

Quantum Mechanics I

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Coding is easy, algebra is easy, calculus is easy, physics is impossible.

– Rajamani S. Narayanan

Preface

This book is written with my lack of knowledge in LaTeX and my notes taken from Professor Rajamani S. Narayanan course on Quantum Mechanics at Florida International University. During these pages I am going to learn how to manage LaTeX as a useful tool while I am dealing with understanding the world of Quantum Mechanics.

I will start with some Classical Mechanics to understand how Quantum Mechanics appeared in the first place.

I also want to thank Abel Rosado who helps me and teaches me on how to use this template.

Adolfo Menéndez Rua

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1.1 The Two-Body Problem

As it is showed in Figure ?? we have two bodies with masses m_1 and m_2 and they interact with a force unknown.

We can define their positions as \vec{r}_1 and \vec{r}_2 and the separation between them as $\vec{r} = \vec{r}_1 - \vec{r}_2$

I said that the force is unknown but we know:

- ▶ $F_{12} = -F_{21}$
- ▶ The force only depends on the distance between the two bodies and the direction is the line between them, i.e. $F = f(r)\hat{r}$ ¹
- ▶ The force can be defined as a central force so we can define a potential as: $f(r) = -\frac{dv(r)}{dr}$

Using Newton 3rd Law we can explain the movement of this two objects by two equations:

$$m_1 \frac{d^2 \vec{r}_1}{dt^2} = f(r)\hat{r} \quad (1.1)$$

$$m_2 \frac{d^2 \vec{r}_2}{dt^2} = -f(r)\hat{r} \quad (1.2)$$

Let's add ?? and ??, this is going to be useful later.

$$m_1 \frac{d^2 \vec{r}_1}{dt^2} + m_2 \frac{d^2 \vec{r}_2}{dt^2} = 0 \quad (1.3)$$

We can not get anything from ?? and ??, so we are going to transform this equations into something we can use to resolve them. First, we define the center of mass and their derivatives among time:

$$r_{CM} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \quad (1.4)$$

$$v_{CM} = \frac{d}{dt} r_{CM} = \frac{1}{m_1 + m_2} \left[m_1 \frac{d\vec{r}_1}{dt} + m_2 \frac{d\vec{r}_2}{dt} \right] \quad (1.5)$$

$$a_{CM} = \frac{d}{dt} v_{CM} = \frac{1}{m_1 + m_2} \left[m_1 \frac{d^2 \vec{r}_1}{dt^2} + m_2 \frac{d^2 \vec{r}_2}{dt^2} \right] \quad (1.6)$$

If we look at ?? we can see something equal to the left term of ?? so we can say that the acceleration of the center of mass is zero, i.e. $a_{CM} = 0$.

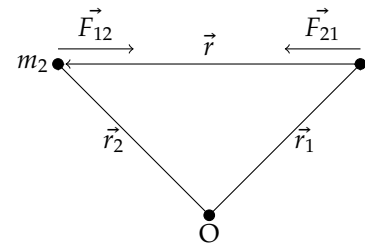
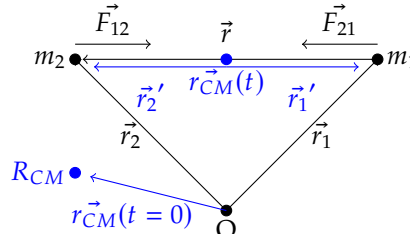


Figure 1.1: Two bodies interacting with some force at some time

This mean that the center of mass moves with a constant velocity and we can define it's move as:

$$r_{CM} = R_{CM} + V_{CM} \cdot t \quad (1.7)$$

Where R_{CM} and V_{CM} are constants defined by initial conditions. Knowing this we can make a change of the coordinates with the next assumption:



$$\vec{r}_1 = \vec{r}'_1 + r_{CM} \quad (1.8)$$

$$\vec{r}_2 = \vec{r}'_2 + r_{CM} \quad (1.9)$$

Now let's calculate the new center of mass r'_{CM} using ?? and ??

Figure 1.2: Two-body problem with the new coordinates

$$\begin{aligned} r'_{CM} &= \frac{m_1 \vec{r}'_1 + m_2 \vec{r}'_2}{m_1 + m_2} \\ &= \frac{1}{m_1 + m_2} [m_1(\vec{r}_1 - r_{CM}) + m_2(\vec{r}_2 - r_{CM})] \\ &= r_{CM} - \frac{1}{m_1 + m_2} [r_{CM}(m_1 + m_2)] \\ &= r_{CM} - r_{CM} = 0 \end{aligned} \quad (1.10)$$

2: The professor demonstrate the same but with the expression for , this is just

As it should be, our new center of mass is the origin for all time because our new referential system is moving with the center of mass. From this assumptions and using the definition of the new center of mass above we can get a new system of equations.

$$\begin{cases} m_1 \vec{r}'_1 + m_2 \vec{r}'_2 = 0 \\ \vec{r}'_1 - \vec{r}'_2 = \vec{r} \end{cases} = \begin{cases} \vec{r}'_1 = \frac{m_2 \vec{r}}{m_1 + m_2} \\ \vec{r}'_2 = \frac{-m_1 \vec{r}}{m_1 + m_2} \end{cases} \quad (1.11)$$

We use what we obtain from ?? in ?? and ??. We can do this because the force does not change in this coordinates because only depend on r and this magnitude is still the same, i.e. $\vec{r} = \vec{r}'$.

$$m \frac{d^2 \vec{r}}{dt^2} = f(r) \vec{r} \quad (1.12)$$

m is the reduce mass define as:

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

We get the same equation from both equations. This expression will be useful to resolve the problem later.

Normally we can use $m \approx m_2$ when $m_1 \gg m_2$, some examples of this are Sun and Earth or proton and electron.

With real values:

$$m_{S-E} = \frac{m_S m_E}{m_S + m_E} \approx 5.96998 \cdot 10^{24} \text{ kg}$$

where $m_S = 2 \cdot 10^{30} \text{ kg}$, $m_E = 5.97 \cdot 10^{24} \text{ kg}$

We define the angular momentum as:

$$\vec{L} = \vec{r} \times \vec{p} = m \vec{r} \times \frac{d\vec{r}}{dt} \quad (1.13)$$

1.2 The Angular Momentum

We want to now how this quantity changes among time.

$$\frac{d\vec{L}}{dt} = m \frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} + m\vec{r} \times \frac{d^2\vec{r}}{dt^2} = 0 + \vec{r} \times f(r)\hat{r} = 0 \quad (1.14)$$

The law of the conservation of angular momentum says that the angular momentum of a system remains constant unless an external rotational force (torque) acts upon it.

As we just demonstrate the angular momentum vector is constant and we can redefine it as:

$$\vec{L} = L\hat{z} \quad (1.15)$$

The decision of L going through the \hat{z} direction is going to be very resourceful.

1.3 Cylindrical coordinates

We need to choose a coordinate system. The problem become easy if we choose cylindrical coordinates.

$$\begin{aligned} \vec{r} &= \rho\hat{\rho} + z\hat{z} \\ \frac{d\vec{r}}{dt} &= \frac{d\rho}{dt}\hat{\rho} + \rho\frac{d\hat{\rho}}{dt} + \frac{dz}{dt}\hat{z} \end{aligned} \quad (1.16)$$

If we use ?? in ?? we can look at \vec{L} components

$$\begin{aligned} \vec{L} &= m(\rho\hat{\rho} + z\hat{z}) \times \left(\frac{d\rho}{dt}\hat{\rho} + \rho\frac{d\hat{\rho}}{dt} + \frac{dz}{dt}\hat{z} \right) = \\ &= -mz\rho\frac{d\hat{\rho}}{dt} + m\left(z\frac{d\rho}{dt} - \rho\frac{dz}{dt}\right)\hat{\phi} + m\rho^2\frac{d\hat{\phi}}{dt}\hat{z} \end{aligned} \quad (1.17)$$

Using the definition in ?? and our new equation we get 4 different equations if we compare the angular momentum in each direction.

$$\begin{aligned} 1. \quad & L \neq 0 \\ 2. \quad & mz\rho\frac{d\hat{\phi}}{dt} = 0 \\ 3. \quad & m\left(z\frac{d\rho}{dt} - \rho\frac{dz}{dt}\right) = 0 \\ 4. \quad & m\rho^2\frac{d\hat{\phi}}{dt} = L \end{aligned} \quad (1.18)$$

This gives you a lot of information:

► $m \neq 0$

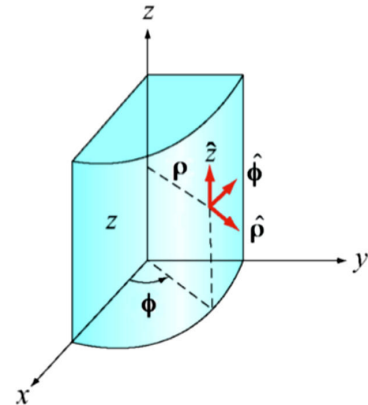


Figure 1.3: Cylindrical coordinates.
<https://www.researchgate.net/publication/334148643/figure/fig1/AS:775915730649091@1562004134358/Diagram-of-a-standard-cylindrical-coordinates-system.jpg>

- $\rho \neq 0$
- $\frac{d\phi}{dt} \neq 0$
- $z = 0$
- $\frac{d\phi}{dt} = \frac{L}{m\rho^2}$

This equations not only give us a fixed value of z it also give us that in the system we choose the value for z is 0 for all time. We also can get a relation between the derivative of ϕ and ρ we are going to use this later to resolve the problem.

If $z=0$ we can redefine \vec{r} from ??.

$$\begin{aligned}\vec{r} &= \vec{\rho} = \rho \hat{\phi} \\ \frac{d^2 \vec{\rho}}{dt^2} &= \left[\frac{d^2 \rho}{dt^2} - \rho \left(\frac{d\phi}{dt} \right)^2 \right] \hat{\rho} + \left[2 \frac{d\rho}{dt} \frac{d\phi}{dt} + \rho \frac{d^2 \phi}{dt^2} \right] \hat{\phi}\end{aligned}\quad (1.19)$$

We loose the therm that goes in the $\hat{\phi}$ direction because is zero. We can see this in two different ways:

- Now our force $F = f(r)\hat{r} = f(\rho)\hat{\rho}$ because ?? so the second derivative of $\vec{\rho}$ only can have a component in the $\hat{\rho}$ direction.
- If we calculate what we have in the $\hat{\phi}$ component using some notions in derivatives and ?? we would get 0³

If we return to ?? with everything we learnt we can get something very interesting, an scalar equation.

$$m \left[\frac{d^2 \rho}{dt^2} - \rho \left(\frac{d\phi}{dt} \right)^2 \right] + \rho \frac{d^2 \phi}{dt^2} = \frac{d}{dt} \left(\rho^2 \frac{d\phi}{dt} \right) = 0 \quad (1.20)$$

3: $\frac{d}{dt} \left(\frac{\rho^2 L}{m} \right) = 0$, because both L and m are constants.

