Path Color Switching

Supervisor: Jean-Charles Régin

Fissore Davide

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Problem Description

> We want to generate sequences of musical "chords" with some known constraints as well as control on the complexity of the sequence.
>
> — Spotify

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Input An oriented graph whose arcs are colored with a set of colors, two nodes of the graphs s and t and a length k.

Output Set single colors to edges to find a path of length k from s to t minimizing the number of color switch.

Definitions & notations

Color switch (CS): given two adjacent arcs a_1 and a_2 colored respectively with c_1 and c_2, we have a color *CS* if $c_1 \neq c_2$.

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$\mathcal{G} = (V, A)$: A directed graph where V is the set of its nodes and A is the set of its arcs.

$\mathcal{C}$: A finite set of colors.

$\mathcal{F}$: The coloring function defined as $\mathcal{F} : A \to 2^{\mathcal{C}}$.

$\mathcal{P} = (v_1, \ldots, v_k)$: A path going from v_1 to v_k.

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$\mathcal{P} = (v_1, \ldots, v_k)$: A path going from v_1 to v_k.

$w(\mathcal{P})$: The cost of the path $\mathcal{P}$ which is given by the sum of its *CS*.

Problem decomposition

The problem can decomposed in small parts:

- Minimize CS on paths;
- Minimize CS on graphs.

Minimize *CS* on Paths

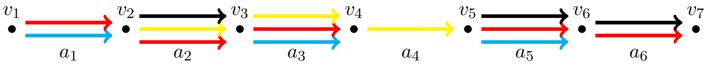

Figure: A path $\mathcal{P}$

What is the color assignation minimizing $w(\mathcal{P})$?

Algorithm

Let $\mathcal{P} = (a_1, \ldots, a_k)$ a path
Let $\mathcal{T} : A \to 2^{\mathcal{C}}$ a function such that:

- $\mathcal{T}(a_1) = \mathcal{F}(a_1)$
- $\mathcal{T}(a_i) = \mathcal{F}(a_i) \cap \mathcal{T}(a_{i-1})$ `if not empty else` $\mathcal{F}(a_i)$

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$\mathcal{H} : A \to \mathcal{C}$ the function minimizing $w(\mathcal{P})$ such that:

- $\mathcal{H}(a_k) =$ `a rnd elt from` $\mathcal{T}(a_k)$
- $\mathcal{H}(a_i) = \mathcal{H}(a_{i+1})$ `if it is in` $\mathcal{T}(a_i)$ `else` $\mathcal{T}(a_i)$`.peek()`

Example run

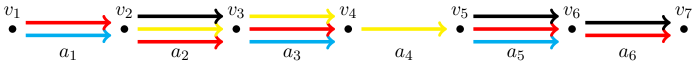


Figure: A path $\mathcal{P}$

Start to compute $\mathcal{T}(\mathcal{P})$

Example run

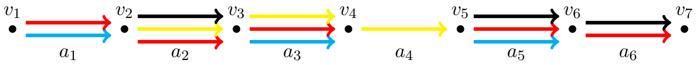


Figure: Computing $\mathcal{T}(\mathcal{P})$

$$\mathcal{T}(a_1) = \mathcal{F}(a_1)$$

Example run

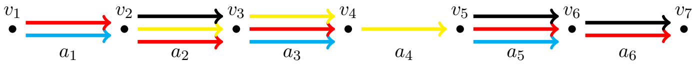
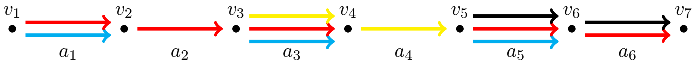


Figure: Computing $\mathcal{T}(\mathcal{P})$

$\mathcal{T}(a_2) = \mathcal{F}(a_2) \cap \mathcal{T}(a_1)$ `since not empty`

Example run



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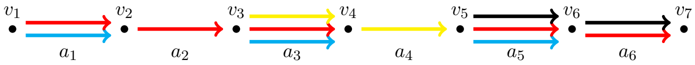

