Higher order unification for free!

Reusing the meta-language unification for the object language

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ABSTRACT

Specifying and implementing a proof system from scratch requires significant effort. Logical Frameworks and Higher Order Logic Programming Languages provide dedicated, high-level Meta Languages (ML) to facilitate this task in two key ways: 1) variable binding and substitution are simplified when ML binders represent object logic ones; 2) proof construction, and even proof search, is greatly simplified by leveraging the unification procedure provided by the ML. Notable examples of ML are Elf [14], Twelf [15], λProlog [10] and Isabelle [22] which have been utilized to implement various formal systems such as First Order Logic [4], Set Theory [13], Higher Order Logic [12], and even the Calculus of Constructions [3].

The object logic we are interested in is Coq's [20] Calculus of Inductive Constructions (CIC), for which we aim to implement a unification procedure $\simeq_o$ using the ML Elpi [2], a dialect of λProlog. Elpi's equational theory comprises $\eta\beta$ equivalence and comes equipped with a higher order unification procedure $\simeq_\lambda$ restricted to the pattern fragment [9]. We want $\simeq_o$ to be as powerful as $\simeq_\lambda$ but on the object logic CIC. Elpi also comes with an encoding for CIC that works well for meta-programming [19, 18, 7, 5]. Unfortunately this encoding, which we refer to as $\mathcal{F}_o$, "underuses" $\simeq_\lambda$ by restricting it to first-order unification problems only. To address this issue, we propose a better-behaved encoding, $\mathcal{H}_o$, demonstrate how to map unification problems in $\mathcal{F}_o$ to related problems in $\mathcal{H}_o$, and illustrate how to map back the unifiers found by $\simeq_\lambda$, effectively implementing $\simeq_o$ on top of $\simeq_\lambda$ for the encoding $\mathcal{F}_o$.

KEYWORDS

Logic Programming, Meta-Programming, Higher-Order Unification, Proof Automation

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1 INTRODUCTION

Specifying and implementing a logic or a proof system from scratch requires significant effort. Logical Frameworks and Higher Order Logic Programming Languages provide dedicated, high-level Meta Languages (ML) to facilitate this task in two key ways. The first and most well know one is that variable binding and substitution can be taken for granted when ML binders represent object logic ones. The second one that comes to mind is unification, the cornerstone for proof construction and proof search, however in this paper we describe how reusing that brick may not be as easy at is seems.

Meta languages such as Elf [14], Twelf [15], λProlog [10] and Isabelle [22] have been utilized to specify various logics [4, 12, 13, 3]. In some cases, the most notable one being Higher Order Logic [12], the ML Isabelle is such a good fit that it implements an interactive proof system for HOL, and not just a specification.

The object logic we are interested in is Coq's [20] Calculus of Inductive Constructions (CIC), and we want to implement a type-class [21] solver for Coq [20] using the Coq-Elpi [19] meta programming language, a dialect of λProlog already used to extend Coq [19, 18, 7, 5]. Type-class solvers are unification based proof search procedures that provide essential automation to widely used Coq libraries. These solvers are reminiscent of Prolog: they back-chain lemmas taken from a designated database of type class instances.

As an example we take the `Decision` type class from the Stdpp [8] library. The class identifies predicates equipped with a decision procedure. The following three designated `Instances` state that: 1) the type `fin n`, of natural numbers smaller than `n`, is `Finite` (another type class); 2) the predicate `nfact n nf`, linking a natural number `n` to its prime factors `nf`, is decidable; 3) the universal closure of a predicate has a decision procedure if its domain is finite and if the predicate is decidable.

```
Instance fin_fin n : Finite (fin n).                (* r1 *)
Instance nfact_dec n nf : Decision (nfact n nf).    (* r2 *)
Instance forall_dec A P : Finite A →                (* r3 *)
  ∀x:A, Decision (P x) → Decision (∀x:A, P x).
```

Given this database of instances a type-class solver is able to prove the following statement automatically, by back-chaining the lemmas above:

```
Check _ : Decision (forall n: fin 7, nfact n 3).         (g)
```

The encoding of CIC provided by Elpi, that we will discuss at length later in sections 3 and 4, is an Higher Order Abstract Syntax (HOAS) datatype `tm` featuring (among others) the following constructors:

```
type lam  tm -> (tm -> tm) -> tm.    % lambda abstraction
type app  list tm -> tm.             % n-ary application
type all  tm -> (tm -> tm) -> tm.    % forall quantifier
type con  string -> tm.              % constants
```

Following standard λProlog [10] the concrete syntax to abstract, at the meta level, an expression `e` over a variable `x` is «`x\ e`», and

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square brackets denote a list of terms separated by comma. As an example we show the encoding of the Coq term «∀y:t, nfact y 3»:

```
all (con"t") y\ app[con"nfact", y, con"3"]
```

We now illustrate the encoding of the three instances above as higher-order logic-programming rules: capital letters denote rule parameters; `:-` separates the rule's head from the premises; `pi w\ p` introduces a fresh nominal constant `w` for the premise `p`.

```
finite (app[con"fin", N]).                                  (r1)
decision (app [con"nfact", N, NF]).                         (r2)
decision (all A x\ app[P, x]) :- finite A,                  (r3)
  pi w\ decision (app[P, w]).
```

Unfortunately this translation of rule $(r3)$ uses the predicate `P` as a first order term: for the meta language its type is `tm`. If we try to backchain the rule $(r3)$ on the encoding of the goal (g) given below

```
decision (all (app[con"fin", con"7"]) y\
  app[con"nfact", y, con"3"]).
```

we obtain an unsolvable unification problem (p): the two lists of terms have different lengths!

```
app[con"nfact", y, con"3"] = app[P, y]                      (p)
```

In this paper we study a more sophisticated encoding of Coq terms allowing us to rephrase the problematic rule $(r3)$ as follows:

why not link?

```
decision (all A x\ Pm x) :- decomp Pm P A, finite A,       (r3a)
  pi x\ decision (app[P, x]).
```

Since `Pm` is an higher-order unification variable of type `tm -> tm`, with `x` in its scope, the unification problem (p') admits one solution:

```
app[con"nfact", y, con"3"] = Pm y                           (p')
Pm = x\ app[con"nfact", x, con"3"]      % assignment for Pm
A = app[con"fin", con"7"]               % assignment for A
```

After unifying the head of rule $(r3a)$ with the goal, Elpi would run the premise «`decomp Pm A P`» in charge of bringing the assignment for `Pm` back to the domain `tm` of Coq terms:

```
P = lam A a\ app[con"nfact", a, con"3"]
```

This simple example is sufficient to show that the encoding we seek is not trivial and does not only concern the head of rules, but the entire sequence of unification problems that constitute the execution of a logic program. In fact the solution for `P` above generates a (Coq) β-redex in the second premise (the predicate under the `pi w\`). We show below the premise before and after the instantiation of `P`:

```
decision (app[                   P                     , w])
decision (app[ lam A (a\ app[con"nfact", a, con"3"]) , w])
```

In turn this redex prevents the rule $(r2)$ to backchain properly since the following unification problem has no solution:

```
app[ lam A (a\ app[con"nfact", a, con"3"]) , x] =
app[ con"nfact"                             , N, NF]
```

The root cause of the problems we sketched in this example is that the unification procedure $\simeq_\lambda$ of the meta language is not aware of the equational theory of the object logic, even if both theories include $\eta\beta$-conversion and admit most general unifiers for unification problems in the pattern fragment $\mathcal{L}_\lambda$ [9].

Contributions. In this paper we discuss alternative encodings of Coq in Elpi (Section 3), then we identify a minimal language $\mathcal{F}_o$ in which the problems sketched here can be fully described. We then detail an encoding `comp` from $\mathcal{F}_o$ to $\mathcal{H}_o$ (the language of the meta language) and a decoding `decomp` to relate the unifiers bla bla..

E:citare Teyjus

. The code discussed in the paper can be accessed at the URL: https://github.com/FissoreD/paper-ho.

2 PROBLEM STATEMENT AND SOLUTION

The equational theory of Coq's Dependent Type Theory is very rich. In addition to the usual $\eta\beta$-equivalence for functions, terms (hence types) are compared up to proposition unfolding and fix-point unrolling. Still, for efficiency and predictability reasons, most form of automatic proof search employ a unification procedure that captures a simpler one, just $\eta\beta$, and that solves higher-order problems restricted to the pattern fragment $\mathcal{L}_\lambda$ [9]. We call this unification procedure $\simeq_o$.

The equational theory of the meta language Elpi is strikingly similar, since it it comprises $\eta\beta$ (for the meta language functions), and the unification procedure $\simeq_\lambda$ solves higher-order problems in $\mathcal{L}_\lambda$.

In spite of the similarity the link between $\simeq_\lambda$ and $\simeq_o$ is not trivial, since the abstraction and application term constructors the two unification procedures deal with are different. For example

`x\ f x`	$\simeq_\lambda$	`f`
`lam A x\ app[con"f", x]`	$\simeq_o$	`con"f"`
`lam A x\ app[con"f", x]`	$\not\simeq_\lambda$	`con"f"`
`P x`	$\simeq_\lambda$	`x`
`app[P, x]`	$\simeq_o$	`x`
`app[P, x]`	$\not\simeq_\lambda$	`x`

One could ignore this similarity, and "just" describe the object language unification procedure in the meta language, that is crafting a `unif` predicate to be used as follows in rule $(r3)$:

```
decision X :- unif X (all A x\ app[P, x]), finite A,
  pi x\ decision (app[P, x]).
```

This choice would underuse the logic programming engine provided by the meta-language since, by removing any datum from the head of rules, indexing degenerates. Moreover the unification procedure `unif` programmed in the meta-language is likely to be n order of magnitude slower than one that is built-in.

To state precisely the problem we solve we need a $\mathcal{F}_o$ representation of CIC terms and a $\mathcal{H}_o$ one. We call $=_o$ the equality over ground terms in $\mathcal{F}_o$, $=_\lambda$ the equality over ground terms in $\mathcal{H}_o$, $\simeq_o$ the unification procedure we want to implement and $\simeq_\lambda$ the one provided by the meta language.

E:extend $=_o$ and $=_\lambda$ with reflexivity on uvars.

We write $t_1 \simeq_\lambda t_2 \mapsto \sigma$ when t_1 and t_2 unify with substitution σ; we write σt for the application of the substitution to t, and $\sigma X = \{\sigma t | t \in X\}$ when X is a set; we write $\sigma \subseteq \sigma'$ when σ is more general than σ'. We assume that the unification of our meta language is correct:

$$t_i \in \mathcal{L}_\lambda \Rightarrow t_1 \simeq_\lambda t_2 \mapsto \rho \Rightarrow \rho t_1 =_\lambda \rho t_2 \quad (1)$$

$$t_i \in \mathcal{L}_\lambda \Rightarrow \rho t_1 =_\lambda \rho t_2 \Rightarrow \exists \rho', t_1 \simeq_\lambda t_2 \mapsto \rho' \wedge \rho' \subseteq \rho \quad (2)$$

We illustrate a compilation $\langle s\rangle \mapsto (t, m, l)$ that maps a term s in $\mathcal{F}_o$ to a term t in $\mathcal{H}_o$, a variable mapping m and list of links l. The variable map connects unification variables in $\mathcal{H}_o$ with variables in $\mathcal{F}_o$ and is used to "decompile" the assignment, $\langle \sigma, m, l\rangle^{-1} \mapsto \rho$. Links represent problematic sub-terms which are linked to the unification variable that stands in their place in the compiled term. These links are checked for or progress

E:XXX improve....

We represent a logic program *run* in $\mathcal{F}_o$ as a list *steps* p of length $\mathcal{N}$. Each step is a unification problem between terms $\mathbb{P}_{p_l}$ and $\mathbb{P}_{p_r}$ taken from the set of all terms $\mathbb{P}$. The composition of these steps starting from the empty substitution ρ_0 produces the final substitution $\rho_{\mathcal{N}}$. [1] The initial here ρ_0 is the empty substitution

$$\text{fstep}(\mathbb{P}, p, \rho) \mapsto \rho'' \stackrel{def}{=} \rho \mathbb{P}_{p_l} \simeq_o \rho \mathbb{P}_{p_r} \mapsto \rho' \wedge \rho'' = \rho \cup \rho'$$
$$\text{frun}(\mathbb{P}, \mathcal{N}) \mapsto \rho_{\mathcal{N}} \stackrel{def}{=} \bigwedge_{p=1}^{\mathcal{N}} \text{fstep}(\mathbb{P}, p, \rho_{p-1}) \mapsto \rho_p$$

We simulate each run in $\mathcal{F}_o$ with a run in $\mathcal{H}_o$ as follows. Note that σ_0 is the empty substitution.

$$\begin{aligned}
&\text{hstep}(\mathbb{T}, p, \sigma, \mathbb{L}) \mapsto (\sigma'', \mathbb{L}') \stackrel{def}{=} \\
&\quad \sigma \mathbb{T}_{p_l} \simeq_\lambda \sigma \mathbb{T}_{p_r} \mapsto \sigma' \wedge \text{progress}(\mathbb{L}, \sigma \cup \sigma') \mapsto (\mathbb{L}', \sigma'') \\
&\text{hrun}(\mathbb{P}, \mathcal{N}) \mapsto \rho_{\mathcal{N}} \stackrel{def}{=} \\
&\quad \mathbb{T} \times \mathbb{M} \times \mathbb{L}_0 = \{(t_j, m_j, l_j) | s_j \in \mathbb{P}, \langle s_j \rangle \mapsto (t_j, m_j, l_j)\} \\
&\quad \bigwedge_{p=1}^{\mathcal{N}} \text{hstep}(\mathbb{T}, p, \sigma_{p-1}, \mathbb{L}_{p-1}) \mapsto (\sigma_p, \mathbb{L}_p) \\
&\quad \langle \sigma_{\mathcal{N}}, \mathbb{M}, \mathbb{L}_{\mathcal{N}} \rangle^{-1} \mapsto \rho_{\mathcal{N}}
\end{aligned}$$

Here hstep is made of two sub-steps: a call to $\simeq_\lambda$ (on the compiled terms) and a call to progress on the set of links. We claim the following:

Proposition 2.1 (Simulation). $\forall \mathbb{P}, \forall \mathcal{N}$,

$$\text{frun}(\mathbb{P}, \mathcal{N}) \mapsto \rho_{\mathcal{N}} \Leftrightarrow \text{hrun}(\mathbb{P}, \mathcal{N}) \mapsto \rho_{\mathcal{N}}$$

That is, the two executions give the same result. Moreover:

Proposition 2.2 (Simulation fidelity). *In the context of* hrun, *if* $\mathbb{T} \subseteq \mathcal{L}_\lambda$ *we have that* $\forall p \in 1 \ldots \mathcal{N}$,

$$\text{fstep}(\mathbb{P}, p, \rho_{p-1}) \mapsto \rho_p \Leftrightarrow \text{hstep}(\mathbb{T}, p, \sigma_{p-1}, \mathbb{L}) \mapsto (\sigma_p, \_)$$

In particular this property guarantees that a *failure* in the $\mathcal{F}_o$ run is matched by a failure in $\mathcal{H}_o$ *at the same step*. We consider this property very important from a practical point of view since it guarantees that the execution traces are strongly related and in turn this enables a user to debug a logic program in $\mathcal{F}_o$ by looking at its execution trace in $\mathcal{H}_o$.