We get a scalar equation with two variables ρ and ϕ , but we can use ?? as before and reduce everything to an one variable scalar equation.

$$m \left[\frac{d^2 \rho}{dt^2} - \rho \frac{L^2}{m^2 \rho^4} \right] = f(\rho) = -\frac{dv(\rho)}{d\rho} \quad (1.21)$$

We can put everything in one side of the equation and multiply both sides by a factor $\frac{d\rho}{dt}$

$$\begin{aligned}m \frac{d^2 \rho}{dt^2} \frac{d\rho}{dt} - \frac{d}{dt} \left(\frac{L^2}{m\rho^3} \right) + \frac{d}{dt} \frac{dv(\rho)}{d\rho} &= 0 \\ \frac{d}{dt} \left(\frac{1}{2} m \left(\frac{d\rho}{dt} \right)^2 + \frac{L^2}{2m\rho^2} + v(\rho) \right) &= 0\end{aligned}\quad (1.22)$$

We know that this magnitude inside the brackets is constant during the time, this magnitude is the total energy (E) but we can divide this energy in three:

Linear Kinetic Energy $K = \frac{1}{2} m \left(\frac{d\rho}{dt} \right)^2$

Angular Kinetic Energy $K_\alpha = \frac{L^2}{2m\rho^2}$

Potential Energy $v(\rho) = \frac{-k}{\rho}$

1.4 Potential wells

We are going to study the angular kinetic energy and the potential energy (both are functions of ρ), we call the addition of these two energies **Effective potential**

In Figure ?? we can see many things:

- ▶ The function have a minimum that is the lowest possible total energy because the linear kinetic energy is always greater or equal to 0, i. e. $K \geq 0$. This energy represents a circular motion.
- ▶ If we have a higher energy but still negative, the movement is an elliptical motion with two values that are the nearest and the furthest point in the ellipse.
- ▶ If the motion is higher than 0 the motion is not more a close orbit and then the motion is an hyperbola.
- ▶ If the total energy is 0, the motion is in the limit between a close and an open orbit and the movement is a parable.

In the next section we are going to give real values to this magnitudes.

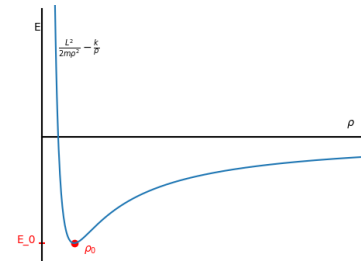


Figure 1.4: Effective Potential

1.5 Physical Examples

1. Earth-Sun problem: If we take the Earth and the Sun as our two bodies we need to know some values. We already know the reduce mass from the previous section. The k value of the potential energy is $k = GM_S m_E$, we know this value from Newton's Gravitational Law.

- ▶ $m_E = 2 \cdot 10^{30} kg$
- ▶ $M_S = 5.97 \cdot 10^{24} kg$
- ▶ $G = 6.67 \cdot 10^{-11} Nm^2/kg^2$

The total energy of the system is:

$$E = \frac{1}{2}m \left(\frac{d\rho}{dt} \right)^2 + \frac{L^2}{2m\rho^2} + \frac{-GM_S m_E}{\rho} \quad (1.23)$$

We can see in Figure ?? that if we represent $U = K_\alpha + v(\rho)$ against ρ we get a similar function to the one in Figure ??, but we can see in the axis that we have large energies and distances. That is reasonable if we think that we are working with massive objects as planets.

We can calculated now the minimum energy the Earth-Sun system can have in which, the Earth would orbit by a circular motion.

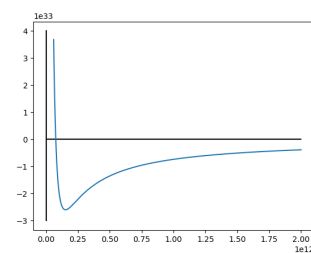


Figure 1.5: Effective Potential for Earth-Sun

To calculate this energy we need to calculate first the value of ρ for this minimum energy (ρ_{min}). This value can be found by matching the first derivative of the function $U(\rho)$ to 0

$$\begin{aligned} \frac{dU}{d\rho} &= \frac{d}{d\rho} \left[\frac{L^2}{2m\rho^2} + \frac{-GM_S m_E}{\rho} \right] = \\ &= \frac{-L^2}{m\rho^3} + \frac{GM_S m_E}{\rho^2} \end{aligned} \quad (1.24)$$

If we match ?? to 0 and solve for ρ we get ρ_{min}

$$\begin{aligned} \frac{-L^2}{m\rho^3} + \frac{GM_S m_E}{\rho^2} &= 0 \\ \frac{GM_S m_E^2 \rho - L^2}{m_E \rho^3} &= 0 \\ \rho_{min} &= \frac{L^2}{GM_S m_E^2} \end{aligned} \quad (1.25)$$

If we use ?? in ?? knowing that our minimum energy implies $K = 0$, we get the minimum energy possible.

$$E = -\frac{1}{2} \frac{G^2 M_S^2 m_E^3}{L^2} = -\frac{1}{2} \frac{GM_S m_E}{\rho_{min}} = \frac{1}{2} v(\rho_{min}) \quad (1.26)$$

If we look into the equation ?? we can demonstrate that the minimum energy is a half of the potential energy.

2. Electron-Proton problem: If we take a proton and an electron we can do the same as before knowing their masses. The problem is very similar, just with different conditions. The reduce mass is going to be the mass of the electron because it is much less than the mass of the proton.

- $m_e = 9.11 \cdot 10^{-31} kg$
- $m_p = 1.67 \cdot 10^{-27} kg$
- $G = 6.67 \cdot 10^{-11} kg$

The total energy of the system is:

$$E = \frac{1}{2} m \left(\frac{d\rho}{dt} \right)^2 + \frac{L^2}{2m\rho^2} + \frac{-GM_p m_e}{\rho} \quad (1.27)$$



Figure 1.6: Effective Potential for Electron-Proton

We can see in Figure ?? that if we represent $U = K_\alpha + v(\rho)$ against ρ we get a similar function to Figure ??, but if we focus in the axis we can see that we have very low energies. That is reasonable if we think that we are working with particles that have very little mass.

I'm not going to do all we did before. In this case, because the energy is so low, is not interesting to study it. But there is another potential that affects electron and proton, the electric potential.

3. Electron-Proton problem with Electric Potential: In this case we need to define the charges of the proton and the electron. Also we need to define a new potential, where $k = \frac{Ze^2}{4\pi\epsilon_0}$ where Z is the atomic number, e is the charge of the electron⁵ and ϵ_0 is the permittivity of free space.

- ⁵ This value was discovered by the Nobel Prize Robert Andrews Millikan by his experimental oil drop experiment
- $e = 1.6 \cdot 10^{-19} \text{ C}$
 - $\epsilon_0 = 8.85 \cdot 10^{-12} \text{ kg}$

The total energy of the system is:

$$E = \frac{1}{2}m \left(\frac{d\rho}{dt} \right)^2 + \frac{L^2}{2m\rho^2} + \frac{-Ze^2}{4\pi\epsilon_0\rho} \quad (1.28)$$

If we look at Figure ?? and compare it with Figure ?? we can see a high difference in the Energy scale. This means that the electrical force is much stronger than the gravitational force.

We can calculate now the minimum energy the Electron-Proton system can have in which, the electron would orbit by a circular motion around the proton.

To calculate this energy we need to calculate first the value of ρ for this minimum energy (ρ_{min}). This value can be found by matching the first derivative of the function in Figure ?? to 0

$$\begin{aligned} \frac{dU}{d\rho} &= \frac{d}{d\rho} \left[\frac{L^2}{2m\rho^2} + \frac{-Ze^2}{\rho} \right] = \\ &= \frac{-L^2}{m\rho^3} + \frac{Ze^2}{\rho^2} \end{aligned} \quad (1.29)$$

If we match ?? to 0 and solve for ρ we get ρ_{min} .

$$\begin{aligned} \frac{-L^2}{m\rho^3} + \frac{Ze^2}{\rho^2} &= 0 \\ Ze^2 m \rho - 4L^2 \pi \epsilon_0 &= 0 \\ \rho_{min} &= \frac{4L^2 \pi \epsilon_0}{Ze^2 m} \end{aligned} \quad (1.30)$$

We can also get the square of the angular momentum as a function of ρ .

$$L^2 = \frac{m \rho_{min} Ze^2}{4\pi\epsilon_0} \quad (1.31)$$

Now, we have two different expressions for the minimum energy.

$$\begin{aligned} 1) E_{min} &= -\frac{Ze^2}{8\pi\epsilon_0\rho^2} \\ 2) E_{min} &= -\frac{Z^2 e^4 m}{2(4\pi\epsilon_0)^2} \frac{1}{L^2} \end{aligned} \quad (1.32)$$

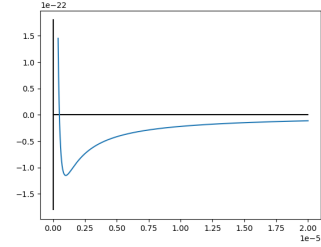


Figure 1.7: Effective Potential for Electron-Proton (Electric Potential)

Looking at the first expression we have the same as the Earth-Proton problem, we get that the minimum energy is half of the potential energy.

The second expression is even more interesting because after measure the length wave of light from an hydrogen atom, Bohr discovered that L was proportional to the energy level of the atom (n). Later they found that the relation is:

$$L = hn \quad (1.33)$$

h is known as the Plank constant.
 $h \approx 6.63 \cdot 10^{-34}$

1.6 Rotation and angular momentum

To finished this chapter we are going to talk about something that is very interesting and purely mathematical.

We know that the rotation in 3D is not commutative and we also know that the definition of the module of \vec{L} is

$$L^2 = |\vec{L}|^2 = L_x^2 + L_y^2 + L_z^2 \quad (1.34)$$

This means that we can have multiple solutions for L using rotations.

We define a vector $|j, m\rangle$ where j is the representation and m is the bases vector of the representation. Then we get this relations:

$$\begin{aligned} 1) L^2 |j, m\rangle &= j(j+1) |j, m\rangle \\ 2) L_z |j, m\rangle &= m |j, m\rangle \end{aligned} \quad (1.35)$$

I'm not proving anything, we will talk about this later.

Waves 2

Waves are some example of classical mechanics, where we have a function that propagates among time. During this chapter we will define their properties and their relation with quantum mechanics.

2.1 The wave function

We define a function that propagates in one dimension (x) among the time (t).

$$\psi(x, t) = e^{i(kx - \omega t)} \quad (2.1)$$

Where ω is the angular velocity of propagation and k is the wave number. We are working with complex numbers but the solution to the function can not be a complex number because it has a physical meaning.

We can define the phase as the function inside the exponential.

$$\phi(x, t) = kx - \omega t \quad (2.2)$$

If we set ϕ constant we will be "riding" the wave.

