MASTER INFORMATIQUE

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Course : TER

Generation of sequences controlled by their "complexity"

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Abstract

We want to generate sequences of musical "chords" (a chord is a set of notes basically) with some known constraints (allDiff, etc.) as well as control on the complexity of the sequence. This complexity in turn is defined by a dynamic programming algorithm working on the instantiated sequence, which makes the whole problem difficult.

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1 Problem description

2 Definitions and notations

In this section we fix some notations that will be reused next.
$G = (V, A)$ is a directed graph where $V = (v_1, \dots, v_n)$ is the set of its vertices and $A = (a_1, \dots, a_m)$ is the set of its arcs. n and m represents the cardinality of respectively V and A. An arc $a_i \in A$ is a pair $(v_i, v_j) \in V^2$ saying that a_i goes from v_i to v_j. The arc (v_i, v_j) is different from (v_j, v_i).
F is the coloring function and takes an arc a as parameters. It returns the set of colors $\mathbb{C}$ associated to a. By abuse of notation we say that $F(a) = F(v_i, v_j)$ if $a = (v_i, v_j)$.
$R : A \to \mathbb{N}$ is the affectation function, and $R(e) = c$ if $c \in F(e)$. For simplicity, if $P = (a_1, \dots, a_k)$ is a list of consecutive arcs whose length is k, then $F(P) = (F(a_1), \dots, F(a_k)) = (\mathbb{C}_1, \dots, \mathbb{C}_k)$ and $R(P) = (R(a_1), \dots, R(a_k)) = (c_1, \dots, c_k)$.
Given a path P of length k and its corresponding affectations $H = R(P)$, the weight of H is the number of its color switches. This cost is computed by $w(R(P))$.

$$w(H) = \sum_{i=1}^{k-1} (c_i \neq c_{i+1}) \tag{1}$$

$w_{OPT}(H)$ is the minimal weight of a path among all the possible affectation H of P, R_{OPT} is an optimal affectation.
Finally, we say that a shortest path from v_i to v_j in a graph G is a path P starting in v_i and ending in v_j whose optimal affectation R_{OPT} is the minimal among all the other possible paths in G.

3 Minimize Switches in Paths

The goal of this section is to provide a greedy algorithm able to compute an optimal affectation H of a given path P. The obtained result, will then be extended to general graphs using the **XXX matrix**.

3.1 Procedure

This problem can be solved through a greedy strategy: taking a path P and a coloring function F, we must delay a color switch as much as possible. At the end we will have selected the biggest $l \in [1, k]$ such that the edges $(e_1, \dots, e_l)$ have at least one color in common. We repeat this procedure from the edge e_{l+1} until reaching the end of our path. An implementation of this algorithm can be found in Algorithm 2.

Proof. Let $R = (c_1, \dots, c_k)$ be a solution returned by our algorithm, we can easily prove by induction on the length of the path that the solution is optimal.

For $k = 1$ we have $w(R) = 0$ by definition of the weight function which is of course the optimal cost.

Let's suppose that the solution R is an optimal one for every path of length at least k. We want to prove that the algorithm is always valid for a path of length $k + 1$, we see that:

- if $F(e_k) \cap F(e_{k+1}) = \varnothing$ then we are forced to do a color switch, for every affectation of the edge $R' = ((c_1, \dots, c_k))$. Since, by hypothesis, the affectation of the edges $w(R')$ is optimal, then it will remain optimal for any affectation of the edge e_{k+1} and $w(R) = w(R') + 1$.

- if $F(e_k) \cap F(e_{k+1}) \neq \varnothing$ we have two cases to treat:

  - if $c_k \in F(e_{k+1})$ then the algorithm we give to e_{k+1} the same color of e_k. This will not increase the number of color switch which will remain optimal.

  - if $c_k \notin F(e_{k+1})$ then the algorithm will force a color switch even if it would have been possible to give them the same color. Despite this, if we decide to give the same colors to e_k and e_{k+1} then we are only anticipating a color switch, and in the end $w(R)$ will remain optimal.