XXX permuting hrun does not change the final result if check does not fail eagerly

XXX if we want to apply heuristics, we can apply them in decomp to avoid committing to a non MGU too early

[1] If the same rule is used multiple time in a run we just consider as many copies as needed of the terms composing the rules, with fresh unification variables each time

We can define $s_1 \simeq_o s_2$ by specializing the code of hrun to $\mathbb{P} = \{s_1, s_2\}$ as follows:

$$\begin{aligned}
&s_1 \simeq_o s_2 \mapsto \rho \stackrel{def}{=} \\
&\quad \langle s_1 \rangle \mapsto (t_1, m_1, l_1) \wedge \langle s_2 \rangle \mapsto (t_2, m_2, l_2) \\
&\quad t_1 \simeq_\lambda t_2 \mapsto \sigma' \wedge \text{progress}(\{l_1, l_2\}, \sigma') \mapsto (L, \sigma'') \wedge \\
&\quad \langle \sigma'', \{m_1, m_2\}, L \rangle^{-1} \mapsto \rho
\end{aligned}$$

Proposition 2.3 (Properties of $\simeq_o$).

$$s_i \in \mathcal{L}_\lambda \Rightarrow s_1 \simeq_o s_2 \mapsto \rho \Rightarrow \rho s_1 =_o \rho s_2 (correct) \quad (3)$$

$$s_i \in \mathcal{L}_\lambda \Rightarrow \rho s_1 =_o \rho s_2 \Rightarrow \exists \rho', s_1 \simeq_o s_2 \mapsto \rho' \wedge \rho' \subseteq \rho (complete) \quad (4)$$

$$\rho s_1 =_o \rho s_2 \Rightarrow \rho' \subseteq \rho \Rightarrow \rho' s_i \in \mathcal{L}_\lambda \Rightarrow \rho' s_1 \simeq_o \rho' s_2 \quad (5)$$

Properties (*correct*) and (*complete*) state, respectively, that in $\mathcal{L}_\lambda$ the implementation of $\simeq_o$ is correct, complete and returns the most general unifier.

E:fix

Property 2.1 states that $\simeq_o$, hence our compilation scheme, is resilient to unification problems outside $\mathcal{L}_\lambda$ solved by a third party. We believe this property is of practical interest since we want the user to be able to add heuristics via hand written rules to the ones obtained by our compilation scheme. A Typical example is the following problem (q) that is outside $\mathcal{L}_\lambda$:

```
app [F, con"a"] = app[con"f", con"a", con"a"]     (q)
F = lam x\ app[con"f",x,x]                        (h)
```

Instead of rejecting it our scheme accepts it and guarantees that if (h) is given (after the compilation part of the scheme, as a run time hint) then ...

2.1 The intuition in a nutshell

A term s is compiled in a term t where every "problematic" sub term p is replaced by a fresh unification variable h and an accessory link that represent a suspended unification problem $h \simeq_\lambda p$. As a result $\simeq_\lambda$ is "well behaved" on t, that is it does not contradict $=_o$ as it would otherwise do on "problematic" terms. We now define "problematic" and "well behaved" more formally.

Definition 2.4 ($\Diamond\eta$). $\Diamond\eta = \{t \mid \exists \rho, \rho t \text{ is an eta expansion}\}$

An example of term t in $\Diamond\eta$ is $\lambda x.\lambda y.F\ y\ x$ since the substitution $\rho = \{F \mapsto \lambda a.\lambda b.f b a\}$ makes $\rho t = \lambda x.\lambda y.f x y$ that is the eta long form of f. This term is problematic since its rigid part, the λ-abstractions, cannot justify a unification failure against, say, a constant.

Definition 2.5 ($\overline{\mathcal{L}_\lambda}$). $\overline{\mathcal{L}_\lambda} = \{X t_1 \ldots t_n \mid X t_1 \ldots t_n \notin \mathcal{L}_\lambda\}$.

An example of t in $\overline{\mathcal{L}_\lambda}$ is Fa for a constant a. Note however tha an oracle could provide an assignment $\rho = \{F \mapsto \lambda x.x\}$ that makes the resulting term fall back in $\mathcal{L}_\lambda$.

Definition 2.6 (Subterms $\mathcal{P}(t)$). The set of sub terms of t is the largest set
subtermt that can be obtained by the following rules.

$$\begin{aligned}
&t \in \mathcal{P}(t) \\
&t = f t_1 \ldots t_n \Rightarrow \mathcal{P}(t_i) \subseteq \mathcal{P}(t) \wedge f \in \mathcal{P}(t) \\
&t = \lambda x.t' \Rightarrow \mathcal{P}(t') \subseteq \mathcal{P}(t)
\end{aligned}$$

We write $\mathcal{P}(X) = \bigcup_{t \in X} \mathcal{P}(t)$ when X is a set of terms.

Definition 2.7 (Well behaved set). Given a set of terms $X \subseteq \mathcal{H}_o$,

$$\mathcal{W}(X) \Leftrightarrow \forall t \in \mathcal{P}(X), t \notin (\overline{\mathcal{L}_\lambda} \cup \diamond\eta)$$

Proposition 2.8 ($\mathcal{W}$-preservation). $\forall \mathbb{T}, \forall \mathbb{L}, \forall p, \forall \sigma, \forall \sigma'$

$$\mathcal{W}(\sigma\mathbb{T}) \land \sigma\mathbb{T}_{p_l} \simeq_\lambda \sigma\mathbb{T}_{p_r} \mapsto \sigma' \Rightarrow \mathcal{W}(\sigma'\mathbb{T})$$
$$\mathcal{W}(\sigma\mathbb{T}) \land \text{progress}(\mathbb{L}, \sigma) \mapsto (\_, \sigma') \Rightarrow \mathcal{W}(\sigma'\mathbb{T})$$

A less formal way to state 2.8 is that hstep and progress never "commit" an unneeded λ-abstraction in σ (a λ that could be erased by an η-contraction), nor put in σ a flexible application outside $\mathcal{L}_\lambda$ (an application node that could be erased by a β-reduction).

Note that proposition 2.8 does not hold for $\simeq_o$ as a whole since decompilation can introduce (actually restore) terms in $\diamond\eta$ or $\overline{\mathcal{L}_\lambda}$ that were move out of the way (put in $\mathbb{L}$) during compilation.

3 OTHER ENCODINGS AND RELATED WORK

Paper [1] introduces semi-shallow.

Our encoding of CIC may look "semi shallow" since we use the meta-language lambda abstraction but not its application (for the terms of type `tm`). A fully shallow encoding unfortunately does not fit our use case, although it would make the running example work:

```
finite (fin N).
decision (nfact N NF).
decision (all A x\ P x) :- finite A, pi x\ decision (P x).
```

There are two reasons for dismissing this encoding. The first one is that in CIC it is not always possible to adopt it since the type system of the meta language is too weak to accommodate terms with a variable arity, like the following example:

```
Fixpoint arr T n := if n is S m then T -> arr T m else T.
Definition sum n : arr nat n := ...
Check sum 2   7 8   : nat.
Check sum 3   7 8 9 : nat.
```

The second reason is the encoding for Coq is used for meta programming the system, hence it must accommodate the manipulation of terms that are now know in advance (not even defined in Coq) without using introspection primitives such as Prologs's `functor` and `arg`.

In the literature we could find a few related encoding of CIC. TODO In [3] is related and make the discrepancy between the types of ML and CIC visible. In this case one needs 4 application nodes. Moreover the objective is an encoding of terms, proofs, not proof search. Also note the conv predicate, akin to the unif we rule out.

TODO This other paper [16] should also be cited.

None of the encodings above provide a solution to our problem.

4 PRELIMINARIES: $\mathcal{F}_o$ AND $\mathcal{H}_o$

In order to reason about unification we provide a description of the $\mathcal{F}_o$ and $\mathcal{H}_o$ languages where unification variables are first class terms, i.e. they have a concrete syntax. We keep these languages minimal, for example, we omit the `all` quantifier of CIC we used in the example in Section 1 together with the type notation of terms carried by the `lam` constructor.

```
kind fm type.                    kind tm  type.
type fapp list fm -> fm.         type app list tm -> tm.
type flam (fm -> fm) -> fm.      type lam (tm -> tm) -> tm.
type fcon string -> fm.          type con string -> tm.
type fuva addr -> fm.            type uva addr -> list tm -> tm.
```

Figure 1: The $\mathcal{F}_o$ and $\mathcal{H}_o$ languages

Unification variables (`fuva` term constructor) in $\mathcal{F}_o$ have no explicit scope: the arguments of an higher order variable are given via the `fapp` constructor. For example the term `P x` is represented as `fapp[fuva N, x]`, where `N` is a memory address and `x` is a bound variable.

In $\mathcal{H}_o$ the representation of `P x` is instead `uva N [x]`, since unification variables come equipped with an explicit scope. We say that the unification variable occurrence `uva N L` is in $\mathcal{L}_\lambda$ if and only if `L` is made of distinct names. The predicate to test this condition is called `pattern-fragment`:

```
type pattern-fragment list A -> o.
```

Natural numbers represent the memory addresses that identify unification variables in both languages. The memory and its associated operations are described below:

```
kind addr type.
type addr nat -> addr.
typeabbrev (mem A) (list (option A)).

type set? addr -> mem A -> A -> o.
type unset? addr -> mem A -> o.
type assign addr -> mem A -> A -> mem A -> o.
type new mem A -> addr -> mem A -> o.
```

If a memory cell is `none`, then the corresponding unification variable is not set. `assign` sets an unset cell to the given value, while `new` finds the first unused address and sets it to `none`.

E:is new used?

Since in $\mathcal{H}_o$ unification variables have a scope, their solution needs to be abstracted over it to enable the instantiation of a single solution to different scopes. This is obtained via the `inctx` container, and in particular via its `abs` binding constructor. On the contrary a solution to a $\mathcal{F}_o$variable is a plain term.

```
typeabbrev fsubst (mem fm).

kind inctx type -> type.                    (· ⊢ ·)
type abs (tm -> inctx A) -> inctx A.
type val A -> inctx A.
typeabbrev assignment (inctx tm).
typeabbrev subst (mem assignment).
```

We call `fsubst` the memory of $\mathcal{F}_o$, while we call `subst` the one of $\mathcal{H}_o$. Both have the invariant that they are not cyclic, TODO: explain.

```
kind arity type.
type arity nat -> arity.

kind fvariable type.
type fv addr -> fvariable.

kind hvariable type.
type hv addr -> arity -> hvariable.
```

```
kind mapping type.
type mapping fvariable -> hvariable -> mapping.
typeabbrev mmap (list mapping).
```

Invariant 1 (Unification variable arity). *Each variable* `A` *in* $\mathcal{H}_o$ *has a (unique) arity* `N` *and each occurrence* `(uva A L)` *is such that* `(len L N)` *holds*

The compiler establishes a mapping between variables of the two languages. In order to preserve invariant 1 we store the arity of each `hvariable` in the mapping and we reuse an existing mapping only if the arity matches.

D:add ref to section 7

```
type m-alloc fvariable -> hvariable -> mmap -> mmap ->
  subst -> subst -> o.                            (malloc)
m-alloc Fv Hv M M S S :- mem M (mapping Fv Hv), !.
m-alloc Fv Hv M [mapping Fv Hv|M] S S1 :- Hv = hv N _,
  alloc S N S1.
```

When a single `fvariable` occurs multiple times with different numbers of arguments the compiler generates multiple mappings for it, on a first approximation, and then makes the mapping bijective by introducing η-`link`; this detail is discussed in section 6.

As we mentioned in section 2.1 the compiler replaces terms in $\Diamond\eta$ and $\overline{\mathcal{L}_\lambda}$ with fresh variables linked to the problematic terms. Each class of problematic terms has a dedicated link.

```
kind baselink type.
type link-eta  tm -> tm -> baselink.
type link-beta tm -> tm -> baselink.
typeabbrev link (inctx baselink).
typeabbrev links (list link).
```

The right hand side of a link, the problematic term, can occur under binders. To accommodate this situation the compiler wraps `baselink` using the `inctx` container (see, $\cdot \vdash \cdot$).

Invariant 2 (Link left hand side). *The left hand side of a suspended link is a variable.*

New links are suspended by construction. If the left hand side variable is assigned during a step, then the link is considered for progress and possibly eliminated. This is discussed in section 6 and section 8.

4.1 Notational conventions

When we write $\mathcal{H}_o$ terms outside code blocks we follow the usual λ-calculus notation, reserving f, g, a, b for constants, x, y, z for bound variables and X, Y, Z, F, G, H for unification variables. However we need to distinguish between the "application" of a unification variable to its scope and the application of a term to a list of arguments. We write the scope of unification variables in subscript while we use juxtaposition for regular application. Here a few examples:

$f \cdot a$	`app[con "f", con "a"]`
$\lambda x.\lambda y.F_{xy}$	`lam x\ lam y\ uva F [x, y]`
$\lambda x.F_x \cdot a$	`lam x\ app[uva F [x], con "a"]`
$\lambda x.F_x \cdot x$	`lam x\ app[uva F [x], x]`

When variables x and y can occur in term t we shall write t_{xy} to stress this fact.

We write $\sigma = \{ A_{xy} \mapsto y \}$ for the assignment `abs x\abs y\y` and $\sigma = \{ A \mapsto \lambda x.\lambda y.y \}$ for `lam x\lam y\y` .

When detailing examples we write links as equations between two terms under a context. The equality sign is subscripted with kind of `baselink`. For example $x \vdash A_x =_\beta F_x \; a$ corresponds to:

```
abs x\ val (link-beta (uva A [x]) (app[uva F [x],con "a"]))
```

When it is clear from the context we shall use the same syntax for $\mathcal{F}_o$ terms (although we never subscripts unification variables).