2.2 Energy and momentum

We are interested in see what happens among time and space in this function, so we are going to differentiate.

$$\begin{aligned} \frac{d}{dt} \psi(x, t) &= -i\omega e^{i(kx - \omega t)} \\ \frac{d}{dx} \psi(x, t) &= ike^{i(kx - \omega t)} \end{aligned} \quad (2.3)$$

We choose to work with one dimension but everything could have been done in three dimensions just using x and k as vectors.

Now we are going to do some assumptions.

$$\begin{aligned} E &= \hbar\omega = i\hbar \frac{d}{dt} \\ P &= \hbar k = -i\hbar \vec{\nabla} \end{aligned} \quad (2.4)$$

We can define the energy as some constant (\hbar) times the frequency. If we use the same constant and multiply it to the wave number we get something with units of momentum, so we can call it momentum.

The second term in both expression is what we call energy and momentum operator.

We now from classical mechanics that energy and momentum are related.

$$E = \frac{P \cdot P}{2m} + v \quad (2.5)$$

In the previous chapter we got the expression of the energy, with which we can get the position at any time if there are some initial conditions given.

In the same way, if $\psi(\vec{x}, 0)$ is given we can compute $\psi(\vec{x}, t)$ using the energy and momentum operators.

2.3 P(x) function I: Definition and properties

Experimentally describe a particle at rest. If we plot the position measure we will get different answers for the same experiment.

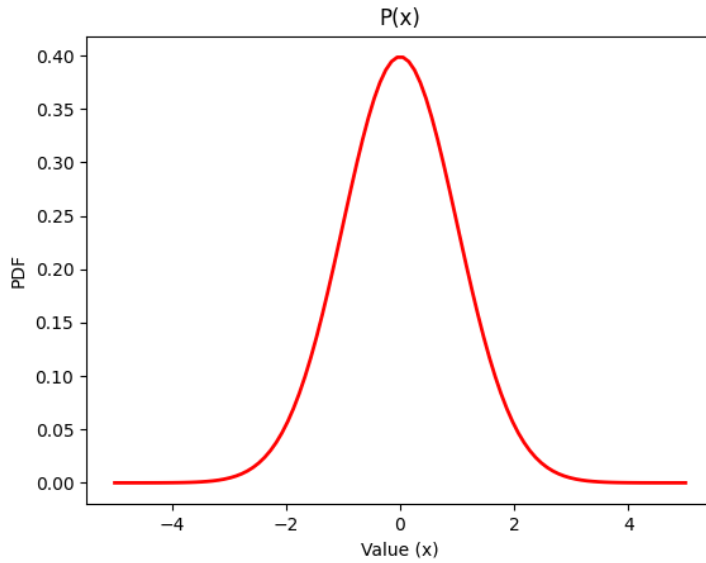


Figure 2.1: Normal distribution for mean=0 and sigma=1, to simulate a experiment measured by different people

*Uncertainty is not probability

In wave mechanics $P(x)$ should be interpreted as $\psi^2(x, 0)$.

We can define $P(x)$ as a Gaussian distribution with an undetermined constant.

$$P(x) = A e^{-\frac{x^2}{2\sigma^2}} \quad (2.6)$$

We also make the next choice to define $P(x)$:

$$\int_{-\infty}^{\infty} P(x) dx = 1 \quad (2.7)$$

We are going to ignore the constant for now to keep things simpler.

$$I = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx \quad (2.8)$$

We need to resolve the integral of $P(x)$, but it is not a direct integral and is difficult to resolve it directly. Luckily we can use a trick to resolve this integral.

$$\Re[\alpha] > 0, \alpha \in \mathbb{C}$$

$$\begin{aligned} I^2 &= \left[\int_{-\infty}^{\infty} e^{-\alpha x^2} dx \right] \left[\int_{-\infty}^{\infty} e^{-\alpha y^2} dy \right] = \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\alpha(x^2+y^2)} dx dy = \\ &= \int_0^{2\pi} d\theta \int_0^{\infty} r e^{-\alpha r^2} dr = \\ &= 2\pi \frac{-1}{2\alpha} \left[e^{-\alpha r^2} \right]_0^{\infty} = \frac{\pi}{\alpha} \end{aligned} \quad (2.9) \quad dx dy = r dr d\theta$$

If we want ?? to be true we need to normalize it using the result from ??.

$$A = \sqrt{\frac{\alpha}{\pi}} \quad \text{where} \quad \alpha = \frac{1}{2\sigma^2} \quad (2.10)$$

We get that the final value of our $P(x)$ function is:

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \quad (2.11)$$

We will continue with this later but first, we need to define Fourier Series.

2.4 Fourier Series

If we have a periodic function with length L so $f(x) = f(x+L)$, then we can rewrite any function in this way:

$$f(x) = \sum_{k=0}^N \tilde{f}_k \psi_k(x) \quad (2.12)$$

Where $\psi_k(x)$ follow the next statements.

$$\begin{aligned} 1) \psi_k(x) &= \frac{1}{\sqrt{L}} e^{i \frac{2\pi k}{L} x} \\ 2) \int_0^L \psi_k^*(x) \psi_{k'}(x) dx &= \delta_{k,k'} \end{aligned} \quad (2.13)$$

δ is a Dirac delta

Using Fourier analysis we can rewrite our wave function into:

$$\psi(x) = \sqrt{P(x)} = \int_{-\infty}^{\infty} \tilde{\psi}(k) e^{ikx} dk \quad (2.14)$$

If now we set some equations for $\psi(x)$.

$$\begin{aligned} 1) \text{Dom}[\psi(x)] &= \left(\frac{-L}{2}, \frac{L}{2} \right) \\ 2) \psi\left(\frac{L}{2}\right) &= \psi\left(\frac{-L}{2}\right) \\ 3) \psi(x) &= e^{ikx} = e^{i\frac{2\pi n}{L}x} \end{aligned} \quad (2.15)$$

We defined a orthogonal function g_n .

$$g_n(x) = \frac{1}{\sqrt{L}} e^{i\frac{2\pi n}{L}x} \quad (2.16)$$

$\delta_{l,n}$ is called Kronecker delta, this function is 1 when $l = n$ and 0 for any other values

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} g_n^*(x) g_l(x) dx = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} e^{i\frac{2\pi x}{L}(l-n)} dx = \delta_{l,n}$$

We can use this function to rewrite $\psi(x)$

$$\begin{aligned} \psi(x) &= \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} \tilde{\psi}_n g_n(x) \\ \psi(x) &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{2\pi}{L} e^{i\frac{2\pi n}{L}x} \tilde{\psi}_n \end{aligned} \quad (2.17)$$

The last expression in ?? is a Riemann sum so we can turn that expression into an integral.

$$\psi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\psi}(k) e^{ikx} dk \quad (2.18)$$

We can find also an expression for $\tilde{\psi}_n$.

$$\begin{aligned}
\int_{-\frac{L}{2}}^{\frac{L}{2}} \psi(x) g_l^*(x) dx &= \\
&= \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} \tilde{\psi}_n \int_{-\frac{L}{2}}^{\frac{L}{2}} g_n(x) g_l^*(x) dx = \\
&= \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} \tilde{\psi}_n \delta_{n,l} = \frac{1}{\sqrt{L}} \tilde{\psi}_l \quad (2.19) \\
\tilde{\psi}_l &= \sqrt{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \psi(x) g_l^*(x) dx \\
\tilde{\psi}_l &= \int_{-\frac{L}{2}}^{\frac{L}{2}} \psi(x) e^{-i \frac{2\pi l}{L} x} dx
\end{aligned}$$

If we extend to the limit where L approximate infinity we can get a general solution for $\tilde{\psi}(x)$.

$$\tilde{\psi}(k) = \int_{-\infty}^{\infty} \psi(x) e^{-ikx} dx \quad (2.20)$$

We have two different relations between $\psi(x)$ and $\tilde{\psi}(k)$, we need to make sure this two relations make sense together. To do this we are going to use ?? and ??.

$$\begin{aligned}
\psi(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \psi(y) e^{-iky} dy \right] e^{ikx} dk = \\
&= \int_{-\infty}^{\infty} \psi(y) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-y)} dk \right] dy \quad (2.21) \\
\psi(x) &= \int_{-\infty}^{\infty} \psi(y) \delta(x-y) dy
\end{aligned}$$

The last equation we get from ?? is the definition of the Dirac delta function itself so we prove that both expressions we have are right.

2.5 P(x) function II: $\psi(x)$ and $\tilde{\psi}(k)$

We recover the equation ?? and now we get $\psi(x)$ remembering that $\psi^2(x) = P(x)$.

$$\psi(x) = \sqrt{P(x)} = \frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}} e^{-\frac{x^2}{4\sigma^2}} \quad (2.22)$$

We can also define $\tilde{\psi}(k)$ with ??.

$$\tilde{\psi}(k) = \frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{4\sigma^2} - ikx} dx \quad (2.23)$$

This integral is not easy to resolve, but we can get something similar to ?? and we already know the solution for that integral.

$$\begin{aligned}\frac{x^2}{4\sigma^2} + ikx &= \frac{1}{4\sigma^2}[x^2 + 4i\sigma^2 kx] = \\ &= \frac{1}{4\sigma^2}[(x + 2i\sigma^2 k)^2 + 4\sigma^4 k^2] \\ e^{\frac{-x^2}{4\sigma^2} - ikx} &= e^{-\sigma^2 k^2} e^{-\frac{1}{4\sigma^2}(x + 2i\sigma^2 k)^2}\end{aligned}\quad (2.24)$$

This expression can be used in ?? to resolve the integral doing a variable change ($\tau = x + 2i\sigma^2 k$).

$$\tilde{\psi}(k) = \frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}} e^{-\sigma^2 k^2} \int_{-\infty}^{\infty} e^{-\frac{\tau^2}{4\sigma^2}} d\tau \quad (2.25)$$

We are not going to explain why $\infty + 2i\sigma^2 k = \infty$ and $-\infty + 2i\sigma^2 k = -\infty$, so the limits do not change. This can be prove with some complex analysis.

Now we can resolve this integral because we already know the solution from ??

$$\tilde{\psi}(k) = \frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}} e^{-\sigma^2 k^2} \sqrt{4\sigma^2\pi} = 2^{\frac{3}{4}} \pi^{\frac{1}{4}} \sigma^{\frac{1}{2}} e^{-\sigma^2 k^2} \quad (2.26)$$

2.6 Free particle

If we set up the problem as a particle.

For $t = 0$:

- $x = 0$
- $v = 0$

Then $x(t) = 0$. Also if E or p are given instead of v we can resolve the problem for $x(t)$

If we interpret the free particle as a wave, using the knowledge from section 2.2 we will find a equation that describe the motion of the wave. First, we want to find a relation between ω and k . We can get this equation using ?? and ?. Because we are working with a free particle $\nu = 0$

$$\omega = \frac{\hbar k^2}{2m} \quad (2.27)$$

The motion function can be found also using ?? and ??

$$\left(i\hbar \frac{d}{dt}\right) \psi = \frac{1}{2m} \left(-i\hbar \frac{d}{dx}\right)^2 \psi \quad (2.28)$$

We define $P(x,0)$ as a gaussian function as we did in previous chapters.

$$P(x, 0) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \quad (2.29)$$

This $P(x,t)$ function is what we called intensity of the wave, the question now is: if the wave is free ($\psi = \psi(x, t)$), what is $P(x,t)$?