□

3.2 Time Complexity

We can analyze the time complexity of this procedure from the implementation proposed in Algorithm 2. We have two loops of size k (the length of the path). Inside them we make intersection between sets of at most s colors, then the intersection between two sets of that size will take $O(s)$. Finally, the global time complexity will be $O(2 * k * s) = O(k * s)$.

3.3 An example run

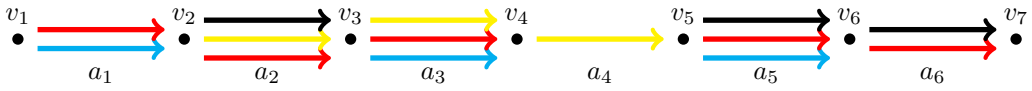

Figure 1: A path example

Let's take Figure 1, where $P = (a_1, \ldots, a_6)$ and F such that

$$\begin{aligned} F(P) = (&\{cyan, red\}, \{red, yellow, black\}, \\ &\{cyan, red, yellow\}, \{yellow\}, \\ &\{cyan, red, black\}, \{red, black\}) \end{aligned}$$

The longest subpath of same color, starting from the vertex v_1, is $P_1 = (a_1, a_2, a_3)$ such that $R(a) = red$ for all $a \in P_1$. Then $R(e_4) = yellow$ and $R(a_5) = R(a_6) = black$. This affectation $H = (red, red, red, yellow, black, black)$ has $w(H) = 2$ and is optimal.

One can note that there can exist other optimal affectations, from Figure 1 we can choose $H' = (cyan, yellow, yellow, yellow, red, red)$, but in any case, any other affectations will not be less than $w(H)$.

3.4 Extension on cycles

A cycle in a path whose starting node coincide with its last one. We see that the previous algorithm is no more effective, since we have to keep into account the potential color switch between the first and the last edge of it. Despite this, the procedure proposed in Section 3.1, can be easily modified to provide an optimal affectation on cycles. Let's take the path of Figure 1 and imagine that nodes n_1 and n_7 coincide. We now see that the affectation H of Section 3.3 is no more optimal: $w(R) = 3$, while the affectation $H' = (red, red, red, yellow, red, red)$ as a cost of 2. In order to take into account this situation, we assign to the first P_1 and the last P_l sub-path of edges with same colors a set of common colors. Finally if the intersection of P_1 and P_l is not empty, we will affect them to a color they share, otherwise, whatever choice of color for P_1 and P_l will not influence the final cost of the chosen affectation.

Concretely, take the example in Figure 1, then $P_1 = (a_1, a_2, a_3)$ and $P_l = (a_5, a_6)$. Let $C_1 = \bigcap_{a \in P_1} R(a)$ and $C_2 = \bigcap_{a \in P_2} R(a)$. We know that both C_1 and C_2 are non-empty. Then since $C_1 \cap C_2 = \{red\}$ then we can set red to all arcs in P_1 and P_2 reducing therefore the overall switch number.

4 Minimize color switches with matrices

The previous section provides a strategy to compute the smallest cost of a given path. The key idea is to delay color switches and in this section we try to rework this algorithm in order to apply it on general directed graph. The goal is to find paths made of a *fixed* number of edges between two vertices in order to minimize the number of color switches.

4.1 Floyd-Warshall algorithm

Floyd [1] and Warshall [2], in respectively 1959 and 1962, gave an implementation [5] of an algorithm able to compute the shortest path of a directed weighted graph.

Let G be a directed graph and c, such that for all couple of vertices i, j of V, if there exists no arc going from i to j in A then $w(i, j) = \infty$. Let M be the $n \times n$ adjacency matrix of G such that each cell M_{ij} equals $w(i, j)$. Note that for each $v \in V$, $w(v, v) = 0$.