4.2 Equational theory and Unification

In order to express properties **??** we need to equip $\mathcal{F}_o$ and $\mathcal{H}_o$ with term equality, substitution application and unification.

Term equality: $=_o$ *vs.* $=_\lambda$. We extend the equational theory over ground terms to the full languages by adding the reflexivity of unification variables (a variable is equal to itself).

The first four rules are common to both equalities and just define the usual congruence over terms, and since we use an HOAS encoding they also capture α-equivalence. In addition to that $=_o$ has rules for η and β-equivalence.

```
type (=o) fm -> fm -> o.                                (=o)
fcon X =o fcon X.
fapp A =o fapp B :- forall2 (=o) A B.
flam F =o flam G :- pi x\ x =o x => F x =o G x.
fuva N =o fuva N.
flam F =o T :-                                         (ηl)
  pi x\ beta T [x] (T' x), x =o x => F x =o T' x.
T =o flam F :-                                         (ηr)
  pi x\ beta T [x] (T' x), x =o x => T' x =o F x.
fapp [flam X | L] =o T :- beta (flam X) L R, R =o T.  (βl)
T =o fapp [flam X | L] :- beta (flam X) L R, T =o R.  (βr)

type (=λ) tm -> tm -> o.
con C =λ fcon C.
app A =λ fapp B :- forall2 (=λ) A B.
lam F =λ flam G :- pi x\ x =λ x => F x =λ G x.
uva N A =λ fuva N B :- forall2 (=λ) A B.
```

The main point in showing these equality tests is to remark how weaker $=_\lambda$ is, and to identify the four rules that need special treatment in the implementation of $\simeq_o$.

For reference, `(beta T A R)` reduces away `lam` nodes in head position in `T` whenever the list `A` provides a corresponding argument.

```
type beta fm -> list fm -> fm -> o.
beta A [] A.
beta (flam Bo) [H | L] R :- napp (Bo H) F, beta F L R.
beta (fapp A) L (fapp X) :- append A L X.
beta (fuva N) L (fapp [fuva N | L]).
beta (fcon H) L (fapp [fcon H | L]).
beta N L (fapp [N | L]) :- name N.

type napp fm -> fm -> o.
napp (fcon C) (fcon C).
napp (fuva A) (fuva A).
napp (flam F) (flam G) :- pi x\ napp (F x) (G x).
napp (fapp [fapp L1 |L2]) T :- !,
  append L1 L2 L3, napp (fapp L3) T.
```

```
napp (fapp L) (fapp L1) :- map napp L L1.
napp N N :- name N.
```

The name predicate holds only on nominal constants (i.e. bound variables).[2] The choice of using n-ary application, rather than binary, is to make it easy to access the application's head. The price we pay is that substituting an application in the head of an application should be amended by "flattening" fapp nodes, that is the job of napp. [3] Finally note that the cut operator is inessential, it could be removed at the cost of a verbose test on the head of L in the second rule about fapp: L's head can be fcon, flam or a name.

Substitution application: ρs and σt. Applying the substitution corresponds to dereferencing a term with respect to the memory. To ease the comparison we split $\mathcal{F}_o$dereferencing into a fder step and a napp one. The former step replaces references to memory cells that are set with their values, and has a corresponding operation in $\mathcal{H}_o$, namely deref. On the contrary napp has no corresponding operation in $\mathcal{H}_o$. The reasons for this asymmetry is that an fapp node with a flexible head is always mapped to a uva (as per section 5.1 and section 8), preventing nested applications to materialize.

```
type fder fsubst -> fm -> fm -> o.
fder _ (fcon C) (fcon C).
fder S (fapp A) (fapp B) :- map (fder S) A B.
fder S (flam F) (flam G) :-
  pi x\ fder S x x => fder S (F x) (G x).
fder S (fuva N) R :- set? N S T, fder S T R.
fder S (fuva N) (fuva N) :- unset? N S.

type fderef fsubst -> fm -> fm -> o.                (ρs)
fderef S T T2 :- fder S T T1, napp T1 T2.
```

Applying the substitution in $\mathcal{H}_o$ is very similar, with the caveat that assignments have to be moved to the current scope, i.e. renaming the abs-bound variables with the names in the scope of the unification variable occurrence.

```
type deref subst -> tm -> tm -> o.                  (σt)
deref _ (con C) (con C).
deref S (app A) (app B) :- map (deref S) A B.
deref S (lam F) (lam G) :-
  pi x\ deref S x x => deref S (F x) (G x).
deref S (uva N L) R :- set? N S A,
  move A L T, deref S T R.
deref S (uva N A) (uva N B) :- unset? N S,
  map (deref S) A B.
```

Note that move strongly relies on invariant 1: the length of the arguments of all occurrences of a unification variable and the number of abstractions in its assignment have to match. In turn this grants that move never fails.

```
type move assignment -> list tm -> tm -> o.
move (abs Bo) [H|L] R :- move (Bo H) L R.
move (val A)  [] A.
```

[2]Elpi provides it as a builtin, but one could implement it by systematically loading the hypothetical rule `name x` every time a nominal constant is postulated via `pi x\`

[3]Note that napp is an artefact of formalization of $\mathcal{F}_o$ we do in this presentation and, as we explain later, no equivalent of napp is needed in $\mathcal{H}_o$.

Term unification: $\simeq_o$ vs. $\simeq_\lambda$. In this paper we assume to have an implementation of $\simeq_\lambda$ that satisfies properties 1 and 2. Although we provide an implementation in the appendix (that we used for testing purposes) we only describe its signature here. Elpi is expected to provide this brick, as well as any other implementation of λProlog.

```
type (≃λ) tm -> tm -> subst -> subst -> o.
```

The only detail worth discussing is the fact that the procedure updates a substitution, rather than just crafting one as presented in section 2. The reason is that the algorithm folds over a term, updating a substitution while it traverses it.

E:explain better

5 BASIC SIMULATION OF $\mathcal{F}_o$ IN $\mathcal{H}_o$

In this section we describe a basic compilation scheme that we refine later, in the following sections. This scheme is sufficient to implement an $\simeq_o$ that respects β-conversion for terms in $\mathcal{L}_\lambda$. The extension to $\eta\beta$-conversion is described in Section 6 and the support for terms outside $\mathcal{L}_\lambda$ in Section 8.

5.1 Compilation

E:manca beta normal in entrata

The main task of the compiler is to recognize $\mathcal{F}_o$ variables standing for functions and map them to higher order variables in $\mathcal{H}_o$. In order to bring back the substitution from $\mathcal{H}_o$ to $\mathcal{F}_o$ the compiler builds a "memory map" connecting the the kind of variables using routine (*malloc*).

The signature of the comp predicate below allows for the generation of links (suspended unification problems) that play no role in this section but play a major role in section 6 and section 8. With respect to section 2 the signature also allows for updates to the substitution. The code below uses that possibility in order to allocate space for the variables, i.e. sets their memory address to none (a details not worth mentioning in the previous discussion).

```
type comp fm -> tm -> mmap -> mmap -> links -> links ->
  subst -> subst -> o.
comp (fcon C) (con C) M M L L S S.
comp (flam F) (lam F1) M1 M2 L1 L2 S1 S2 :-          (cλ)
  comp-lam F F1 M1 M2 L1 L2 S1 S2.
comp (fuva A) (uva B []) M1 M2 L L S1 S2 :-
  m-alloc (fv A) (hv B (arity z)) M1 M2 S1 S2.
comp (fapp [fuva A|Ag]) (uva B Ag1) M1 M2 L L S1 S2 :-
  pattern-fragment Ag, !,
    fold6 comp Ag Ag1 M1 M1 L L S1 S1,
    len Ag Arity,
    m-alloc (fv A) (hv B (arity Arity)) M1 M2 S1 S2.
comp (fapp A) (app A1) M1 M2 L1 L2 S1 S2 :-
  fold6 comp A A1 M1 M2 L1 L2 S1 S2.
```

This preliminary version of comp recognizes $\mathcal{F}_o$ variables applied to a (possibly empty) duplicate free list of names. Note that compiling Ag cannot create new mappings nor links, since Ag is made of bound variables and the hypothetical rule loaded by comp-lam (see below) grants this property.

```
type comp-lam (fm -> fm) -> (tm -> tm) ->
  mmap -> mmap -> links -> links -> subst -> subst -> o.
```

```
comp-lam F G M1 M2 L1 L3 S1 S2 :-
  pi x y\ (pi M L S\ comp x y M M L L S S) =>
    comp (F x) (G y) M1 M2 L1 (L2 y) S1 S2,
  close-links L2 L3.
```

In the code above the syntax `pi x y\..` is syntactic sugar for iterated `pi` abstraction, as in `pi x\ pi y\...`

The auxiliary function `close-links` tests if the bound variable `v` really occurs in the link. If it is the case the link is wrapped into an additional `abs` node binding `v`. In this way links generated deep inside the compiled terms can be moved outside their original context of binders.

```
type close-links (tm -> links) -> links -> o.
close-links (v\[X  |L v]) [X|R] :- !, close-links L R.
close-links (v\[X v|L v]) [abs X|R] :- close-links L R.
close-links (_\[]) [].
```

Note that we could remove the first rule, whose solve purpose is to make links more readable by pruning unused context entries.

5.2 Execution

A step in $\mathcal{H}_o$ consists in unifying two terms and reconsidering all links for progress. If any of the two tasks fail we say that the entire step fails, and it is at this granularity that we can relate steps in the two languages.

```
type hstep tm -> tm -> links -> links -> subst -> subst -> o.
hstep T1 T2 L1 L2 S1 S3 :-
  (T1 ≃λ T2) S1 S2,
  progress L1 L2 S2 S3.
```

Note tha the infix notation `((A ≃λ B) C D)` is syntactic sugar for `((≃λ) A B C D)`.

Reconsidering links is a fixpoint, since the progress of a link can update the substitution and in turn enable another link to progress.

```
type progress links -> links -> subst -> subst -> o.
progress L L2 S1 S3 :-
  progress1 L L1 S1 S2,
  occur-check-links L1,
  if (L = L1, S1 = S2)
     (L2 = L1, S3 = S1)
     (progress L1 L2 S2 S3).
```

In the base compilation scheme `progress1` is the identity on both the links and the substitution, so the fixpoint trivially terminates. Sections 6 and 8 add rules to `progress1` and justify why the don't hinder termination. For brevity we omit the code that applies the substitution `S1` to all terms in $\mathbb{L}$.

Since compilation moves problematic terms out of the sigh of $\simeq_\lambda$, that procedure can only perform a partial occur check. For example the unification problem $X \simeq_\lambda f\ Y$ cannot generate a cyclic substitution alone, but should be disallowed if a $\mathbb{L}$ contains a link like $\vdash Y =_\eta \lambda z.X_z$: We don't know yet if Y will feature a lambda in head position, but we surely know it contains X, hence $f\ Y$ and that fails the occur check. The procedure `occur-check-links` is in charge of ensuring that each link does not represent a (suspended) unification problem doomed to fail because of occur check. This check is needed in order to guarantee proposition 2.2 (simulation fidelity).

5.3 Substitution decompilation

Decompiling the substitution requires to first force the progress of links and then allocating new unassigned variables in the substitution for $\mathcal{F}_o$ and finally decompiling all assignments. Note that invariant 2 and the occur check allows us to update the subst.

```
type decompile mmap -> links -> subst ->
  fsubst -> fsubst -> o.
decompile M1 L S F1 F3 :-
  commit-links L S S1,
  complete-mapping S1 S1 M1 M2 F1 F2,
  decompm M2 M2 S1 F2 F3.
```

E:What is commit-links and complete-mapping?

Decompiling an assignment requires to turn abstractions into lambdas. For aesthetic purposes we also eta-contract the result (not needed since $\mathcal{F}_o$ equality can do that)

```
type decompm mmap -> mmap -> subst -> fsubst -> fsubst -> o.
decompm _ [] _ F F.
decompm M [mapping (fv V) (hv H _)|MS] S F1 F3 :- set? H S A,
  deref-assmt S A A1,
  abs->lam A1 T, decomp M T T1,
  eta-contract T1 T2,
  assign V F1 T2 F2,
  decompm M MS S F2 F3.
decompm M [mapping _ (hv H _)|MS] S F1 F2 :- unset? H S,
  decompm M MS S F1 F2.
```

Finally decompiling a term is trivial, now that we have an extended mapping containing all unassigned variables $\simeq_\lambda$ may have introduced.

```
type decomp mmap -> tm -> fm -> o.
decomp _ (con C)  (fcon C).
decomp M (app A) (fapp B) :- map (decomp M) A B.
decomp M (lam F) (flam G) :-
  pi x y\ (pi M\ decomp M x y) => decomp M (F x) (G y).
decomp M (uva Hv Ag) R :-
  mem M (mapping (fv Fv) (hv Hv _)),
  map (decomp M) Ag Bg,
  beta (fuva Fv) Bg R.
```

Note that we use beta to build fapp nodes when needed (if Ag is empty no `fapp` node should appear).

Invariant 3. *TODO: dire che il mapping è bijective*

5.4 Definition of $\simeq_o$ and its properties

```
type (≃o) fm -> fm -> fsubst -> o.
(A ≃o B) F :-
  fo.beta-reduce A A',
  fo.beta-reduce B B',
  comp A' A'' [] M1 [] [] [] S1,
  comp B' B'' M1 M2 [] [] S1 S2,
  hstep A'' B'' [] [] S2 S3,
  decompm M2 M2 S3 [] F.
```

The code given so far applies to terms in $\beta\eta$-normal form where unification variables in $\mathcal{F}_o$ can occur non linearly but always with the same number of arguments, and where their arguments are distinct names (as per $\mathcal{L}_\lambda$).

Lemma 5.1 (Compilation round trip). *If* `comp S T [] M [] _ [] _` *then* `decomp M T S`

Proof sketch. trivial, since the terms are beta normal beta just builds an app. □

D:Reformulate?

Lemma 5.2. *Properties* (*correct*) *and* (*complete*) *hold for the implementation of* $\simeq_o$ *above*

Proof sketch. In this setting $=_\lambda$ is as strong as $=_o$ on ground terms. What we have to show is that whenever two different $\mathcal{F}_o$ terms can be made equal by a substitution ρ (plus the β_l and β_r if needed) we can find this ρ by finding a σ via $\simeq_\lambda$ on the corresponding $\mathcal{H}_o$ terms and by decompiling it. If we look at the $\mathcal{F}_o$ terms, the are two interesting cases:

- `fuva X` $\simeq_o$ `s`. In this case after `comp` we have $Y \simeq_\lambda t$ that succeeds with $\sigma = \{Y \mapsto t\}$ and σ is decompiled to $\rho = \{Y \mapsto s\}$.
- `fapp[fuva X|L]` $\simeq_o$ `s`. In this case we have $Y_{\vec{x}} \simeq_\lambda t$ that succeeds with $\sigma = \{\vec{y} \vdash Y \mapsto t[\vec{x}/\vec{y}]\}$ that in turn is decompiled to $\rho = \{Y \mapsto \lambda\vec{y}.s[\vec{x}/\vec{y}]\}$. Thanks to β_l $(\lambda\vec{y}.s[\vec{x}/\vec{y}])\ \vec{x} =_o s$.