We know from the previous chapter the solutions for $\psi(x, 0)$ and $\tilde{\psi}(x, 0)$ when $P(x,0)$ is a gaussian.

$$\psi(x, 0) = \frac{1}{\sqrt[4]{2\pi\sigma^2}} e^{-\frac{x^2}{4\sigma^2}} \quad (2.30)$$

$$\tilde{\psi}(k) = 2^{\frac{3}{4}} \pi^{\frac{1}{4}} \sigma^{\frac{1}{2}} e^{-\sigma^2 k^2} \quad (2.31)$$

If $\psi(x, 0)$ is interpreted as amplitude at x , $\tilde{\psi}(k)$ can be interpreted as amplitude at k . Also, we can define a intensity of k where $P(k) = \tilde{\psi}^2(k)$

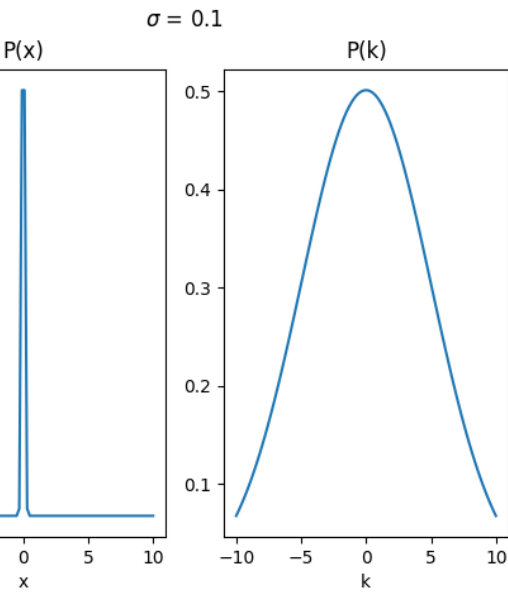


Figure 2.2: $P(x)$ and $P(k)$ for $\sigma = 0.1$.

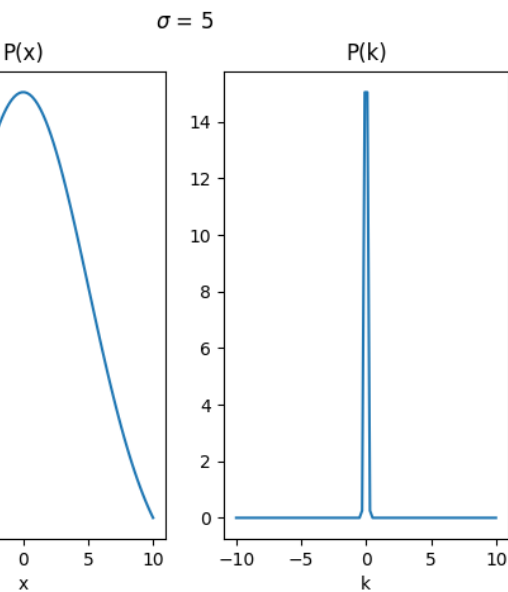


Figure 2.3: $P(x)$ and $P(k)$ for $\sigma = 5$.

In this page we can see two different figures. In Figure ?? we have a very

accurate result for the x value and a more deviated value for k while in Figure ?? is completely the opposite. This is a preamble to the uncertainty principle, however we still haven't talk about quantum physics, this is just wave mechanics.

We need to get the expression among the time.

$$\psi(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2^{\frac{3}{4}} \pi^{\frac{1}{4}} \sigma^{\frac{1}{2}} e^{-\sigma^2 k^2} e^{ikx} e^{-i\frac{\hbar k^2}{2m}t} dk \quad (2.32)$$

This integral is not easy to resolve so we need to work the exponent.

$$\begin{aligned} & -\sigma^2 k^2 - i\frac{\hbar k^2}{2m}t + ikx = \\ & = -(\sigma^2 + \frac{i\hbar t}{2m})k^2 + ikx = \\ & = -(\sigma^2 + \frac{i\hbar t}{2m})(k^2 - \frac{ikx}{\sigma^2 + \frac{i\hbar t}{2m}}) = \\ & = -(\sigma^2 + \frac{i\hbar t}{2m})(k - \frac{ix}{2(\sigma^2 + \frac{i\hbar t}{2m})})^2 + \frac{x^2}{4(\sigma^2 + \frac{i\hbar t}{2m})^2} \end{aligned} \quad (2.33)$$

We get a similar exponent to the ones we work with where $\alpha = -(\sigma^2 + \frac{i\hbar t}{2m})$ and my variable is $\tau = k - \frac{ix}{2(\sigma^2 + \frac{i\hbar t}{2m})}$

Using this to resolve ?? we can get a final expression for $\psi(x, t)$.

$$\begin{aligned} \psi(x, t) &= \frac{1}{2\pi} 2^{\frac{3}{4}} \pi^{\frac{1}{4}} \sigma^{\frac{1}{2}} \frac{\pi^{\frac{1}{2}}}{(\sigma^2 + \frac{i\hbar t}{2m})^{\frac{1}{2}}} e^{\frac{-x^2}{4(\sigma^2 + \frac{i\hbar t}{2m})}} = \\ &= \left(\frac{\sigma^2}{2\pi(\sigma^2 + \frac{i\hbar t}{2m})^2} \right)^{\frac{1}{4}} e^{-\frac{x^2}{4(\sigma^2 + \frac{i\hbar t}{2m})}} \end{aligned} \quad (2.34)$$

We can calculate $P(x, t)$ knowing $\psi(x, t)$.

$$\begin{aligned} P(x, t) &= \frac{\sigma}{\sqrt{2\pi}(\sigma^4 + \frac{\hbar^2 t^2}{4m^2})^{\frac{1}{2}}} e^{\frac{-x^2}{4} \left(\frac{1}{\sigma^2 + \frac{i\hbar t}{2m}} + \frac{1}{\sigma^2 - \frac{i\hbar t}{2m}} \right)} = \\ &= \frac{1}{\sqrt{2\pi\sigma^2(1 + \frac{\hbar^2 t^2}{4m^2\sigma^4})}} e^{\frac{-x^2}{2\sigma^2} \left(\frac{1}{1 + \frac{\hbar^2 t^2}{4m^2\sigma^4}} \right)} \end{aligned} \quad (2.35)$$

We can get the final result of $P(x, t)$ from this equations.

$$P(x, t) = \frac{1}{\sqrt{2\pi\sigma^2(t)}} e^{-\frac{x^2}{2\sigma^2(t)}} \quad (2.36)$$

$$\sigma(t) = \sigma \sqrt{1 + \frac{\hbar^2 t^2}{4m^2\sigma^4}} \quad (2.37)$$

We define the characteristic time (τ) as a property of the experiment itself.

$$\tau = \frac{m\sigma^2}{\hbar} \quad (2.38)$$

2.7 Examples

As we did in the first chapter we want to get some real values for this equations. We will find and plot the $P(x,t)$ and $\sigma(t)$ for 3 problems: the earth, the electron and a neutrino.

Earth problem: We want to measure the position of the planet earth. Our data for this problem is:

- ▶ $m = 5.97 \cdot 10^{24} \text{ kg}$
- ▶ $\sigma_0 = 10\%$
- ▶ $\hbar = 1.05 \cdot 10^{-34} \text{ J/s}$

We want to know the characteristic time first.

$$\tau = \frac{5.97 \cdot 10^{24} \cdot 0.1^2}{1.05 \cdot 10^{-34}} = 5.69 \cdot 10^{56} \text{ s} \quad (2.39)$$

We can plot now in steps of τ the function $\sigma(t)$.

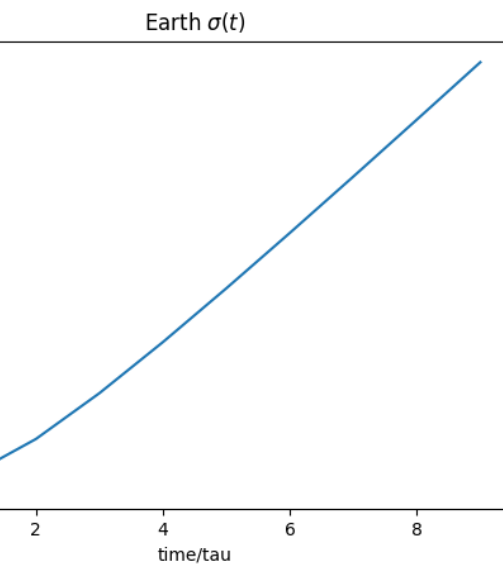


Figure 2.4: Function $\sigma(t)$ for the Earth experiment

Figure 2.5: Gif for $P(x,t)$ of the Earth experiment

Electron problem: We want to measure the position of an electron. Our data for this problem is:

- ▶ $m = 9.11 \cdot 10^{-31} \text{ kg}$
- ▶ $\sigma_0 = 10\%$
- ▶ $\hbar = 1.05 \cdot 10^{-34} \text{ J/s}$

We want to know the characteristic time first.

$$\tau = \frac{9.11 \cdot 10^{-31} \cdot 0.1^2}{1.05 \cdot 10^{-34}} = 86.76 \text{ s} \quad (2.40)$$

We can plot now in steps of τ the function $\sigma(t)$.

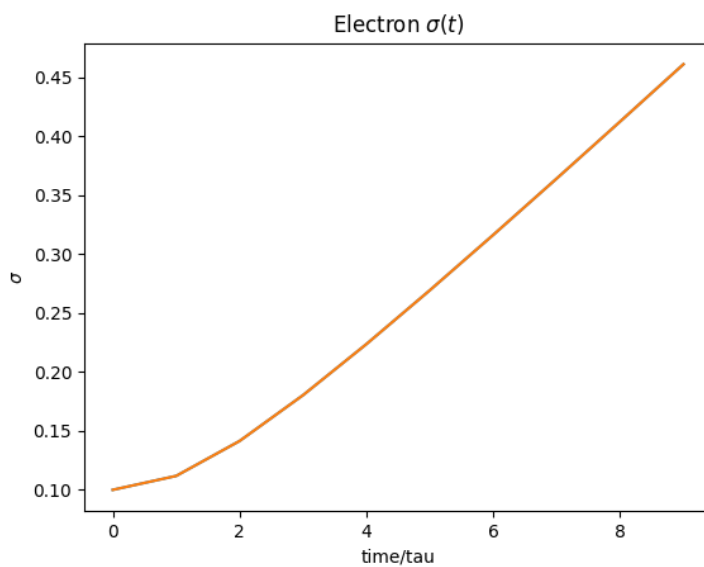


Figure 2.6: Function $\sigma(t)$ for the Electron experiment

Figure 2.7: Gif for $P(x,t)$ of the Electron experiment

At a first view the figures for the Earth and the Electron seems similar but we have to think that we are using the characteristic time as a step so the scale is completely different, while the electron functions evolve significantly in less than 100 seconds, the Earth function is changing in an order of 10^{56} seconds (MORE THAN THE AGE OF THE UNIVERSE!!).

