- **Lecture 2: Conditional Probability**

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DEFINITIONS & EXAMPLES

1. Definition of Conditional Probability
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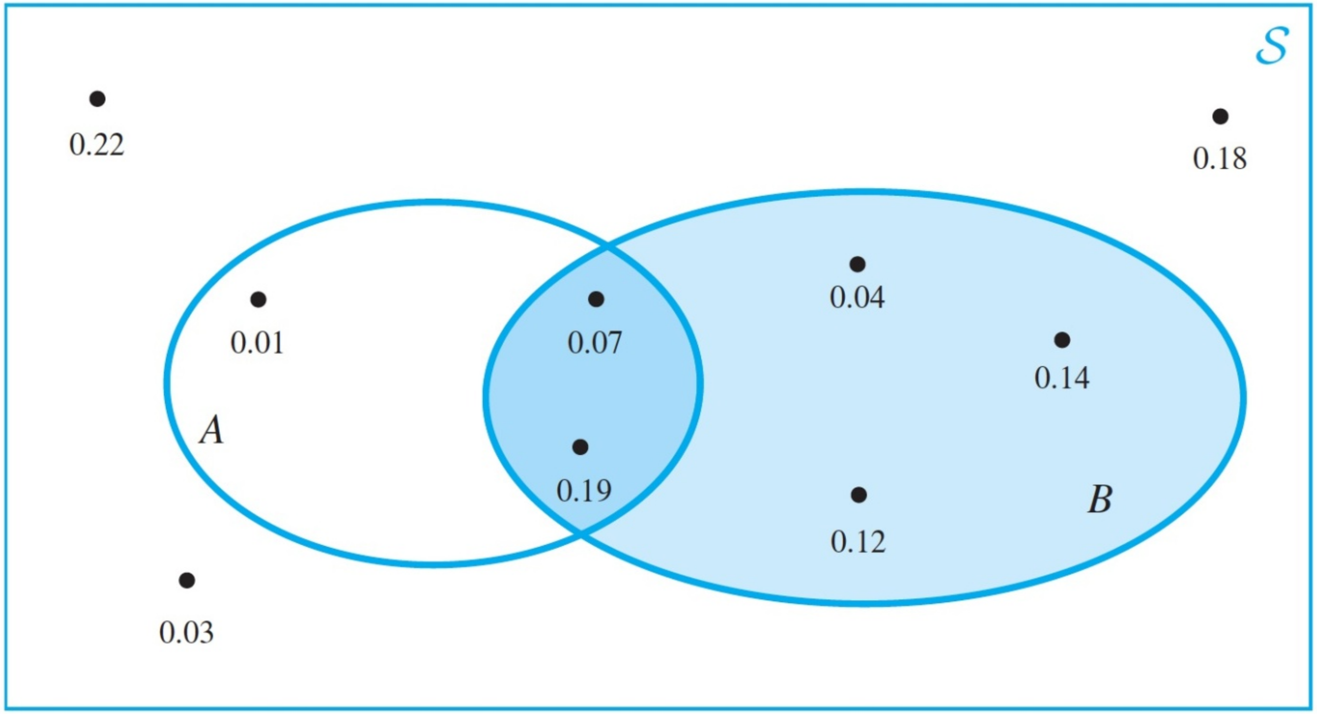
Definitions

The conditional probablity of event A conditional on event B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

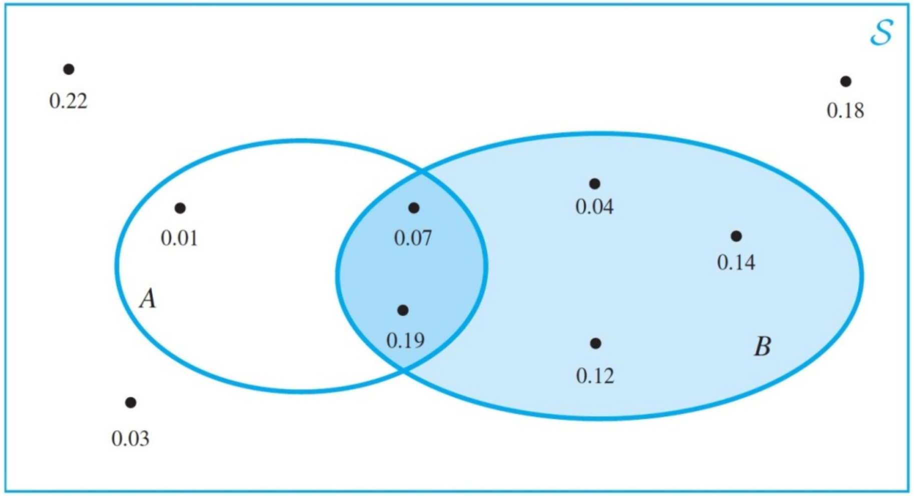
for $P(B) > 0$. It measures the probability that event A occurs when it is known that event B occurs.