Since the mapping is a bijection occur check in $\mathcal{H}_o$ corresponds to occur check in $\mathcal{F}_o$. □

Lemma 5.3. *Properties simulation (2.1) and fidelity (2.2) hold*

Proof sketch. Since `progress1` is trivial fstep and hstep are the same, that is in this context where input terms are $\beta\eta$-normal and we disregard η-equivalence $\simeq_\lambda$ is equivalent to $\simeq_o$. □

5.5 Limitations of by this basic scheme

The basic compilation scheme is not about to deal wit the following problem:

$$\mathbb{P} = \{\ \lambda xy.X \cdot y \cdot x \ \simeq_o\ \lambda xy.x \qquad \lambda x.f \cdot (X \cdot x) \cdot x \ \simeq_o\ Y\ \}$$

Note that here X is used with different arities, moreover in the second problem the left hand side happens to be an eta expansion (of $f\,(\lambda y.y)$) only after we discover (at run time) that $X = \lambda x \lambda y.y$ (i.e. that X discards the x argument). Both problems are addressed in the next two sections.

6 HANDLING OF $\Diamond\eta$

η-reduction is an equivalence relation where a term of the form $\lambda x.t \cdot x$ can be converted to t any time x does not occur as a free variable in t. We call t the η-contraction of $\lambda x.t \cdot x$.

Following the compilation scheme of section 5.1 the unification problem $\mathbb{P}$ is compiled as follows:

$$\begin{aligned}\mathbb{P} &= \{\ \lambda x.X \cdot x \ \simeq_o\ f\ \}\\ \mathbb{T} &= \{\ \ \lambda x.A_x \ \simeq_\lambda\ f\ \}\\ \mathbb{M} &= \{\ \ X \mapsto A^1\ \ \}\end{aligned}$$

While $\lambda x.X \cdot x \simeq_o f$ does admit the solution $\rho = \{\ X \mapsto f\ \}$, the corresponding problem in $\mathbb{T}$ does not: `lam x\ uva A [x]` and `con"f"` start with different, rigid, term constructors hence $\simeq_\lambda$ fails.

In order to guarantee proposition 2.1 we detect lambdas that can disappear by eta contraction (section 6.1) and we modify the compiled terms by putting fresh unification variables in their place: the problematic term is moved from $\mathbb{T}$ to $\mathbb{L}$ (section 6.2). The compilation of the problem $\mathbb{P}$ above is refined to:

$$\begin{aligned}\mathbb{P} &= \{\ \lambda x.X \cdot x \ \simeq_o\ f\ \}\\ \mathbb{T} &= \{\qquad\ \ A \ \simeq_\lambda\ f\ \}\\ \mathbb{M} &= \{\ \ X \mapsto B^1\ \ \}\\ \mathbb{L} &= \{\ \vdash A =_\eta \lambda x.B_x\ \ \}\end{aligned}$$

As per invariant 2 the term on the left is a variable, and its right counterpart is the term in $\Diamond\eta$. That term has the following property:

Invariant 4 (η-`link` rhs). *The* rhs *of any* η-`link` *has the shape* $\lambda x.t$ *and* t *is not a lambda.*

η-`link` are kept in the link store $\mathbb{L}$ during execution and activated when some conditions hold on lhs or rhs. Link activation is implemented by extending the `progress1` predicate (defined in section 5.2).

6.1 Detection of $\Diamond\eta$

When compiling a term t we need to determine if any subterm $s \in \mathcal{P}(t)$ that is of the form $\lambda x.r$, where x occurs in r, can be a η-expansion, i.e. if there exists a substitution ρ such that $\rho(\lambda x.r) =_o s$. The detection of lambda abstractions that can "disappear" is not as trivial as it may seems, here a few examples:

$$\begin{array}{lll}\lambda x.f \cdot (A \cdot x) & \in \Diamond\eta & \rho = \{\ A \mapsto \lambda x.x\ \}\\ \lambda x.f \cdot (A \cdot x) \cdot x & \in \Diamond\eta & \rho = \{\ A \mapsto \lambda x.a\ \}\\ \lambda x.f \cdot x \cdot (A \cdot x) & \notin \Diamond\eta & \\ \lambda x.\lambda y.f \cdot (A \cdot x) \cdot (B \cdot y \cdot x) & \in \Diamond\eta & \rho = \{\ A \mapsto \lambda x.x,\ B \mapsto \lambda y.\lambda x.y\ \}\end{array}$$

The first two examples are easy, and show how a unification variable can expose or erase a variable in their scope and turn the resulting term in an η-expansion or not.
The third example shows that when a variable occurs outside the scope of a unification variable it cannot be erased and can hence prevent a term from being an η-expansion.
The last example shows the recursive nature of the check we need to implement. The term starts with a spine of two lambdas hence the whole term is in $\Diamond\eta$ iff the inner term $\lambda y.f \cdot (A \cdot x) \cdot (B \cdot y \cdot x)$ is in $\Diamond\eta$ itself. If it is, it could η-contract to $f \cdot (A \cdot x)$ making $\lambda x.f \cdot (A \cdot x)$ a potential η-expansion.
We can now define more formally how $\Diamond\eta$ terms are detected together with its auxiliary functions:

Definition 6.1 (may-contract-to). A β-normal term s *may-contract-to* a name x if there exists a substitution ρ such that $\rho s =_o x$.

Lemma 6.2. *A β-normal term $s = \lambda x_1 \ldots x_n.t$* may-contract-to *$x$ only if one of the following three conditions holds:*

1. *$n = 0$ and $t = x$;*
2. *t is the application of x to a list of terms l and each l_i* may-contract-to *x_i (e.g. $\lambda x_1 \ldots x_n.x \cdot x_1 \ldots x_n =_o x$);*
3. *t is a unification variable with scope W, and for any $v \in \{x, x_1 \ldots x_n\}$, there exists a $w_i \in W$, such that w_i* may-contract-to *v (if $n = 0$ this is equivalent to $x \in W$).*

Proof sketch. Since our terms are in β-normal form there is only one rule that can play a role (namely η_l), hence if the term s is not exactly x (case 1) it can only be an η-expansion of x, or a unification variable that can be assigned to x, or a combination of both. If s begins with a lambda, then the lambda can only disappear by η contraction. In that case the term t is under the spine of binders $x_1 \dots x_n$, t can either be x applied to terms that can *may-contract-to* these variables (case 2), or a unification variable that can be assigned to that application (case 3). □

Definition 6.3 (occurs-rigidly). A name x *occurs-rigidly* in a β-normal term t, if $\forall \rho, x \in \mathcal{P}(\rho t)$

D:Not too clear?

In other words x occurs-rigidly in t if it occurs in t outside of the scope of unification variables since an instantiation is allowed to discard x from the scope of the unification variable. Note that η-contraction cannot make x disappear, since the variables being erased by η-contraction are locally bound inside t.

We can now derive the implementation for $\Diamond\eta$ detection:

Definition 6.4 (maybe-eta). Given a β-normal term $s = \lambda x_1 \dots x_n.t$, *maybe-eta* s holds if any of the following holds:

1. t is a constant or a name applied to the arguments $l_1 \dots l_m$ such that $m \geq n$ and for every i such that $m - n < i \leq m$ the term l_i *may-contract-to* x_i, and no x_i *occurs-rigidly* in $l_1 \dots l_{m-n}$;
2. t is a unification variable with scope W and for each x_i there exists a $w_j \in W$ such that w_j *may-contract-to* x_i.

Lemma 6.5 ($\Diamond\eta$ detection). *If t is a β-normal term and* maybe-eta t *holds, then $t \in \Diamond\eta$.*

Proof sketch. Follows from definition 6.3 and lemma 6.2 □

Remark that the converse of lemma 6.5 does not hold: there exists a term t satisfying the criteria (1) of definition 6.4 that is not in $\Diamond\eta$, i.e. there exists no substitution ρ such that ρt is an η-expansion. A simple counter example is $\lambda x.f \cdot (A \cdot x) \cdot (A \cdot x)$ since x does not *occur-rigidly* in the first argument of f, and the second argument of f *may-contract-to* x. In other words $A \cdot x$ may either use or discard x, but our analysis does not take into account that *the same term* cannot have two contrasting behaviors.

As we will see in the rest of this section this is not a problem since it does not break proposition 2.1 nor proposition 2.2.

6.2 Compilation and decompilation

Compilation. The following rule is inserted just before rule (c_λ) from the code in section 5.1.

```
comp (flam F) (uva A Scope) M1 M2 L1 L3 S1 S3 :-
  maybe-eta (flam F) [], !,
    alloc S1 A S2,
    comp-lam F F1 M1 M2 L1 L2 S2 S3,
    get-scope (lam F1) Scope,
    L3 = [val (link-eta (uva A Scope) (lam F1)) | L2].
```

The rule triggers when the input term `flam F` is in $\Diamond\eta$. It compiles `flam F` to `lam F1` but puts the fresh variable `A` in its place. This variable sees all the names free in `lam F1`. The critical part of this rule is the creation of the η-`link`, which relates the variable `A` with `lam F1`. This link clearly validates invariant 2.

Corollary 6.6. *The* rhs *of any η-*`link` *has exactly one lambda abstraction, hence the rule above respects invariant 4.*

Proof sketch. By contradiction, suppose that the rule above is triggered and that the rhs of the link is $\lambda x.\lambda y.t_{xy}$. If *maybe-eta* $\lambda y.t_{xy}$ holds the recursive call to `comp` (made by `comp-lam`) must have put a fresh variable in its place, so this case is impossible. Otherwise, if *maybe-eta* $\lambda y.t_{xy}$ does not hold, also *maybe-eta* $\lambda x.\lambda y.t_{xy}$ does not hold, contradicting the assumption that the rule triggered. □

Decompilation. Decompilation of η-`link` is performed by adding new rules to the `commit-link` predicate. In particular, given $\Gamma \vdash X =_\eta t$, we can note that this unification never fails, since X is a flexible term and no other η-`link` has X has lhs (by definition 6.9). The link is remove from $\mathbb{L}$ and `commit-links` terminates.

6.3 Progress

η-`link` are meant to delay the unification of "problematic" terms until we know for sure if the term has to be η-contracted or not.

Definition 6.7 (progress-η-left). A link $\Gamma \vdash X =_\eta T$ is removed from $\mathbb{L}$ when X becomes rigid. Let $y \in \Gamma$, there are two cases:

1. if $X = a$ or $X = y$ or $X = f \cdot a_1 \dots a_n$ we unify the η-expansion of X with T, that is we run $\lambda x.X \cdot x \simeq_\lambda T$
2. if $X = \lambda x.t$ we run $X \simeq_\lambda T$.

Definition 6.8 (progress-η-right). A link $\Gamma \vdash X =_\eta T$ is removed from $\mathbb{L}$ when either 1) *maybe-eta* T does not hold (anymore) or 2) by η-contracting T to T', T' is a term not starting with the `lam` constructor. In the first case, X is unified with T and in the second one, X is unified with T' (under the context Γ).

There is a third case in which a link is removed from $\mathbb{L}$, namely when the lhs is assigned to a variable that is the lhs of another η-`link`.

Definition 6.9 (progress-η-deduplicate). A link $\Gamma \vdash X_{\vec{s}} =_\eta T$ is removed from $\mathbb{L}$ when another link $\Delta \vdash X_{\vec{r}} =_\eta T'$ is in $\mathbb{L}$. By invariant 1 the length of $\vec{s}$ and $\vec{r}$ is the same hence we can move the term T' from Δ to Γ by renaming its bound variables, i.e. $T'' = T'[\vec{r}/\vec{s}]$. We then run $T \simeq_\lambda T''$ (under the context Γ).

D:Below the proof of proposition 2.8, ho usato 3 lemmi ausiliari, forse si può compattare in una prova più piccola?

Lemma 6.10. *Given a η-*`link` *l, the unification done by* progress-η-left *is between terms in $\mathcal{W}$*

Proof sketch. Let σ be the substitution, such that $\mathcal{W}(\sigma)$. lhs $\in \sigma$, therefore $\mathcal{W}$(lhs). By definition 6.7, if 1) lhs is a name, a constant of an application, then, lhs is unified with the η-reduced term t obtain from rhs. By corollary 6.6, rhs has one lambda, therefore $\mathcal{W}(t)$. Otherwise, 2) lhs has `lam` as functor, rhs should not be an η-expansion ans, so, $\mathcal{W}$(rhs). In both cases, unification is performed between terms in $\mathcal{W}$. □

Lemma 6.11. *Given a η-*`link` *l, the unification done by* progress-η-right *is between terms in $\mathcal{W}$.*

Proof sketch. lhs is variable, and, by definition 6.8, rhs is either no more a $\Diamond\eta$, i.e. rhs is not a η-expansion and, so, $\mathcal{W}$(rhs). Otherwise, rhs can reduce to a term which cannot be a η-expansion, and, so, $\mathcal{W}$(rhs). In both cases, unification is done between terms in $\mathcal{W}$. □

Lemma 6.12. *Given a η-`link` l, the unification done by* progress-η-deduplicate*is between terms in $\mathcal{W}$.*

Proof. Trivial, since the unification is done between unification variables, which are by definition in $\mathcal{W}$. □

Lemma 6.13. *Proposition 2.8 holds, i.e., given a substitution σ and a η-`link` l, after the activation of l, $\mathcal{W}(\sigma)$ holds.*

Proof sketch. By lemmas 6.10 to 6.12, every unification performed by the activation of a η-`link` is performed between terms in $\mathcal{W}$, therefore, the substitution remains $\mathcal{W}$.

