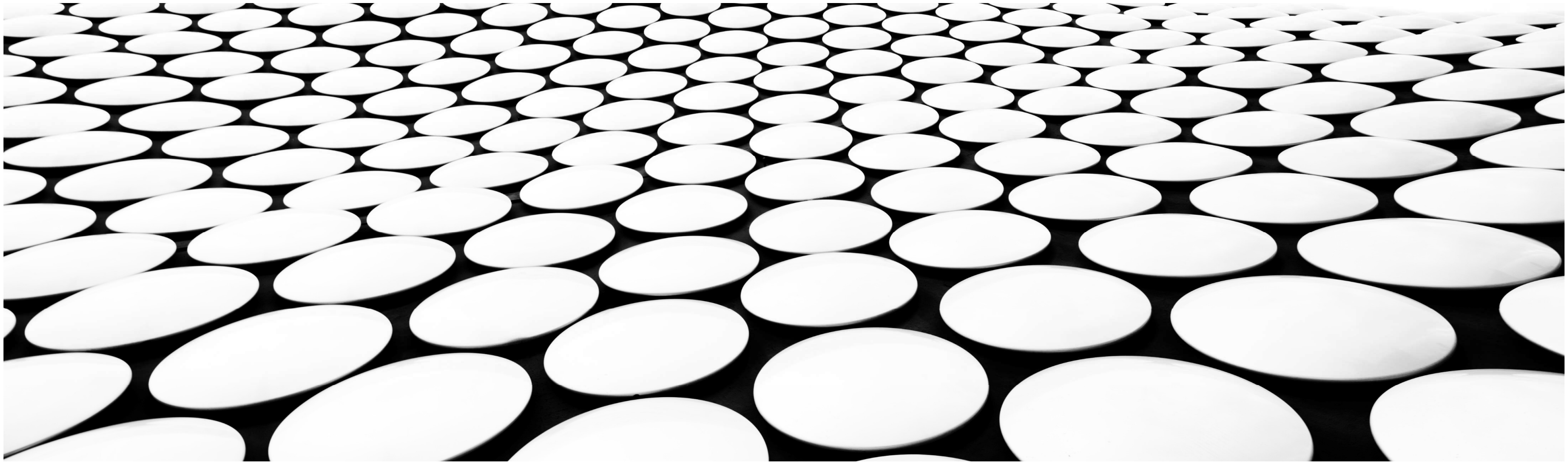
EQUIVALENCE PDA AND CFG

IF 2124 TEORI BAHASA FORMAL OTOMATA



Equivalence of PDA's and CFG's

A language is

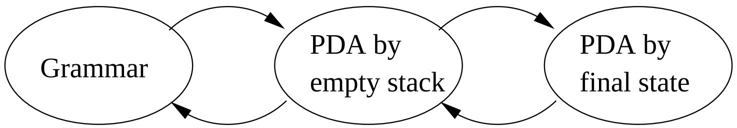
generated by a CFG

if and only if it is

accepted by a PDA by empty stack

if and only if it is

accepted by a PDA by final state



We already know how to go between null stack and final state.

From CFG's to PDA's

Given G, we construct a PDA that simulates $\overset{*}{\underset{lm}{\Rightarrow}}$.

We write left-sentential forms as

$$xA\alpha$$

where A is the leftmost variable in the form. For instance,

$$\underbrace{(a+}_{x}\underbrace{\underbrace{E}_{A}\underbrace{)}_{\alpha}}_{\text{tail}}$$

Let $xA\alpha \underset{lm}{\Rightarrow} x\beta\alpha$. This corresponds to the PDA first having consumed x and having $A\alpha$ on the stack, and then on ϵ it pops A and pushes β.

More fomally, let y, s.t. $w = xy$. Then the PDA goes non-deterministically from configuration $(q, y, A\alpha)$ to configuration $(q, y, \beta\alpha)$.

At $(q, y, \beta\alpha)$ the PDA behaves as before, unless there are terminals in the prefix of β. In that case, the PDA pops them, provided it can consume matching input.

If all guesses are right, the PDA ends up with empty stack and input.

Formally, let $G = (V, T, Q, S)$ be a CFG. Define P_G as

$$(\{q\}, T, V \cup T, \delta, q, S),$$

where

$$\delta(q, \epsilon, A) = \{(q, \beta) : A \to \beta \in Q\},$$

for $A \in V$, and

$$\delta(q, a, a) = \{(q, \epsilon)\},$$

for $a \in T$.