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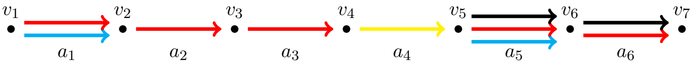
$$\mathcal{T}(a_3) = \mathcal{F}(a_3) \cap \mathcal{T}(a_2) \texttt{ since not empty}$$

Example run



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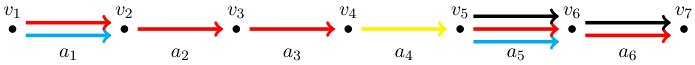


Figure: Computing $\mathcal{T}(\mathcal{P})$

$$\mathcal{T}(a_4) = \mathcal{F}(a_4) \ \texttt{since} \ \mathcal{F}(a_4) \cap \mathcal{T}(a_3) = \varnothing$$

Example run

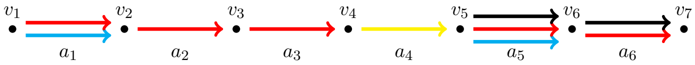


Figure: Computing $\mathcal{T}(\mathcal{P})$

$$\mathcal{T}(a_5) = \mathcal{F}(a_5) \ \texttt{since} \ \mathcal{F}(a_5) \cap \mathcal{T}(a_4) = \varnothing$$

Example run

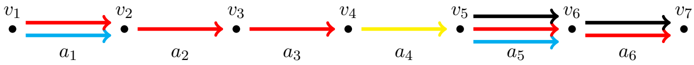


Figure: Computing $\mathcal{T}(\mathcal{P})$

$$\mathcal{T}(a_6) = \mathcal{F}(a_6) \cap \mathcal{T}(a_5) \texttt{ since not empty}$$

Example run

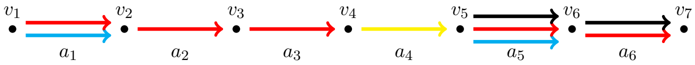


Figure: Computing $\mathcal{T}(\mathcal{P})$

Start to compute $\mathcal{H}(\mathcal{P})$

Example run

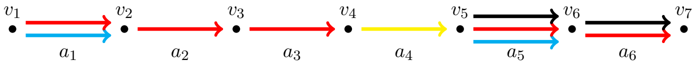
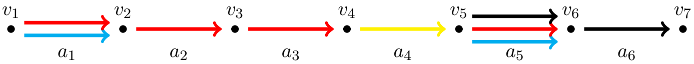


Figure: Computing $\mathcal{H}(\mathcal{P})$

$$\mathcal{H}(a_6) = black$$

Example run



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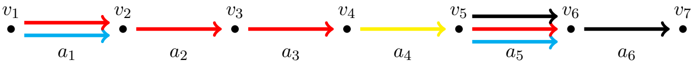
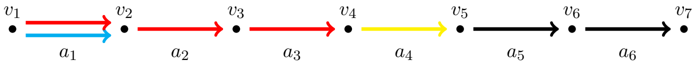


Figure: Computing $\mathcal{H}(\mathcal{P})$

$$\mathcal{H}(a_5) = black \ \texttt{since} \ black \in \mathcal{T}(a_5)$$

Example run



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Example run

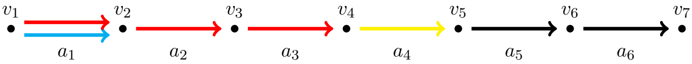

Figure: Computing $\mathcal{H}(\mathcal{P})$

Nothing to do for a_4, a_3 and a_2 since they only have 1 color

Example run

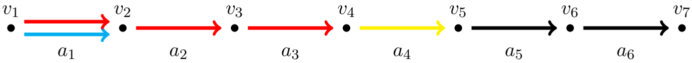

Figure: Computing $\mathcal{H}(\mathcal{P})$

$$\mathcal{H}(a_1) = \mathcal{H}(a_2) \ \texttt{since} \ red \in \mathcal{T}(a_1)$$

Example run

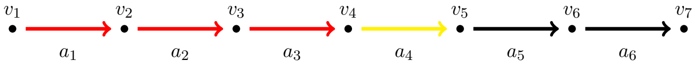

Figure: Minimum cost assignation

Example run

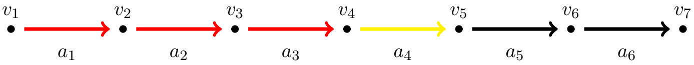

Figure: Minimum cost assignation

$$w(\mathcal{P}) = 2$$

Proof sketch

Algo Part 1.

