Operational semantics for Prolog with Cut in Rocq and its application to determinacy analysis

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Abstract

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1 Introduction

Elpi is a dialect of λProlog (see [14, 15, 7, 12]) used as an extension language for the Rocq prover (formerly the Coq proof assistant). Elpi has become an important infrastructure component: several projects and libraries depend on it [13, 3, 4, 19, 8, 9]. Examples include the Hierarchy-Builder library-structuring tool [5] and Derive [17, 18, 11], a program-and-proof synthesis framework with industrial applications at SkyLabs AI.

Starting with version 3, Elpi gained a static analysis for determinacy [10] to help users tame backtracking. Rocq users are familiar with functional programming but not necessarily with logic programming and uncontrolled backtracking is a common source of inefficiency and makes debugging harder. The determinacy checkers identifies predicates that behave like functions, i.e., predicates that commit to their first solution and leave no *choice points* (places where backtracking could resume).

This paper reports our first steps towards a mechanization, in the Rocq prover, of the determinacy analysis from [10]. We focus on the control operator *cut*, which is useful to restrict backtracking but makes the semantic depart from a pure logical reading.

We formalize two operational semantics for Prolog with cut. The first is a stack-based semantics that closely models Elpi's implementation and is similar to the semantics mechanized by Pusch in Isabelle/HOL [16] and to the model of Debray and Mishra [6, Sec. 4.3]. This stack-based semantics is a good starting point to study further optimizations used by standard Prolog abstract machines [20, 1], but it makes reasoning about the scope of *cut* difficult. To address that limitation we introduce a tree-based semantics in which the branches pruned by *cut* are explicit and we prove the two semantics equivalent. Using the

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tree-based semantics we then show that if every rule of a predicate passes the determinacy analysis, the call to a deterministic predicate does not leave any choice points.

2 Common code: the language

```
Inductive Tm :=
  | Tm_Kp     : Kp -> Tm
  | Tm_Kd     : Kd -> Tm
  | Tm_V      : V  -> Tm
  | Tm_Comb   : Tm -> Tm -> Tm.

Inductive Callable :=
  | Callable_Kp   : Kp -> Callable
  | Callable_V    : V -> Callable
  | Callable_Comb : Callable -> Tm -> Callable.

Inductive RCallable :=
  | RCallable_Kp   : Kp -> RCallable
  | RCallable_Comb : RCallable -> Tm -> RCallable.
```

A callable term is a term without a data constructor as functor.
An rcallable is a term with rigid head.

```
Inductive A := cut | call : Callable -> A.
```

An atom is the smallest syntactic unit that can be executed in a prolog program $\mathcal{P}$.

```
Record R := mkR { head : RCallable; premises : list A }.
```

We exploit the typing system to ensure that the head of a "valid" rule is a term with rigid head.

```
(*simpler than in the code: signatures of preds are hidden*)
Definition program := seq R.
```

A program is made by a list of rules. Rules in $\mathcal{P}$ are indexed by their position in the list. Given a list of rules $\mathcal{R}$ and two indexes i and j, s.t. $i \neq j$ then, $\mathcal{R}_i$ has a higher priority then $\mathcal{R}_j$.

```
f 1 2.   f 2 3.   r 2 4.   r 2 8.
g X X.                     % r1
g X Z :- r X Z, !.         % r2
g X Z :- f X Y, f Y Z.     % r3
```

Figure 1 Small program example

The elpi program above would be translated as a list of 6 elements where the heads and body are translated in the natural way.
Sigma is a substitution mapping variables to their term instantiation.

```
Definition Sigma := {fmap V -> Tm}.
```

The backchaining algorithm is the function $\mathcal{B}$ aims to filter only the rules in the program $\mathcal{P}$ having rules unifying with the current query q in a given subtitution σ using the list of modes m. In particular $\mathcal{B}$ returns for each selected rule r a substitution σ' that is the substitution obtained by the unification of the query and the head of r.

$$\mathcal{B} \ : \ (\mathcal{P}, \sigma, q) \to \texttt{seq}(\sigma * R)$$

2.1 The cut operator

The semantics of the cut operator we have chosen in the Elpi language is the hard cut operator used in standard SWI-Prolog. It has two main roles: it eliminates alternatives that are chonologically created both at the same moment as, and after, the creation of the cut operator in the execution state.

As a small example of this high-level definition. Let's take the program in Figure 1 and the query $q =$ `g 2 Z`. All the 3 rules for `g` can be used on the q. They are executed in order of the definition in the progrma, i.e., r1 is tried first then r2 and finally r3.

The first rule has no premises returns the assignement `Z = 2`. We however are not finished, there are still two non-explored alternatives consiting in the premises of $r2$ and $r3$.