Example: On blackboard in class.

Theorem 6.13: $N(P_G) = L(G)$.

Proof:

($\supseteq$-*direction*.) Let $w \in L(G)$. Then

$$S = \gamma_1 \underset{lm}{\Rightarrow} \gamma_2 \underset{lm}{\Rightarrow} \cdots \underset{lm}{\Rightarrow} \gamma_n = w$$

Let $\gamma_i = x_i \alpha_i$. We show by induction on i that if

$$S \overset{*}{\underset{lm}{\Rightarrow}} \gamma_i,$$

then

$$(q, w, S) \overset{*}{\vdash} (q, y_i, \alpha_i),$$

where $w = x_i y_i$.

Basis: For $i = 1, \gamma_1 = S$. Thus $x_1 = \epsilon$, and $y_1 = w$. Clearly $(q, w, S) \overset{*}{\vdash} (q, w, S)$.

Induction: IH is $(q, w, S) \overset{*}{\vdash} (q, y_i, \alpha_i)$. We have to show that

$$(q, y_i, \alpha_i) \vdash (q, y_{i+1}, \alpha_{i+1})$$

Now α_i begins with a variable A, and we have the form

$$\underbrace{x_i A \chi}_{\gamma_i} \underset{lm}{\Rightarrow} \underbrace{x_{i+1} \beta \chi}_{\gamma_{i+1}}$$

By IH $A\chi$ is on the stack, and y_i is unconsumed. From the construction of P_G is follows that we can make the move

$$(q, y_i, \chi) \vdash (q, y_i, \beta\chi).$$

If β has a prefix of terminals, we can pop them with matching terminals in a prefix of y_i, ending up in configuration $(q, y_{i+1}, \alpha_{i+1})$, where $\alpha_{i+1} = \beta\chi$, which is the tail of the sentential $x_i \beta \chi = \gamma_{i+1}$.

Finally, since $\gamma_n = w$, we have $\alpha_n = \epsilon$, and $y_n = \epsilon$, and thus $(q, w, S) \overset{*}{\vdash} (q, \epsilon, \epsilon)$, i.e. $w \in N(P_G)$

($\subseteq$-*direction.*) We shall show by an induction on the length of $\overset{*}{\vdash}$, that

($\clubsuit$) If $(q, x, A) \overset{*}{\vdash} (q, \epsilon, \epsilon)$, then $A \overset{*}{\Rightarrow} x$.

Basis: Length 1. Then it must be that $A \to \epsilon$ is in G, and we have $(q, \epsilon) \in \delta(q, \epsilon, A)$. Thus $A \overset{*}{\Rightarrow} \epsilon$.

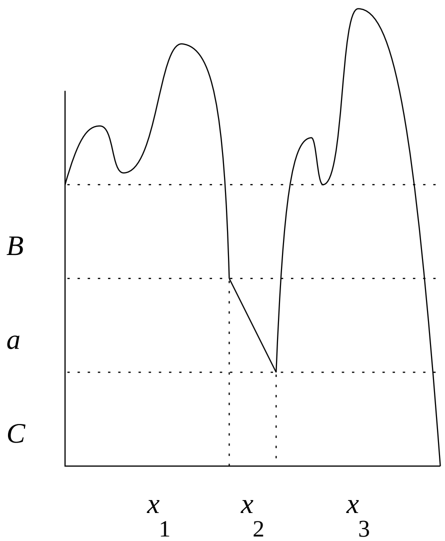
Induction: Length is $n > 1$, and the IH holds for lengths $< n$.

Since A is a variable, we must have

$$(q, x, A) \vdash (q, x, Y_1 Y_2 \cdots Y_k) \vdash \cdots \vdash (q, \epsilon, \epsilon)$$

where $A \to Y_1 Y_2 \cdots Y_k$ is in G.

We can now write x as $x_1 x_2 \cdots x_n$, according to the figure below, where $Y_1 = B, Y_2 = a$, and $Y_3 = C$.