D:Bisogna aggiungere un lemma nella section 2.1 che dice che unificare due termini in $\mathcal{W}$, in una σ, tale che $\mathcal{W}(\sigma)$, non invalida $\mathcal{W}$

□

Lemma 6.14. `progress` *terminates.*

Proof sketch. Rules definitions 6.7 and 6.8 and definition 6.9 remove one link from $\mathbb{L}$, hence they cannot be applied indefinitely. Moreover each rule only relies on terminating operations such as $\simeq_\lambda$, η-contraction, η-expansion, relocation (a recursive copy of a finite term). □

D:Proove simulation fidelity, dicendo che *progress-η-right* è inutile

Example of progress-η-left. The example at the beginning of section 6, once $\sigma = \{\, A \mapsto f \,\}$, triggers this rule since the link becomes $\vdash f =_\eta \lambda x.B_x$ and the lhs is a constant. In turn the rule runs $\lambda x.f\cdot x \simeq_\lambda \lambda x.B_x$, resulting in $\sigma = \{\, A \mapsto f \,;\, B_x \mapsto f \,\}$. Decompilation the generates $\rho = \{\, X \mapsto f \,\}$, since X is mapped to B and f is the η-contracted version of $\lambda x.f\cdot x$.

Example of progress-η-deduplicate. A very basic example of η-`link` deduplication, is given below:

$$\begin{array}{l}
\mathbb{P} = \{\ \lambda x.(X \cdot x) \ \simeq_o\ \lambda x.(Y \cdot x)\ \} \\
\mathbb{T} = \{\ \qquad A \ \simeq_\lambda\ C \qquad \} \\
\mathbb{M} = \{\ X \mapsto B^1 \quad Y \mapsto D^1\ \} \\
\mathbb{L} = \{\ \vdash A =_\eta \lambda x.B_x \quad \vdash C =_\eta \lambda x.D_x\ \}
\end{array}$$

The result of $A \simeq_\lambda C$ is that the two η-`link` share the same lhs. By unifying the two rhs we get $\sigma = \{A \mapsto C, B \mapsto D\,\}$. In turn, given the map $\mathbb{M}$, this second assignment is decompiled to $\rho = \{\, X \mapsto Y \,\}$ as expected.

We delay at the end of next section an example of η-`link` progression due to *progress-η-right*

7 ENFORCING INVARIANT 1

We report here the problem given in section 5.5 where X is used with two different arities and the output of the compilation does not respect invariant 3 (merging the two mappings for s would break invariant 1). In this section we explain how to replace the duplicate mapping with some η-`link` in order to restore the invariants.

$$\begin{array}{l}
\mathbb{P} = \{\ \lambda x.\lambda y.(X \cdot y \cdot x) \ \simeq_o\ \lambda x.\lambda y.x \qquad \lambda x.(f \cdot (X \cdot x) \cdot x) \ \simeq_o\ Y\ \} \\
\mathbb{T} = \{\ \qquad A \ \simeq_\lambda\ \lambda x.\lambda y.x \qquad D \ \simeq_\lambda\ F\ \} \\
\mathbb{M} = \{\ X \mapsto E^1 \quad Y \mapsto F^0 \quad X \mapsto C^2\ \} \\
\mathbb{L} = \left\{ \begin{array}{ll} \vdash D =_\eta \lambda x.(f \cdot E_x \cdot x) & \vdash A =_\eta \lambda x.B_x \\ x \vdash B_x =_\eta \lambda y.C_{yx} & \end{array} \right\}
\end{array}$$

We see that the `maybe-eta` as identified $\lambda xy.X \cdot y \cdot x$ and $\lambda x.f \cdot (X \cdot x) \cdot x$ and the compiler has replaced them with A and D respectively. However, the mapping $\mathbb{M}$ breaks invariant 3: the $\mathcal{F}_o$ variable X is mapped to two different $\mathcal{H}_o$ variables. To address this problem we adjust the compiler's output with a `map-deduplication` procedure.

Definition 7.1 (align-arity). Given two mappings $m_1 : X \mapsto A^m$ and $m_2 : X \mapsto C^n$ where $m < n$ and $d = n - m$, *align-arity* m_1 m_2 generates the following d links, one for each i such that $0 \le i < d$,

$$x_0 \ldots x_{m+i} \vdash B^i_{x_0 \ldots x_{m+i}} =_\eta \lambda x_{m+i+1}.B^{i+1}_{x_0 \ldots x_{m+i+1}}$$

where B^i is a fresh variable of arity $m + i$, and $B^0 = A$ as well as $B^d = C$.

The intuition is that we η-expand the occurrence of the variable with lower arity to match the higher arity. Since each η-`link` can add exactly one lambda, we need as many links as the difference between the two arities.

Definition 7.2 (map-deduplication). Forall mappings $m_1, m_2 \in \mathbb{M}$ such that $m_1 : X \mapsto A^m$ and $m_2 : X \mapsto C^n$ and $m < n$ we remove m_1 from $\mathbb{M}$ and add to $\mathbb{L}$ the result of *align-arity* m_1 m_2.

If we look back the example give at the beginning of this section, we can deduplicate $X \mapsto E^1, X \mapsto C^2$ by removing the first mapping and adding the auxiliary η-`link`: $x \vdash E_x =_\eta \lambda y.C_{xy}$. After deduplication the compiler output is as follows:

$$\begin{array}{l}
\mathbb{P} = \{\ \lambda x.\lambda y.(X \cdot y \cdot x) \ \simeq_o\ \lambda x.\lambda y.x \qquad \lambda x.(f \cdot (X \cdot x) \cdot x) \ \simeq_o\ Y\ \} \\
\mathbb{T} = \{\ \qquad A \ \simeq_\lambda\ \lambda x.\lambda y.x \qquad D \ \simeq_\lambda\ F\ \} \\
\mathbb{M} = \{\ Y \mapsto F^0 \quad X \mapsto C^2\ \} \\
\mathbb{L} = \left\{ \begin{array}{ll} x \vdash E_x =_\eta \lambda y.C_{xy} & \vdash D =_\eta \lambda x.(f \cdot E_x \cdot x) \\ \vdash A =_\eta \lambda x.B_x & x \vdash B_x =_\eta \lambda y.C_{yx} \end{array} \right\}
\end{array}$$

In this example, $\mathbb{T}_1$ assigns A which triggers $\mathbb{L}_3$ and then $\mathbb{L}_4$ by definition 6.7. C_{yx} is therefore assigned to x (the second variable of its scope). We can finally see the *progress-η-right* of $\mathbb{L}_1$: its rhs is now $\lambda y.y$ (C_{xy} gives y). Since it is no more in $\diamond\eta$, $\lambda y.y$ is unified with E_x. Moreover, $\mathbb{L}_2$ is also triggered due to definition 6.8: $\lambda x.(f \cdot (\lambda y.y) \cdot x)$ is η-reducible to $f \cdot (\lambda y.y)$ which is a term not starting with the `lam` constructor.

8 HANDLING OF $\overline{\mathcal{L}_\lambda}$

D:I've rewritten it, it is clearer?

Until now, we have only dealt we unification of terms in $\mathcal{L}_\lambda$. However, we want the unification relation to be more robust so that it can work with terms in $\overline{\mathcal{L}_\lambda}$. In general, unification in $\overline{\mathcal{L}_\lambda}$ admits more then one solution and committing one of them in the substitution does not guarantee prop. (*complete*). For instance, $X \cdot a \simeq_o a$ is a unification problem admits two different substitutions: $\rho_1 = \{X \mapsto \lambda x.x\}$ and $\rho_2 = \{X \mapsto \lambda\_.a\}$. Prefer one over the other may break future unifications.

It is the case that, given a list of unification problems, $\mathbb{P}_1 \dots \mathbb{P}_n$ with $\mathbb{P}_n$ in $\overline{\mathcal{L}_\lambda}$, the resolution of $\bigwedge_{i=0}^{n-1} \mathbb{P}_i$ gives a partial substitution ρ, such that $\rho\mathbb{P}_n$ falls again in $\mathcal{L}_\lambda$.

$$\begin{array}{l} \mathbb{P} = \{\ X \simeq_o \lambda x.Y \quad (X \cdot a) \simeq_o a\ \} \\ \mathbb{T} = \{\ A \simeq_\lambda \lambda x.B \quad (A \cdot a) \simeq_\lambda a\ \} \\ \mathbb{M} = \{\ Y \mapsto B^0 \quad X \mapsto A^0\ \} \end{array}$$

In the example above, we see that $\mathbb{P}_1$ instantiates X so that $\mathbb{P}_2$, can be solved in $\mathcal{L}_\lambda$.

E:it is even a ground term, there is no unification left to perform actually

D:i don't understand the note

On the other hand, we see that, $\simeq_\lambda$ can't solve the compiled problems $\mathbb{T}$. In fact, the resolution of $\mathbb{T}_1$ gives the substitution $\sigma = \{A \mapsto \lambda x.B\}$, but the dereferencing of $\mathbb{T}_2$ gives the non-unifiable problem $(\lambda x.B) \cdot a \simeq_\lambda a$.

To address this unification problem, term compilation should capture the terms t in $\overline{\mathcal{L}_\lambda}$ and replace them with fresh variables X. The variables X and the terms t are linked through a β-`link`.

β-`link` guarantees invariant 2 and the term on the rhs has the following property:

D:Is it clearer?

Invariant 5 (β-`link` rhs). *The* rhs *of any* β-`link` *has the shape* $X_{s_1 \dots s_n} \cdot t_1 \dots t_m$ *such that* X *is a unification variable with scope* $s_1 \dots s_n$ *and* $t_1 \dots t_m$ *is a list of terms. This is equivalent to* `app[uva X S | L]`, *where* `S` $= s_1 \dots s_n$ *and* `L` $= t_1 \dots t_m$.

Lemma 8.1. *If the* lhs *of a* β-`link` *is instantiated to a rigid term and its* rhs *counterpart is still in* $\overline{\mathcal{L}_\lambda}$, *the original unification problem is not in* $\mathcal{L}_\lambda$ *and the unification fails.*

Proof sketch. Given $X \cdot t_1 \dots t_n \simeq_\lambda t$ where t is a rigid term and $t_1 \dots t_n$ is not in $\mathcal{L}_\lambda$. By construction, $X \cdot t_1 \dots t_n$ is replaced with a variable Y, and the β-`link` $\Gamma \vdash Y =_\beta X \cdot t_1 \dots t_n$ is created. The unification instantiates Y to t, making the lhs of the link a rigid term, while rhs is still in $\overline{\mathcal{L}_\lambda}$. The original problem is in fact outside $\mathcal{L}_\lambda$. □

8.1 Compilation and decompilation

Compilation. Detection of $\overline{\mathcal{L}_\lambda}$ is quite simple to implement in the compiler, since it is sufficient to detect applications with flexible head and argument that are not in $\mathcal{L}_\lambda$.

```
comp (fapp [fuva A|Ag]) (uva B Scope) M1 M3 L1 L3 S1 S4 :- !,
  pattern-fragment-prefix Ag Pf Extra,
  len Pf Arity,
  alloc S1 B S2,
  m-alloc (fv A) (hv C (arity Arity)) M1 M2 S2 S3,
  fold6 comp Pf    Pf1    M2 M2 L1 L1 S3 S3,
  fold6 comp Extra Extra1 M2 M3 L1 L2 S3 S4,
  Beta = app [uva C Pf1 | Extra1],
  get-scope Beta Scope,
  L3 = [val (link-beta (uva B Scope) Beta) | L2].
```

The list `Ag` is split into the list `Pf` and `Extra` such that `append Pf Extra Ag` and `Pf` is the largest prefix of `Ag` such that `Pf` is in $\mathcal{L}_\lambda$. The rhs of the β-`link` is the application of a fresh variable `C` having in scope all the free variables appearing in the compiled version of `Pf` and `Extra`. The variable `B`, returned has the compiled term, is a fresh variable having in scope all the free variables occurring in `Pf1` and `Extra1`

Invariant 6. *The* rhs *of a* β-`link` *has the shape* $X_{s_1 \dots s_n} \cdot t_1 \dots t_m$.

Corollary 8.2. *Let* $X_{s_1 \dots s_n} \cdot t_1 \dots t_m$ *be the* rhs *of a* β-`link`, *then* $m > 0$.

Proof sketch. Assume we have a β-`link`, by contradiction, if $m = 0$, then the original $\mathcal{F}_o$ term has the shape `fapp[fuva M | Ag]` where `Ag` is a list of distinct names (i.e. the list `Extra` is empty). This case is however captured by rule (c_λ) (from section 5.1) and no β-`link` is produced which contradicts our initial assumption. □

Corollary 8.3. *Let* $X_{s_1 \dots s_n} \cdot t_1 \dots t_m$ *be the* rhs *of a* β-`link`, *then* t_1 *either appears in* $s_1 \dots s_n$ *or it is not a name.*

Proof sketch. By construction, the lists $s_1 \dots s_n$ and $t_1 \dots t_m$ are built by splitting the list `Ag` from the original term `fapp [fuva A|Ag]`. $s_1 \dots s_n$ is the longest prefix of the compiled terms in `Ag` which is in $\mathcal{L}_\lambda$. Therefore, by definition of $\mathcal{L}_\lambda$, t_1 must appear in $s_1 \dots s_n$, otherwise $s_1 \dots s_n$ is not the longest prefix in $\mathcal{L}_\lambda$, or it is a term with a constructor of `tm` as functor. □

E:Dire che maybe eta fa il detect anche su termini che non sono il llambda, oppure dirlo in section of maybeeta + dare un esempio?

Decompilation. During `progress`, as claimed in invariant 5, the decompilation can only have β-`link` with not instantiated lhs. In this case, lhs is unified with rhs.

D:not really sure of this, we can have $F\ a = \lambda x.Gx$. In this case when do we fail: for sure in decompile. But to respect fidelity, we should fail immediately: we have a β-`link` and a η-`link` with same lhs

8.2 Progress

The activation of a β-`link` is performed when its rhs falls under $\mathcal{L}_\lambda$ under a given substitution.

Definition 8.4 (progress-beta-$\mathcal{L}_\lambda$). Given a substitution σ and a β-`link` $\Gamma \vdash T =_\beta X_{s_1 \dots s_n} \cdot t_1 \dots t_m$ such that σt_1 is a name, say t, and $t \notin s_1 \dots s_n$. If $m = 0$, then the β-`link` is removed and lhs is unified with $X_{s_1 \dots s_n}$. If $m > 0$, then the β-`link` is replaced by a refined version $\Gamma \vdash T =_\beta Y_{s_1 \dots s_n, t} \cdot t_2 \dots t_m$ with reduced list of arguments and Y being a fresh variable. Moreover, the new link $\Gamma \vdash X_{s_1 \dots s_n} =_\eta \lambda x.Y_{s_1 \dots s_n, x}$ is added to $\mathbb{L}$.

D:L'ho riformulato

Definition 8.5 (progress-beta-rigid-head). A link $\Gamma \vdash X =_\beta X_{s_1 \dots s_n} \cdot t_1 \dots t_m$ is removed from $\mathbb{L}$ if $X_{s_1 \dots s_n}$ is instantiated to a term t and the β-reduced term t' obtained from the application of t to $l_1 \dots l_m$ is in $\mathcal{L}_\lambda$. Moreover, X is unified to t.