Neutron problem: In this experiment we are going to measure the position of a neutron before it decays. A neutron has an average life time of 879 seconds. In this case because we know the scale of time we want to work with we won't calculate the characteristic time now. The data for this experiment is:

- ▶ $m = 1.67 \cdot 10^{-27} kg$
- ▶ $\sigma_0 = 10\%$
- ▶ $\hbar = 1.05 \cdot 10^{-34} J/s$

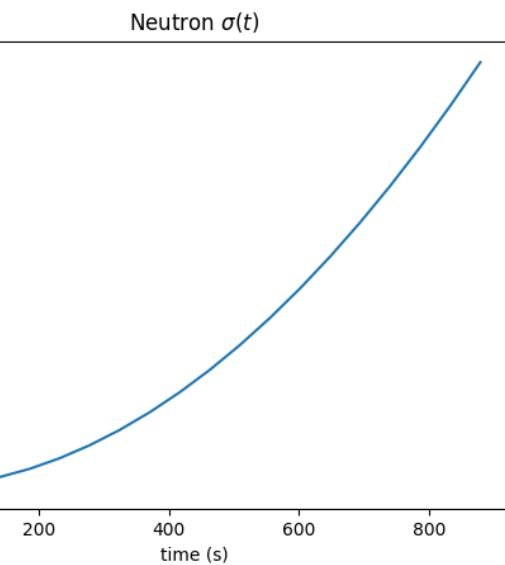


Figure 2.8: Function $\sigma(t)$ for the Neutron decay

Figure 2.9: Gif for $P(x,t)$ of the Neutron decay

In this case we can not appreciate a change in this scale of time. The neutron decays too fast to get an important change in his intensity. But how fast, is too fast? We can calculate the characteristic time to answer this question.

$$\tau = \frac{1.67 \cdot 10^{-27} 0.1^2}{1.05 \cdot 10^{-34}} = 1.59 \cdot 10^5 s \quad (2.41)$$

We can appreciate in this equation that the characteristic time is almost 10^3 times bigger than the decay time, that is why we don't get to see a change in the density function among time.

Extra problem: We want to do one more experiment. We can use this functions to know how much it would change the measure of a human. The data is:

- $m = 70 kg$
- $\sigma_0 = 10\%$
- $\hbar = 1.05 \cdot 10^{-34} J/s$

We want to know the characteristic time first.

$$\tau = \frac{70 \cdot 0.1^2}{1.05 \cdot 10^{-34}} = 6.67 \cdot 10^{33} s \quad (2.42)$$

We can plot now in steps of τ the function $\sigma(t)$.

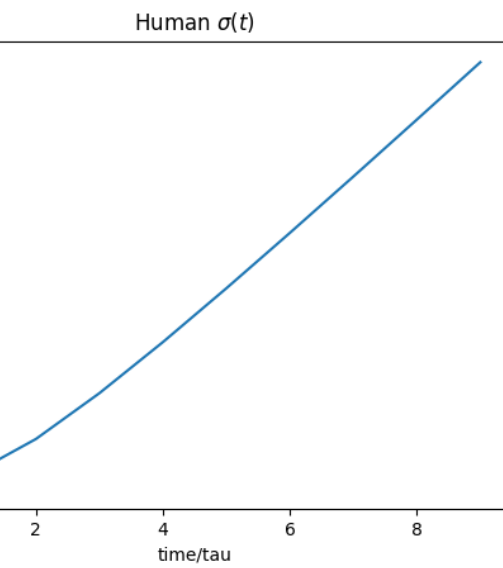


Figure 2.10: Function $\sigma(t)$ for the Electron experiment

Figure 2.11: Gif for $P(x,t)$ of the Electron experiment

Again we get something similar to the Earth and the Electron problems, but this time our step is an order of $10^{33}s$, this is still larger than the age of the universe.

A free particle is an easy example but we will start with something more difficult in the next chapter.

In this chapter we will find the intensity and the ψ function of a particle under a potential.

3.1 Potential in 3 dimensions

First, we need to define our energy and our equations in 3D, this equations can be obtained from the second chapter generalizing to 3 dimensions.

$$\frac{p_x^2 + p_y^2 + p_z^2}{2m} + V(x, y, z) = E \quad (3.1)$$

$$\left[-\frac{\hbar^2}{2m} \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] + V(x, y, z) \right] \psi = i\hbar \frac{\partial}{\partial t} \psi \quad (3.2)$$

Where ψ is a function of space and time, i.e. $\psi = \psi(x, y, z, t)$. Before doing some operations with this equations we want to define a boundary for the intensity function. When we talk about this function in the second chapter we define a boundary in ??, we need to do the same but in 3 dimensions.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(x, y, z) dx dy dz = 1 \quad (3.3)$$

For this equation to be satisfied we need to achieve this properties:

- ▶ $\lim_{x \rightarrow \infty} P(x, y, z) = 0 \quad \forall y, z$
- ▶ $\lim_{y \rightarrow \infty} P(x, y, z) = 0 \quad \forall x, z$
- ▶ $\lim_{z \rightarrow \infty} P(x, y, z) = 0 \quad \forall x, y$

The intensity is define as $P(x, y, z, t) = \psi \psi^*$. We have defined the intensity and we can continue now with Equation ???. To resolve this we will use the equation and it's conjugate and we will multiply for ψ or ψ^* .

$$\begin{aligned} \psi^* \left[-\frac{\hbar^2}{2m} \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] + V \right] \psi &= i\psi^* \hbar \frac{\partial}{\partial t} \psi \\ \psi \left[-\frac{\hbar^2}{2m} \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] + V \right] \psi^* &= -i\psi \hbar \frac{\partial}{\partial t} \psi^* \end{aligned} \quad (3.4)$$

To continue we will subtract the first equation with the second one.

$$\begin{aligned} \frac{\hbar^2}{2m} \left[\psi \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] \psi^* - \psi^* \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] \psi \right] &= i\hbar \left[\psi \frac{\partial}{\partial t} \psi^* + \psi^* \frac{\partial}{\partial t} \psi \right] \\ \frac{\hbar^2}{2m} \left[\frac{\partial}{\partial x} \left(\psi \frac{\partial}{\partial x} \psi^* - \psi^* \frac{\partial}{\partial x} \psi \right) + \frac{\partial}{\partial y} \left(\psi \frac{\partial}{\partial y} \psi^* - \psi^* \frac{\partial}{\partial y} \psi \right) + \right. \\ &\left. + \frac{\partial}{\partial z} \left(\psi \frac{\partial}{\partial z} \psi^* - \psi^* \frac{\partial}{\partial z} \psi \right) \right] = i\hbar \left[\psi \frac{\partial}{\partial t} \psi^* + \psi^* \frac{\partial}{\partial t} \psi \right] \end{aligned} \quad (3.5)$$

If we integrate both sides we get that the left side is 0.

$$\begin{aligned} i\hbar \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \left[\psi \frac{\partial}{\partial t} \psi^* + \psi^* \frac{\partial}{\partial t} \psi \right] &= 0 \\ \frac{d}{dt} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \psi \psi^* &= 0 \end{aligned} \quad (3.6)$$

This equation tell us that the total intensity must be constant among the time, so now we know the conditions of the intensity for space from ?? and for time.

3.2 Discontinuous Functions



Figure 3.1: Discontinuous function with a finite jump in $x = 0$

We will work in one dimension only for now on to get things easier. Some potentials are discontinuous function, one of the most common example is the square well potential. We want to study this discontinuous functions and for that we will recover the delta function.

We want to prove that:

$$\frac{d\theta}{dx} = C\delta(x) \quad (3.7)$$

This equation is true if:

$$\int_{-\infty}^{\infty} f(x) \frac{d\theta}{dx} dx = f(0)C \quad (3.8)$$

Because of the definition of the delta function. To prove this and also to

get the value of C we need to operate carefully this integral.

$$\begin{aligned}
 & \int_{-\infty}^{\infty} \left(\frac{d}{dx} [f(x)\theta(x)] - \theta(x) \frac{df}{dx} \right) dx = \\
 & = [f(x)\theta(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d\theta}{dx} \theta(x) dx = \\
 & = f(\infty)\theta_+ + f(-\infty)\theta_- + \theta_- \int_{-\infty}^0 \frac{df}{dx} dx - \theta_+ \int_0^{\infty} \frac{df}{dx} dx = \\
 & = f(\infty)\theta_+ + f(-\infty)\theta_- + \theta_- (f(0) - f(-\infty)) - \theta_+ (f(\infty) - f(0)) = \\
 & = f(0)[\theta_- + \theta_+]
 \end{aligned} \tag{3.9}$$

We prove that ?? is true and also that C is the "jump" distance in the discontinuous function.

3.3 Square Potential Well

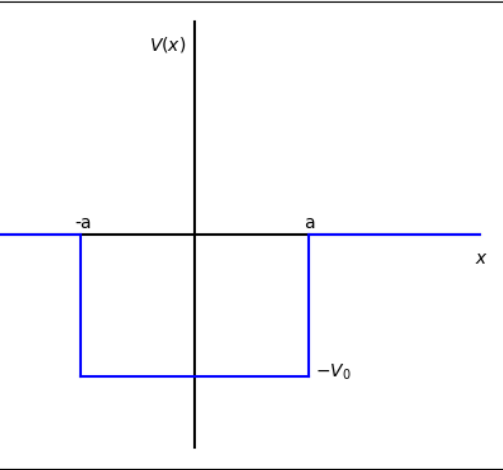


Figure 3.2: Square Potential Well

We have a potential as the one in Figure ?? and we want to study the intensity and the ψ function of a particle among space and time. First i have to define the potential.

$$f(x) = \frac{-dV}{dx} \tag{3.10}$$

In particle mechanics the problem will be solved with the equation of the energy.

$$E = \frac{1}{2}m \left(\frac{dx}{dt} \right)^2 + V(x) = constant \tag{3.11}$$

In wave mechanics is more complex we will get the Schrödinger Wave Equation for one dimension.

$$i\hbar \frac{\partial \psi(x, t)}{\partial t} = \frac{-\hbar^2}{2m} \frac{\partial^2 \psi(x, t)}{\partial x^2} + V(x)\psi(x, t) \quad (3.12)$$

Let's consider one solution to this equation.

$$\psi(x, t) = \phi(x)e^{-i\frac{E}{\hbar}t} \quad (3.13)$$

Where $\phi(x)$ is a real function, if we remember we solved $\phi(x)$ for the free particle problem and it was a exponential.

$$\phi(x) = e^{ikx} \quad (3.14)$$

This function does not fulfill equation ??.

$$\int_{-\infty}^{\infty} \phi^*(x)\phi(x)dx = \infty \quad (3.15)$$

This means that we only have solutions if $V(x) \neq 0$.