Now we can conclude that

$$(q, x_i x_{i+1} \cdots x_k, Y_i) \overset{*}{\vdash} (q, x_{i+1} \cdots x_k, \epsilon)$$

is less than n steps, for all $i \in \{1, \ldots, k\}$. If Y_i is a variable we have by the IH and Theorem 6.6 that

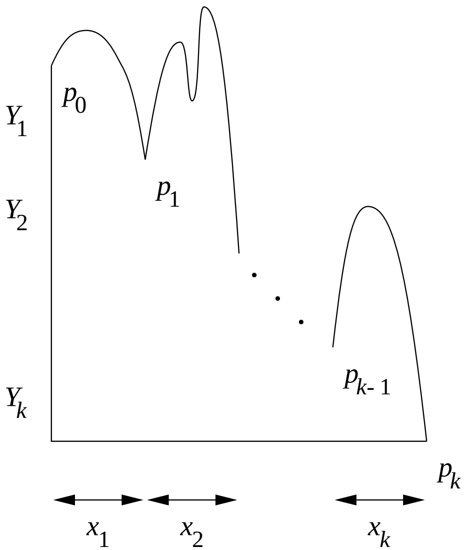
$$Y_i \overset{*}{\Rightarrow} x_i$$

If Y_i is a terminal, we have $|x_i| = 1$, and $Y_i = x_i$. Thus $Y_i \overset{*}{\Rightarrow} x_i$ by the reflexivity of $\overset{*}{\Rightarrow}$.

The claim of the theorem now follows by choosing $A = S$, and $x = w$. Suppose $w \in N(P)$. Then $(q, w, S) \overset{*}{\vdash} (q, \epsilon, \epsilon)$, and by ($\clubsuit$), we have $S \overset{*}{\Rightarrow} w$, meaning $w \in L(G)$.

From PDA's to CFG's

Let's look at how a PDA can consume $x = x_1 x_2 \cdots x_k$ and empty the stack.

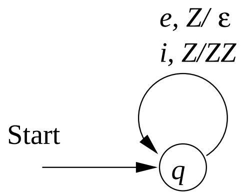


We shall define a grammar with variables of the form $[p_{i-1} Y_i p_i]$ representing going from p_{i-1} to p_i with net effect of popping Y_i.

Formally, let $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0)$ be a PDA. Define $G = (V, \Sigma, R, S)$, where

$$
\begin{aligned}
V &= \{[pXq] : \{p, q\} \subseteq Q, X \in \Gamma\} \cup \{S\} \\
R &= \{S \to [q_0 Z_0 p] : p \in Q\} \cup \\
&\quad \{[\mathbf{q} X r_k] \to a[\mathbf{r} Y_1 r_1] \cdots [r_{k-1} Y_k r_k] : \\
&\qquad\qquad a \in \Sigma \cup \{\epsilon\}, \\
&\qquad\qquad \{r_1, \ldots, r_k\} \subseteq Q, \\
&\qquad\qquad (\mathbf{r}, Y_1 Y_2 \cdots Y_k) \in \delta(\mathbf{q}, a, X)\}
\end{aligned}
$$

Example: Let's convert



$$P_N = (\{q\}, \{i, e\}, \{Z\}, \delta_N, q, Z),$$

where $\delta_N(q, i, Z) = \{(q, ZZ)\}$,
and $\delta_N(q, e, Z) = \{(q, \epsilon)\}$ to a grammar

$$G = (V, \{i, e\}, R, S),$$

where $V = \{[qZq], S\}$, and
$R = \{[qZq] \to i[qZq][qZq], [qZq] \to e\}$.

If we replace $[qZq]$ by A we get the productions
$S \to A$ and $A \to iAA|e$.

Example: Let $P = (\{p, q\}, \{0, 1\}, \{X, Z_0\}, \delta, q, Z_0)$, where δ is given by

1. $\delta(q, 1, Z_0) = \{(q, XZ_0)\}$
2. $\delta(q, 1, X) = \{(q, XX)\}$
3. $\delta(q, 0, X) = \{(p, X)\}$
4. $\delta(q, \epsilon, X) = \{(q, \epsilon)\}$
5. $\delta(p, 1, X) = \{(p, \epsilon)\}$
6. $\delta(p, 0, Z_0) = \{(q, Z_0)\}$

to a CFG.

We get $G = (V, \{0, 1\}, R, S)$, where

$$V = \{[pXp], [pXq], [pZ_0p], [pZ_0q], S\}$$

and the productions in R are

$S \to [qZ_0q] | [qZ_0p]$

From rule (1):

$[qZ_0q] \to 1[qXq][qZ_0q]$
$[qZ_0q] \to 1[qXp][pZ_0q]$
$[qZ_0p] \to 1[qXq][qZ_0p]$
$[qZ_0p] \to 1[qXp][pZ_0p]$

From rule (2):

$[qXq] \to 1[qXq][qXq]$
$[qXq] \to 1[qXp][pXq]$
$[qXp] \to 1[qXq][qXp]$
$[qXp] \to 1[qXp][pXp]$

