Operational semantics for Prolog with Cut in Rocq and its application to determinacy analysis

Jane Open Access
Dummy University Computing Laboratory, [optional: Address], Country
My second affiliation, Country

Joan R. Public[1]
Department of Informatics, Dummy College, [optional: Address], Country

Abstract

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1 Introduction

Elpi is a dialect of λProlog (see [14, 15, 7, 12]) used as an extension language for the Rocq prover (formerly the Coq proof assistant). Elpi has become an important infrastructure component: several projects and libraries depend on it [13, 3, 4, 19, 8, 9]. Examples include the Hierarchy-Builder library-structuring tool [5] and Derive [17, 18, 11], a program-and-proof synthesis framework with industrial applications at SkyLabs AI.

Starting with version 3, Elpi gained a static analysis for determinacy [10] to help users tame backtracking. Rocq users are familiar with functional programming but not necessarily with logic programming and uncontrolled backtracking is a common source of inefficiency and makes debugging harder. The determinacy checkers identifies predicates that behave like functions, i.e., predicates that commit to their first solution and leave no *choice points* (places where backtracking could resume).

This paper reports our first steps towards a mechanization, in the Rocq prover, of the determinacy analysis from [10]. We focus on the control operator *cut*, which is useful to restrict backtracking but makes the semantic depart from a pure logical reading.

We formalize two operational semantics for Prolog with cut. The first is a stack-based semantics that closely models Elpi's implementation and is similar to the semantics mechanized by Pusch in Isabelle/HOL [16] and to the model of Debray and Mishra [6, Sec. 4.3]. This stack-based semantics is a good starting point to study further optimizations used by standard Prolog abstract machines [20, 1], but it makes reasoning about the scope of *cut* difficult. To address that limitation we introduce a tree-based semantics in which the branches pruned by *cut* are explicit and we prove the two semantics equivalent. Using the

[1] Optional footnote, e.g. to mark corresponding author

```
Inductive P := IP of nat. Inductive D := ID of nat. Inductive V := IV of nat.

Inductive Tm :=                                  Inductive Callable :=
  | Tm_P of P       | Tm_D    of D                 | Callable_P    of P
  | Tm_V of V       | Tm_App  of Tm & Tm.          | Callable_App of Callable & Tm.
```

Figure 1 Tm and Callable types

tree-based semantics we then show that if every rule of a predicate passes the determinacy analysis, the call to a deterministic predicate does not leave any choice points.

2 Common code: the language

Before going to the two semantcis, we show the piece of data structure that are shared by the them. The smallest unit of code that we can use in the langauge is an atom. The atom inductive (see Type 1) is either a cut or a call. A call carries a callable term (see Figure 1). A term (Tm) is either a predicate, a datum, a variable or the binary application of a term to another. A Callable is a term accepting predicates only predicates as functors.

```
Inductive A := cut | call : Callable -> A.                          (1)
Record R := mkR { head : Callable; premises : list A }.             (2)
Record program := { rules : index; sig : sigT }.                    (3)
Definition Sigma := {fmap V -> Tm}.                                 (4)
```

We exploit the typing system to ensure that the head of a "valid" rule is a term with rigid head.

A program is made by a list of rules. Rules in $\mathcal{P}$ are indexed by their position in the list. Given a list of rules $\mathcal{R}$ and two indexes i and j, s.t. $i \neq j$ then, $\mathcal{R}_i$ has a higher priority then $\mathcal{R}_j$.

```
f 1 2.   f 2 3.   r 2 4.   r 2 8.
g X X.                     % r1
g X Z :- r X Z, !.         % r2
g X Z :- f X Y, f Y Z.     % r3
```

Figure 2 Small program example

The elpi program above would be translated as a list of 6 elements where the heads and body are translated in the natural way.

Sigma is a substitution mapping variables to their term instantiation.

The backchaining algorithm is the function $\mathcal{B}$ aims to filter only the rules in the program $\mathcal{P}$ having rules unifying with the current query q in a given subtitution σ using the list of modes m. In particular $\mathcal{B}$ returns for each selected rule r a substitution σ' that is the substitution obtained by the unification of the query and the head of r.

$$\mathcal{B} \;:\; (\mathcal{P}, \sigma, q) \to \mathtt{seq}(\sigma * R)$$

2.1 The cut operator

The semantics of the cut operator we have chosen in the Elpi language is the hard cut operator used in standard SWI-Prolog. It has two main roles: it eliminates alternatives that are chonologically created both at the same moment as, and after, the creation of the cut operator in the execution state.