The goal of the algorithm is to update the weight of each cell for every iteration. In particular at time 0, we have $M^1 = M$ representing all the shortest path of length *at most* 1 between two vertex. To

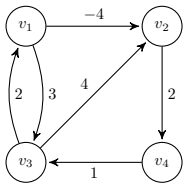

Figure 2: A directed weighted graph example

	v_1	v_2	v_3	v_4
v_1	0	−4	3	∞
v_2	∞	0	∞	2
v_3	2	4	0	∞
v_4	∞	∞	1	0

(a) Iteration 1

	v_1	v_2	v_3	v_4
v_1	0	−4	3	−2
v_2	∞	0	∞	2
v_3	2	−2	0	0
v_4	∞	∞	1	0

(b) Iteration 2

	v_1	v_2	v_3	v_4
v_1	0	−4	3	−2
v_2	∞	0	∞	2
v_3	2	−2	0	0
v_4	3	−1	1	0

(c) Iteration 3

	v_1	v_2	v_3	v_4
v_1	0	−4	−1	−2
v_2	5	0	3	2
v_3	2	−2	0	0
v_4	3	−1	1	0

(d) Iteration 4

Table 1: Floyd-Warshall algorithm execution of Figure 2

improve the information about the global shortest path, we look for path from every couple $i, j \in V^2$ passing through a third vertex k and take the minimum distance. Therefore M^2 will indicate all shortest path for every pair of vertices of length *at most* 2.

$$M_{ij} = \min_{k \in V}(M_{ik} + M_{kj}) \tag{2}$$

Globally, the matrix should be updated n times. In fact, except for negative cycles, every shortest path between two vertices will pass through every vertex *at most* one time.

Let's take the directed graph represented in Figure 2. The corresponding adjacency matrix is indicated in Table 1a. At the iteration 2, the distance from vertex v_3 to vertex v_2 is updated to -2 since, while looking for all possible paths passing through an intermediate vertex, there is the path P' going from n_3 to n_1 and then from n_1 to n_2. The overall cost P' is made of $c_{3,1} + c_{1,2} = 2 + (-4) = -2$ which is less than the direct path v_3 to v_2 depicted in the adjacency matrix.

4.2 Paths of fixed length with minimum cost

As explained in [6] and [7], the Floyd-Warshall algorithm can be generalized in order to compute shortest paths on directed weighted graphs having a fixed number of edges. This approach is based on the theory of semi-rings [3].

Let M be the adjacency matrix of a graph whose cells on the diagonal have an infinity weights if there is no self-loop on the considered vertex. We say that M^k is the k-matrix where each cell M_{ij} contains the cost of the shortest path from i to j with *exactly*[1] k edges.

The update function of this generalized approach is slightly different from Equation (2) since the cost of the cell c_{ij} will depend of its cost at the previous iteration and the original adjacency matrix.

$$M^k_{ij} = \min_{v \in V}(M^{k-1}_{in} + M_{nj}) \tag{3}$$

An important remark of this approach is that while finding path with fixed length k, we consent to pass more than one time through each vertices (*i.e.*we can have non-simple path).

4.3 Minimize color switches with matrices

In this section we propose an adaptation of the generalized Floyd-Warshall algorithm in order to solve the shortest path problem minimizing the number of color switches in oriented graphs.

The main strategy as proved in Section 3.1 will be to delay color switches as further as possible by keeping trace of the set of colors for every edge.

What will change is the construction of the adjacency matrix M since we do not have exact costs associated to edges, but instead a set of colors. The weight of this edge will depend by the chosen affectation compared to the one of its neighbors. M is a $n \times n$-matrix where

[1] Note that in Section 4.1 we spoke about path of *at most* k edges.

$$M_{ij} = \begin{cases} F(ij), \text{ if } ij \in A \\ \varnothing, \text{ otherwise} \end{cases} \tag{4}$$

The cells of the M^k matrix is a pair *(w, cols)* where: w is the cost of the path and *cols* is the set of colors to choose in cell ij minimizing the number of color switches for the current path.