Induction proof on the length k of $\mathcal{P}$.

$\square$

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Induction proof on the length k of $\mathcal{P}$.
If $k = 1$ then $w(\mathcal{P}) = 0$ which is optimal. $\square$

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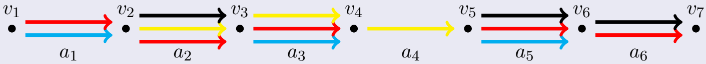
Algo Part 1.

Induction proof on the length k of $\mathcal{P}$.
We suppose the algo to be true for an arbitrary length k.
If $\mathcal{F}(a_k) \cap \mathcal{F}(a_{k+1}) = \varnothing$

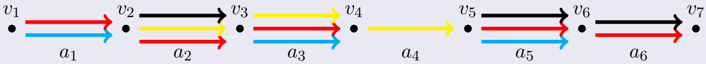


□

Proof sketch

Algo Part 1.

Induction proof on the length k of $\mathcal{P}$.
We suppose the algo to be true for an arbitrary length k.
If $\mathcal{T}(a_k) \cap \mathcal{F}(a_{k+1}) \neq \varnothing$

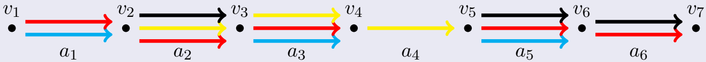


$\square$

Proof sketch

Algo Part 1.

Induction proof on the length k of $\mathcal{P}$.
We suppose the algo to be true for an arbitrary length k.
If $\mathcal{T}(a_k) \cap \mathcal{F}(a_{k+1}) = \varnothing$ and $\mathcal{F}(a_k) \cap \mathcal{F}(a_{k+1}) \neq \varnothing$



$\square$

Proof sketch

Algo Part 1.

Induction proof on the length k of $\mathcal{P}$.
Done $\square$

Algo Part 2.

The number of CS inside $\mathcal{T}$ is the same as the number of CS inside $\mathcal{H}$. $\square$

Time Complexity

The algo is made by two sub-procedures:

Recall the first part:

- $\mathcal{T}(a_1) = \mathcal{F}(a_1)$
- $\mathcal{T}(a_i) = \mathcal{F}(a_i) \cap \mathcal{T}(a_{i-1})$ `if not empty else` $\mathcal{F}(a_i)$

Complexity:

- First part : $\mathcal{O}(k * |\mathcal{C}|)$

Time Complexity

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Recall the second part:

- $\mathcal{H}(a_k) = \texttt{a rnd elt from } \mathcal{T}(a_k)$
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Complexity:

- First part : $\mathcal{O}(k * |\mathcal{C}|)$
- Second part : $\mathcal{O}(k * \log |\mathcal{C}|)$

Time Complexity

Complexity:

- First part : $\mathcal{O}(k * |\mathcal{C}|)$
- Second part : $\mathcal{O}(k * \log |\mathcal{C}|)$

Global complexity: $\mathcal{O}(k * |\mathcal{C}|)$.
This complexity is optimal wrt the entry of the problem.

Minimize *CS* in Graph

Strategy: Use the *MDD* data structure

A state of a *MDD* is:

{name: String, cost: Int, colors: Set of Colors}

Algorithm

- The root = {name: s, cost: 0, colors: $\mathcal{C}$}

Algorithm 1: Construction of the layer $\mathcal{L}_{i+1}$

```
for ∀st ∈ L_i do
    for ∀v ∈ succ(st.name) do
        col = F(st.name, v);
        inter ← col ∩ st.colors;
        if inter = ∅ then
            L_{i+1}.add({name: v, cost: st.cost +
             1, colors: col})
        else
            L_{i+1}.add({name: v, cost: st.cost, colors: inter})
        end
    end
end
```

MDD reduction

Let s_1 and s_2 two state on the same layer $\mathcal{L}$, having same name.