As a small example of this high-level definition. Let's take the program in Figure 2 and the query $q =$ `g 2 Z`. All the 3 rules for `g` can be used on the q. They are executed in order of the definition in the progrma, i.e., r1 is tried first then r2 and finally r3.

The first rule has no premises returns the assignement `Z = 2`. We however are not finished, there are still two non-explored alternatives consiting in the premises of $r2$ and $r3$.

The premises of $r2$ are "`r 2 Z, !`". In this sequent the role of the cut become evident: if it is executed, i.e. `r 2 Z` succeeds, then the premises of $r3$ will be cut away, since they have been created at the same time of the creation of the cut in the alternatives list; moreover, if the call `r 2 Z` leaves alternatives, only the first is committed and the other are discarded, since these alternatives would have a depper depth then the cut itself.

Concretely speaking, `r 2 Z` will provide two alternatives, assigning `Z` respectively to `4` and `8`. The second solution is discaded by the cut.

3 Semantics intro

We propose two operational semantics for a logic program with cut. The two semantics are based on different syntaxes, the first syntax (called tree) exploits a tree-like structure and is ideal both to have a graphical view of its evolution while the state is being intepreted and to prove lemmas over it. The second syntax, called elpi, is the elpi's syntax and has the advantage of reducing the computational cost of cutting and backtracking alternatives by using shared pointers. We aim to prove the equivalence of the two semantics together with some interesting lemmas of the cut behavior.

3.1 Tree semantics

```
Inductive tree :=
  | KO | OK | Dead
  | TA : A -> tree
  | Or  : tree -> Sigma -> tree -> tree
  | And : tree -> seq A -> tree -> tree.
```

In the tree we distinguish 6 main cases: KO, OK, and $Dead$ are special meta-symbols representing, respectively, a failed, a successful, and a dead terminal. These symbols are considered meta because they are internal intermediate symbols used to give structure to the tree. While the first two symbols are of immiediate understanding, we use $Dead$ to represent ghost state, that is, the $Dead$ symbol is always ignored by the tree interpreter.

TA (acronym for tree-atom) is the constructor of atoms in the tree.

The two recursive cases of a tree are the Or and And non-terminals. The Or non-terminal $A \vee B_\sigma$ denotes a disjunction between two trees A and B. The second branch is annotated with a suspended substitution σ so that, upon backtracking to B, σ is used as the initial substitution for the execution of B.

The And non-terminal $A \wedge_{B_0} B$ represents a conjunction of two trees A and B. We call B_0 the reset point for B; it is used to restore the state of B to its initial form if a backtracking operation occurs on A. Intuitively in prolog-like syntax, in a tree $A \wedge_{B_0} B$, if $t2l$ is the function flattening the tree in a list of sequents disjnction and $t2l(A) = A_1, \ldots, A_n$, then we would have $(A_1, t2l\ B); (A_2, B0); \ldots; (A_n, B0)$.

A graphical representation of the tree is shown in Figure 4a. To make the graph more compact, the And and Or non-terminals are n-ary rather than binary, with right-binding

```
Fixpoint is_dead A :=
  match A with
  | Dead => true
  | OK | KO | TA _ => false
  | And A B0 B => is_dead A
  | Or A s B => is_dead A && is_dead B
  end.
```

(a) Defintion of *is_dead*

```
Fixpoint path_end A :=
  match A with
  | Dead | OK | KO | TA _ => A
  | Or A _ B =>
    if is_dead A then path_end B
    else path_end A
  | And A B0 B =>
    match path_end A with
    | OK => path_end B
    | A => A
    end
  end.
```

(b) Defintion of *path_end*

priority. The *KO* and *Dead* terminals act as the neutral elements in the *Or* list, while *OK* is the neutral element of the *And* list.