Example 1: Suppose that event *B* is known to occur. In other words, suppose that it is known that the outcome occurring is one of the five outcomes contained within the event B. What then is the conditional probability of event *A* occurring?



Solution 1: There are two outcomes in A ∩ B, the conditional probability is the probability that one of these two outcomes occurs rather than one of the other three outcomes (which are in A ∩ B). This probability is calculated to be

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.26}{0.56} = 0.464$$

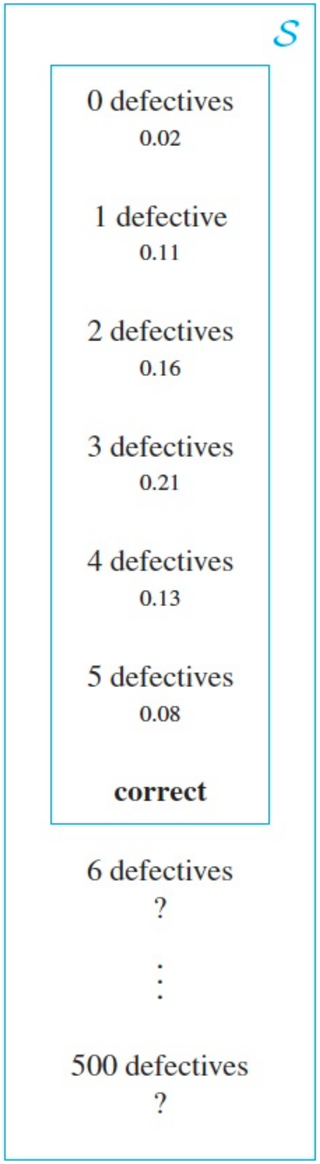


If event *B* is known not to occur, then

$$P(A|B') = \frac{P(A \cap B')}{P(B)} = 0.023$$

$$P(A|B) + P(A'|B) = 1 \ ?$$

Examples



Example 2: Number of defective chips in a box of 500 chips has a sample space as figure. The event *correct*, with, $P(\text{correct}) = 0.71$ consists of 6 outcomes corresponding to no more than 5 defectives. The probability that a box has no defective chips is

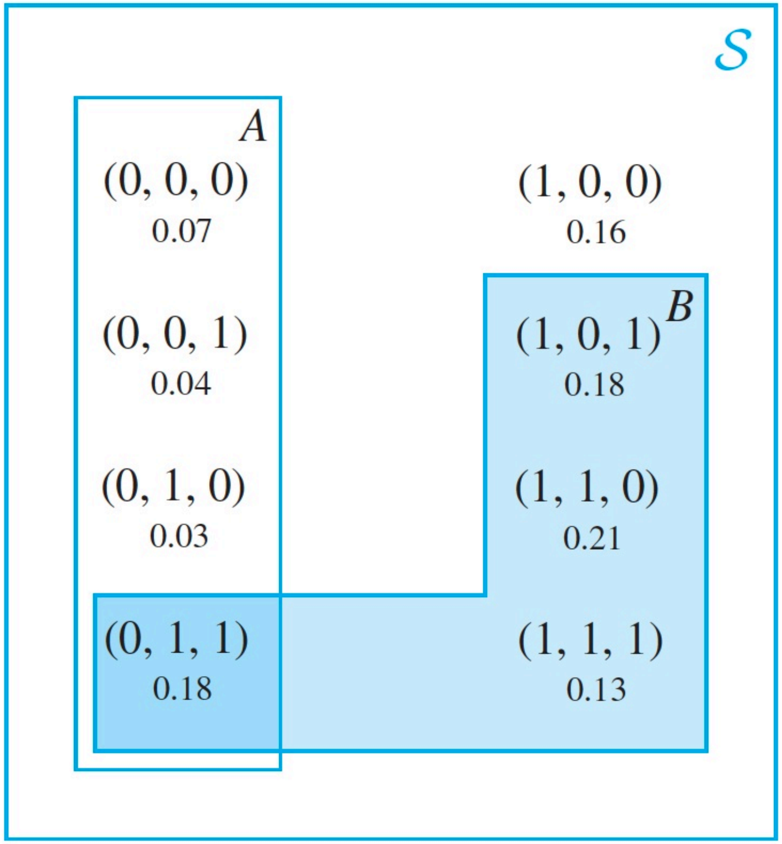
$$P(0 \text{ defective}) = 0.02$$

The probability of no defectives conditional on there being no more than 5 defectives, which is

$$P(0 \text{ defective}|\text{correct}) = \frac{P(0 \text{ defectives} \cap \text{correct})}{P(\text{correct})}$$

$$= \frac{P(0 \text{ defectives})}{P(\text{correct})} = \frac{0.02}{0.71} = 0.028$$