Using our knowledge on the previous section we can get an expression for $f(x)$.

$$f(x) = \frac{-dV}{dx} = -[(-V_0 - 0)\delta(-a) + (0 - (-V_0))\delta(a)] = V_0\delta(-a) - V_0\delta(a) \quad (3.16)$$

We want to redefine the energy, so we do not get stuck with the signs.

Energy = - E where $E > 0$

For a bound state $0 < E < V_0$ must be true. This is a bound state because is similar to the first chapter example when we said that a negative energy implies a closed orbit because the particle must be between the r_{min} and the r_{max} for the energy to be constant, this is exactly the same, but because is in one dimension the particle should bounce between the potential well in a classical approach.

We need now to rewrite the wave equation and the solution.

$$\psi(x, t) = \phi(x)e^{i\frac{E}{\hbar}t} \quad (3.17)$$

$$-E\phi(x) = \frac{-\hbar^2}{2m} \frac{d^2\phi(x)}{dx^2} + V(x)\phi(x) \quad (3.18)$$

So the equation for the three different parts of the potential are:

$$\begin{aligned}\frac{d^2\phi}{dx^2} &= \frac{2mE}{\hbar^2}\phi & \text{where } x < -a \\ \frac{d^2\phi}{dx^2} &= -\frac{2m(V_0 - E)}{\hbar^2}\phi & \text{where } -a < x < a \\ \frac{d^2\phi}{dx^2} &= \frac{2mE}{\hbar^2}\phi & \text{where } x > a\end{aligned}\quad (3.19)$$

We need to resolve this equations. The solution to this equations are exponential for the first and the third intervals and an addition of a sine and a cosine function.

For the exponential functions we are going to define the exponent as α . Because the differential equation implies a second derivative this means that the exponential can be $-\alpha$ or $+\alpha$, this sign is going to depend on the equation ???. We are also going to define β as the argument of the sine and cosine functions.

$$\begin{aligned}\alpha &= a\sqrt{\frac{2mE}{\hbar^2}} \\ \beta &= a\sqrt{\frac{2m(V_0 - E)}{\hbar^2}} \\ \gamma^2 &= \beta^2 + \alpha^2 = a^2\frac{2mV_0}{\hbar^2}\end{aligned}\quad (3.20)$$

We have also defined gamma, that is a constant of the problem (does not depend on the energy) and is going to be useful to determine the values of α and β after we solve the equations.

We can solve now our differential equations.

$$\begin{aligned}Ae^{\frac{-\alpha}{a}x} & & x > a \\ \psi(x) &= B\sin\left(\frac{\beta}{a}x\right) + C\cos\left(\frac{\beta}{a}x\right) & -a < x < a \\ De^{\frac{\alpha}{a}x} & & x < -a\end{aligned}\quad (3.21)$$

Now we need to use the boundary conditions. We already know that ??? must be true and we will use that condition later, but we also know that ϕ must be continuous because we define it as a continuous function and the first derivative must be continuous also because if not, then we will have Dirac's delta functions in the wave equation that can not cancel with anything. So we have two boundary conditions that will became 4 equations. But before we will do some changes in ??? to get

easier equations later.

$$\begin{aligned} Ae^{\frac{-\alpha}{a}x} & \quad x > a \\ \psi(x) = Be^{-\alpha} \sin\left(\frac{\beta}{a}x\right) + Ce^{-\alpha} \cos\left(\frac{\beta}{a}x\right) & \quad -a < x < a \\ De^{\frac{\alpha}{a}x} & \quad x < -a \end{aligned} \quad (3.22)$$

The continuity conditions can be set now.

$$\begin{aligned} D &= -B\sin(\beta) + C\cos(\beta) \\ A &= B\sin(\beta) + C\cos(\beta) \\ D\alpha &= B\beta\cos(\beta) + C\beta\sin(\beta) \\ -A\alpha &= B\beta\cos(\beta) - C\beta\sin(\beta) \end{aligned} \quad (3.23)$$

This is a homogeneous system, so one of the solutions is $A=B=C=D=0$ but this solution is not allowed. If we solved for this we will get two equations that actually imply 4 equations.

$$\begin{aligned} C(\alpha\cos(\beta) - \beta\sin(\beta)) &= 0 \\ B(\alpha\sin(\beta) + \beta\cos(\beta)) &= 0 \end{aligned} \quad (3.24)$$

If $B = C = 0$, A and D will be 0 also and we already said that is not an option, so we have three possibilities. Let's try doing both parenthesis equal to 0 and add them multiplying by some factor.

$$\begin{aligned} \sin(\beta)(\alpha\sin(\beta) + \beta\cos(\beta)) + \cos(\beta)(\alpha\cos(\beta) - \beta\sin(\beta)) &= 0 \\ \alpha\sin^2(\beta) + \alpha\cos^2(\beta) &= 0 \end{aligned} \quad (3.25)$$

With our choice the solution is $\alpha = 0$ but α can not be 0, so we only have two choices both of them are right.

If we take a look at this dependency between α and β and we include our gamma equation to the chart we would see something interesting.

$$\begin{aligned} \alpha &= -\beta \cot(\beta) \\ \alpha &= \beta \tan(\beta) \\ \gamma^2 &= \alpha^2 + \beta^2 \end{aligned} \quad (3.26)$$

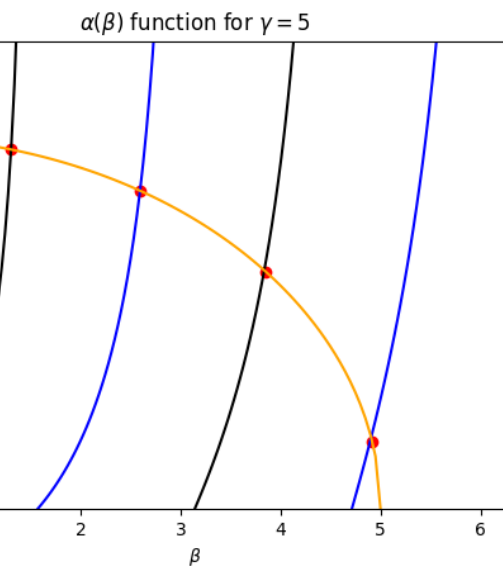


Figure 3.3: α as a function of β for $\gamma = 5$

This proves that the energy is quantized because only some values for α (that is a function of Energy) are possible.

And the solutions are:

Solution for $C = 0$

$$\begin{aligned} A &= B \sin(\beta) \\ B &= B \\ C &= 0 \\ D &= -B \sin(\beta) \end{aligned} \quad (3.27)$$

Solution for $B = 0$

$$\begin{aligned} A &= C \cos(\beta) \\ B &= 0 \\ C &= C \\ D &= C \cos(\beta) \end{aligned} \quad (3.28)$$

We still have one variable left to resolve, to do this we will need to use ???. We will resolve only for the case $B = 0$, the other one will turn the

same.

$$\begin{aligned}
 \int_{-\infty}^{\infty} \phi^2(x) dx &= \\
 &= C^2 \left[\cos^2(\beta) \int_{-\infty}^{-a} e^{\frac{2\alpha}{a}x} dx + e^{-2\alpha} \int_{-a}^a \cos^2\left(\frac{\beta}{a}x\right) dx + \cos^2(\beta) \int_a^{\infty} e^{-\frac{2\alpha}{a}x} dx \right] = \\
 &= C^2 \left[\cos^2(\beta) \frac{a}{2\alpha} \left[e^{\frac{2\alpha}{a}x} \right]_{-\infty}^{-a} + \frac{e^{-2\alpha}}{2} \left[\frac{a}{2\beta} \sin\left(\frac{2\beta}{a}x\right) + x \right]_{-a}^a - \cos^2(\beta) \frac{a}{2\alpha} \left[e^{-\frac{2\alpha}{a}x} \right]_a^{\infty} \right] = \\
 &= C^2 a e^{-2\alpha} \left[\frac{\cos^2(\beta)}{\beta \tan(\beta)} + \frac{\sin(2\beta)}{2\beta} + 1 \right] = \\
 &= C^2 a e^{-2\alpha} \left[\frac{1}{\alpha} + 1 \right] = 1
 \end{aligned} \tag{3.29}$$

Now we need to resolve for C in the last step.

$$C^2 = \frac{\alpha e^{2\alpha}}{a(1 + \alpha)} \tag{3.30}$$

If $C=0$ then, B^2 is equal to the expression above.

Now we can solve for everything in the problem for a given gamma.

3.4 Examples

We will plot some of the solutions for different gamma and for the largest gamma we will try to approach and explain a classical behaviour.

First example: In this case we will took gamma = 10, and plot two of the solutions, one for $B=0$ and the other one for $C=0$.

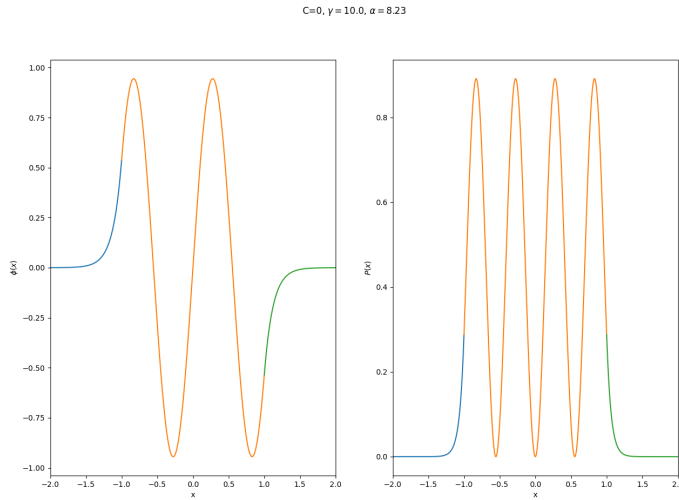


Figure 3.4: $\phi(x)$ and $P(x)$ functions for gamma = 10

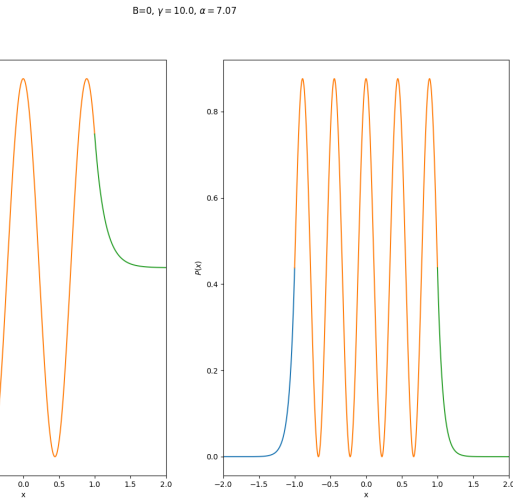


Figure 3.5: $\phi(x)$ and $P(x)$ functions for $\gamma = 10$

Now we will try to achieve a similar behaviour to the well-known particle approach, where all the probability is inside the square root potential and all of it is equally likely to have the particle. We can achieve this finding the $P(x)$ function for a bigger γ and a small α .