- *Dominated states:* s_1 dominates s_2 if the cost of s_1 is smaller than the cost of s_2
- *s-compatible states:* s_1 and s_2 are s-compatible if they have same cost. In this case, s_1 and s_2 are removed from $\mathcal{L}$ and $s_3 = \{\text{name: } s_1\text{.name, cost: } s_1\text{.cost, colors: } s_1\text{.colors} \cup s_2\text{.colors}\}$ is added to $\mathcal{L}$

Complexity

- Each layer at most has $|V|$ states;
- The height of the *MDD* is k;

The overall complexity is therefore

$$\mathcal{O}(k * |V|^2 * |\mathcal{C}|)$$

Since $|\mathcal{C}|$ is constant we can simplify and get:

$$\mathcal{O}(k * |V|^2)$$

An example

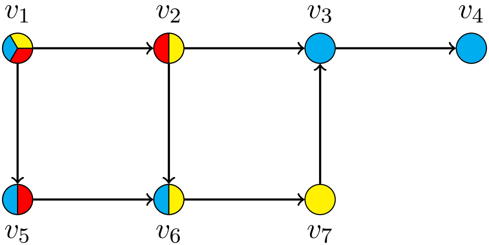
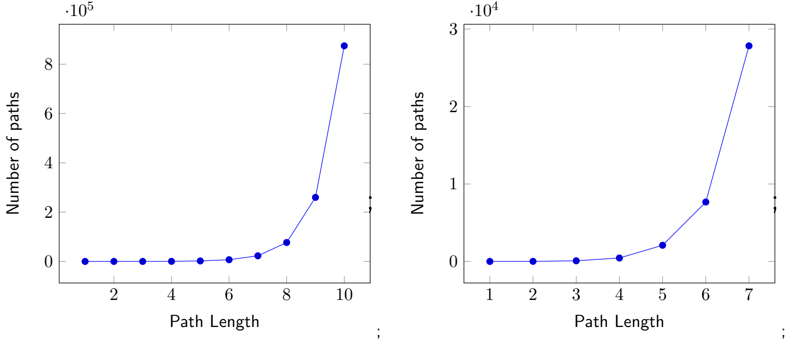


Figure: A colored graph example

Let's take $k = 5$

Benchmark: Solutions



(a) Classic algorithm

(b) *allDiff* constraint

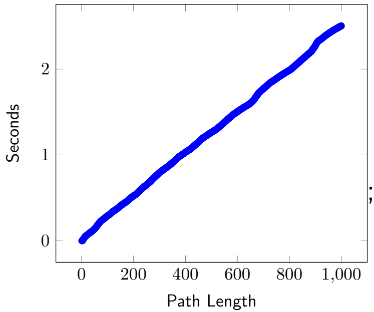
Figure: Number of paths of a given length from the node 1

The *allDiff* variant

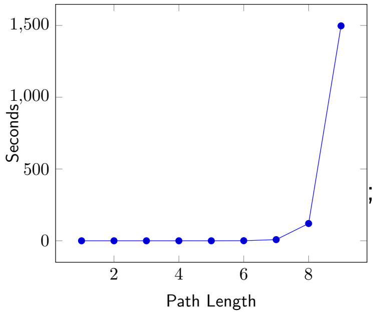
A variant we can add to the problem is the introduction of the *allDiff* constraint to the nodes of the path.

- This modification entails:
    - Maintain a trace of the fathers of the current state,
    - Two states can be reduced only if they have same fathers.
- The second point causes a complexity blow up: the layer size can be $2^{|V|}$.
- The algorithm has now an exponential complexity.

Benchmark: Time



(a) Classic algorithm



(b) *allDiff* constraint

Figure: Time taken to compute paths of a given length from the node 1

Conclusion

We have worked in order to:

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Thanks!