D:added the def below to respect fidelity

Definition 8.6 (progress-beta-dedup). Given two β-`link` $l1$ and $l2$, if they share the same lhs, unification fails.

Lemma 8.7. `progress` *terminates*

Proof sketch. Let l a β-`link` in the store $\mathbb{L}$. If l is activated by *progress-beta-rigid-head*, then it disappears from $\mathbb{L}$ and `progress` terminates. Otherwise, the rhs of l is made by a variable applied to m arguments. At each activation of *progress-beta-*$\mathcal{L}_\lambda$, l is replaced by a new β-`link` l^1 having $m-1$ arguments. At the m^{th} iteration, the β-`link` l^m has no more arguments and is removed from $\mathbb{L}$. Note that at the m^{th} iteration, m new η-`link` have been added to $\mathbb{L}$, however, by lemma 6.14, the algorithm terminates. □

D:L'ho riscritto

E:funziona. per essere più precisi io parlerei di ordine lessicografico (tipico ordine ben fondato usato per dimostrare terminazione). Nl nostro caso è la tripla (argomenti extra dei beta, numero di beta, numero di eta).

Corollary 8.8. *Given a β-`link`, the variables occurring in its rhs are in $\mathcal{L}_\lambda$.*

D:is it clearer?

Proof sketch. By construction, the rhs of β-`link` has the shape $X_{s_1 \ldots s_n} \cdot t_1 \ldots t_m$, $s_1 \ldots s_n$ is in $\mathcal{L}_\lambda$ and all the terms $t_1 \ldots t_n$ are in $\mathcal{L}_\lambda$, too. If a β-`link` is triggered by *progress-beta-rigid-head*, then, by definition 8.5, that link is removed by $\mathbb{L}$, and the property is satisfied. If the η-`link` is activated by *progress-beta-*$\mathcal{L}_\lambda$, then, by definition 8.4, the new β-`link` as a variable as a scope which is still in $\mathcal{L}_\lambda$. □

Lemma 8.9. *Given a $\mathbb{T}$ and a substitution σ then the resolution of $\sigma\mathbb{T}$ guarantees proposition 2.2*

Proof sketch. If $\sigma\mathbb{T}$ is in $\mathcal{L}_\lambda$, then by definitions 8.4 and 8.5, then β-`link` disappear and the unification done between terms in $\mathcal{L}_\lambda$. This problem unifies iff its corresponding $\mathcal{F}_o$ problem unifies too. If $\sigma\mathbb{T}$ is in $\overline{\mathcal{L}_\lambda}$, then, by lemma 8.1, the unification fails, as per the corresponding unification in $\mathcal{F}_o$. □

Example of progress-beta-$\mathcal{L}_\lambda$. Consider the β-`link` below:

$$
\begin{array}{l}
\mathbb{P} = \{\ X \simeq_o \lambda x.x \quad \lambda x.(Y \cdot (X \cdot x)) \simeq_o f\ \} \\
\mathbb{T} = \{\ A \simeq_\lambda \lambda x.x \qquad\qquad\qquad B \simeq_\lambda f\ \} \\
\mathbb{M} = \{\ Y \mapsto D^0 \quad X \mapsto A^0\ \} \\
\mathbb{L} = \left\{ \begin{array}{ll} \vdash A =_\eta \lambda x.E_x & \vdash B =_\eta \lambda x.C_x \\ x \vdash C_x =_\beta (D \cdot E_x) & \end{array} \right\}
\end{array}
$$

Initially the β-`link` rhs is a variable D applied to the E_x. The first unification problem results in $\sigma = \{A \mapsto \lambda x.x\}$. In turn this instantiation triggers $\mathbb{L}_1$ by *progress-η-left* and E_x is assigned to x. Under this substitution the β-`link` becomes $x \vdash C_x =_\beta (D \cdot x)$, and by *progress-beta-*$\mathcal{L}_\lambda$it is replaced with the link: $\vdash E =_\eta \lambda x.D_x$, while C_x is unified with D_x. The second unification problem assigns f to B, that in turn activates the second η-`link` (f is assigned to C), and then all the remaining links are solved. The final $\mathcal{H}_o$ substitution is $\sigma = \{A \mapsto \lambda x.x, B \mapsto f, C_x \mapsto (f \cdot x), D \mapsto f, E_x \mapsto x, F_x \mapsto C_x\}$ and is decompiled into $\rho = \{X \mapsto \lambda x.x, Y \mapsto f\}$.

Example of progress-beta-rigid-head. We can take the example provided in section 8. The problem is compiled into:

$$
\begin{array}{l}
\mathbb{P} = \{\ X \simeq_o \lambda x.Y \quad (X \cdot a) \simeq_o a\ \} \\
\mathbb{T} = \{\ A \simeq_\lambda \lambda x.B \qquad C \simeq_\lambda a\ \} \\
\mathbb{M} = \{\ Y \mapsto B^0 \quad X \mapsto A^0\ \} \\
\mathbb{L} = \{\ \vdash C =_\beta (A \cdot a)\ \}
\end{array}
$$

The first unification problems is solved by the substitution $\sigma = \{A \mapsto \lambda x.B\}$. The β-`link` becomes $\vdash C =_\beta ((\lambda x.B) \cdot a)$ whose rhs can be β-reduced to B. B is in $\mathcal{L}_\lambda$ and is unified with C. The resolution of the second unification problem gives the final substitution $\sigma = \{A \mapsto \lambda x.B, B \mapsto C, C \mapsto a\}$ which is decompiled into $\rho = \{X \mapsto \lambda x.a, Y \mapsto a\}$.

8.3 Relaxing lemma 8.1 : tobe renamed

Working with terms in $\mathcal{L}_\lambda$ is sometime too restrictive. There exists systems such as λProlog [11], Abella [6], which delay the resolution of $\overline{\mathcal{L}_\lambda}$ unification problems if the substitution is not able to put them in $\mathcal{L}_\lambda$.

$$
\mathbb{P} = \{\ (X \cdot a) \simeq_o a \quad X \simeq_o \lambda x.Y\ \}
$$

In the example above, $\mathbb{P}_1$ is in $\overline{\mathcal{L}_\lambda}$ and the object language cannot solve it, and, by proposition 2.2, the meta language neither. However, we can be more permissive, and relax lemma 8.1. This modification is quite simple to manage: we are introducing a new $\overline{\mathcal{L}_\lambda}$ progress rule, say *progress-beta-*$\overline{\mathcal{L}_\lambda}$, by which, if lhs is rigid and rhs is flexible, the considered β-`link` is kept in the store and no progression is done[4]. *progress-beta-*$\overline{\mathcal{L}_\lambda}$makes `occur-check-links` partial, since the check is possible only on links with a variable on the lhs. This means that we can have two links $\vdash X =_\beta Y \cdot a$ and $\vdash f \cdot X =_\beta Y \cdot a$ where the occur check does not throw an error. Note however, that the decompilation of the two links will force the unification of X to $Y \cdot a$ and then the unification of $f \cdot (Y \cdot a)$ to $Y \cdot a$, which fails by the occur check of $\simeq_\lambda$.

A second strategy to deal with problem that are in $\overline{\mathcal{L}_\lambda}$ is to make some approximation. This is the case for example of the unification algorithm of Coq used in its type class solver [17]. The approximation consists in forcing a choice (among the others) when the unification problem is in $\overline{\mathcal{L}_\lambda}$. For instance, in $X \cdot a \cdot b = Y \cdot b$, the last argument of the two terms is the same, therefore Y is assigned to Xa. Note that this is of course an approximation, since $\sigma = \{X = \lambda x.Y, Y = \_\}$ is another valid substitution for the original problem. This approximation can be easily introduced in our unification procedure, by adding new custom β-`link` progress rules.

D:Paragraph for decompile in case of $\overline{\mathcal{L}_\lambda}$

The `commit-link` predicate can be extended to add heuristics if during the decompilation phase β-`link` remain. For example, the same approximation explained above can be delayed and applied only if the terms in $\overline{\mathcal{L}_\lambda}$ never falls in $\mathcal{L}_\lambda$ after the execution of all the unification problems. We want to point out, that we call this *approximation*, since we are making a choice among all the possible unifiers and therefore, we can pick the wrong one.

9 ACTUAL IMPLEMENTATION IN ELPI

In this paper we show a minimized example. The full code is there. But we also have to code things in Coq-Elpi.

[4]This new rule trivially guarantees the termination of progress

In Elpi we don't have a main loop, we rely on the interpreter one. So links are constraints and progress are CHR rules. Constraints are suspended goals that are resumend when some unif variable is assigned.

```
link-eta bla :- suspend-condition Holes, !, declare_constraint (link-eta bla) Holes.
link-eta bla :- progress. % solves the goal, hence the constraint is not more in L.
```

This matches with all progress rules but for the ones considering two links. For these we need to resort to CHR to manipulate L. For example deduplicate eta

```
constraints link-eta {
  rule (N1 : G1 ?- link-eta (uvar X LX1) T1) % match
    /  (N2 : G2 ?- link-eta (uvar X LX2) T2) % remove
    |  (relocate LX1 LX2 T2 T2')             % condition
   <=> (N1 : G1 ?- T1 = T2').                % new goal
}
```

Remark how the invariant about uvar arity makes this easy, since LX1 and LX2 have the same length. Also note that N1 only contains the names of the first link (while relocate runs in the disjoint union) and Elpi ensures that T2' can live in N1.

10 CONCLUSION

Benefits: less work, reuse efficient ho unif (3x faster), indexing,

Future: tabling and static analysis (reuse for ML again).

Very little is Coq specific. Applies to all OL that are not a subsystem of HOL, or for ML that are used for meta programming.

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APPENDIX

This appendix contains the entire code described in this paper. The code can also be accessed at the URL: https://github.com/FissoreD/paper-ho

Note that `(a infix b) c d` de-sugars to `(infix) a b c d`.

Explain builtin name (can be implemented by loading name after each pi)

11 THE MEMORY

```
kind addr type.
type addr nat -> addr.
typeabbrev (mem A) (list (option A)).

type set? addr -> mem A -> A -> o.
set? (addr A) Mem Val :- get A Mem Val.

type unset? addr -> mem A -> o.
unset? Addr Mem :- not (set? Addr Mem _).

type assign-aux nat -> mem A -> A -> mem A -> o.
assign-aux z (none :: L) Y (some Y :: L).
assign-aux (s N) (X :: L) Y (X :: L1) :- assign-aux N L Y L1.

type assign addr -> mem A -> A -> mem A -> o.
assign (addr A) Mem1 Val Mem2 :- assign-aux A Mem1 Val Mem2.

type get nat -> mem A -> A -> o.
get z (some Y :: _) Y.
get (s N) (_ :: L) X :- get N L X.

type alloc-aux nat -> mem A -> mem A -> o.
alloc-aux z [] [none] :- !.
alloc-aux z L L.
alloc-aux (s N) [] [none | M] :- alloc-aux N [] M.
alloc-aux (s N) [X | L] [X | M] :- alloc-aux N L M.

type alloc addr -> mem A -> mem A -> o.
alloc (addr A as Ad) Mem1 Mem2 :- unset? Ad Mem1,
  alloc-aux A Mem1 Mem2.

type new-aux mem A -> nat -> mem A -> o.
new-aux [] z [none].
new-aux [A | As] (s N) [A | Bs] :- new-aux As N Bs.

type new mem A -> addr -> mem A -> o.
new Mem1 (addr Ad) Mem2 :- new-aux Mem1 Ad Mem2.
```

12 THE OBJECT LANGUAGE

```
kind fm type.
type fapp list fm -> fm.
type flam (fm -> fm) -> fm.
type fcon string -> fm.
type fuva addr -> fm.

typeabbrev fsubst (mem fm).

type fder fsubst -> fm -> fm -> o.
fder _ (fcon C) (fcon C).
fder S (fapp A) (fapp B) :- map (fder S) A B.
fder S (flam F) (flam G) :-
  pi x\ fder S x x => fder S (F x) (G x).
fder S (fuva N) R :- set? N S T, fder S T R.
fder S (fuva N) (fuva N) :- unset? N S.

type fderef fsubst -> fm -> fm -> o.                  (ρs)
fderef S T T2 :- fder S T T1, napp T1 T2.


type (=_o) fm -> fm -> o.                             (=_o)
fcon X =_o fcon X.
fapp A =_o fapp B :- forall2 (=_o) A B.
flam F =_o flam G :- pi x\ x =_o x => F x =_o G x.
fuva N =_o fuva N.
flam F =_o T :-                                       (η_l)
  pi x\ beta T [x] (T' x), x =_o x => F x =_o T' x.
T =_o flam F :-                                       (η_r)
  pi x\ beta T [x] (T' x), x =_o x => T' x =_o F x.
fapp [flam X | L] =_o T :- beta (flam X) L R, R =_o T.  (β_l)
T =_o fapp [flam X | L] :- beta (flam X) L R, T =_o R.  (β_r)

type extend-subst fm -> fsubst -> fsubst -> o.
extend-subst (fuva N) S S' :- mem.alloc N S S'.
extend-subst (flam F) S S' :-
  pi x\ (pi S\extend-subst x S S) => extend-subst (F x) S S'.
extend-subst (fcon _) S S.
extend-subst (fapp L) S S1 :- fold extend-subst L S S1.

type beta fm -> list fm -> fm -> o.
beta A [] A.
beta (flam Bo) [H | L] R :- napp (Bo H) F, beta F L R.
beta (fapp A) L (fapp X) :- append A L X.
beta (fuva N) L (fapp [fuva N | L]).
beta (fcon H) L (fapp [fcon H | L]).
beta N L (fapp [N | L]) :- name N.

type napp fm -> fm -> o.
napp (fcon C) (fcon C).
napp (fuva A) (fuva A).
napp (flam F) (flam G) :- pi x\ napp (F x) (G x).
napp (fapp [fapp L1 |L2]) T :- !,
  append L1 L2 L3, napp (fapp L3) T.
napp (fapp L) (fapp L1) :- map napp L L1.
napp N N :- name N.

type beta-reduce fm -> fm -> o.
beta-reduce (uvar _ _) _ :- halt "Passed uvar to beta-reduce".
beta-reduce A A :- name A.
beta-reduce (fcon A) (fcon A).
beta-reduce (fuva A) (fuva A).
beta-reduce (flam A) (flam B) :-
  pi x\ beta-reduce (A x) (B x).
beta-reduce (fapp [flam B | L]) T2 :- !,
```

```
  beta (flam B) L T1, beta-reduce T1 T2.
beta-reduce (fapp L) (fapp L1) :-
  map beta-reduce L L1.

type mk-app fm -> list fm -> fm -> o.
mk-app T L S :- beta T L S.

type eta-contract fm -> fm -> o.
eta-contract (fcon X) (fcon X).
eta-contract (fapp L) (fapp L1) :- map eta-contract L L1.
eta-contract (flam F) T :- eta-contract-aux [] (flam F) T.
eta-contract (flam F) (flam F1) :-
  pi x\ eta-contract x x => eta-contract (F x) (F1 x).
eta-contract (fuva X) (fuva X).
eta-contract X X :- name X.

type eta-contract-aux list fm -> fm -> fm -> o.
eta-contract-aux L (flam F) T :-
  pi x\ eta-contract-aux [x|L] (F x) T. % also checks H Prefix does not see x
eta-contract-aux L (fapp [H|Args]) T :-
  rev L LRev, append Prefix LRev Args,
  if (Prefix = []) (T = H) (T = fapp [H|Prefix]).
```