Examples



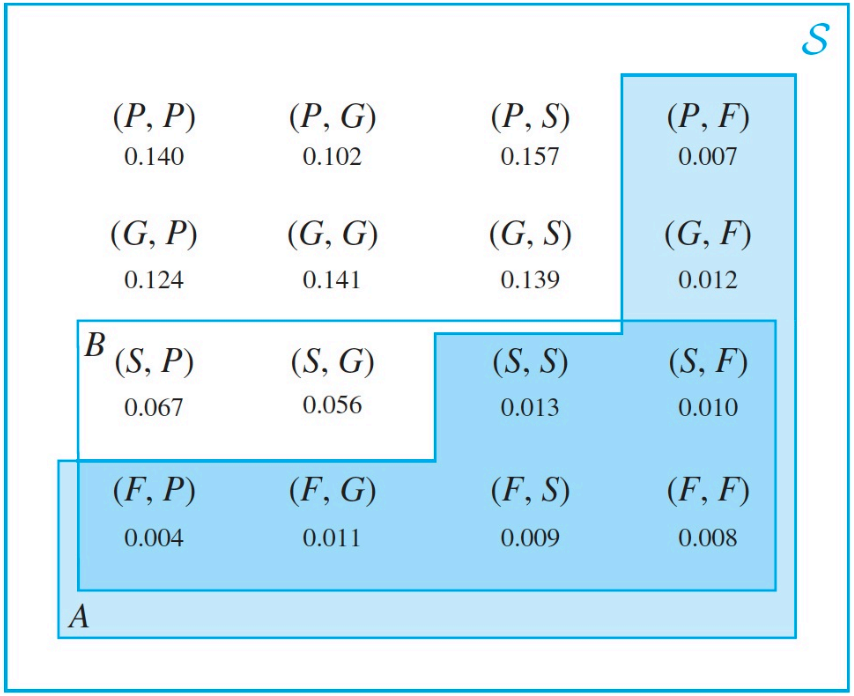
Example 4: The probability that plant *X* is idle is *P(A) = 0.32.* However, suppose it is known that at least 2 out of the 3 plants are generating electricity (event *B*). How does this change the probability of event A occurs? The probability that event A occurs with the condition of event B occurs is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.18}{0.70} = 0.257$$

Therefore, whereas plant X is idle about 32% of the time, it is idle only about 25.7% of the time when at least two of the plants are generating electricity.

Examples

Example 5: The probability that an appliance has a picture graded = *S*(*Satisfactory)* *or* *F*(*Fail)* is *P(B) = 0.178*. However, suppose that a technician takes a television set from a pile of sets that could not be shipped (picture=*S*,*F*;appearance=*S*,*F*). What is the probability that the appliance taken by the technician has a picture graded=*S* or *F* ?