The interpretation of a tree is performed by two main routines: *step* and *next_alt* that traverse the tree depth-first, left-to-right. Then, then *run* inductive makes the transitive closure of step *step* and *next_alt*: it iterates the calls to its auxiliary functions. In Types 6–8 we give the types contrats of these symbols where `fv` is a set of variable names.

```
Inductive step_tag := Expanded | CutBrothers | Failed | Success.                (5)
Definition step : program -> fv -> Sigma -> tree -> (fv * step_tag * tree) :=  (6)
Definition next_alt : bool -> tree -> option tree :=                            (7)
Inductive run (p : program): {fset V} -> Sigma -> tree -> option Sigma -> tree -> bool -> Prop :=   (8)
```

A particular tree we want to identify is a *is_dead* tree (defined in Figure 3a). This tree has the property to never produce a solution: it is eiher the *Dead* tree or both branches of *Or* are dead, or the lhs of *And* is dead. In the latter case, we note that B can be non-dead, but this is not a problem since the interpreter can run B only if A is non-dead.

The prolog interpreter explores the state in DFS strategy, it finds the “first-to-be-explored” (ftbe) atom of the tree and then interpretes it. In a non-*is_dead* tree, we get the ftbe node via *path_end*, shown in Figure 3b. The *path_end* is either the tree itself if the tree is a leaf. Otherwise, if the tree is a disjunction, the path continues on the left- or the right-subtree depending of if the the lhs is a *is_dead* tree. In the *Or* case we are clearing ignoring the dead (ghost) state.

In the case of a conjunction, it is more interesting to see what happens. If the *path_end* p of the lhs is a success then we look for the *path_end* in the rhs, otherwise we return p. In Figure 4a the *path_end* of the tree is `g X`.

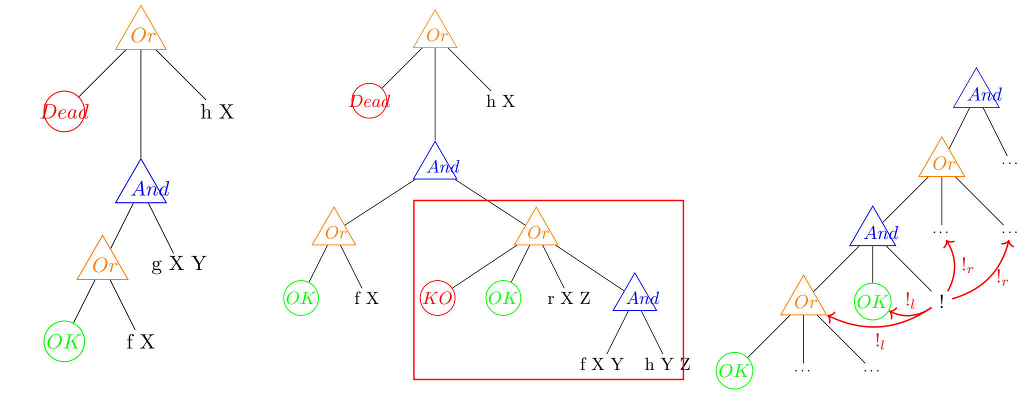
Below we define two special kind of trees depending on their pathend.

```
Definition success A := path_end A == OK.                          (1)
Definition failed A := (path_end A == KO) || (path_end A == Dead). (2)
```

The *step* procedure takes as input a program, a set of free variables (fv), a substitution, and a tree, and returns an updated set of free variables, a *step_tag*, and an updated tree.

Free variables are those variables that appear in a tree; they are used in the backchaining operation to refresh the variables in the program.

The *step_tag* indicates the type of internal tree step that has been performed. `CutBrothers` denotes the interpretation of a superficial cut, i.e., a cut whose parent nodes are all *And*-nodes. `Expanded` denotes the interpretation of non-superficial cuts or predicate calls. `Failure` and



(a) Path end := g X Y **(b)** Step on g X Y using Figure 2 **(c)** Scope of the cut

Figure 4 Some tree representations

`Success` are returned for, respectively, $failed$ and $success$ trees.

The step procedure is intended to interpretate atoms, that is, it returnes the identity for $success$ and $failed$ tree.

```
Lemma success_step u p fv s A: success A -> step u p fv s A = (fv, Success, A). (1)
Lemma failed_step u p fv s1 A: failed A -> step u p fv  s1 A = (fv, Failed, A). (2)
```

Therefore, the two interesting cases of a tree the interpretation of trees with path-end equal to a call or a cut atom.