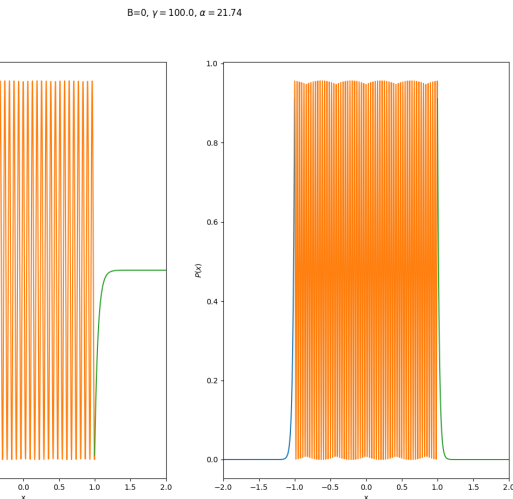


Figure 3.6: $\phi(x)$ and $P(x)$ functions for $\gamma = 100$

In Figure ?? we can see that we get what we expect for a particle behaviour but we still have that quantum behaviour because the integral outside of the square potential is not zero as it should be in the particle behaviour.

Delta function potential

4

We want to understand solids and their properties, for example why copper is a conductor and wood is not? For this and other questions we will propose our next problem, the delta function potential problem.

4.1 Definition of the potential

We define our potential this time as

$$V(x) = -2C\delta(x) \quad (4.1)$$

Where $C = V_0a$.

Because we have delta functions in our equations we can not call them normal equations so we will use quotes to differentiate them from real equations.

$$'' + \frac{\hbar^2}{2m} \frac{\partial^2 \phi}{\partial x^2} + 2C\delta(x)\phi(x) = +E\phi(x)'' \quad (4.2)$$

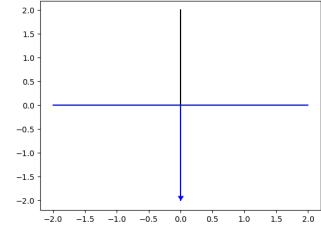


Figure 4.1: Delta function potential

4.2 Solution

The solution for this equation is:

$$\psi(x, t) = e^{i\frac{E}{\hbar}t} \phi(x) \quad (4.3)$$

Now we need to solve our equation for $\phi(x)$.

$$\begin{aligned} '' \frac{\partial^2 \phi}{\partial x^2} + \frac{4mC}{\hbar^2} \delta(x)\phi(x) &= \frac{2mE}{\hbar^2} \phi(x)'' \\ '' \frac{\partial^2 \phi}{\partial x^2} + \frac{4mV_0a}{\hbar^2} \delta(x)\phi(x) &= \frac{2mE}{\hbar^2} \phi(x)'' \end{aligned} \quad (4.4)$$

We need to define two variables to resolve this equations, α and g (This variables are completely different to the variables in the previous chapter).

$$\begin{aligned} g &= \frac{2mV_0a}{\hbar^2} \\ \alpha &= \sqrt{\frac{2mE}{\hbar^2}} \end{aligned} \quad (4.5)$$

As we did in the previous chapter we have to think about the continuity in $\phi(x)$. Can $\phi(x)$ be discontinuous? The answer is no. $\phi(x)$ is continuous because if it weren't continuous the second derivative would look like the derivative of a delta function and the equations don't make sense with that assumption. However, in this case the first derivative of $\phi(x)$ must be discontinuous because we need the second derivative to go as a delta function.

We solve for $\phi(x) \neq 0$.

$$\frac{\partial^2 \phi(x)}{\partial x^2} = \alpha^2 \phi(x) \quad (4.6)$$

The solution to this differential equation is:

$$\phi(x) = \begin{cases} Ae^{\alpha x} & \text{where } x \leq 0 \\ Ae^{-\alpha x} & \text{where } x \geq 0 \end{cases} \quad (4.7)$$

And the derivatives are going to be:

$$\frac{\partial \phi(x)}{\partial x} = \begin{cases} A\alpha e^{\alpha x} & \text{where } x \leq 0 \\ -A\alpha e^{-\alpha x} & \text{where } x \geq 0 \end{cases} \quad (4.8)$$

$$\frac{\partial^2 \phi(x)}{\partial x^2} = \begin{cases} A\alpha^2 e^{\alpha x} & \text{where } x \leq 0 \\ A\alpha^2 e^{-\alpha x} & \text{where } x \geq 0 \end{cases} + B\delta(x) \quad (4.9)$$

To solve for A we are going to use the continuity conditions we mention before and the fundamental theorem of calculus.

$$\int_{-\infty}^{\infty} \frac{\partial^2 \phi(x)}{\partial x^2} dx = \left[\frac{\partial \phi}{\partial x} \right]_{x=-\infty} - \left[\frac{\partial \phi}{\partial x} \right]_{x=\infty}$$

$$A\alpha^2 \int_{-\infty}^0 e^{\alpha x} dx + A\alpha^2 \int_0^{\infty} e^{-\alpha x} dx + B \int_{-\infty}^{\infty} \delta(x) dx = 0 \quad (4.10)$$

$$A\alpha + A\alpha + B = 0$$

$$B = -2A\alpha$$

We want to integrate ?? for $x=0$ to remove the quotes.

$$A\alpha^2 \int_{-\infty}^0 e^{\alpha x} dx + A\alpha^2 \int_0^{\infty} e^{-\alpha x} dx + = \int_{-\infty}^{\infty} \frac{\partial^2 \phi(x)}{\partial x^2} dx + 2g \int_{-\infty}^{\infty} \delta(x) \phi(x) dx$$

$$[\alpha Ae^{\alpha x}]_{-\infty}^0 - [\alpha Ae^{-\alpha x}]_0^{\infty} = 0 + 2g\phi(0)$$

$$2\alpha A = 2gA$$

$$\alpha = g$$

(4.11)

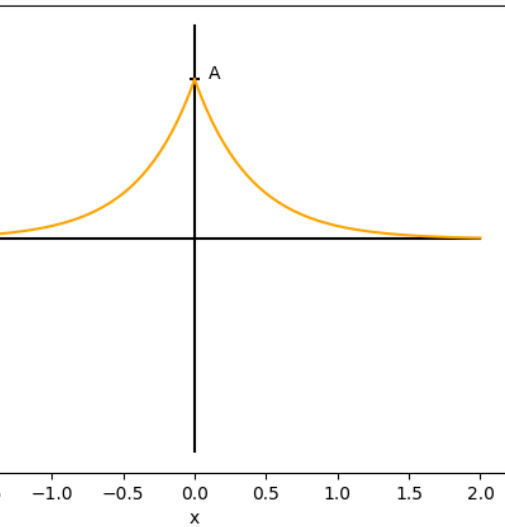


Figure 4.2: Solution for $g=2.5$ of $\phi(x)$ for the delta function potential.

In this case we only have one possible value for α so only one Energy level is possible. The solution to the problem is:

$$\phi(x) = \begin{cases} Ae^{gx} & x < 0 \\ Ae^{-gx} & x > 0 \end{cases} \quad (4.12)$$

$$P(x) = \begin{cases} A^2 e^{2gx} & x < 0 \\ A^2 e^{-2gx} & x > 0 \end{cases} \quad (4.13)$$

To solve for A we need to use the property from ??.

$$P(x) = \begin{cases} A^2 e^{2gx} & x < 0 \\ A^2 e^{-2gx} & x > 0 \end{cases} \quad (4.14)$$

$$\int_{-\infty}^{\infty} P(x) dx = 1$$

$$\frac{A^2}{g} = 1 \quad (4.15)$$

$$A^2 = g$$

We have finally solved the problem and the solution is:

$$\phi(x) = \begin{cases} \sqrt{g} e^{gx} & x < 0 \\ \sqrt{g} e^{-gx} & x > 0 \end{cases} \quad (4.16)$$

We will not do examples using this model because is not that usefull, but it will help us understand the next problem better.

Multiple potentials

5

Until this moment we have only work with 1 potential well but if our goal is to understand solids and their properties we need to figure out how our quantum world works with multiple potential.

5.1 Definition

In this problem we will have N delta function potentials, as the one in the previous chapter, separated by a distance d, with a fix value for g of g=1. The potential can be defined as:

$$V(x) = -2V_0a \left[\sum_{n=1}^N \delta(x - nd) \right] \quad (5.1)$$

Using ?? we get the wave equation for this problem.

$$\alpha^2 \phi(x) = \frac{\partial^2 \phi(x)}{\partial x^2} + 2g \left[\sum_{n=1}^N \delta(x - nd) \right] \phi(x) \quad (5.2)$$

Outside of the potentials the equation to solve will be the same as ??, our goal is going to be trying to find the relation between the A and B coefficients at both sides of a potential as we can see in Figure ??

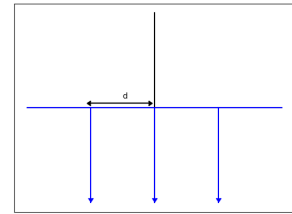


Figure 5.1: Multiple delta function potentials problem.

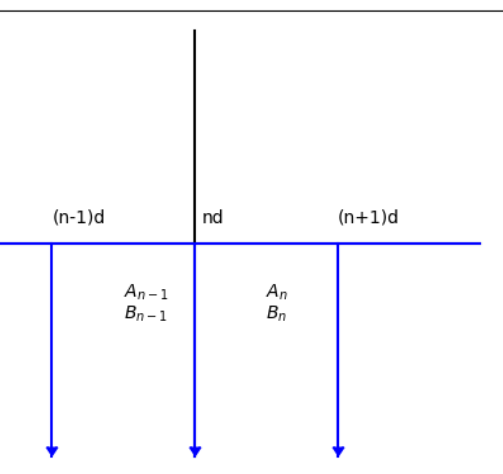


Figure 5.2: Coefficients outside of the potentials near $x=nd$.

5.2 Solution

Our potential at $x = nd$ will look like Figure ?? and the solution at the left and right side of the potential in $x=nd$ will be:

$$\begin{aligned}\phi(x) &= A_{n-1}e^{-\alpha(x-[nd-\frac{d}{2}])} + B_{n-1}e^{\alpha(x-[nd-\frac{d}{2}])} & x < nd \\ \phi(x) &= A_n e^{-\alpha(x-[nd+\frac{d}{2}])} + B_n e^{\alpha(x-[nd+\frac{d}{2}])} & x > nd\end{aligned}\quad (5.3)$$

First, if we look at the limits when $n=1$ and $n=N$ we get the solutions from the previous chapter.

$$\begin{aligned}A_0 &= 0 \\ B_N &= 0\end{aligned}\quad (5.4)$$

We will use the continuity of the function and its derivatives, as we already explain in Chapter 4, to relate the A and B coefficients.