13 THE META LANGUAGE

$(\cdot \vdash \cdot)$

```
kind inctx type -> type.
type abs (tm -> inctx A) -> inctx A.
type val A -> inctx A.
typeabbrev assignment (inctx tm).
typeabbrev subst (mem assignment).

kind tm  type.
type app list tm -> tm.
type lam (tm -> tm) -> tm.
type con string -> tm.
type uva addr -> list tm -> tm.

type (≃λ) tm -> tm -> subst -> subst -> o.
(con C ≃λ con C) S S.
(app L1 ≃λ app L2) S S1 :- fold2 (≃λ) L1 L2 S S1.
(lam F1 ≃λ lam F2) S S1 :-
  pi x\ (pi S\ (x ≃λ x) S S) => (F1 x ≃λ F2 x) S S1.
(uva N Args ≃λ T) S S1 :-
  set? N S F,!, move F Args T1, (T1 ≃λ T) S S1.
(T ≃λ uva N Args) S S1 :-
  set? N S F,!, move F Args T1, (T ≃λ T1) S S1.
(uva M A1 ≃λ uva N A2) S1 S2 :- !,
  pattern-fragment A1, pattern-fragment A2,
  prune! M A1 N A2 S1 S2.
(uva N Args ≃λ T) S S1 :- not_occ N S T, pattern-fragment Args,
  bind T Args T1, assign N S T1 S1.
(T ≃λ uva N Args) S S1 :- not_occ N S T, pattern-fragment Args,
  bind T Args T1, assign N S T1 S1.

type prune! addr -> list tm -> addr ->
            list tm -> subst -> subst -> o.
/* no pruning needed */
prune! N A  N A  S S :- !.
prune! M A  N A  S1 S2 :- !, bind (uva M A) A Ass,
  assign N S1 Ass S2.
/* prune different arguments */
prune! N A1 N A2 S1 S3 :- !,
  new S1 W S2, prune-same-variable W A1 A2 [] Ass,
  assign N S2 Ass S3.
/* prune to the intersection of scopes */
prune! N A1 M A2 S1 S4 :- !,
  new S1 W S2, prune-diff-variables W A1 A2 Ass1 Ass2,
  assign N S2 Ass1 S3,
  assign M S3 Ass2 S4.

type prune-same-variable addr -> list tm -> list tm ->
                             list tm -> assignment -> o.
prune-same-variable N [] [] ACC (val (uva N Args)) :-
  rev ACC Args.
prune-same-variable N [X|XS] [X|YS] ACC (abs F) :-
  pi x\ prune-same-variable N XS YS [x|ACC] (F x).
prune-same-variable N [_|XS] [_|YS] ACC (abs F) :-
  pi x\ prune-same-variable N XS YS ACC (F x).

type permute list nat -> list tm -> list tm -> o.
permute [] _ [].
permute [P|PS] Args [T|TS] :-
  nth P Args T,
  permute PS Args TS.

type build-perm-assign addr -> list tm -> list bool ->
                       list nat -> assignment -> o.
build-perm-assign N ArgsR [] Perm (val (uva N PermutedArgs)) :-
  rev ArgsR Args, permute Perm Args PermutedArgs.
build-perm-assign N Acc [tt|L] Perm (abs T) :-
  pi x\ build-perm-assign N [x|Acc] L Perm (T x).
build-perm-assign N Acc [ff|L] Perm (abs T) :-
  pi x\ build-perm-assign N Acc L Perm (T x).

type keep list A -> A -> bool -> o.
keep L A tt :- mem L A, !.
keep _ _ ff.

type prune-diff-variables addr -> list tm -> list tm ->
                             assignment -> assignment -> o.
prune-diff-variables N Args1 Args2 Ass1 Ass2 :-
  map (keep Args2) Args1 Bits1,
  map (keep Args1) Args2 Bits2,
  filter Args1 (mem Args2) ToKeep1,
  filter Args2 (mem Args1) ToKeep2,
  map (index ToKeep1) ToKeep1 IdPerm,
  map (index ToKeep1) ToKeep2 Perm21,
  build-perm-assign N [] Bits1 IdPerm Ass1,
  build-perm-assign N [] Bits2 Perm21 Ass2.

type beta tm -> list tm -> tm -> o.
beta A [] A :- !.
beta (lam Bo) [H | L] R :- beta (Bo H) L R1, beta-aux R1 R.
beta (app A) L (app X) :- append A L X.
```

```
beta (con H) L (app [con H | L]).
beta X L (app[X|L]) :- name X.

type beta-aux tm -> tm -> o.
beta-aux (app [HD|TL]) R :- !, beta HD TL R.
beta-aux A A.

/* occur check for N before crossing a functor */
type not_occ  addr -> subst -> tm -> o.
not_occ N S (uva M Args) :- set? M S F,
  move F Args T, not_occ N S T.
not_occ N S (uva M Args) :- unset? M S, not (M = N),
  forall1 (not_occ_aux N S) Args.
not_occ _ _ (con _).
not_occ N S (app L) :- not_occ_aux N S (app L).
/* Note: lam is a functor for the meta language! */
not_occ N S (lam L) :- pi x\ not_occ_aux N S (L x).
not_occ _ _ X :- name X.
/* finding N is ok */
not_occ N _ (uva N _).

/* occur check for X after crossing a functor */
type not_occ_aux addr -> subst -> tm -> o.
not_occ_aux N S (uva M _) :- unset? M S, not (N = M).
not_occ_aux N S (uva M Args) :- set? M S F,
  move F Args T, not_occ_aux N S T.
not_occ_aux N S (app L) :- forall1 (not_occ_aux N S) L.
not_occ_aux N S (lam F) :- pi x\ not_occ_aux N S (F x).
not_occ_aux _ _ (con _).
not_occ_aux _ _ X :- name X.
/* finding N is ko, hence no rule */

/* copy T T' vails if T contains a free variable, i.e. it
   performs scope checking for bind */
type copy tm -> tm -> o.
copy (con C)   (con C).
copy (app L)   (app L') :- map copy L L'.
copy (lam T)   (lam T') :- pi x\ copy x x => copy (T x) (T' x).
copy (uva A L) (uva A L') :- map copy L L'.

type bind tm -> list tm -> assignment -> o.
bind T [] (val T') :- copy T T'.
bind T [X | TL] (abs T') :- pi x\ copy X x => bind T TL (T' x).

type deref subst -> tm -> tm -> o.                    (σt)
deref _ (con C) (con C).
deref S (app A) (app B) :- map (deref S) A B.
deref S (lam F) (lam G) :-
  pi x\ deref S x x => deref S (F x) (G x).
deref S (uva N L) R :- set? N S A,
  move A L T, deref S T R.
deref S (uva N A) (uva N B) :- unset? N S,
  map (deref S) A B.

type move assignment -> list tm -> tm -> o.
move (abs Bo) [H|L] R :- move (Bo H) L R.
move (val A)  [] A.
```

```
type deref-assmt subst -> assignment -> assignment -> o.
deref-assmt S (abs T) (abs R) :- pi x\ deref-assmt S (T x) (R x).
deref-assmt S (val T) (val R) :- deref S T R.
```

14 THE COMPILER

```
kind arity type.
type arity nat -> arity.

kind fvariable type.
type fv addr -> fvariable.

kind hvariable type.
type hv addr -> arity -> hvariable.

kind mapping type.
type mapping fvariable -> hvariable -> mapping.
typeabbrev mmap (list mapping).

typeabbrev scope (list tm).
typeabbrev inctx ho.inctx.
kind baselink type.
type link-eta  tm -> tm -> baselink.
type link-beta tm -> tm -> baselink.
typeabbrev link (inctx baselink).
typeabbrev links (list link).

macro @val-link-eta T1 T2 :- ho.val (link-eta T1 T2).
macro @val-link-beta T1 T2 :- ho.val (link-beta T1 T2).
```

```
type occurs-rigidly fm -> fm -> o.
occurs-rigidly N N.
occurs-rigidly _ (fapp [fuva _|_]) :- !, fail.
occurs-rigidly N (fapp L) :- exists (occurs-rigidly N) L.
occurs-rigidly N (flam B) :- pi x\ occurs-rigidly N (B x).

type reducible-to list fm -> fm -> fm -> o.
reducible-to _ N N :- !.
reducible-to L N (fapp[fuva _|Args]) :- !,
  forall1 (x\ exists (reducible-to [] x) Args) [N|L].
reducible-to L N (flam B) :- !,
  pi x\ reducible-to [x | L] N (B x).
reducible-to L N (fapp [N|Args]) :-
  last-n {len L} Args R,
  forall2 (reducible-to []) R {rev L}.

type maybe-eta fm -> list fm -> o.                     (◇η)
maybe-eta (fapp[fuva _|Args]) L :- !,
  forall1 (x\ exists (reducible-to [] x) Args) L, !.
maybe-eta (flam B) L :- !, pi x\ maybe-eta (B x) [x | L].
maybe-eta (fapp [T|Args]) L :- (name T; T = fcon _),
  split-last-n {len L} Args First Last,
  none (x\ exists (y\ occurs-rigidly x y) First) L,
```

```
    forall2 (reducible-to []) {rev L} Last.


  type locally-bound tm -> o.
  type get-scope-aux tm -> list tm -> o.
  get-scope-aux (con _) [].
  get-scope-aux (uva _ L) L1 :-
    forall2 get-scope-aux L R,
    flatten R L1.
  get-scope-aux (lam B) L1 :-
    pi x\ locally-bound x => get-scope-aux (B x) L1.
  get-scope-aux (app L) L1 :-
    forall2 get-scope-aux L R,
    flatten R L1.
  get-scope-aux X [X] :- name X, not (locally-bound X).
  get-scope-aux X [] :- name X, (locally-bound X).

  type names1 list tm -> o.
  names1 L :-
    names L1,
    new_int N,
    if (1 is N mod 2) (L1 = L) (rev L1 L).

  type get-scope tm -> list tm -> o.
  get-scope T Scope :-
    get-scope-aux T ScopeDuplicata,
    undup ScopeDuplicata Scope.
  type rigid fm -> o.
  rigid X :- not (X = fuva _).

  type comp-lam (fm -> fm) -> (tm -> tm) ->
    mmap -> mmap -> links -> links -> subst -> subst -> o.
  comp-lam F G M1 M2 L1 L3 S1 S2 :-
    pi x y\ (pi M L S\ comp x y M M L L S S) =>
      comp (F x) (G y) M1 M2 L1 (L2 y) S1 S2,
    close-links L2 L3.

  type close-links (tm -> links) -> links -> o.
  close-links (v\[X  |L v]) [X|R] :- !, close-links L R.
  close-links (v\[X v|L v]) [abs X|R] :- close-links L R.
  close-links (_\[]) [].
  type comp fm -> tm -> mmap -> mmap -> links -> links ->
    subst -> subst -> o.
  comp (fcon C) (con C) M M L L S S.
  comp (flam F) (uva A Scope) M1 M2 L1 L3 S1 S3 :-
    maybe-eta (flam F) [], !,
      alloc S1 A S2,
      comp-lam F F1 M1 M2 L1 L2 S2 S3,
      get-scope (lam F1) Scope,
      L3 = [val (link-eta (uva A Scope) (lam F1)) | L2].
  comp (flam F) (lam F1) M1 M2 L1 L2 S1 S2 :-            (cλ)
    comp-lam F F1 M1 M2 L1 L2 S1 S2.
  comp (fuva A) (uva B []) M1 M2 L L S1 S2 :-
    m-alloc (fv A) (hv B (arity z)) M1 M2 S1 S2.
  comp (fapp [fuva A|Ag]) (uva B Ag1) M1 M2 L L S1 S2 :-
    pattern-fragment Ag, !,
      fold6 comp Ag Ag1 M1 M1 L L S1 S1,
      len Ag Arity,
      m-alloc (fv A) (hv B (arity Arity)) M1 M2 S1 S2.
  comp (fapp [fuva A|Ag]) (uva B Scope) M1 M3 L1 L3 S1 S4 :- !,
    pattern-fragment-prefix Ag Pf Extra,
    len Pf Arity,
    alloc S1 B S2,
    m-alloc (fv A) (hv C (arity Arity)) M1 M2 S2 S3,
    fold6 comp Pf    Pf1    M2 M2 L1 L1 S3 S3,
    fold6 comp Extra Extra1 M2 M3 L1 L2 S3 S4,
    Beta = app [uva C Pf1 | Extra1],
    get-scope Beta Scope,
    L3 = [val (link-beta (uva B Scope) Beta) | L2].
  comp (fapp A) (app A1) M1 M2 L1 L2 S1 S2 :-
    fold6 comp A A1 M1 M2 L1 L2 S1 S2.

  type alloc mem A -> addr -> mem A -> o.
  alloc S N S1 :- mem.new S N S1.

  type compile-terms-diagnostic
    triple diagnostic fm fm ->
    triple diagnostic tm tm ->
    mmap -> mmap ->
    links -> links ->
    subst -> subst -> o.
  compile-terms-diagnostic (triple D FO1 FO2) (triple D HO1 HO2) M1 M3 L1
    fo.beta-reduce FO1 FO1',
    fo.beta-reduce FO2 FO2',
    comp FO1' HO1 M1 M2 L1 L2 S1 S2,
    comp FO2' HO2 M2 M3 L2 L3 S2 S3.

  type compile-terms
    list (triple diagnostic fm fm) ->
    list (triple diagnostic tm tm) ->
    mmap -> links -> subst -> o.
  compile-terms T H M L S :-
    fold6 compile-terms-diagnostic T H [] M_ [] L_ [] S_,
    print-compil-result T H L_ M_,
    deduplicate-map M_ M S_ S L_ L.

  type make-eta-link-aux nat -> addr -> addr ->
    list tm -> links -> subst -> subst -> o.
  make-eta-link-aux z Ad1 Ad2 Scope1 L H1 H1  :-
    rev Scope1 Scope, eta-expand (uva Ad2 Scope) T1,
    L = [val (link-eta (uva Ad1 Scope) T1)].
  make-eta-link-aux (s N) Ad1 Ad2 Scope1 L H1 H3 :-
    rev Scope1 Scope, alloc H1 Ad H2,
    eta-expand (uva Ad Scope) T2,
    (pi x\ make-eta-link-aux N Ad Ad2 [x|Scope1] (L1 x) H2 H3),
    close-links L1 L2,
    L = [val (link-eta (uva Ad1 Scope) T2) | L2].

  type make-eta-link nat -> nat -> addr -> addr ->
        list tm -> links -> subst -> subst -> o.
  make-eta-link (s N) z Ad1 Ad2 Vars L H H1 :-
    make-eta-link-aux N Ad2 Ad1 Vars L H H1.
  make-eta-link z (s N) Ad1 Ad2 Vars L H H1 :-
    make-eta-link-aux N Ad1 Ad2 Vars L H H1.
```

```
make-eta-link (s N) (s M) Ad1 Ad2 Vars Links H H1 :-
  (pi x\ make-eta-link N M Ad1 Ad2 [x|Vars] (L x) H H1),
  close-links L Links.

type deduplicate-map mmap -> mmap ->
    subst -> subst -> links -> links -> o.
deduplicate-map [] [] H H L L.
deduplicate-map [(mapping (fv O) (hv M (arity LenM)) as X1) | Map1] Map2 H1 H3 L1 L3 :-
  take-list Map1 (mapping (fv O) (hv M' (arity LenM'))) _, !,
  std.assert! (not (LenM = LenM')) "Deduplicate map, there is a bug",
  print "arity-fix links:" {ppmapping X1} "~!~" {ppmapping (mapping (fv O) (hv M' (arity LenM')))},
  make-eta-link LenM LenM' M M' [] New H1 H2,
  print "new eta link" {pplinks New},
  append New L1 L2,
  deduplicate-map Map1 Map2 H2 H3 L2 L3.
deduplicate-map [A|As] [A|Bs] H1 H2 L1 L2 :-
  deduplicate-map As Bs H1 H2 L1 L2, !.
deduplicate-map [A|_] _ H _ _ _ :-
  halt "deduplicating mapping error" {ppmapping A} {ho.ppsubst H}.
```