Call step The interpretation of a call c is performed by replacing the call wrt the result of the $\mathcal{B}$ c, then if l is empty then KO tree is returned, otherwise the call is replaced by right-skewed tree made of n inner Or nodes, where n is the length of l. The root has KO as left child. The lhs of the other nodes is a right-skewed tree of And nodes. The And nodes are again a right-seked tree containing then atoms (either cut or call) taken from the list l. dire dei reset point

A step in the tree of Figure 4a makes a backchain operation over the query `g X Y` and, in the program defined in Figure 2, the new tree would be the one in Figure 4b. We have put a red border aroung the new generated subtree. It is a disjunction of four subtrees: the first node is the KO node (by default), the second is OK, since r1 has no premises, the third and the fourth contains the premises of respectively r2 and r3.

dire che le sostituzioni del backchain sono importanti e dove sono mess

Cut step The latter case is delicate since interpreting a cut in a tree has three main impacts: at first it is replaced by the OK node, then some special subtrees, in the scope of the Cut, are cut away: in particular we need to soft-kill the left-siblings of the Cut and hard-kill the right-uncles of the the Cut.

▶ **Definition 1** (Left-siblings (resp. right-sibling)). *Given a node A, the left-siblings (resp. right-sibling) of A are the list of subtrees sharing the same parent of A and that appear on its left (resp. right).*

▶ **Definition 2** (Right-uncles). *Given a node A, the right-uncles of A are the list of right-sibling of the father of A.*

▶ **Definition 3** (Soft-kill). *Given a tree t, soft-kill replaces all the leaves of the tree with the node KO except for the leaves that are part of the path p of t.*

▶ **Definition 4** (Hard-kill). *Given a tree t, hard-kill replaces all the leaves of the tree with the node KO*

An example of the impact of the cut is show in Figure 4c. The step routine interprets the cut if it is at the end of the current path. In the example we have tagged in red the arrow $!_l$ indicating which sub-trees is soft-killed and $!_r$ indicated which is sub-trees are to be hard-killed.

3.1.1 Execution example

3.1.2 Valid tree

3.2 Elpi semantics

TODO: dire che la semantica ad albero è puù faicle per le prove

The Elpi interpreter is based on an operational semantics close to the one picked by Pusch in [16], in turn closely related to the one given by Debray and Mishra in [6, Section 4.3]. Push mechanized the semantics in Isabelle/HOL together with some optimizations that are present in the Warren Abstract Machine [20, 1].

In these operational semantics we need to decorate the cut atom with a list of alternative, morally a pointer to a sub-list of the overall alternatives. An atom in the elpi semantcis is defined as follows:

```
Inductive alts :=
  | no_alt
  | more_alt : (Sigma * goals) -> alts -> alts
with goals :=
  | no_goals
  | more_goals : (A * alts) -> goals -> goals .
```

We are completely loosing the tree structure. There are no clean reset points. The backtracking operation is simpler: it is the tail function. The cutr and cutl operations disappears: the alternatives are stored directly in the cutE terminal.

The elpi interpreter is as follows:

```
(*TODO: add system of rules*)
Inductive nur : Sigma -> goals ->  alts -> Sigma -> alts -> Type :=
  | StopE s a : nur s nilC a s a
  | CutE s s1 a ca r gl : nur s gl ca s1 r -> nur s ((cutE ca) ::: gl) a s1 r
  | CallE p s s1 a b bs gl r t :
    F u p t s = [:: b & bs ] ->
      nur b.1 (save_goals a gl (a2gs1 p b)) (save_alts a gl ((aa2gs p) bs) ++ a) s1 r ->
        nur s ((callE p t) ::: gl) a s1 r
  | FailE p s s1 s2 t gl a al r :
    F u p t s = [::] -> nur s1 a al s2 r -> nur s ((callE p t) ::: gl) ((s1, a) ::: al) s2 r.
```

The translation of a tree to a list is as follows:

```
Fixpoint t2l (A: tree) s (bt : alts) : alts :=
match A with
| OK          => [:: (s, [::]) ]
| (KO | Dead) => [::]
| TA a        => [:: (s, [:: (a,[::]) ]) ]
| Or A s1 B   =>
    let lB := t2l B s1 [::] in
    let lA := t2l A s lB in
```

$$
\begin{tikzcd}
t_1 \arrow[r, "*"] \arrow[d, "\mathrm{t2l}"] & t_2 \arrow[d, "\mathrm{t2l}"] \\
e_1 \arrow[r, "*"] \arrow[ur, "\mathrm{induction}"] & e_2
\end{tikzcd}
$$