The premises of $r2$ are "`r 2 Z, !`". In this sequent the role of the cut become evident: if it is executed, i.e. `r 2 Z` succeeds, then the premises of $r3$ will be cut away, since they have been created at the same time of the creation of the cut in the alternatives list; moreover, if the call `r 2 Z` leaves alternatives, only the first is committed and the other are discarded, since these alternatives would have a depper depth then the cut itself.

Concretely speaking, `r 2 Z` will provide two alternatives, assigning `Z` respectively to `4` and `8`. The second solution is discaded by the cut.

3 Semantics intro

We propose two operational semantics for a logic program with cut. The two semantics are based on different syntaxes, the first syntax (called tree) exploits a tree-like structure and is ideal both to have a graphical view of its evolution while the state is being intepreted and to prove lemmas over it. The second syntax, called elpi, is the elpi's syntax and has the advantage of reducing the computational cost of cutting and backtracking alternatives by using shared pointers. We aim to prove the equivalence of the two semantics together with some interesting lemmas of the cut behavior.

3.1 Tree semantics

```
Inductive tree :=
  | KO | OK | Dead
  | TA : A -> tree
  | Or  : tree -> Sigma -> tree -> tree
  | And : tree -> seq A -> tree -> tree.
```

In the tree we distinguish 6 main cases: `KO`, `OK`, and `Dead` are special meta-symbols representing, respectively, a failed, a successful, and a dead terminal. These symbols are considered meta because they are internal intermediate symbols used to give structure to the tree. While the first two symbols are of immiediate understanding, we use `Dead` to represent ghost state, that is, the `Dead` symbol is always ignored by the tree interpreter.

`TA` (acronym for tree-atom) is the constructor of atoms in the tree.

```
Fixpoint is_dead A :=
  match A with
  | Dead => true
  | OK | KO | TA _ => false
  | And A B0 B => is_dead A
  | Or A s B => is_dead A && is_dead B
  end.
```

```
Fixpoint path_end A :=
  match A with
  | Dead | OK | KO | TA _ => A
  | Or A _ B =>
    if is_dead A then path_end B
    else path_end A
  | And A B0 B =>
    match path_end A with
    | OK => path_end B
    | A => A
    end
  end.
```

Figure 2 isdead and pathend

The two recursive cases of a tree are the `Or` and `And` non-terminals. The `Or` non-terminal $A \vee B_\sigma$ denotes a disjunction between two trees A and B. The second branch is annotated with a suspended substitution σ so that, upon backtracking to B, σ is used as the initial substitution for the execution of B.

The `And` non-terminal $A \wedge_{B_0} B$ represents a conjunction of two trees A and B. We call B_0 the reset point for B; it is used to restore the state of B to its initial form if a backtracking operation occurs on A.

A graphical representation of the tree is shown in Figure 3a. To make the graph more compact, the `And` and `Or` non-terminals are n-ary rather than binary, with right-binding priority. The `KO` and `Dead` terminals act as the neutral elements in the `Or` list, while `OK` is the neutral element of the `And` list.

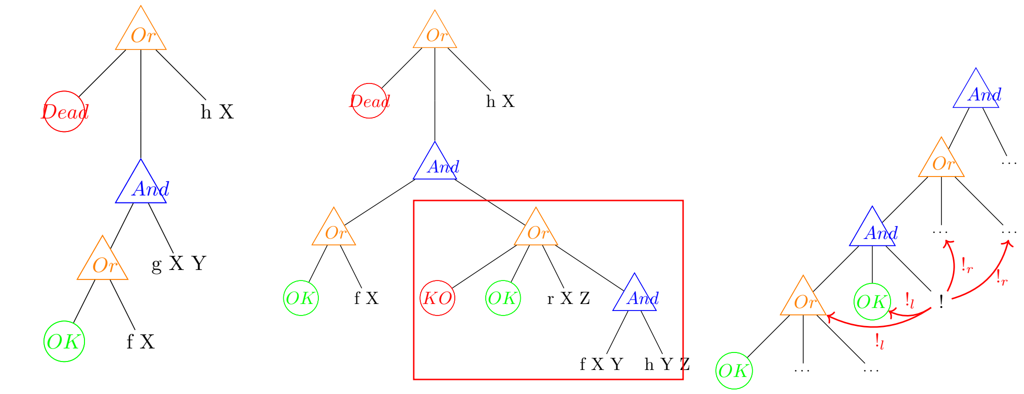
The interpretation of a tree is performed by two main routines: `step` and `next_alt` that traverse the tree depth-first, left-to-right. Then, then `run` inductive makes the transitive closure of step `step` and `next_alt`: it iterates the calls to its auxiliary functions. In Equations (2)–(4) we give the types contrats of these symbols.

```
Inductive step_tag := Expanded | CutBrothers| Failure | Success.                (1)
Definition step : program -> sigma -> tree -> (step_tag * tree) := ...          (2)
Definition next_alt : bool -> tree -> option tree := ...                        (3)
Inductive run (p : program): Sigma -> tree -> Sigma -> tree -> bool -> Type := ...
                                                                                (4)
```