From rule (3):

$[qXq] \rightarrow 0[pXq]$
$[qXp] \rightarrow 0[pXp]$

From rule (4):

$[qXq] \rightarrow \epsilon$

From rule (5):

$[pXp] \rightarrow 1$

From rule (6):

$[pZ_0q] \rightarrow 0[qZ_0q]$
$[pZ_0p] \rightarrow 0[qZ_0p]$

Theorem 6.14: Let G be constructed from a PDA P as above. Then $L(G) = N(P)$

Proof:

($\supseteq$-*direction.*) We shall show by an induction on the length of the sequence $\overset{*}{\vdash}$ that

(♠) If $(q, w, X) \overset{*}{\vdash} (p, \epsilon, \epsilon)$ then $[qXp] \overset{*}{\Rightarrow} w$.

Basis: Length 1. Then w is an a or ϵ, and $(p, \epsilon) \in \delta(q, w, X)$. By the construction of G we have $[qXp] \to w$ and thus $[qXp] \overset{*}{\Rightarrow} w$.

Induction: Length is $n > 1$, and ♠ holds for lengths $< n$. We must have

$$(q, w, X) \vdash (r_0, x, Y_1 Y_2 \cdots Y_k) \vdash \cdots \vdash (p, \epsilon, \epsilon),$$

where $w = ax$ or $w = \epsilon x$. It follows that $(r_0, Y_1 Y_2 \cdots Y_k) \in \delta(q, a, X)$. Then we have a production

$$[qXr_k] \to a[r_0 Y_1 r_1] \cdots [r_{k-1} Y_k r_k],$$

for all $\{r_1, \ldots, r_k\} \subset Q$.

We may now choose r_i to be the state in the sequence $\overset{*}{\vdash}$ when Y_i is popped. Let $w = w_1 w_2 \cdots w_k$, where w_i is consumed while Y_i is popped. Then

$$(r_{i-1}, w_i, Y_i) \overset{*}{\vdash} (r_i, \epsilon, \epsilon).$$

By the IH we get

$$[r_{i-1}, Y, r_i] \overset{*}{\Rightarrow} w_i$$

We then get the following derivation sequence:

$$
\begin{gathered}
[qXr_k] \Rightarrow a[r_0Y_1r_1]\cdots[r_{k-1}Y_kr_k] \stackrel{*}{\Rightarrow} \\
aw_1[r_1Y_2r_2][r_2Y_3r_3]\cdots[r_{k-1}Y_kr_k] \stackrel{*}{\Rightarrow} \\
aw_1w_2[r_2Y_3r_3]\cdots[r_{k-1}Y_kr_k] \stackrel{*}{\Rightarrow} \\
\cdots \\
aw_1w_2\cdots w_k = w
\end{gathered}
$$

($\supseteq$-*direction.*) We shall show by an induction on the length of the derivation $\stackrel{*}{\Rightarrow}$ that

($\heartsuit$) If $[qXp] \stackrel{*}{\Rightarrow} w$ then $(q, w, X) \stackrel{*}{\vdash} (p, \epsilon, \epsilon)$

Basis: One step. Then we have a production $[qXp] \rightarrow w$. From the construction of G it follows that $(p, \epsilon) \in \delta(q, a, X)$, where $w = a$. But then $(q, w, X) \stackrel{*}{\vdash} (p, \epsilon, \epsilon)$.

Induction: Length of $\stackrel{*}{\Rightarrow}$ is $n > 1$, and $\heartsuit$ holds for lengths $< n$. Then we must have

$$[qXr_k] \Rightarrow a[r_0Y_1r_1][r_1Y_2r_2] \cdots [r_{k-1}Y_kr_k] \stackrel{*}{\Rightarrow} w$$

We can break w into $aw_2 \cdots w_k$ such that $[r_{i-1}Y_ir_i] \stackrel{*}{\Rightarrow} w_i$. From the IH we get

$$(r_{i-1}, w_i, Y_i) \stackrel{*}{\vdash} (r_i, \epsilon, \epsilon)$$

From Theorem 6.5 we get
$(r_{i-1}, w_i w_{i+1} \cdots w_k, Y_i Y_{i+1} \cdots Y_k) \overset{*}{\vdash}$
$(r_i, w_{i+1} \cdots w_k, Y_{i+1} \cdots Y_k)$

Since this holds for all $i \in \{1, \ldots, k\}$, we get
$(q, aw_1 w_2 \cdots w_k, X) \vdash$
$(r_0, w_1 w_2 \cdots w_k, Y_1 Y_2 \cdots Y_k) \overset{*}{\vdash}$
$(r_1, w_2 \cdots w_k, Y_2 \cdots Y_k) \overset{*}{\vdash}$
$(r_2, w_3 \cdots w_k, Y_3 \cdots Y_k) \overset{*}{\vdash}$
(p, ϵ, ϵ).