The first condition is that $\phi(x)$ is continuous. To simplify we will use $v = e^{-\alpha\frac{d}{2}}$.

$$A_{n-1}v + \frac{B_{n-1}}{v} = \frac{A_n}{v} + B_nv \quad (5.5)$$

The second condition is our relation between the derivatives in $x = nd$ shown in ??

$$\begin{aligned}[-\alpha A_{n-1}v + \alpha \frac{B_{n-1}}{v}] - [-\alpha \frac{A_n}{v} + \alpha B_nv] &= 2[A_{n-1}v + \frac{B_{n-1}}{v}] \\ A_{n-1}(-\alpha v - 2v) + B_{n-1}(\frac{\alpha}{v} - \frac{2}{v}) &= -\alpha A_n \frac{1}{v} + \alpha B_nv \\ -A_n \frac{1}{v} + B_nv &= -vA_{n-1}(1 + \frac{2}{\alpha}) + \frac{1}{v}B_{n-1}(1 - \frac{2}{\alpha})\end{aligned}\quad (5.6)$$

If we add and substract both conditions we will get the next equations:

$$\begin{aligned}B_n &= -\frac{1}{\alpha}A_{n-1} + \frac{1}{v^2}(1 - \frac{1}{\alpha})B_{n-1} \\ A_n &= v^2(1 + \frac{1}{\alpha})A_{n-1} + \frac{1}{\alpha}B_{n-1}\end{aligned}\quad (5.7)$$

We want to show this like a matrix.

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{bmatrix} v^2(1 + \frac{1}{\alpha}) & \frac{1}{\alpha} \\ -\frac{1}{v} & \frac{1}{v^2}(1 - \frac{1}{\alpha}) \end{bmatrix} \begin{pmatrix} A_{n-1} \\ B_{n-1} \end{pmatrix} \quad (5.8)$$

We will called this matrix T. Now we can relate any index with each other because T does not depend on n, i.e. $T = T(\alpha, d)$. Now we will

find the solutions for alpha using the relation between $n=1$ and $n=N$ coefficients.

$$\begin{pmatrix} A_N \\ B_N \end{pmatrix} = [T]^N \begin{pmatrix} A_0 \\ B_0 \end{pmatrix} \quad (5.9)$$

To solve for alpha we will use ??.

$$\begin{pmatrix} A_N \\ 0 \end{pmatrix} = \begin{bmatrix} (T^N)_{11} & (T^N)_{12} \\ (T^N)_{21} & (T^N)_{22} \end{bmatrix} \begin{pmatrix} 0 \\ B_0 \end{pmatrix} \quad (5.10)$$

For this equations to be true we get the condition, $(T^N)_{22} = 0$. This will give the solutions for alpha and will depend only on d.

5.3 Solutions for $N = 1, 2$

Now that we have the general solution, we can try to get the simplest examples.

For $N = 1$ it will be extremely simple.

$$\frac{1}{v^2} \left(1 - \frac{1}{\alpha}\right) = 0 \quad (5.11)$$

Solving for α we get $\alpha = 1$ that exactly the value for g, this tell us that when $N=1$ we recover the solution from the previous chapter, wich make sense.

For $N = 2$ we need to square the matrix.

$$\begin{aligned} T^2 &= \begin{bmatrix} v^2(1 + \frac{1}{\alpha}) & \frac{1}{\alpha} \\ -\frac{1}{\alpha} & \frac{1}{v^2}(1 - \frac{1}{\alpha}) \end{bmatrix}^2 = \\ &= \begin{bmatrix} v^2(1 + \frac{1}{\alpha}) & \frac{1}{\alpha} \\ -\frac{1}{\alpha} & \frac{1}{v^2}(1 - \frac{1}{\alpha}) \end{bmatrix} \begin{bmatrix} v^2(1 + \frac{1}{\alpha}) & \frac{1}{\alpha} \\ -\frac{1}{\alpha} & \frac{1}{v^2}(1 - \frac{1}{\alpha}) \end{bmatrix} = \\ &= \begin{bmatrix} v^4(1 + \frac{1}{\alpha})^2 - \frac{1}{\alpha^2} & \frac{1}{\alpha}[v^2(1 + \frac{1}{\alpha}) + \frac{1}{v}(1 - \frac{1}{\alpha})] \\ -\frac{1}{\alpha}[v^2(1 + \frac{1}{\alpha}) + \frac{1}{v}(1 - \frac{1}{\alpha})] & -\frac{1}{\alpha^2} + \frac{1}{v^4}(1 - \frac{1}{\alpha})^2 \end{bmatrix} \end{aligned} \quad (5.12)$$

From the expresion of T_{22} we can get the solutions for alpha.

$$\begin{aligned} -\frac{1}{\alpha^2} + \frac{1}{v^4}(1 - \frac{1}{\alpha})^2 &= 0 \\ \pm \frac{1}{\alpha} &= e^{\alpha d} (1 - \frac{1}{\alpha}) \end{aligned} \quad (5.13)$$

We get two different solutions from this equation.

$$\begin{aligned}\alpha &= 1 + e^{-\alpha d} \\ \alpha &= 1 - e^{-\alpha d}\end{aligned}\tag{5.14}$$

If we plot both sides of the equations we can see two different results depending on d . If d is greater than 1 we will get two solutions if d is least than 1 we will only find one solution. This can be observe in Figure ?? and Figure ??.

As N increase with a fix d the number of alphas will increase until the discrete turns into a almost continuous behaviour. With N from 1 to 40 we can see the behaviour for alpha when $d=2$.

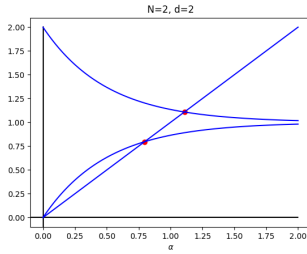


Figure 5.3: Solutions for $(T^N)_{22} = 0$ for $N=2$ and $d=2$

The solutions have been found using numerical analysis.

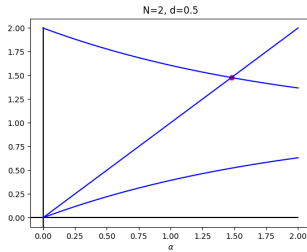


Figure 5.4: Solutions for $(T^N)_{22} = 0$ for $N=2$ and $d=0.5$

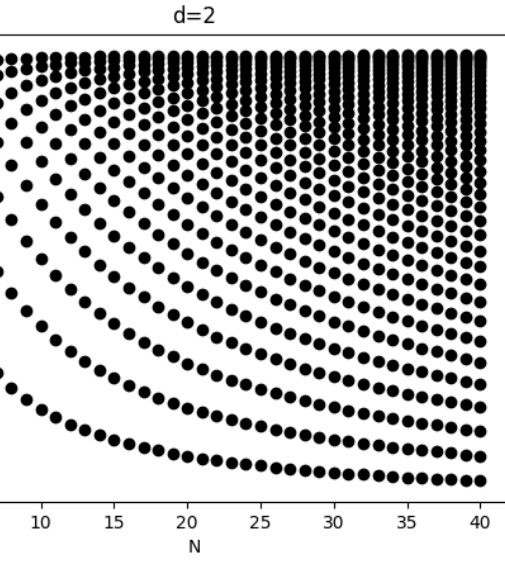


Figure 5.5: alpha as a function of N for $d = 2$.

5.4 Solving for x as a circumference

If we assume x axis is not a straight line, and instead it is inside a circumference of length L , like Figure ??, the math will look better and we will get some interesting new results.

Because we have N potentials separated a distance d , we can say that $L = Nd$. When N is infinite we will have again that L is approximately infinite and therefore the circumference will recover the straight line shape.

One of the consequences of this new approach is that now E is not confined to only negative values. We know get a new wave equation where E is the energy.

$$-\frac{\hbar^2}{2m} \frac{d^2\phi}{dx^2} + V(x)\phi(x) = E\phi(x) \quad (5.15)$$

We have to solve this equation with our boundaries as before, but we have new boundaries as a result of our transformation. We can use the rotation simetry to say that now our function is periodic. This give us new conditions:

$$\begin{aligned} \phi(x + L) &= \phi(x) \\ \frac{d\phi}{dx}(x + L) &= \frac{d\phi}{dx}(x) \\ V(x + L) &= V(x) \end{aligned} \quad (5.16)$$

We want to go even further and solve problems where $V(x)$ is periodic in d instead of L , i.e. $V(x + d) = V(x)$.

In this case we say, and it can be prove, that $\phi(x)$ is quasi-periodic.

$$\phi(x + d) = C\phi(x) \quad (5.17)$$



Figure 5.6: x as a circumference of distance L .

After the N potentials $\phi(x)$ must be the same again according to ??.

$$\phi(x + Nd) = C^N \phi(x) = \phi(x + L) = \phi(x) \quad (5.18)$$

This means that $C^N = 1$ and this lead us to N solutions for C .

$$C = e^{i\frac{2\pi}{N}k} \quad k = 0, 1, 2, \dots, N - 1 \quad (5.19)$$

We will solve the problem between $d/2$ and $-d/2$ as we can see in Figure ??

Outside of the potential the wave equation is:

$$\frac{d^2\phi(x)}{dx^2} = \begin{cases} \alpha^2\phi(x) & E < 0 \\ \beta^2\phi(x) & E > 0 \end{cases} \quad (5.20)$$

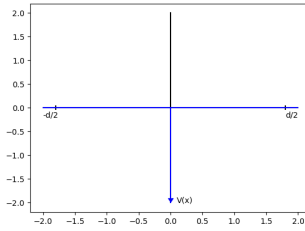


Figure 5.7: Potential between $-d/2$ and $d/2$

APPENDIX

Notation

The next list describes several symbols that will be later used within the body of the document.

- c Speed of light in a vacuum inertial frame
- h Planck constant

Greek Letters with Pronunciations

Character	Name	Character	Name
α	alpha <i>AL-fuh</i>	ν	nu <i>NEW</i>
β	beta <i>BAY-tuh</i>	ξ, Ξ	xi <i>KSIGH</i>
γ, Γ	gamma <i>GAM-muh</i>	\omicron	omicron <i>OM-uh-CRON</i>
δ, Δ	delta <i>DEL-tuh</i>	π, Π	pi <i>PIE</i>
ϵ	epsilon <i>EP-suh-lon</i>	ρ	rho <i>ROW</i>
ζ	zeta <i>ZAY-tuh</i>	σ, Σ	sigma <i>SIG-muh</i>
η	eta <i>AY-tuh</i>	τ	tau <i>TOW (as in cow)</i>
θ, Θ	theta <i>THAY-tuh</i>	υ, Υ	upsilon <i>OOP-suh-LON</i>
ι	iota <i>eye-OH-tuh</i>	ϕ, Φ	phi <i>FEE, or FI (as in hi)</i>
κ	kappa <i>KAP-uh</i>	χ	chi <i>KI (as in hi)</i>
λ, Λ	lambda <i>LAM-duh</i>	ψ, Ψ	psi <i>SIGH, or PSIGH</i>
μ	mu <i>MEW</i>	ω, Ω	omega <i>oh-MAY-guh</i>

Capitals shown are the ones that differ from Roman capitals.

Alphabetical Index

preface, v