A particular tree we want to identify is a is_dead tree. This tree has the property to never produce a solution and deals with terminal ending in Dead states. Its definition is in Section 3.1. In a non-dead tree, we get the first-to-beexplored node via path-end the path_end routine shown in Section 3.1. The path-end is either the tree itself if the tree is a terminal. Otherwise, if the tree is a disjunction, the path continues on the left- or the right-subtree depending of if the the lhs is a Dead node. In the case of a conjunction, it is more interesting to see what happens. If the path-end p of the lhs is a success then we look for the path-end in the rhs, otherwise we return p. In Figure 3a the path-end g X.

Below we define two special kind of trees depending on their path-end

```
Definition successT A := path_end A == OK.
```



(a) Path end := g X Y **(b)** Step on g X Y using Figure 1 **(c)** Scope of the cut

Figure 3 Some tree representations

```
Definition failedT A := (path_end A == KO) || (path_end A == Dead).
```

The `step` procedure takes a program a substitution and a tree and returns a `step_tag` together with the oupdated tree. The `step_tag` is a tag telling what kind of internal tree step has been performed. It is either a call expansion (`Expanded`) or the evaluation of an internal cut (i.e. a cut appering below a Or), a supeficial cut evaluation (`CutBrothers`), i.e. a cut having only And-nodes as fathers, `Failure` or `Success` if the tree is either successT or failedT. Therefore, the two interesting cases of a tree step are the step of a call and the step of a cut.

Call step In the former case the call node is replace with a new subtree made by the rules returned by the $\mathcal{B}$ function. If $\mathcal{B}$ returns a list l, if l is empty then KO tree is returned, otherwise the call is replaced by right-skewed tree made of n inner Or nodes, where n is the length of l. The root Or-node has KO as left child. The lhs of the other nodes is a right-skewed tree of And nodes. The And nodes are again a right-seked tree containing then atoms (either cut or call) taken from the list l. dire dei reset point

A step in the tree of Figure 3a makes a backchain operation over the query `g X Y` and, in the program defined in Figure 1, the new tree would be the one in Figure 3b. We have put a red border aroung the new generated subtree. It is a disjunction of four subtrees: the first node is the KO node (by default), the second is OK, since r1 has no premises, the third and the fourth contains the premises of respectively r2 and r3. dire che le sostituzioni del backchain sono importanti e dove sono mess

Cut step The latter case is delicate since interpreting a cut in a tree has three main impacts: at first the cut node is replaced by a OK node, but then we need to cut-away the subtrees that are in the scope of the cut: in particular we need to soft-kill the left-siblings of the Cut and hard-kill the right-uncles of the the Cut.

▶ **Definition 1** (Left-siblings (resp. right-sibling)). *Given a node A, the left-siblings (resp. right-sibling) of A are the list of subtrees sharing the same parent of A and that appear on its left (resp. right).*

▶ **Definition 2** (Right-uncles). *Given a node A, the right-uncles of A are the list of right-sibling of the father of A.*

▶ **Definition 3** (Soft-kill). *Given a tree t, soft-kill replaces all the leaves of the tree with the node KO except for the leaves that are part of the path p of t.*

▶ **Definition 4** (Hard-kill). *Given a tree t, hard-kill replaces all the leaves of the tree with the node KO*

An example of the impact of the cut is show in Figure 3c. The step routine interprets the cut if it is at the end of the current path. In the example we have tagged in red the arrow $!_l$ indicating which sub-trees is soft-killed and $!_r$ indicated which is sub-trees are to be hard-killed.

3.1.1 Execution example

3.1.2 Valid tree

3.2 Elpi semantics

TODO: dire che la semantica ad albero è puù faicle per le prove

The Elpi interpreter is based on an operational semantics close to the one picked by Pusch in [16], in turn closely related to the one given by Debray and Mishra in [6, Section 4.3]. Push mechanized the semantics in Isabelle/HOL together with some optimizations that are present in the Warren Abstract Machine [20, 1].