Deterministic PDA's

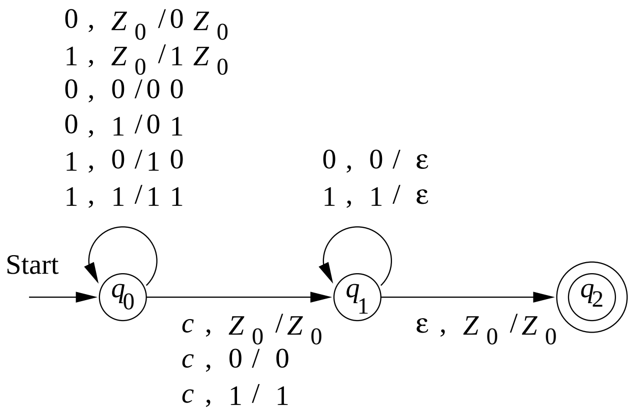
A PDA $P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ is *deterministic* iff

1. $\delta(q, a, X)$ is always empty or a singleton.
2. If $\delta(q, a, X)$ is nonempty, then $\delta(q, \epsilon, X)$ must be empty.

Example: Let us define

$$L_{wcwr} = \{wcw^R : w \in \{0, 1\}^*\}$$

Then L_{wcwr} is recognized by the following DPDA



We'll show that Regular$\subset L$(DPDA) $\subset$ CFL

Theorem 6.17: If L is regular, then $L = L(P)$ for some DPDA P.

Proof: Since L is regular there is a DFA A s.t. $L = L(A)$. Let

$$A = (Q, \Sigma, \delta_A, q_0, F)$$

We define the DPDA

$$P = (Q, \Sigma, \{Z_0\}, \delta_P, q_0, Z_0, F),$$

where

$$\delta_P(q, a, Z_0) = \{(\delta_A(q, a), Z_0)\},$$

for all $p, q \in Q$, and $a \in \Sigma$.

An easy induction (do it!) on $|w|$ gives

$$(q_0, w, Z_0) \overset{*}{\vdash} (p, \epsilon, Z_0) \Leftrightarrow \widehat{\delta_A}(q_0, w) = p$$

The theorem then follows (why?)

What about DPDA's that accept by null stack?

They can recognize only CFL's with the prefix property.

A language L has the *prefix property* if there are no two distinct strings in L, such that one is a prefix of the other.

Example: L_{wcwr} has the prefix property.

Example: $\{0\}^*$ does not have the prefix property.

Theorem 6.19: L is $N(P)$ for some DPDA P if and only if L has the prefix property and L is $L(P')$ for some DPDA P'.

Proof: Homework

- We have seen that Regular$\subseteq L$(DPDA).

- $L_{wcwr} \in L$(DPDA)$\setminus$ Regular

- Are there languages in CFL$\setminus L$(DPDA).

  Yes, for example L_{wwr}.

- What about DPDA's and Ambiguous Grammars?

  L_{wwr} has unamb. grammar $S \rightarrow 0S0|1S1|\epsilon$
  but is not L(DPDA).

For the converse we have

Theorem 6.20: If $L = N(P)$ for some DPDA P, then L has an unambiguous CFG.

Proof: By inspecting the proof of Theorem 6.14 we see that if the construction is applied to a DPDA the result is a CFG with unique leftmost derivations.

Theorem 6.20 can actually be strengthen as follows

Theorem 6.21: If $L = L(P)$ for some DPDA P, then L has an unambiguous CFG.

Proof: Let \$ be a symbol outside the alphabet of L, and let $L' = L\$$.
It is easy to see that L' has the prefix property.
By Theorem 6.20 we have $L' = N(P')$ for some DPDA P'.
By Theorem 6.20 $N(P')$ can be generated by an unambiguous CFG G'
Modify G' into G, s.t. $L(G) = L$, by adding the production

$$\$ \rightarrow \epsilon$$

Since G' has unique leftmost derivations, G' also has unique lm's, since the only new thing we're doing is adding derivations

$$w\$ \underset{lm}{\Rightarrow} w$$

to the end.