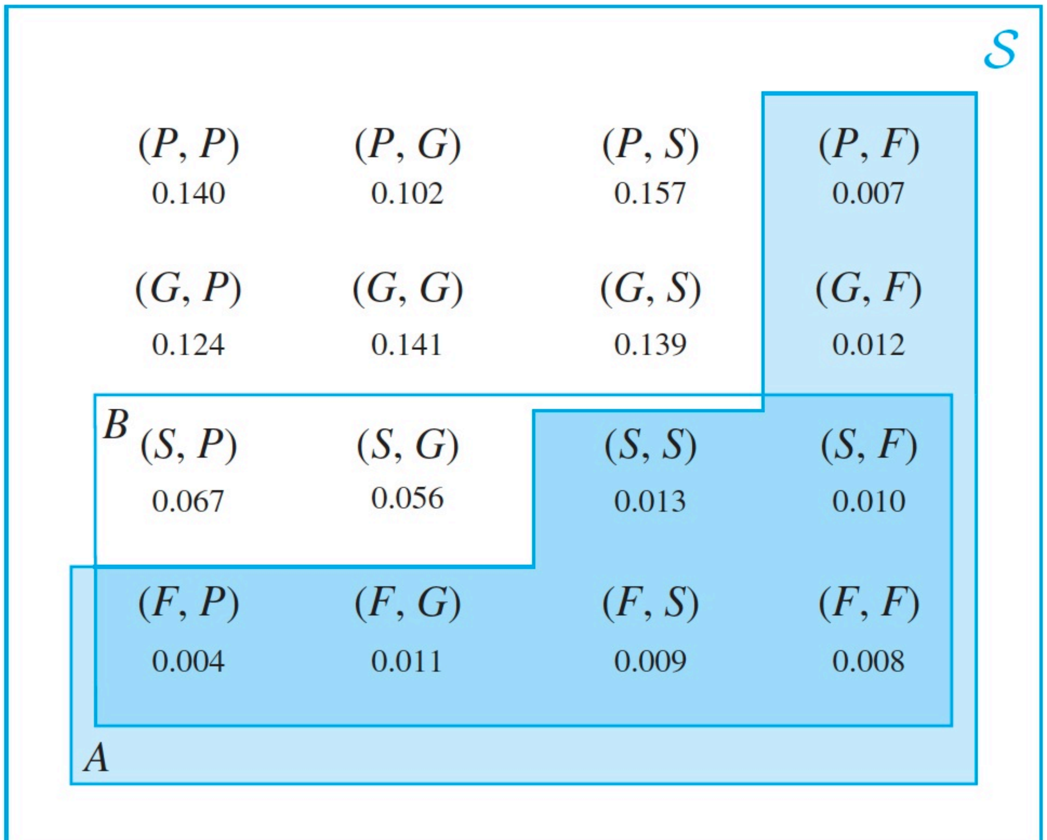


The required probability is the probability that an appliance has a picture graded as either *Satisfactory or Fail (event B)* *conditional on the appliance not being shipped (event A)*. This can be calculated to be

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.055}{0.074} = 0.743$$

Examples

so that whereas only about 17.8% of all the appliances manufactured have a picture graded as either *Satisfactory or Fail*, 74.3% of the appliances that cannot be shipped have a picture graded as either *Satisfactory* or *Fail*.



$$\begin{aligned} P(B|A) &= \frac{P(A \cap B)}{P(A)} \\ &= \frac{0.055}{0.074} \\ &= 0.743 \end{aligned}$$

Examples

Example 6: If a red die and a blue die are thrown, with each of the 36 outcomes being equally likely, let *A* be the event that the red die scores a 6, let *B* be the event that at least one 6 is obtained on the two dice. We have

$$P(A) = \frac{6}{36} = \frac{1}{6}$$

$$P(B) = \frac{11}{36}$$

Suppose that somebody rolls the two dice without showing you, but announces that at least one 6 has been scored. What then is the probability that the red die scored a 6?

Examples

this conditional probability is calculated to be

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{1/6}{11/36} = \frac{6}{11}$$

As expected, this conditional probability is larger than $P(A) = 1/6$.

Examples

Example 7: If a red die and a blue die are thrown, with each of the 36 outcomes being equally likely, let *A* be the event that the red die scores a 6, let *C* be the event that exactly one 6 is obtained on the two dice. We have

$$P(A) = \frac{5}{36}$$

$$P(C) = \frac{10}{36}$$

Suppose that somebody rolls the two dice without showing you, but announces that exactly one 6 has been scored. What then is the probability that the red die scored a 6?