Similarly to the matrix computation illustrated in the previous section, M^k is computed from the matrix at time $k-1$ and the adjacency matrix M. Particularly, for all $i, j \in V$, $M^1_{ij} = \{w \leftarrow 0 \text{ if } M_{ij} \neq \varnothing;\ cols \leftarrow M_{ij}\}$ and the other matrices are computed thanks Algorithm 1

Algorithm 1: Compute M^{k+1}

Input: M^k, M respectively the matrix at time k and the adjacency matrix
Output: M^{k+1} the matrix at time $k+1$

$n = len(M)$;
// Matrix initialization
$M^{k+1} \leftarrow$ new $n \times n$ matrix ;
$\forall i, j \in [0..n]^2 : M^{k+1}_{ij} \leftarrow \{w \leftarrow \infty;\ cols \leftarrow \varnothing\}$;
// Procedure start
for $i = 1$ **to** n **do**
  for $j = 1$ **to** n **do**
    for $p = 1$ **to** n **do**
      $inter\_prov \leftarrow M^k_{ip}.cols \cap M_{pj}$;
      $cost \leftarrow M^k_{ip}.w +$ (if $inter\_prov = \varnothing$ then 1 else 0);
      $inter \leftarrow$ (if $inter\_prov = \varnothing$ then M_{pj} else $inter\_prov$);
      if $cost < M^{k+1}_{ij}.w$ **then**
        $M^{k+1}_{ij} \leftarrow \{w \leftarrow cost;\ cols \leftarrow inter\}$;
      else if $cost = M^{k+1}_{ij}.w$ **then**
        $M^{k+1}_{ij}.cols \leftarrow inter \cup M^{k+1}_{ij}.cols$;
      end
    end
  end
end
return M^{k+1};

Firstly, the matrix M^{k+1} initialized and each cell as an empty set of colors and an infinity cost. In a second moment for each cell ij we try to find a minimal path passing through a vertex $p \in M$. If such path exists (*i.e.* the distance is not ∞), we have to check the intersection I of the color set of the cell ip at time k and the set of colors returned by the coloring function for the edge pj. The cost of the edge is either 0 if I is non-empty, 1 otherwise. The subset of colors associated to the edge pj at position k is I if it is non-empty, otherwise $F(pj)$.

The shortest path is obtained by finding the path ipk with minimum associated cost. If there are multiple minima then the colors we can use at vertex j is made by the union of all the colors associated to the vertices with the minimum cost.

Why take the union, and why it is the good strategy to find the minimum paths ?

Add binary exponentiation

5 Minimize colors switches with MDD

5.1 Multi-Valued Decision Diagram

A *Multi-Valued Decision Diagram* is

7 References

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[7] *Number of paths of fixed length / Shortest paths of fixed length*. June 8, 2022. URL: https://cp-algorithms.com/graph/fixed_length_paths.html.

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A Algorithms

A.1 Minimize color switches in a path

Algorithm 2: Shortest path of a path

Input: $P = (a_1, \ldots, a_k), F :=$ a path and the color function
Output: $H :=$ a path affectation minimizing the color switches

$colSet \leftarrow [F(a_i)$ for $i \in [1..k]]$;
for $i \leftarrow 2$ **to** k **do**
  $inter \leftarrow colSet[i-1] \cap colSet[i]$; `// Delay a color switch`
  if $inter \neq \varnothing$ **then**
    $colSet[i] \leftarrow inter$;
  end
end
$H \leftarrow [colSet[i].choose()$ for $i \in [1..k]]$;
for $i = k-2$ **downto** 1 **do**
  if $H[i+1] \in colSet[i] \wedge H[i] \neq H[i+1]$ **then**
    $H[i] \leftarrow H[i+1]$; `// If possible the` $R(e_i)$ `equals` $R(e_{i+1})$
  end
end
return H;