15 THE PROGRESS FUNCTION

```
macro @one :- s z.

type contract-rigid list ho.tm -> ho.tm -> ho.tm -> o.
contract-rigid L (ho.lam F) T :-
  pi x\ contract-rigid [x|L] (F x) T. % also checks H Prefix does not see x
contract-rigid L (ho.app [H|Args]) T :-
  rev L LRev, append Prefix LRev Args,
  if (Prefix = []) (T = H) (T = ho.app [H|Prefix]).

type progress-eta-link ho.tm -> ho.tm -> ho.subst -> ho.subst -> links -> o.
progress-eta-link (ho.app _ as T) (ho.lam x\ _ as T1) H H1 [] :- !,
  ({eta-expand T @one} ==l T1) H H1.
progress-eta-link (ho.con _ as T) (ho.lam x\ _ as T1) H H1 [] :- !,
  ({eta-expand T @one} ==l T1) H H1.
progress-eta-link (ho.lam _ as T) T1 H H1 [] :- !,
  (T ==l T1) H H1.
progress-eta-link (ho.uva _ _ as X) T H H1 [] :-
  contract-rigid [] T T1, !, (X ==l T1) H H1.
progress-eta-link (ho.uva Ad _ as T1) T2 H H [@val-link-eta T1 T2] :- !,
  if (ho.not_occ Ad H T2) true fail.

type is-in-pf ho.tm -> o.
is-in-pf (ho.app [ho.uva _ _ | _]) :- !, fail.
is-in-pf (ho.lam B) :- !, pi x\ is-in-pf (B x).
is-in-pf (ho.con _).
is-in-pf (ho.app L) :- forall1 is-in-pf L.
is-in-pf N :- name N.
is-in-pf (ho.uva _ L) :- pattern-fragment L.

type arity ho.tm -> nat -> o.
arity (ho.con _) z.
arity (ho.app L) A :- len L A.

type occur-check-err ho.tm -> ho.tm -> ho.subst -> o.
occur-check-err (ho.con _) _ _ :- !.
occur-check-err (ho.app _) _ _ :- !.
occur-check-err (ho.lam _) _ _ :- !.
occur-check-err (ho.uva Ad _) T S :-
  not (ho.not_occ Ad S T).

type progress-beta-link-aux ho.tm -> ho.tm ->
      ho.subst -> ho.subst -> links -> o.
progress-beta-link-aux T1 T2 S1 S2 [] :-  is-in-pf T2, !,
  (T1 ==l T2) S1 S2.
progress-beta-link-aux T1 T2 S S [@val-link-beta T1 T2] :- !.

type progress-beta-link ho.tm -> ho.tm -> ho.subst ->
     ho.subst -> links -> o.
progress-beta-link T (ho.app[ho.uva V Scope | L] as T2) S S2 [@val-link-
  arity T Arity, len L ArgsNb, ArgsNb >n Arity, !,
  minus ArgsNb Arity Diff, mem.new S V1 S1,
  eta-expand (ho.uva V1 Scope) Diff T1,
  ((ho.uva V Scope) ==l T1) S1 S2.

progress-beta-link (ho.uva _ _ as T) (ho.app[ho.uva Ad1 Scope1 | L1] as
  append Scope1 L1 Scope1L,
  pattern-fragment-prefix Scope1L Scope2 L2,
  not (Scope1 = Scope2), !,
  mem.new S1 Ad2 S2,
  len Scope1 Scope1Len,
  len Scope2 Scope2Len,
  make-eta-link Scope1Len Scope2Len Ad1 Ad2 [] LinkEta S2 S3,
  if (L2 = []) (NewLinks = LinkEta, T2 = ho.uva Ad2 Scope2)
    (T2 = ho.app [ho.uva Ad2 Scope2 | L2],
    NewLinks = [@val-link-beta T T2 | LinkEta]).

progress-beta-link T1 (ho.app[ho.uva _ _ | _] as T2) _ _ _ :-
  not (T1 = ho.uva _ _), !, fail.

progress-beta-link (ho.uva _ _ as T) (ho.app[ho.uva _ _ | _] as T2) S1 _
  occur-check-err T T2 S1, !, fail.

progress-beta-link T1 (ho.app[ho.uva _ _ | _] as T2) H H [@val-link-beta

progress-beta-link T1 (ho.app [Hd | Tl]) S1 S2 B :-
  ho.beta Hd Tl T3,
  progress-beta-link-aux T1 T3 S1 S2 B.

type solve-link-abs link -> links -> ho.subst -> ho.subst -> o.
solve-link-abs (ho.abs X) R H H1 :-
  pi x\ ho.copy x x => (pi S\ ho.deref S x x) =>
    solve-link-abs (X x) (R' x) H H1,
  close-links R' R.

solve-link-abs (@val-link-eta A B) NewLinks S S1 :- !,
  progress-eta-link A B S S1 NewLinks.

solve-link-abs (@val-link-beta A B) NewLinks S S1 :- !,
  progress-beta-link A B S S1 NewLinks.

type take-link link -> links -> link -> links -> o.
take-link A [B|XS] B XS :- link-abs-same-lhs A B, !.
```

```
take-link A [L|XS] B [L|YS] :- take-link A XS B YS.

type link-abs-same-lhs link -> link -> o.
link-abs-same-lhs (ho.abs F) B :-
  pi x\ link-abs-same-lhs (F x) B.
link-abs-same-lhs A (ho.abs G) :-
  pi x\ link-abs-same-lhs A (G x).
link-abs-same-lhs (@val-link-eta (ho.uva N _) _) (@val-link-eta (ho.uva N _) _).

type same-link-eta link -> link -> ho.subst -> ho.subst -> o.
same-link-eta (ho.abs F) B H H1 :- !, pi x\ same-link-eta (F x) B H H1.
same-link-eta A (ho.abs G) H H1 :- !, pi x\ same-link-eta A (G x) H H1.
same-link-eta (@val-link-eta (ho.uva N S1) A)
              (@val-link-eta (ho.uva N S2) B) H H1 :-
  std.map2 S1 S2 (x\y\r\ r = ho.copy x y) Perm,
  Perm => ho.copy A A',
  (A' ==l B) H H1.

type progress1 links -> links -> ho.subst -> ho.subst -> o.
progress1 [] [] X X.
progress1 [A|L1] [A|L3] S S2 :- take-link A L1 B L2, !,
  same-link-eta A B S S1,
  progress1 L2 L3 S1 S2.
progress1 [L0|L1] L3 S S2 :- deref-link S L0 L,
  solve-link-abs L R S S1, !,
  progress1 L1 L2 S1 S2, append R L2 L3.
```

16 THE DECOMPILER

```
type abs->lam ho.assignment -> ho.tm -> o.
abs->lam (ho.abs T) (ho.lam R)  :- !, pi x\ abs->lam (T x) (R x).
abs->lam (ho.val A) A.

type commit-links-aux link -> ho.subst -> ho.subst -> o.
commit-links-aux (@val-link-eta T1 T2) H1 H2 :-
  ho.deref H1 T1 T1', ho.deref H1 T2 T2',
  (T1' ==l T2') H1 H2.
commit-links-aux (@val-link-beta T1 T2) H1 H2 :-
  ho.deref H1 T1 T1', ho.deref H1 T2 T2',
  (T1' ==l T2') H1 H2.
commit-links-aux (ho.abs B) H H1 :-
  pi x\ commit-links-aux (B x) H H1.

type commit-links links -> links -> ho.subst -> ho.subst -> o.
commit-links [] [] H H.
commit-links [Abs | Links] L H H2 :-
  commit-links-aux Abs H H1, !, commit-links Links L H1 H2.

type decompl-subst map -> map -> ho.subst ->
  fo.fsubst -> fo.fsubst -> o.
decompl-subst _ [A|_] _ _ _ :- fail.
decompl-subst _ [] _ F F.
decompl-subst Map [mapping (fv VO) (hv VM _)|Tl] H F F2 :-
  mem.set? VM H T, !,
  ho.deref-assmt H T TTT,
  abs->lam TTT T', tm->fm Map T' T1,
  fo.eta-contract T1 T2, mem.assign VO F T2 F1,
  decompl-subst Map Tl H F1 F2.
decompl-subst Map [mapping _ (hv VM _)|Tl] H F F2 :-
  mem.unset? VM H, decompl-subst Map Tl H F F2.

type tm->fm map -> ho.tm -> fo.fm -> o.
tm->fm _ (ho.con C)  (fo.fcon C).
tm->fm L (ho.lam B1) (fo.flam B2) :-
  pi x\ y\ tm->fm _ x y => tm->fm L (B1 x) (B2 y).
tm->fm L (ho.app L1) T :- map (tm->fm L) L1 [Hd|Tl],
  fo.mk-app Hd Tl T.
tm->fm L (ho.uva VM TL) T :- mem L (mapping (fv VO) (hv VM _)),
  map (tm->fm L) TL T1, fo.mk-app (fo.fuva VO) T1 T.

type add-new-map-aux ho.subst -> list ho.tm -> map ->
     map ->  fo.fsubst -> fo.fsubst -> o.
add-new-map-aux _ [] _ [] S S.
add-new-map-aux H [T|Ts] L L2 S S2 :-
  add-new-map H T L L1 S S1,
  add-new-map-aux H Ts L1 L2 S1 S2.

type add-new-map ho.subst -> ho.tm -> map ->
    map ->  fo.fsubst -> fo.fsubst -> o.
add-new-map _ (ho.uva N _) Map [] F1 F1 :-
  mem Map (mapping _ (hv N _)), !.
add-new-map H (ho.uva N L) Map [Map1 | MapL] F1 F3 :-
  mem.new F1 M F2,
  len L Arity, Map1 = mapping (fv M) (hv N (arity Arity)),
  add-new-map H (ho.app L) [Map1 | Map] MapL F2 F3.
add-new-map H (ho.lam B) Map NewMap F1 F2 :-
  pi x\ add-new-map H (B x) Map NewMap F1 F2.
add-new-map H (ho.app L) Map NewMap F1 F3 :-
  add-new-map-aux H L Map NewMap F1 F3.
add-new-map _ (ho.con _) _ [] F F :- !.
add-new-map _ N _ [] F F :- name N.

type complete-mapping-under-ass ho.subst -> ho.assignment ->
  map -> map ->  fo.fsubst -> fo.fsubst -> o.
complete-mapping-under-ass H (ho.val Val) Map1 Map2 F1 F2 :-
  add-new-map H Val Map1 Map2 F1 F2.
complete-mapping-under-ass H (ho.abs Abs) Map1 Map2 F1 F2 :-
  pi x\ complete-mapping-under-ass H (Abs x) Map1 Map2 F1 F2.

type complete-mapping ho.subst -> ho.subst ->
  map -> map -> fo.fsubst -> fo.fsubst -> o.
complete-mapping _ [] L L F F.
complete-mapping H [none | Tl] L1 L2 F1 F2 :-
  complete-mapping H Tl L1 L2 F1 F2.
complete-mapping H [some T0 | Tl] L1 L3 F1 F3 :-
  ho.deref-assmt H T0 T,
  complete-mapping-under-ass H T L1 L2 F1 F2,
  append L1 L2 LAll,
  complete-mapping H Tl LAll L3 F2 F3.

type decompile map -> links -> ho.subst ->
  fo.fsubst -> fo.fsubst -> o.
decompile Map1 L HO FO FO2 :-
  commit-links L L1_ HO HO1, !,
```

```
  complete-mapping HO1 HO1 Map1 Map2 FO FO1,
  decompl-subst Map2 Map2 HO1 FO1 FO2.
```

17 AUXILIARY FUNCTIONS

```
type fold4 (A -> A1 -> B -> B -> C -> C -> o) -> list A ->
  list A1 -> B -> B -> C -> C -> o.
fold4 _ [] [] A A B B.
fold4 F [X|XS] [Y|YS] A A1 B B1 :- F X Y A A0 B B0,
  fold4 F XS YS A0 A1 B0 B1.

type len list A -> nat -> o.
len [] z.
len [_|L] (s X) :- len L X.
```