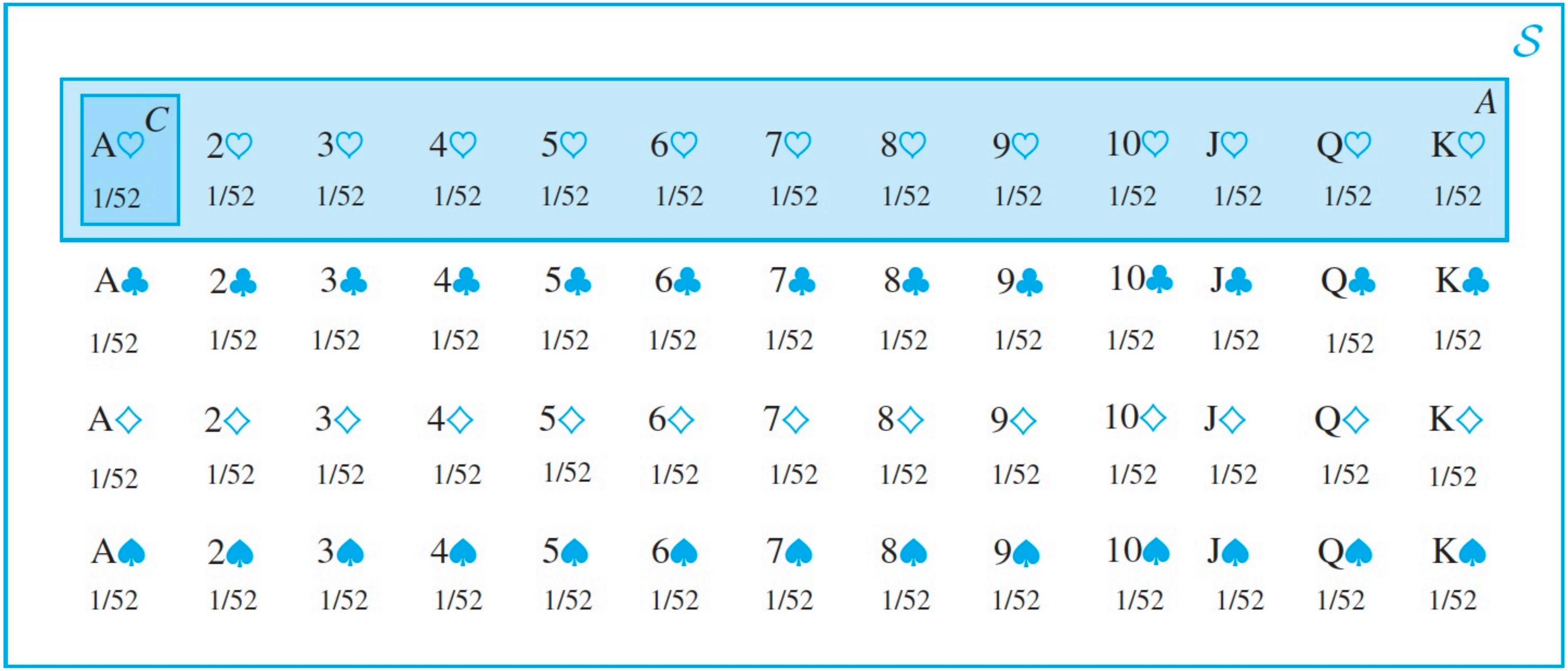
so that the conditional probability

$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{P(A)}{P(C)} = \frac{5/6}{10/36} = \frac{1}{2}$$

As expected, this conditional probability is *P(A|C)* = 1/2<P(A|B)

Examples

Example 8: If a card is drawn from a pack of cards, let *A* be the event that a card from the heart suit is obtained, and let *B* be the event that a picture card is drawn. Recall that $P(A) = 13/52 = 1/4$ and $P(B) = 12/52 = 3/13$. Also, the event $A \cap B$, the event that a picture card from the heart suit is drawn $P(A \cap B) = 3/52$.



Examples

Suppose that somebody draws a card and announces that it is from the heart suit. What then is the probability that it is a picture card? This conditional probability is

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A)}{P(C)} = \frac{3/52}{1/4} = \frac{3}{13}$$

Notice that, *P(B|A) = P(B)* because the proportion of picture cards in the heart suit is identical to the proportion of picture cards in the whole pack.

Let *C* be the event that the *A*♥ is chosen, *with P(C) = 1/52.* If it is known that a card from the heart suit is obtained, then intuitively the conditional probability of the card being *A*♥ is 1/13 since there are 13 equally likely cards in the heart suit.

$$P(C|A) = \frac{P(C \cap A)}{P(A)} = \frac{P(C)}{P(A)} = \frac{1/52}{1/4} = \frac{1}{13}$$

Problems

Problem 1: A car repair is either on time or late and either satisfactory or unsatisfactory. If a repair is made on time, then there is a probability of 0.85 that it is satisfactory. There is a probability of 0.77 that a repair will be made on time. What is the probability that a repair is made on time and is satisfactory?

Solution: Let the event O be an on time repair and the event S be a satisfactory repair. It is known that $P(S|O) = 0.85$, and $P(O) = 0.77$. The question asks for $P(S \cap O)$

$$\begin{aligned} P(S|O) &= \frac{P(S \cap O)}{P(O)} \rightarrow P(S \cap O) = P(S|O) \times P(O) \\ &= 0.85 \times 0.77 = 0.6545 \end{aligned}$$

Problems

Problem 2: With the two assembly lines.
Calculate the probabilities:

S

(S, S) 0.02	(S, P) 0.06	(S, F) 0.05
(P, S) 0.07	(P, P) 0.14	(P, F) 0.20
(F, S) 0.06	(F, P) 0.21	(F, F) 0.19

The probability values for assembly line operations.

(a) Both lines are at full capacity conditional on neither line being shut down.
(b) At least one line is at full capacity conditional on neither line being shut down.
(c) One line is at full capacity conditional on exactly one line being shut down.
(d) Neither line is at full capacity conditional on at least one line operating at partial capacity.

Solution: (a) Let *A* be the event both lines at full capacity consiting of the outcome $\{(\mathrm{F},\mathrm{F})\}$.