Figure 5 Induction scheme for Theorem 6

```
    add_ca_deep bt (lA ++ lB)
| And A B0 B   =>
    let lB0 : goals := r2l B0 in
    let lA  := t2l A s bt in
    if lA is [:: (slA, x) & xs] then
      let xz := add_deepG bt lB0 x in
      let xs := add_deep bt lB0 xs in
      let xs := make_lB0 xs lB0 in
      let lB := t2l B slA (xs ++ bt) in
      (make_lB01 lB xz) ++ xs
    else [::]
end.
```

▶ **Theorem 5** (`tree_to_elpi`).

$$
\begin{aligned}
&\forall A\ \sigma_1\ B\ \sigma_2\ b\ \sigma_0, vt\ A \to \\
&run_u\ \sigma_1\ A\ (Some\ \sigma_2)\ B\ b \to \\
&\exists x\ xs, t2l\ A\ \sigma_1\ \varnothing = x ::: xs \wedge nur_u\ x.1\ x.2\ xs\ \sigma_2\ (t2l\ B\ \sigma_0\ \varnothing).
\end{aligned}
$$

▶ **Theorem 6** (`elpi_to_tree`).

$$
\begin{aligned}
&\forall \sigma_1\ \sigma_2\ a\ na\ g, \\
&nur_u\ \sigma_1\ g\ a\ \sigma_2\ na \to \\
&\forall \sigma_0\ t, vt\ t \to (t2l\ t\ \sigma_0\ \varnothing) = ((\sigma_1, g) ::: a) \to \\
&\exists t'\ n, run_u\ \sigma_0\ t\ (Some\ \sigma_2)\ t'\ n \wedge t2l\ t'\ \sigma_0\ \varnothing = na.
\end{aligned}
$$

The proof of Theorem 6 is based on the idea explained in [2, Section 3.3]. An ideal statement for this lemma would be: given a function `l2t` transforming an elpi state to a tree, we would have have that the the execution of an elpi state e is the same as executing run on the tree resulting from `l2t`(e). However, it is difficult to retrive the strucutre of an elpi state and create a tree from it. This is because, in an elpi state, we have no clear information about the scope of an atom inside the list and, therefore, no evident clue about where this atom should be place in the tree.

Our theorem states that, starting from a valid state t which translates to a list of alternatives $(\sigma_1, g) :: a$. If we run in elpi the list of alternatives, then the execution of the tree t returns the same result as the execution in elpi. The proof is performed by induction on the derivations of the elpi execution. We have 4 derivations.

We have 4 case to analyse:

References

1 Hassan Aït-Kaci. *Warren's Abstract Machine: A Tutorial Reconstruction*. The MIT Press, 08 1991. doi:10.7551/mitpress/7160.001.0001.

2 Yves Bertot. A certified compiler for an imperative language. Technical Report RR-3488, INRIA, September 1998. URL: https://inria.hal.science/inria-00073199v1.

3 Valentin Blot, Denis Cousineau, Enzo Crance, Louise Dubois de Prisque, Chantal Keller, Assia Mahboubi, and Pierre Vial. Compositional pre-processing for automated reasoning in dependent type theory. In Robbert Krebbers, Dmitriy Traytel, Brigitte Pientka, and Steve Zdancewic, editors, *Proceedings of the 12th ACM SIGPLAN International Conference on Certified Programs and Proofs, CPP 2023, Boston, MA, USA, January 16-17, 2023*, pages 63–77. ACM, 2023. doi:10.1145/3573105.3575676.

4 Cyril Cohen, Enzo Crance, and Assia Mahboubi. Trocq: Proof transfer for free, with or without univalence. In Stephanie Weirich, editor, *Programming Languages and Systems*, pages 239–268, Cham, 2024. Springer Nature Switzerland.

5 Cyril Cohen, Kazuhiko Sakaguchi, and Enrico Tassi. Hierarchy Builder: Algebraic hierarchies Made Easy in Coq with Elpi. In *Proceedings of FSCD*, volume 167 of *LIPIcs*, pages 34:1–34:21, 2020. URL: https://drops.dagstuhl.de/entities/document/10.4230/LIPIcs.FSCD.2020.34, doi:10.4230/LIPIcs.FSCD.2020.34.

