> link -> links -> o.
take-link A [B|XS] B XS :- link-abs-same-lhs A B, !.
take-link A [L|XS] B [L|YS] :- take-link A XS B YS.

type link-abs-same-lhs link -> link -> o.
link-abs-same-lhs (ho.abs F) B :-
  pi x\ link-abs-same-lhs (F x) B.
link-abs-same-lhs A (ho.abs G) :-
  pi x\ link-abs-same-lhs A (G x).
link-abs-same-lhs (@val-link-eta (ho.uva N _) _) (@val-link-eta (ho.uva

type same-link-eta link -> link -> ho.subst -> ho.subst -> o.
same-link-eta (ho.abs F) B H H1 :- !, pi x\ same-link-eta (F x) B H H1.
same-link-eta A (ho.abs G) H H1 :- !, pi x\ same-link-eta A (G x) H H1.
same-link-eta (@val-link-eta (ho.uva N S1) A)
              (@val-link-eta (ho.uva N S2) B) H H1 :-
  std.map2 S1 S2 (x\y\r\ r = ho.copy x y) Perm,
  Perm => ho.copy A A',
  (A' ==l B) H H1.

type progress1 links -> links -> ho.subst -> ho.subst -> o.
progress1 [] [] X X.
```

```
progress1 [A|L1] [A|L3] S S2 :- take-link A L1 B L2, !,
  same-link-eta A B S S1,
  progress1 L2 L3 S1 S2.
progress1 [L0|L1] L3 S S2 :- deref-link S L0 L,
  solve-link-abs L R S S1, !,
  progress1 L1 L2 S1 S2, append R L2 L3.
```

18 THE DECOMPILER

```
type abs->lam ho.assignment -> ho.tm -> o.
abs->lam (ho.abs T) (ho.lam R)  :- !, pi x\ abs->lam (T x) (R x).
abs->lam (ho.val A) A.

type commit-links-aux link -> ho.subst -> ho.subst -> o.
commit-links-aux (@val-link-eta T1 T2) H1 H2 :-
  ho.deref H1 T1 T1', ho.deref H1 T2 T2',
  (T1' ==l T2') H1 H2.
commit-links-aux (@val-link-beta T1 T2) H1 H2 :-
  ho.deref H1 T1 T1', ho.deref H1 T2 T2',
  (T1' ==l T2') H1 H2.
commit-links-aux (ho.abs B) H H1 :-
  pi x\ commit-links-aux (B x) H H1.

type commit-links links -> links -> ho.subst -> ho.subst -> o.
commit-links [] [] H H.
commit-links [Abs | Links] L H H2 :-
  commit-links-aux Abs H H1, !, commit-links Links L H1 H2.

type decompl-subst map -> map -> ho.subst ->
  fo.fsubst -> fo.fsubst -> o.
decompl-subst _ [A|_] _ _ _ :- fail.
decompl-subst _ [] _ F F.
decompl-subst Map [mapping (fv VO) (hv VM _)|Tl] H F F2 :-
  mem.set? VM H T, !,
  ho.deref-assmt H T TTT,
  abs->lam TTT T', tm->fm Map T' T1,
  fo.eta-contract T1 T2, mem.assign VO F T2 F1,
  decompl-subst Map Tl H F1 F2.
decompl-subst Map [mapping _ (hv VM _)|Tl] H F F2 :-
  mem.unset? VM H, decompl-subst Map Tl H F F2.

type tm->fm map -> ho.tm -> fo.fm -> o.
tm->fm _ (ho.con C)  (fo.fcon C).
tm->fm L (ho.lam B1) (fo.flam B2) :-
  pi x y\ tm->fm _ x y => tm->fm L (B1 x) (B2 y).
tm->fm L (ho.app L1) T :- map (tm->fm L) L1 [Hd|Tl],
  fo.mk-app Hd Tl T.
tm->fm L (ho.uva VM TL) T :- mem L (mapping (fv VO) (hv VM _)),
  map (tm->fm L) TL T1, fo.mk-app (fo.fuva VO) T1 T.

type add-new-map-aux ho.subst -> list ho.tm -> map ->
     map ->  fo.fsubst -> fo.fsubst -> o.
add-new-map-aux _ [] _ [] S S.
add-new-map-aux H [T|Ts] L L2 S S2 :-
  add-new-map H T L L1 S S1,
  add-new-map-aux H Ts L1 L2 S1 S2.

type add-new-map ho.subst -> ho.tm -> map ->
   map ->  fo.fsubst -> fo.fsubst -> o.
add-new-map _ (ho.uva N _) Map [] F1 F1 :-
  mem Map (mapping _ (hv N _)), !.
add-new-map H (ho.uva N L) Map [Map1 | MapL] F1 F3 :-
  mem.new F1 M F2,
  len L Arity, Map1 = mapping (fv M) (hv N (arity Arity)),
  add-new-map H (ho.app L) [Map1 | Map] MapL F2 F3.
add-new-map H (ho.lam B) Map NewMap F1 F2 :-
  pi x\ add-new-map H (B x) Map NewMap F1 F2.
add-new-map H (ho.app L) Map NewMap F1 F3 :-
  add-new-map-aux H L Map NewMap F1 F3.
add-new-map _ (ho.con _) _ [] F F :- !.
add-new-map _ N _ [] F F :- name N.

type complete-mapping-under-ass ho.subst -> ho.assignment ->
  map -> map ->  fo.fsubst -> fo.fsubst -> o.
complete-mapping-under-ass H (ho.val Val) Map1 Map2 F1 F2 :-
  add-new-map H Val Map1 Map2 F1 F2.
complete-mapping-under-ass H (ho.abs Abs) Map1 Map2 F1 F2 :-
  pi x\ complete-mapping-under-ass H (Abs x) Map1 Map2 F1 F2.

type complete-mapping ho.subst -> ho.subst ->
  map -> map -> fo.fsubst -> fo.fsubst -> o.
complete-mapping _ [] L L F F.
complete-mapping H [none | Tl] L1 L2 F1 F2 :-
  complete-mapping H Tl L1 L2 F1 F2.
complete-mapping H [some T0 | Tl] L1 L3 F1 F3 :-
  ho.deref-assmt H T0 T,
  complete-mapping-under-ass H T L1 L2 F1 F2,
  append L1 L2 LAll,
  complete-mapping H Tl LAll L3 F2 F3.

type decompile map -> links -> ho.subst ->
  fo.fsubst -> fo.fsubst -> o.
decompile Map1 L HO FO FO2 :-
  commit-links L L1_ HO HO1, !,
  complete-mapping HO1 HO1 Map1 Map2 FO FO1,
  decompl-subst Map2 Map2 HO1 FO1 FO2.
```

19 AUXILIARY FUNCTIONS

```
type fold4 (A -> A1 -> B -> B -> C -> C -> o) -> list A ->
  list A1 -> B -> B -> C -> C -> o.
fold4 _ [] [] A A B B.
fold4 F [X|XS] [Y|YS] A A1 B B1 :- F X Y A A0 B B0,
  fold4 F XS YS A0 A1 B0 B1.

type len list A -> nat -> o.
len [] z.
len [_|L] (s X) :- len L X.
```