S

(S, S) 0.02	(S, P) 0.06	(S, F) 0.05
(P, S) 0.07	(P, P) 0.14	(P, F) 0.20
(F, S) 0.06	(F, P) 0.21	(F, F) 0.19

The probability values for assembly line operations.

Let *B* be the event neither line is shutdown consiting of the outcomes $\{(\mathrm{P},\mathrm{P}),(\mathrm{P},\mathrm{F}),(\mathrm{F},\mathrm{P}),(\mathrm{F},\mathrm{F})\}$.
Therefore $A \cap B = \{(\mathrm{F},\mathrm{F})\}$
hence

$$
\begin{aligned}
P(A|B) &= \frac{P(A \cap B)}{P(B)} \\
&= \frac{0.19}{(0.14 + 0.2 + 0.21 + 0.19)} \\
&= 0.257
\end{aligned}
$$

Solution: (b) Let C be the event at least one line at full capacity consiting of the outcome $\{(\mathrm{F,P}),(\mathrm{F,S}),(\mathrm{F,F}),(\mathrm{S,F}),(\mathrm{P,F})\}$.

(S, S) 0.02	(S, P) 0.06	(S, F) 0.05
(P, S) 0.07	(P, P) 0.14	(P, F) 0.20
(F, S) 0.06	(F, P) 0.21	(F, F) 0.19

S

The probability values for assembly line operations.

Then $C \cap B = \{(\mathrm{F,P}),(\mathrm{F,F}),(\mathrm{P,F})\}$ and hence

$$P(C|B) = \frac{P(C \cap B)}{P(B)} = \frac{0.21 + 0.19 + 0.2}{0.74} = 0.811$$

Problems

Solution: (c) Let *D* be the event that one line is at full capacity consiting of the outcome $\{(\mathrm{F,P}),(\mathrm{F,S}),(\mathrm{P,F}),(\mathrm{S,F})\}$.

$\mathcal{S}$

(S, S) 0.02	(S, P) 0.06	(S, F) 0.05
(P, S) 0.07	(P, P) 0.14	(P, F) 0.20
(F, S) 0.06	(F, P) 0.21	(F, F) 0.19

The probability values for assembly line operations.

Let *E* be the event one line is shutdown consiting of the outcomes

$$\{(\mathrm{S,P}),(\mathrm{S,F}),(\mathrm{P,S}),(\mathrm{F,S})\}$$

Then

$$D \cap E = \{(\mathrm{F,S}),(\mathrm{S,F})\}$$

and hence

$$\begin{aligned} P(D|E) &= \frac{P(D \cap E)}{P(E)} \\ &= \frac{0.06 + 0.05}{0.06 + 0.05 + 0.07 + 0.06} \\ &= 0.458 \end{aligned}$$

Problems

Solution: (d) Let G be the event that neither line is at full capacity consiting of the outcome $\{(\mathrm{S,S}),(\mathrm{S,P}),(\mathrm{P,S}),(\mathrm{P,P})\}$.

S

(S, S) 0.02	(S, P) 0.06	(S, F) 0.05
(P, S) 0.07	(P, P) 0.14	(P, F) 0.20
(F, S) 0.06	(F, P) 0.21	(F, F) 0.19

The probability values for assembly line operations.

Let H be the event at least one line is at partial capacity consiting of the outcomes

$$\{(\mathrm{S,P}),(\mathrm{P,S}),(\mathrm{P,P}),(\mathrm{P,F}),(\mathrm{F,P})\}$$

Then $G \cap H = \{(\mathrm{S,P}),(\mathrm{P,S}),(\mathrm{P,P})\}$ and hence

$$P(G|H) = \frac{P(G \cap H)}{P(H)}$$

$$= \frac{0.06 + 0.07 + 0.14}{0.06 + 0.07 + 0.14 + 0.2 + 0.21} = 0.397$$

INDEPENDENT EVENTS

1. Definition of Independence
2. Multiplication Rule
3. Problems

Definition of Independence Events

Two events *A* and *B* are independent if and only if

$$P(A|B) = P(A) \text{ and } P(B|A) = P(B)$$

or $$P(A \cap B) = P(A)P(B)$$

asuming the existences of the conditional probabilities. Otherwise, *A* and *B* are dependent.

Independence of Two Events

If two events *A* and *B* are considered to be independent because the events are physically unrelated, and if the probabilities $P(A)$ and $P(B)$ are known, then the definition can be used to assign a value to $P(A \cap B)$.