In these operational semantics we need to decorate the cut atom with a list of alternative, morally a pointer to a sub-list of the overall alternatives. An atom in the elpi semantcis is defined as follows:

```
Inductive alts :=
  | no_alt
  | more_alt : (Sigma * goals) -> alts -> alts
with goals :=
  | no_goals
  | more_goals : (A * alts) -> goals -> goals .
```

We are completely loosing the tree structure. There are no clean reset points. The backtracking operation is simpler: it is the tail function. The cutr and cutl operations disappears: the alternatives are stored directly in the cutE terminal.

The elpi interpreter is as follows:

```
(*TODO: add system of rules*)
Inductive nur : Sigma -> goals ->  alts -> Sigma -> alts -> Type :=
  | StopE s a : nur s nilC a s a
  | CutE s s1 a ca r gl : nur s gl ca s1 r -> nur s ((cutE ca) ::: gl) a s1 r
  | CallE p s s1 a b bs gl r t :
    F u p t s = [:: b & bs ] ->
      nur b.1 (save_goals a gl (a2gs1 p b)) (save_alts a gl ((aa2gs p) bs) ++ a) s1 r ->
        nur s ((callE p t) ::: gl) a s1 r
  | FailE p s s1 s2 t gl a al r :
    F u p t s = [::] -> nur s1 a al s2 r -> nur s ((callE p t) ::: gl) ((s1, a) ::: al) s2 r.
```

The translation of a tree to a list is as follows:

```
Fixpoint t2l (A: tree) s (bt : alts) : alts :=
match A with
| OK          => [:: (s, [::]) ]
| (KO | Dead) => [::]
| TA a        => [:: (s, [:: (a,[::]) ]) ]
```

```
  | Or A s1 B     =>
      let lB := t2l B s1 [::] in
      let lA := t2l A s lB in
      add_ca_deep bt (lA ++ lB)
  | And A B0 B    =>
      let lB0 : goals := r2l B0 in
      let lA   := t2l A s bt in
      if lA is [:: (slA, x) & xs] then
        let xz := add_deepG bt lB0 x in
        let xs := add_deep bt lB0 xs in
        let xs := make_lB0 xs lB0 in
        let lB := t2l B slA (xs ++ bt) in
        (make_lB01 lB xz) ++ xs
      else [::]
  end.
```

▶ **Theorem 5** (`tree_to_elpi`).

$$\forall A\ \sigma_1\ B\ \sigma_2\ b\ \sigma_0, \boldsymbol{vt}\ A \to$$
$$\boldsymbol{run}_u\ \sigma_1\ A\ (Some\ \sigma_2)\ B\ b \to$$
$$\exists x\ xs, \boldsymbol{t2l}\ A\ \sigma_1\ \varnothing = x ::: xs \land \boldsymbol{nur}_u\ x.1\ x.2\ xs\ \sigma_2\ (\boldsymbol{t2l}\ B\ \sigma_0\ \varnothing).$$

▶ **Theorem 6** (`elpi_to_tree`).

$$\forall \sigma_1\ \sigma_2\ a\ na\ g,$$
$$\boldsymbol{nur}_u\ \sigma_1\ g\ a\ \sigma_2\ na \to$$
$$\forall \sigma_0\ t, \boldsymbol{vt}\ t \to (\boldsymbol{t2l}\ t\ \sigma_0\ \varnothing) = ((\sigma_1, g) ::: a) \to$$
$$\exists t'\ n, \boldsymbol{run}_u\ \sigma_0\ t\ (Some\ \sigma_2)\ t'\ n \land \boldsymbol{t2l}\ t'\ \sigma_0\ \varnothing = na.$$

The proof of Theorem 6 is based on the idea explained in [2, Section 3.3]. An ideal statement for this lemma would be: given a function `l2t` transforming an elpi state to a tree, we would have have that the the execution of an elpi state e is the same as executing `run` on the tree resulting from `l2t`(e). However, it is difficult to retrive the strucutre of an elpi state and create a tree from it. This is because, in an elpi state, we have no clear information about the scope of an atom inside the list and, therefore, no evident clue about where this atom should be place in the tree.

Our theorem states that, starting from a valid state t which translates to a list of alternatives $(\sigma_1, g) :: a$. If we run in elpi the list of alternatives, then the execution of the tree t returns the same result as the execution in elpi. The proof is performed by induction on the derivations of the elpi execution. We have 4 derivations.

We have 4 case to analyse:

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$$\begin{tikzcd} t_1 \arrow[r, "*"] \arrow[d, "\mathrm{t2l}"] & t_2 \arrow[d, "\mathrm{t2l}"] \\ e_1 \arrow[r, "*"] \arrow[ur, "\text{induction}"] & e_2 \end{tikzcd}$$

Figure 4 Induction scheme for Theorem 6

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