6 Saumya K. Debray and Prateek Mishra. Denotational and operational semantics for prolog. *J. Log. Program.*, 5(1):61–91, March 1988. doi:10.1016/0743-1066(88)90007-6.

7 Cvetan Dunchev, Ferruccio Guidi, Claudio Sacerdoti Coen, and Enrico Tassi. ELPI: fast, embeddable, λProlog interpreter. In *Proceedings of LPAR*, volume 9450 of *LNCS*, pages 460–468. Springer, 2015. URL: https://inria.hal.science/hal-01176856v1, doi:10.1007/978-3-662-48899-7_32.

8 Davide Fissore and Enrico Tassi. A new Type-Class solver for Coq in Elpi. In *The Coq Workshop*, July 2023. URL: https://inria.hal.science/hal-04467855.

9 Davide Fissore and Enrico Tassi. Higher-order unification for free!: Reusing the meta-language unification for the object language. In *Proceedings of PPDP*, pages 1–13. ACM, 2024. doi:10.1145/3678232.3678233.

10 Davide Fissore and Enrico Tassi. Determinacy checking for elpi: an higher-order logic programming language with cut. In *Practical Aspects of Declarative Languages: 28th International Symposium, PADL 2026, Rennes, France, January 12–13, 2026, Proceedings*, pages 77–95, Berlin, Heidelberg, 2026. Springer-Verlag. doi:10.1007/978-3-032-15981-6_5.

11 Benjamin Grégoire, Jean-Christophe Léchenet, and Enrico Tassi. Practical and sound equality tests, automatically. In *Proceedings of CPP*, page 167–181. Association for Computing Machinery, 2023. doi:10.1145/3573105.3575683.

12 Ferruccio Guidi, Claudio Sacerdoti Coen, and Enrico Tassi. Implementing type theory in higher order constraint logic programming. In *Mathematical Structures in Computer Science*, volume 29, pages 1125–1150. Cambridge University Press, 2019. doi:10.1017/S0960129518000427.

13 Robbert Krebbers, Luko van der Maas, and Enrico Tassi. Inductive Predicates via Least Fixpoints in Higher-Order Separation Logic. In Yannick Forster and Chantal Keller, editors, *16th International Conference on Interactive Theorem Proving (ITP 2025)*, volume 352 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 27:1–27:21, Dagstuhl, Germany, 2025. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. URL: https://drops.dagstuhl.de/entities/document/10.4230/LIPIcs.ITP.2025.27, doi:10.4230/LIPIcs.ITP.2025.27.

14 Dale Miller. A logic programming language with lambda-abstraction, function variables, and simple unification. In *Extensions of Logic Programming*, pages 253–281. Springer, 1991.

15 Dale Miller and Gopalan Nadathur. *Programming with Higher-Order Logic*. Cambridge University Press, 2012.

16 Cornelia Pusch. Verification of compiler correctness for the wam. In Gerhard Goos, Juris Hartmanis, Jan van Leeuwen, Joakim von Wright, Jim Grundy, and John Harrison, editors, *Theorem Proving in Higher Order Logics*, pages 347–361, Berlin, Heidelberg, 1996. Springer Berlin Heidelberg.

17 Enrico Tassi. Elpi: an extension language for Coq (Metaprogramming Coq in the Elpi λProlog dialect). In *The Fourth International Workshop on Coq for Programming Languages*, January 2018. URL: `https://inria.hal.science/hal-01637063`.

18 Enrico Tassi. Deriving proved equality tests in Coq-Elpi. In *Proceedings of ITP*, volume 141 of *LIPIcs*, pages 29:1–29:18, September 2019. URL: `https://inria.hal.science/hal-01897468`, `doi:10.4230/LIPIcs.CVIT.2016.23`.

19 Luko van der Maas. Extending the Iris Proof Mode with inductive predicates using Elpi. Master's thesis, Radboud University Nijmegen, 2024. `doi:10.5281/zenodo.12568604`.

20 David H.D. Warren. An Abstract Prolog Instruction Set. Technical Report Technical Note 309, SRI International, Artificial Intelligence Center, Computer Science and Technology Division, Menlo Park, CA, USA, October 1983. URL: `https://www.sri.com/wp-content/uploads/2021/12/641.pdf`.