Independence of Two Events

Example 9: Two machines 1 and 2 in a factory are operated independently of each other. Let *A* be the event that machine 1 will become inoperative during a given 8-hour period, let *B* be the event that machine 2 will become inoperative during the same period, and suppose that P(A) = 1/3 and P(B) = 1/4. We shall determine the probability that at least one of the machines will become inoperative during the given period.

The probability P(A ∩ B) that both machines will become inoperative during the period is

$$P(A \cap B) = P(A)P(B) = \left(\frac{1}{3}\right) \times \left(\frac{1}{4}\right) = \frac{1}{12}$$

Multiplicative Rule

- If in an experiment the events A and B can both occur, then

$$P(A \cap B) = P(A)P(B|A), \text{ provided } P(A) > 0$$

- Thus, the probability that both A and B occur is equal to the probability that A occurs multiplied by the conditional probability that B occurs, given that A occurs. Since

$$A \cap B = B \cap A$$

- then we can write

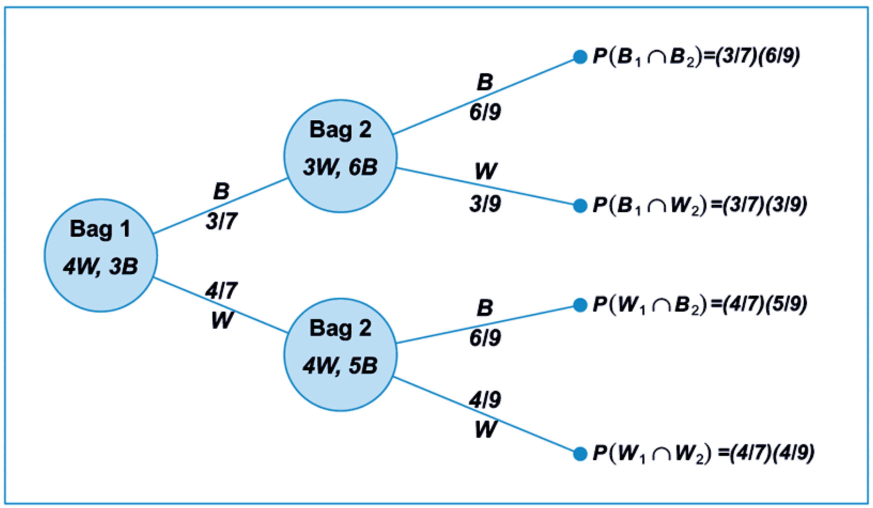
$$P(A \cap B) = P(B \cap A) = P(B)P(A|B)$$

Example 10: One bag contains 4 white balls and 3 black balls, and a second bag contains 3 white balls and 5 black balls. One ball is drawn from the first bag and placed unseen in the second bag.

What is the probability that a ball now drawn from the second bag is black?

Multiplication Rule

Solution: Let *B1*, *B2*, and *W1* represent, respectively, the drawing of a black ball from bag 1, a black ball from bag 2, and a white ball from bag 1. We are interested in the union of the mutually exclusive events *B1 ∩ B2 and W1 ∩ B2.*

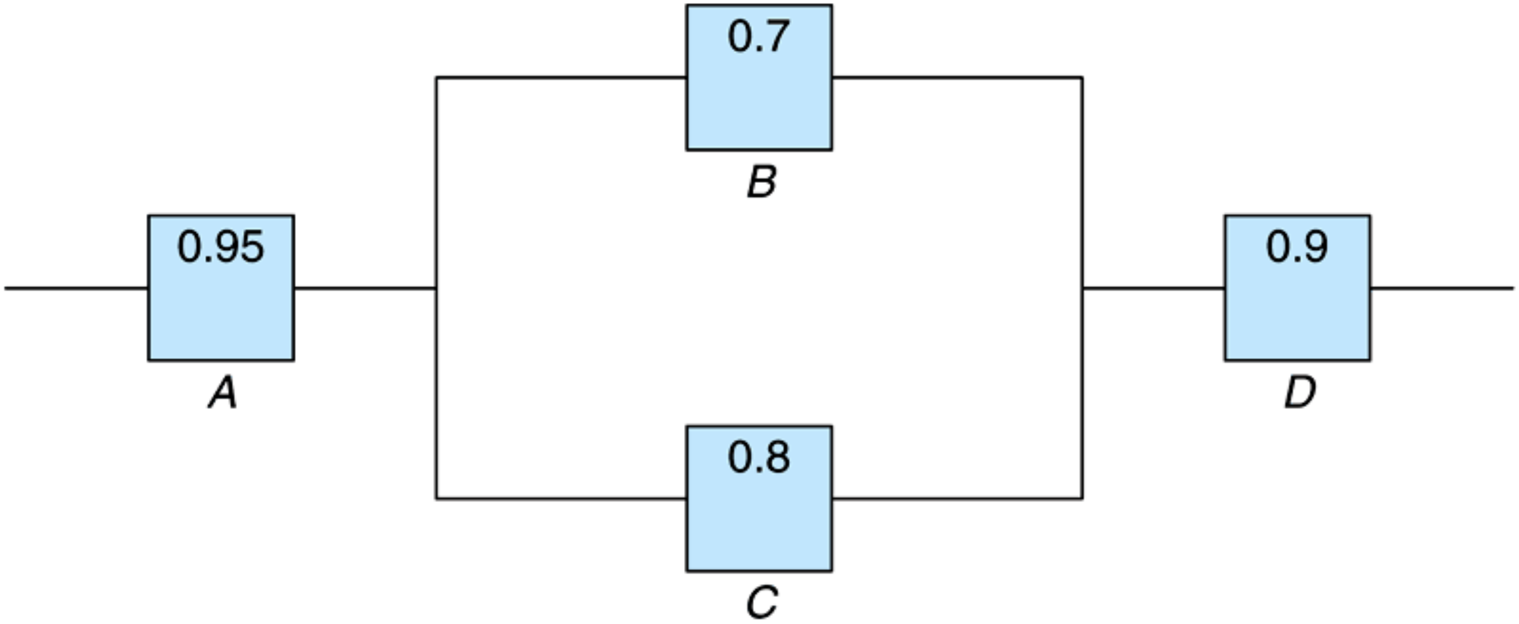


$$
\begin{aligned}
&P[(B_1 \cap B_2) \vee (W_1 \cap B_2)] \\
&= P(B_1 \cap B_2) + P(W_1 \cap B_2) \\
&= P(B_1)P(B_2|B_1) + P(W_1)P(B_2|W_1) \\
&= \left(\frac{3}{7}\right)\left(\frac{6}{9}\right) + \left(\frac{4}{7}\right)\left(\frac{5}{9}\right) = \frac{38}{63}
\end{aligned}
$$

The various posibilities and probabilites

Problems

Problem 3: Suppose the diagram of an electrical system is as given in the Figure below. What is the probability that the system works? Assume the components fail independently.



$$P(\text{work}) = (0.95)[1 - (1 - 0.7)(1 - 0.8)]0.9 = 0.8037$$

BAYES' THEOREM

Bayes' Theorem

- Suppose that we are interested in which of several disjoint events $B_1, \ldots, B_k$ will occur and that we will get to observe some other event A. If $P(A|B_i)$ is available for each i, then Bayes' theorem is a useful formula for computing the conditional probabilities of the B_i events given A.

Bayes' Theorem (cont'd)

Let the events $B_1, \ldots, B_k$ form a partition of the space $\mathcal{S}$ such that $P(B_j) > 0$ for $j = 1, \ldots, k$ and let *A* be an event such that $P(A) > 0$. Then, for $i = 1, \ldots, k$

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{\sum_{j=1}^{k} P(B_j)P(A|B_j)}$$

Bayes' Theorem (cont'd)

Test for a Disease: Suppose that you are walking down the street and notice that the Department of Public Health is giving a free medical test for a certain disease. The test is 90 percent reliable in the following sense: If a person has the disease, there is a probability of 0.9 that the test will give a positive response; whereas, if a person does not have the disease, there is a probability of only 0.1 that the test will give a positive response.

Data indicate that your chances of having the disease are only 1 in 10,000.

However, since the test costs you nothing, and is fast and harmless, you decide to stop and take the test. A few days later you learn that you had a positive response to the test.

Now, what is the probability that you have the disease?

Bayes' Theorem (cont'd)

Test for a Disease: We have tested positive for a disease. The test was 90 percent reliable. We want to know the probability that we have the disease after we learn that the result of the test is positive.

Some readers may feel that this probability should be about 0.9. However, this feeling completely ignores the small probability of 0.0001 that you had the disease before taking the test.

We shall let B_1 denote the event that you have the disease, and let B_2 denote the event that you do not have the disease. The events B_1 and B_2 form a partition. Also, let A denote the event that the response to the test is positive. The event A is information we will learn that tells us something about the partition elements. Then, by Bayes' theorem,

$$\begin{aligned}
P(B_1|A) &= \frac{P(B_1)P(A|B_1)}{P(B_1)P(A|B_1)+P(B_2)P(A|B_2)} \\
&= \frac{(0.0001)(0.9)}{(0.0001)(0.9)+(0.9999)(0.1)} \\
&= 0.00090
\end{aligned}$$

Summary

1. Definition & Examples
2. Independent Events
3. Bayes' Theorem